Supplementary Material: Bulk-edge correspondence and long range hopping in the topological plasmonic chain

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More bulk boundary breakdown plots

Behaviour of $\beta < 1$ systems

Figure S1 shows $|E|$ for $N = 600$ particle finite plasmonic chains with (a) $\beta = 1.4$ and (b) $\beta = 0.6$ with changing $k_{sp}d$. Zak phase transitions occur for the chain at $\beta = 1$ and at values symmetric around $\beta = 1$, so that symmetrically either side of $\beta = 1$ the Zak phase is exactly opposite. Comparing the chiral (red) case for (a) and (b) we see that topologically protected edge modes exist in the appropriate regions given the changing Zak phase. The non-chiral (blue) full dipolar case highlights that although BBC breakdown leads to the disappearance of edge modes in the $\beta > 1$ case (a), it does not lead to the appearance of edge modes for the opposite $\beta < 1$ in (b). In addition to this we see that for $\beta = 1.4$ the movement of the bulk dominates BEC breakdown in (a), where the edge modes are nearly zero until they enter the bulk for the first time.

As mentioned in the main text, we observe in the (blue) full dipolar case the presence of modes outside the bulk in the $\beta = 1.4$ case (which do not exist for $\beta = 0.6$) for approximately $k_{sp}d/\pi > 0.9$. As discussed in the main text these modes and are localised to the edges of the chain but their existence does not agree with the Zak phase, they are far from $|E| = 0$, and do not appear to be well protected from disorder. As such we do not label these as topologically protected edge states and consider the system to still have broken bulk-edge correspondence. Although it appears that these are approaching the $|E| = 0$ point in this plot these modes actually move back up into the bulk as $k_{sp}d/\pi$ increases. This return and loss of modes outside of the band which do not correspond to the Zak phase continues to happen as $k_{sp}d/\pi$ increases.

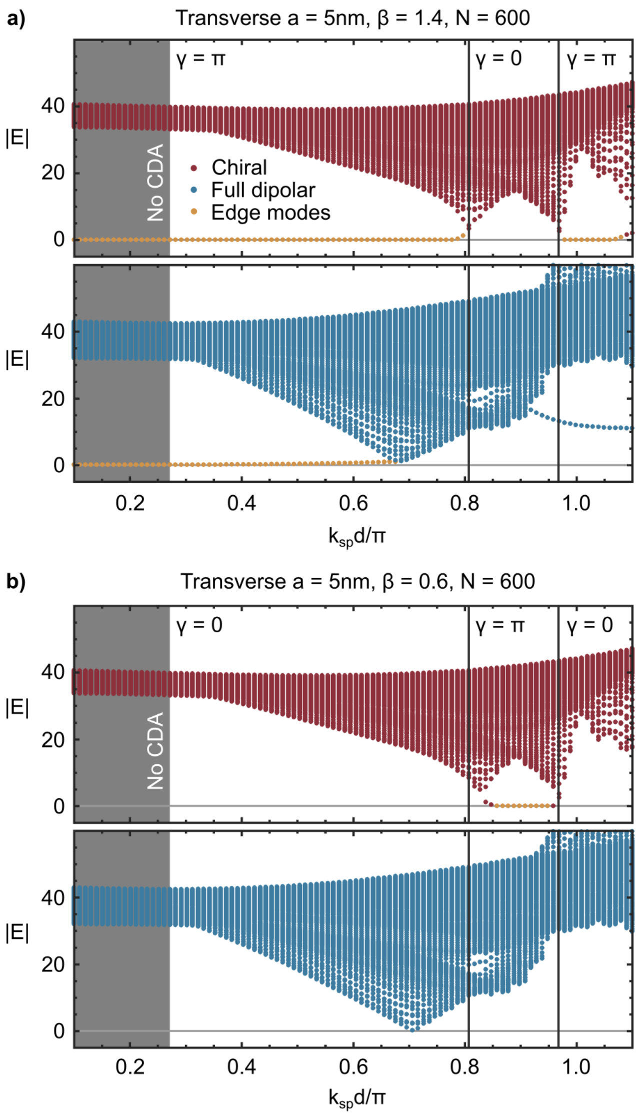

Figure S1: Eigenvalues of the chiral (red) and full dipolar (blue) topological plasmonic chain with changing $k_{sp}d$ for the transverse polarisation with (a) $\beta = 1.4$ and (b) $beta = 0.6$. Topologically protected edge modes are yellow. The dark grey area indicates the region where the CDA is not valid as the particles are too closely spaced. Vertical black lines indicate Zak phase transitions as predicted by the closing of the bulk gap.

Real and imaginary parts

Figure S2 shows the real and imaginary parts of the frequency ω for figure S1(a), similar to 3(b) in the main text but for $\beta = 1.4$, using $E = d^3/\alpha(\omega)$. In the non-chiral (blue) case edge modes disappear as the bulk crosses their path, as shown in the $|E|$ plots featured

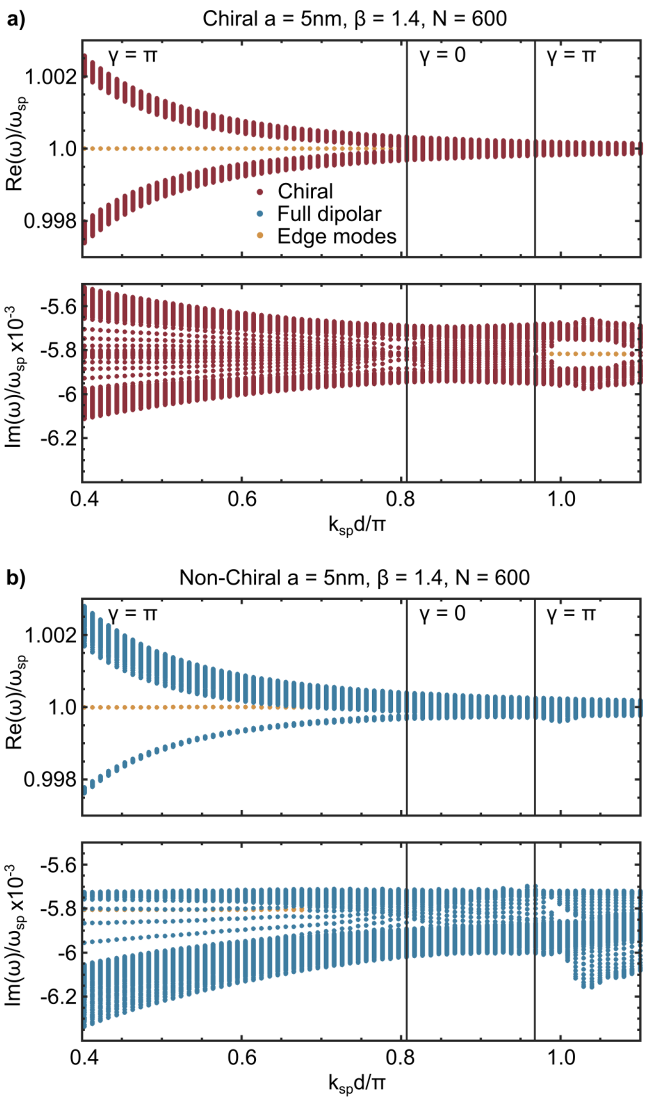

Figure S2: Frequency values of the chiral (red) and full dipolar (blue) topological plasmonic chain with changing $k_{sp}d$ for the transverse polarisations. Topologically protected edge modes are yellow. Vertical black lines indicate Zak phase transitions as predicted by the closing of the bulk gap.

in the main text.

Figure S3 shows the real and imaginary parts of the eigenvalues for figure 4(b) and (c) in the main text. For the real part in Figure S3(b) the bulk obscures the edge modes, but BEC correspondence doesn't break down until the edge modes disappear in the imaginary part plot.

Perturbation theory for NNN edge state energies

We start with a finite SSH model with complex hoppings v and w with symmetric but non-Hermitian Hamiltonian of the form,

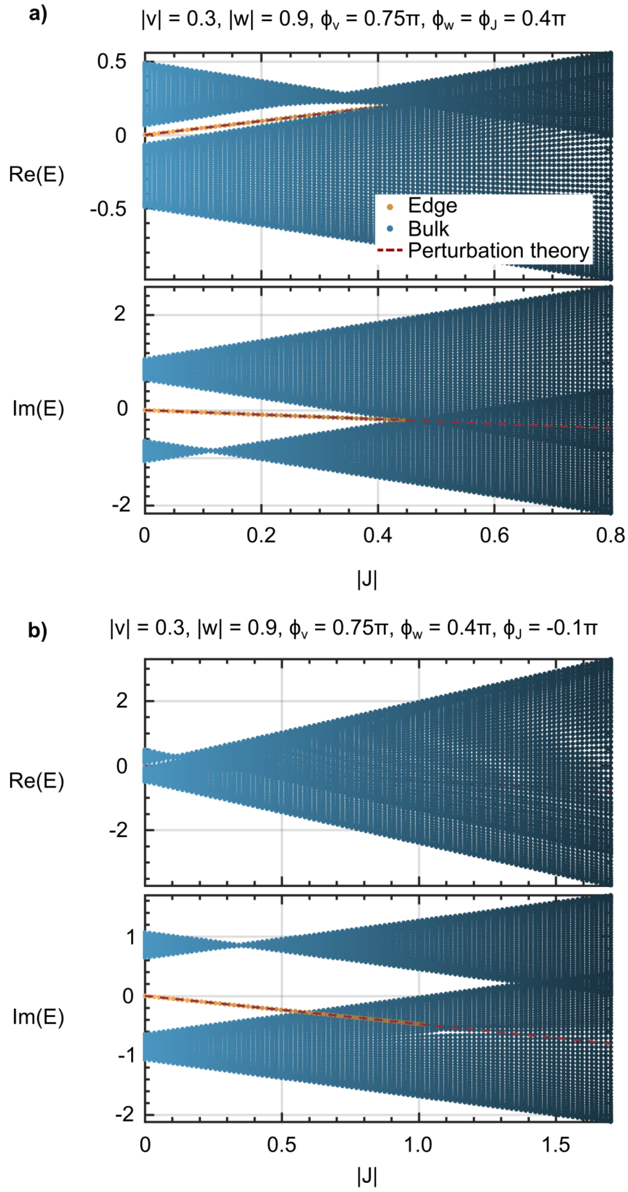

Figure S3: (a) and (b) Bulk (blue) and edge mode (yellow) eigenvalues of the non-Hermitian NNN SSH model for changing values of $|J|$, for different choices of hopping parameters and phases corresponding to figures 4(b) and (c) respectively in the main text.

$$\begin{aligned}\mathcal{H}^0 =& v\sum_{n=1}^{N}\left[|n,A\rangle\langle n,B| + H.c.\right] \\ &+ w\sum_{n=1}^{N-1}\left[|n+1,A\rangle\langle n,B| + H.c.\right]. \end{aligned} \tag{1}$$

We treat the addition of next nearest neighbour hoppings as a perturbation by an operator $\mathcal{J}$, which features A to A and B to B hoppings and is of the form,

$$\mathcal{J} = J\sum_{n=1}^{N-1}\left[|n+1,A\rangle\langle n,A| + |n+1,B\rangle\langle n,B| + H.c.\right]. \tag{2}$$

We wish to find what happens to the zero energy edge modes of $\mathcal{H}^0$ as we increase the strength of the next nearest neighbour hopping, by examining the matrix $\mathcal{H}^1 = \mathcal{H}^0 + \mathcal{J}$, where we treat J as the small parameter which turns on the next nearest neighbour hopping. Since $\mathcal{H}^0$ is symmetric its right and left eigenvectors are related simply,

$$\mathcal{H}^0 \mathbf{v}_n^0 = E_n^0 \mathbf{v}_n^0, \tag{3}$$

$$(\mathbf{v}_n^0)^T \mathcal{H}^0 = E_n^0 (\mathbf{v}_n^0)^T, \tag{4}$$

with the orthogonality condition that $(\mathbf{v}_m^0)^T \mathbf{v}_n^0 = 0$ if $m \neq n$. We are interested in looking at the edge modes which have $E^0 = 0$, so we can work through the usual perturbation theory using the symmetry properties to show that, to first order, the energy of the perturbed edge modes will be given by

$$E^1 = \frac{(\mathbf{v}^0)^T \mathcal{J} \mathbf{v}^0}{(\mathbf{v}^0)^T \mathbf{v}^0}. \tag{5}$$

According to section 1.5.6 of *A Short Course On Topological Insulators* [1], in the thermodynamic limit the left and right edge modes of the unperturbed SSH chain are approximately given by the following:

$$|L\rangle = \sum_{m=1}^{N} a_m |m, A\rangle, \qquad |R\rangle = \sum_{m=1}^{N} b_m |m, B\rangle, \tag{6}$$

where a_m and b_m are given by

$$a_m = a_1 \left(\frac{-v}{w}\right)^{m-1} \tag{7}$$

$$b_m = b_N \left(\frac{-v}{w}\right)^{N-m} \qquad \forall m \in \{1, \ldots, N\}, \tag{8}$$

and a_1 and b_N fix normalisation.

First we calculate the numerator of equation 5

$$|L\rangle^T \mathcal{J} |L\rangle = 2J \sum_{m=1}^{N-1} a_m a_{m+1}, \tag{9}$$

and use equation 7 to see that

$$|L\rangle^T \mathcal{J} |L\rangle = -2J \frac{v}{w} a_1^2 \sum_{m=1}^{N-1} \left(\frac{v}{w}\right)^{2m-2}. \tag{10}$$

Next we calculate the denominator of equation 5. For the left eigenvalues this is given by

$$|L\rangle^T |L\rangle = \sum_{m=1}^{N} a_m^2, \tag{11}$$

$$= a_1^2 \sum_{m=1}^{N} \left(\frac{v}{m}\right)^{2m-2}, \tag{12}$$

So the energy of the left edge mode for small next nearest neighbour hopping is given by

$$E_L = -2J \frac{v}{w} \frac{\sum_{m=1}^{N-1} \left(\frac{v}{m}\right)^{2m-2}}{\sum_{m=1}^{N} \left(\frac{v}{m}\right)^{2m-2}} \tag{13}$$

In the thermodynamic limit $N \to \infty$, given that $|v| < |w|$ (which is required for edge modes to exist), this converges to

$$E_L = -2J \frac{v}{w}, \tag{14}$$

as predicted for the Hermitian case by numerical fit [2].

For completeness we perform the same calculation for the right edge mode. We have, for the numerator,

$$|R\rangle^T \mathcal{J} |R\rangle = 2J \sum_{m=1}^{N-1} b_m b_{m+1}, \tag{15}$$

which when combined with equation 8 has

$$|R\rangle^T \mathcal{J} |R\rangle = 2J b_N^2 \sum_{m=1}^{N-1} \left(\frac{-v}{w}\right)^{2N-2m-1} \tag{16}$$

Next we rearrange the sum by making the substitution $j = N - m$, at which point it becomes clear that the calculation is the same as for the left edge modes,

$$|R\rangle^T \mathcal{J} |R\rangle = -2J \frac{v}{w} b_N^2 \sum_{m=1}^{N-1} \left(\frac{v}{w}\right)^{2m-2}. \tag{17}$$

For the right edge modes the denominator of equation 5. is given by

$$|R\rangle^T |R\rangle = \sum_{m=1}^{N} b_m^2, \tag{18}$$

$$= b_N^2 \sum_{m=1}^{N} \left(\frac{v}{m}\right)^{2N-2m}, \tag{19}$$

$$= b_N^2 \sum_{m=1}^{N} \left(\frac{v}{m}\right)^{2m-2}, \tag{20}$$

$$\tag{21}$$

As with the left eigenmode the energy for small next nearest neighbour hopping is given by

$$E_R = -2J \frac{v}{w} \frac{\sum_{m=1}^{N-1} \left(\frac{v}{m}\right)^{2m-2}}{\sum_{m=1}^{N} \left(\frac{v}{m}\right)^{2m-2}} \tag{22}$$

Again, in the thermodynamic limit $N \to \infty$, given that $|v| < |w|$ (which is required for edge modes to exist), this converges to

$$E_R = -2J \frac{v}{w}, \tag{23}$$

so as expected for this symmetric Hamiltonian both edge modes have the same path in complex space as the next nearest neighbour hopping J changes.

References

[1] János K. Asbóth, László Oroszlány, and András Pályi. *A Short Course on Topological Insulators*. Springer International Publishing, Switzerland, 2016.

[2] Beatriz Pérez-González, Miguel Bello, Álvaro Gómez-León, and Gloria Platero. Ssh model with long-range hoppings: topology, driving and disorder, 2018. arXiv:1802.03973.