Dynamic Factor Models with Infinite-Dimensional Factor Space: Asymptotic Analysis

Mario Forni[*]

Università di Modena e Reggio Emilia, CEPR and RECent

Marc Hallin[†‡]

ECARES, Université Libre de Bruxelles and ORFE, Princeton University

Marco Lippi[§]

Università di Roma La Sapienza and EIEF

Paolo Zaffaroni[¶]

Imperial College London and Università di Roma La Sapienza

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Abstract. Factor models, all particular cases of the Generalized Dynamic Factor Model (GDFM) introduced in Forni, Hallin, Lippi and Reichlin (2000), have become extremely popular in the theory and practice of large panels of time series data. The asymptotic properties (consistency and rates) of the corresponding estimators have been studied in Forni, Hallin, Lippi and Reichlin (2004). Those estimators, however, rely on Brillinger's *dynamic* principal components, and thus involve two-sided filters, which leads to rather poort forecasting performances. No such problem arises with estimators based on standard *(static)* principal components, which have been dominant in this literature. On the other hand, the consistency of those *static* estimators requires the assumption that the space spanned by the factors has finite dimension, which severely restricts the generality afforded by the GDFM. This paper derives the asymptotic properties of a semiparametric estimator of the loadings and common shocks based on one-sided filters recently proposed by Forni, Hallin, Lippi and Zaffaroni (2015). Consistency and exact rates of convergence are obtained for this estimator, under a general class of GDFMs that does not require a finite-dimensional factor space. A Monte Carlo experiment corroborates those theoretical results and demonstrates the excellent performance of those estimators in out-of-sample forecasting.

1 Introduction

In the present paper, we provide consistency results and consistency rates for the estimators recently proposed by Forni, Hallin, Lippi and Zaffaroni (2015) (hereafter, FHLZ) for the *Generalized Dynamic Factor Model* (GDFM).

A GDFM, as introduced in Forni *et al.* (2000) and Forni and Lippi (2001), is a countably infinite set of observable stochastic processes x_{it} admitting a decomposition of the form

$$x_{it} = \chi_{it} + \xi_{it} = b_{i1}(L)u_{1t} + b_{i2}(L)u_{2t} + \cdots + b_{iq}(L)u_{qt} + \xi_{it}, \quad i \in \mathbb{N},\ t \in \mathbb{Z}, \qquad (1.1)$$

where $\mathbf{u}_t = (u_{1t}\ u_{2t}\ \cdots\ u_{qt})'$ is unobservable q-dimensional orthonormal white noise and the filters $b_{if}(L)$, $i \in \mathbb{N}$, $f = 1, \ldots, q$, are square-summable (L, as usual, stands for the lag oper-

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ator): χ_{it} and ξ_{it} are called the *common* and *idiosyncratic components*, respectively. Detailed assumptions on (1.1) are given below. Let us only recall here that the idiosyncratic components ξ_{it} and the *common shocks* u_{ft} are mutually orthogonal at any lead and lag, and that the idiosyncratic components are "weakly" cross-correlated (cross-sectional orthogonality being an extreme case).

Much of the literature on Dynamic Factor Models is based on (1.1) under the assumption that the space spanned by the stochastic variables χ_{it}, for t given and $i \in \mathbb{N}$, is *finite-dimensional*.[1] Under that assumption, model (1.1) can be rewritten in the so-called *static representation*

$$
\begin{aligned}
x_{it} &= \lambda_{i1}F_{1t} + \lambda_{i2}F_{2t} + \cdots + \lambda_{ir}F_{rt} + \xi_{it} \\
\mathbf{F}_t &= (F_{1t} \; \ldots \; F_{rt})' = \mathbf{N}(L)\mathbf{u}_t.
\end{aligned}
\tag{1.2}
$$

Criteria to determine r consistently have been given in Bai and Ng (2002) and, more recently, in Alessi et al. (2010), Onatski (2010), and Ahn and Horenstein (2013). The vectors $\mathbf{F}_t$ and the loadings λ_{ij} can be estimated consistently using the first r standard principal components, see Stock and Watson (2002a,b), Bai and Ng (2002). Moreover, the second equation in (1.2) is usually specified as a possibly singular VAR, so that (1.2) becomes

$$
\begin{aligned}
x_{it} = \lambda_{i1}F_{1t} + \lambda_{i2}F_{2t} + \cdots + \lambda_{ir}F_{rt} \;&+\; \xi_{it} \\
\mathbf{D}(L)\mathbf{F}_t = (\mathbf{I} - \mathbf{D}_1 L - \mathbf{D}_2 L^2 - \ldots - \mathbf{D}_p L^p)\mathbf{F}_t \;&=\; \mathbf{K}\mathbf{u}_t,
\end{aligned}
\tag{1.3}
$$

where the matrices $\mathbf{D}_j$ are $r \times r$ while $\mathbf{K}$ is $r \times q$, $r \geq q$. Under (1.3), Bai and Ng (2007) and Amengual and Watson (2007) provide consistent criteria to determine q.

The assumption of a finite-dimensional factor space, however, is far from being innocuous. For instance, (1.2) is so restrictive that even the very elementary model

$$
x_{it} = a_i(1 - \alpha_i L)^{-1}u_t + \xi_{it},
\tag{1.4}
$$

where $q = 1$, u_t is scalar white noise, and the coefficients α_i are drawn from a uniform distribution over the stationary region, is ruled out. In this case, the space spanned, for given t, by the common components χ_{it}, $i \in \mathbb{N}$, is easily seen to be infinite-dimensional unless the α_i's take only a finite number of values.

The problem is that, in the absence of the finite-dimensionality assumption, estimation of model (1.1) cannot be based on a finite number r of standard principal components. That

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[1]The definition of χ_{it} obviously implies that this dimension does not depend on t.

situation is the one studied in Forni et al. (2000), who are using q principal components in the frequency domain (Brillinger's *dynamic principal components*; see Brillinger (1981)) to esti- mate the common components χ_{it}.[2] However, their estimators involve the application of two-sided filters acting on the observations x_{it}, and hence perform poorly at the end/beginning of the observation period. As a consequence, they are of little help for prediction.

In FHLZ, we show how one-sided estimators without the finite-dimensionality assumption can be obtained, under the additional condition that the common components have a *rational spectral density*, that is, each filter $b_{if}(L)$ in (1.1) is a ratio of polynomials in L:

$$\chi_{it} \;\; = \;\; \frac{c_{i1}(L)}{d_{i1}(L)}u_{1t} + \frac{c_{i2}(L)}{d_{i2}(L)}u_{2t} + \cdots + \frac{c_{iq}(L)}{d_{iq}(L)}u_{qt}, \quad i \in \mathbb{N}, \; f = 1, 2, \ldots, q, \quad (1.5)$$

where

$$c_{if}(L) = c_{if,0} + c_{if,1}L + \ldots + c_{if,s_1}L^{s_1} \quad \text{and} \quad d_{if}(L) = d_{if,0} + d_{if,1}L + \ldots + d_{if,s_2}L^{s_2}$$

(the degrees s_1 and s_2 of the polynomials are assumed to be independent of i and f for the sake of simplicity).

Denote by $\mathbf{x}_t$, $\boldsymbol{\chi}_t$, $\boldsymbol{\xi}_t$ the infinite-dimensional column vectors with components x_{it}, χ_{it}, ξ_{it}, respectively. Elaborating upon recent results by Anderson and Deistler (2008a, b), FHLZ prove that, for generic values of the parameters $c_{if,k}$ and $d_{if,k}$ (i.e. apart from a lower-dimensional subset in the parameter space, see FHLZ for details), the infinite-dimensional idiosyncratic vector $\boldsymbol{\chi}_t = (\chi_{1t}\ \chi_{2t}\ \cdots\ \chi_{nt}\ \cdots)'$ admits *a unique autoregressive representation with block structure* of the form

$$\begin{pmatrix} \mathbf{A}^1(L) & 0 & \cdots & 0 & \cdots \\ 0 & \mathbf{A}^2(L) & \cdots & 0 & \\ & & \ddots & & \\ 0 & 0 & \cdots & \mathbf{A}^k(L) & \\ \vdots & & & & \ddots \end{pmatrix} \boldsymbol{\chi}_t = \begin{pmatrix} \mathbf{R}^1 \\ \mathbf{R}^2 \\ \vdots \\ \mathbf{R}^k \\ \vdots \end{pmatrix} \mathbf{u}_t, \quad (1.6)$$

where $\mathbf{A}^k(L)$ is a $(q+1) \times (q+1)$ polynomial matrix *with finite degree* and $\mathbf{R}^k$ is $(q+1) \times q$. Denoting by $\underline{\mathbf{A}}(L)$ and $\underline{\mathbf{R}}$ the (infinite) matrices on the left- and right-hand sides of (1.6),

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[2]Criteria to determine q without assuming (1.2) or (1.3) are obtained in Hallin and Liška, 2007 and Onatski, 2009.

respectively, and letting $\mathbf{Z}_t = \underline{\mathbf{A}}(L)\mathbf{x}_t$, it follows that

$$\mathbf{Z}_t = \underline{\mathbf{R}}\mathbf{u}_t + \underline{\mathbf{A}}(L)\boldsymbol{\xi}_t. \tag{1.7}$$

Under the assumptions of the present paper, the term $\underline{\mathbf{A}}(L)\boldsymbol{\xi}_t$ is still idiosyncratic, so that (1.7) is a static representation of the form (1.3), with $\mathbf{D}(L) = \mathbf{I}$. That static representation can be estimated via traditional principal components, which does not require two-sided filters.

FHLZ thus obtain one-sided estimators for the common components without imposing the standard finite-dimension restriction. Moreover, the high-dimensional VAR (1.6) is obtained by piecing tothether the low-dimensional matrices $\mathbf{A}^k(L)$, each one depending only on the covariances of $q+1$ common components. Therefore, no curse of dimensionality occurs with the procedure. Estimation of the common components χ_{it}, the shocks $\mathbf{u}_t$ and the filters $b_{if}(L)$ is based on the sample analogues of representations (1.6) and (1.7):

(i) We start with a lag-window estimator of the spectral density matrix of the observed vector $\mathbf{x}_{nt} = (x_{1t} \; x_{2t} \; \cdots \; x_{nt})$, call it $\hat{\boldsymbol{\Sigma}}_n^x(\theta)$.

(ii) Using the first q frequency domain principal components of $\hat{\boldsymbol{\Sigma}}_n^x(\theta)$, we construct an estimator of the spectral density of $\boldsymbol{\chi}_{nt} = (\chi_{1t} \; \chi_{2t} \; \cdots \; \chi_{nt})$, call it $\hat{\boldsymbol{\Sigma}}_n^\chi(\theta)$. Estimators of the autocovariances of $\boldsymbol{\chi}_{nt}$ are then obtained from $\hat{\boldsymbol{\Sigma}}_n^\chi(\theta)$; call $\hat{\boldsymbol{\Gamma}}_{n,h}^\chi$ the estimator of the covariance between $\boldsymbol{\chi}_{nt}$ and $\boldsymbol{\chi}_{n,t-h}$. Those $\hat{\boldsymbol{\Gamma}}_{n,h}^\chi$'s are used, in a traditional, low-dimensional way, to construct the autoregressive estimators $\hat{\mathbf{A}}^k(L)$.

(iii) Blockwise estimators of the variables Z_{jt} are obtained by applying the finite-degree filters $\hat{\mathbf{A}}^k(L)$ to the observed variables x_{it}, while inverting the same $\hat{\mathbf{A}}^k(L)$'s provides estimators for the filters $b_{if}(L)$. Estimators for the shocks u_{ft} and the matrix $\mathbf{R}^k$ are obtained by using the first q traditional principal components of the variables Z_{it}.

Our consistency results for the estimators described in (ii) and (iii) above are based on recent results on lag-window spectral estimators in Shao and Wu (2007) and Liu and Wu (2010), as extended to the multivariate case by Wu and Zaffaroni (2014). Starting with the observable time series x_{it}, denoting by T the number of observations for each series and by $\hat{\sigma}_{ij}(\theta)$ a lag-window estimator of the cross-spectrum between x_{it} and x_{jt}, the (i, j)

entry of $\hat{\mathbf{\Sigma}}(\theta)$, under quite general assumptions on the processes x_{it}, x_{jt} and the kernel, these papers prove that $\hat{\sigma}_{ij}(\theta)$ is consistent, as $T \to \infty$, *uniformly* with respect to θ, with rate $\sqrt{B_T \log B_T / T}$, where B_T is the size of the lag window. As an important innovation with respect to the previous literature on spectral estimation, these results are obtained without assuming linearity or Gaussianity of the processes x_{it}.

Using of those results here, however, requires some enhancement of the FHLZ assumptions on the common shocks and the idiosyncratic components. In particular, the vector $\mathbf{u}_t$, which is second-order white noise in FHLZ, is i.i.d. here. This, as well as some other changes in the FHLZ assumptions, is discussed in detail in Section 2. Under this enhanced set of assumptions, we prove that the estimators $\hat{\mathbf{\Sigma}}^\chi(\theta)$, $\hat{\mathbf{\Gamma}}^\chi$ and $\hat{\mathbf{A}}^k(L)$ are consistent with rate

$$\zeta_{nT} = \max \left(\sqrt{n^{-1}}, \sqrt{T^{-1} B_T \log B_T} \right), \tag{1.8}$$

where B_T diverges as T^δ, with $1/3 < \delta < 1$. Establishing those rates, raises some nontrivial difficulties. Although model (1.7) is finite-dimensional, indeed, the series Z_{it} are estimated, not observed. As a consequence, the well-known results from static-factor literature (Stock and Watson, 2002a and b, Bai and Ng, 2002) do not readily apply, and proving that consistency holds with the same rates ζ_{nT} as if Z_{it} were observed requires non negligible efforts.

As pointed out in FHLZ (end of Section 4.5) despite the fact that the dynamic model studied in this paper is more general than model (1.3), when a dataset is given, with finite n and T, the static approach might perform well even tough the required finite-dimension assumptions are not satisfied. A Monte Carlo study is provided in Section 4, in which the static and dynamic methods have been applied to simulated data. A very short summary of our results is that (i) when the data are generated by infinite-dimensional models which are simple generalizations of (1.4), the estimation of impulse-response functions and predictions via the dynamic method is by far better than those obtained via the static one; (ii) even when the data are generated by (1.3), still the dynamic method performs slightly better. Though not conclusive, our Monte Carlo results strongly suggest that the model proposed in the present paper may be uniformly competitive.

The paper is organized as follows. In Sections 2, we present and comment the main assumptions to be made throughout. Section 3 provides the main asymptotic results. Section 4 gives a detailed description of the Monte Carlo experiments, and their analysis, and Section 5

concludes. Short proofs are given in the body of the paper, the longer ones in the Appendix.

2 Main assumptions and some preliminary results

2.1 Common and idiosyncratic components

The Dynamic Factor Model studied in the present paper is a decomposition, of the form

$$x_{it} = \chi_{it} + \xi_{it}, \quad i \in \mathbb{N},\ t \in \mathbb{Z}$$

of an observed variable x_{it} into a nonobserved *common* component χ_{it} and a nonobserved *idiosyncratic* component ξ_{it}. Throughout, we are assuming that the family of stochastic variables

$$\{x_{it},\ \chi_{it},\ \xi_{it},\ i \in \mathbb{N},\ t \in \mathbb{Z}\},$$

fulfills the assumptions listed below as Assumptions 1 through 10.

Assumption 1 *There exist a natural number $q > 0$ and*

(1) a q-dimensional stochastic zero-mean process $\mathbf{u}_t = (u_{1t}\ u_{2t}\ \cdots\ u_{qt})'$, $t \in \mathbb{Z}$, and an infinite-dimensional stochastic process $\boldsymbol{\eta}_t = (\eta_{1t}\ \eta_{2t}\ \cdots)'$, $t \in \mathbb{Z}$;

(2) square-summable filters $b_{if}(L)$, $i \in \mathbb{N}$, $f = 1, \ldots, q$;

(3) coefficients $\beta_{ij,k}$, for $i, j \in \mathbb{N}$, $k = 0, 1, \ldots, \infty$, where $\sum_{j=1}^{\infty} \sum_{k=0}^{\infty} \beta_{ij,k}^2 < \infty$ for all $i \in \mathbb{N}$;

such that

(i) the vectors $\mathbf{S}_t = (\mathbf{u}_t'\ \boldsymbol{\eta}_t')'$, $t \in \mathbb{Z}$, are i.i.d. and orthonormal, i.e. $E(\mathbf{S}_t \mathbf{S}_t') = \mathbf{I}_\infty$; in particular, $\mathrm{cov}(u_{ft}, \eta_{j,t-k}) = 0$, $f = 1, \ldots, q$, $j \in \mathbb{N}$, $k = 0, 1, \ldots, \infty$;

(ii)
$$\chi_{it} = b_{i1}(L)u_{1t} + b_{i2}(L)u_{2t} + \cdots + b_{iq}(L)u_{qt}$$
$$\xi_{it} = \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} \beta_{ij,k} \eta_{j,t-k}. \tag{2.1}$$

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Clearly, neither $\mathbf{u}_t$ nor the polynomials $b_{if}(L)$ are identified. Indeed, rewriting the first equation in (2.1) as $\chi_{it} = \mathbf{b}_i(L)\mathbf{u}_t$, for any orthogonal matrix $\mathbf{Q}$, the common component χ_{it} has the alternative representation $\chi_{it} = \left[\mathbf{b}_i(L)\mathbf{Q}^{-1}\right]\left[\mathbf{Q}\mathbf{u}_t\right] = \mathbf{b}_i^*(L)\mathbf{u}_t^*$. Note that (i) and (2.1) imply $\mathrm{cov}(u_{ft}, \xi_{i,t-k}) = 0$ for all f, i, k.

Two differences with respect to FHLZ must be pointed out. Firstly, here $\mathbf{u}_t$ is i.i.d., not just second-order white noise as in FHLZ. Secondly, unlike in FHLZ, the idiosyncratic components are modeled as (infinite-order) moving averages of the infinite-dimensional i.i.d. vector $\boldsymbol{\eta}_t$.

Assumption 2 *Conditions on the filters $b_{if}(L)$.*
(i) The filters $b_{if}(L)$ are rational. More precisely, there exist natural numbers s_1, s_2 such that $b_{if}(L) = c_{if}(L)/d_{if}(L)$, where

$$c_{if}(L) = c_{if,0} + c_{if,1}L + \cdots + c_{if,s_1}L^{s_1} \quad and \quad d_{if}(L) = 1 + d_{if,1}L + \cdots + d_{if,s_2}L^{s_2}, \quad (2.2)$$

for $i \in \mathbb{N}$, $f = 1, \ldots, q$.
(ii) There exists $\phi > 1$ such that none of the roots of $d_{if}(L)$ is less than ϕ in modulus, for $i \in \mathbb{N}$, $f = 1, \ldots, q$.
(iii) There exists B^χ, $0 < B^\chi < \infty$, such that $|c_{if,j}| \leq B^\chi$, $i \in \mathbb{N}$, $f = 1, \ldots, q$, $j = 0, \ldots, s_1$.

Under Assumption 2, the vector $\boldsymbol{\chi}_{nt} = (\chi_{1t} \ \chi_{2t} \ \cdots \ \chi_{nt})'$ has a rational spectral density matrix $\boldsymbol{\Sigma}_n^\chi(\theta)$; denote by $\lambda_{nj}^\chi(\theta)$ its j-th eigenvalue (in decreasing order).

Assumption 3 *Common component spectral density eigenvalues: divergence and separation.*
There exist continuous functions

$$\alpha_f^\chi(\theta), \quad f = 1, \ldots, q \quad and \quad \beta_f^\chi(\theta), \quad f = 0, \ldots, q-1, \quad \theta \in [-\pi, \ \pi],$$

and a positive integer n^χ such that, for $n > n^\chi$,

$$\beta_0^\chi(\theta) \geq \frac{\lambda_{n1}^\chi(\theta)}{n} \geq \alpha_1^\chi(\theta) > \beta_1^\chi(\theta) \geq \frac{\lambda_{n2}^\chi(\theta)}{n} \geq \cdots \geq \alpha_{q-1}^\chi(\theta) > \beta_{q-1}^\chi(\theta) \geq \frac{\lambda_{nq}^\chi(\theta)}{n} \geq \alpha_q^\chi(\theta) > 0,$$

for all $\theta \in [-\pi, \ \pi]$.

Assumption 3 is an enhancement of the standard assumption on the eigenvalues of common components. It will be used in our consistency proof: see, in particular, Lemma 3, Appendix B.

Assumption 4 *Serial dependence of idiosyncratic components.*

There exists finite positive numbers B, B_{is}, $i \in \mathbb{N}$, $s \in \mathbb{N}$, and ρ, $0 \leq \rho < 1$, such that

$$\sum_{s=1}^{\infty} B_{is} \leq B, \quad \text{for all } i \in \mathbb{N} \tag{2.3}$$

$$\sum_{i=1}^{\infty} B_{is} \leq B, \quad \text{for all } s \in \mathbb{N} \tag{2.4}$$

$$|\beta_{is,k}| \leq B_{is}\rho^k, \quad \text{for all } i, s \in \mathbb{N} \text{ and } k = 0, 1, \ldots \tag{2.5}$$

An immediate consequence of (2.3) and (2.4) is that

$$\sum_{i=1}^{\infty} \sum_{s=1}^{\infty} B_{is} B_{js} \leq B^2, \quad \text{for all } j \in \mathbb{N}. \tag{2.6}$$

Conditions (2.3) and (2.4) are quite obviously satisfied in the "purely idiosyncratic" case $\xi_{it} = \eta_{it}$, and for finite "cross-sectional moving averages" such as $\xi_{it} = \eta_{it} + \eta_{i+1,t}$. By condition (2.5), the time dependence of the variables ξ_{it} declines geometrically, at common rate ρ.

Under Assumption 4, setting $\beta_{is}(L) = \sum_{k=0}^{\infty} \beta_{is,k} L^k$ and $\xi_{it} = \sum_{s=1}^{\infty} \beta_{is}(L)\eta_{st}$, and denoting by $\imath$ the imaginary unit,

$$|\beta_{is}(e^{-\imath\theta})| = \left| \sum_{k=0}^{\infty} \beta_{is,k} e^{-\imath k\theta} \right| \leq \sum_{k=0}^{\infty} |\beta_{is,k}| \leq \sum_{k=0}^{\infty} B_{is}\rho^k \leq B_{is}\frac{1}{1-\rho}.$$

Therefore, letting $\sigma_{ij}^{\xi}(\theta)$ be the cross-spectral density of ξ_{it} and ξ_{jt},

$$\sum_{i=1}^{\infty} |\sigma_{ij}^{\xi}(\theta)| \leq \frac{1}{2\pi} \sum_{i=1}^{\infty} \sum_{s=1}^{\infty} |\beta_{is}(e^{-\imath\theta})\overline{\beta_{js}(e^{-\imath\theta})}| \leq \frac{1}{2\pi(1-\rho)^2} \sum_{i=1}^{\infty} \sum_{s=1}^{\infty} B_{is} B_{js}$$
$$\leq B^2 \frac{1}{2\pi(1-\rho)^2}, \tag{2.7}$$

by (2.6). Assumption 4 thus implies that the cross-spectra $\sigma_{ij}^{\xi}(\theta)$ are bounded, in θ, uniformly in i and j. On the other hand, Assumption 2, (ii) and (iii), implies that $\sigma_{ij}^{\chi}(\theta)$ is bounded, in θ, uniformly in i and j. Therefore, $\sigma_{ij}^{x}(\theta) = \sigma_{ij}^{\chi}(\theta) + \sigma_{ij}^{\xi}(\theta)$ is bounded, in θ, uniformly in i and j.

The spectral density matrices of the ξ's and the x's, and their eigenvalues, ordered in decreasing order, are denoted by $\mathbf{\Sigma}_n^{\xi}(\theta)$, $\mathbf{\Sigma}_n^{x}(\theta)$, $\lambda_{nj}^{\xi}(\theta)$ and $\lambda_{nj}^{x}(\theta)$, respectively; under the above assumptions, they satisfy the following properties.

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Proposition 1 *Under Assumptions 1 through 4,*

(i) there exists $B^\xi > 0$ such that $\lambda^\xi_{n1}(\theta) \le B^\xi$ for all $n \in \mathbb{N}$ and $\theta \in [-\pi, \pi]$ (thus, the ξ's are idiosyncratic, see FHLZ, Section 2.2);

(ii) there exists $n^x \in \mathbb{N}$ such that, for $n > n^x$ and all $\theta \in [-\pi, \pi]$,

$$\frac{\lambda^x_{n1}(\theta)}{n} > \alpha^\chi_1(\theta) > \frac{\lambda^x_{n2}(\theta)}{n} > \cdots > \alpha^\chi_{q-1}(\theta) > \frac{\lambda^x_{nq}(\theta)}{n} > \alpha^\chi_q(\theta),$$

where the functions $\alpha^\chi_j(\theta)$ are defined in Assumption 3;

(iii) there exists $B^x > 0$ such that $\lambda^x_{n,q+1}(\theta) \le B^x$ for all $n \in \mathbb{N}$ and $\theta \in [-\pi, \pi]$.

PROOF. The column and row norms of $\boldsymbol{\Sigma}^\xi_n(\theta)$ are equal, and, by (2.7), satisfy

$$\max_{j=1,2,\ldots,n} \sum_{i=1}^{n} |\sigma^\xi_{ij}(\theta)| \le \max_{j=1,2,\ldots,n} \sum_{i=1}^{\infty} |\sigma^\xi_{ij}(\theta)| \le B^2 \frac{1}{2\pi(1-\rho)^2}.$$

On the other hand, the product of the row and the column norms, the square of the column norm in our case, is greater than or equal to the square of the spectral norm, see Lancaster and Tismenetsky (1985), p. 366, Exercise 11. As a consequence, setting $B^\xi = B^2 1/2\pi(1-\rho)^2$, we have $\lambda^\xi_{n1}(\theta) \le B^\xi$ for all n and θ.

Regarding (ii), $\boldsymbol{\Sigma}^x_n(\theta) = \boldsymbol{\Sigma}^\chi_n(\theta) + \boldsymbol{\Sigma}^\xi_n(\theta)$ implies that

$$\lambda^x_{nf}(\theta) \ge \lambda^\chi_{nf}(\theta) + \lambda^\xi_{nn}(\theta) \quad \text{and} \quad \lambda^x_{nf}(\theta) \le \lambda^\chi_{nf}(\theta) + \lambda^\xi_{n1}(\theta)$$

(these are two of the Weyl's inequalities, see Franklin (2000), p. 157, Theorem 1; see also Appendix B). By Assumption 3,

$$\frac{\lambda^x_{nf}(\theta)}{n} \ge \frac{\lambda^\chi_{nf}(\theta) + \lambda^\xi_{nn}(\theta)}{n} > \alpha^\chi_f(\theta),$$

for $f = 1, \ldots, q$, and, for $f = 2, \ldots, q$,

$$\frac{\lambda^x_{nf}(\theta)}{n} \le \frac{\lambda^\chi_{nf}(\theta) + \lambda^\xi_{n1}(\theta)}{n} \le \frac{\lambda^\chi_{nf}(\theta)}{n} + \frac{B^\xi}{n} \le \beta^\chi_{f-1}(\theta) + \frac{B^\xi}{n} < \alpha^\chi_{f-1}(\theta),$$

for $n > n^\chi$, n^χ being such that $B^\xi/n^\chi < \min_{f=1,2,\ldots,q} \left[\min_{\theta \in [-\pi, \pi]} (\alpha^\chi_f(\theta) - \beta^\chi_f(\theta))\right]$.

As for (iii), $\lambda^x_{n,q+1} \le \lambda^\chi_{n,q+1}(\theta) + \lambda^\xi_{n1}(\theta)$. On the other hand, $\lambda^\chi_{n,q+1}(\theta) = 0$ for all θ. The result then follows from (i). $\square$

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Proposition 2 *Under Assumptions 1 through 4, the cross-spectral densities $\sigma_{ij}^x(\theta)$ possess derivatives of any order and are of bounded variation uniformly in $i, j \in \mathbb{N}$; namely, there exists $A^x > 0$ such that*

$$\sum_{h=1}^{\nu} |\sigma_{ij}^x(\theta_h) - \sigma_{ij}^x(\theta_{h-1})| \leq A^x$$

for all $i, j, \nu \in \mathbb{N}$ and all partitions

$$-\pi = \theta_0 < \theta_1 < \cdots < \theta_{\nu-1} < \theta_\nu = \pi$$

of the interval $[-\pi, \ \pi]$.

PROOF. Denoting by $\gamma_{ij,h}^\xi$, $h \geq 0$, the covariance between ξ_{it} and $\xi_{j,t-h}$,

$$|\gamma_{ij,h}^\xi| = \left| \sum_{k=0}^{\infty} \sum_{s=1}^{\infty} \beta_{is,k} \beta_{js,k+h} \right| \leq \sum_{k=0}^{\infty} \sum_{s=1}^{\infty} B_{is} B_{js} \rho^k \rho^{k+h} \leq \rho^h \sum_{k=0}^{\infty} \rho^{2k} \sum_{s=1}^{\infty} B_{is} B_{js} \leq \rho^h \frac{B^2}{1-\rho^2},$$

by (2.6). For $h < 0$, $\gamma_{ij,h}^\xi = \gamma_{ji,-h}^\xi$, so that $|\gamma_{ij,h}^\xi| \leq \rho^{|h|} B^2/(1-\rho^2)$. This implies that

$$\sigma_{ij}^\xi(\theta) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \gamma_{ij,h}^\xi e^{-\imath h\theta}$$

has derivatives of all orders. Moreover,

$$|\sigma_{ij}^{\xi\,\prime}(\theta)| = \frac{1}{2\pi} \left| \sum_{h=-\infty}^{\infty} (-\imath h) \gamma_{ij,h}^\xi e^{-\imath h\theta} \right| \leq \frac{B^2}{\pi(1-\rho^2)} \sum_{h=1}^{\infty} h\rho^h = \frac{B^2}{\pi(1-\rho^2)(1-\rho)^2},$$

which entails bounded variation of $\sigma_{ij}^\xi(\theta)$ uniformly in i and j. Bounded variation of $\sigma_{ij}^\chi(\theta)$, uniformly in i and j, is an obvious consequence of Assumption 2. The conclusion follows from the fact that $\sigma_{ij}^x(\theta) = \sigma_{ij}^\chi(\theta) + \sigma_{ij}^\xi(\theta)$. □

2.2 Autoregressive representation of the χ's

In FHLZ we prove that, for *generic values* of the parameters $c_{if,k}$ and $d_{if,k}$ in (2.2), the space spanned by $u_{f,t-k}$, $f = 1, 2, \ldots, q$, $k \geq 0$, is equal to the space spanned by any $(q+1)$-dimensional subvector of $\boldsymbol{\chi}_t$ and its lags. In other words, $\mathbf{u}_t$ is *fundamental* for all the $(q+1)$-dimensional subvectors of $\boldsymbol{\chi}_t$ (but not for all q-dimensional ones). Moreover, we prove that, *generically,* the $(q+1)$-dimensional subvectors of $\boldsymbol{\chi}_t$ admit a *finite and unique* autoregressive

representation (see, in particular, Section 4.1, Lemma 3). Following FHLZ, we use these genericity results as a motivation for assuming that each of the vectors

$$\begin{pmatrix} \chi_{1t} & \chi_{2t} & \cdots & \chi_{q+1,t} \end{pmatrix}, \quad \begin{pmatrix} \chi_{q+2,t} & \chi_{q+3,t} & \cdots & \chi_{2(q+1),t} \end{pmatrix}, \quad \cdots,$$

that is, each of the vectors

$$\boldsymbol{\chi}_t^k = \begin{pmatrix} \chi_{(k-1)(q+1)+1,t} & \cdots & \chi_{k(q+1),t} \end{pmatrix}, \quad k \in \mathbb{N},$$

and its lags spans the space spanned by the u's and has a unique finite autoregressive representation.

Assumption 5 *Each vector $\boldsymbol{\chi}_t^k$, $k \in \mathbb{N}$, has an autoregressive representation*

$$\mathbf{A}^k(L)\boldsymbol{\chi}_t^k = \mathbf{R}^k \mathbf{u}_t, \tag{2.8}$$

where

(i) $\mathbf{A}^k$ *is* $(q+1) \times (q+1)$, *of degree not greater than* $S = qs_1 + q^2 s_2$, *and* $\mathbf{A}^k(0) = \mathbf{I}_{q+1}$;

(ii) $\mathbf{R}^k$ *is* $(q+1) \times q$ *and has rank* q;

(iii) *the representation (2.8) is unique among the autoregressive representations of order not greater than S, i.e. if $\mathbf{B}(L)\boldsymbol{\chi}_t^k = \tilde{\mathbf{R}}\mathbf{u}_t$, where the degree of $\mathbf{B}(L)$ does not exceed S and $\mathbf{B}(0) = \mathbf{I}_{q+1}$, then $\mathbf{B}(L) = \mathbf{A}^k(L)$ and $\tilde{\mathbf{R}} = \mathbf{R}^k$.*

Representation (2.8) is a specification of (1.6) (the degrees of the polynomial matrices $\mathbf{A}^k(L)$ and their uniqueness).[3] Writing $\underline{\mathbf{A}}(L)$ for the (infinite) block-diagonal matrix with diagonal blocks $\mathbf{A}^1(L), \mathbf{A}^2(L), \ldots$, and letting $\underline{\mathbf{R}} = (\mathbf{R}^{1\prime}, \mathbf{R}^{2\prime}, \cdots)'$, we thus have

$$\underline{\mathbf{A}}(L)\boldsymbol{\chi}_t = \underline{\mathbf{R}}\mathbf{u}_t. \tag{2.9}$$

The upper $n \times n$ submatrix of $\underline{\mathbf{A}}(L)$ and the upper $n \times q$ submatrix of $\underline{\mathbf{R}}$ are denoted by $\mathbf{A}_n(L)$ and $\mathbf{R}_n$ respectively. If $n = m(q+1)$, so that the first m blocks of size $q+1$ are included,

$$\mathbf{A}_n(L)\boldsymbol{\chi}_{nt} = \mathbf{R}_n\mathbf{u}_t. \tag{2.10}$$

---

[3]Based on a genericity argument, FHLZ assume that (2.8) holds for any $(q+1)$-dimensional vector $(\chi_{i_1,t} \ \chi_{i_2,t} \ \cdots \ \chi_{i_{q+1},t})$, see Assumption A.3. The weaker version in Assumption 5 above is sufficient for our purposes.

The following proposition is an immediate consequence of the fact that (2.9) is the difference between $\boldsymbol{\chi}_t$ and its orthogonal projection on its past values; details are left to the reader.

Proposition 3 *Let Assumptions 1 through 5 hold.*

(i) *Let $\underline{\mathbf{A}}^*(L)\boldsymbol{\chi}_t = \underline{\mathbf{R}}^*\mathbf{v}_t$, where the degree of $\underline{\mathbf{A}}^*(L)$ is at most S: then, $\underline{\mathbf{A}}^*(L) = \underline{\mathbf{A}}(L)$, and there exists a $q \times q$ orthogonal matrix $\mathbf{Q}$ such that $\underline{\mathbf{R}}^* = \underline{\mathbf{R}}\mathbf{Q}'$ and $\mathbf{v}_t = \mathbf{Q}\mathbf{u}_t$.*

(ii) *Let $\mathbf{r} = (r_1 \;\; \cdots \;\; r_q)$ be the row of $\underline{\mathbf{R}}$ (the row of $\mathbf{R}^k$) corresponding to χ_{it}: then, $r_f = c_{if}(0) = c_{if,0}, \; f = 1, \ldots, q, \; i \in \mathbb{N}$.*

Letting $\boldsymbol{\Psi}_t = \underline{\mathbf{A}}(L)\boldsymbol{\chi}_t = \underline{\mathbf{R}}\mathbf{u}_t$, denote by $\boldsymbol{\Gamma}_n^{\psi}$ the variance-covariance matrix of $\boldsymbol{\Psi}_{nt}$, with eigenvalues $\mu_{nj}^{\psi}, \; j = 1, \ldots, n$, in decreasing order.

Assumption 6 *There exist real numbers $\alpha_f^{\psi}, \; f = 1, \ldots, q, \; \beta_f^{\psi}, \; f = 0, \ldots, q - 1$, and a positive integer n^{ψ} such that, for $n > n^{\psi}$,*

$$\beta_0^{\psi} \geq \frac{\mu_{n1}^{\psi}}{n} \geq \alpha_1^{\psi} > \beta_1^{\psi} \geq \frac{\mu_{n2}^{\psi}}{n} \geq \alpha_2^{\psi} > \beta_2^{\psi} \geq \cdots \geq \alpha_{q-1}^{\psi} > \beta_{q-1}^{\psi} \geq \frac{\mu_{nq}^{\psi}}{n} \geq \alpha_q^{\psi} > 0.$$

Note that the eigenvalues μ_{nf}^{ψ} depend on the coefficients $c_{if,0}$, see Proposition 3(ii), but are invariant if $\underline{\mathbf{R}}$ and $\mathbf{u}_t$ are replaced by $\underline{\mathbf{R}}\mathbf{Q}'$ and $\mathbf{Q}\mathbf{u}_t$ respectively.

We now show how the matrices $\mathbf{A}^k(L)$ appearing in (2.8) can be constructed from the spectral density of the χ's. This construction, with the population quantities replaced by their estimates, leads to our estimator as explained in Section 3. It proceeds in two steps:

(i) Denoting by $\boldsymbol{\Sigma}_{jk}^X(\theta)$ the $(q+1) \times (q+1)$ cross-spectral density between $\boldsymbol{\chi}_t^j$ and $\boldsymbol{\chi}_t^k$, and by $\boldsymbol{\Gamma}_{jk,s}^X$ the covariance between $\boldsymbol{\chi}_t^j$ and $\boldsymbol{\chi}_{t-s}^k$, we have

$$\boldsymbol{\Gamma}_{jk,s}^X = \mathrm{E}\left(\boldsymbol{\chi}_t^j\boldsymbol{\chi}_{t-s}^{k}{}'\right) = \int_{-\pi}^{\pi} e^{\imath s\theta}\boldsymbol{\Sigma}_{jk}^X(\theta)d\theta. \tag{2.11}$$

(ii) The minimum-lag matrix polynomial $\mathbf{A}^k(L)$ and the variance-covariance function of the unobservable vectors

$$\boldsymbol{\Psi}_t^1 = \mathbf{A}^1(L)\boldsymbol{\chi}_t^1, \quad \boldsymbol{\Psi}_t^2 = \mathbf{A}^2(L)\boldsymbol{\chi}_t^2 \quad \cdots \tag{2.12}$$

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follow from that autocovariance function $\mathbf{\Gamma}^\chi_{kk,s}$. Indeed, defining

$$\mathbf{A}^k(L) = \mathbf{I}_{q+1} - \mathbf{A}^k_1 L - \cdots - \mathbf{A}^k_S L^S,$$

$$\mathbf{A}^{[k]} = \left(\mathbf{A}^k_1 \ \mathbf{A}^k_2 \ \cdots \ \mathbf{A}^k_S \right), \qquad \mathbf{B}^\chi_k = \left(\mathbf{\Gamma}^\chi_{kk,1} \ \mathbf{\Gamma}^\chi_{kk,2} \ \cdots \ \mathbf{\Gamma}^\chi_{kk,S} \right) \qquad (2.13)$$

and

$$\mathbf{C}^\chi_{jk} = \begin{pmatrix} \mathbf{\Gamma}^\chi_{jk,0} & \mathbf{\Gamma}^\chi_{jk,1} & \cdots & \mathbf{\Gamma}^\chi_{jk,S-1} \\ \mathbf{\Gamma}^\chi_{jk,-1} & \mathbf{\Gamma}^\chi_{jk,0} & \cdots & \mathbf{\Gamma}^\chi_{jk,S-2} \\ \vdots & & & \vdots \\ \mathbf{\Gamma}^\chi_{jk,-S+1} & \mathbf{\Gamma}^\chi_{jk,-S+2} & \cdots & \mathbf{\Gamma}^\chi_{jk,0} \end{pmatrix}, \qquad (2.14)$$

we have

$$\mathbf{A}^{[k]} = \mathbf{B}^\chi_k \left(\mathbf{C}^\chi_{kk} \right)^{-1} = \mathbf{B}^\chi_k \left(\mathbf{C}^\chi_{kk} \right)_{\mathrm{ad}} \det \left(\mathbf{C}^\chi_{kk} \right)^{-1}, \qquad (2.15)$$

where $\left(\mathbf{C}^\chi_{kk} \right)_{\mathrm{ad}}$ stands for the adjoint of the square matrix $\mathbf{C}^\chi_{kk}$.

Non-singularity of $\mathbf{C}^\chi_{kk}$ is necessary for the uniqueness of the $\mathbf{A}^{[k]}$'s, and it therefore is implied by Assumption 5. However, we require a slightly stronger condition to ensure that the $\mathbf{A}^{[k]}$'s are (uniformly) bounded, in norm, as n tends to infinity.

Assumption 7 *There exists a real $d > 0$ such that $\det \mathbf{C}^\chi_{kk} > d$ for all $k \in \mathbb{N}$.*

Letting $\mathbf{Z}_t = \underline{\mathbf{A}}(L)\mathbf{x}_t$, we have

$$\mathbf{Z}_t = \mathbf{\Psi}_t + \mathbf{\Phi}_t$$
$$\text{with} \quad \mathbf{\Psi}_t = \underline{\mathbf{R}}\mathbf{u}_t \quad \mathbf{\Phi}_t \ = \underline{\mathbf{A}}(L)\mathbf{\xi}_t. \qquad (2.16)$$

Denote by $\mathbf{\Gamma}^\Phi_n$ the variance-covariance matrix of $\mathbf{\Phi}_{nt} = (\Phi_{1t} \ \Phi_{2t} \ \cdots \ \Phi_{nt})$ and by μ^Φ_{nj} its j-th eigenvalue: the following holds

Proposition 4 *Under Assumptions 1 through 7, there exists $B^\Phi > 0$ such that $\mu^\Phi_{n1} \leq B^\Phi$ for all $n \in \mathbb{N}$.*

PROOF. Let $\lambda^\Phi_{nj}(\theta)$ be the j-th eigenvalue of the spectral density matrix of $\mathbf{\Phi}_{nt}$. Let us show that there exists a constant C^Φ such that $\lambda^\Phi_{n1}(\theta) \leq C^\Phi$ for all n and θ. Because $\lambda^\Phi_{n1}(\theta)$, for all θ, is non-decreasing with n (see Forni and Lippi, 2001), we can assume without loss of generality that $n = m(q+1)$. The spectral density of $\mathbf{\Phi}_{nt}$ is

$$\mathbf{A}_n(e^{-\imath\theta})\mathbf{\Sigma}^\xi_n(\theta)\mathbf{A}'_n(e^{\imath\theta}),$$

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where $\mathbf{A}_n(L)$ (see equation (2.10)) has the matrices $\mathbf{A}^k(L)$ on the diagonal. If $\mathbf{a}(\theta)$ is an n-dimensional complex column vector such that $\mathbf{a}(\theta)'\overline{\mathbf{a}(\theta)} = 1$ for all θ, we have

$$\mathbf{a}(\theta)'\mathbf{A}_n(e^{-\imath\theta})\mathbf{\Sigma}_n^\xi(\theta)\mathbf{A}_n'(e^{\imath\theta})\overline{\mathbf{a}(\theta)} \leq \lambda_{n1}^\xi(\theta)\left(\mathbf{a}'(\theta)\mathbf{A}_n(e^{-\imath\theta})\mathbf{A}_n'(e^{\imath\theta})\overline{\mathbf{a}(\theta)}\right) \leq \lambda_{n1}^\xi(\theta)\lambda_1^{A_n}(\theta),$$

where $\lambda_1^{A_n}(\theta)$ is the first eigenvalue of $\mathbf{A}_n(e^{-\imath\theta})\mathbf{A}_n'(e^{\imath\theta})$, which is Hermitian, non-negative definite. By Proposition 1 $\sup_n \lambda_{n1}^\xi(\theta) \leq B^\xi$. Moreover, given the diagonal structure of $\mathbf{A}_n(L)$, $\lambda_1^{A_n}(\theta) = \sup_{k=1,2,\ldots,m} \lambda_1^{A^k}(\theta) \leq \sup_{k\in\mathbb{N}} \lambda_1^{A^k}(\theta)$, where $\lambda_1^{A^k}(\theta)$ is the first eigenvalue of $\mathbf{A}^k(e^{-\imath\theta})\mathbf{A}^{k'}(e^{\imath\theta})$. Assumptions 2 and 7 imply that $\sup_{k\in\mathbb{N}} \lambda_1^{A^k}(\theta) \leq D^\Phi$ for some $D^\Phi > 0$ and all θ. On the other hand,

$$\lambda_{n1}^\Phi(\theta) = \sup \mathbf{a}(\theta)'\mathbf{A}_n(e^{-\imath\theta})\mathbf{\Sigma}_n^\xi(\theta)\mathbf{A}_n'(e^{\imath\theta})\overline{\mathbf{a}(\theta)} \leq B^\xi D^\Phi,$$

the sup being over all the vectors $\mathbf{a}(\theta)$ such that $\mathbf{a}(\theta)'\overline{\mathbf{a}(\theta)} = 1$. Lastly,

$$\mu_{n1}^\Phi = \sup \mathbf{b}'\mathbf{\Gamma}_n^\Phi\mathbf{b} = \int_{-\pi}^{\pi} \left(\mathbf{b}'\mathbf{\Sigma}_n^\Phi(\theta)\mathbf{b}\right) d\theta \leq \int_{-\pi}^{\pi} \lambda_{n1}^\Phi(\theta)d\theta \leq 2\pi B^\xi D^\Phi,$$

the sup being over all the n-dimensional column vectors $\mathbf{b}$ such that $\mathbf{b}'\mathbf{b} = 1$. $\quad\square$

Note that $\mathbf{\Phi}_t$ and $\mathbf{\Psi}_t$ are mutually orthogonal, a consequence of Assumption 1(i). In view of Assumption 6 and Proposition 4, the model (2.16) is thus a static factor model—a special case of (1.3), with $r = q$ and $\mathbf{N}(L) = \mathbf{I}_q$.

3 Estimation: asymptotics

Our estimation procedure follows the same steps as the population construction in Section 2.2, with the population spectral density of the x's replaced with an estimator $\hat{\mathbf{\Sigma}}_n^x(\theta)$ fulfilling Assumption 9 below. Based on Forni et al. (2000), we obtain the estimator $\hat{\mathbf{\Sigma}}_n^\chi(\theta)$ by means of the first q frequency-domain principal components of the x's (using the first q eigenvectors of $\hat{\mathbf{\Sigma}}_n^x(\theta)$). Then the matrices $\hat{\mathbf{\Gamma}}_{jk}^\chi$, $\hat{\mathbf{B}}_{jk}^\chi$, $\hat{\mathbf{C}}_{jk}^\chi$ and $\hat{\mathbf{A}}_n(L)$ are computed as natural counterparts of their population versions in Section 2.2. Finally, estimators for $\mathbf{R}_n$ and $\mathbf{u}_t$ are obtained via a standard principal component analysis of $\hat{\mathbf{Z}}_{nt} = \hat{\mathbf{A}}(L)\mathbf{x}_{nt}$. Consistency with exact rate of convergence ζ_{nT}, as defined in equation (1.8), for all the above estimators are provided in Propositions 7 through 11.

Explicit dependence on the index n has been necessary in Section 2. From now on, it will be convenient to introduce a minor change in notation, dropping n whenever possible. In particular,

(i) $\boldsymbol{\Sigma}^x(\theta) = \left(\sigma_{ij}^x(\theta)\right)_{i,j=1,\ldots,n}$ and $\lambda_f^x(\theta)$ replace $\boldsymbol{\Sigma}_n^x(\theta)$ and $\lambda_{nf}^x(\theta)$, respectively.

(ii) $\boldsymbol{\Lambda}^x(\theta)$ denotes the $q \times q$ diagonal matrix with diagonal elements $\lambda_f^x(\theta)$.

(iii) $\mathbf{P}^x(\theta)$ denotes the $n \times q$ matrix the q columns of which are the unit-modulus eigenvectors corresponding to $\boldsymbol{\Sigma}^x(\theta)$'s first q eigenvalues. The columns and entries of $\mathbf{P}^x(\theta)$ are denoted by $\mathbf{P}_f^x(\theta)$ and $p_{if}^x(\theta)$, $f = 1,\ldots,q$, $i = 1,\ldots,n$, respectively.

(iv) $\boldsymbol{\Sigma}^\chi(\theta) = \left(\sigma_{ij}^\chi(\theta)\right)_{i,j=1,\ldots,n}$, $\lambda_f^\chi(\theta)$, $\boldsymbol{\Lambda}^\chi(\theta)$, $\mathbf{P}^\chi(\theta)$, $\boldsymbol{\Sigma}^\xi(\theta)$, etc. are defined as in (i).

(v) All the above matrices and scalars depend on n; the corresponding estimators,

$$\hat{\boldsymbol{\Sigma}}^x(\theta), \ \ \hat{\lambda}_f^x(\theta), \ \ \hat{\boldsymbol{\Lambda}}^x(\theta), \ \ \hat{\mathbf{P}}^x(\theta) \ \ \text{and} \ \ \hat{\boldsymbol{\Sigma}}^\chi(\theta), \ \ \hat{\lambda}_f^\chi(\theta), \ \ \hat{\boldsymbol{\Lambda}}^\chi(\theta), \ \ \hat{\mathbf{P}}^\chi(\theta)$$

(precise definitions are provided below) depend both on n and the observed values x_{it}, $i = 1,\ldots,n$, $t = 1,\ldots,T$. For simplicity, we say that they depend on n and T.

(vi) The same notational change applies to $\boldsymbol{\Gamma}_n^\psi$ and related eigenvalues and eigenvectors.

(vii) $\mathbf{A}(L)$ and $\mathbf{R}$, denoting the upper left $n \times n$ and $n \times q$ submatrices of $\underline{\mathbf{A}}(L)$ and $\underline{\mathbf{R}}$, respectively, are used instead of $\mathbf{A}_n(L)$ and $\mathbf{R}_n$; $\hat{\mathbf{A}}(L)$ and $\hat{\mathbf{R}}$ stand for their estimated counterparts.

(viii) To avoid confusion, however, we keep explicit reference to n in $\mathbf{x}_{nt}$, $\boldsymbol{\chi}_{nt}$, $\mathbf{Z}_{nt}$ etc., with estimated counterparts of the form $\hat{\boldsymbol{\chi}}_{nt}$, $\hat{\mathbf{Z}}_{nt}$, etc.; thus, we write, for instance, $\mathbf{Z}_{nt} = \mathbf{A}(L)\mathbf{x}_{nt} = \mathbf{R}\mathbf{u}_t + \boldsymbol{\Phi}_{nt}$.

(ix) Lastly, if $\mathbf{F}$ is a matrix, we denote by $\tilde{\mathbf{F}}$ its conjugate transpose, and by $||\mathbf{F}||$ its spectral norm (see Appendix B).

3.1 Estimation of $\boldsymbol{\Sigma}^x(\theta)$

The following definition, coined by Wu (2005), generalizes the usual measures of time dependence for stochastic processes.

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Definition 1 *Physical dependence. Let $\boldsymbol{\epsilon}_t$ be an i.i.d. stochastic vector process, possibly infinite-dimensional, and let $z_t = F(\boldsymbol{\epsilon}_t, \boldsymbol{\epsilon}_{t-1}, \ldots)$, where $F : [\mathbb{R} \times \mathbb{R} \times \cdots] \to \mathbb{R}$ is a measurable function; assume that z_t has finite p moment for $p > 0$. Let $\boldsymbol{\epsilon}^*$ be a stochastic vector with the same dimension and distribution as the $\boldsymbol{\epsilon}_t$'s, such that $\boldsymbol{\epsilon}^*$ and $\boldsymbol{\epsilon}_t$ are independent for all t. For $k \geq 0$ the physical dependence $\delta_{kp}^{[z_t]}$ is defined as*

$$\delta_{kp}^{[z_t]} = (E\,(|F(\boldsymbol{\epsilon}_k, \ldots, \boldsymbol{\epsilon}_0, \boldsymbol{\epsilon}_{-1}, \ldots) - F(\boldsymbol{\epsilon}_k, \ldots, \boldsymbol{\epsilon}^*, \boldsymbol{\epsilon}_{-1}, \ldots)|^p))^{1/p}\,.$$

Assumption 8 *There exist p, A, with $p > 4$, $0 < A < \infty$, such that*

$$E\,(|u_{ft}|^p) \leq A, \quad E\,(|\eta_{it}|^p) \leq A, \tag{3.1}$$

for all $i \in \mathbb{N}$ and $f = 1, \ldots, q$.

The main result of the section, that the estimate of the cross-spectral density between x_{it} and x_{jt} converges uniformly with respect to the frequency and to i and j, see Proposition 6, requires the following results on the p-th moments and the physical dependence of the x's.

Proposition 5 *Under Assumptions 1 through 8, there exist $\rho_1 \in (0,1)$ and $A_1 \in (0,\infty)$ such that, for all $i \in \mathbb{N}$,*

$$E\,(|x_{it}|^p) \leq A_1 \quad and \quad \delta_{kp}^{[x_{it}]} \leq A_1 \rho_1^k. \tag{3.2}$$

PROOF. By the Minkovski inequality,

$$(E\,(|x_{it}|^p))^{\frac{1}{p}} = (E\,(|\chi_{it} + \xi_{it}|^p))^{\frac{1}{p}} \leq (E\,(|\chi_{it}|^p))^{\frac{1}{p}} + (E\,(|\xi_{it}|^p))^{\frac{1}{p}}\,.$$

Using the Minkovski inequality again, condition (2.3) and Assumption 8, we obtain

$$(E\,(|\xi_{it}|^p))^{\frac{1}{p}} = \left(E\left(\left|\sum_{s=1}^{\infty}\sum_{k=0}^{\infty} \beta_{is,k}\eta_{s,t-k}\right|^p\right)\right)^{\frac{1}{p}} \leq \sum_{s=1}^{\infty}\sum_{k=0}^{\infty}(E\,(|\beta_{is,k}\eta_{s,t-k}|)^p)^{\frac{1}{p}}$$

$$\leq \sum_{s=1}^{\infty}\sum_{k=0}^{\infty}|\beta_{is,k}|E\,(|\eta_{s,t-k}|^p)^{\frac{1}{p}} \leq A^{\frac{1}{p}}\sum_{s=1}^{\infty}\sum_{k=0}^{\infty}B_{is}\rho^k \leq A^{\frac{1}{p}}B\frac{1}{1-\rho}.$$

An analogous inequality can be obtained for the common components, using Assumption 2 and the first of inequalities (3.1). The first inequality in (3.2) follows.

Turning to the second inequality, for $k \geq 0$,

$$\xi_{ik} - \xi_{ik}^* = \sum_{s=1}^{\infty} \beta_{is,k}(\eta_{sk} - \eta_s^*),$$

where ξ_{ik}^* has the same definition as ξ_{ik}, with η_{s0} replaced by η_s^*. The Minkovski inequality, condition (2.3) and Assumption 8 imply

$$\delta_{k,p}^{[\xi_{it}]} = \left(E \left(\left| \sum_{s=1}^{\infty} \beta_{is,k}(\eta_{sk} - \eta_s^*) \right|^p \right) \right)^{\frac{1}{p}} \leq \sum_{s=1}^{\infty} \left(E \left(|\beta_{is,k}(\eta_{sk} - \eta_s^*)|^p \right) \right)^{\frac{1}{p}}$$

$$\leq \rho^k \sum_{s=1}^{\infty} B_{is} \left(E \left(|\eta_{sk} - \eta_s^*|^p \right) \right)^{\frac{1}{p}} \leq \rho^k 2 B A^{\frac{1}{p}}.$$

An analogous inequality can be ontained for the common components, using Assumption 2 and the first of inequalities (3.1), with ρ replaced by ϕ^{-1}, ϕ being defined in Assumption 2. Then,

$$\delta_{kp}^{[x_{it}]} = (E |x_{it} - x_{it}^*|^p)^{\frac{1}{p}} = \left(E \left(|(\chi_{it} - \chi_{it}^*) + (\xi_{it} - \xi_{it}^*)|^p)^{\frac{1}{p}} \right)$$

$$\leq (E \left(|\chi_{it} - \chi_{it}^*| \right)^p)^{\frac{1}{p}} + (E \left(|\xi_{it} - \xi_{it}^*| \right)^p)^{\frac{1}{p}} = \delta_{kp}^{[\chi_{it}]} + \delta_{kp}^{[\xi_{it}]}.$$

The conclusion follows. $\qquad\square$

Consider now the lag-window estimator

$$\hat{\boldsymbol{\Sigma}}^x(\theta) = \frac{1}{2\pi} \sum_{k=-T+1}^{T-1} K\left(\frac{k}{B_T}\right) e^{-\imath k\theta} \hat{\boldsymbol{\Gamma}}_k^x, \tag{3.3}$$

of the spectral density $\boldsymbol{\Sigma}^x(\theta)$, where $\hat{\boldsymbol{\Gamma}}_k^x = \frac{1}{T} \sum_{t=|k|+1}^{T} \mathbf{x}_t \mathbf{x}_{t-|k|}$.

Assumption 9 *Lag-window estimation of $\boldsymbol{\Sigma}^x(\theta)$.*

(i) *The kernel function K is even, bounded, with support $[-1,1]$; moreover,*

    (1) *for some $\kappa > 0$, $\lim_{u \to 0} |K(u) - 1| = O(|u|^\kappa)$,*

    (2) *$\int_{-\infty}^{\infty} K^2(u)du < \infty$,*

    (3) *$\sum_{j \in \mathbb{Z}} \sup_{|s-j| \leq 1} |K(jw) - K(sw)| = O(1)$ as $w \to 0$;*

(ii) *For some $c_1, c_2 > 0$, δ and $\underline{\delta}$ such that $0 < \underline{\delta} < \delta < 1 < \underline{\delta}(2\kappa + 1)$, $c_1 T^{\underline{\delta}} \leq B_T \leq c_2 T^\delta$.*

Proposition 6 *Under Assumptions 1 through 9, there exists $C > 0$ such that*

$$E\left(\max_{|h| \leq B_T} \left| \hat{\sigma}_{ij}^x(\theta_h^*) - \sigma_{ij}^x(\theta_h^*) \right|^2 \right) \leq C \left(T^{-1} B_T \log B_T \right), \tag{3.4}$$

where $\theta_h^* = \pi h / B_T$, for all T, i and j in $\mathbb{N}$.[MF: I would write θ_h instead of θ_h^* both here and in Prop. 7]

See Appendix A for the proof.

3.2 Estimation of $\sigma_{ij}^{\chi}(\theta)$ and $\gamma_{ij,k}^{\chi}$

Our estimator of the spectral density matrix of the common components $\boldsymbol{\chi}_{nt}$ is the Forni et al. (2000) estimator $\hat{\boldsymbol{\Sigma}}^{\chi}(\theta) = \hat{\mathbf{P}}^{x}(\theta)\hat{\boldsymbol{\Lambda}}^{x}(\theta)\tilde{\hat{\mathbf{P}}}^{x}(\theta_{h})$.

Proposition 7 *Under Assumptions 1 through 7,*

$$\max_{|h| \le B_T} |\hat{\sigma}_{ij}^{\chi}(\theta_h^*) - \sigma_{ij}^{\chi}(\theta_h^*)| = O_P\left(\zeta_{nT}\right),$$

where $\theta_h^* = \pi h/B_T$, as $T \to \infty$ and $n \to \infty$, uniformly in i and j. Precisely, for any $\epsilon > 0$, there exists $\eta(\epsilon)$, independent of i and j, such that, for all n and T,

$$P\left(\frac{\max_{h \le B_T} |\hat{\sigma}_{ii}^{\chi}(\theta_h^*) - \sigma_{ii}^{\chi}(\theta_h^*)|}{\zeta_{nT}} \ge \eta(\epsilon)\right) < \epsilon.$$

See Appendix B for the proof.

Our estimator of the covariance $\gamma_{ij,\ell}^{\chi}$ of χ_{it} and $\chi_{j,t-\ell}$ is, as in Forni et al. (2005),

$$\hat{\gamma}_{ij,\ell}^{\chi} = \frac{\pi}{B_T} \sum_{|h| \le B_T} e^{\imath \ell \theta_h^*} \hat{\sigma}_{ij}^{\chi}(\theta_h^*), \tag{3.5}$$

where $\theta_h^* = \pi h/B_T$. Recalling that $\gamma_{ij,\ell}^{\chi} = \int_{-\pi}^{\pi} e^{\imath \ell \theta} \sigma_{ij}^{\chi}(\theta) d\theta$, we have

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$$
\begin{aligned}
|\hat{\gamma}_{ij,\ell}^{\chi} - \gamma_{ij,\ell}^{\chi}| \;\leq\; & \frac{\pi}{B_T} \sum_{|h|\leq B_T} |e^{\imath\ell\theta_h^*}\hat{\sigma}_{ij}^{\chi}(\theta_h^*) - e^{\imath\ell\theta_h^*}\sigma_{ij}^{\chi}(\theta_h^*)| \\
& + \left| \frac{\pi}{B_T} \sum_{|h|\leq B_T} e^{\imath\ell\theta_h^*}\sigma_{ij}^{\chi}(\theta_h^*) - \int_{-\pi}^{\pi} e^{\imath\ell\theta}\sigma_{ij}^{\chi}(\theta)d\theta \right| \\
\leq\; & \frac{\pi}{B_T} \sum_{|h|\leq B_T} |\hat{\sigma}_{ij}^{\chi}(\theta_h^*) - \sigma_{ij}^{\chi}(\theta_h^*)| \\
& + \frac{\pi}{B_T} \sum_{|h|\leq B_T} \max_{\theta_{h-1}^*\leq\theta\leq\theta_h^*} |e^{\imath\ell\theta_h^*}\sigma_{ij}^{\chi}(\theta_h^*) - e^{\imath\ell\theta}\sigma_{ij}^{\chi}(\theta)| \\
\leq\; & \pi \max_{|h|\leq B_T} |\hat{\sigma}_{ij}^{\chi}(\theta_h^*) - \sigma_{ij}^{\chi}(\theta_h^*)| + \frac{\pi B}{B_T} \sum_{|h|\leq B_T} \max_{\theta_{h-1}^*\leq\theta\leq\theta_h^*} |e^{\imath\ell\theta_h^*} - e^{\imath\ell\theta}| \\
& + \frac{\pi}{B_T} \sum_{|h|\leq B_T} \max_{\theta_{h-1}^*\leq\theta\leq\theta_h^*} |\sigma_{ij}^{\chi}(\theta_h^*) - \sigma_{ij}^{\chi}(\theta)| \\
\leq\; & \pi \max_{|h|\leq B_T} |\hat{\sigma}_{ij}^{\chi}(\theta_h^*) - \sigma_{ij}^{\chi}(\theta_h^*)| \qquad\qquad\qquad\qquad (3.6) \\
& + \frac{\pi B}{B_T} \sum_{|h|\leq B_T} \left(|e^{\imath\ell\theta_{h-1}^*} - e^{\imath\ell\tilde{\theta}_{h-1}^*}| + |e^{\imath\ell\tilde{\theta}_{h-1}^*} - e^{\imath\ell\theta_{h-1}^*}| \right) \\
& + \frac{\pi}{B_T} \sum_{|h|\leq B_T} \left(|\sigma_{ij}^{\chi}(\theta_{h-1}^*) - \sigma_{ij}^{\chi}(\check{\theta}_{h-1}^*)| + |\sigma_{ij}^{\chi}(\check{\theta}_{h-1}^*) - \sigma_{ij}^{\chi}(\theta_h^*)| \right),
\end{aligned}
$$

where B is the bound in Proposition 1(i), and $\tilde{\theta}_{h-1}^*$ and $\check{\theta}_{h-1}^*$ are points in the interval $[\theta_{h-1}, \theta_h]$ where the functions of θ, $|e^{\imath\ell\theta_s^*} - e^{\imath\ell\theta}|$ and $|\sigma_{ij}(\theta_s^*) - \sigma_{ij}(\theta)|$, respectively, attain a maximum. Of course, the function $e^{\imath\ell\theta}$ is of bounded variation, while the functions $\sigma_{ij}^{\chi}(\theta)$ are of bounded variation by Assumption 2, so that the second and third terms are $O(1/B_T)$.

Using Proposition 7, we obtain that $|\hat{\gamma}_{ij,\ell}^{\chi} - \gamma_{ij,\ell}^{\chi}|$ is $O_P(\zeta_{nT}) + O(1/B_T)$. Since $\zeta_{nT} = \max(1/\sqrt{n},\ 1/\sqrt{T/B_T \log T})$, the latter term is absorbed in the former under Assumption 10 below. Proposition 8 follows.

Assumption 10 *The lower bound $\underline{\delta}$ in Assumption 9 satisfies $\underline{\delta} > 1/3$.*

Proposition 8 *Under Assumptions 1 through 10, for each $\ell \geq 0$,*

$$
|\hat{\gamma}_{ij,\ell}^{\chi} - \gamma_{ij,\ell}^{\chi}| = O_P(\zeta_{nT}), \qquad\qquad (3.7)
$$

as $T \to \infty$ and $n \to \infty$.

3.3   Estimation of $\mathbf{A}^k(L)$

Under our assumptions, the common component admits the block-diagonal vector autoregressive representation (1.5) of finite order. If the χ_t's were observed, estimation by OLS would be appropriate. However, although we do not observe the χ_t's, we do have (consistent) estimates of their autocovariance function. This naturally leads to a Yule-Walker estimator of the autoregressive coefficients and the innovation covariance matrix. The definition of $\hat{\mathbf{A}}^{[k]}$ then is straightforward from (2.13), (2.14) and (2.15).

Proposition 9   *Under Assumptions 1 through 10, $\|\hat{\mathbf{A}}^{[k]} - \mathbf{A}^{[k]}\| = O_P(\zeta_{nT})$ as $T \to \infty$ and $n \to \infty$.*

See Appendix C for the proof.

3.4   Estimation of R and $\mathbf{u}_t$

We start with $\mathbf{Z}_{nt} = \mathbf{\Psi}_{nt} + \mathbf{\Phi}_{nt} = \mathbf{R}\mathbf{u}_t + \mathbf{\Phi}_{nt}$. The covariance matrix of $\mathbf{\Psi}_{nt}$ is

$$\mathbf{R}\mathbf{R}' = \mathbf{P}^\psi \mathbf{\Lambda}^\psi \mathbf{P}^{\psi\prime} = \mathbf{P}^\psi (\mathbf{\Lambda}^\psi)^{1/2} (\mathbf{\Lambda}^\psi)^{1/2} \mathbf{P}^{\psi\prime},$$

where $\mathbf{\Lambda}^\psi$ is $q \times q$ with the non-zero eigenvalues of $\mathbf{R}\mathbf{R}'$ on the main diagonal, while $\mathbf{P}^\psi$ is $n \times q$ with the corresponding eigenvectors on the columns. Thus, we have the representation

$$\mathbf{Z}_{nt} = \mathbf{P}^\psi (\mathbf{\Lambda}^\psi)^{1/2} \mathbf{v}_t + \mathbf{\Phi}_{nt} = \mathcal{R}\mathbf{v}_t + \mathbf{\Phi}_{nt},$$

say, where $\mathbf{v}_t = \mathbf{H}\mathbf{u}_t$, with $\mathbf{H}$ orthogonal. Note that, for given i and f, the (i, f) entry of $\mathcal{R}$ depends on n, so that the matrices $\mathcal{R}$ are not nested; nor is $\mathbf{v}_t$ independent of n. However, the product of each row of $\mathcal{R}$ by $\mathbf{v}_t$ yields the corresponding coordinate of $\mathbf{\Psi}_{nt}$ which does not depend on n.

Our estimator of $\mathcal{R} = \mathbf{P}^\psi (\mathbf{\Lambda}^\psi)^{1/2}$ is $\hat{\mathcal{R}} = \hat{\mathbf{P}}^z (\hat{\mathbf{\Lambda}}^z)^{1/2}$, where $\hat{\mathbf{P}}^z$ and $\hat{\mathbf{\Lambda}}^z$ are the eigenvectors and eigenvalues, respectively, of the empirical variance-covariance matrix of $\hat{\mathbf{Z}}_{nt} = \hat{\mathbf{A}}(L)\mathbf{x}_{nt}$, that is, $\mathbf{x}_{nt}$ filtered with the *estimated* matrices $\hat{\mathbf{A}}(L)$. This, as already observed, is the reason for the complications we have to deal with in Appendix D.

Proposition 10   *Under Assumptions 1 through 10, $\|\hat{\mathcal{R}}_i - \mathcal{R}_i \hat{\mathbf{W}}_q\| = O_P(\zeta_{nT})$, as $T \to \infty$ and $n \to \infty$, where $\mathcal{R}_i$ is the i-th row of $\mathcal{R}$, and $\hat{\mathbf{W}}_q$ is a $q \times q$ diagonal matrix, depending on n and T, whose diagonal entries are either 1 or -1.*

See Appendix D for the proof.

Let us point out again that the i-th row of $\mathcal{R}$ depends on n. Therefore, Proposition 10 only states that the difference between the estimated entries of $\hat{\mathcal{R}}$ and the entries of $\mathcal{R}$ converges to zero (upon sign correction), not that the estimated entries converge. Now, suppose that the common shocks can be identified by means of economically meaningful statements. For example, suppose that we have good reasons to claim that the upper $q \times q$ matrix of the "structural" representation is lower triangular with positive diagonal entries (an iterative scheme for the first q common components). As is well known, such conditions determine a unique representation, denote it by $\mathbf{Z}_t = \underline{\mathbf{R}}^* \mathbf{u}_t^* + \mathbf{\Phi}_t$, or $\mathbf{Z}_{nt} = \mathbf{R}^* \mathbf{u}_t^* + \mathbf{\Phi}_t$, where the $n \times q$ matrices $\mathbf{R}^*$ are nested. In particular, starting with $\mathbf{Z}_{nt} = \mathcal{R} \mathbf{v}_t + \mathbf{\Phi}_{nt}$, there exists exactly one orthogonal matrix $\mathbf{G}(\mathcal{R})$ (actually $\mathbf{G}(\mathcal{R})$ only depends on the $q \times q$ upper submatrix of $\mathcal{R}$) such that $\mathbf{R}^* = \mathcal{R}\mathbf{G}(\mathcal{R})$. Thus, while the entries of $\mathcal{R}$ depend on n, the entries of $\mathcal{R}\mathbf{G}(\mathcal{R})$ do not.

Applying the same rule to $\hat{\mathcal{R}}$ we obtain the matrices $\hat{\mathbf{R}}^* = \hat{\mathcal{R}}\mathbf{G}(\hat{\mathcal{R}})$. It is easily seen that each entry of $\hat{\mathbf{R}}^*$ (depending on n and T) converges to the corresponding entry of $\mathbf{R}^*$ (independent of n and T) at rate ζ_{nT}.

Lastly, define the population *impulse-response functions* as the entries of the $n \times q$ matrix $\mathbf{B}^*(L) = \mathbf{A}(L)^{-1}\mathbf{R}^*$, and their estimators as those of $\hat{\mathbf{B}}^*(L) = \hat{\mathbf{A}}(L)^{-1}\hat{\mathbf{R}}^*$. Denoting by $b_{if}^*(L) = b_{if,0}^* + b_{if,1}^* L + \cdots$ and $\hat{b}_{if}^*(L) = \hat{b}_{if,0}^* + \hat{b}_{if,1}^* L + \cdots$, respectively, such entries, Propositions 9 and 10 imply that $|\hat{b}_{if,k}^* - b_{if,k}^*| = O_P(\zeta_{nT})$ for all i, f and k.

An iterative identification scheme will be used in Section 4 to compare different estimates of the impulse-response functions.[4]

Our estimator of $\mathbf{v}_t$ is simply the projection of $\hat{\mathbf{z}}_t$ on $\hat{\mathbf{P}}^z(\hat{\mathbf{\Lambda}}^z)^{-1/2}$, namely,

$$\hat{\mathbf{v}}_t = ((\hat{\mathbf{\Lambda}}^z)^{1/2}\hat{\mathbf{P}}^{z\prime}\hat{\mathbf{P}}^z(\hat{\mathbf{\Lambda}}^z)^{1/2})^{-1}(\hat{\mathbf{\Lambda}}^z)^{1/2}\hat{\mathbf{P}}^{z\prime}\hat{\mathbf{z}}_t = (\hat{\mathbf{\Lambda}}^z)^{-1/2}\hat{\mathbf{P}}^{z\prime}\hat{\mathbf{z}}_t.$$

For that estimation $\hat{\mathbf{v}}_t$ we have the following consistency result.

Proposition 11 *Under Assumptions 1 through 10, $\|\hat{\mathbf{v}}_t - \hat{\mathbf{W}}_q \mathbf{v}_t\| = O_P(\zeta_{nT})$, as $T \to \infty$ and $n \to \infty$, where $\hat{\mathbf{W}}_q$ is a $q \times q$ diagonal matrix, depending on n and T, whose diagonal*

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[4]All just-identifying rules considered in the SVAR literature can be dealt with along the same lines, see Forni et al. (2009).

entries equal either 1 *or* −1.

See Appendix E for the proof.

[PZ The finite-dimensional representation in the $\mathbf{Z}_{nt}$ allows to re-write the model is state space. Therefore, the loadings $\mathcal{R}$ and the dynamic factors $\mathbf{v}_t$ can be estimated by a quasi maximum likelihood estimation approach based on the Kalman recursion. Doz, Giannone and Reichlin (2011,2012) establish the asymptotic theory for this estimation procedure when the $\mathbf{Z}_{nt}$ are observed. Besides the additional difficulty of dealing with estimated $\mathbf{Z}_{nt}$, possible gains from using the Kalman filter in our context are likely to be rather limited in practice: as discussed in Doz et al (2012), making use of the state space of the finite-dimensionl GDFM is justified whenever the common factors are truly dynamic, such as for instance when these follow a vector autoregression. In our case, the finite-dimensional representation of the GDFM in the Z_{nt} has white-noise common factors. Hence, the static principal estimator, here studied, and the quasi maximum likelihood estimator will exhibit in practice very similar statistical properties. [5]]

3.5   Estimation and cross-sectional ordering

A fundamental feature of econometric panel data is that cross-sectional ordering is completely arbitrary. Hence, sensible concepts, and sensible inference methods, as a rule, should be invariant under the $n!$ permutations of cross-sectional units. While all definitions above (common and idiosyncratic components, dynamic eigenvalues and principal components, etc.), as well as the estimation method proposed in Forni et al. (2000), clearly are insensitive to such permutations, the FHLZ one-sided estimation method, as just described, is not. The ordering of the panel indeed has a crucial impact on the selection of the $(q+1)$-dimensional blocks in the autoregressive representation of Section 2.2.

In principle, any permutation of the n cross-sectional items—more precisely, any of the $n!/(n/(q+1))!(q+1)!^{n/(q+1)}$ permutations that lead to distinct partitions of $\{1, 2, \ldots, n\}$ into $n/(q+1)$ subsets of size $(q+1)$—yield distinct estimators, all enjoying the properties we have been establishing for those associated with the "original" ordering. Those estimators

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[5]Indeed, the static principal components coincide precisely with the maximum likelihood estimator when the factors are white-noise and the idiosyncratic component has a spherical covariance matrix.

somehow should be aggregated into a unique one, which should improve performances while restoring permutational invariance.

If we assume that the n cross-sectional indices are determined randomly, and that all $n!$ orderings are equally probable, averaging the the $n!$ resulting estimators seems quite natural. An arbitrarily ordered panel—for instance, the one resulting from ordering the observations at time $t = 1$ from smallest to largest—then is a sufficient statistic. If that statistic also can be assumed *complete*, the *Rao-Blackwellization* of the associated estimators leads to an arithmetic mean of the $n!$ estimators resulting from the $n!$ possible cross-sectional orderings (see Chapter 2 of Lehmann and Casella 1998). The celebrated Lehmann-Scheffé Theorem then tells us that this arithmetic mean minimizes the expected square deviation among all estimators sharing the same bias (as the original one; ibid., page 88).

Now, performing $n!$ dynamic factor analyses and estimations on an n-dimensional panel, even for moderately large values of n, is infeasible. However, in practice, it appears that, selecting a few permutations at random and averaging the corresponding estimators leads to rapidly stabilizing results: see Section 4 for practical details and a numerical illustration.

4 A simulation exercise

In this section, we evaluate numerically the performance of the estimation methods studied in the previous sections. We focus on (i) estimation of impulse response function, (ii) estimation of structural shocks and (iii) one-step-ahead forecasts. Regarding (i) and (ii), we compare FHLZ with the method proposed in Forni et al. (2009), referred to as FGLR. As regards (iii), the results of FHLZ are compared to the method in Stock and Watson (2002a), referred to as SW. Let us recall that both FGLR and SW assume the existence of the static factor representation (1.3), and are based on ordinary principal components. We generate artificial data according to two simple models: (I) a dynamic factor model with no static factor model representation (so that neither FGLR nor SW are consistent) and (II) a model admitting a static factor model representation (under which all methods are consistent).

4.1 Data-generating processes

We consider the following data-generating processes.

Model I (no static factor model representation)

$$x_{it} = a_{i1}(1 - \alpha_{i1}L)^{-1}u_{1t} + a_{i1}(1 - \alpha_{i2}L)^{-1}u_{2t} + \xi_{it}.$$

We generate u_{jt}, $j = 1, 2$ and ξ_{it}, $i = 1, \ldots, n$, $t = 1, \ldots, T$ as i.i.d. standard Gaussian variables; a_{ji} as independent variables, uniformly distributed on the interval $[-1, 1]$; α_{ji} as independent variables, uniformly distributed on the interval $[-0.8, 0.8]$.

Estimation of the shocks and the impulse-response functions requires an identification rule. Our exercise is based on a Choleski identification scheme on the first q variables. Precisely, denote by $\mathbf{B}_q(0)$ the matrix with $b_{if}(0)$, $i = 1, 2, \ldots, q$, $f = 1, 2, \ldots, q$, in the (i, f) entry, and let $\mathbf{H}$ be the lower triangular matrix with positive diagonal entries such that $\mathbf{HH}' = \mathbf{B}_q(0)\mathbf{B}_q(0)'$. Then, the "structural" shocks, denoted by $\mathbf{u}_t^*$, and the impulse-response functions, denoted by $\mathbf{b}_i^*(L)$, are $\mathbf{b}_i^*(L) = \mathbf{b}_i(L)\mathbf{B}_q(0)^{-1}\mathbf{H}$ and $\mathbf{u}_t^* = \mathbf{H}'\mathbf{B}_q(0)'\mathbf{u}_t$, respectively.

Model II (with static factor representation)

$$
\begin{aligned}
x_{it} &= \lambda_{i1}F_{1t} + \lambda_{i2}F_{2t} + \cdots + \lambda_{ir}F_{rt} + \xi_{it} \\
\mathbf{F}_t &= \mathbf{D}\mathbf{F}_{t-1} + \mathbf{K}\mathbf{u}_t.
\end{aligned}
$$

Here $\mathbf{F}_t = (F_{1t} \ \ldots \ F_{rt})'$ and $\mathbf{u}_t = (u_{1t} \ \ldots \ u_{qt})'$, $\mathbf{D}$ is $r \times r$ and $\mathbf{K}$ is $r \times q$. Again, u_{jt}, $j = 1, \ldots, q$ and ξ_{it}, $i = 1, \ldots, n$, $t = 1, \ldots, T$ are i.i.d. standard Gaussian and mutually independent white noises. Moreover, λ_{hi}, $h = 1, \ldots, r$, $i = 1, \ldots, n$ and the entries of $\mathbf{K}$ are independently, uniformly distributed on the interval $[-1, 1]$. Finally, the entries of $\mathbf{D}$ are generated as follows: first we generated entries independently, uniformly distributed on the interval $[-1, 1]$; second, we divided the resulting matrix by its spectral norm to obtain unit norm; third, we multiplied the resulting matrix by a random variable uniformly distributed on the interval $[0.4, 0.9]$, to ensure stationarity while preserving sizable dynamic responses. Precisely, $\mathbf{b}_i(L) = \boldsymbol{\lambda}_i(\mathbf{I} - \mathbf{D}L)^{-1}\mathbf{K}$, $\boldsymbol{\lambda}_n$ being the $1 \times r$ matrix having λ_{ih} as its (i, h) entry. To identify the "structural" shocks $\mathbf{u}_t^*$ and the corresponding impulse response functions $\mathbf{b}_i^*(L)$ we impose a Cholesky identification scheme on the first q variables as in Model I.

4.2 Estimation details and accuracy evaluation

Let $b_{if}^*(L) = \sum_{k=0}^\infty b_{if,k}^* L^k$ be the f-th entry of $\mathbf{b}_i^*(L)$. Our target is estimation of $b_{if,k}^*$, $i = 1, \ldots, n$, $f = 1, \ldots, q$, $k = 0, \ldots, K$ and u_{ft}^*, $f = 1, \ldots, q$, $t = 1, \ldots, T$, as well as forecasting of x_{iT+1}, $i = 1, \ldots, n$.

The structural impulse response functions and the structural shocks are estimated by the FHLZ and the FGLR method. For FHLZ, the number of lags for each $(q + 1)$-dimensional VAR is determined by the BIC criterion. The contemporaneous and lagged covariances of the common components needed to compute the VAR coefficients are estimated by the FHLR (2000) dynamic principal component method, with a Bartlett lag window of size $B_T = \sqrt{T}$. As regards FGLR, we estimate a VAR for the principal components of the data. The number of principal components is either assumed known or determined by Bai and Ng's IC_{p2} criterion, the number of lags is determined by the BIC criterion. The number of structural shocks is assumed to be known: such condition is obviously needed when estimating the structural shocks and impulse response functions. Identification is obtained by imposing the Cholesky scheme above.

Regarding prediction, FHLZ forecasts are computed by filtering the estimated shocks with the estimated impulse response functions:

$$\hat{x}_{i,T+1} = \sum_{f=1}^q \left(\hat{b}_{if,1}^* \hat{u}_{fT}^* + \hat{b}_{if,2}^* \hat{u}_{f,T-1}^* + \cdots \right).$$

The number of structural shocks is no longer assumed known. Rather, it is estimated by the Hallin and Liška (2007) method.[6] SW forecasts are obtained by regressing $x_{i,T+1}$ onto either the ordinary principal components at T and x_{iT}, or the principal components at T alone. The former method corresponds to the original Stock and Watson (2002a) method; the latter is motivated by the fact that in both of the models above the idiosyncratic components are serially uncorrelated. The number of principal components is determined with Bai and Ng's IC_{p2} criterion.

The estimation error for the impulse-response functions is defined as the normalized sum of the squared deviations of the estimated from the "structural" impulse response coefficients.

---

[6]We used the log criterion $IC_{2;n}^T$ with penalty function p_1 and lag window equal to $\sqrt{T}$. The "second stability interval" was evaluated over the grid $n_j = \lfloor (3n/4 + jn/40) \rfloor$, $T_j = T$, $j = 1, \ldots, 10$.

Precisely, let $\hat{b}^*_{if,k}$ be the estimated impulse-response coefficient of variable i, shock f, lag k: the estimation error on the impulse response functions is measured by

$$\frac{\sum_{i=1}^{n} \sum_{f=1}^{q} \sum_{k=0}^{K} \left(\hat{b}^*_{if,h} - b^*_{if,h}\right)^2}{\sum_{i=1}^{n} \sum_{f=1}^{q} \sum_{k=0}^{K} (b^*_{if,k})^2}.$$

The truncation lag K is set to 60. Similarly, denoting by $\hat{u}^*_{ft}$ the estimate of u^*_{ft}, the estimation error on the "structural" shocks is measured by

$$\frac{\sum_{f=1}^{q} \sum_{t=1}^{T} \left(\hat{u}^*_{ft} - u^*_{ft}\right)^2}{\sum_{f=1}^{q} \sum_{t=1}^{T} (u^*_{ft})^2}.$$

Finally, the accuracy of the forecast is measured by the sum of the squared deviations of the forecasts from the unfeasible forecasts obtained by filtering the true structural shocks with the true structural impulse response functions, i.e. $x^P_{iT+1} = \sum_{f=1}^{q} \sum_{k=1}^{T} b^*_{if,k} u^*_{fT+1-k}$. Again, we normalize by dividing by the sum of the squared targets:

$$\frac{\sum_{i=1}^{n} \left(\hat{x}_{iT+1} - x^P_{iT+1}\right)^2}{\sum_{i=1}^{n} (x^P_{iT+1})^2}.$$

Model I is evaluated for different sample size combinations, with $n = 30, 60, 120, 240$ and $T = 60, 120, 240, 480$. Model II is evaluated for a fixed sample size of $n = 120$ and $T = 240$, but different configurations of q and r, i.e. $r = 4, 6, 8, 12$ and $q = 2, 4, 6$, $r > q$.[7] For each couple (n, T), Model I, and (r, q), Model II, we generated 500 data sets and computed the average MSE.

4.3 Cross-sectional permutations

As explained in Section 3.5, the estimators obtained via the FHLZ method should be averaged over $n!$ permutations of cross-sectional items. In order to study the influence of such permutations, we simulated 500 datasets from Model I and various values of n and T. For each of the resulting panels, we computed (with the Choleski identification rule described in Section 4.1) the estimated impulse response functions averaged over $m = 1, \ldots, M$ randomly chosen permutations. For each value of m, the MSEs (over the 500 replications) of the averaged estimators wee recorded, leading to the following conclusions:

---

[7]We impose $r > q$ since for the case $r = q$, method FHLZ, the regressors of the $q + 1$-dimensional VARs are asymptotically collinear.

Figure 1: Model I. Average MSE of estimated impulse response functions over 500 experiments, as a function of the number of random reorderings of the variables used in estimation.
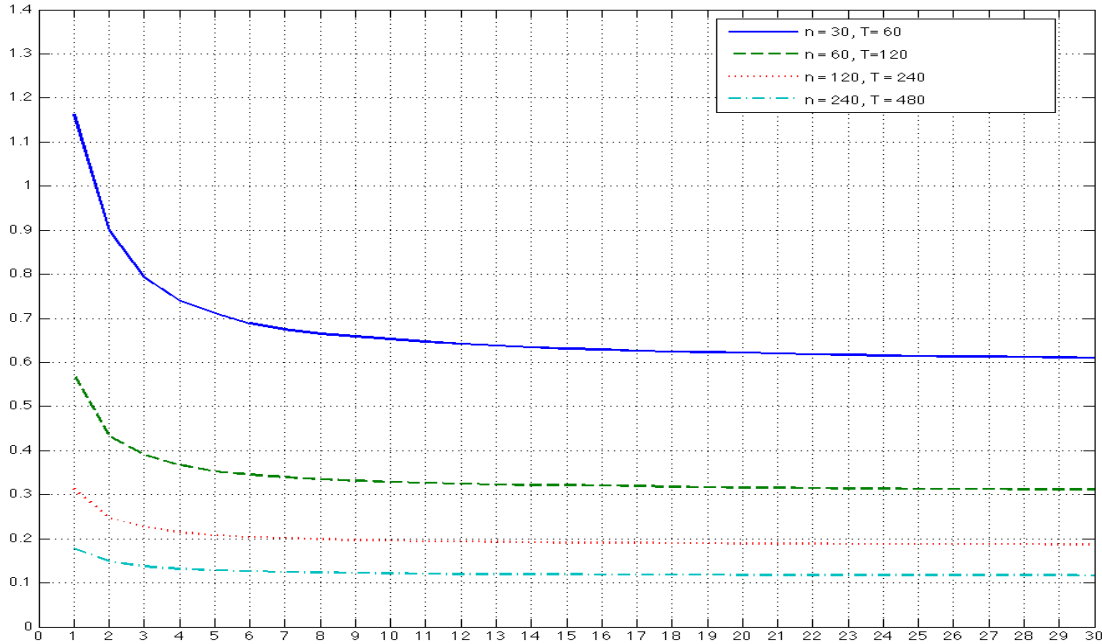
(i)   as expected, estimates corresponding to different random permutations do differ;

(ii)   averaging those estimates yields a clear improvement in the MSE;

(iii)   the rate of that improvement declines steadily as the number m of permutations increases, and rapidly stabilizes until additional permutations produce negligible effect;

(iv)   as n and T increase, the improvement decreases, both in absolute and relative terms, and the number of permutations required for "stabilization" decreases: 10 for ($n = 60$, $T = 120$), only 5 for ($n = 240$, $T = 480$).

Results are reported in Figure 1.

Summing up, averaging over random permutations until the resulting estimates stabilize is essentially equivalent to averaging over the $n!$ possible permutations, hence restores the independence of the FHLZ method with respect to the panel ordering, while significantly improving the small-sample performance of FHLZ. Such averaging moreover does not modify the asymptotic results of Section 3.

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4.4 Results

We now turn to a performance comparison between the FHLZ method and its competitors.

Table 1 reports the results for the estimation of impulse response functions and structural shocks, Model I. The upper panel reports results for the FHLZ method without averaging; the central panel for the FHLZ with averaging over 30 reorderings; the lower panel for the FGLR method. The estimates obtained with FGLR, despite being theoretically inconsistent, approach the target as n and T get larger. This is because the number of estimated static factors increases with n and T, so that the static model achieves a fairly good approximation of the underlying "infinite-factor" model.[8] However, FHLZ clearly outperforms FGLR. Regarding impulse response functions, FHLZ, with and without averaging, dominates the static method for all n-T configurations. The error is up to 50-60% smaller than the one of FGLR. As for the shocks, the performance of FHLZ with averaging is similar to that of FGLR for large T, but dominates FGLR for small T.

Forecast results are reported in Table 2. Not surprisingly, the SW method (central and lower panels) performs better when lagged x's are not included among the regressors, owing to the fact that the idiosyncratic components are serially uncorrelated. Indeed, we are comparing forecasts of the common components of the x's, i.e. the χ's, rather than the x's themselves. FHLZ forecasts (with averaging) outperforms SW for all (n, T) configurations, with an improvement ranging from 20 to 40%.[9] Observe that here we no longer impose the correct q, but estimate it with Hallin and Liška's (2007) criterion, so that both forecasts in the upper an central panels are feasible.

Table 3 reports results for Model II, estimation of impulse response functions and structural shocks. Here both FHLZ and FGLR are consistent. Somewhat surprisingly, FHLZ (with averaging, upper panel) over-performs FGLR for all (r, q) configurations. With this model, Bai and Ng's criterion tends to underestimate the number of factors.[10] Hence, we computed the (unfeasible) FGLR estimation obtained by imposing the correct r (lower panel), to see whether the above result can be ascribed to underestimation of r. In general, FGLR performs better when imposing the correct number of factors; nonetheless, FHLZ still exhibits the best

---

[8]The average $\hat{r}$ is 2.01 for $n = 30, T = 60$ and 4.00 for $n = 240, T = 480$.

[9]FHLZ without averaging, not reported here, performs better than SW but worse than FHLZ with averaging, in line with the results in Table 1.

[10]On average, $\hat{r}$ is smaller than r for all n and T configurations.

Table 1: Model I, estimated impulse response functions and structural shocks. Average and standard deviation (in brackets) of normalized MSE across 500 data sets of different size. For the static method, the number of static factors is determined by Bai and Ng's IC_{p2} criterion.

	Impulse response functions				*Structural shocks*			
T	60	120	240	480	60	120	240	480
n				*FHLZ method , no averaging*				
30	0.93 (0.51)	0.54 (0.34)	0.32 (0.11)	0.22 (0.11)	0.84 (0.48)	0.59 (0.31)	0.48 (0.11)	0.45 (0.06)
60	0.68 (0.39)	0.41 (0.26)	0.25 (0.11)	0.17 (0.04)	0.55 (0.45)	0.38 (0.26)	0.30 (0.12)	0.28 (0.04)
120	0.58 (0.34)	0.34 (0.11)	0.22 (0.04)	0.15 (0.03)	0.38 (0.42)	0.23 (0.13)	0.18 (0.05)	0.16 (0.03)
240	0.55 (0.32)	0.33 (0.12)	0.21 (0.04)	0.15 (0.02)	0.31 (0.40)	0.17 (0.15)	0.11 (0.05)	0.10 (0.02)
n				*FHLZ method , with averaging*				
30	0.66 (0.31)	0.39 (0.19)	0.23 (0.06)	0.15 (0.04)	0.64 (0.30)	0.49 (0.18)	0.41 (0.07)	0.39 (0.05)
60	0.54 (0.29)	0.32 (0.15)	0.18 (0.05)	0.12 (0.03)	0.45 (0.32)	0.32 (0.16)	0.25 (0.05)	0.24 (0.03)
120	0.48 (0.24)	0.28 (0.10)	0.17 (0.04)	0.11 (0.02)	0.32 (0.29)	0.21 (0.11)	0.15 (0.04)	0.14 (0.02)
240	0.46 (0.23)	0.27 (0.09)	0.16 (0.04)	0.10 (0.02)	0.25 (0.29)	0.15 (0.10)	0.10 (0.04)	0.08 (0.02)
n				*static factor method (FGLR)*				
30	0.86 (0.45)	0.52 (0.20)	0.38 (0.07)	0.31 (0.05)	0.74 (0.45)	0.51 (0.20)	0.42 (0.08)	0.38 (0.06)
60	0.71 (0.38)	0.45 (0.13)	0.34 (0.05)	0.20 (0.03)	0.54 (0.44)	0.34 (0.15)	0.27 (0.05)	0.23 (0.04)
120	0.63 (0.32)	0.42 (0.09)	0.33 (0.04)	0.28 (0.03)	0.40 (0.38)	0.24 (0.11)	0.17 (0.05)	0.13 (0.03)
240	0.60 (0.28)	0.42 (0.10)	0.32 (0.03)	0.27 (0.02)	0.32 (0.34)	0.19 (0.13)	0.12 (0.04)	0.08 (0.02)

performance in most cases.

Forecasts errors, reported in Table 4, confirm the result that FHLZ performs better than SW for most (r, q) configurations.

5 Conclusions

An estimate of the common-component spectral density matrix $\hat{\boldsymbol{\Sigma}}^{\chi}$ is obtained using the frequency-domain principal components of the observations x_{it}. The central idea of the present paper is that, because $\hat{\boldsymbol{\Sigma}}^{\chi}$ has large dimension but small rank q, a factorization of $\hat{\boldsymbol{\Sigma}}^{\chi}$ can be obtained piecewise. Precisely, the factorization of $\hat{\boldsymbol{\Sigma}}^{\chi}$ only requires the factorization of $(q + 1)$-dimensional subvectors of $\boldsymbol{\chi}_t$. Under our assumption of rational spectral density for the common components, this implies that the number of parameters to estimate grows as n, not n^2.

Table 2: Model I, one-step-ahead forecasts. Average and standard deviation (in brackets) of the normalized mean square deviation from the population forecasts across 500 data sets of different size. For the dynamic method, the number of dynamic factors is determined by Hallin and Liška's log criterion. For the static method, the number of static factors is determined by Bai and Ng's IC_{p2} criterion.

	$T = 60$	$T = 120$	$T = 240$	$T = 480$
FHLZ method, with averaging				
$n = 30$	0.97 (0.65)	0.91 (1.06)	0.73 (0.61)	0.74 (0.67)
$n = 60$	0.82 (0.32)	0.68 (0.35)	0.59 (0.95)	0.50 (0.35)
$n = 120$	0.74 (0.21)	0.58 (0.16)	0.47 (0.27)	0.39 (0.22)
$n = 240$	0.70 (0.18)	0.53 (0.14)	0.41 (0.14)	0.33 (0.14)
static factor method (SW), with lagged x's				
$n = 30$	2.58 (3.46)	1.65 (2.99)	1.12 (1.81)	0.89 (0.88)
$n = 60$	2.17 (2.22)	1.28 (1.00)	0.99 (2.31)	0.73 (0.61)
$n = 120$	1.94 (1.53)	1.16 (0.72)	0.83 (0.90)	0.64 (0.43)
$n = 240$	1.87 (1.51)	1.08 (0.62)	0.75 (0.47)	0.60 (0.35)
static factor method (SW), no lagged x's				
$n = 30$	1.90 (2.62)	1.33 (2.05)	0.94 (0.95)	0.80 (0.74)
$n = 60$	1.52 (1.54)	1.02 (0.75)	0.86 (1.84)	0.68 (0.54)
$n = 120$	1.32 (0.89)	0.89 (0.48)	0.72 (0.66)	0.61 (0.39)
$n = 240$	1.24 (0.69)	0.82 (0.41)	0.64 (0.38)	0.56 (0.33)

The rational spectral density assumption also has the important consequences that $\boldsymbol{\chi}_t$ has a finite autoregressive representation and that the dynamic factor model can be transformed into the static model $\mathbf{z}_t = \mathbf{R}\mathbf{v}_t + \boldsymbol{\phi}_t$, where $\mathbf{z}_t = \mathbf{A}(L)\mathbf{x}_t$. We construct estimators for $\mathbf{A}(L)$, $\mathbf{R}$ and $\mathbf{v}_t$ starting with a standard non-parametric estimator of the spectral density of the x's. This implies a slower rate of convergence as compared to the usual $T^{-1/2}$. However, in Section 3, we prove that our estimators for $\mathbf{A}(L)$, $\mathbf{R}$ and $\mathbf{v}_t$ do not undergo any further reduction in their speed of convergence.

The main difference of the present paper with respect to previous literature on GDFM's is that although we make use of a parametric structure for the common components, we do not make the standard, but quite restrictive assumption that our dynamic factor model has a static representation of the form (1.3). Section 4 provides important empirical support to the richer dynamic structure of unrestricted GDFM's.

Table 3: Model II, estimated impulse response functions and structural shocks. Average and standard deviation (in brackets) of the normalized MSE across 500 data sets with different configurations of static and dynamic factors. For the static method, the number of static factors is determined by Bai and Ng's IC_{p2} criterion.

	Impulse response functions				Structural shocks			
r	4	6	8	12	4	6	8	12
q				FHLZ method, with averaging				
2	0.13 (0.05)	0.11 (0.05)	0.10 (0.05)	0.09 (0.07)	0.17 (0.08)	0.12 (0.07)	0.10 (0.06)	0.08 (0.07)
4		0.15 (0.09)	0.15 (0.11)	0.14 (0.15)		0.27 (0.15)	0.22 (0.16)	0.17 (0.17)
6			0.17 (0.09)	0.15 (0.10)			0.34 (0.13)	0.24 (0.16)
				FGLR method, r determined with IC_{p2}				
2	0.16 (0.14)	0.16 (0.13)	0.15 (0.13)	0.12 (0.14)	0.21 (0.25)	0.17 (0.26)	0.12 (0.19)	0.08 (0.13)
4		0.18 (0.15)	0.19 (0.16)	0.20 (0.23)		0.35 (0.26)	0.31 (0.28)	0.23 (0.28)
6			0.20 (0.13)	0.22 (0.15)			0.43 (0.22)	0.35 (0.25)
				FGLR method , r assumed known				
2	0.10 (0.07)	0.09 (0.07)	0.08 (0.07)	0.07 (0.06)	0.17 (0.14)	0.13 (0.13)	0.12 (0.10)	0.11 (0.06)
4		0.14 (0.12)	0.15 (0.14)	0.14 (0.19)		0.28 (0.22)	0.25 (0.23)	0.21 (0.23)
6			0.18 (0.13)	0.17 (0.14)			0.38 (0.21)	0.28 (0.21)

Table 4: Model II, one-step-ahead forecasts. Average and standard deviation (in brackets) of the normalized mean square deviations from the population forecasts, across 500 data sets with different configurations of static and dynamic factors. For the dynamic method, the number of dynamic factors is determined by Hallin and Liška's log criterion. For the static method, the number of static factors is determined by Bai and Ng's IC_{p2} criterion.

	$r = 4$	$r = 6$	$r = 8$	$r = 12$
FHLZ method, with averaging				
$q = 2$	0.79 (1.59)	0.68 (0.75)	0.59 (0.97)	0.56 (0.52)
$q = 4$		0.44 (0.36)	0.44 (0.28)	0.40 (0.20)
$q = 6$			0.40 (0.28)	0.38 (0.18)
static factor method (SW), no lagged x's				
$q = 2$	1.00 (2.10)	0.67 (1.04)	0.52 (64)	0.49 (0.66)
$q = 4$		0.61 (1.37)	0.53 (0.67)	0.43 (0.37)
$q = 6$			0.50 (0.58)	0.42 (0.34)

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Appendix

A   Proof of Proposition 6

Adding and subtracting $E(\hat{\sigma}_{ij}^x(\theta_h^*))$ within the absolute value in $E\left(\max_{|h|\leq B_T}\left|\hat{\sigma}_{ij}^x(\theta_h^*) - \sigma_{ij}^x(\theta_h^*)\right|^2\right)$ and re-arranging gives

$$E\left(\max_{|h|\leq B_T}\left|\hat{\sigma}_{ij}^x(\theta_h^*) - \sigma_{ij}^x(\theta_h^*)\right|^2\right) \leq E\left(\max_{|h|\leq B_T}\left|\hat{\sigma}_{ij}^x(\theta_h^*) - E\hat{\sigma}_{ij}^x(\theta_h^*)\right|^2\right) + \left(\max_{|h|\leq B_T}\left|E\hat{\sigma}_{ij}^x(\theta_h^*) - \sigma_{ij}^x(\theta_h^*)\right|^2\right).$$

The first term (variance) on the right hand satisfies

$$E\left(\max_{|h|\leq B_T}\left|\hat{\sigma}_{ij}^x(\theta_h^*) - E\hat{\sigma}_{ij}^x(\theta_h^*)\right|^2\right) \leq C(B_T log B_T/T),$$

by Lemma 10 of Wu and Zaffaroni (2015) with $\nu = 2$. Specifically, assumptions 1 and 2 of Wu and Zaffaroni (2015) are implied by ours and, moreover, $\delta_{kp}^{[x_{it}]} \leq A_1 \rho_1^k$, by our Proposition 5, implying $\Theta_{0,4}^{[i]} \leq A_1/(1-\rho_1) < \infty$.

As for the second (squared bias) term, simple calculations give

$$E\hat{\sigma}_{ij}^x(\theta) - \sigma_{ij}^x(\theta) = \frac{1}{2\pi}\left(\sum_{u=-T+1}^{T-1} e^{-\imath u\theta}(1-\frac{u}{T})\gamma_{ij,u}^x K(\frac{u}{B_T}) - \sum_{u=-\infty}^{\infty} e^{-\imath u\theta}\gamma_{ij,u}^x\right).$$

Smoothness of the cross-spectra $\sigma_{ij}^x(\theta)$ and standard arguments (see for instance Theorem V.4.10, p.282, of Hannan, 1970) imply that

$$\max_{|h|\leq B_T}\left|E\hat{\sigma}_{ij}^x(\theta_h^*) - \sigma_{ij}^x(\theta_h^*)\right|^2 \leq CB_T^{-2\kappa} \leq CT^{-2\underline{\delta}\kappa}.$$

The constant C is independent of i,j since, for some $0 < \rho_1 < 1$ and constant C_1, for every $i,j \in \mathcal{N}$

$$|\gamma_{ij,u}^x| \leq |\gamma_{ij,u}^\chi| + |\gamma_{ij,u}^\xi| \leq C_1\rho_1^u$$

by Assumption 2 and Proposition 2. Finally, note that the squared bias term tends to zero faster than $O(B_T \log B_T/T)$ whenever $1 < \underline{\delta}(2\kappa + 1)$.   □

B   Proof of Proposition 7

The proof below closely follows Forni et al. (2009). Denote by $\mu_j(\mathbf{A})$, $j = 1, 2, \ldots, s$, the (real) eigenvalues, in decreasing order, of a complex $s \times s$ Hermitian matrix $\mathbf{A}$, and by

$\|\mathbf{B}\| = \sqrt{\mu_1(\tilde{\mathbf{B}}\mathbf{B})}$ the spectral norm of an $s_1 \times s_2$ matrix $\mathbf{B}$. The norm $\|\mathbf{B}\|$ coincides with the Euclidean norm of $\mathbf{B}$ when $\mathbf{B}$ is a column matrix and is equal to $|\mu_1(\mathbf{B})|$ when $\mathbf{B}$ is square and Hermitian. Recall that, if $\mathbf{B}_1$ is $s_1 \times s_2$ and $\mathbf{B}_2$ is $s_2 \times s_3$, then

$$\|\mathbf{B}_1\mathbf{B}_2\| \leq \|\mathbf{B}_1\|\|\mathbf{B}_2\|. \tag{B.1}$$

We will use the fact that, for any two $s \times s$ Hermitian matrices $\mathbf{A}_1$ and $\mathbf{A}_2$,

$$|\mu_j(\mathbf{A}_1 + \mathbf{A}_2) - \mu_j(\mathbf{A}_1)| \leq \|\mathbf{A}_2\|, \quad j = 1, \ldots, s. \tag{B.2}$$

This fact is an obvious consequence of Weyl's inequality $\mu_j(\mathbf{A}_1 + \mathbf{A}_2) \leq \mu_j(\mathbf{A}_1) + \mu_1(\mathbf{A}_2)$ (Franklin, 2000, p. 157, Theorem 1).

The proof of Proposition 7 is divided into several intermediate propositions. Denote by $\mathcal{S}_i$ the $n \times 1$ matrix with 1 in entries $(i, 1)$ and 0 elsewhere, so that $\mathcal{S}_1'\mathbf{A}$ is the i-th row of $\mathbf{A}$, and define $\rho_T = T/B_T \log B_T$.

As most of the arguments below depend on equalities and inequalities that hold for all $\theta \in [-\pi, \pi]$, the notation has been simplified by dropping θ. Properties holding for $\max_{|h| \leq B_T} F(\theta_h)$, where F is some function of θ, are often phrased as holding for F *uniformly in θ*. The meaning of uniformity in i, or i and j, has been clarified in the statement of Proposition 7.

All lemmas in this Appendix hold and are proved under Assumptions 1 through 10.

Lemma 1 *As $T \to \infty$ and $n \to \infty$,*

(i) $\max_{|h| \leq B_T} n^{-1}\|\hat{\mathbf{\Sigma}}^x - \mathbf{\Sigma}^x\| = O_P(\rho_T^{-1/2})$;

(ii) $\max_{|h| \leq B_T} n^{-1/2}\|\mathcal{S}_i'(\hat{\mathbf{\Sigma}}^x - \mathbf{\Sigma}^x)\| = O_P(\rho_T^{-1/2})$ *uniformly in i;*

(iii) $\max_{|h| \leq B_T} n^{-1}\|\hat{\mathbf{\Sigma}}^x - \mathbf{\Sigma}^\chi\| = O_P(\max(n^{-1}, \rho_T^{-1/2}))$;

(iv) $\max_{|h| \leq B_T} n^{-1/2}\|\mathcal{S}_i'(\hat{\mathbf{\Sigma}}^x - \mathbf{\Sigma}^\chi)\| = O_P(\max(n^{-1/2}, \rho_T^{-1/2})) = O_P(\zeta_{nT})$ *uniformly in i.*

PROOF. We have

$$\mu_1((\hat{\mathbf{\Sigma}}^x - \mathbf{\Sigma}^x)(\tilde{\hat{\mathbf{\Sigma}}}^x - \tilde{\mathbf{\Sigma}}^x)) \leq \text{trace}((\hat{\mathbf{\Sigma}}^x - \mathbf{\Sigma}^x)(\tilde{\hat{\mathbf{\Sigma}}}^x - \tilde{\mathbf{\Sigma}}^x)) = \sum_{i=1}^n \sum_{j=1}^n |\hat{\sigma}_{ij}^x - \sigma_{ij}^x|^2.$$

Using (3.4) and the Markov inquality,

$$n^{-2} \max_{|h| \leq B_T} \sum_{i=1}^n \sum_{j=1}^n |\hat{\sigma}_{ij}^x - \sigma_{ij}^x|^2 \leq n^{-2} \sum_{i=1}^n \sum_{j=1}^n \max_{|h| \leq B_T} |\hat{\sigma}_{ij}^x - \sigma_{ij}^x|^2 \leq C\rho_T^{-1}.$$

Statement (i) follows. In the same way,

$$n^{-1}\mathcal{S}_i'(\hat{\boldsymbol{\Sigma}}^x - \boldsymbol{\Sigma}^x)(\tilde{\hat{\boldsymbol{\Sigma}}}^x - \tilde{\boldsymbol{\Sigma}}^x)\mathcal{S}_i = n^{-1}\sum_{j=1}^n |\hat{\sigma}_{ij}^x - \sigma_{ij}^x|^2 \leq C\rho_T^{-1},$$

where C is independent of i. Statement (ii) follows. As regards (iii), $\boldsymbol{\Sigma}^x = \boldsymbol{\Sigma}^\chi + \boldsymbol{\Sigma}^\xi$ implies that $\hat{\boldsymbol{\Sigma}}^x - \boldsymbol{\Sigma}^\chi = \hat{\boldsymbol{\Sigma}}^x - \boldsymbol{\Sigma}^x + \boldsymbol{\Sigma}^\xi$, so that, by the triangle inequality for matrix norm,

$$\|\hat{\boldsymbol{\Sigma}}^x - \boldsymbol{\Sigma}^\chi\| \leq \|\hat{\boldsymbol{\Sigma}}^x - \boldsymbol{\Sigma}^x\| + \|\boldsymbol{\Sigma}^\xi\|.$$

The statement follows from (i) and the fact that $\|\boldsymbol{\Sigma}^\xi\| = \lambda_1^\xi$ is bounded. Statement (iv) is obtained in a similar way, using (ii) instead of (i). $\qquad\square$

Lemma 2 *As $T \to \infty$ and $n \to \infty$,*

(i) $\max_{|h| \leq B_T} n^{-1}\left|\hat{\lambda}_f^x - \lambda_f^\chi\right| = O_P(\max(n^{-1}, \rho_T^{-1/2}))$ *for $f = 1, 2, \ldots, q$;*

(ii) letting

$$\mathbf{G}^\chi = \begin{cases} \mathbf{I}_q & \text{if } \lambda_q^\chi = 0, \\ n(\boldsymbol{\Lambda}^\chi)^{-1} & \text{otherwise,} \end{cases} \quad \text{and} \quad \hat{\mathbf{G}}^x = \begin{cases} \mathbf{I}_q & \text{if } \hat{\lambda}_q^x = 0, \\ n(\hat{\boldsymbol{\Lambda}}^x)^{-1} & \text{otherwise,} \end{cases},$$

$\max_{|h| \leq B_T} n^{-1}\|\boldsymbol{\Lambda}^\chi\|$ *and* $\max_{|h| \leq B_T} \|\mathbf{G}^\chi\|$ *are $O(1)$, $\max_{|h| \leq B_T} n^{-1}\|\hat{\boldsymbol{\Lambda}}^x\|$ and $\max_{|h| \leq B_T} \|\hat{\mathbf{G}}^x\|$ are $O_P(1)$.*

PROOF. Setting $\mathbf{A}_1 = \boldsymbol{\Sigma}^\chi$ and $\mathbf{A}_2 = \hat{\boldsymbol{\Sigma}}^x - \boldsymbol{\Sigma}^\chi$, (B.2) yields $|\hat{\lambda}_f^x - \lambda_f^\chi| \leq \|\hat{\boldsymbol{\Sigma}}^x - \boldsymbol{\Sigma}^\chi\|$; hence, statement (i) follows from Lemma 1 (iii). Boundedness of $n^{-1}\|\boldsymbol{\Lambda}^\chi\|$ and $\|\mathbf{G}^\chi\|$, uniformly in θ, is a consequence of Assumption 3. Boundedness in probability of $n^{-1}\|\hat{\boldsymbol{\Lambda}}^x\|$ and $\|\hat{\mathbf{G}}^x\|$, uniformly in θ, follows from statement (i). $\qquad\square$

Lemma 3 *As $T \to \infty$ and $n \to \infty$,*

(i) $\max_{|h| \leq B_T} n^{-1}\|\tilde{\mathbf{P}}^\chi\hat{\mathbf{P}}^x\hat{\boldsymbol{\Lambda}}^x - \boldsymbol{\Lambda}^\chi\tilde{\mathbf{P}}^\chi\hat{\mathbf{P}}^x\| = O_P(\max(n^{-1}, \rho_T^{-1/2}))$;

(ii) $\max_{|h| \leq B_T} \|\tilde{\hat{\mathbf{P}}}^x\mathbf{P}^\chi\tilde{\mathbf{P}}^\chi\hat{\mathbf{P}}^x - \mathbf{I}_q\| = O_P(\max(n^{-1}, \rho_T^{-1/2}))$;

(iii) there exist diagonal complex orthogonal matrices $\hat{\mathbf{W}}_q = \text{diag}(\hat{w}_1 \ \hat{w}_2 \ \cdots \ \hat{w}_q)$, $|\hat{w}_j|^2 = 1$, $j = 1, \ldots, q$ depending on n and T, such that $\max_{|h| \leq B_T} \|\tilde{\hat{\mathbf{P}}}^x\mathbf{P}^\chi - \hat{\mathbf{W}}_q\| = O_P(\max(n^{-1}, \rho_T^{-1/2}))$.

PROOF. Using inequality (B.1) and the fact that $\|\tilde{\mathbf{P}}^\chi\| = \|\hat{\mathbf{P}}^x\| = 1$, we have

$$\|\tilde{\mathbf{P}}^\chi\hat{\mathbf{P}}^x\hat{\boldsymbol{\Lambda}}^x - \boldsymbol{\Lambda}^\chi\tilde{\mathbf{P}}^\chi\hat{\mathbf{P}}^x\| = \|\tilde{\mathbf{P}}^\chi(\hat{\boldsymbol{\Sigma}}^x - \boldsymbol{\Sigma}^\chi)\hat{\mathbf{P}}^x\| \leq \|\hat{\boldsymbol{\Sigma}}^x - \boldsymbol{\Sigma}^\chi\|.$$

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Statement (i) thus follows from Lemma 1 (iii). Turning to (ii), set

$$\mathbf{a} = \tilde{\tilde{\mathbf{P}}}^x \mathbf{P}^\chi \tilde{\mathbf{P}}^\chi \hat{\mathbf{P}}^x, \qquad \mathbf{b} = \left[\tilde{\tilde{\mathbf{P}}}^x \mathbf{P}^\chi \tilde{\mathbf{P}}^\chi \hat{\mathbf{P}}^x\right] n^{-1} \hat{\mathbf{\Lambda}}^x \hat{\mathbf{G}}^x = \tilde{\tilde{\mathbf{P}}}^x \mathbf{P}^\chi \left[\tilde{\mathbf{P}}^\chi \hat{\mathbf{P}}^x n^{-1} \hat{\mathbf{\Lambda}}^x\right] \hat{\mathbf{G}}^x,$$

$$\mathbf{c} = \tilde{\tilde{\mathbf{P}}}^x \mathbf{P}^\chi \left[n^{-1} \mathbf{\Lambda}^\chi \tilde{\mathbf{P}}^\chi \hat{\mathbf{P}}^x\right] \hat{\mathbf{G}}^x = \left[n^{-1} \tilde{\tilde{\mathbf{P}}}^x \mathbf{\Sigma}^\chi \hat{\mathbf{P}}^x\right] \hat{\mathbf{G}}^x, \quad \mathbf{d} = \left[n^{-1} \tilde{\tilde{\mathbf{P}}}^x \hat{\mathbf{\Sigma}}^x \hat{\mathbf{P}}^x\right] \hat{\mathbf{G}}^x = n^{-1} \hat{\mathbf{\Lambda}}^x \hat{\mathbf{G}}^x,$$

and $\mathbf{f} = \mathbf{I}_q$; we have

$$\left\|\tilde{\tilde{\mathbf{P}}}^x \mathbf{P}^\chi \tilde{\mathbf{P}}^\chi \hat{\mathbf{P}}^x - \mathbf{I}_q\right\| \le \|\mathbf{a} - \mathbf{b}\| + \|\mathbf{b} - \mathbf{c}\| + \|\mathbf{c} - \mathbf{d}\| + \|\mathbf{d} - \mathbf{f}\|. \tag{B.3}$$

Using Lemma 2, statement (i), and the boundedness in probability, uniformly in θ, of $\|\tilde{\tilde{\mathbf{P}}}^x \mathbf{P}^\chi\|$, $\|\hat{\mathbf{G}}^x\|$ and $\|\tilde{\tilde{\mathbf{P}}}^x \mathbf{P}^\chi \tilde{\mathbf{P}}^\chi \hat{\mathbf{P}}^x\|$, all terms on the right-hand side of inequality (B.3) can be shown to be $O_P(\max(n^{-1}, \rho_T^{-1/2}))$, uniformly in θ.

Turning to (iii), note that, from statement (i), $n^{-1} \tilde{\tilde{\mathbf{P}}}_h^x \mathbf{P}_k^\chi (\lambda_k^\chi - \hat{\lambda}_h^x) = O_P(\max(n^{-1}, \rho_T^{-1/2}))$. Assumption 3 (asymptotic separation of the eigenvalues $\lambda_f^\chi(\theta)$) implies that, for $h \ne k$, $\tilde{\tilde{\mathbf{P}}}_h^x \mathbf{P}_k^\chi = O_P(\max(n^{-1}, \rho_T^{-1/2}))$. Moreover, $\sum_{f=1}^q |\tilde{\tilde{\mathbf{P}}}_h^x \mathbf{P}_f^\chi|^2 - 1 = O_P(\max(n^{-1}, \rho_T^{-1/2}))$ from statement (ii). Therefore,

$$|\tilde{\tilde{\mathbf{P}}}_h^x \mathbf{P}_h^\chi|^2 - 1 = (|\tilde{\tilde{\mathbf{P}}}_h^x \mathbf{P}_h^\chi| - 1)(|\tilde{\mathbf{P}}_h^\chi \hat{\mathbf{P}}_h^x| + 1) = O_P(\max(n^{-1}, \rho_T^{-1/2})).$$

The conclusion follows. □

Note that Lemma 3 clearly also holds for $n^{-1}\|\tilde{\tilde{\mathbf{P}}}^x \mathbf{P}^\chi \mathbf{\Lambda}^\chi - \hat{\mathbf{\Lambda}}^x \tilde{\tilde{\mathbf{P}}}^x \mathbf{P}^\chi\|$, $\|\tilde{\mathbf{P}}^\chi \hat{\mathbf{P}}^x \tilde{\tilde{\mathbf{P}}}^x \mathbf{P}^\chi - \mathbf{I}_q\|$ and $\|\tilde{\tilde{\mathbf{P}}}^\chi \hat{\mathbf{P}}^x - \tilde{\tilde{\mathbf{W}}}_q\|$.

Lemma 4 *As $T \to \infty$ and $n \to \infty$,*

$$\max_{|h| \le B_T} \|\mathcal{S}_i'(\mathbf{P}^\chi (\mathbf{\Lambda}^\chi)^{1/2} \hat{\mathbf{W}}_q - \hat{\mathbf{P}}^x (\hat{\mathbf{\Lambda}}^x)^{1/2})\| = O_P(\zeta_{nT}), \tag{B.4}$$

uniformly in i.

PROOF. We have

$$\|\mathcal{S}_i'(\mathbf{P}^\chi (\mathbf{\Lambda}^\chi)^{1/2} \hat{\mathbf{W}}_q - \hat{\mathbf{P}}^x (\hat{\mathbf{\Lambda}}^x)^{1/2})\| \le \|\mathcal{S}_i'(n^{1/2}\mathbf{P}^\chi \hat{\mathbf{W}}_q - n^{1/2}\hat{\mathbf{P}}^x)(n^{-1}\mathbf{\Lambda}^\chi)^{1/2}\|$$
$$+ \|\mathcal{S}_i' \hat{\mathbf{P}}^x (n^{-1/2}(\mathbf{\Lambda}^\chi)^{1/2} - n^{-1/2}(\hat{\mathbf{\Lambda}}^x)^{1/2})\|.$$

By Lemma 2 (i), thus, we only need to prove that

$$\|n^{1/2}\mathcal{S}_i'\mathbf{P}^\chi \hat{\mathbf{W}}_q - n^{1/2}\mathcal{S}_i'\hat{\mathbf{P}}^x\| = O_P(\max(n^{-1/2}, \rho_T^{-1/2})).$$

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Firstly, we show that

$$\|n^{1/2}\mathcal{S}'_i\mathbf{P}^\chi\| \leq \mathcal{A}, \tag{B.5}$$

for some $\mathcal{A}$ and all θ and i. Assumption 2 implies that $\sigma^\chi_{ii} = \sum_{f=1}^q \lambda^\chi_f |p^\chi_{if}|^2 \leq \mathcal{B}$, for some $\mathcal{B}$ and all θ and i. As all the terms in the sum are positive, $\lambda^\chi_f |p^\chi_{if}|^2 = (\lambda^\chi_f/n)n|p^\chi_{if}|^2 \leq \mathcal{B}$, for all θ and i. By Assumption 3, $\lambda^\chi_f/n \geq \mathcal{C} > 0$ for all θ and f, so that $n|p^\chi_{if}|^2 \leq \mathcal{D}$ for all θ and i. Hence, $n\, \mathcal{S}'_i\mathbf{P}^\chi\tilde{\mathbf{P}}^\chi\mathcal{S}_i$ is bounded uniformly in θ and i; (B.5) follows. Next, define

$$\mathbf{g} = n^{1/2}\mathcal{S}'_i\mathbf{P}^\chi\left[\hat{\mathbf{W}}_q\right], \qquad \mathbf{h} = n^{1/2}\mathcal{S}'_i\mathbf{P}^\chi\left[\tilde{\mathbf{P}}^\chi\hat{\mathbf{P}}^x\right] = n^{1/2}\mathcal{S}'_i\mathbf{P}^\chi[\tilde{\mathbf{P}}^\chi\hat{\mathbf{P}}^x\hat{\mathbf{\Lambda}}^x/n](\hat{\mathbf{\Lambda}}^x/n)^{-1},$$

$$\mathbf{i} = n^{1/2}\mathcal{S}'_i\mathbf{P}^\chi[(\mathbf{\Lambda}^\chi/n)\tilde{\mathbf{P}}^\chi\hat{\mathbf{P}}^x](\hat{\mathbf{\Lambda}}^x/n)^{-1} = [n^{-1/2}\mathcal{S}'_i\mathbf{\Sigma}^\chi]\hat{\mathbf{P}}^x(\hat{\mathbf{\Lambda}}^x/n)^{-1},$$

and

$$\mathbf{j} = [n^{-1/2}\mathcal{S}'_i\hat{\mathbf{\Sigma}}^x]\hat{\mathbf{P}}^x(\hat{\mathbf{\Lambda}}^x/n)^{-1} = n^{1/2}\mathcal{S}'_i\hat{\mathbf{P}}^x.$$

Lemma 3(iii) and inequality (B.5) imply that $\|\mathbf{g}-\mathbf{h}\|$ is $O_P(\max(n^{-1}, \rho_T^{-1/2}))$ uniformly in θ and i. Inequality (B.5), Lemma 3(i) and Lemma 2(ii) imply that $\|\mathbf{h}-\mathbf{i}\|$ is $O_P(\max(n^{-1}, \rho_T^{-1/2}))$ uniformly in θ and i. Moreover, $\|\hat{\mathbf{P}}^x(\hat{\mathbf{\Lambda}}^x/n)^{-1}\| = O_P(1)$, uniformly in θ, by Lemma 2(ii) and the fact that $\|\hat{\mathbf{P}}^x\| = 1$. Thus, using Lemma 1(iv), it is seen that, uniformly in θ and i, $\|\mathbf{i}-\mathbf{j}\|$ is $O_P(\max(n^{-1/2}, \rho_T^{-1/2}))$. The result follows. $\quad\square$

Proposition 7 now follows from

$$\hat{\mathbf{\Sigma}}^\chi = \left[\hat{\mathbf{P}}^x\left(\hat{\mathbf{\Lambda}}^x\right)^{1/2}\right]\left[\left(\hat{\mathbf{\Lambda}}^x\right)^{1/2}\tilde{\hat{\mathbf{P}}}^x\right] = \hat{\mathbf{P}}^\chi\hat{\mathbf{\Lambda}}^\chi\tilde{\hat{\mathbf{P}}}^\chi.$$

and

$$\mathbf{\Sigma}^\chi = \left[\mathbf{P}^x\left(\mathbf{\Lambda}^x\right)^{1/2}\hat{\mathbf{W}}_q\right]\left[\tilde{\hat{\mathbf{W}}}_q\left(\mathbf{\Lambda}^x\right)^{1/2}\tilde{\mathbf{P}}^x\right] = \mathbf{P}^\chi\mathbf{\Lambda}^\chi\tilde{\mathbf{P}}^\chi.$$

$\quad\square$

Note that the eigenvectors $\mathbf{P}^\chi$ are defined up to post-multiplication by a complex diagonal matrix with unit modulus diagonal entries. In particular, using the eigenvectors $\mathbf{\Pi}^\chi = \mathbf{P}^\chi\hat{\mathbf{W}}_q$, (B.4) would hold for $\mathbf{\Pi}^\chi(\mathbf{\Lambda}^\chi)^{1/2} - \hat{\mathbf{P}}^x(\hat{\mathbf{\Lambda}}^x)^{1/2}$. For the sake of simplicity, we avoid introducing a new symbol and henceforth refer to the result of Lemma 4 as

$$\max_{|h|\leq B_T} \|\mathcal{S}'_1(\mathbf{P}^\chi(\mathbf{\Lambda}^\chi)^{1/2} - \hat{\mathbf{P}}^x(\hat{\mathbf{\Lambda}}^x)^{1/2})\| = O_P(\max(n^{-1/2}, \rho_T^{-1/2})) \tag{B.6}$$

and the result of Lemma 3(iii) as

$$\|\tilde{\hat{\mathbf{P}}}^x\mathbf{P}^\chi - \mathbf{I}_q\| = O_P(\max(n^{-1}, \rho_T^{-1/2})).$$

In the same way, we drop $\hat{\mathbf{W}}_q$ in Lemmas 6, 7, 8, though not in the conclusion of Appendix D, nor in Appendix E.

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C   Proof of Proposition 9

To start with, note that, as the extreme right-hand side in (3.6) contains the term

$$\frac{\pi B}{B_T} \sum_{|h| \leq B_T} \left(|e^{\imath \ell \theta^*_{h-1}} - e^{\imath \ell \tilde{\theta}^*_{h-1}}| + |e^{\imath \ell \tilde{\theta}^*_{h-1}} - e^{\imath \ell \theta^*_{h-1}}| \right),$$

convergence in (3.7) is not uniform with respect to ℓ. However, estimation of the matrices $\mathbf{B}^\chi_k$ and $\mathbf{C}^\chi_{jk}$ only requires the covariances $\hat{\gamma}^\chi_{ij,\ell}$ with $\ell \leq S$, where S is finite. Therefore, Proposition 8 implies that $\|\hat{\mathbf{B}}^\chi_k - \mathbf{B}^\chi_k\|$ and $\|\hat{\mathbf{C}}^\chi_{jk} - \mathbf{C}^\chi_{jk}\|$ are $O_P(\max(n^{-1/2}, \rho_T^{-1/2}))$. From (2.15), applying (B.1),

$$\|\hat{\mathbf{A}}^{[k]} - \mathbf{A}^{[k]}\| \leq \|\hat{\mathbf{B}}^\chi_k\|\|(\hat{\mathbf{C}}^\chi_{kk})^{-1} - (\mathbf{C}^\chi_{kk})^{-1}\| + \|\hat{\mathbf{B}}^\chi_k - \mathbf{B}^\chi_k\|\| (\mathbf{C}^\chi_{kk})^{-1}\|.$$

By Assumption 2, $\|\mathbf{B}^\chi_k\| \leq W$ for some constant $W > 0$, so that $\|\hat{\mathbf{B}}^\chi_k\|$ is bounded in probability. By Assumptions 2 and 7, $\| (\mathbf{C}^\chi_{kk})^{-1} \| \leq W_1$ for some $W_1 > 0$. Observing that the entries of $(\mathbf{C}^\chi_{kk})^{-1}$ are rational functions of the entries of $\mathbf{C}^\chi_{kk}$, and that $\det (\mathbf{C}^\chi_{kk}) > 0$ by Assumption 7, Proposition 8 implies that $\|(\hat{\mathbf{C}}^\chi_{kk})^{-1} - (\mathbf{C}^\chi_{kk})^{-1}\|$ is $O_P(\max(n^{-1/2}, \rho_T^{-1/2}))$. The conclusion follows. $\qquad\qquad\qquad\square$

D   Proof of Proposition 10

Consider the static model $\mathbf{Z}_{nt} = \mathcal{R}\mathbf{v}_t + \mathbf{\Phi}_{nt}$. If $\mathbf{Z}_{nt} = \mathbf{A}(L)\mathbf{x}_{nt}$ were observed, i.e. if the matrices $\mathbf{A}(L)$ were known, then Proposition 10, with an estimator of $\mathcal{R}$ based on the empirical covariance $\mathbf{\Gamma}^z$ of the $\mathbf{Z}_{nt}$, would be straightforward. However, we only have access to $\hat{\mathbf{Z}}_{nt} = \hat{\mathbf{A}}(L)\mathbf{x}_t$ and its empirical covariance matrix $\hat{\mathbf{\Gamma}}^z$, which makes the estimation of $\mathcal{R}$ significantly more difficult. The consistency properties of our estimator follow from the convergence result (D.4) in Lemma 11, which establishes the asymptotic behavior of the difference $\mathbf{\Gamma}^z - \hat{\mathbf{\Gamma}}^z$; Lemmas 5 through 10 are but a preparation for that key result. All lemmas in this Appendix hold, and are proved under Assumptions 1 through 10.

Lemma 5   *For $f = 1, \ldots, q$, as $T \to \infty$ and $n \to \infty$,*

(i) $| p^\chi_{if} | = O(n^{-1/2})$ and $| \hat{p}^x_{if} | = O_P(n^{-1/2})$, uniformly in θ and i;

(ii) for any positive integer d, $n^{-1} \sum_{i=1}^n | p^\chi_{if} |^d$ and $n^{-1} \sum_{i=1}^n | \hat{p}^x_{if} |^d$ are $O(n^{-d/2})$ and $O_P(n^{-d/2})$, respectively, uniformly in θ.

PROOF. The first part of (i) follows from B.5. As regards the second part, let us first prove that $\hat{\sigma}_{ii}^x$ is $O_P(1)$ uniformly in θ and i. We have

$$\max_h \hat{\sigma}_{ii}^x(\theta_h) \leq \max_h \sigma_{ii}^x(\theta_h) + \max_h |\hat{\sigma}_{ii}^x(\theta_h) - \sigma_{ii}^x(\theta_h)|.$$

By Assumptions 2 and 4, the first term on the right-hand side is bounded uniformly in i. By the Markov inequality and (3.4),

$$P(\max_h |\hat{\sigma}_{ii}^x(\theta_h) - \sigma_{ii}^x(\theta_h)| \geq \eta) \leq \eta^{-2} E \left(\max_{|h| \leq B_T} |\hat{\sigma}_{ii}^x(\theta_h^*) - \sigma_{ii}^x(\theta_h^*)|^2 \right)$$

$$\leq \eta^{-2} C(T^{-1} B_T \log B_T)$$

Thus, for any $\epsilon > 0$, we can set

$$\eta(\epsilon) \geq \left[\frac{\max_T C(T^{-1} B_T \log B_T)}{\epsilon} \right]^{1/2},$$

irrespectively of θ_h and i. Because $\hat{\sigma}_{ii}^\chi \leq \hat{\sigma}_{ii}^x$, we have that $\hat{\sigma}_{ii}^\chi = \sum_{f=1}^q \hat{\lambda}_f^x |\hat{p}_{if}^x|^2 = O_P(1)$ uniformly in θ and i. As all the terms in the sum are positive, $\hat{\lambda}_f^x |\hat{p}_{if}^x|^2 = (\hat{\lambda}_f^x/n) n |\hat{p}_{if}^x|^2$ is $O_P(1)$ as well, uniformly in θ and i. Lemma 2 (i) and Assumption 3 imply that $\hat{\lambda}_f^x/n$ is $O_P(1)$ and bounded away from zero in probability uniformly in θ. The conclusion follows.

Statement (ii) is proved by induction. Consider $\mathbf{P}_f^\chi$. It follows from statement (i) that $n^{-1} \sum_{i=1}^n |p_{if}^\chi|$ is $O(n^{-1/2})$, uniformly in θ. Assume now that the result holds for $d - 1$, with $d \geq 2$. Using the first part of (i), uniformity in i in particular, we have

$$n^{-1} \sum_{i=1}^n |p_{if}^\chi|^d = n^{-1} \sum_{i=1}^n |p_{if}^\chi|^{d-1} |p_{if}^\chi|$$

$$\leq (\max_{i \leq n} |p_{if}^\chi|) n^{-1} \sum_{i=1}^n |p_{if}^\chi|^{d-1} = O(n^{-1/2} n^{-(d-1)/2}) = O\left(n^{-d/2}\right).$$

The same argument applies to $\hat{\mathbf{P}}_f^x$. $\qquad \square$

Lemma 6 *As $T \to \infty$ and $n \to \infty$,*

$$\max_{|h| \leq B_T} \left\| \mathbf{P}^\chi (\mathbf{\Lambda}^\chi)^{1/2} - \hat{\mathbf{P}}^x (\hat{\mathbf{\Lambda}}^x)^{1/2} \right\| = O_P(n^{1/2} \max(n^{-1}, \rho_T^{-1/2})). \tag{D.1}$$

PROOF. The left-hand side of (D.1) equals the left-hand side of (B.4) when $\mathcal{S}_i$ is replaced by $\mathbf{I}_n$. The proof goes along the same lines as that of Lemma 4. Firstly, $\|n^{1/2} \mathbf{P}^\chi\|$ is $O(n^{1/2})$.

Both $\|\mathbf{g} - \mathbf{h}\|$ and $\|\mathbf{h} - \mathbf{i}\|$ are $O_P(n^{1/2} \max(n^{-1}, \rho_T^{-1/2}))$. As for $\|\mathbf{i} - \mathbf{j}\|$, the conclusion follows from Lemma 1 (iii). □

Lemma 7 *For $f = 1, \ldots, q$, as $T \to \infty$ and $n \to \infty$, $|p_{if}^{\chi} - \hat{p}_{if}^{x}| = O_P(n^{-1/2} \max(n^{-1/2}, \rho_T^{-1/2}))$, uniformly in θ and i.*

PROOF. By (B.6), $p_{if}^{\chi}(\lambda_f^{\chi})^{1/2} - \hat{p}_{if}^{x}(\hat{\lambda}_f^{x})^{1/2} = O_P(\max(n^{-1/2}, \rho_T^{-1/2}))$, uniformly in θ and i. Now,

$$p_{if}^{\chi}(\lambda_f^{\chi})^{1/2} - \hat{p}^{x}(\hat{\lambda}_f^{x})^{1/2} = p_{if}^{\chi} \left((\lambda_f^{\chi})^{1/2} - (\hat{\lambda}_f^{x})^{1/2} \right) + (\hat{\lambda}_f^{x})^{1/2} \left(p_{if}^{\chi} - \hat{p}_{if}^{x} \right). \quad (D.2)$$

The former term on the right-hand side can be written as

$$n^{1/2} p_{if}^{\chi} \frac{(\lambda_f^{\chi} - \hat{\lambda}_f^{x})/n}{((\lambda_f^{\chi})^{1/2} + (\hat{\lambda}_f^{x})^{1/2})/n^{1/2}},$$

which is $O_P(\max(n^{-1}, \rho_T^{-1/2}))$, uniformly in θ and i, since the numerator is $O_P(\max(n^{-1}, \rho_T^{-1/2}))$, uniformly in θ, by Lemma 2(i); the denominator is bounded away from zero, uniformly in θ, by Assumption 3 and $n^{1/2} p_{if}^{\chi}$ is $O(1)$, uniformly in θ and i, by Lemma 5(i). It follows that the latter term in (D.2), $(\hat{\lambda}_f^{x})^{1/2} \left(p_{if}^{\chi} - \hat{p}_{if}^{x} \right)$, is $O_P(\max(n^{-1/2}, \rho_T^{1/2}))$, uniformly in θ and i. By Lemma 2(ii), $n^{-1/2}(\hat{\lambda}_f^{x})^{1/2}$ is bounded away from zero in probability, uniformly in θ. The result follows. □

Lemma 8 *For any integer $d \in \mathbb{N}$, for $f = 1, \ldots, q$, as $T \to \infty$ and $n \to \infty$,*

$$n^{-1} \sum_{i=1}^{n} |p_{if}^{\chi} - \hat{p}_{if}^{x}|^d = O_P((n^{-1} \max(n^{-1}, \rho_T^{-1}))^{d/2}), \quad (D.3)$$

uniformly in θ.

PROOF. Lemma 7 implies that $\left(\max_{i \le n} |p_{if}^{\chi} - \hat{p}_{if}^{x}| \right)$, and therefore $n^{-1} \sum_{i=1}^{n} |\hat{p}_{if}^{x} - p_{if}^{\chi}|$, are $O_P((n^{-1} \max(n^{-1}, \rho_T^{-1}))^{1/2})$, uniformly in θ. By induction, assume now that the result holds for $d - 1$, $d \ge 2$. We have

$$
\begin{aligned}
n^{-1} \sum_{i=1}^{n} |p_{if}^{\chi} - \hat{p}_{if}^{x}|^d &= n^{-1} \sum_{i=1}^{n} |p_{if}^{\chi} - \hat{p}_{if}^{x}|^{d-1} |p_{if}^{\chi} - \hat{p}_{if}^{x}| \\
&\le \left(\max_{i \le n} |p_{if}^{\chi} - \hat{p}_{if}^{x}| \right) n^{-1} \sum_{i=1}^{n} |p_{if}^{\chi} - \hat{p}_{if}^{x}|^{d-1} \\
&= O_P((n^{-1} \max(n^{-1}, \rho_T^{-1}))^{1/2}) \, O_P((n^{-1} \max(n^{-1}, \rho_T^{-1}))^{(d-1)/2}),
\end{aligned}
$$

uniformly in θ, as was to be shown. □

Lemma 9 *For $n \to \infty$ and $T \to \infty$, uniformly in θ,*

(i) $n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{n} |\hat{\sigma}_{ij}^{\chi}(\theta) - \sigma_{ij}^{\chi}(\theta)|^d = O_P((\max(n^{-1}, \rho_T^{-1}))^{d/2});$

(ii) $n^{-1} \sum_{i=1}^{n} |\hat{\sigma}_{ij}^{\chi}(\theta) - \sigma_{ij}^{\chi}(\theta)|^d = O_P((\max(n^{-1}, \rho_T^{-1}))^{d/2})$ *for any* $1 \le j \le n;$

(iii) $n^{-1} \sum_{i=1}^{n} |\hat{\sigma}_{ii}^{\chi}(\theta) - \sigma_{ii}^{\chi}(\theta)|^d = O_P((\max(n^{-1}, \rho_T^{-1}))^{d/2}).$

PROOF. We have

$$
\hat{\sigma}_{ij}^{\chi} - \sigma_{ij}^{\chi} = (\hat{\lambda}_1^x - \lambda_1^{\chi})\hat{p}_{i1}^x \bar{\hat{p}}_{j1}^x + \cdots + (\hat{\lambda}_q^x - \lambda_q^{\chi})\hat{p}_{iq}^x \bar{\hat{p}}_{jq}^x + \lambda_1^{\chi} \hat{p}_{i1}^x (\bar{\hat{p}}_{j1}^x - \bar{p}_{j1}^{\chi})
$$

$$
+ \lambda_1^{\chi} \bar{p}_{j1}^{\chi}(\hat{p}_{i1}^x - p_{i1}^{\chi}) + \ldots + \lambda_q^{\chi} \hat{p}_{iq}^x (\bar{\hat{p}}_{jq}^x - \bar{p}_{jq}^{\chi}) + \lambda_q^{\chi} \bar{p}_{jq}^{\chi}(\hat{p}_{iq}^x - p_{iq}^{\chi}).
$$

Using the triangular and C_r inequalities, by Lemmas 2, 5 and 8,

$$
n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{n} |\hat{\sigma}_{ij}^{\chi} - \sigma_{ij}^{\chi}|^d
$$

$$
\le (3q)^{d-1} \left(|\lambda_1^{\chi} - \hat{\lambda}_1^x|^d \left(n^{-1} \sum_{i=1}^{n} |\hat{p}_{i1}^x|^d \right)^2 + \cdots + |\lambda_q^{\chi} - \hat{\lambda}_q^x|^d \left(n^{-1} \sum_{i=1}^{n} |\hat{p}_{iq}^x|^d \right)^2 \right)
$$

$$
+ (3q)^{d-1}(\lambda_1^{\chi})^d \left(n^{-2} \sum_{i=1}^{n} |\hat{p}_{i1}^x|^d \sum_{j=1}^{n} |p_{j1}^{\chi} - \hat{p}_{j1}^x|^d + n^{-2} \sum_{j=1}^{n} |p_{j1}^{\chi}|^d \sum_{i=1}^{n} |p_{i1}^{\chi} - \hat{p}_{i1}^x|^d \right)
$$

$$
+ \cdots
$$

$$
+ (3q)^{d-1}(\lambda_q^{\chi})^d \left(n^{-2} \sum_{i=1}^{n} |\hat{p}_{iq}^x|^d \sum_{j=1}^{n} |p_{jq}^{\chi} - \hat{p}_{jq}^x|^d + n^{-2} \sum_{j=1}^{n} |p_{jq}^{\chi}|^d \sum_{i=1}^{n} |p_{iq}^{\chi} - \hat{p}_{iq}^x|^d \right)
$$

$$
= O_P((\max(n^{-1}, \rho_T^{-1/2}))^d) + O_P((\max(n^{-1}, \rho_T^{-1}))^{d/2}) = O_P((\max(n^{-1}, \rho_T^{-1}))^{d/2}).
$$

Statement (i) follows. For statement (ii),

$$
n^{-1} \sum_{i=1}^{n} |\hat{\sigma}_{ij}^{\chi} - \sigma_{ij}^{\chi}|^d
$$

$$
\le (3q)^{d-1} \left(|\lambda_1^{\chi} - \hat{\lambda}_1^x|^d |\hat{p}_{j1}^x|^d \, n^{-1} \sum_{i=1}^{n} |\hat{p}_{i1}^x|^d + \cdots + |\lambda_q^{\chi} - \hat{\lambda}_q^x|^d |\hat{p}_{jq}^x|^d \, n^{-1} \sum_{i=1}^{n} |\hat{p}_{iq}^x|^d \right)
$$

$$
+ (3q)^{d-1}(\lambda_1^{\chi})^d \left(\left| p_{j1}^{\chi} - \hat{p}_{j1}^x \right|^d n^{-1} \sum_{i=1}^{n} |\hat{p}_{i1}^x|^d + \left| p_{j1}^{\chi} \right|^d n^{-1} \sum_{i=1}^{n} |p_{i1}^{\chi} - \hat{p}_{i1}^x|^d \right)
$$

$$
+ \cdots
$$

$$
+ (3q)^{d-1}(\lambda_q^{\chi})^d \left(\left| p_{jq}^{\chi} - \hat{p}_{jq}^x \right|^d n^{-1} \sum_{i=1}^{n} |\hat{p}_{iq}^x|^d + \left| p_{jq}^{\chi} \right|^d n^{-1} \sum_{i=1}^{n} |p_{iq}^{\chi} - \hat{p}_{iq}^x|^d \right)
$$

$$
= O_P((\max(n^{-1}, \rho_T^{-1/2}))^d) + O_P((\max(n^{-1}, \rho_T^{-1}))^{d/2}) = O_P((\max(n^{-1}, \rho_T^{-1}))^{d/2}).
$$

Statement (iii) follows along the same lines, by setting $j = i$. $\qquad \square$

Lemma 10 *For $n \to \infty$ and $T \to \infty$, $n^{-2} \sum_{\ell=0}^{S} \sum_{i=1}^{n} \sum_{j=1}^{n} |\hat{\gamma}_{ij,\ell}^{\chi} - \gamma_{ij,\ell}^{\chi}|^{d}$ and, for any given j in $\{1,\ldots,n\}$, $n^{-1} \sum_{\ell=0}^{S} \sum_{i=1}^{n} |\hat{\gamma}_{ij,\ell}^{\chi} - \gamma_{ij,\ell}^{\chi}|^{d}$, are $O_P \big((\max(n^{-1}, \rho_T^{-1}))^{d/2} \big)$.*

PROOF. We have $|\hat{\gamma}_{ij,\ell}^{\chi} - \gamma_{ij,\ell}^{\chi}| \le \mathcal{U}_{ij} + \mathcal{V}_{\ell} + \mathcal{W}_{ij}$, where $\mathcal{U}_{ij}$, $\mathcal{V}_{\ell}$ and $\mathcal{W}_{ij}$ are the terms in the extreme right-hand side of (3.6). Using the C_r inequality, we get

$$n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{n} |\hat{\gamma}_{ij,0}^{\chi} - \gamma_{ij,0}^{\chi}|^{d} \le n^{-2} 3^{d-1} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathcal{U}_{ij}^{d} + n^{-2} 3^{d-1} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathcal{V}_{\ell}^{d} + n^{-2} 3^{d-1} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathcal{W}_{ij}^{d}.$$

The first term on the right-hand side is bounded in view of Lemma 9. Since ℓ takes only a finite number of values, the second term is $O(B_T^{-d})$ (see the proof of Proposition 9). Because the functions σ_{ij}^{x} are of bounded variation uniformly in i and j, see Proposition 2, the third term is $O(B_T^{-d})$. The same argument used to obtain Proposition 8 applies. The second statement is proved in the same way. $\qquad\square$

We are now able to state and prove the main lemma of this section. Assume, without loss of generality, that n increases by blocks of size $q + 1$, so that $n = m(q + 1)$.

Lemma 11 *Denoting by $\hat{\mathbf{Z}}$ the $T \times n$ matrix with $\hat{Z}_{it}$ in entry (t, i), let $\hat{\mathbf{\Gamma}}^{z} = \hat{\mathbf{Z}}' \hat{\mathbf{Z}}/T$. Then, as $n \to \infty$ and $T \to \infty$,*

$$n^{-1} \|\hat{\mathbf{\Gamma}}^{z} - \mathbf{\Gamma}^{z}\| = O_P(\zeta_{nT}) \quad and \quad n^{-1/2} \|\mathcal{S}_i'(\hat{\mathbf{\Gamma}}^{z} - \mathbf{\Gamma}^{z})\| = O_P(\zeta_{nT}), \tag{D.4}$$

where $\mathbf{\Gamma}^{z}$ is the population covariance matrix of $\mathbf{Z}_{nt}$.

PROOF. Denote by $\check{\mathbf{\Gamma}}^{z} = \mathbf{Z}' \mathbf{Z}/T$ the empirical covariance matrix we would compute from the $\mathbf{Z}_{nt}$'s, were the matrices $\mathbf{A}(L)$ known. We have

$$\|\hat{\mathbf{\Gamma}}^{z} - \mathbf{\Gamma}^{z}\| \le \|\hat{\mathbf{\Gamma}}^{z} - \check{\mathbf{\Gamma}}^{z}\| + \|\check{\mathbf{\Gamma}}^{z} - \mathbf{\Gamma}^{z}\|, \tag{D.5}$$

so that the lemma can be proved by showing that (D.4) holds with $\|\hat{\mathbf{\Gamma}}^{z} - \mathbf{\Gamma}^{z}\|$ replaced by any of the two terms on the right-hand side of (D.5).

First consider $\|\check{\mathbf{\Gamma}}^{z} - \mathbf{\Gamma}^{z}\|$. Since $\mathbf{A}(L) = \mathbf{I}_n - \mathbf{A}_1 L - \cdots - \mathbf{A}_S L^S$, where

$$\mathbf{A}_s = \begin{pmatrix} \mathbf{A}_s^1 & 0 & \cdots & 0 \\ 0 & \mathbf{A}_s^2 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & \mathbf{A}_s^m \end{pmatrix}, \qquad s = 1, \ldots, S$$

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and $\mathbf{A}_0 = \mathbf{I}_n$, we obtain

$$\|\check{\mathbf{\Gamma}}^z - \mathbf{\Gamma}^z\|^2 \leq \sum_{s=0}^{S}\sum_{r=0}^{S} \|\mathbf{A}_s\hat{\mathbf{\Gamma}}^x_{s-r}\mathbf{A}'_r - \mathbf{A}_s\mathbf{\Gamma}^x_{s-r}\mathbf{A}'_r\|^2 = \sum_{s=0}^{S}\sum_{r=0}^{S} \|\mathbf{A}_s\left(\hat{\mathbf{\Gamma}}^x_{s-r} - \mathbf{\Gamma}^x_{s-r}\right)\mathbf{A}'_r\|^2, \quad \text{(D.6)}$$

which is a sum of $(S+1)^2$ terms, where we set $\hat{\mathbf{\Gamma}}^x_{s-r} = T^{-1}\sum_{t=1}^{T}\mathbf{x}_{t-r}\mathbf{x}'_{t-s}$. Inspection of the right-hand side of (D.6) shows that (D.4) holds, with $\|\hat{\mathbf{\Gamma}}^z - \mathbf{\Gamma}^z\|$ replaced with $\|\check{\mathbf{\Gamma}}^z - \mathbf{\Gamma}^z\|$, under Assumptions 2 and 7, and in view of Propositions 2 and 6.

Turning to $\|\hat{\mathbf{\Gamma}}^z - \check{\mathbf{\Gamma}}^z\|$, since

$$\|\hat{\mathbf{\Gamma}}^z - \check{\mathbf{\Gamma}}^z\|^2 \leq \sum_{s=0}^{S}\sum_{r=0}^{S} \|\hat{\mathbf{A}}_s\hat{\mathbf{\Gamma}}^x_{s-r}\hat{\mathbf{A}}'_r - \mathbf{A}_s\hat{\mathbf{\Gamma}}^x_{s-r}\mathbf{A}'_r\|^2,$$

it is sufficient to prove that (D.4) still holds with $\|\hat{\mathbf{\Gamma}}^z - \mathbf{\Gamma}^z\|$ replaced with any of the $\|\hat{\mathbf{A}}_s\hat{\mathbf{\Gamma}}^x_{s-r}\hat{\mathbf{A}}'_r - \mathbf{A}_s\hat{\mathbf{\Gamma}}^x_{s-r}\mathbf{A}'_r\|$'s. Denoting by $\mathbf{a}^j_{s\alpha}$, $1 \leq \alpha \leq q+1$, the α-th column of $\mathbf{A}^{j'}_s$, we have

$$\begin{aligned}
\|\hat{\mathbf{A}}^s\hat{\mathbf{\Gamma}}^x_{s-r}\hat{\mathbf{A}}^{r\prime} - \mathbf{A}^s\hat{\mathbf{\Gamma}}^x_{s-r}\mathbf{A}^{r\prime}\|^2 \ &\leq \sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{\alpha=1}^{q+1}\sum_{\beta=1}^{q+1} \left(\hat{\mathbf{a}}^{j\prime}_{s\alpha}\hat{\mathbf{\Gamma}}^x_{jk,s-r}\hat{\mathbf{a}}^k_{r\beta} - \mathbf{a}^{j\prime}_{s\alpha}\hat{\mathbf{\Gamma}}^x_{jk,s-r}\mathbf{a}^k_{r\beta}\right)^2 \\
&\leq 2\sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{\alpha=1}^{q+1}\sum_{\beta=1}^{q+1} \left((\hat{\mathbf{a}}^j_{s\alpha} - \mathbf{a}^{j\prime}_{s\alpha})\hat{\mathbf{\Gamma}}^x_{jk,s-r}\hat{\mathbf{a}}^k_{r\beta}\right)^2 \qquad \text{(D.7)} \\
&\quad + 2\sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{\alpha=1}^{q+1}\sum_{\beta=1}^{q+1} \left(\mathbf{a}^{j\prime}_{s\alpha}\hat{\mathbf{\Gamma}}^x_{jk,s-r}(\hat{\mathbf{a}}^k_{r\beta} - \mathbf{a}^k_{r\beta})\right)^2,
\end{aligned}$$

where $\hat{\mathbf{\Gamma}}^x_{jk,s-r}$ is the (j,k)-block of $\hat{\mathbf{\Gamma}}^x_{s-r}$, and the second inequality follows from applying the C_r inequality to each term of the form

$$(\hat{\mathbf{a}}^{j\prime}_{s\alpha}\hat{\mathbf{\Gamma}}^x_{jk,s-r}\hat{\mathbf{a}}^k_{r\beta} - \mathbf{a}^{j\prime}_{s\alpha}\hat{\mathbf{\Gamma}}^x_{jk,s-r}\mathbf{a}^k_{r\beta})^2 = ((\hat{\mathbf{a}}^j_{s\alpha} - \mathbf{a}^j_{s\alpha})'\hat{\mathbf{\Gamma}}^x_{jk,s-r}\hat{\mathbf{a}}^k_{r\beta} - \mathbf{a}^{j\prime}_{s\alpha}\hat{\mathbf{\Gamma}}^x_{jk,s-r}(\hat{\mathbf{a}}^k_{r\beta} - \mathbf{a}^k_{r\beta}))^2.$$

The two terms on the right-hand side of (D.7) can be dealt with in the same way. Let us focus on the first of them. Using twice the Cauchy-Schwartz inequality, then subsequently the C_r and Jensen inequalities, we obtain

$$\sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{\alpha=1}^{q+1}\sum_{\beta=1}^{q+1} ((\hat{\mathbf{a}}^j_{s\alpha} - \mathbf{a}^j_{s\alpha})'\hat{\mathbf{\Gamma}}^x_{jk,s-r}\hat{\mathbf{a}}^k_{r\beta})^2$$

$$
\leq \sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{\alpha=1}^{q+1}\sum_{\beta=1}^{q+1}((\hat{\mathbf{a}}_{s\alpha}^{j}-\mathbf{a}_{s\alpha}^{j})'(\hat{\mathbf{a}}_{s\alpha}^{j}-\mathbf{a}_{s\alpha}^{j})\hat{\mathbf{a}}_{r\beta}^{k\prime}\hat{\mathbf{\Gamma}}_{jk,s-r}^{x\prime}\hat{\mathbf{\Gamma}}_{jk,s-r}^{x}\hat{\mathbf{a}}_{r\beta}^{k}
$$

$$
= \sum_{k=1}^{m}\sum_{\beta=1}^{q+1}\sum_{j=1}^{m}\sum_{\alpha=1}^{q+1}(\hat{\mathbf{a}}_{s\alpha}^{j}-\mathbf{a}_{s\alpha}^{j})'(\hat{\mathbf{a}}_{s\alpha}^{j}-\mathbf{a}_{s\alpha}^{j})\hat{\mathbf{a}}_{r\beta}^{k\prime}\hat{\mathbf{\Gamma}}_{jk,s-r}^{x\prime}\hat{\mathbf{\Gamma}}_{jk,s-r}^{x}\hat{\mathbf{a}}_{r\beta}^{k}
$$

$$
\leq \sum_{k=1}^{m}\sum_{\beta=1}^{q+1}\big[\sum_{j=1}^{m}\big[\sum_{\alpha=1}^{q+1}(\hat{\mathbf{a}}_{s\alpha}^{j}-\mathbf{a}_{s\alpha}^{j})'(\hat{\mathbf{a}}_{s\alpha}^{j}-\mathbf{a}_{s\alpha}^{j})\big]^{2}\big]^{1/2}\big[\sum_{j=1}^{m}\big(\hat{\mathbf{a}}_{r\beta}^{k}{}'\hat{\mathbf{\Gamma}}_{jk,s-r}^{x}{}'\hat{\mathbf{\Gamma}}_{jk,s-r}^{x}\hat{\mathbf{a}}_{r\beta}^{k}\big)^{2}\big]^{1/2}
$$

$$
= m\big[\sum_{j=1}^{m}\big[\sum_{\alpha=1}^{q+1}(\hat{\mathbf{a}}_{s\alpha}^{j}-\mathbf{a}_{s\alpha}^{j\prime})(\hat{\mathbf{a}}_{s\alpha}^{j}-\mathbf{a}_{s\alpha}^{j})\big]^{2}\big]^{1/2}\frac{1}{m}\sum_{k=1}^{m}\sum_{\beta=1}^{q+1}\big[\sum_{j=1}^{m}\big(\hat{\mathbf{a}}_{r\beta}^{k\prime}\hat{\mathbf{\Gamma}}_{jk,s-r}^{x\prime}\hat{\mathbf{\Gamma}}_{jk,s-r}^{x}\hat{\mathbf{a}}_{r\beta}^{k}\big)^{2}\big]^{1/2}
$$

$$
\leq \mathcal{AB}, \quad \text{say,}
$$

where

$$
\mathcal{A} = m(q+1)^{1/2}\big[\sum_{j=1}^{m}\sum_{\alpha=1}^{q+1}\big((\hat{\mathbf{a}}_{s\alpha}^{j}-\mathbf{a}_{s\alpha}^{j})'(\hat{\mathbf{a}}_{s\alpha}^{j}-\mathbf{a}_{s\alpha}^{j})\big)^{2}\big]^{1/2}
$$

and

$$
\mathcal{B} = \frac{1}{m}\sum_{k=1}^{m}\sum_{\beta=1}^{q+1}\big[\sum_{j=1}^{m}\big(\hat{\mathbf{a}}_{r\beta}^{k\prime}\hat{\mathbf{\Gamma}}_{jk,s-r}^{x\prime}\hat{\mathbf{\Gamma}}_{jk,s-r}^{x}\hat{\mathbf{a}}_{r\beta}^{k}\big)^{2}\big]^{1/2}
$$

$$
\leq \big[(q+1)/m\sum_{k=1}^{m}\sum_{\beta=1}^{q+1}\sum_{j=1}^{m}\big(\hat{\mathbf{a}}_{r\beta}^{k\prime}\hat{\mathbf{\Gamma}}_{jk,s-r}^{x\prime}\hat{\mathbf{\Gamma}}_{jk,s-r}^{x}\hat{\mathbf{a}}_{r\beta}^{k}\big)^{2}\big]^{1/2} = \mathcal{C}, \quad \text{say.}
$$

First consider $\mathcal{A}$. Letting $\mathbf{a}_{s\alpha}^{j\prime} = (a_{s\alpha,1}^{j}\ a_{s\alpha,2}^{j}\ \cdots\ a_{s\alpha,q+1}^{j})$, note that $a_{s\alpha,\delta}^{j} = \mathbf{e}_{\alpha}'\mathbf{A}^{[j]}\mathbf{g}_{s\delta}$, where $\mathbf{e}_{\alpha}$ and $\mathbf{g}_{s\delta}$ stand for the α-th and $(s-1)(q+1)+\delta$-th unit vectors in the $(q+1)$- and $(q+1)S$-dimensional canonical bases, respectively. Writing, for the sake of simplicity, $\mathbf{B}_{j}$ and $\mathbf{C}_{j}$ instead of $\mathbf{B}_{j}^{\chi}$ and $\mathbf{C}_{jj}^{\chi}$, as defined in (2.13) and (2.14), we obtain, from (B.1), and applying subsequently the C_{r}, the triangular, the C_{r} again and then twice the Cauchy-Schwartz inequalities,

$$
\left[\sum_{j=1}^{m}\sum_{\alpha=1}^{q+1}\big((\hat{\mathbf{a}}_{s\alpha}^{j}-\mathbf{a}_{s\alpha}^{j})'(\hat{\mathbf{a}}_{s\alpha}^{j}-\mathbf{a}_{s\alpha}^{j})\big)^{2}\right]^{1/2}
$$

$$\leq (q+1)^{1/2}\Big(\sum_{j=1}^{m}\sum_{\alpha=1}^{q+1}\sum_{\delta=1}^{q+1}(\hat{a}^j_{s\alpha,\delta}-a^j_{s\alpha,\delta})^4\Big)^{1/2}$$

$$= (q+1)^{1/2}\Big(\sum_{j=1}^{m}\sum_{\alpha=1}^{q+1}\sum_{\delta=1}^{q+1}\Big[\mathbf{e}_\alpha\Big((\hat{\mathbf{B}}_j-\mathbf{B}_j)\hat{\mathbf{C}}_j^{-1}+\mathbf{B}_j\hat{\mathbf{C}}_j^{-1}(\hat{\mathbf{C}}_j-\mathbf{C}_j)\mathbf{C}_j^{-1}\Big)\mathbf{g}_{s\delta}\Big]^4\Big)^{1/2}$$

$$\leq 2^{3/2}(q+1)^{3/2}\Big(\sum_{j=1}^{m}\|(\hat{\mathbf{B}}_j-\mathbf{B}_j)\hat{\mathbf{C}}_j^{-1}\|^4+\|\mathbf{B}_j\hat{\mathbf{C}}_j^{-1}(\hat{\mathbf{C}}_j-\mathbf{C}_j)\mathbf{C}_j^{-1}\|^4\Big)^{1/2}$$

$$\leq 2^{3/2}(q+1)^{3/2}\Big(\Big[\sum_{j=1}^{m}\|\hat{\mathbf{B}}_j-\mathbf{B}_j\|^8\Big]^{1/2}\Big[\sum_{j=1}^{m}\|\hat{\mathbf{C}}_j^{-1}\|^8\Big]^{1/2}$$

$$+\Big[\sum_{j=1}^{m}\|\hat{\mathbf{C}}_j-\mathbf{C}_j\|^8\Big]^{1/2}\Big[\sum_{j=1}^{m}\|\hat{\mathbf{B}}_j\hat{\mathbf{C}}_j^{-1}\|^8\|\mathbf{C}_j^{-1}\|^8\Big]^{1/2}\Big)^{1/2}$$

$$\leq 2^{3/2}(q+1)^{3/2}\Big(\Big[\sum_{j=1}^{m}\|\hat{\mathbf{B}}_j-\mathbf{B}_j\|^8\Big]^{1/2}\Big[\sum_{j=1}^{m}\|\hat{\mathbf{C}}_j^{-1}\|^8\Big]^{1/2}$$

$$+\Big[\sum_{j=1}^{m}\|\hat{\mathbf{C}}_j-\mathbf{C}_j\|^8\Big]^{1/2}\Big[\sum_{j=1}^{m}\|\hat{\mathbf{B}}_j\|^{16}\Big]^{\frac{1}{4}}\Big[\sum_{j=1}^{m}\|\hat{\mathbf{C}}_j^{-1}\|^{16}\|\mathbf{C}_j^{-1}\|^{16}\Big]^{\frac{1}{4}}\Big)^{1/2}.$$

Denoting by $b^j_{i\delta}$ the entries of $\mathbf{B}_j$, $i=1,\ldots,q+1$, $\delta=1,\ldots,S(q+1)$, the C_r inequality and Lemma 10 entail

$$\sum_{j=1}^{m}\|\hat{\mathbf{B}}_j-\mathbf{B}_j\|^8 \leq \sum_{j=1}^{m}\Big(\sum_{i=1}^{q+1}\sum_{\delta=1}^{S(q+1)}(\hat{b}^j_{i\delta}-b^j_{i\delta})^2\Big)^4$$

$$\leq (q+1)^6 S^3\sum_{j=1}^{m}\sum_{i=1}^{q+1}\sum_{\delta=1}^{S(q+1)}(\hat{b}^j_{i\delta}-b^j_{i\delta})^8 = O_P(m(\max(n^{-1},\rho_T^{-1}))^4).$$

In a similar way, one can prove that $\sum_{j=1}^{m}\|\hat{\mathbf{C}}_j-\mathbf{C}_j\|^8$ is $O_P(m(\max(n^{-1},\rho_T^{-1}))^4)$. Moreover, Assumptions 2 and 7 together with Lemma 10 imply that $\sum_{j=1}^{m}\|\hat{\mathbf{B}}_j\|^{16}$ and $\sum_{j=1}^{m}\|\mathbf{C}_j^{-1}\|^{16}$, as well as $\sum_{j=1}^{m}\|\hat{\mathbf{C}}_j^{-1}\|^8$ and $\sum_{j=1}^{m}\|\hat{\mathbf{C}}_j^{-1}\|^{16}$, are $O_P(m)$.

Collecting terms yields

$$\mathcal{A} = m(q+1)^{1/2}\Big[\sum_{j=1}^{m}\sum_{\alpha=1}^{q+1}\big((\hat{\mathbf{a}}^j_{s\alpha}-\mathbf{a}^j_{s\alpha})'(\hat{\mathbf{a}}^j_{s\alpha}-\mathbf{a}^j_{s\alpha})\big)^2\Big]^{1/2}$$

$$\leq 2^{3/2}(q+1)^2 m\Big(\sum_{i=1}^{m}\|\hat{\mathbf{A}}^i_s-\mathbf{A}^i_s\|^4\Big)^{1/2} = O_P\big(m^{3/2}\max(n^{-1},\rho_T^{-1})\big). \tag{D.8}$$

Turning to $\mathcal{C}$, we obtain, by means of similar methods,

$$
\begin{aligned}
\mathcal{C} &\leq ((q+1)/m)^{1/2}\Big\{\Big[\sum_{k=1}^{m}\Big(\sum_{\beta=1}^{q+1}(\hat{\mathbf{a}}_{r\beta}^{k\prime}\hat{\mathbf{a}}_{r\beta}^{k})^2\Big)^2\Big]^{1/2}\Big[\sum_{j=1}^{m}\Big(\sum_{k=1}^{m}\big(\text{trace}[\hat{\mathbf{\Gamma}}_{jk,s-r}^{x\prime}\hat{\mathbf{\Gamma}}_{jk,s-r}^{x}]\big)^4\Big)^{1/2}\Big]\Big\}^{1/2} \\
&\leq ((q+1)/m)^{1/2}\Big\{\Big[(q+1)\sum_{k=1}^{m}\sum_{\beta=1}^{q+1}(\hat{\mathbf{a}}_{r\beta}^{k\prime}\hat{\mathbf{a}}_{r\beta}^{k})^4\Big]^{1/2}\Big[\sum_{j=1}^{m}\Big(\sum_{k=1}^{m}\big(\text{trace}[\hat{\mathbf{\Gamma}}_{jk,s-r}^{x\prime}\hat{\mathbf{\Gamma}}_{jk,s-r}^{x}]\big)^4\Big)^{1/2}\Big]\Big\}^{1/2} \\
&\leq (q+1)^{1/2}\Big[(q+1)^4\sum_{k=1}^{m}\sum_{\alpha=1}^{q+1}\sum_{\beta=1}^{q+1}(\hat{a}_{r,\alpha\beta}^{k})^8\Big]^{1/4}\Big[m^{-1}\sum_{j=1}^{m}\sum_{k=1}^{m}\big(\text{trace}[\hat{\mathbf{\Gamma}}_{jk,s-r}^{x\prime}\hat{\mathbf{\Gamma}}_{jk,s-r}^{x}]\big)^4\Big]^{1/4} \\
&\leq (q+1)^{3/2}\Big[\sum_{k=1}^{m}\sum_{\alpha=1}^{q+1}\sum_{\beta=1}^{q+1}(\hat{a}_{r,\alpha\beta}^{k})^8\Big]^{1/4}\Big[((q+1)^6/m)\sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{\alpha=1}^{q+1}\sum_{\beta=1}^{q+1}(\hat{\gamma}_{jk,\alpha\beta}^{x}(s-r))^8\Big]^{\frac{1}{4}} \\
&= O_P(m^{1/2}),
\end{aligned}
$$

where $\hat{\gamma}_{jk,\alpha\beta}^{x}(s-r)$ stands for the (α,β) entry of $\hat{\mathbf{\Gamma}}_{jk,s-r}^{x}$. Collecting terms again, we get

$$
m^{-1}\|\hat{\mathbf{A}}_s\hat{\mathbf{\Gamma}}_{s-r}^{x}\hat{\mathbf{A}}_r' - \mathbf{A}_s\hat{\mathbf{\Gamma}}_{s-r}^{x}\mathbf{A}_r'\| \leq \Big(\frac{1}{m^2}\mathcal{A}\mathcal{C}\Big)^{1/2} = O_P(\zeta_{nT}), \quad r,s=0,...,S.
$$

Now consider the second statement in (D.4). Again, it is sufficient to prove that it holds with $\|\hat{\mathbf{\Gamma}}^z - \mathbf{\Gamma}^z\|$ replaced with any of the $\|\hat{\mathbf{A}}_s\hat{\mathbf{\Gamma}}_{s-r}^{x}\hat{\mathbf{A}}_r' - \mathbf{A}_s\hat{\mathbf{\Gamma}}_{s-r}^{x}\mathbf{A}_r'\|$'s. The two terms on the right-hand side of (D.7) must be dealt with separately. In the first of those two terms, dropping one of the summations for $k=1,\dots,m$ and setting $k=i$,

$$
\sum_{j=1}^{m}\sum_{\alpha=1}^{q+1}\sum_{\beta=1}^{q+1}\Big((\hat{\mathbf{a}}_{s\alpha}^{j} - \mathbf{a}_{s\alpha}^{j})'\hat{\mathbf{\Gamma}}_{ji,s-r}^{x}\hat{\mathbf{a}}_{r\beta}^{i}\Big)^2 = O_P\big(m(\max(n^{-1},\rho_T^{-1}))\big).
$$

Indeed, the left-hand side is bounded by a product $\mathcal{D}\mathcal{E}$, say, where

$$
\mathcal{D} = m^{1/2}(q+1)^{1/2}\Big[\sum_{j=1}^{m}\sum_{\alpha=1}^{q+1}\big((\hat{\mathbf{a}}_{s\alpha}^{j} - \mathbf{a}_{s\alpha}^{j})'(\hat{\mathbf{a}}_{s\alpha}^{j} - \mathbf{a}_{s\alpha}^{j})\big)^2\Big]^{1/2}
$$

and

$$
\mathcal{E} = \sum_{\beta=1}^{q+1}\Big(\frac{1}{m}\sum_{j=1}^{m}\big(\hat{\mathbf{a}}_{r\beta}^{i\prime}\hat{\mathbf{\Gamma}}_{ji,s-r}^{x\prime}\hat{\mathbf{\Gamma}}_{jk,s-r}^{x}\hat{\mathbf{a}}_{r\beta}^{i}\big)^2\Big)^{1/2}
$$

can be bounded along the same lines as $\mathcal{A}$ and $\mathcal{B}$ in the proof of the first statement.

As for the second term of (D.7), using arguments similar to those used in the first part of the proof, we obtain

$$
\begin{aligned}
\sum_{j=1}^{m}\sum_{\alpha=1}^{q+1}\sum_{\beta=1}^{q+1}&\Big((\hat{\mathbf{a}}_{s\alpha}^{i} - \mathbf{a}_{s\alpha}^{i})'\hat{\mathbf{\Gamma}}_{jk,s-r}^{x\prime}\mathbf{a}_{r\beta}^{j}\Big)^2 \\
&\leq m\Big[\Big[\sum_{\alpha=1}^{q+1}(\hat{\mathbf{a}}_{s\alpha}^{i} - \mathbf{a}_{s\alpha}^{i})'(\hat{\mathbf{a}}_{s\alpha}^{i} - \mathbf{a}_{s\alpha}^{i})\Big]^2\Big]^{1/2}\Big[\frac{1}{m}\sum_{j=1}^{m}\sum_{\beta=1}^{q+1}(\mathbf{a}_{r\beta}^{j\prime}\hat{\mathbf{\Gamma}}_{ji,s-r}^{x}\hat{\mathbf{\Gamma}}_{ji,s-r}^{x\prime}\mathbf{a}_{r\beta}^{j}) \\
&= \mathcal{F}\mathcal{G}, \text{ say.}
\end{aligned}
$$

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It easily follows from Proposition 9 that $\mathcal{F} = O_P(m\,\zeta_{nT}^2)$, while $\mathcal{G} = O_P(1)$ can be obtained from the arguments used to bound $\mathcal{C}$ in the proof of the first statement. Collecting terms, we obtain, as desired,

$$m^{-1/2}\|\mathcal{S}_i'(\hat{\mathbf{A}}_s\hat{\boldsymbol{\Gamma}}_{s-r}^x\hat{\mathbf{A}}_r' - \mathbf{A}_s\hat{\boldsymbol{\Gamma}}_{s-r}^x\mathbf{A}_r')\| = O_p\left(\zeta_{nT}\right), \quad r, s = 0, ..., S. \qquad \square$$

Starting with Lemma 11, which plays here the same role as Proposition 6 does for the proof of Proposition 7, we can easily prove statements that replicate in this context Lemmas 1, 2, 3 and 4, using the same arguments as in Section B, with x, χ and ξ replaced by Z, Ψ and Φ, respectively. More precisely,

(I) In the results corresponding to Lemma 1 we obtain the rate ζ_{nT} for (i), (ii), (iii) and (iv). Note that no reduction from $1/n$ to $1/\sqrt{n}$ occurs between (iii) and (iv), as in Lemma 1. For, (iii) has $O_P(\zeta_{nT}) + O(1/n) = O_P(\zeta_{nT})$, while (iv) has $O_P(\zeta_{nT}) + O(1/\sqrt{n})$, which is $O_P(\zeta_{nT})$.

(II) The same rate ζ_{nT} is obtained for the results of Lemma 2.

(III) The same holds for Lemma 3. The orthogonal matrix in point (iii), call it again $\hat{\mathbf{W}}_q$, has either 1 or -1 on the diagonal; thus $\tilde{\hat{\mathbf{W}}}_q = \hat{\mathbf{W}}_q$.

(IV) Lastly, Lemma 4 becomes

$$\left\|\mathcal{S}_i'\left(\hat{\mathbf{P}}^z\left(\hat{\boldsymbol{\Lambda}}^z\right)^{1/2} - \mathbf{P}^\psi\left(\boldsymbol{\Lambda}^\psi\right)^{1/2}\hat{\mathbf{W}}_q\right)\right\| = \left\|\hat{\mathcal{R}}_i - \mathcal{R}_i\hat{\mathbf{W}}_q\right\| = O_P\left(\zeta_{nT}\right). \qquad (\text{D.9})$$

Going over the proof of Lemma 4, we see that $\|\mathbf{i} - \mathbf{j}\|$ has the worst rate, whereas here $\|\mathbf{g} - \mathbf{h}\|$, $\|\mathbf{h} - \mathbf{i}\|$ and $\|\mathbf{i} - \mathbf{j}\|$ all have rate $O_P(\zeta_{nT})$. This completes the proof of Proposition 10. $\qquad \square$

Finallly, in the same way as the proof of Lemma 4 can be replicated to obtain (D.9), the proof of Lemma 6 can be replicated to obtain

$$\|\hat{\mathbf{P}}^z(\hat{\boldsymbol{\Lambda}}^z)^{1/2} - \mathbf{P}^\psi(\boldsymbol{\Lambda}^\psi)^{1/2}\hat{\mathbf{W}}_q\| = O_P\left(n^{1/2}\zeta_{nT}\right). \qquad (\text{D.10})$$

E Proof of Proposition 11

We have

$$
\begin{aligned}
\hat{\mathbf{v}}_t &= \big((\hat{\mathbf{\Lambda}}^z)^{1/2}\hat{\mathbf{P}}^{z\prime}\hat{\mathbf{P}}^z(\hat{\mathbf{\Lambda}}^z)^{1/2}\big)^{-1}(\hat{\mathbf{\Lambda}}^z)^{1/2}\hat{\mathbf{P}}^{z\prime}\hat{\mathbf{Z}}_t = (\hat{\mathbf{\Lambda}}^z)^{-1/2}\hat{\mathbf{P}}^{z\prime}\hat{\mathbf{Z}}_t \\
&= (\hat{\mathbf{\Lambda}}^z)^{-1/2}\hat{\mathbf{P}}^{z\prime}\big(\hat{\mathbf{A}}(L)-\mathbf{A}(L)\big)\mathbf{x}_t + \big((\hat{\mathbf{\Lambda}}^z)^{-1/2}\hat{\mathbf{P}}^{z\prime} - \hat{\mathbf{W}}_q(\mathbf{\Lambda}^\psi)^{-1/2}\mathbf{P}^{\psi\prime}\big)\mathbf{A}(L)\mathbf{x}_t \\
&\quad + \hat{\mathbf{W}}_q(\mathbf{\Lambda}^\psi)^{-1/2}\mathbf{P}^{\psi\prime}\mathbf{A}(L)\xi_t + \hat{\mathbf{W}}_q(\mathbf{\Lambda}^\psi)^{-1/2}\mathbf{P}^{\psi\prime}\mathbf{P}^\psi(\mathbf{\Lambda}^\psi)^{1/2}\mathbf{v}_t. \qquad\text{(E.11)}
\end{aligned}
$$

Considering the first term on the right-hand side of (E.11),

$$
\begin{aligned}
\|(\hat{\mathbf{\Lambda}}^z)^{-1/2}\hat{\mathbf{P}}^{z\prime}\big(\hat{\mathbf{A}}(L)-\mathbf{A}(L)\big)\mathbf{x}_t\| &= \|(\hat{\mathbf{\Lambda}}^z/n)^{-1/2}\hat{\mathbf{P}}^{z\prime}n^{-1/2}\big(\hat{\mathbf{A}}(L)-\mathbf{A}(L)\big)\mathbf{x}_t\| \\
&\leq \|(\hat{\mathbf{\Lambda}}^z/n)^{-1/2}\|\|\hat{\mathbf{P}}^{z\prime}\|\|n^{-1/2}\big(\hat{\mathbf{A}}(L)-\mathbf{A}(L)\big)\mathbf{x}_t\|.
\end{aligned}
$$

Since $\|(\hat{\mathbf{\Lambda}}^z/n)^{-1/2}\| = O_P(1)$ and $\|\hat{\mathbf{P}}^z\| = 1$, by (D.8), we get

$$
\begin{aligned}
\|n^{-1/2}\big(\hat{\mathbf{A}}(L)-\mathbf{A}(L)\big)\mathbf{x}_t\| &\leq n^{-1/2}\sum_{r=0}^{p}\Big(\sum_{i=1}^{m}\mathbf{x}_{t-r}^{i\prime}(\hat{\mathbf{A}}_r^i - \mathbf{A}_r^i)^\prime(\hat{\mathbf{A}}_r^i - \mathbf{A}_r^i)\mathbf{x}_{t-r}^i\Big)^{1/2} \\
&\leq \sum_{r=0}^{p}\big(n^{-1}\sum_{i=1}^{m}(\mathbf{x}_{t-r}^{i\prime}\mathbf{x}_{t-r}^i)^2\big)^{1/4}\Big(n^{-1}\sum_{i=1}^{m}\big(\sum_{j=1}^{q+1}\sum_{h=1}^{q+1}(\hat{a}_{r,jh}^i - a_{r,jh}^i)^2\big)^2\Big)^{1/4} \\
&\leq \sum_{r=0}^{p}\big(n^{-1}\sum_{i=1}^{m}(\mathbf{x}_{t-r}^{i\prime}\mathbf{x}_{t-r}^i)^2\big)^{1/4}\big((q+1)^3 n^{-1}\sum_{i=1}^{m}\|\hat{\mathbf{A}}_r^i - \mathbf{A}_r^i\|^4\big)^{1/4} \\
&= O_P(\zeta_{nT})
\end{aligned}
$$

where $\mathbf{x}_t = (\mathbf{x}_t^{1\prime}...\mathbf{x}_t^{i\prime}...\mathbf{x}_t^{m\prime})^\prime$ stands for sub-vectors $\mathbf{x}_t^i$ of size $(q+1)\times 1$.

Next, considering the second term on the right-hand side of (E.11),

$$
\begin{aligned}
&\Big\|\big((\hat{\mathbf{\Lambda}}^z)^{-1/2}\hat{\mathbf{P}}^{z\prime} - \hat{\mathbf{W}}_q(\mathbf{\Lambda}^\psi)^{-1/2}\mathbf{P}^{\psi\prime}\big)\mathbf{A}(L)\mathbf{x}_t\Big\| \\
&= \Big\|(\hat{\mathbf{\Lambda}}^z/n)^{-1}\big((\hat{\mathbf{\Lambda}}^z)^{1/2}\hat{\mathbf{P}}^{z\prime} - \hat{\mathbf{W}}_q\hat{\mathbf{\Lambda}}^z(\mathbf{\Lambda}^\psi)^{-1/2}\mathbf{P}^{\psi\prime}\big)\mathbf{A}(L)\mathbf{x}_t/n\Big\| \\
&= \Big\|(\hat{\mathbf{\Lambda}}^z/n)^{-1}\big((\hat{\mathbf{\Lambda}}^z)^{1/2}\hat{\mathbf{P}}^{z\prime} - \hat{\mathbf{W}}_q[\hat{\mathbf{\Lambda}}^z - \mathbf{\Lambda}^\psi + \mathbf{\Lambda}^\psi](\mathbf{\Lambda}^\psi)^{-1/2}\mathbf{P}^{\psi\prime}\big)\mathbf{A}(L)\mathbf{x}_t/n\Big\| \\
&\leq \|(\hat{\mathbf{\Lambda}}^z/n)^{-1}\|\Big\|\big((\hat{\mathbf{\Lambda}}^z)^{1/2}\hat{\mathbf{P}}^{z\prime} - \hat{\mathbf{W}}_q(\mathbf{\Lambda}^\psi)^{1/2}\mathbf{P}^{\psi\prime}\big)\Big\|\|\mathbf{A}(L)\mathbf{x}_t/n\| \\
&\quad + \|(\hat{\mathbf{\Lambda}}^z/n)^{-1}\|\|\hat{\mathbf{W}}_q(\hat{\mathbf{\Lambda}}^z - \mathbf{\Lambda}^\psi)(\mathbf{\Lambda}^\psi)^{-1/2}\mathbf{P}^{\psi\prime}\|\|\mathbf{A}(L)\mathbf{x}_t/n\| = O_P(\zeta_{nT}),
\end{aligned}
$$

since, by (D.10), $\|(\hat{\mathbf{P}}^z(\hat{\mathbf{\Lambda}}^z)^{1/2} - \mathbf{P}^\psi(\mathbf{\Lambda}^\psi)^{1/2}\hat{\mathbf{W}}_q)\| = O_P(n^{1/2}\zeta_{nT})$, and

$$
\begin{aligned}
\|\hat{\mathbf{A}}(L)\mathbf{x}_t/n\| &= n^{-1/2}\big(\mathbf{x}_t^\prime\hat{\mathbf{A}}^\prime(L)\hat{\mathbf{A}}(L)\mathbf{x}_t/n\big)^{1/2} \\
&\leq n^{-1/2}\sum_{r=0}^{p}\big(\mathbf{x}_{t-r}^\prime\hat{\mathbf{A}}_r^\prime\hat{\mathbf{A}}_r\mathbf{x}_{t-r}/n\big)^{1/2} \\
&\leq n^{-1/2}\sum_{r=0}^{p}(\mathbf{x}_{t-r}^\prime\mathbf{x}_{t-r}/n)^{1/2}(\lambda_1(\hat{\mathbf{A}}_r^\prime\hat{\mathbf{A}}_r))^{1/2} = O_P(n^{-1/2}),
\end{aligned}
$$

boundedness of $\lambda_1(\hat{\mathbf{A}}_r'\hat{\mathbf{A}}_r)$ being a consequence of Assumptions 2 and 7. As for the third term on the right-hand side of (E.11), $(\mathbf{\Lambda}^\psi)^{-1/2}\mathbf{P}^{\psi\prime}\mathbf{A}(L)\xi_t$ is $O_P(n^{-1/2})$. To conclude, note that the last term $\hat{\mathbf{W}}_q(\mathbf{\Lambda}^\psi)^{-1/2}\mathbf{P}^{\psi\prime}\mathbf{P}^\psi(\mathbf{\Lambda}^\psi)^{1/2}\mathbf{v}_t$ is equal to $\hat{\mathbf{W}}_q\mathbf{v}_t$. The conclusion follows. $\square$