Prediction of Acoustic Resonances in Core Volumes

by

Armel de Montgros

A thesis submitted to Imperial College London for the degree of

Doctor of Philosophy
February 2011

Department of Mechanical Engineering
Imperial College London
London SW7 2AZ
Statement of Originality

The work presented in the thesis is, to the best of the candidate’s knowledge and belief, original and the candidate’s own work, except as acknowledged in the text. The material has not been submitted, either in whole or in part, for a degree or comparable award of Imperial College or any other university or institution.

Armel de Montgros
2011
To Joanna and Lucas
Abstract

In numerous industrial systems, acoustic resonances are synonymous with loud noise and dramatic structural vibrations. They take their origin from trapped weak pressure waves, but can sometimes develop into complex instabilities. With specific emphasis on turbomachines applications, two computational methods have been developed for the prediction of acoustic resonances in core volumes. The first method, called the averaged response function, consists of combined time-domain and frequency-domain approaches. A Favre-averaged RANS solver is used to model the response of a system to a controlled chirp excitation. The response is then analysed using well-known modal analysis tools in order to extract the system’s acoustic characteristics. The second method, called the Arnoldi method, focuses on the stability of the CFD solver. The system is transformed, using a linear Euler solver, into a classic matrix stability problem. The eigenpairs of the matrix, corresponding to the acoustic modeshapes of the system, are then extracted thanks to the iterative Arnoldi method. The two methods are validated and compared on a wide range of cases such as enclosures, open geometries and flow applications. Such a thorough study first provides the reader with a deeper insight into acoustic phenomena by considering classic acoustic examples such as the end correction concept, the Doppler effect and the ”lock-in“ phenomenon. This study also inves-
tigates the limitations and qualities of the implemented methods which are seen to behave very well when compared to theory and experiments. They give accurate results in predicting the three components of the acoustic resonance: the frequency, the damping and the modeshape. As a result, the methods implemented are considered to be mature and can be used to study either the complete acoustic map of the system across a wide range of frequencies or specific acoustic instabilities in a narrow frequency range.
Acknowledgements

I would first like to thank Jo for her constant support, trust and patience during these years of research. Thanks for not asking too many questions. Thanks to my son Lucas for just being Lucas and to my family Xavier, Christine, Stephane and Severine for contributing to what I am today.

I would like to thank my supervisors Prof. Mehmet Imregun and Dr Luca di Mare for their help and patience. For making this project as easy as it could be and letting me direct this thesis the way I wanted while providing constant and always useful ideas and recommendations. This thesis would not have been completed without their trust and knowledge. I would also like to thank Prof. Imregun and Mr Peter Higgs for easing all administrative work throughout my time at the VUTC.

I would like to thank Dr Gabriel Saiz and Mr Constantinos Zegos for helping me when I first started. I really enjoyed our coffees and discussions. I would like to thank Dr Zacharias-Ioannis Zachariadis who has been so patient, making sure that I could get access to all resources needed at any time and day. And I would like to thank everyone at the VUTC for making it such a nice place to work and come
back to. At Imperial College, I would finally like to thanks Simon Burbridge for his help with all the HPC resources. A large amount of the computations presented here have been acheived thanks to him.

I am grateful to Rolls-Royce for sponsoring this project along with the EPSRC and for their strong support throughout. In particular, I would like to thank Dr John Marshall for allowing me to do my Master’s project with the company. Thanks to Dr Francois Moyroud for starting the thesis and to Dr Jeff Green for keeping a good eye on my progress and easing my contact with the company. I would also like to thank Dr Kevin Menzies and Dr Sue Broomfield who are currently studying the applications of the thesis on real geometries. I have worked hard to give Rolls-Royce an efficient tool. I hope Rolls-Royce will find it of good use.

I would finally like to thank Prof. Lars-Erik Eriksson from Chalmers University for kindly giving unconditional access to his work.
## Contents

1 General Acoustics 31

1.1 Introduction .................................................. 32

1.2 Sound waves properties ...................................... 35
    1.2.1 The definition of sound waves ....................... 35
    1.2.2 The wave equation .................................... 37
    1.2.3 The speed of sound waves ............................. 39
    1.2.4 The equations and parameters of sound propagation . 40
    1.2.5 The inviscid characteristic of wave propagation quantified . 42
    1.2.6 The energy of sound waves ............................ 42

1.3 Simple resonances ............................................ 45
    1.3.1 Acoustic resonances in enclosures .................... 46
    1.3.2 The organ pipe ........................................ 49
    1.3.3 Cut-on/cut-off acoustic modes in a waveguide ........ 50
    1.3.4 The Rijke tube ........................................ 53

1.4 Conclusion .................................................... 54

2 Acoustic resonances in the literature 57

2.1 Experimental acoustics ...................................... 58
    2.1.1 Enclosure resonances .................................. 58
    2.1.2 Flow induced resonances ............................... 59
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1.3</td>
<td>Thermo-acoustic resonances</td>
<td>65</td>
</tr>
<tr>
<td>2.2</td>
<td>Analytical acoustics</td>
<td>66</td>
</tr>
<tr>
<td>2.2.1</td>
<td>The equations of sound</td>
<td>66</td>
</tr>
<tr>
<td>2.2.2</td>
<td>A brief overview of analytical methods</td>
<td>70</td>
</tr>
<tr>
<td>2.3</td>
<td>Computational acoustics</td>
<td>75</td>
</tr>
<tr>
<td>2.3.1</td>
<td>Modal analysis methods</td>
<td>76</td>
</tr>
<tr>
<td>2.3.2</td>
<td>Eigenvalue methods</td>
<td>76</td>
</tr>
<tr>
<td>2.4</td>
<td>Conclusion</td>
<td>82</td>
</tr>
<tr>
<td>3</td>
<td>The averaged response function method</td>
<td>85</td>
</tr>
<tr>
<td>3.1</td>
<td>The ARF method implementation</td>
<td>86</td>
</tr>
<tr>
<td>3.1.1</td>
<td>The equations, the solver and the system</td>
<td>86</td>
</tr>
<tr>
<td>3.1.2</td>
<td>The set up of the computation</td>
<td>92</td>
</tr>
<tr>
<td>3.1.3</td>
<td>The preliminary post-processing</td>
<td>96</td>
</tr>
<tr>
<td>3.1.4</td>
<td>The final post-processing</td>
<td>102</td>
</tr>
<tr>
<td>3.2</td>
<td>Validation on a closed circular cylinder</td>
<td>108</td>
</tr>
<tr>
<td>3.2.1</td>
<td>The computation of reference</td>
<td>108</td>
</tr>
<tr>
<td>3.2.2</td>
<td>Impact of the solver</td>
<td>110</td>
</tr>
<tr>
<td>3.2.3</td>
<td>Impact of the excitation</td>
<td>114</td>
</tr>
<tr>
<td>3.2.4</td>
<td>Impact of the meshes</td>
<td>118</td>
</tr>
<tr>
<td>3.3</td>
<td>Conclusion</td>
<td>120</td>
</tr>
<tr>
<td>4</td>
<td>The Arnoldi method</td>
<td>125</td>
</tr>
<tr>
<td>4.1</td>
<td>Theory of the Arnoldi method</td>
<td>126</td>
</tr>
<tr>
<td>4.1.1</td>
<td>The Arnoldi algorithm</td>
<td>127</td>
</tr>
<tr>
<td>4.1.2</td>
<td>Arnoldi limitations</td>
<td>130</td>
</tr>
<tr>
<td>4.1.3</td>
<td>The latest Arnoldi implementations</td>
<td>131</td>
</tr>
<tr>
<td>4.2</td>
<td>Eriksson’s method</td>
<td>134</td>
</tr>
</tbody>
</table>
4.2.1 Theory .............................................. 134
4.2.2 Application to a closed circular cylinder ............... 138
4.2.3 Parametric study ..................................... 142
4.2.4 Conclusion on Eriksson’s method ....................... 143

4.3 Our implementation .................................. 145
4.3.1 Frequency and damping estimates ...................... 145
4.3.2 Convergence criterion ................................ 146
4.3.3 Implementation of an unstructured solver .............. 147
4.3.4 Conclusion on the new implementation ................. 147

4.4 Validation of the new Arnoldi method ................. 148
4.4.1 Comparison on a structured hexahedral mesh ...... 149
4.4.2 Comparison on an unstructured tetrahedral mesh ... 152

4.5 Parametric study of the new Arnoldi solver .......... 153
4.5.1 The convergence criterion ........................... 156
4.5.2 Impact of the mesh refinement ....................... 156
4.5.3 Parametric study ..................................... 158

4.6 Conclusion ............................................. 160

5 The ARF method vs. the Arnoldi method ............... 165
5.1 Presentation of the geometry .......................... 166
5.1.1 The geometry and the mesh ........................ 166
5.1.2 The analytical solution ............................ 167

5.2 The ARF method normalization ......................... 169
5.2.1 Building the acoustic characteristics from the time history . 169
5.2.2 The modeshape estimate ............................ 171
5.2.3 The normalization ................................... 173

5.3 The Arnoldi method normalization ..................... 175
5.3.1 The Arnoldi results .................................. 175
5.3.2 The Arnoldi modeshapes and normalization .............. 185
5.4 Comparison of the two methods .......................... 188
  5.4.1 Comparison in terms of the number of modes .......... 189
  5.4.2 Comparison in terms of accuracy in frequency and damping 190
  5.4.3 Comparison in terms of computational efficiency ......... 196
5.5 Conclusion ............................................. 200

6 The end correction and Doppler effects ...................... 205
  6.1 Study of open geometries and end corrections ............ 205
    6.1.1 The theory of end corrections ...................... 206
    6.1.2 The ARF method analysis .......................... 209
    6.1.3 The Arnoldi method ................................ 219
  6.2 Impact of the Doppler effect on acoustic resonances ...... 229
    6.2.1 Theory ........................................... 231
    6.2.2 The ARF method .................................. 231
  6.3 Conclusion ............................................. 237

7 The lock-in phenomenon ..................................... 241
  7.1 Theory of the lock-in phenomenon ......................... 242
    7.1.1 Presentation of the system ......................... 242
    7.1.2 Sidebranches systems: the lock-in phenomenon ....... 243
    7.1.3 Euler solvers and vortex shedding modelling ......... 245
  7.2 No flow study ........................................... 247
    7.2.1 Introduction ...................................... 247
    7.2.2 The set up ....................................... 248
    7.2.3 The Arnoldi method ................................ 250
    7.2.4 The roll-posts and its subsystems .................... 250
  7.3 Flow study .............................................. 260
Contents

7.3.1 The set up ........................................................................... 260
7.3.2 The ARF method ................................................................. 261
7.3.3 The Arnoldi method ............................................................. 265
7.4 Conclusion ............................................................................ 270

8 Conclusions and recommendations ........................................... 273
  8.1 The ARF method .................................................................. 274
  8.1.1 Implementation ............................................................... 274
  8.1.2 Results ......................................................................... 274
  8.1.3 Recommended use .......................................................... 275
  8.2 The Arnoldi method .............................................................. 276
  8.2.1 Implementation ............................................................... 276
  8.2.2 Results ......................................................................... 276
  8.2.3 Recommended use .......................................................... 277
  8.3 Further work ..................................................................... 279
  8.3.1 Computational methods .................................................. 279
  8.3.2 Computational testcases ................................................... 281
  8.3.3 Experimental and analytical studies ................................. 281

Appendices ................................................................................. 283

A From the Navier-Stokes equations to the wave equation ............ 285

B The Arnoldi eigenvalue method ............................................... 289
  B.1 The Lanczos algorithm ....................................................... 290
     B.1.1 The theory ................................................................. 290
     B.1.2 The Lanczos convergence, implementation and limitations 292
  B.2 The Arnoldi method ............................................................. 294
     B.2.1 The theory ................................................................. 294
B.2.2 The implementation ........................................ 295

C The unstructured second order TVD Roe scheme .......................... 299
  C.1 The theory .................................................. 299
    C.1.1 The first order Roe Scheme .............................. 299
    C.1.2 The second order TVD Roe Scheme ...................... 300
  C.2 The validation on a shock-tube application ......................... 302

D Raw Arnoldi results .................................................. 307
  D.1 Chapter 4 .................................................. 308
  D.2 Chapter 5 .................................................. 309
  D.3 Chapter 6 .................................................. 310
  D.4 Chapter 7 .................................................. 312
List of Figures

1.1 Example of noise regulations ........................................... 33
1.2 The control volume example for the definition of the one dimen-
sional wave equation ................................................. 37
1.3 A simple non directional sound source radiating W watts producing
a sound intensity I ......................................................... 43
1.4 A set of four simple geometries where acoustic resonances are seen
to occur ................................................................. 47

2.1 Vortex shedding excitation at the leading edge of cavities. ........... 60
2.2 Sidebranches systems. .................................................. 62
2.3 Lock-in phenomenon around flat plates. ................................ 64
2.4 CFD Studies of acoustic resonances in different systems ............ 77
2.5 The problem of structures and acoustics interactions: computa-
tional studies .......................................................... 79
2.6 Acoustic resonances and flow applications ............................ 81
2.7 Cutaway of a military jet engine and areas of possible acoustic res-
onance development .................................................. 82

3.1 Implementation of the ARF method ..................................... 87
3.1 Implementation of the ARF method ..................................... 88
3.1 Implementation of the ARF method ..................................... 89
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>Implementation of the ARF method</td>
<td>90</td>
</tr>
<tr>
<td>3.2</td>
<td>Initialization of the method</td>
<td>93</td>
</tr>
<tr>
<td>3.3</td>
<td>Parallel edges on primitive elements. Parallel edges share no vertices, but connect the same faces. If a set of parallel edges is collapsed, the element disappear [1]</td>
<td>95</td>
</tr>
<tr>
<td>3.4</td>
<td>The first stage of post-processing</td>
<td>97</td>
</tr>
<tr>
<td>3.5</td>
<td>The post-processing tools of the ARF method</td>
<td>103</td>
</tr>
<tr>
<td>3.6</td>
<td>Line-fit method for the calculation of frequency and damping with accuracy</td>
<td>105</td>
</tr>
<tr>
<td>3.7</td>
<td>Study of the influence of the number of iterations on the ARF method accuracy</td>
<td>111</td>
</tr>
<tr>
<td>3.8</td>
<td>Study of the influence of the timestep on the ARF method accuracy</td>
<td>113</td>
</tr>
<tr>
<td>3.9</td>
<td>Study of the influence of the amplitude of the excitation on the ARF method accuracy</td>
<td>115</td>
</tr>
<tr>
<td>3.10</td>
<td>Study of the influence of the frequency range on the ARF method accuracy</td>
<td>117</td>
</tr>
<tr>
<td>3.11</td>
<td>Study of the influence of the refinement of the geometry’s mesh on the ARF method accuracy</td>
<td>119</td>
</tr>
<tr>
<td>3.12</td>
<td>Study of the influence of the refinement of the transducers mesh on the ARF method accuracy</td>
<td>121</td>
</tr>
<tr>
<td>4.1</td>
<td>Spectrum transformation resulting from preconditioning of the matrix A</td>
<td>136</td>
</tr>
<tr>
<td>4.2</td>
<td>Closed circular cylinder with two-block structured hexahedral mesh</td>
<td>138</td>
</tr>
<tr>
<td>4.3</td>
<td>Acoustic prediction of Eriksson’s method on a closed cylinder</td>
<td>140</td>
</tr>
<tr>
<td>4.4</td>
<td>Parametric study of Eriksson’s method</td>
<td>144</td>
</tr>
<tr>
<td>4.5</td>
<td>Closed circular cylinder with hexahedral mesh</td>
<td>149</td>
</tr>
</tbody>
</table>
4.6 Acoustic prediction of the new solver on a closed circular cylinder using hexahedral mesh ........................................ 151
4.7 Closed circular cylinder with tetrahedral mesh .................. 152
4.8 Acoustic prediction of the new solver on closed circular cylinder using tetrahedral mesh ........................................ 154
4.8 Acoustic prediction of the new solver on a closed circular cylinder using tetrahedral mesh ........................................ 155
4.9 Impact of the convergence criterion on the convergence of the acoustic modes ........................................ 157
4.10 Impact of the mesh refinement on the convergence of the acoustic modes ........................................ 159
4.11 New solver’s parameters study ........................................ 161
4.12 Parameters impact on frequency and damping ................... 162

5.1 The closed circular cylinder geometry for the methods’ comparison ........................................ 167
5.2 The ARF results: the averaged response function, the multivariate mode indicator function and the acoustic node maps ........................................ 170
5.3 The modeshapes of the first four fundamental acoustic modes ........................................ 174
5.4 Spectrum in the complex plane of the closed circular cylinder ........................................ 177
5.5 Acoustic prediction of the Arnoldi method on a closed circular cylinder using tetrahedral mesh ........................................ 179
5.5 Acoustic prediction of the Arnoldi solver on closed circular cylinder using tetrahedral mesh ........................................ 180
5.5 Acoustic prediction of the Arnoldi solver on closed circular cylinder using tetrahedral mesh ........................................ 181
5.6 The closed cylinder geometry with increasing mesh refinements ........................................ 183
5.7 Impact of the mesh refinement on frequency and damping accuracy ........................................ 184
5.8 The identification of the fundamental modeshape for the Arnoldi method, using the Argand plot ........................................... 186
5.9 The evolution of the number of acoustic modes captured with the mesh refinement when all other input parameters are fixed ........ 190
5.10 The response function plotted at five different transducers ........ 192
5.11 Averaged response function over the whole geometry ............ 193
5.12 Impact of the mesh refinement on the frequency error and numerical damping for both methods and the first four acoustic modes .... 197
5.13 Evolution of the computational time and data size with mesh refinement for both methods .............................................. 199

6.1 The theory of flanged cylinder end corrections ..................... 207
6.2 The end correction (ec) for an unflanged cylinder of radius \( a \), function of \( 2\pi a/\lambda \) .................................................. 209
6.3 Meshes for the semi-open, flanged and unflanged cylinders ........ 210
6.4 The ARF set of results: the averaged response function, the multivariate mode indicator function and the acoustic node maps .... 212
6.5 The ARF input parameters study on the semi-open cylinder ........ 214
6.6 The averaged response function for the semi-open, flanged and unflanged cylinder ......................................................... 216
6.7 The acoustic node maps for the semi-open, flanged and unflanged cylinder ................................................................. 217
6.8 Acoustic prediction of the semi-open cylinder using the Arnoldi method ................................................................. 221
6.8 Acoustic prediction of the semi-open cylinder using the Arnoldi method ................................................................. 222
6.9 The flanged and unflanged meshes for the Arnoldi study ........... 224
6.10 Acoustic prediction of the flanged and unflanged cylinders using the Arnoldi method .......................... 225
6.10 Acoustic prediction of the flanged and unflanged cylinders using the Arnoldi method .......................... 226
6.10 Acoustic prediction of the flanged and unflanged cylinders using the Arnoldi method .......................... 227
6.11 The flanged and unflanged additional study ...................... 230
6.12 The coaxial cylinders mesh, 140 000 nodes ..................... 232
6.13 The averaged response function plot and the acoustic node maps for the coaxial cylinders ..................... 234
6.14 Steady state represented in terms of the Mach number corresponding to an inlet velocity of 100 \text{m.s}^{-1} .............. 235
6.15 The evolution of the averaged response function and acoustic frequencies with a uniform flow ............. 236

7.1 The experimental prototype for the lock-in study ............... 242
7.2 The single, tandem and coaxial sidebranches .................. 244
7.3 The modelled roll-posts lock-in phenomenon ................... 246
7.4 The roll-posts and its subsystems ............................... 248
7.5 The roll-posts geometry and meshes ............................ 249
7.6 The ARF set of results: the averaged response function, the multivariate mode indicator function and the acoustic node maps ........ 251
7.7 Spectrum in the complex plane .................................. 253
7.8 Acoustic prediction of the roll-posts system using the Arnoldi method ......................................... 254
7.8 Acoustic prediction of the roll-posts system using the Arnoldi method ......................................... 255
7.9 Comparison of ARF and Arnoldi frequencies with analytical frequencies and experiments .................. 260
7.10 Steady flow computation of full roll-posts system with inlet velocity of 50 $m.s^{-1}$ ........................................ 262
7.11 The evolution of acoustic resonances with inlet velocity both with the ARF method and the work by Bravo ................ 263
7.12 Evolution of the roll-posts spectrum with flow .................. 265
7.13 The Arnoldi study of roll-posts system with flow ................. 267
7.13 The Arnoldi study of roll-posts system with flow ................. 269

C.1 Hexahedral and tetrahedral meshes of the waveguide used in the Riemann problem ........................................ 303
C.2 Propagation of a wave as modelled by a first and second order TVD Roe scheme ........................................ 304
C.2 Propagation of a wave as modelled by a first and second order TVD Roe scheme ........................................ 305

D.1 Acoustic modeshapes of the roll-posts system using the Arnoldi method ........................................ 313
D.1 Acoustic modeshapes of the roll-posts system using the Arnoldi method ........................................ 314
List of Tables

1.1 Sound-power levels and sound-pressure levels (at the distance of 1m from the source) as well as equivalent pressure fluctuations for different sources of sound .................................................. 45

2.1 Flow equations and corresponding acoustic assumptions ............ 68

3.1 Theoretical frequencies of the lowest frequency acoustic modes for a closed circular cylinder of length 0.693m and radius 0.0191m at atmospheric conditions, $c = 340m.s^{-1}$. ........................................ 92

3.2 Frequency and damping obtained thanks to the line-fit method compared with theory for the computation of reference ................. 110

3.3 Frequency and damping obtained thanks to the line-fit compared with theory for $N = 1048576$ ........................................ 112

3.4 Frequency and damping obtained thanks to the line-fit method compared with theory for $dt = 10^{-6}s$ .............................. 114

3.5 Frequency and damping obtained thanks to the line-fit method compared with theory for a frequency range between 1 and 2000 Hz ... 118

3.6 Recapitulative table of the impact of the different parameters on the ARF method accuracy ............................................. 123

5.1 Characteristic values $q_{n_1,n_2}$ for the cylindrical first order Bessel function ............................................................ 168
5.2 Theoretical frequencies of the first acoustic modes for a closed circular cylinder of length $0.693m$ and radius $0.0191m$ at atmospheric conditions, $c = 340m.s^{-1}$.

5.3 Comparison between the ARF method and the Arnoldi method in terms of frequency and damping accuracy.

5.4 Recapitulative comparison of the two acoustic methods.

6.1 Line-fit frequency and damping compared with theory for the semi-open cylinder.

6.2 Line-fit frequency and damping compared with theory for change in time-step and number of iterations.

6.3 Line-fit frequencies and dampings compared with theory for the semi-open, flanged and unflanged cylinders with time-step $dt = 1.10^{-5}s$.

6.4 Acoustic frequencies compared with theory for the semi-open, flanged and unflanged cylinders with time-step $dt = 1.10^{-6}s$.

6.5 Line-fit frequency and damping, compared with theory, for the set of coaxial cylinders.

6.6 Recapitulative comparison of the two acoustic methods.

7.1 Line-fit frequency and damping for the roll-posts system.

7.2 Estimated frequencies of the acoustic modes predicted with the ARF and Arnoldi methods. Mode defined as branch/outer cylinder/inner cylinder. The error in brackets corresponds to the error against analytical frequencies of a single branch or the coaxial cylinders.

8.1 Recapitulative comparison of the two acoustic methods.
D.1  Data for the closed cylinder study with hexahedral mesh in Chapter 4 308
D.2  Data for the closed cylinder study with tetrahedral mesh in Chapter 4 308
D.3  Data for the closed cylinder study in Chapter 5 309
D.4  Data for the semi-opened cylinder study in Chapter 7 310
D.5  Data for the flanged cylinder study in Chapter 7 310
D.6  Data for the unflanged cylinder study in Chapter 7 311
D.7  Data for the roll-posts system study in Chapter 7 312
Nomenclature

\( D_o \)  Diameter coaxial cylinders
\( E \)  Total energy
\( G_{xp}(f) \)  Cross-spectral function
\( G_{xx}(f) \)  Auto-spectral function
\( H \)  Total enthalpy
\( H(f) \)  Response function
\( I \)  Acoustic intensity
\( J_{ni} \)  Bessel function of first order
\( L \)  Characteristic length
\( L_b \)  Length side-branch
\( N \)  Number computational iterations
\( P(f) \)  Fourier Transform of response
\( PWL \)  Sound power level
\( R \)  Gas constant, \( 287.058 \text{ J.kg}^{-1}.\text{K}^{-1} \)
\( Re \)  Reynolds number
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S$</td>
<td>Unit surface</td>
</tr>
<tr>
<td>$SPL$</td>
<td>Sound pressure level</td>
</tr>
<tr>
<td>$St$</td>
<td>Strouhal number</td>
</tr>
<tr>
<td>$T$</td>
<td>Period</td>
</tr>
<tr>
<td>$T_s$</td>
<td>Temperature</td>
</tr>
<tr>
<td>$T_s$</td>
<td>Sampling time - virtual time</td>
</tr>
<tr>
<td>$U$</td>
<td>Vector of unsteady unknowns</td>
</tr>
<tr>
<td>$W$</td>
<td>Acoustic power</td>
</tr>
<tr>
<td>$X(f)$</td>
<td>Fourier Transform of excitation</td>
</tr>
<tr>
<td>$Z$</td>
<td>Impedance</td>
</tr>
<tr>
<td>$\Delta_c$</td>
<td>End correction</td>
</tr>
<tr>
<td>$\delta_r$</td>
<td>Damping ratio for mode $r$</td>
</tr>
<tr>
<td>$\eta_r$</td>
<td>Structural damping for mode $r$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Heat capacity ratio, 1.4</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Wavelength</td>
</tr>
<tr>
<td>$A$</td>
<td>Solver Matrix</td>
</tr>
<tr>
<td>$B$</td>
<td>Preconditionning matrix</td>
</tr>
<tr>
<td>$H$</td>
<td>Hessenberg matrix</td>
</tr>
<tr>
<td>$Q$</td>
<td>Matrix of Arnoldi vectors</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
</tr>
<tr>
<td>-------</td>
<td>--------------------------------------</td>
</tr>
<tr>
<td>f</td>
<td>External forces</td>
</tr>
<tr>
<td>v</td>
<td>Velocity vector</td>
</tr>
<tr>
<td>μ</td>
<td>Viscosity</td>
</tr>
<tr>
<td>∇</td>
<td>Gradient</td>
</tr>
<tr>
<td>∇²</td>
<td>Laplacian</td>
</tr>
<tr>
<td>ω</td>
<td>Pulsation</td>
</tr>
<tr>
<td>ω'_r</td>
<td>Local unsteady heat release</td>
</tr>
<tr>
<td>ω_r</td>
<td>Resonant pulsation for mode r</td>
</tr>
<tr>
<td>φ_r</td>
<td>Mass-normalized modeshape for mode r</td>
</tr>
<tr>
<td>ψ_r</td>
<td>Modeshape for mode r</td>
</tr>
<tr>
<td>ρ</td>
<td>Density</td>
</tr>
<tr>
<td>ρ'</td>
<td>Acoustic density</td>
</tr>
<tr>
<td>ρ_o</td>
<td>Steady density</td>
</tr>
<tr>
<td>×</td>
<td>Tensor product</td>
</tr>
<tr>
<td>a</td>
<td>Radius</td>
</tr>
<tr>
<td>a_r + ib_r</td>
<td>Constant of response function for mode r</td>
</tr>
<tr>
<td>c</td>
<td>Speed of sound</td>
</tr>
<tr>
<td>c_o</td>
<td>Convergence criterion</td>
</tr>
<tr>
<td>df</td>
<td>Frequency resolution</td>
</tr>
</tbody>
</table>
dt  Timestep

f  Frequency

$f_{Nyquist}$  Nyquist frequency

k  Wave number

$l_i$  Axial length in ith direction

m  Number of computed Arnoldi vectors

$m_r$  Modal mass for mode r

$n_r$  Number of nodes in radial direction

$n_t$  Number of nodes in circumferential direction

$n_x$  Number of nodes in axial direction

p  Pressure

$p'$  Acoustic pressure

$p_o$  Steady pressure

$q_{m_i,n_r}$  Characteristic values of the first order Bessel function

$r_k$  Residual after k iterations

u  Velocity

$u'$  Acoustic velocity

$u_o$  Steady velocity
Chapter 1

General Acoustics

Sounds made by a guitar, a beer bottle and your old tap’s valve are all the result of acoustic resonances. They are an everyday life phenomenon that we have often learned to appreciate. Acoustic resonances are present in the most simple of objects, like the whistle on a football pitch, and in the most complex of technologies, such as jet engines. But in such a hi-tech system, acoustic resonances are much more of a concern as they are the source of loud noise and dramatic system failures due to excessive vibrations.

How and when are acoustic resonances developing? What are their corresponding frequencies, damping and modeshapes? These are the questions addressed in this thesis. To achieve this goal, two computational methods were developed predicting acoustic resonances in core volumes. They will be detailed in the following chapters.

But before we enter the nitty gritty of the problem, it is interesting to focus on the
fundamentals of acoustics in order to better understand the phenomenon in hand and to lay the foundations of the two computational methods implemented. Therefore, after a short introduction on the objectives of this thesis, the main characteristics of the acoustic field will be presented.

1.1 Introduction

Looking back in time, it is striking to see how long has acoustics been the ugly duckling of science. It was developed long after optics, solid mechanics and even astronomy. This early oversight came from the fact that the ancient Greeks did already understand the basics after all [2]. Sound waves are created by motion of solids, they propagate through a medium, most often air, and finally impact on your eardrums. Thus, it is only when Lord Rayleigh started considering acoustics from a more mathematical point of view [2, 3] that the field established its worth.

The paradox between the late development of the theory and the number of applications is striking. The advent of machines expanded the acoustic field beyond Lord Rayleigh’s primary subject of musical instruments. And the offsprings of his theory now explain, model and characterise numerous sound applications in the most complex of environments. With a mix of combustion, high pressure and highly turbulent flows, jet engines are one of the latter. They will be the focus of this thesis.

The primary impact of acoustic resonances is noise, and noise has been identified as one of the most stressful forms of pollution [5, 6]. The most striking example, which often makes the headlines, is noise generated around airports. As a consequence, jet engine manufacturers now need to meet stricter and stricter regulations
1.1. INTRODUCTION

(a) The definition of the ICAO regulations criterion

(b) Certification flyover (take-off) noise levels for ICAO Annex 16 depending on the plane size

(c) Breakdown of the noise sources of a plane at take off

Figure 1.1. Example of noise regulations [4]
limiting noise level. The International Civil Aviation Organisation (ICAO) [4] describes three criteria a plane has to fulfill if it is to pass regulations: the lateral, flyover and approach criteria measure the level of noise in decibels at three different points around the runway (Fig. 1.1a). It must be lower than the regulated limits and modern planes achieve this objective (Fig. 1.1b). Large improvement have already been made with for example high bypass ratio engines, but looking at the breakdown of the sources of noise in Fig. 1.1c, it is obvious that the engines still play an important role as they produce about half the sound emitted. Therefore, every effort should be made to dampen noise emissions.

The second reason behind the interest in acoustic resonances may be less obvious but it is as important. Acoustic resonances are characterised by confined high pressure oscillations. These oscillations can be very damaging for the structural integrity of the system in which they develop. The high standards in safety and reliability imposed on jet engine manufacturers do not allow this kind of failure. But nowadays, the control for acoustic resonances is mostly done at the end of the design by testing prototypes. In these times, where low cost and short development times are important factors in the success of an engine, having a tool that can be used at the early design stage will be a great asset.

Better understanding the development of acoustic resonances, where they occur and at what frequency, is then crucial in the design of the next generation of engines. This is the aim of this thesis, whose lay out is as follows; Chapters 2 will draw a picture of resonances applications from the literature available as well as present the current method of prediction. Chapters 3 and 4 will introduce in detail the two methods identified as potential solutions: respectively the averaged response function method and the Arnoldi method. Chapter 5 will compare the two methods
1.2. Sound waves properties

The specific acoustic properties, compared with general fluid phenomena, will play an important role in the design of a quick, efficient and reliable method of prediction. The reader will now be presented with the fundamental properties of sound wave propagation and energy. This will help understand simple resonance applications presented in the following section as well as the implementation of computational methods.

1.2.1 The definition of sound waves

Sound waves are pressure waves. Pressure perturbations are created by the motion of a solid or an instability in a flow. They propagate through a medium like gas, liquid, or solids and do not impact on, or interact with, the mean state of the medium. Pressure perturbations are therefore small compared to the mean state. For exam-
ple, when you listen to a conversation, you do not feel incoming waves. In the rest of this thesis, this pressure perturbation around a mean state $p_0$ will be defined as $p'$. $p'$ is often called the acoustic pressure.

Disconnecting the perturbation from the mean state corresponds to linearising the acoustic field around the mean flow. This is known as the acoustic approximation. It implies that the perturbation unknowns, such as density, velocity and pressure, will all be proportional to each other, and that two sound waves will not interact with each other while propagating. For example, during a concert, the human ear can clearly differentiate between the violin and the clarinet lines.

Acoustic waves are therefore created by a small pressure perturbation, but how is this perturbation propagating? First, sound waves cannot propagate in a void. They rely therefore on the inertia and elasticity of the medium to propagate. When a particle of air is excited with a back and forth displacement, inertia advocates the oscillations of the particle and compressibility ensures that neighbouring particles will receive some of its kinetic energy. As soon as energy is transmitted a wave is propagating. On the other hand, sound waves propagating through air implies that they are longitudinal waves. Indeed, air cannot handle shear stress without motion and the displacement of the air particles will have to move in the direction of propagation of the wave.

In the end, generating a sound wave only requires a to-and-fro motion or a compression-rarefaction input from a source. The latter can then be provided by any solid vibrations, such as a piston, or fluid interactions, such as mixing jets from a jet engine. The medium will then ensure that the wave and its energy are propagating. This propagation is modelled by a simple set of equations easily deduced from the fun-
1.2. SOUND WAVES PROPERTIES

1.2.2 The wave equation

Consider a control volume of air, as pictured in Fig. 1.2, and assume the problem to be one-dimensional. The flow variables are defined using the acoustic approximation and the fluid is assumed to be stagnant. The acoustic approximation implies that the flow field can be linearised and therefore the density, velocity and pressure can be expressed by defining a perturbation around a mean state: $\rho = \rho_o + \rho'$, $u = u_o + u'$ and $p = p_o + p'$. Three physics fundamentals will then be used to define the equation of propagation of the sound waves [7, pp. 16–17].

The conservation of mass

The conservation of mass states that the rate of change of mass within the volume is equal to the difference of the massflow in and out of the volume:

$$\frac{\partial \rho'}{\partial t} dxS = ((\rho_o + \rho') u') (x, t)S - ((\rho_o + \rho') u') (x + dx, t)S$$ (1.1)

Here the fluid is assumed stagnant and therefore $u = u'$. According to the acoustic approximation $\rho' u'$ will be small and therefore negligible compared to the other
quantities. This leads to the linearised one dimensional mass conservation equation:

\[
\frac{\partial \rho'}{\partial t} + \rho_o \frac{\partial u'}{\partial x} = 0
\]  

**(1.2)**

**Newton’s principle**

Newton’s principle states that the acceleration of the fluid particles inside the volume is equal to the forces exerted on the volume. Here gravity is ignored and only pressure forces are considered.

\[
\rho_o \frac{\partial u'}{\partial t} \, dx \, S = (p'(x,t) - p'(x + dx,t)) \, S
\]  

**(1.3)**

or

\[
\rho_o \frac{\partial u'}{\partial t} + \frac{\partial p'}{\partial x} = 0
\]  

**(1.4)**

Combining (1.2) and (1.4) allows the elimination of the velocity unknown from the system:

\[
\frac{\partial^2 \rho'}{\partial t^2} - \frac{\partial^2 p'}{\partial x^2} = 0
\]  

**(1.5)**

**The fluid state relationship**

In any fluid, the evolution of pressure will be linked to the evolution of density by what is called a state equation. Assuming the acoustic approximation holds, it is possible to linearise that relationship such as:

\[
p = p_o + (\rho - \rho_o) \frac{dp}{d\rho} (\rho_o) + o(d\rho)
\]  

**(1.6)**

This implies that:

\[
p' = \rho' \frac{dp}{d\rho} (\rho_o)
\]  

**(1.7)**
We can then define a variable $c$ such that

$$c^2 = \frac{dp}{d\rho}(\rho_o) \quad (1.8)$$

and

$$p' = c^2 \rho' \quad (1.9)$$

(1.2), (1.4) and (1.9), deduced from the three fundamental principles above, can now be combined to define the following one dimensional equation in pressure:

$$\frac{1}{c^2} \frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x^2} = 0 \quad (1.10)$$

(1.10) is the wave equation in pressure. The dependence of time and space implies that small pressure perturbations propagate as waves along the axis $x$ and at a speed $c$ defined by (1.8). From the one dimensional equation, it is possible to extrapolate the three dimensional wave equation:

$$\frac{1}{c^2} \frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = 0 \quad (1.11)$$

where $\nabla^2$ is the Laplacian operator.

### 1.2.3 The speed of sound waves

The wave equation (1.11) defines $c$ as the velocity of propagation of the acoustic waves. It was first measured accurately by a group of scientists of the French Academy. In 1738, they fired canons and found that, at 0°C, the sound propagated at $337 \text{m.s}^{-1}$ [2, pp. 2]. Newton originally gave a first analytical estimate of the speed of sound by using Boyle’s Law for perfect gas. The latter relates the change in pressure and density to the temperature:

$$\frac{p}{\rho} = RT \quad (1.12)$$
where $R$ is the gas constant $R = 287.058 J/kg^1.K^{-1}$. If we assume the transformation isothermal, the value of $c$ equals:

$$c = \sqrt{\frac{dp}{d\rho}} = \sqrt{\frac{p_o}{\rho_o}} = 280 m.s^{-1}$$ \hspace{1cm} (1.13)

This result did not agree with initial experiments, and it is only when Laplace discovered, in 1816, that the compression in sound waves was too quick to exchange heat, that an accurate estimate was calculated. Indeed, for an adiabatic expansion in a perfect gas, the ratio $\frac{p}{\rho}$ is constant. Hence:

$$c = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s} = \sqrt{\frac{\gamma p_o}{\rho_o}} = 331 m.s^{-1}$$ \hspace{1cm} (1.14)

This is a better estimate of the speed of sound as observed in experiments [7, pp. 20–21].

### 1.2.4 The equations and parameters of sound propagation

Now that the speed of sound has been defined, we can solve the wave equation (1.10). For a plane one-dimensional wave, the solution is:

$$p'(x, t) = f \left(t - \frac{x}{c}\right) + g \left(t + \frac{x}{c}\right)$$ \hspace{1cm} (1.15)

where the functions $f$ and $g$ correspond to a wave propagating in respectively the forward and backward directions. The simplest way to represent vibrational motion is to use sine and cosine functions. This is known as the simple harmonic representation. For a forward propagating wave, we define:

$$p'(x, t) = p_{max} \cdot \cos \left[\omega \left(t - \frac{x}{c}\right)\right]$$ \hspace{1cm} (1.16)

From that definition, four fundamental parameters of wave propagation can be established:
1.2. SOUND WAVES PROPERTIES

- The wavelength $\lambda$ is the length the wave has to travel before reaching an equivalent state. It is expressed in meters.
- The period $T$ is the time between two equivalent states. It is expressed in seconds.
- The frequency $f$ is the number of oscillations per second, i.e.: $f = \frac{1}{T}$. It is expressed in Hertz.
- The pulsation $\omega$ is the equivalent of the frequency but projected on a circle. $\omega = 2\pi f$. It is expressed in radians per seconds.

From the wave definition (1.16), we define the dispersion equation relating the speed of sound to the frequency and wavelength:

$$c = \lambda f$$  \hspace{1cm} (1.17)

Using (1.4), it is also possible to express the fluid velocity as a function of the pressure:

$$u'(x, t) = \pm \frac{1}{\rho_0 c} p'(x, t)$$  \hspace{1cm} (1.18)

The $\pm$ sign depends on the direction of the propagating wave. The ratio:

$$\frac{p'}{u'} = \pm \rho_0 c$$  \hspace{1cm} (1.19)

is known as the impedance of the system.

Finally, for later reference, a spherical wave can also be represented in the spherical referential [8, pp. 28] by:

$$p'(r, t) = \frac{p_{\text{max}}}{r} \cos \left[ \omega \left( t - \frac{r}{c} \right) \right]$$  \hspace{1cm} (1.20)

We can deduce from (1.16) and (1.20) that the amplitude of the acoustic waves stays constant in a longitudinal plane wave but decreases with the radius in a spherical wave. As shown later in Sec.1.2.6, this has a very interesting impact on the energy propagating with the wave.
1.2.5 The inviscid characteristic of wave propagation quantified

The nondimensional Reynolds number is the ratio between the inertia forces of the convective fluxes and the viscosity forces. If it is small, the fluid is said to be incompressible and viscous. If it is large, the fluid is compressible and inviscid. The acoustic Reynolds number is defined by:

\[ Re = \frac{\rho c \lambda}{\mu} \]  

(1.21)

The density and viscosity for air at atmospheric conditions are respectively \( \rho = 1.266 \text{kg.m}^{-3} \) and \( \mu = 2 \times 10^{-5} \text{Pa.s} \). The range of frequency a human ear can hear sounds is between 20 Hz and 20 kHz, and therefore a 1 kHz sound wave is considered. If the speed of sound is taken equal to \( c = 340 \text{m.s}^{-1} \) (value taken at 25°C), then (1.17) implies that the wavelength is equal to \( \lambda = 0.034 \text{m} \) and the corresponding Reynolds number is \( Re = 7 \times 10^6 \). The Reynolds number is largely over unity. The phenomenon is considered inviscid.

1.2.6 The energy of sound waves

Sound waves propagate in a fluid just like a wave can travel through springs, it transfers motion from one particle to another. Acoustic waves therefore propagate a form of energy which will now be defined and quantified.

The acoustic energy is defined by the work done by pressure forces on air particles. The rate of acoustic energy which crosses a unit area is known as the intensity. It is expressed as a mean value using (1.18) [8, pp. 27]:

\[ I = \overline{p' u'} = \frac{\overline{p^2}}{\rho_o c} \]  

(1.22)
This expression is valid for both plane and spherical waves. The corresponding power $W$ of the wave is defined as the total energy passing through the surface $S$ surrounding a source (Fig.1.3). That is to say:

$$W = I.S$$ \hspace{1cm} (1.23)

In a gas like air, it is assumed that no phenomenon is able to damp the sound wave power. Energy is therefore conserved such that all of the power radiated must pass through any surface enclosing the source. As a result, if a plane wave propagates in a cylinder, the intensity of the wave will stay constant because the surface of the wave front stays constant, while if the wave is spherical, the intensity of the wave will be inversely proportional to the squared distance from the source. This is in accordance with (1.16) and (1.20) defining the propagation of plane and spherical waves. This difference in propagating patterns explains for example the use of conic megaphones.
The power of a sound source is now to be determined. Because of the large range of amplitudes, the log scale is adopted and the sound-power level is defined as [8, pp. 47]:

\[ PWL = 10 \log_{10} \frac{W}{W_{ref}} \] (1.24)

The use of the logarithmic scale implies that we need to define a reference power \( W_{ref} \) so that the ratio is a dimensionless quantity. The dimension of the sound-power level is therefore the acoustic decibel (dB) relative to \( W_{ref} \). \( W_{ref} \) is often defined as equal to \( 10^{-12} W \) for reasons we will see shortly.

Power measurement can nevertheless be sometimes difficult and directly measuring the pressure of the wave is often preferred. Using (1.22), we can define another indicator of power, the sound-pressure level or SPL [8, pp. 53]:

\[ SPL = 20 \log_{10} \frac{p_{rms}}{p_{ref}} \] (1.25)

This is also expressed in acoustic decibels but relative to \( p_{ref} \). \( p_{ref} \) is set to \( 2 \times 10^{-5} Pa \) as this is the threshold of hearing for the human ear for a 1 kHz wave. It is important to notice here that while the PWL is constant, the SPL for a single source will decrease linearly with the distance from the source. The earlier value \( W_{ref} \) comes from the fact that if we impose the two levels to be equal for a surface of \( 1m^2 \), \( W_{ref} \) as to be equal to \( 10^{-12} W \).

Now that the different ways to measure noise have been defined, it is interesting to compare them for known sources of sound. Table.1.1 shows that the rocket engine sound, corresponding to SPL=180dB, corresponds to a pressure perturbation of 0.2 atmosphere at a 1m distance. A conversation corresponds to sound waves whose amplitudes are \( 10^{-7} \) atmosphere, while the threshold of hearing corresponds to \( 10^{-10} \) atmosphere. This clearly shows that the acoustic approximation holds, as
1.3. SIMPLE RESONANCES

<table>
<thead>
<tr>
<th>Sources of sound</th>
<th>PWL (dB)</th>
<th>SPL (dB)</th>
<th>p (atm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Large rocket engine</td>
<td>190</td>
<td>180</td>
<td>2.10^{-1}</td>
</tr>
<tr>
<td>Military turbojet engine</td>
<td>170</td>
<td>160</td>
<td>2.10^{-2}</td>
</tr>
<tr>
<td>4 propeller airlinerg</td>
<td>160</td>
<td>150</td>
<td>6.10^{-3}</td>
</tr>
<tr>
<td>Threshold of pain</td>
<td>150</td>
<td>140</td>
<td>2.10^{-3}</td>
</tr>
<tr>
<td>75-piece orchestra</td>
<td>130</td>
<td>120</td>
<td>2.10^{-4}</td>
</tr>
<tr>
<td>Small aircraft engine</td>
<td>130</td>
<td>120</td>
<td>2.10^{-4}</td>
</tr>
<tr>
<td>Auto on motorway[50mph]</td>
<td>110</td>
<td>100</td>
<td>2.10^{-5}</td>
</tr>
<tr>
<td>Voice shouting</td>
<td>90</td>
<td>80</td>
<td>2.10^{-6}</td>
</tr>
<tr>
<td>Appliances</td>
<td>80</td>
<td>70</td>
<td>6.10^{-7}</td>
</tr>
<tr>
<td>Voice conversation</td>
<td>70</td>
<td>60</td>
<td>2.10^{-7}</td>
</tr>
<tr>
<td>Voice soft whisper</td>
<td>30</td>
<td>20</td>
<td>2.10^{-9}</td>
</tr>
<tr>
<td>Threshold of hearing</td>
<td>10</td>
<td>0</td>
<td>2.10^{-10}</td>
</tr>
</tbody>
</table>

Table 1.1. Sound-power levels and sound-pressure Levels (at the distance of 1m from the source) as well as equivalent pressure fluctuations for different sources of sound [8, pp. 45]

...the acoustic oscillations are small compared to the mean state. The fact that the acoustic field can be linearised comes as a consequence.

As a conclusion to this section, acoustic waves are weak pressure perturbations propagating inviscidly through a medium. Wave propagation can be modelled using the wave equation (1.11) and its velocity can be accurately calculated using (1.14). The solutions of the wave equation are periodic and have a specific wavelength defined by the dispersion equation (1.17). Why then can such weak phenomenon have dramatic consequences?

1.3 Simple resonances

Acoustic waves are weak pressure waves of amplitude $10^{-5}$ atmosphere or so. They should therefore not have any impact on structures. But the acoustic energy, combining kinetic and potential energy, is travelling with the sound waves. Though it is small, if this energy stops propagating, and if the fluid is excited continually, strong
pressure oscillations will develop in the system. These are acoustic resonances. After looking at how they form in a cylindrical enclosure, three simple examples will be introduced, highlighting acoustic resonances properties. These three examples are known as the organ pipe, the cut-on/cut-off modes in a waveguide and the Rijke tube.

1.3.1 Acoustic resonances in enclosures

Let’s first consider a simple geometry which will be used a great deal in the rest of the thesis, an enclosed circular cylinder of axis $x$, presented in Fig.1.4a, whose radius $a$ is assumed small compared to its length $l_x$. Thus only an axial wave will be able to propagate. Let the end walls be infinitely rigid so that a simple harmonic forward propagating wave $f(t - \frac{x}{c}) = p_{\text{max}} \cos \left(\omega \left(t - \frac{x}{c}\right)\right)$ will be reflected back on the endwall under the form of a backward propagating wave $g(t + \frac{x}{c}) = p_{\text{max}} \cos \left[\omega \left(t + \frac{x}{c}\right)\right]$. Due to the linear approximation, the resulting acoustic pressure field will correspond to the sum of those two waves:

$$p_0(x,t) = f(t - \frac{x}{c}) + g(t + \frac{x}{c}),$$

(1.26)

Simple trigonometry yields:

$$p_0(x,t) = 2p_{\text{max}} \cos(\omega t) \cos\left(\omega \frac{x}{c}\right)$$

(1.27)

The temporal and spatial variables are now independent. The acoustic wave, and the corresponding energy, are therefore not propagating. Such a wave is known as a standing wave. In a rigid enclosure, the energy cannot escape. If the fluid is excited, the wave amplitude, under the form of pressure oscillations, will grow. This corresponds to an acoustic resonance.

Acoustic resonances are defined by three parameters: the frequency, the modeshape
1.3. SIMPLE RESONANCES

Figure 1.4. A set of four simple geometries where acoustic resonances are seen to occur.
and the damping. On one hand, (1.27), along with appropriate boundary conditions, will define the frequency and modeshape of the acoustic mode excited. On the other hand, the damping will be influenced by the boundary conditions characteristics such as non-rigid walls or open boundaries.

The boundary conditions for rigid walls imposes that the normal velocity of the fluid to the surface is equal to zero, or equivalently, that the pressure gradient normal to the wall is zero:

$$\nabla p'.n = 0$$ \hfill (1.28)

Looking at (1.27), the boundary condition at $x = l_x$ imposes that:

$$\sin \left( \frac{\omega x}{c} \right) = 0$$ \hfill (1.29)

This is only possible for a discrete range of frequencies $(f_n)_{n \in \mathbb{N}}$

$$f_n = \frac{1}{2\pi} \frac{\omega_n}{c} = \frac{n}{2l_x}$$ \hfill (1.30)

The corresponding wavelength is:

$$\lambda_n = \frac{c}{f_n} = \frac{\pi n}{l_x}$$ \hfill (1.31)

The pressure modeshape is expressed as:

$$p'(x, t)_n = 2p_{max} \cdot \cos (\omega_n t) \cos \left( \frac{2\pi x}{\lambda_n} \right)$$ \hfill (1.32)

At this stage, it is possible to define another parameter, the wave number, corresponding to a spatial pulsation:

$$k_n = \frac{\omega_n}{c} = \frac{2l_x}{n}$$ \hfill (1.33)
such that:

\[ p'(x, t)_n = 2p_{max} \cos(\omega_n t) \cos(k_n x) \]  \hspace{1cm} (1.34)

If the radius \( a \) is not considered negligible with regards to the length of the cylinder anymore, non axial modes will also develop. The theoretical modeshapes in cylindrical coordinates, solving the wave equation in the cylindrical referential, are defined by the Bessel functions [9]:

\[ p'(x, \theta, r)_{n_x, n_t, n_r} = \cos(k_{n_x} x) \cos(n_t \theta) J_{n_t} \left( \frac{\pi q_{n_t, n_r} r}{a} \right) \]  \hspace{1cm} (1.35)

\( J_{n_t} \) is the Bessel function of first order, \( n_t \) the number of circumferential nodes and \( n_r \) the number of radial nodes. Concerning the frequency of the modes, rigid wall boundary conditions force the gradient of pressure normal to the wall to be equal to zero. This defines the zeros of the Bessel function and \( q_{n_t, n_r} \). The frequency of the modes is then expressed as:

\[ \omega_{n_x, n_t, n_r} = \sqrt{\left( \frac{\pi n_x}{l_x} \right)^2 + \left( \frac{\pi q_{n_t, n_r}}{a} \right)^2} \]  \hspace{1cm} (1.36)

(1.36) will be used throughout this thesis for the validation of the two computational methods implemented.

### 1.3.2 The organ pipe

The organ pipe is an example of a circular cylinder geometry where the end walls are now removed and the fluid is connected to the outside domain (Fig.1.4b). The boundary conditions imply that the pressure at each end of the cylinder is atmospheric, that is to say that the two ends of the cylinder correspond to a node of the acoustic modes. This implies that the modes frequencies will be expressed as:

\[ f_n = n \frac{c}{2l_x} \]  \hspace{1cm} (1.37)
The corresponding wavelength is:

\[ \lambda_n = \frac{c}{f_n} = \frac{2l_x}{n} \quad (1.38) \]

And because of the new boundary conditions, the modeshapes will now be, if we consider the radius \( a \) negligible:

\[ p'(x, t)_n = 2p_{\text{max}} \sin (\omega_n t) \sin \left( 2\pi \frac{x}{\lambda_n} \right) \quad (1.39) \]

For open pipes, these boundary conditions are however simplistic and do not take into account the fact that the wave changes from an axial to a spherical behaviour, at the mouth of the cylinder. An end-correction is therefore introduced to represent the phenomenon with more accuracy. For example, the length of unflanged cylinders should be increased by the end correction \( \Delta_c \) equal to:

\[ \Delta_c = 0.61a \quad (1.40) \]

In terms of the organ pipe considered, the wavelength and frequencies should be expressed as:

\[ \lambda_n = \frac{c}{f_n} = \frac{2(l_x + 2\Delta_c)}{n} \quad (1.41) \]

and

\[ f_n = \frac{\omega_n}{2\pi} = n \frac{c}{2(l_x + 1.2a)} \quad (1.42) \]

A further study of the impact of end corrections will be presented in Chapter 6 using the two different computational methods developed.

### 1.3.3 Cut-on/cut-off acoustic modes in a waveguide

The development of acoustic resonances in both systems above are not surprising as either the end walls, in the case of the enclosure, or the pipe openings, in the case of
the organ pipe, act as respectively total or partial reflectors. The acoustic energy is then reflected back and trapped within the system. But acoustic resonances are also sometimes observed in open pipes, where the energy should be able to propagate along the main axis [9, 10].

In order to explain, let’s consider an infinite rectangular pipe, as presented in Fig. 1.4c, of dimension \((l_y, l_z)\) in the direction \((y, z)\) and of axis \(x\). The wave equation (1.11) is still valid and the acoustic pressure can be expressed, using the harmonic representation, as \(p'(x, t) = f(x)g(y)h(z)e^{i\omega t}\). Plane waves will propagate in the \(y\) and \(z\) directions in the same way they were propagating in the \(x\) direction when considering the enclosure, that is:

\[
g^{(m)}(y) = \cos\left(\frac{2\pi y}{\lambda_y^{(m)}}\right), \quad h^{(n)}(z) = \cos\left(\frac{2\pi z}{\lambda_z^{(n)}}\right)
\]

with:

\[
\lambda_y^{(m)} = \frac{2l_y}{m}, \quad \lambda_z^{(n)} = \frac{2l_z}{n}
\]

The pair \((m, n)\) identifies the \(m^{th}\) and \(n^{th}\) mode in respectively the \(y\) and \(z\) direction. Substituting the functions \(g\) and \(h\) into the wave equation (1.11) leads to an ordinary differential equation regarding the function \(f\):

\[
\frac{d^2 f(x)}{dt^2} + \left(\frac{\omega^2}{c^2} - 4\pi^2\left(\frac{1}{\lambda_y^{(m)^2}} + \frac{1}{\lambda_z^{(n)^2}}\right)\right) f(x) = 0
\]

Equation (1.45) has for solution:

\[
f^{(m,n)}(x) = A_1e^{-jk^{(m,n)}x} + A_2e^{jk^{(m,n)}x}
\]

where the wave number \(k^{(m,n)}\) is defined by:

\[
k^{(m,n)} = \sqrt{\frac{\omega^2}{c^2} - 4\pi^2\left(\frac{1}{\lambda_y^{(m)^2}} + \frac{1}{\lambda_z^{(n)^2}}\right)}
\]
To simplify the previous relationship, we define a cut-off frequency as:

\[ f_{c}^{(m,n)} = c \sqrt{\frac{1}{\lambda_{y}^{(m)^2}} + \frac{1}{\lambda_{z}^{(n)^2}}} \]  

(1.48)

The wave number \( k^{(m,n)} \) then takes the form:

\[ k^{(m,n)} = \frac{2\pi}{c} \sqrt{f^2 - f_{c}^{(m,n)^2}} \]  

(1.49)

Depending on the propagating wave frequency, we can identify two cases for the expression of the wave number:

- The driving frequency is above the cut-off frequency. The wave number is real and the wave is allowed to propagate. The resulting pressure distribution is such that:

\[ p'(x, t) = p'_{\text{max}} \cos \left( \frac{2\pi y}{\lambda_{y}^{(m)}} \right) \cos \left( \frac{2\pi z}{\lambda_{z}^{(n)}} \right) \cos \left( k^{(m,n)} x - \omega t \right) \]  

(1.50)

- The driving frequency is under the cut-off frequency. The wave number is purely imaginary and the wave is heavily damped. The resulting pressure distribution is such that:

\[ p'(x, t) = p'_{\text{max}} \cos \left( \frac{2\pi y}{\lambda_{y}^{(m)}} \right) \cos \left( \frac{2\pi z}{\lambda_{z}^{(n)}} \right) e^{-k^{(m,n)} x} \cos (\omega t) \]  

(1.51)

It is now worth examining the definition of the cut-off frequency (1.48) in more detail. Firstly, every longitudinal mode, for which \( m = 0 \) and \( n = 0 \), will propagate, while other modes will only propagate if the driving frequency is above the cut-off frequency \( f_{c}^{(m,n)} \). Secondly, we can see that not only \( f_{c}^{(m,n)} \) is dependent on the fluid properties, it is also dependent on the waveguide’s section dimensions. The smaller the section, the larger is the cut-off frequency. Therefore, in varying section ducts, an interface can be defined between the cut-off and cut-on domains. Energy
1.3. SIMPLE RESONANCES

will not be able to propagate from the cut-on to the cut-off region and a reflective wave will be generated. As a result, superposition of the incoming and reflective waves are likely to occur and produce acoustic resonances. Such phenomena have been seen to occur, for example, in jet engine intakes where waves are confined between the fan and the cut-off interface.

1.3.4 The Rijke tube

The Rijke tube experiment was first developed by Rijke in 1850. As presented in Fig.1.4d, it consists of a vertical organ pipe with a heated grid [7]. Place the heated grid in the bottom half of the cylinder and the fundamental mode of the cylinder will be excited. Place it in the upper half and no sound is generated. This phenomenon was first explained by Lord Rayleigh [2] and illustrates perfectly the strong link, expressed in the Rayleigh criterion, between the heat release rate and acoustic pressure fluctuations. Lord Rayleigh noticed that if heat is added to the sound wave when it is at the high pressure phase of the cycle, energy is fed into the acoustic disturbance, otherwise the acoustic resonance is damped.

To explain the phenomenon, two distinct air displacements have to be taken into account. First, air heated by the grid will rise in the pipe due to convection. But displacement will also be induced by the fundamental acoustic modeshape. When the acoustic displacement is upwards, the two effects will combine and a large amount of air will be passing through the grid. As a result, the heat release rate will increase. On the contrary, if the acoustic displacement is downwards, the convection and acoustics will counteract and little heat will be exchanged.

According to the Rayleigh criterion, the fundamental acoustic mode will be excited if the heat release rate, and therefore the particle displacement, is in phase with the
acoustic pressure. Looking at the expression of the acoustic pressure for an organ pipe in (1.39), we have:

\[ p'(x, t)_n = 2p_{\text{max}} \cdot \sin (\omega_n t) \sin (k_n x) \]  

(1.52)

The corresponding velocity field in the pipe can be deduced from (1.4):

\[ u_n = \frac{2p_{\text{max}}}{\rho_0 c} \cos (\omega_n t) \cos (k_n x) \]  

(1.53)

And the particle displacement is defined by:

\[ \eta_n = \frac{2p_{\text{max}}}{\rho_0 c \omega_n} \sin (\omega_n t) \cos (k_n x) \]  

(1.54)

Looking at the pressure and air displacement, we get:

\[ \frac{p'_n}{\rho_0 c^2} = \frac{\omega_n \eta_n}{c} \tan (k_n x) \]  

(1.55)

Considering the fundamental acoustic mode, (1.55) shows that the pressure and the displacement are in phase if \( \tan \left( \frac{\pi x}{l_x} \right) \) is positive, that is to say, if \( 0 < x < \frac{l_x}{2} \). The fundamental mode will therefore only be excited if the grid is placed in the bottom half of the cylinder.

### 1.4 Conclusion

The purpose of the chapter was to establish the fundamentals of acoustics that will be used throughout this thesis. It was seen that acoustic resonances are present in numerous systems and that many industrial applications are prone to acoustic resonances. Jet engines, which will be the focus of this thesis, are particularly sensitive to such issues because of tougher and tougher regulations on noise emissions and strong interactions between fluid and structures in a very confined and complex environment.
Acoustic waves have been shown to be weak pressure waves propagating inviscidly and longitudinally in a medium such as air. The amplitude of pressure oscillations are of the order of $10^{-5}$ atmosphere which is very small compared to usual flow patterns in a jet engine. An acoustic approximation is therefore defined, allowing the linearisation of the acoustic field about the steady state. This important property will be used throughout the thesis. The acoustic waves are defined by the wave equation:

$$\frac{1}{c^2} \frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = 0$$

(1.56)

The solution to the wave equation can be put under the form of a harmonic forward and backward propagating wave:

$$p'(x, t) = p_{\text{max}} \cos \left[ \omega \left( t - \frac{x}{c} \right) \right] + p_{\text{max}} \cos \left[ \omega \left( t + \frac{x}{c} \right) \right]$$

(1.57)

The two waves propagate at the speed of sound, defined as:

$$c = \sqrt{\left( \frac{\partial p}{\partial \rho} \right)_s} = \sqrt{\frac{\gamma p_o}{\rho_o}}$$

(1.58)

Acoustic resonances develop when the forward and backward waves are confined within a volume. This results in a standing wave where the acoustic energy cannot dissipate. If energy is then fed into the system, large pressure oscillations will develop. Acoustic resonances are described by three parameters: the frequency, the damping and the modeshape. Once these parameters are known, the acoustic characteristics of the system are known.

Simple analytical methods can model the development of acoustic resonances in simple applications such as enclosures, the organ pipe, simple waveguides and the Rijke tube. But acoustic resonances have been seen to develop in much more complex systems where simple analytical methods are not powerful enough. Enhanced
methods have therefore been developed. Before entering into the nitty gritty of the two computational methods we have implemented, it is interesting to have a look at how previous experimental, analytical and computational methods have managed to improve our understanding of the phenomenon.
Chapter 2

Acoustic resonances in the literature

For a long time, the only known phenomenon of acoustic resonances concerned the sound produced by musical instruments such as the organ pipe or violins. Rayleigh [2] first theoretized acoustics and the development of acoustic resonances by studying music. With the industrial revolution and the advent of machines, occurrences of acoustic resonances become widespread, prompting the development of methods to better understand, better model and better predict the phenomenon.

Experimental acoustic studies led the way by creating a wealth of testcases. Analytical methods then followed, using the experimental database as a means of validation. From the use of simple mathematics as presented in Chapter 1, the analytical methods improved, using more and more complex models, in order to match what the experiments would observe. When CFD started to appear in the 1950s, it quickly backed up analytical methods to offer a deeper insight into more and more complex studies. In the end, experiments as well as analytical and computational
methods are still used nowadays, side by side, to give a better understanding on the
development of acoustic resonances. The development of those methods will be
presented here with an specific emphasis on turbomachinery applications.

2.1 Experimental acoustics

Over the last decades, experiments have highlighted the development of acoustic
resonances in many applications. It first focused on simple enclosures, like cars, as
well as open systems, like instruments. It then studied flow induced and combus-
tion induced acoustic resonances in more and more complex geometries. We will
present here a non exhaustive list of studies in order to show how wide the range of
acoustic resonance applications really is.

2.1.1 Enclosure resonances

If energy is given to the fluid inside an enclosure, usually under the form of a
periodic vibration, acoustic resonances will develop. Their characteristics, like the
frequency and modeshape, will depend on the dimensions of the system studied
(see Section.1.3).

For example, the vibrations of the string of a violin are transmitted to the air in the
resonant box via the bridge. Acoustic modes develop in the box and sound prop-
grates through the f-holes [11]. A similar experiment was carried out to study the
interaction between the fluid inside a concrete block and the vibration of one of
its walls [12]. It highlights the coupling between the wall used for excitation and
the fluid excited. Such a phenomenon can for example be found in the passenger
compartment of a car. Vortices are shed from the wing-mirrors and cause the vibra-
tion of the driver’s window. The latter interacts with the acoustics properties of the passenger enclosure. Prediction of acoustic resonances in cars was the first testcase for early analytical and numerical methods [13, 14, 15].

Such enclosure resonances are now found in many applications. In jet engines, acoustic resonances have been seen to develop along the main shaft where many closed volumes are located. In those cases, the excitation can come from a vibrating wall but experiment have shown that the energy of excitation could also come from flow instabilities.

2.1.2 Flow induced resonances

In this section, the excitation is not induced by solid vibrations but by flow instabilities, generating a periodic cycle of pressure compression and rarefaction. Such phenomenon can be observed for example on flow over cavities, in rocket boosters, around a plate in a wind tunnel and in more complex systems such as the compressor stages of a jet engine.

2.1.2.1 Flow over a cavity

The first set of applications focuses on acoustic resonances in flow cavities. The flow over a cavity sheds vortices periodically at the cavity’s upwind edge. The resulting vortices then impact on the downwind edge. Smoke visualizations, presented in Fig.2.1a, have clearly captured the phenomenon. The interactions is recaptured in Fig.2.1b. If the frequency of the vortex shedding is close to the acoustic frequencies, high pressure oscillations will then develop [16, 18, 17]. But the resonances can be avoided if care is taken on the design of the edges of those cavities [19, 20].
(a) Flow visualization photographs of vortex formation at the mouths of tandem side-branches [16]

(b) Principal elements of self-sustained oscillations in flow over a cavity [17]

**Figure 2.1.** Vortex shedding excitation at the leading edge of cavities.
The acoustic modes excited depend essentially on the ratio between the length $L$ and opening $D$ of the cavity:

- Shallow cavities, for which the ratio $L$ over $D$ is less than one, will mostly develop acoustic modes in the direction of $D$. These are known as the Rossiter modes [21, 22]. Understanding such a phenomenon is critical for example when designing weapons bays. Testcases have highlighted the dramatic degradation of the downwind edge under the effect of constant vortices impact.

- Deep cavities, like sidebranches, for which the ratio $L$ over $D$ is greater than one, will mostly have acoustic modes developing in the direction of $L$ [23]. This type of geometry is the most prone to resonances and is often observed in piping systems. It can lead to complex resonant patterns if for example two side-branches are coupled together in a system as descibed in Fig.2.2a [24, 25]. Such an application, representative of the roll-posts system presented in Fig.2.2b, has been thoroughly studied to highlight the lock-in phenomenon existing between the vortex shedding and the acoustic resonances [26].

Shed vortices are also responsible for exciting acoustic resonances of large systems. The most striking example is the rocket propellant boosters [27, 28, 29]. Vortices are shed from the cavities of the wall appearing once the fuel has been burned, and propagate until impacting the outlet nozzle of the engine. This periodic excitation, in the form of a pressure perturbation in the vicinity of the outlet nozzle, is then able to excite the fundamental modes of the booster [30].
(a) Different systems of coupled sidebranches with associated fundamental modes [24]

(b) Study of a modelled roll-posts system [26].

Figure 2.2. Sidebranches systems.
2.1. EXPERIMENTAL ACOUSTICS

2.1.2.2 The lock-in phenomenon

Cavity resonances are created by the periodic impinging of vortices, shedded from the front edge of the cavity, onto the back edge of the cavity. But interactions between the acoustic field and vortex shedding can also exist with no impinging of the separated flow. The phenomenon is known as the lock-in phenomenon.

The lock-in phenomenon consists of the interaction of the flow and the acoustic field around a body. Indeed, if a body is immersed into a flow, vortices will be shed at the trailing edge of the system, at a specific frequency. This frequency is determined by the Strouhal number:

\[
St = \frac{fL}{u}
\]  

(2.1)

where \( L \) is the characteristic length of the system and \( u \) the characteristic velocity of the fluid. If the shedding frequency is close to the excitation frequency of one of the modes, the vortex shedding process and the acoustic field will be coupled. The result is recapitulated in Fig.2.3a. By lining up the shedding frequency with the acoustic frequency, the acoustic field will strengthen and harmonise the vortex shedding along the span of the body. The vortex shedding will then provide the energy necessary for the acoustic mode to develop. Such a phenomenon has been observed in many examples such as systems of bluff bodies or cascades of plates.

Acoustic modes have first been seen to develop in the vicinity of a cylinder and then of a tandem of cylinders [34, 31]. These two papers show how the acoustic field stabilises the vortex shedding along the full span of the system. Thick plates, with different leading and trailing edge were also studied [35, 36]. But the most striking example remains the one involving thin slender plates.
Figure 2.3. Lock-in phenomenon around flat plates.
The lock-in phenomenon was observed in a cascade of unstaggered flat plates, whose system is represented in Fig.2.3b [32]. The resulting acoustic modes are known as Parker modes, in reference to the first researcher who identified them [37]. Parker observed four distinct modes known as $\alpha$, $\beta$, $\gamma$ and $\delta$. Since then, plates in tandem have been studied [38] and to better understand the lock-in mechanism, the phenomenon has been decomposed in two steps. First, vortex shedding excites the acoustic modes [39]. Then, the acoustic modes will control the vortex shedding frequency [40, 41]. Fig.2.3c clearly shows the uniformizing impact of the acoustic field on the vortex shedding [33].

**2.1.2.3 Acoustic resonances in industrial systems**

Far from being confined to laboratory experiments, such acoustic resonances are now observed in full-scale industrial applications [42, 43]. Heat exchanger geometries were considered by studying arrays of circular cylinders [44, 45, 46, 47, 48, 49, 50]. Compressor-like systems were also studied, with first the study of an annular passage with cantilevered static flat blades [51] and then a single stage axial flow compressor [52, 53, 54, 55]. The latest research has then focused on full compressor geometries: the acoustics of a 35 MW turbo-compressor and a four-stage high speed axial compressor have been investigated [56, 57]. In such geometries, it is likely that either the vibration of a solid, i.e. blade or casing, or the flow instabilities such as vortex shedding are the source of acoustic resonances.

**2.1.3 Thermo-acoustic resonances**

The last example of acoustic resonances to be introduced is known as the thermo-acoustic instability. It consists in an interaction between the combustion instabilities and the pressure field leading to high pressure oscillations [2]. It follows the
principle of the Rijke tube [58], detailed in Section 1.3.4 and is explained by the Rayleigh criterion stating that acoustic resonances will develop if the heat release rate and acoustic pressure fluctuations are in phase. The Rijke tube is a genuine experiment, but now consider an environment as complex as a jet engine in which, the development of such resonances could lead to catastrophic consequences. Two parts of the engine are particularly concerned by these thermo-acoustic instabilities: the combustion chamber [59, 60, 61, 62, 58] and the afterburner [63, 64], where low frequency axial modes, also known as rumble modes, as well as high frequency radial and circumferential modes, known as screech modes, have been observed.

2.2 Analytical acoustics

Analytical methods followed the steps of the experimental studies in order to better understand and predict acoustic resonances. They rely on mathematical models. We will first present these mathematical models before briefly explaining the current analytical methods and findings.

2.2.1 The equations of sound

Modelling the physical world requires the formulation of a set of equations. We are here mostly interested in the development of acoustic resonances in a fluid. The flow variables will therefore be defined by the Navier-Stokes equations [65]. They are highly nonlinear and, even with current knowledge, solving them requires weeks of computations. Nevertheless, the specific properties of the sound field, as seen in Section 1.2.1, will simplify those equations, from the complex non-linear Navier-Stokes equations to the simple wave equation. The following section is summed up in Table 2.1 and a more detailed explanation on the assumptions made is presented.
First, the fact that the propagation of an acoustic wave is an inviscid and adiabatic process simplifies the Navier-Stokes equations into the Euler equations:

\[
\begin{align*}
\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) &= 0 \quad (2.2a) \\
\partial_t (\rho \mathbf{v}) + \nabla \cdot (p \mathbf{I} + \rho \mathbf{v} \times \mathbf{v}) &= \mathbf{f} \quad (2.2b) \\
\partial_t (\rho E) + \nabla \cdot (\rho \mathbf{v} H) &= \mathbf{f} \cdot \mathbf{v} \quad (2.2c)
\end{align*}
\]

The variables \( \rho, \mathbf{v}, E \text{ and } H \) are respectively the density, the velocity, the total energy and total enthalpy of the fluid, while \( \mathbf{f} \) denotes the external forces applied. Equations (2.2a), (2.2b) and (2.2c) express the conservation of respectively the mass, the momentum and the energy of the fluid. These equations will allow modelling of flows in complex geometries.

Earlier studies focused on simpler enclosed volumes of fluid. For such systems, it is possible to further simplify the Euler equations using the following assumptions:

- No external forces are applied on the fluid \( \mathbf{f} = 0 \). This implies that the fluid is isentropic.
- The acoustic perturbations are very small compared to the mean variable, which implies both that the convective terms of the Euler equations \( \rho \mathbf{v} \times \mathbf{v} \) are negligible and that the equations can be linearised. As a result, all variables can be define as the sum of mean and a perturbation component, such that: \( p = p_o + p' \).
- The fluid is stagnant \( \mathbf{v}_o = 0 \). 

<table>
<thead>
<tr>
<th>System of equations</th>
<th>Equations</th>
<th>Assumptions</th>
</tr>
</thead>
</table>
| Full Navier-Stokes Equations | \[
\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) &= 0 \\
\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \times \mathbf{v} - \bar{\sigma}) &= \mathbf{f} \quad \text{with} \quad \bar{\sigma} = \bar{\tau} - p \bar{l} \\
\frac{\partial \rho E}{\partial t} + \nabla \cdot (\rho \mathbf{v} H - \bar{\tau} \cdot \mathbf{v}) &= -\nabla \cdot \mathbf{q} + \mathbf{f} \cdot \mathbf{v}
\end{align*}
\] | • Newtonian Fluid  
\( \tau_{ij} = \eta \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) - \frac{2}{3} \eta \left( \frac{\partial v_k}{\partial x_k} \right) \delta_{ij} \)  
• Thermodynamic equilibrium  
\( \mathbf{q} = -\lambda \nabla T \) |
| Euler’s equations           | \[
\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) &= 0 \\
\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (p \bar{l} + \rho \mathbf{v} \times \mathbf{v}) &= \mathbf{f} \\
\frac{\partial (\rho E)}{\partial t} + \nabla \cdot (\rho \mathbf{v} H) &= \mathbf{f} \cdot \mathbf{v}
\end{align*}
\] | • Inviscid  
• Adiabatic  
• No external forces |
| Wave equation               | \[
\begin{align*}
\nabla^2 p' - \frac{1}{c^2} \frac{\partial^2 p'}{\partial t^2} &= 0 \\
\nabla^2 \mathbf{v}' - \frac{1}{c^2} \frac{\partial^2 \mathbf{v}'}{\partial t^2} &= 0
\end{align*}
\] | • Irrotational  
• Stagnant fluid  
• Linear acoustic field |
| Helmholtz equation          | \[
\nabla^2 p' + \frac{\omega^2}{c^2} \frac{\partial^2 p'}{\partial t^2} = 0
\] | • Harmonic definition |
This leads to the establishment of the known acoustic wave in pressure and velocity already presented in Section 1.3:

\[
\nabla^2 p' - \frac{1}{c^2} \frac{\partial^2 p'}{\partial t^2} = 0 \quad (2.3a)
\]

\[
\nabla^2 v' - \frac{1}{c^2} \frac{\partial^2 v'}{\partial t^2} = 0 \quad (2.3b)
\]

Finally, if we assume an harmonic behaviour of the variables, the pressure is expressed as \( p'(x, t) = p'(x)e^{j\omega t}, \) the wave equation becomes the well known Helmholtz equation:

\[
\nabla^2 p' + \frac{\omega^2}{c_o^2} \frac{\partial^2 p'}{\partial t^2} = 0 \quad (2.4)
\]

In order to study thermo-acoustic instabilities it is possible to linearise the Navier-Stokes for reacting flows. We can then define in the same way as above the wave equation for reacting flows [66]:

\[
\rho_o c \nabla \left( \frac{1}{\rho_o} \nabla p' \right) - \frac{\partial^2 p'}{\partial t^2} = -(\gamma - 1) \frac{\partial \omega'T}{\partial t} \quad (2.5)
\]

where \( \omega'T \) is the local unsteady heat release. The right hand side of the equation corresponds to the Rayleigh criterion defined when looking at experimental studies.

Depending on the system studied, various models with different levels of complexity and inherent assumptions will be adopted. Earlier studies were focusing on the wave equation. But methods using the Euler equations have been recently developed. How those equations are used to described the world of acoustics is what we are looking at in the next section.
2.2.2 A brief overview of analytical methods

Analytical methods have the advantage of being exact as well as giving a good insight on the propagation and development of acoustic resonances. But their scope is often limited to specific systems. Therefore they will not be presented in this section in detail. The interested reader should have a look at detailed reviews [67, 68].

2.2.2.1 Acoustic resonances in enclosures

In earlier times, the principal aim of analytical methods was to determine the fluid-structure interaction of a cavity with no mean flow and compressible inviscid fluid. The problem was defined in the frequency domain and was therefore using the Helmholtz equation (2.4) for either the acoustic pressure or the velocity potential.

A very efficient and widely used method is the method of separation of variables [67]. The system equation is broken up into a set of ordinary differential equations, as seen in Section 1.2.1, each involving just one independent variable. The method does not have the universality of the integral methods as described later, and can therefore be applied to a limited amount of cases. It is nevertheless interesting to see that some analytical methods have been developed following this path, giving satisfactory results. Acoustic resonances in heat exchanger tubes [45], and the coupling between a vibrating panel and closed rectangular cavity [69, 70, 71, 72, 73, 73] were, for example, modelled accurately.

Another early group of methods uses a variational approach consisting in minimising, or maximising, a quantity described in its integral form. They are therefore known as integral methods [67], and not only allowed the study of more complex
2.2. ANALYTICAL ACOUSTICS

geometries [74, 75], but also layed the foundations of the first numerical methods [76, 13, 14].

2.2.2.2 Acoustic resonances in ducts, the cut-on/cut-off effect

An important step was reached when analytical methods managed to model flow-induced acoustic resonances. Those modes are generally observed in all ducted or laterally periodic arrays and are called trapped modes. They develop around obstacles and do not have any damping. They are associated with the cut-on/cut-off effect presented in Section 1.3.3 for a simple rectangular waveguide, and can be represented in the mathematical domain as point eigenvalues of the differential operator. The concept of trapped modes was originally introduced in the 1950s for water waves [77, 78], but can also be applied to acoustics.

Analytically, the propagation of waves can easily be predicted in simple constant-section ducts as presented in Section 1.3.3. Determining the evolution of the cut-off frequency [10] in a duct of varying section, such as an engine intake, is nevertheless more complicated. In a hard-walled duct, varying sections could induce a point where the mode switches from a cut-on to a cut-off behaviour. This interface creates reflective waves which could lead to the development of acoustic resonances. In the design of rotors and stators, imposing the first acoustic waves as cut-off would allow the reduction of the sound output of the engine. But it may also generate trapped modes inside a section of the duct possibly leading to acoustic resonances and instabilities.

The method often used to model such phenomena is known as the method of multiple scales [79] and consists in expressing the flow variables using the curvature of the pipe. This method has been recently validated against finite element com-
putations on a jet engine intake [80] and was applied on first an axisymmetric duct without mean flow [81], with mean irrotational flow [82], mean swirling flow [83]. Then followed studies on non-axisymmetric ducts [84]. The interface between cut-on and cut-off behaviour, also known as a turning point, has received particular attention both without [81] and with flow [85]. Those turning points, acting as acoustic reflectors, will play a key role in the development of acoustic resonances in specific sections of the jet engine. Those modes can either be three dimensional, trapped in the vicinity of an obstacle, or two dimensional, localized near the wall surface of the duct. The former, first observed by Parker around a flat plate in a tunnel [32], has been known for a long time and will be detailed in the following paragraphs. The latter has just started to be studied [86, 87, 88, 89].

2.2.2.3 Trapped and embedded modes

For specific systems, several methods have been used to predict acoustic trapped modes. A numerical relaxation method [37], a variational approach [90] and a mode matching method [91] have been able to predict the Parker modes with accuracy. As a reminder, Parker modes are trapped modes in the vicinity of a cascade of flat plates. The first inclusion of mean flow [92], as well as the introduction of a third dimension [93], had to wait for the use of the Wiener-Hopf technique [67]. Edging closer and closer to a real jet engine geometry, a variational formulation [94] and a matched eigenfunction expansion method [95], were used to calculate the frequencies of acoustic resonances in a circular cylindrical waveguide, in the presence of thin radial fins of finite length. The Wiener-Hopf technique also allowed the modeling with accuracy of the trapped modes for a two-bladerow system, with a rotational and non rotational rotor, as well as unstaggered and staggered blades [96, 97], while the method of multiple scales modelled acoustic modes in a jet engine intakes with swirling flows [79]. The studies are nevertheless not limited
2.2. ANALYTICAL ACOUSTICS

to plates and trapped modes for a thick plate [98], a small cylinder [99] and a submerged vertical cylinder [100] were also accurately predicted.

To study more general shapes, it is necessary to consider a more general mathematical approach: the study of the spectrum of the operator. To better understand the principles behind it, a simple example is needed. A two dimensional waveguide is defined by the surface $0 \leq y \leq d$ and $-\infty \leq x \leq +\infty$. The acoustic pressure distribution is described by the Helmholtz operator:

$$\nabla^2 p_{0} + k^2 p_{0} = 0$$

(2.6)

Two possibilities of boundaries will be studied.

• If the rigid boundary condition is imposed on both walls, i.e. the $y$-componant of $p$ is equal to zero, the solutions to (2.6) are under the form of:

$$e^{\pm k_{n}x} \cos \left[ n\pi (d - y)/d \right]$$

(2.7)

where:

$$k_{n} = \sqrt{n^2 \pi^2 / d^2 - k^2}$$

(2.8)

$n \in \mathbb{N}$. Therefore, for any eigenvalue $k \in [0, +\infty[$ there exists a wave propagating in the waveguide. It is then possible to represent the propagating modes as the continuous spectrum for the Helmholtz solver.

With rigid walls, the continuous spectrum is the semi-interval $[0, +\infty[$.

This is the classical case of a rigid wall as presented in Section 1.3.3.

• If the rigid wall is imposed for $y = d$, and a soft boundary such as $p' = 0$, for $y = 0$, the solutions to (2.6) take the form of:

$$e^{\pm k_{n}x} \sin \left[ (n - 1/2)\pi y/d \right]$$

(2.9)

where:

$$k_{n} = \sqrt{(n - 1/2)^2 \pi^2 / d^2 - k^2}$$

(2.10)
Such boundary conditions correspond to the study of the acoustics of a waveguide of height $2d$ restricted to the antisymmetric waves. In such a system, the pressure waves can only propagate if, and only if, $k$ is superior to the cut-off frequency $\pi/(2d)$. The continuous spectrum for such a system is now represented by the semi-interval $[\pi/(2d), +\infty]$.

Trapped modes, as identified in previous paragraphs, correspond to point eigenvalues of the Helmholtz operator. These trapped modes are embedded in the continuous spectrum. The continuous spectrum is defined as modes able to propagate independently of their frequency. From a mathematical point of view, eigenvalues embedded in a continuous spectrum are hard to identify and require special care, hence the use of symmetry properties to identify modes under the cut-off frequency of the antisymmetric modes. Using this symmetry argument, it has been proved that if any symmetric general shaped obstacle is present within the waveguide, there exists at least one mode of oscillation, antisymmetric about the centreline [101, 102]. The extension to general bluff bodies was soon established as trapped modes were proved to exist in their vicinity [99, 103, 104].

If now the frequency of the trapped mode is above the first cut-off frequency of the waveguide, or if the obstacle is not symmetric with respect to the waveguide axis, the acoustic modes are more difficult to isolate as the point eigenvalue mode is included in the continuous propagating spectrum of the operator. These modes are known in the literature as “the embedded trapped acoustic modes”. An embedded trapped mode is defined as a point eigenvalue contained in the continuous spectrum. The first example is given when considering a plate off-centred that shows a trapped mode existing embedded in the continuous spectrum [105]. Indeed, trapped modes
exist as soon as a thin obstacle placed in a waveguide has its normal everywhere perpendicular to the generators of the cylinder [106, 107]. Modes were found not only near an obstacle in a channel but also near indentations in the channel wall for modes above the first cut-off frequency [108, 102, 109, 110].

In the end, analytical methods give a valuable insight on the cause and development of acoustic modes. They allow the identification of turning points, the possibilities of acoustic reflection and the section where acoustic resonances are likely to develop. Used efficiently, they can quickly give an idea of the acoustic map of a specific system. Nevertheless, they are often limited to two dimensional cases and often rely on simplifying assumptions. With the advent of computers, CFD methods have been increasingly used for more and more complex systems and flows.

2.3 Computational acoustics

The advent of computers saw the development of a new family of predicting methods known as numerical methods. These methods rely on discretizing the system by a set of nodes on which the modeling equations are solved. The first computational methods focused on structural systems but they were soon extended to fluid dynamics. The methods, depending on how the system is discretized, are known as finite element, finite difference or finite volume. In the literature, two families of methods, designed specifically for the prediction of acoustic resonances have been identified. They correspond to what we will call modal analysis methods and eigenvalue methods.
2.3.1 Modal analysis methods

Because acoustic resonances are fluid phenomena, they can be modelled by the Navier-Stokes equations. Numerical studies have therefore been carried out to model what was observed in experiments (Fig.2.4). The systems studied include the Rijke tube [111] and a lean premixed system [112].

In such studies, a Navier Stokes solver, such as a Reynolds averaged solver, is used to study the interactions between the acoustic characteristics and either the heat release rate [111, 112]. The system is excited by an initial perturbation or relies on self-excitation, under the form of flow or combustion instabilities. The computation is then left to run until the monitored increase in pressure oscillations reaches a limit cycle.

Such studies show that it is possible to excite the system and model its response using the latest CFD tools. If we now excite the system over a wide range of frequencies, we will thus be able to draw its acoustic map. Such a method, mimicking the modal analysis of structural systems, has first been developed by Chassaing et al. [114, 113].

2.3.2 Eigenvalue methods

This section presents a short review of computational acoustic resonances prediction methods using an eigenvalue extraction method. The earlier studies focused on predicting acoustic resonances in enclosures such as cars and concert halls. It is only recently that applications with flow and combustion have been considered due to increasing computational power available and the refining of the acoustic methods.
2.3. COMPUTATIONAL ACOUSTICS

Figure 2.4. CFD Studies of acoustic resonances in different systems

(a) Modeling of the Rijke tube [111]

(b) Modeling of a model combustor [112]

(c) Modeling of a convergent-divergent nozzle [113]
2.3.2.1 **Eigenanalysis of acoustic resonance in enclosures**

Methods for the prediction of acoustic resonances in enclosures have been developed more than a decade ago now. Applications are usually limited to reducing noise levels in car passenger cabins or improving the sound quality in concert halls. The primary concern in those applications is to study the interaction between the structures and the acoustic properties of the enclosures. The first analytical studies considered a parallelepiped with a membrane, as presented in Fig. 2.5a [69, 115, 70]. Such examples, combined with the first definition of the variational principle, also known as the method of weighted residuals [74, 75], paved the way for numerical studies. Because the computations involved structural dynamics, early numerical methods used a finite element approach [117] in order to form the matrix of the system and solve the resulting general eigenvalue problem [76, 13, 118, 14, 119, 15]. An example of possible application is given in Fig. 2.5b, which shows the fundamental modes of the car’s cabin without and with seats. They made full use of the banded, symmetric and frequency independent character of the finite element matrix [68] to use linear algebra methods such as Gauss elimination, QR iterations or Lanczos algorithms.

The necessity to study more and more complex geometries pushed for the development of a quicker and more memory efficient method. The boundary element approach [120] tried to solve the problem by restricting the mesh to the boundaries. The early limitations of the method, predominantly regarding the nonsymmetric, fully populated and frequency dependent system matrix [68], were soon resolved by a set of new methods [121] under the names of the Dual Reciprocity Method [122, 123, 124] and the Particular Integral Method [125, 126, 116]. The latter have been able to model the acoustic mode of the car’s interior presented in Fig. 2.5c.
2.3. COMPUTATIONAL ACOUSTICS

(a) Example of preliminary studies of the interactions between a panel and the fluid in an enclosure [115]

(b) Example of finite element studies for the acoustics of a car’s passengers compartment [15]

(c) Example of boundary element studies for the acoustics of a car’s passengers compartment [116]

Figure 2.5. The problem of structures and acoustics interactions: computational studies.
The boundary element and the finite element methods focused on the Helmholtz equations and proved to give very accurate results for arbitrarily shaped enclosures. Nevertheless, if flow is considered, the Helmholtz equation is not enough to describe the development of acoustic resonances. A method, better adapted to fluid dynamics, has to be considered. This promoted the development of resonance prediction with finite volume CFD solvers.

2.3.2.2 Eigenanalysis of acoustic resonances in flow/combustion applications

The earliest attempt to consider flow applications concerned the compressor stages of a jet engines. Parker [54] solved the wave equation and altered the speed of sound in order to represent the impact of the blades on the wave propagation. In order to model more complex applications, with possibly flow and combustion, Navier-Stokes linearized methods have emerged in a number of papers. The steady state is first calculated using a Reynold averaged Navier-Stokes or LES solver. The acoustic approximation then superimposes the acoustic field onto the mean state. The acoustic field is represented by the classical wave equation [127, 128, 129, 130], the wave equation for reactive gas [66, 131, 132] or the linearized Euler equations [130, 133]. The problem can be solved in the frequency [127, 128, 129, 130, 131, 132] or time domain [133], using an eigenvalue extraction method known as the Arnoldi method. The latter will be explained in more detail in Chapter 4.

Nowadays, thermo-acoustic instabilities are the most sensitive acoustic phenomenon in jet-engines. It is therefore no surprise that simplified nozzles [130], afterburners [130, 133] and combustion chambers [134, 129, 130, 66, 131, 132] have been the subject of many research papers. But the method has also been applied to pure flow application such as high lift systems [127] and pipe systems [128].
2.3. COMPUTATIONAL ACOUSTICS

(a) Study of acoustic resonances in a compressor [54]

(b) Study of acoustic resonances in an afterburner [133]

(c) Study of acoustic resonances in a combustion chamber [131]

(d) Study of acoustic resonances in high lift systems [127]

Figure 2.6. Acoustic resonances and flow applications
These methods are able to model complex acoustic modes with accuracy but the fact that they differentiate between the steady state and the acoustic field means that interactions between the two will not be modelled. This could be an important criterion in the future developments of these methods.

2.4 Conclusion

Acoustic resonances have been found to develop in many systems, from simple enclosures to the most complex systems. Modes are sometimes better known by their discoverer's name as if they were strange phenomena. We might recall the Parker modes for acoustic modes around plates in cascades or Rossiter modes defining acoustic instabilities of sub-cavities. Experiments have also shown that acoustic resonances can be excited by either vibration of the structure, flow instabilities or combustion patterns. They seem universal and, if not controlled, could lead to rapid system failures.
In a jet engine, all the different kinds of acoustic resonances, may develop (Fig. 2.7). Enclosure resonances may occur in the numerous cavities along the drive-shaft. Flow induced resonances may happen in the compressor stages and cut-on/cut-off modes are likely to develop in the engine intake or core. Finally, thermo-acoustic resonances have been observed in either the combustion chamber or the afterburner. The jet engine therefore presents, with the added interest of being a real commercial application, a comprehensive and complex set of characteristics ideal to test any acoustic predicting methods.

To model acoustic resonances, a flurry of analytical methods have first been created. The state of the art of these methods are now able, for example, to predict cut-on and cut-off properties of an arbitrary shaped duct such as jet intakes. Their developments brings a clearer view on how acoustic systems develop. This view is now completed by the growing number of computational methods. The early CFD methods considered enclosures such as the passenger’s compartment of a car, but the need for a predicting acoustic tools drives the development of more and more sophisticated alternatives. First using FEM and BEM models, the latest methods now consider finite volume solvers in order to model complex flow with turbulence and combustion.

In the end, an efficient computational method will need to meet three clear objectives. It has to be:

- applicable to any geometry.
- able to study a system with complex flows and possibly combustion.
- fast, accurate and reliable.

Experiments allow a better understanding of the physical phenomenon. Analytical methods translate the physical world into the mathematical domain and give
a valuable insight on the fundamental characteristics of acoustic resonances. But the sheer range of applications with possibly complex flows and combustion increases the need for computational predicting methods. In the rest of the thesis, two computational methods will be presented as two solutions answering the required objectives.
Chapter 3

The averaged response function method

The objective of the thesis is to implement a CFD method able to predict acoustic resonances in core volumes. This chapter presents a first solution to the problem. The method is called the “averaged response function method”, or ARF method, and is directly derived from the modal analysis theory first developed in the 1950s. Modal analysis consists in extracting the mathematical model of structures from their measured response to an excitation. It relies on the fact that the response of the system is composed of its specific structural modes. The aim of this chapter is to study if this structural dynamics theory can be applied to flows in core geometries by using the latest development in computational fluid dynamics.

The combination of time-domain computations and frequency-domain post-processing will prove to give an accurate acoustic map of the system studied. But before we
jump to conclusions, the implementation of the method is first detailed and the method is validated against theory on a simple closed geometry.

3.1 The ARF method implementation

The averaged response function method applies to CFD what has been widely used in structural dynamics over the last decades. It draws on the work done by Chassaing et al. [114, 113]. The process is summed up in Fig.3.1.

First, the physical principle behind the method is illustrated in Fig.3.1a. In structural dynamics, if a system is excited, for example a beam hit by a hammer, the excitation causes the beam to oscillate around its original state. That response is defined by the intrinsic characteristics of the system. In this thesis, we will see that the same phenomenon occurs in fluid applications. The fluid, after excitation, will revert to its original state with a behaviour defined by the system’s acoustic modes. The idea is therefore to reproduce the experiment in the numerical domain using CFD solvers. The system considered can then be represented by Fig.3.1b, where the controlled excitation is defined using boundary conditions, the system is represented by a mesh and the behaviour of the system is defined by the Navier-Stokes equations. The method implemented will be presented step by step in this section, from the equations and solver, to the initialization and the two-step post-processing.

3.1.1 The equations, the solver and the system

Acoustic resonances are an inviscid fluid phenomenon and their properties will be ruled by the Euler equations. In order to have optimal performance, the solver used is the one developed by the Imperial College London VUTC [135]. It consists of an
3.1. THE ARF METHOD IMPLEMENTATION

Figure 3.1. Implementation of the ARF method
Figure 3.1. Implementation of the ARF method
Figure 3.1. Implementation of the ARF method
Figure 3.1. Implementation of the ARF method
unstructured Favre-averaged Euler solver. The resulting equations are discretized using a node centered finite volume scheme relying on representing the mesh using an edge based data structure. A dual time-stepping technique, ensuring time and space accuracy by using inner Jacobi and outer Newton iterations, is applied to a point implicit formulation [135]. The interest here is that the mesh is unstructured, allowing a better representation of complex geometries. It was also first designed in order to conduct forced response and flutter analyses thanks to the implementation of mesh deformation. The latter property will be heavily used here.

In order to present the implementation of the ARF method, an example is often more illustrative than lengthy explanations. A simple geometry is therefore considered throughout the chapter. The system considered is a closed circular cylinder of length $l_x = 0.693\, m$ and radius $a = 0.0191\, m$. The acoustic modes for such a geometry are defined by the equation (see Section 1.3):

$$\rho' (x, \theta, r, t)_{n_x, n_t, n_r} = \cos(k_{n_x} x) \cos(n_t \theta) J_{n_t} \left( \frac{\pi q_{n_t, n_r}}{a} \right)$$ (3.1)

The corresponding acoustic frequencies are:

$$\omega_{n_x, n_t, n_r} = \sqrt{\left( \frac{\pi n_x}{l_x} \right)^2 + \left( \frac{\pi q_{n_t, n_r}}{a} \right)^2}$$ (3.2)

The frequencies of the first acoustic modes are presented in Table 3.1. Due to the fact that the diameter of the cylinder is much smaller than its length, the lowest frequency acoustic modes are the axial modes. The first circumferential mode has a frequency higher than 5 000 Hz.

Once the equations, solver and system have been defined (Fig 3.1b), it is necessary to set up the computation in order to reproduce the structural dynamics experiments as faithfully as possible.
<table>
<thead>
<tr>
<th>n_x</th>
<th>n_t</th>
<th>n_r</th>
<th>Frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>245</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>490</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>736</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>981</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>1226</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>1472</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>0</td>
<td>1717</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>0</td>
<td>1963</td>
</tr>
<tr>
<td>9</td>
<td>0</td>
<td>0</td>
<td>2208</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>0</td>
<td>2453</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>21</td>
<td>0</td>
<td>0</td>
<td>5152</td>
</tr>
<tr>
<td>20</td>
<td>1</td>
<td>0</td>
<td>5218</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>5224</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>5241</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
<td>5269</td>
</tr>
</tbody>
</table>

Table 3.1. Theoretical frequencies of the lowest frequency acoustic modes for a closed circular cylinder of length 0.693m and radius 0.0191m at atmospheric conditions, \( c = 340 m/s \).

3.1.2 The set up of the computation

Setting up the ARF method takes three distinctive steps that have been identified in Fig.3.1b. First, we must define the excitation. Then, we must consider the way the response is captured through the location of pressure transducers. Finally, we must define the solver’s input parameters taking into account the modal analysis requirements.

The excitation requirements are twofold. Decisions have to be made on where and how the flow will be excited:

- The excitation has to be as smooth and representative of reality as possible in order for the runs to be accurate and stable. Because of viscosity, and contrary to structures where excitation can be defined at one point, exciting a fluid will require a much broader area. The idea is then to make one of the system’s walls to vibrate. In our example, the end-wall of the closed cylinder makes an ideal candidate. It will be modelled
3.1. THE ARF METHOD IMPLEMENTATION

(a) Membrane’s deflection

(b) Time histories of the excitation in terms of membrane’s displacement, velocity and acceleration

(c) Geometry’s mesh, 12300 nodes
(d) Transducers mesh, 2900 nodes

Figure 3.2. Initialization of the method
as a membrane, vibrating around its initial state with a modeshape as presented in Fig.3.2a and defined by [9, p.211]:

\[ x(r, \theta) = \cos(m\theta)J_m \left( \frac{2\pi q_{m,n}r}{a} \right) \]  \hspace{1cm} (3.3)

\((r, \theta)\) are cylindrical coordinates and \(J_m\) the Bessel function of first order.

- Now that the excitation has been defined spatially, we need to define the excitation in time. A white noise excitation, as used by Chassaing [114], has the advantage of exciting a large range of acoustic modes. But this is at the price of loosing control over the excitation. It has been observed that the solver better behaves, in terms of stability, if a smooth excitation is used. That is why a chirp excitation has been preferred. In the later, the displacement of the membrane is defined by:

\[ x(t) = x_{max} \sin(2\pi f(t)t) \]  \hspace{1cm} (3.4)

The maximum amplitude \(x_{max}\) of the displacement stays constant while the frequency \(f(t)\) varies linearly in time. Both the maximum amplitude and frequency range are defined by the operator. This ensures a much better control of the run in terms of accuracy and reliability as well as a much smoother way to excite the fluid. Profiles of the maximum displacement, velocity and acceleration of the membrane are presented in Fig.3.2b.

Once the excitation is defined, we need to make sure that the system’s response is captured accurately. The response of the system corresponds to pressure oscillations and therefore, virtual pressure transducers are located throughout the geometry. Using the complete CFD mesh would mean that too much and probably
3.1. THE ARF METHOD IMPLEMENTATION

Figure 3.3. Parallel edges on primitive elements. Parallel edges share no vertices, but connect the same faces. If a set of parallel edges is collapsed, the element disappear [1].

redundant information would be stored and analysed. Therefore a submesh of transducers has to be created.

As presented above, the CFD solver uses unstructured meshes. The latter makes the creation of a submesh more difficult than with structured meshes as the location of its nodes are not known in advance. The method used is called the collapse of edges and consists in reducing the number of nodes by combining neighbouring edges [1]. The smallest element is made to collapse by successively collapsing its shortest edges and creating a node at its midpoint. In order to ensure that a volume is collapsed smoothly, parallel edges are collapsed together as represented in Fig.3.3, and the number of collapse per cell is limited. The resulting meshes, for respectively the closed cylinder system and the corresponding set of transducers, are presented in Fig.3.2c and Fig.3.2d.

Final considerations on the way the excitation and response will be analysed during post-processing imposes conditions on the computation’s timestep $dt$ and number of iterations $N$. Indeed, the post-processing will rely on frequency-domain tools
such as the Fast Fourier Transform. In modal analysis, the FFT resolution $df$ is directly linked to the time of the experiment, here represented by the virtual time $Ndt$. According to theory, $df$ is such that:

$$
df = \frac{1}{Ndt} \quad (3.5)
$$

In other words, the response resolution will be sharper if $Ndt$ is large. Due to CFD considerations, the timestep $dt$ needs to stay within the solver’s stability conditions. As a result, the number of iterations $N$ will have to be large. This is the first glimpse at the compromise that will have to be made between fast computations and refined and accurate results.

All input parameters have now been defined, and the computations are ready to start. On one hand, the system is excited by controlling the displacement of a membrane and transducers are placed in the system to capture the pressure response. Solver’s parameters are also defined so that post-processing analysis will give the best possible results. The post-processing will actually be carried out in two steps. First, each transducer will be considered individually. Then the system will be considered as a whole entity.

### 3.1.3 The preliminary post-processing

#### 3.1.3.1 The individual response functions and Bode plots

Now that the system has been excited and its response captured, a first stage of processing will be conducted in order to better represent the behaviour of the fluid at each transducer. As presented in Fig.3.1c, the excitation, under the form of the maximum displacement $x(t)$, and the pressure history at the transducer (i) $p_i(t)$ are analysed in the frequency domain. Post-processing tools are based on the well
3.1. THE ARF METHOD IMPLEMENTATION

Figure 3.4. The first stage of post-processing
Figure 3.4. The first stage of post-processing
known Fast Fourier Transform (FFT). The FFT calculates from the time histories the excitation and response spectra, respectively defined as $X(f)$ and $P_i(f)$. The use of the Fast Fourier Transform imposes two conditions on the number of iterations $N$ and the time step $dt$ of the computational run:

- First the maximum frequency of interest has to be less than the Nyquist frequency defined as:

$$f_{\text{Nyquist}} = \frac{1}{2dt} \quad (3.6)$$

This implies that to capture phenomenon of frequency $1000\text{Hz}$, the time step $dt$ has to be less than $5 \times 10^{-4}\text{s}$.

- Then in order to get accurate results, a refined spectrum is needed. That is to say that the frequency interval of the spectrum $df$ has to be small. Because $dt$ is limited by the Nyquist frequency, the number of iterations $N$ has to be large. Indeed, theory states that:

$$df = \frac{1}{Ndt} \quad (3.7)$$

Having taken this into consideration, the frequency response function for the transducer (i) $H_i(f)$ is defined as:

$$H_i(f) = \frac{P_i(f)}{X(f)} \quad (3.8)$$

Throughout the thesis, we will use a more accurate definition, relying on the auto-correlation and cross-correlations coefficients, and using the auto-spectral $G_{xx}$ and cross-spectral $G_{xp_i}$ functions [136]:

$$H_i(f) = \frac{G_{xp_i}(f)}{G_{xx}(f)} \quad (3.9)$$
The frequency response function can be represented in different ways. The Bode plot, as presented in Fig.3.4a, will be the preferred representation in this thesis. It represents the modulus, or gain, and the phase of $H_i(f)$ as a function of the frequency. For a specific transducer, it allows highlighting resonances defined as a gain peak and a corresponding 180 degrees phase shift. Fig.3.4a clearly shows four resonances, each followed by four anti-resonances. It can be seen that the lower frequencies of the response function are polluted. We believe it could be either due to numerical errors creeping in with small excitation and response levels, or be the consequence of a short sampling time resulting in a poor resolution of low frequencies. In this study, the sampling time is equal to 2.62s which results in the lowest acquirable frequency equal to 0.38Hz. We therefore think the noise preferably will come from numerical errors.

3.1.3.2 The leakage error

The leakage error results from the fact that the Fast Fourier Transform assumes that the signal is periodic, that is, that if the run is taking a time $T$, then it will be reproduced every $nT$. If the signal does not go to zero at $T$, a discontinuity is introduced. The latter is equivalent to high frequency noise. There are three ways to correct these discrepancies. The corresponding impacts on the excitation spectrum are presented in Fig.3.4b.

- The excitation itself can be altered in order to incorporate a varying amplitude or, as we will call it, an “envelope”. In order to study the efficiency of such a method, a quadratic envelope has been implemented. It is defined by:

$$x(t) = x_o \frac{4t(T_s - t)}{T^2} \sin(2\pi f(t)t)$$  \hspace{1cm} (3.10)

- The time histories of the response can also be altered after capture. The treatment, in the time domain, corresponds to applying a correcting
window on the data. This is known as tapering. The corrective window we will be considering is the Hanning window defined as:

$$x_{taper}(t) = x(t) \cdot (1 - \cos(\pi \frac{t}{T_s})^2)$$

(3.11)

This often corrects the leakage errors but it is worth bearing in mind that by directly altering the amplitude of the signal, it will effectively reduce the frequency resolution from $$df = 1/T_s$$ to $$df = 4/T_s$$.

- Because of the alteration of the studied signal when tappering, it is sometimes preferred to alter the signal in the frequency-domain rather than in the time-domain. The technique is called frequency averaging and consists in averaging the spectrum of the signal of frequency interval $$df$$ over a coarser $$p df$$ interval. It allows the capture of rapidly changing spectra while damping leakage [136], but it also requires longer computations as the resolution of the spectrum will be decreased.

Fig.3.4b shows that the best representation of the excitation spectrum is the frequency averaged spectrum. Indeed, it represents spectrum discontinuities more accurately but this is at the cost of longer runs. We see that again, a compromise will have to be made between accuracy and speed. In this thesis, depending on the system studied, we will therefore either use the time-domain tapering or the frequency-domain averaging.

In the end, the first stage of post-processing focuses on individual transducers and creates a collection of frequency response functions. The resulting system is represented in Fig.3.1d. It is called a single input (the excitation taken as the maximum displacement of the membrane) multiple output (the pressure history captured at each transducer) system or SIMO. Examples of the corresponding set of response functions can be seen in Fig.3.4c. The frequency response function, here represented by its gain, clearly evolves depending on the location of the transducer with
respect to the excitation location. If the latter is placed at a node of the mode the response function will show no peak while if it is placed at an antinode, the peak will have a maximum amplitude. Now, a second stage of post-treatment is needed in order to extract, from this set of response functions, the general acoustic characteristics of the system.

3.1.4 The final post-processing

This section will answer the final questions on how to capture the acoustic characteristics of the whole system. It will introduce the reader to the concept of averaged response function, multivariate mode indicator function, a spatial interpretation of the spectrum and the line-fit method.

3.1.4.1 The averaged response function and multivariate mode indicator function

The first tool used is the averaging of the set of response functions obtained from the family of transducers above. The averaging will cancel the phase of the system but the gain of the averaged response function will indicate the modes with their specific frequencies. If the transducer’s mesh contains $N_{transducers}$ nodes, capturing the family of response functions $(H_i)_{i \in [1,N_{transducers}]}$, the gain of the averaged response function is defined as:

$$|H(f)| = \sum_{i=1}^{N_{transducers}} \frac{|H_i(f)|}{N_{transducers}}$$  \hspace{1cm} (3.12)

The resulting averaged response function, represented in Fig.3.5a, has a general gain level equivalent to the individual response functions. It also shows similar resonant frequency and damping. For the closed cylinder system, the averaged response function is a clean spectrum identifying the four acoustic modes predicted by theory in the 1 to 1000 Hz frequency window (see Table.3.1).
3.1. THE ARF METHOD IMPLEMENTATION

Figure 3.5. The post-processing tools of the ARF method
Another representation, adapted to single input multi output systems, has also been considered. It is known as the multivariate mode indicator function or MMIF [137, p.302] and is defined by:

$$MMIF(f) = \frac{\sum_{i=1}^{N_{\text{transducers}}} Re(H_i(f))^2}{\sum_{i=1}^{N_{\text{transducers}}}|H_i(f)|^2}$$  (3.13)

This function is comprised within the [0, 1] interval. Modes are indicated by peaks close to 0, and, as presented in Fig.3.5b, the first four acoustic modes are clearly identified. The MMIF is therefore consistent with the averaged response function and will be used as a confirmation tool.

### 3.1.4.2 The acoustic node map

Now that acoustic frequencies have been identified, the corresponding modeshapes will be considered. At each identified resonant frequency, the pressure spectrum is plotted at each transducer and the geometry of the system is rebuilt. The example given in Fig.3.5c shows that modeshapes are in this way clearly defined by representing the nodes (in green), where no pressure oscillation is observed, and anti-nodes (in red), where maximum pressure oscillations are captured, of the standing wave. The method does not represent the modeshape directly and we will therefore refer to this representation in this thesis as the acoustic node maps of the system.

### 3.1.4.3 The line-fit method

The final step aims to determine with more accuracy the frequency and damping of each mode. To that purpose, the widely used method, known as the line-fit method [137, p.309-325], has been implemented. In the latter, each acoustic peak is isolated so that the system can be considered equivalent to a single degree of
Figure 3.6. Line-fit method for the calculation of frequency and damping with accuracy
freedom (SDOF) system. In such a system, the response function for each acoustic mode \( r \) can be represented as:

\[
H_r(\omega) = \frac{a_r + ib_r}{\omega_r^2 - \omega^2 + i\eta_r\omega_r^2}
\]  

(3.14)

where \( a_r \) and \( b_r \) are the coefficients of the individual response function, and \( \eta_r \) and \( \omega_r \) are respectively the structural damping and resonant frequency of the mode. The complete response function will simply be the sum over the number of modes of the SDOF response functions.

In order to estimate for each mode: \( a_r, b_r, \omega_r \) and \( \eta_r \), the method consists in using the linear behaviour of \( \text{Re} \left( \frac{\omega^2}{H(\omega)} \right) \) and \( \text{Im} \left( \frac{\omega^2}{H(\omega)} \right) \) against the square of the frequency. The process is represented in Fig.3.6. The two plots on top represent the real and imaginary part of the intermediate function defined by:

\[
\Delta_\Omega(\omega) = \omega^2 - \Omega^2 
\]  

(3.15)

Where \( \Omega \) is a frequency chosen in the vicinity of each acoustic peak. The line-fit analysis then have three steps.

- As presented in the top plots of Fig.3.6, the functions \( \text{Re} \left( \Delta_\Omega(\omega) \right) \) and \( \text{Im} \left( \Delta_\Omega(\omega) \right) \), where \( \Omega \) is fixed within the vicinity of the mode, creates two families of linear functions against \( \omega^2 \). New variables can be defined \( m_r, c_r, m_i \) and \( c_i \) such that:

\[
\text{Re} \left( \Delta_\Omega \right)(\omega) = m_r(\Omega)\omega^2 + c_r(\Omega) \]  

(3.16)

\[
\text{Im} \left( \Delta_\Omega \right)(\omega) = m_i(\Omega)\omega^2 + c_i(\Omega) \]  

(3.17)

- As presented in the bottom plots of Fig.3.6, the slopes of the functions defined by (3.16) and (3.17), are linear against \( \Omega^2 \). New variables can be defined:

\[
m_r(\Omega) = n_r\Omega^2 + d_r
\]  

(3.18)
\[ m_i(\Omega) = n_i \Omega^2 + d_i \quad (3.19) \]

- The two ratios \( p = \frac{n_i}{n_r} \) and \( q = \frac{d_i}{d_r} \) are then used to deduce the characteristics of each modes:

\[ \eta_r = \frac{q - p}{1 + pq}, \quad \omega_r^2 = \frac{d_r}{(pn_r - 1)n_r}, \quad (3.20) \]
\[ a_r = \frac{\omega_r^2(pn_r - 1)}{(1 + p^2)d_r}, \quad b_r = -a_r p \]

In order to get the overall characteristic of the system, the line-fit study is carried out for each transducer and the mode characteristics are then averaged. In order to verify the accuracy of the mathematical model, and along with a standard deviation analysis, it is possible to compare the initial averaged response function to what is often called the regenerated response function. The latter is defined as:

\[ H_{\text{regenerated}}(\omega) = \sum_{r=1}^{m_{\text{modes}}} \frac{a_r + ib_r}{\omega_r^2 - \omega^2 + i\eta_r\omega_r^2} \quad (3.21) \]

Fig.3.5d shows that there is a good agreement between the calculated averaged response function and the regenerated response function.

The complete set of tools implemented constitutes an efficient method to predict the three properties of acoustic resonances: frequency, damping and modeshape. In the end, the whole method can largely be regarded as a black box (Fig.3.1e). The input parameters are defining the excitation and the geometry of interest. And, with marginal input from the operator, the post-processing tools will extract, from the computational run, the set of acoustic characteristics of the system.
3.2 Validation on a closed circular cylinder

If a method is to be successful, it has to be reliable. This will be true if the accuracy of the ARF results, in terms of frequency, damping and modeshape, are not dependent on the input parameters. In this study, the input parameters in question have been combined in three subgroups named as: the solver, the excitation and the mesh related parameters. The study is done on the same explanatory closed cylinder presented earlier. The corresponding results will be used here as a reference for our study.

3.2.1 The computation of reference

The geometry studied is a closed circular cylinder. Acoustic waves are made to propagate in a fluid at atmospheric conditions. For the steady state, the mean density and static pressure are respectively equal to 1.226 $kg.m^{-3}$ and 101300 $Pa$. No flow is present in the geometry. The input parameters for the reference computation are:

- The number of iterations $N = 262144$. The requirements are that $N$ has to be of the form $2^p$ for Fast Fourier Transform calculation. The value of 262 144 iterations has been chosen as it allows a fast computation and does not require too much memory space. But it is large enough to obtain a refined response function.
- The time-step $dt = 1.10^{-5}s$. The timestep should be chosen so that the solver’s stability is ensured and that the maximum excitation frequency is accurately modelled. For this benchmark, $dt = 1.10^{-5}s$ seemed a good compromise with regard to the stability of the solver, the modelling of the excitation as well as the refinement of the response function.
The maximum amplitude of excitation $x_o = 5.10^{-5}$ m. The maximum amplitude of the excitation is kept constant during the run. The excitation profile is represented in Fig.3.2b. The small amplitude ensures that the pressure oscillations are confined to the linear domain where the amplitude of acoustic pressure wave is less than 1% of the atmospheric pressure.

- The frequency range $f_{range} = [1, 1000]$ Hz. The frequency range is defined to include the first four axial acoustic modes of the geometry studied.

- The membrane’s modeshape corresponds to the end wall fundamental mode of vibration. It is defined by Bessel functionas and is represented in Fig.3.2a.

- The geometry’s mesh includes $N_{mesh} = 12300$ nodes. This corresponds to a mesh refined enough to prevent excessive numerical damping and dispersive error for the frequency range chosen (see Fig.3.2c).

- The transducers mesh is made of $N_{transducers} = 2900$ transducers. This ensures that the first four acoustic modes will be correctly captured (see Fig.3.2d).

The results of the run are presented in Fig.3.5. They show four clean peaks, on both the averaged response function (Fig.3.5a) and the MMIF (Fig.3.5b), corresponding to the first four axial acoustic modes of the closed cylinder. The four peaks highlight the excitation of the first four acoustic modes whose acoustic node maps are represented in Fig.3.5c. Finally, the line-fit method gives an accurate estimate of the frequency and the damping, as recapitulated in (Table.3.2).
Table 3.2. Frequency and damping obtained thanks to the line-fit method compared with theory for the computation of reference.

With this set of parameters, the error in frequency is contained within 5% which is acceptable, considering that the run only took 10 hours on 16 processors. The damping ratio estimates are also kept small as one would expect for an enclosed geometry. The regenerated response function presented in Fig.3.5d, in combination with the standard deviation values in Table.3.2, show that the damping and frequency values are accurate.

3.2.2 Impact of the solver

The first set of parameters studied will concern the post-processing quality and refinement, i.e. whether the Fourier transform will be refined enough to capture the different modes. Simple Fourier transform theory tells us that the frequency step of the Fourier transform is such that:

\[ df = \frac{1}{N dt} \]  

(3.22)

where \( N \) is the number of data points and \( dt \) is the time interval between two successive points. From a CFD point of view, it corresponds respectively to the number of iterations and the timestep of the solver.
3.2. VALIDATION ON A CLOSED CIRCULAR CYLINDER

Figure 3.7. Study of the influence of the number of iterations on the ARF method accuracy
Table 3.3. Frequency and damping obtained thanks to the line-fit compared with theory for $N = 1048576$.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Frequency (Hz)</th>
<th>Standard deviation</th>
<th>Theoretical frequency (Hz)</th>
<th>Error (%)</th>
<th>Damping ratio (%)</th>
<th>Standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>235.67</td>
<td>3.56 $10^{-2}$</td>
<td>245.31</td>
<td>3.92</td>
<td>2.83 $10^{-2}$</td>
<td>2.79 $10^{-4}$</td>
</tr>
<tr>
<td>2</td>
<td>471.36</td>
<td>1.47 $10^{-2}$</td>
<td>490.62</td>
<td>3.92</td>
<td>8.39 $10^{-2}$</td>
<td>5.17 $10^{-5}$</td>
</tr>
<tr>
<td>3</td>
<td>705.50</td>
<td>6.44 $10^{-3}$</td>
<td>735.93</td>
<td>4.13</td>
<td>1.83 $10^{-1}$</td>
<td>2.42 $10^{-5}$</td>
</tr>
<tr>
<td>4</td>
<td>938.48</td>
<td>7.67 $10^{-2}$</td>
<td>981.24</td>
<td>4.35</td>
<td>3.08 $10^{-1}$</td>
<td>1.35 $10^{-4}$</td>
</tr>
</tbody>
</table>

3.2.2.1 The number of iterations $N$

The number of iterations, provided the time-step is fixed, has an important impact on the results (Fig.3.7). In addition to the $N = 262144$ reference run studied above, results for $N = 16384$, $N = 65536$ and $N = 1048576$ are presented. A first look at the averaged response functions (Fig.3.7a) and the MMIFs (Fig.3.7b) shows that $N$ will influence the look of the curves by directly reducing or improving their refinements. Keeping the refinement issue in mind, the frequencies and the modeshapes are nevertheless well predicted, and this even for $N = 16384$ (Fig.3.7c). The line-fit method requires a fairly well refined response function though and therefore will not be suited to $N = 16384$ and $N = 65536$ runs. To conduct a clean analysis, the number of iterations has to be superior to 262144. The impact of increasing the number of iterations on the standard deviation (Table.3.3) implies that the line-fit method becomes more and more accurate.

In the end, the operator has to strike a compromise between a fast or a refined analysis (Fig.3.7d). $N = 16384$ will be computed in less than 1h on 16CPUs but some information will be lost, specifically at higher frequencies. In contrast, $N = 1048576$ will take more than 40h on 16CPUs, but all the modes will be clearly identified and frequency and damping will be accurately predicted.
3.2. VALIDATION ON A CLOSED CIRCULAR CYLINDER

3.2.2.2 The timestep $dt$

The time step is another critical parameter. The Bode plot (Fig. 3.8a) and the MMIF (Fig. 3.8b) show how much the frequency and damping of the modes are dependent on it. A small timestep is preferred as each period of oscillation will be represented by a larger number of points. If the maximum frequency of interest is 1000 Hz, a timestep of $5 \times 10^{-5}$ s will only represent the shortest cycle with 20 points. This is often not enough in CFD applications. $dt = 1.0 \times 10^{-5}$ s looks like a good compromise, but for perfectly accurate results, in complete agreement with theory, only time-steps of around $1.0 \times 10^{-6}$ s seem to pass the mark (Table 3.4). Indeed, for $dt = 1.0 \times 10^{-6}$ s, the error in frequency is then less than 0.5% and is constant across the frequency range studied. Such a time-step describes the highest frequencies of
Table 3.4. Frequency and damping obtained thanks to the line-fit method compared with theory for $dt = 10^{-6} s$

excitation with 1000 points, which explains less damping and dispersive errors.

Hence, while the number of iterations $N$, influences only the refinement of the response functions and MMIF, the time-step plays a more crucial role. As you would expect, it directly impacts on the performance of the solver and of the averaged response function method, in terms of frequency and damping prediction.

3.2.3 Impact of the excitation

Now that the solver parameters have been studied, it is time to consider the excitation. Let’s look at the excitation’s amplitude and frequency range.

3.2.3.1 The amplitude of excitation

The time histories of the acoustic pressure Fig.3.9a, from a transducer located at two thirds on the axis of the cylinder, show that, provided the computation is kept within the linear domain in terms of pressure waves, the increase in the excitation amplitude results in a proportional increase in the amplitude of the pressure oscillations.

The Bode plot (Fig.3.9b) and the MMIF (Fig.3.9c) confirm the linear behaviour
Figure 3.9. Study of the influence of the amplitude of the excitation on the ARF method accuracy
of the system as the response function level does not change with the excitation amplitude except for the higher frequencies, too close to the end of the excitation frequency window to be accurate. It is also shown that the application of an envelope on the excitation does not provide any improvement on the results. In the end, provided the study is done in the linear domain of the system, the amplitude of the excitation will have no impact on the analysis.

3.2.3.2 The frequency range of the excitation

In order to study the behaviour of the method regarding the excitation frequency window, we will extend the frequency range to 2000Hz. Fig.3.10a shows that, with a timestep of $1.10^{-5}$s, over damping and frequency lag creeps in for higher frequency modes. This could have been predicted by the earlier study of the timestep. For the highest frequencies, each cycle of oscillations is now only represented by 50 points. Nevertheless, Fig.3.10a also shows that both runs are exactly equivalent within the 1 to 1000Hz window.

In order to get more accurate results for higher frequencies, it is necessary to reduce the timestep. A time step of $1.10^{-6}$s is considered. This allows the capture of a much better looking response functions for frequencies close to 2000Hz (Fig.3.10b and Fig.3.10c). The first eight acoustic modes are now predicted with accuracy in frequency and damping (Table.3.5) as well as in modesphahes (Fig.3.10d to Fig.3.10k).

In the end, the timestep will have an important role on the range of frequencies that can be studied. The time steps should allow to represent the highest frequency modes with at least 200 points. That is why the chirp excitation has been chosen earlier. There is no need to use white noise excitation as it would make the com-
3.2. VALIDATION ON A CLOSED CIRCULAR CYLINDER

(a) Gain of the averaged response function with a $1 \times 10^{-5}$ s timestep and comparison on the frequency range

(b) Gain of the averaged response function with comparison on different timestep

(c) Multivariate mode indicator function with a $1 \times 10^{-6}$ s timestep

(d) 245 Hz
(e) 488 Hz
(f) 732 Hz
(g) 974 Hz

(h) 1218 Hz
(i) 1458 Hz
(j) 1699 Hz
(k) 1936 Hz

Figure 3.10. Study of the influence of the frequency range on the ARF method accuracy
Table 3.5. Frequency and damping obtained thanks to the line-fit method compared with theory for a frequency range between 1 and 2000 Hz.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Frequency (Hz)</th>
<th>Standard deviation</th>
<th>Theoretical frequency (Hz)</th>
<th>Error (%)</th>
<th>Damping ratio (%)</th>
<th>Standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>245.51</td>
<td>7.34.10^{-1}</td>
<td>245.31</td>
<td>0.08</td>
<td>3.41.10^{-1}</td>
<td>4.89.10^{-2}</td>
</tr>
<tr>
<td>2</td>
<td>488.00</td>
<td>3.76.10^{-1}</td>
<td>490.62</td>
<td>0.53</td>
<td>1.97.10^{-1}</td>
<td>1.26.10^{-2}</td>
</tr>
<tr>
<td>3</td>
<td>732.47</td>
<td>3.55.10^{-1}</td>
<td>735.93</td>
<td>0.47</td>
<td>2.47.10^{-1}</td>
<td>1.08.10^{-2}</td>
</tr>
<tr>
<td>4</td>
<td>974.07</td>
<td>2.69.10^0</td>
<td>981.24</td>
<td>0.73</td>
<td>6.93.10^{-1}</td>
<td>7.78.10^{-2}</td>
</tr>
<tr>
<td>5</td>
<td>1218.44</td>
<td>1.51.10^{-1}</td>
<td>1226.55</td>
<td>0.66</td>
<td>6.94.10^{-2}</td>
<td>1.43.10^{-3}</td>
</tr>
<tr>
<td>6</td>
<td>1458.62</td>
<td>1.74.10^{-1}</td>
<td>1471.86</td>
<td>0.89</td>
<td>1.32.10^{-1}</td>
<td>6.47.10^{-3}</td>
</tr>
<tr>
<td>7</td>
<td>1699.21</td>
<td>5.97.10^{-1}</td>
<td>1717.17</td>
<td>1.04</td>
<td>8.12.10^{-2}</td>
<td>6.69.10^{-3}</td>
</tr>
<tr>
<td>8</td>
<td>1936.32</td>
<td>2.51.10^{-1}</td>
<td>1962.48</td>
<td>1.33</td>
<td>1.69.10^{-1}</td>
<td>2.09.10^{-3}</td>
</tr>
</tbody>
</table>

Impact of the meshes

The last step of the validation concerns the study of the impact of both the geometry’s and transducers meshes. We will see that provided that they are refined enough to model the acoustic phenomena within the frequency range of the excitation, they will not have a strong impact on the quality of the results.

3.2.4.1 The mesh of the geometry

In CFD, the mesh refinement should have consequences on the accuracy of results. And this study is no exception. But the impact seems actually pretty small. Three meshes have been used, containing \( N_{\text{mesh}} = 289 \), \( N_{\text{mesh}} = 1729 \) and \( N_{\text{mesh}} = 12300 \) (Fig.3.11a). The Bode plot (Fig.3.11b) and MMIF (Fig.3.11c) show that the refinement of the mesh does not greatly impact on the results as the four acoustic modes are predicted in frequency, damping and even modeshape (Fig.3.11d).

Looking at the higher frequencies, we can nevertheless see the impact of the mesh.
3.2. VALIDATION ON A CLOSED CIRCULAR CYLINDER

(a) Three different meshes with, from left to right, $N_{\text{mesh}} = 289$, $N_{\text{mesh}} = 1729$ and $N_{\text{mesh}} = 123010$ nodes

(b) Gain of the averaged response function

(c) Multivariate mode indicator function

(d) Fourth acoustic mode with, from left to right, $N_{\text{mesh}} = 289$, $N_{\text{mesh}} = 1729$ and $N_{\text{mesh}} = 12301$ nodes

Figure 3.11. Study of the influence of the refinement of the geometry’s mesh on the ARF method accuracy
refinement creeping in. Indeed numerical damping can be observed for the fourth acoustic mode on the coarser mesh. This is explained by the fact that the mesh does not have enough nodes to represent the acoustic wavelengths. This problem is seen in many CFD solvers, and it is generally accepted that the mesh should contain about 40 nodes per wavelength. Depending on the frequency range studied, the user will have to make sure that his mesh is refined enough to model the modes of interest.

3.2.4.2 The mesh of transducers

Similarly, the transducers mesh does not have any impact on the results. Three different transducers meshes have been used (Fig.3.12a). The Bode plot (Fig.3.12b) and MMIF (Fig.3.12c) show perfectly identical results. This is true of course as soon as the transducers mesh is refined enough to capture the acoustic mode of interest.

3.3 Conclusion

The objective of the thesis is to create a fast and reliable method able to predict acoustic resonances. By associating classical structural dynamics tools with the latest development in CFD, a combined time and frequency-domain method has been developed.

The fluid inside a system is excited by the vibration of one of the walls using a chirp excitation profile. The excitation frequency is made to vary linearly in order to excite successive acoustic resonances. The resulting pressure perturbations are captured thanks to an array of virtual transducers. This data is then analysed us-
3.3. CONCLUSION

(a) Geometry’s mesh with $N_{\text{mesh}} = 12301$ nodes and three transducer’s mesh with respectively $N_{\text{transducers}} = 1480$, $N_{\text{transducers}} = 2907$ and $N_{\text{transducers}} = 7308$ nodes

(b) Gain of the averaged response function

(c) Multivariate mode indicator function

Figure 3.12. Study of the influence of the refinement of the transducers mesh on the ARF method accuracy
ing modal analysis tools such as the Fast Fourier transform in order to extract the acoustic modes of the system.

The method was originally defined in a paper by Chassaing et al. [113]. In the latter paper, only simple geometries were considered. The method has here been expanded in order to quickly study complex systems. To that purpose, a set of transducers is automatically defined given a specific mesh and more refined post-processing tools are used. The latter are able to directly predict frequency and damping values as well as modeshape.

Beyond giving accurate results, the method has three advantages. Firstly, it is easy to implement, provided there is an existing CFD solver able to model the propagation of acoustic waves. Secondly, it can provide very quick results for the fundamental, low frequency modes of a system. Thirdly, the operator has full control on how to strike the best compromise between getting quick results or drawing the complete and very accurate map of acoustic resonances developing in the system.

The method has been seen to be dependent of various input parameters. As shown in Table.3.6, the most important parameters are those linked to the CFD solver, that is, the number of iterations and solver’s time-step. These two will directly impact on both the speed of computation and quality of the results. The other parameters, such as the excitation parameters and the meshes’ refinements, will not impact on the results as soon as the excitation is limited to the linear domain and the mesh contains more than 30 nodes per modes’ wavelengths.

The ARF method is a method mimicking physical experiments. The ease of implementation and analysis are its strongest arguments. Its main drawback is the
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Impact on refinement of the response function</th>
<th>Impact on the speed of computation</th>
<th>Impact on the frequency of the modes</th>
<th>Impact on the damping of the modes</th>
<th>Impact on the modeshape of the modes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of iterations $N$</td>
<td>Increase in $N$ allows the refinement of the response function</td>
<td>Increase in $N$ slows the computation</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Time step $dt$</td>
<td>Increase in $dt$ allows the refinement of the response function</td>
<td>No direct impact, but to ensure accuracy in frequency, a decrease in the time step requires an increase in the number of iterations, slowing down the computation</td>
<td>A timestep too large results in a loss of accuracy in frequency</td>
<td>A timestep too large results in extra damping of the higher frequency modes</td>
<td>-</td>
</tr>
<tr>
<td>Amplitude of excitation</td>
<td>-</td>
<td>No, provided the excitation remains in the linear domain</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Frequency range of excitation</td>
<td>-</td>
<td>-</td>
<td>- impact provided $dt$ is adapted</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Geometry’s mesh</td>
<td>-</td>
<td>-</td>
<td>No impact provided the mesh is refined enough for the modes excited</td>
<td>No impact provided the mesh is refined enough for the modes excited</td>
<td>No impact provided the mesh is refined enough for the modes excited</td>
</tr>
<tr>
<td>Transducers’ mesh</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 3.6. Recapitulative table of the impact of the different parameters on the ARF method accuracy.
restriction on the frequency range. This implies that, to clear a system over a large range of frequencies, several computations, with different frequency windows, will have to be run. We will now consider another approach to the prediction of acoustic resonances which does not have such a problem. It is known as the Arnoldi method.
Chapter 4

The Arnoldi method

In computational methods, a system is modelled with a set of differential equations. It is therefore possible to know the characteristics of a system by only studying the characteristics of the set equations. This relies on studying the spectrum of the equations using eigenvalue extraction methods. The spectrum is essentially a projection of the system onto a more comprehensive basis. In fluid applications, the basis of interest is the one formed by the acoustic modes of the system from which the system’s behaviour can be deduced. Such eigenvalue methods have been widely used in structural dynamics. In this chapter, we will present a possible implementation for fluid applications combining a CFD solver and the eigenvalue extraction method known as the Arnoldi method.

The reader will find in this chapter that the Arnoldi method is a quick, reliable and accurate method. The first part of the chapter presents the Arnoldi algorithm, its benefits and its latest developments. The chapter will then focus on one specific
implementation, developed by Eriksson [133], and kindly given to the author for study. We will then present our implementation of the method, which will be compared to Eriksson’s method before being validated on a closed geometry.

4.1 Theory of the Arnoldi method

Predicting acoustic resonances using CFD computations and eigenvalue extraction methods poses two challenges:

- Solving the Euler equations involves solving a system of five unknowns over a mesh sometimes composed of millions of nodes. Each unknown has to be expressed at each node. The system therefore has, in comparison to structural systems, a large number of degrees of freedom.
- The method of choice for CFD computations is known as the finite volume method. It is preferred to finite element and finite difference schemes as the conservation equations of mass, momentum and energy are implicitly expressed leading to a simpler implementation. But contrary to finite element or finite difference methods, the finite volume scheme does not explicitly define the matrix of the system.

Such characteristics limit the choice of eigenvalue extraction methods [138, 139]. Indeed, the size of the matrix involved excludes all direct methods such as the well known QR Iteration. Iterative methods are then preferred as only a subset of the spectrum, for example, the eigenvalues of the least stable modes, can be extracted. Iterative methods will also allow the expression of the matrix under the form of a matrix-vector multiplication, and the system’s matrix will not have to be explicitly defined. Iterative methods are therefore bound to find applications in
Iterative methods for symmetric matrices, such as the Lanczos method, are now well understood and mature. Sadly, the discretized Euler equations do not give the luxury of symmetry. The development of similar methods for the nonsymmetric case are still under development and a large number of improvements are published every year giving an array of stable, efficient methods. The methods in competition are named the nonsymmetric Lanczos algorithm [138], Davidson’s algorithm [140], the Jacobi-Davidson algorithm [141] and the Arnoldi method [142]. To the writer’s knowledge, the most widely used scheme is the Arnoldi method [142, 138] as it is derived from the Lanczos method and inherits the Lanczos method’s good convergence properties. This section introduces briefly the Arnoldi algorithm with its advantages and limitations. It then presents its latest developments. The interested reader can find a more extensive review in Appendix B.

### 4.1.1 The Arnoldi algorithm

The Arnoldi method is based on the Lanczos algorithm, first developed by Lanczos in 1950 [143, 138]. The aim was to iteratively reduce a symmetric matrix $A$ to a triangular matrix $T$, whose eigenvalues would be much easier to extract. While the first aim of Lanczos was to calculate the complete matrix $T$, it was proved that during the Lanczos iterations, the eigenvalues of the iterative triangular matrix were fast converging to the extremum eigenvalues of $A$ [144, 145]. The algorithm is therefore ideal to compute only a selected part of the spectrum of the matrix $A$.

The Lanczos algorithm was extended to nonsymmetric matrices by Arnoldi [142]. He states that the non-symmetric matrix $A$ can be reduced to an Hessenberg matrix...
H by using a basis of orthonormal vectors \((q_i)_{i \in \mathbb{N}}\) spanning the Krylov subspace \(\mathcal{K}(A, q_1, n) = \text{span}\{q_1, Aq_1, \ldots, A^{n-1}q_1\}\). The Hessenberg matrix, defined as an upper-triangular matrix with an additional subdiagonal, can then be interpreted as the orthonormal projection of \(A\) onto this Krylov subspace. In mathematical terms, there exists a transitional orthonormal matrix \(Q\) such that \([138, \text{pp.499}]\):

\[Q^T A Q = H\]  
(4.1)

If the matrix \(A\) in \(\mathbb{R}^{n \times n}\) is large, sparse and non-symmetric, the family of vectors forming the matrix \(Q\), known as the Arnoldi or Krylov vectors, can be defined iteratively. Indeed, using the Hessenberg definition of the matrix \(H = [h_{i,j}]_{(i,j) \in \mathbb{N}^2}\), it is possible to build the orthonormal matrix \(Q_k = [q_1, \ldots, q_k]\) iteratively, column after column. After \(k\) iterations, the vector \(q_{k+1}\) can be expressed, using (4.1), in terms of its predecessor:

\[h_{k+1,k} q_{k+1} = A.q_k - \sum_{i=1}^{k} h_{i,k} q_i = r_k\]  
(4.2)

where \(r_k\) is known as the residual.

The Hessenberg coefficients can then easily be computed by using (4.2) in combination with the orthonormal properties of the Arnoldi vectors \((q_i)_{i \in \mathbb{N}, k \in \mathbb{N}}\):

\[h_{i,k} = q_i^T A.q_k \quad \text{for } 1 \leq i \leq k\]  
(4.3)

At this stage, an additional step is often carried out in order to ensure that the new \(k+1\) vector is orthogonal to all others. It is known as the Gramm-Schmidt orthogonalization:

\[r_k = r_k - \sum_{i=1}^{k} (r_k^T q_i) .q_i\]  
(4.4)
If the residual is non zero, the new unit-2 norm Arnoldi vector is defined as:

\[ q_{k+1} = \frac{r_k}{h_{k+1,k}} , \quad h_{k+1,k} = \| r_k \|_2 \]  

(4.5)

This simple procedure can be translated into a simple algorithm. It will be used throughout the thesis and is known as the Gramm-Schmidt modified Arnoldi algorithm listed below:

**Algorithm 4.1** The Arnoldi Algorithm with modified Gramm-Schmidt re-orthogonalization

1. \( r_0 = q_1 \)
2. \( h_{1,0} = 1.0 \)
3. \( k = 0 \)
4. **while** \( h_{k+1,k} \neq 0 \) **do**
   
   - **Loop on the number of Arnoldi vectors**
     
     - \( q_{k+1} = \frac{r_k}{h_{k+1,k}} \)
     - \( k = k + 1 \)
     - \( r_k = A.q_k \)

   - **for** \( i = 1 \text{ to } k \) **do**
     
     - Building the Hessenberg matrix
       
       - \( h_{i,k} = q_i^T . r_k \)
       - \( r_k = r_k - h_{i,k}.q_k \)
   
   - **end for**

   - **for** \( i = 1 \text{ to } k \) **do**
     
     - Gramm-Schmidt re-orthogonalization
       
       - \( s_i = q_i^T . r_k \)
       - \( r_k = r_k - s_i.q_i \)

   - **end for**

   - \( h_{k+1,k} = \| r_k \|_2 \)

5. **end while**

In the end, the simple implementation of the Arnoldi scheme, together with the
CFD requirements introduced earlier, explain why the method has gained more and more advocates for predicting acoustic resonances:

- Arnoldi allows the reduction of the eigenvalue extraction to a much smaller size. This is due to the good convergence properties of the algorithm to the outermost eigenvalues of $A$ ([146] and Appendix B).
- Arnoldi does not need the explicit definition of the differential operator. This is ideal if a finite volume method is to be used.
- Arnoldi allows the isolation of part of the spectrum, whose properties will be of interest: largest modulus, largest real part, smallest imaginary part. This is interesting if the study is focused for example on the stability of the differential operator.
- Arnoldi is a fast and cheap process in terms of computation and memory.

### 4.1.2 Arnoldi limitations

In our quest for an effective eigenvalue extraction method to combine with finite volume CFD solvers, there are strong arguments for the Arnoldi method. But, as with any computational methods, it also has some limitations. These limitations are two-fold.

First, the Arnoldi algorithm will converge to eigenvalues located furthest from the rest of the spectrum. Therefore all is well if the matrix $A$ has a well-separated spectrum, but the convergence properties might drop if the matrix has a heavily clustered spectrum.

Second, the Arnoldi algorithm suffers from a problem of orthogonalization of the Arnoldi vectors which could lead to large errors in eigenvalue’s estimates. The
issue, illustrated in the definition of the residual $r_k$ (see (4.2)), is intrinsically linked to the convergence of the method: the better the convergence, the higher is the probability of stalling due to orthogonalization issues.

These limitations can be corrected by a set of methods that will now be presented. They are known as the preconditioning and restarting techniques.

**4.1.3 The latest Arnoldi implementations**

**4.1.3.1 The preconditioning method**

The Arnoldi algorithm first converges to the outermost eigenvalues of the operators spectrum. But often the modes of interest are gathered in the most clustered part of the spectrum. The problem of clustered spectrum can be solved by a technique called preconditioning. Before the Arnoldi iterations, the matrix $A$ is transformed into a matrix $B$ whose spectrum is more adapted to Arnoldi’s convergence properties.

To achieve better convergence, a polynomial function of order $p$, such that $B = p(A)$, is often used. If $A$ is real, the eigenvalues of $B$ are such that $\lambda_B = p(\lambda_A)$ and the eigenvectors of $B$ will be equal to the eigenvectors of $A$. To deduce the eigenvalues of $A$ from the approximated eigenvalues of $B$, it is possible to either solve the polynomial equation, or compute the eigenvalues of $A$ by using the eigenvectors $(x_i)_{i \in \mathbb{N}}$ of $B$. Solving the polynomial equation is tedious for polynomial functions of order larger than 3. Using the eigenvectors requires more computations and might not be the best approach, as we will typically need to compute 100 eigenvectors, and recomputing $A.x_i$ will extend the computational time.

Preconditioning therefore offers a quick remedy to improve Arnoldi’s convergence.
In parallel, convergence can also be improved by considering the quality of the initial Arnoldi vector. This is achieved thanks to the restarting methods.

### 4.1.3.2 The restarting method

Arnoldi’s convergence is largely influenced by the choice of the initial vector. By restarting the algorithm with an improved estimate of the family of modes of interest, it is possible to direct the convergence towards this area of the spectrum.

After $m$ iterations of the Arnoldi algorithm, $r$ eigenvalues are chosen in the spectrum of the Hessenberg matrix according to a predefined criterion: largest real part, smallest modulus... The algorithm then restarts with an updated vector $q_1'$ such that $q_1' = p(A).q_1$ where $p$ is a polynomial function. If the vector $q_1$ is expressed in the basis of the eigenvectors of $A$, $(x_i)_{1 \leq i \leq n}$, such that $q_1 = a_1x_1 + \cdots + a_nx_n$ and if the matrix $A$ is real, the updated starting vector $q_1'$ can be defined as:

$$q_1' = a_1p(\lambda_1)x_1 + \cdots + a_np(\lambda_n)x_n$$  \hspace{1cm} (4.6)$$

where $(\lambda_i)_{1 \leq i \leq n}$ is the family of associated eigenvalues. The Krylov subspace $K(A, q_{1u}, m)$ is enriched in eigenvectors emphasized by $p(\lambda)$. Hence, if $p(\lambda_{\text{wanted}})$ is much larger than $p(\lambda_{\text{unwanted}})$, Arnoldi will converge much faster to the set of desired eigenvalues $(\lambda_i)_{1 \leq i \leq r}$.

Restarting the algorithm serves two purposes. First, a good polynomial filter will tune out the unwanted portions of the spectrum of $A$. If the starting vector $q_1$ is a linear combination of the wanted eigenvectors $(v_i)_{1 \leq i \leq r}$, the residual of the Arnoldi method will converge to zero and the Hessenberg matrix will be the orthogonal projection of $A$ onto $\text{span}(v_1, \cdots, v_r)$. Second, limiting the number of iterations reduces the risk of orthogonalization issues associated with convergence.
Two approaches of restarting methods exist depending on the way the polynomial filter is defined:

1. The explicitly restarted Arnoldi methods
   The most effective explicitly restarting methods use the Chebyshev polynomials [147, 148, 149] as they can define an ellipse in which to enclose the set of unwanted eigenvalues [150, 151]. Such methods use an iterative process looking at the best rate of convergence of the different eigenvalues.

2. The implicitly restarted Arnoldi methods
   The polynomial filter can also be defined implicitly [152, 153, 154]. The new starting vector is defined using QR iterations with shifts. The idea comes from the closeness of Arnoldi and the QR iterations. After m Arnoldi iterations, p QR iterations are done with a set of shifts \((\mu_i)_{1 \leq i \leq p}\). This is equivalent to using a filter polynomial whose zeros are the actual shifts: \(p(\lambda) = \prod_{i=1}^{p} (\lambda - \mu_i)\). Therefore, an interesting choice for the shifts of the QR iterations are the unwanted eigenvalues of the Hessenberg matrix.

4.1.3.3 Conclusion on the Arnoldi theory
As a conclusion, the Arnoldi algorithm seems fully suited for the prediction of acoustic resonances because only a small part of the spectrum can be studied and because the matrix itself does not have to be explicitly determined. Structural dynamics computations could do without it because the matrix is simpler and smaller. While early Arnoldi implementations suffered from convergence limitations, recent implementations, such as preconditioning and restarting, greatly improved the accuracy of the solver, making the Arnoldi method a more and more appealing alternative. That is why, while earlier computational studies, focusing on enclosures
resonances, used direct QR methods, the current state of the art in terms of prediction of acoustic resonances uses the Arnoldi algorithm.

A number of studies, recapitulated in Section 2.3, have been carried out in the last decades, combining Arnoldi with CFD solvers, in order to study the acoustic characteristics of complex systems. We are particularly interested in the work done by Eriksson and his coworkers as it has been proved to provide fast and accurate results on geometries like afterburners. Thanks to his help, we have been able to use and develop his method in order to create a complete acoustic resonance prediction tool. The next section will present Eriksson’s work.

4.2 Eriksson’s method

Eriksson’s method [133] for the prediction of acoustic modes uses a time-domain solver on a structured mesh. His work is a first attempt towards a completely reliable, fast and accurate tool for the prediction of acoustic resonances. We will here present the theory behind Eriksson’s method, give example of results on a simple closed system, study the impact of the input parameters on the method’s results and finally conclude on the achievements and limitations of the method.

4.2.1 Theory

Eriksson’s method proceeds in two steps. First the acoustic propagation of waves is modelled using a CFD solver. Then the Arnoldi algorithm is used to generate the successive initial states known as the Arnoldi vectors. The combination of the two will build the Hessenberg matrix, approximation of the system’s matrix.
4.2.1.1 Eriksson’s CFD solver

In order to model the flow, Eriksson opted for a time-domain approach. Block structured meshes were used to quicken and simplify the development of the solver. This combination allows the modelling of a large range of applications with a fairly simple solver. The acoustic field is expressed using the linearized Euler equations around a mean flow previously calculated by a traditional RANS solver. This linearization is justified if the acoustic approximation, stating that the acoustic field is small compared to the steady state, is verified. The discretization of the Euler equations allows the definition of the problem as a pure stability problem expressed by the matrix $A$:

$$\frac{dU}{dt} = A.U$$ (4.7)

where $U$ defines the vector of unsteady unknowns.

4.2.1.2 Eriksson’s Arnoldi implementation

Following Arnoldi’s theory presented in Section 4.1.1, the Hessenberg matrix $H$, representative of the matrix $A$, is formed thanks to the definition of successive Arnoldi vectors. In order to define $H$, Eriksson uses a Gramm-Schmidt modified Arnoldi method [155]. He thus ensures that the algorithm does not stall due to Arnoldi’s orthogonalization issue. The eigenvalues of the matrix $A$ are approximated by the eigenvalues of $H$. They define the acoustic properties of the system.

The eigenvalues of a stability problem such as (4.7) lie in the left hand side of the complex domain (Fig.4.1a) as the real part of the eigenvalues corresponds to the damping of the mode and must be negative for stable physical modes. The closest the real part is to zero, the less stable is that mode. On one hand, the Arnoldi
algorithm first converges to the eigenvalues that are furthest from each other, and therefore it will focus on the left-hand part of the spectrum. On the other hand, the eigenvalues of greatest interest are located in the most clustered part of the spectrum, near the imaginary axis. Eriksson therefore chooses to use a preconditioning matrix. The preconditioned matrix $B$ is defined from $A$ by using a series of polynomial functions $p_k$ such that:

$$B = p_k(A) = \left( I + \Delta t A + \frac{1}{2}(\Delta t A)^2 + \frac{1}{4}(\Delta t A)^3 \right)^k \quad (4.8)$$

The polynomial function defined above is equivalent to using a three-stage Runge-Kutta scheme for time integration. The one used by Eriksson is known as the scheme of Gary [133]. $\Delta t$ is the time-step. $k$ is the number of iterations.

Because matrix $A$ is real, so is matrix $B$. The eigenvectors of $B$ will be the eigenvectors of $A$ and the eigenvalues of $B$, $(\lambda^B_i)_{i \in J_1,nK}$, are linked to the eigenvalues of $A$, $(\lambda^A_i)_{i \in J_1,nK}$, by:

$$\lambda^B_i = \left( 1 + \Delta t \lambda^A_i + \frac{1}{2}(\Delta t \lambda^A_i)^2 + \frac{1}{4}(\Delta t \lambda^A_i)^3 \right)^k \quad (4.9)$$
4.2. ERIKSSON’S METHOD

We recognize here the first terms of the Taylor expansion of the exponential function. Therefore, if the number of iterations $k$ is large, the eigenvalue $\lambda_i^B$ will be approximated by:

$$\lambda_i^B \approx e^{\lambda_i^A \cdot k \cdot \Delta t}$$  \hspace{1cm} (4.10)

The eigenvalues of $A$ are therefore transformed into a spectrum located in the unit circle of the complex plane (Fig.4.1b). The least stable eigenmodes, of main interest to us, are located close to the unit circle, on the outskirt of the spectrum. The objective of the preconditioning is then fulfilled as the Arnoldi algorithm will first converge towards the least stable modes.

4.2.1.3 Eriksson’s post-processing

Once the eigenvalues of $H$ and therefore $A$ are known, it is possible to extract both the damping and frequency of the acoustic modes by a simple post-processing analysis.

The eigenvalues of matrix $A$ contain information on both the frequency and damping of the acoustic modes. These can respectively be expressed in terms of the angular frequency $\omega_n$ and the damping ratio $\delta_n$. For low damping modes, we have:

$$\lambda^A = \delta_n \omega_n + j \omega_n \sqrt{1 - \delta_n^2} \approx \delta_n \omega_n + j \omega_n$$  \hspace{1cm} (4.11)

The approximate definition of the eigenvalues of $B$ in (4.10) implies two interesting properties. First, the natural frequency of the modes can be deduced from the argument of the eigenvalue. It is define modulo $2\pi$ by the relationship:

$$\arctan(\lambda^B) = \omega_n \cdot k \cdot \Delta t + 2m\pi$$  \hspace{1cm} (4.12)
Extracting the frequency requires that either the frequency is priorly known, or that two computations are done, with a different number of iterations, in order to pinpoint the acoustic frequency from its $2\pi$ equivalents. Second, once the natural frequency is known, the associated damping ratio can be deduced from the radius of the eigenvalue $R_n$, using:

$$R_n = |\lambda^B| = |e^{-\delta_n \omega_n k A t}|$$

(4.13)

### 4.2.2 Application to a closed circular cylinder

#### 4.2.2.1 The system and the theory

Eriksson’s method is now tested on a simple closed circular cylinder of length $l_x = 0.693m$ and radius $a = 0.3465m$. The results are compared with theory. The latter predicts that the acoustic modes shapes of the system are defined by (see Section 1.3):

$$p'(x, \theta, r)_{n_x, n_t, n_r} = \cos(n_t \theta) J_{n_t} \left( \frac{\pi q_{n_t, n_r} r}{a} \right) \cos(k_{n_r} x)$$

(4.14)
4.2. ERIKSSON’S METHOD

With their corresponding acoustic frequencies equal to:

\[ \omega_{n_x, n_t, n_r} = \sqrt{\left(\frac{\pi n_x}{l_x}\right)^2 + \left(\frac{\pi q_{n_x, n_t, n_r}}{a}\right)^2} \]  

(4.15)

In order to accurately represent the system, the cylinder is discretized with a two-block structured hexahedral mesh. The latter, presented in Fig.4.2, contains about 101 400 nodes.

4.2.2.2 The results

The computation is considering \( m = 100 \) Arnoldi vectors with \( N = 1000 \) iterations and time-step \( dt = 1.10^{-6} \) s. The results, presented in Fig.4.3 shows that the method predicts a large number of modes with accuracy in both frequency and damping. The analysis will take three steps: the study of the Hessenberg spectrum, the study of the estimated frequency and damping, and the study of the acoustic modeshapes.

First, the spectrum of the Hessenberg matrix, and therefore the approximated eigenvalues of the preconditionned matrix \( B \), is represented in Fig.4.3a. The figure shows that all modes are contained, as expected, within the unit circle. The spectrum contains 100 eigenvalues corresponding to the size of the Hessenberg matrix and the number of Arnoldi vectors. It is symmetric with respect to the real axis as the spectrum contains both conjugates of the same acoustic modes. A few real modes can be observed corresponding to unphysical modes. The other complex eigenvalues stay close to the unit circle, exhibiting low damping values. This is expected for a closed system with rigid walls.

Second, acoustic damping and frequency estimates can be deduced from the spectrum by using the definitions (4.12) and (4.13). With (4.12), and because the theoretical frequencies of each mode are known, the predicted acoustic frequencies
Figure 4.3. Acoustic prediction of Eriksson’s method on a closed cylinder
can be differentiated from their modulo $2\pi$ equivalents. The resulting estimates are shown in Fig. 4.3b to be in very good agreement with theory. Once the acoustic frequency $\omega_n$ is known, the damping ratio $\delta_n$ is deduced using (4.13). It is also recapitulated in Fig. 4.3b. The damping ratio is seen to slowly increase with frequency. This is to be expected and is a classic behaviour of CFD solvers. The solver is indeed not able to handle higher frequency modes as the mesh does not have enough nodes to represent the corresponding shorter wavelengths. Very good damping estimates are nevertheless achieved for modes with a frequency under 2000Hz.

Finally, the last acoustic characteristic left to study are the acoustic modeshapes. They are deduced from the eigenvectors of the Hessenberg matrix $(y_i)_{i \in J_1, nK}$ using classic algebra. If there exist an orthonormal transformation matrix $Q$ such that:

$$Q^T_m A Q_m = H_m$$  (4.16)

Then, if $A$ is real, the eigenvectors of matrix $A$, $(x_i)_{i \in J_1, nK}$, corresponding to the acoustic modeshapes, are expressed as:

$$x_i = Q y_i \text{ for } 1 \leq i \leq n$$  (4.17)

The extracted acoustic modes, presented in Fig. 4.3c, are in agreement with theory. They describe known acoustic modes. For higher frequency modes, discrepancies can sometimes be observed when the acoustic mode has not completely converged.

In the end, Arnoldi predicted 20 acoustic modes out of 50 possible with accuracy in frequency, damping. The modeshapes extracted also agrees with theory. The results correspond to a ratio of 40% of converged acoustic modes which is usually seen as a good compromise between number of modes predicted and computational
requirements. This ratio is mainly dependent on the input parameters of the method. Their impact will be studied in the next section.

### 4.2.3 Parametric study

After focusing on the results of one run, let us consider the influence of the different input parameters such as the number of iterations $N$, the time-step $dt$, the simulation time $T = Ndt$ and the number of Arnoldi vectors $m$. The impact of the different parameters is studied by looking at both the percentage of the physical modes extracted against the number of Arnoldi vectors and the averaged error in frequency, between the estimated frequency and the analytical frequency expressed in (4.12).

Fig.4.4a and Fig.4.4b show that increasing either the number of iterations or the time-step allow the prediction of more acoustic modes. The percentage of acoustic modes reaches up to about 60% of the Arnoldi vectors, while the averaged error in frequency stays constantly less than 1%. This is expected as by doing so, the simulation time $T = Ndt$ is increased improving the convergence of the modes.

If the simulation time is now kept constant and the ratio between $N$ and $dt$ is changed, the results in percentage of acoustic modes and averaged frequency error stays strikingly constant (Fig.4.4c). This is interesting as it shows that only the simulation time will have an important impact on the results. A compromise has therefore to be struck between a small enough time-step, ensuring stability, and a large number of iterations, increasing the computational time.

Finally, the number of Arnoldi vectors $m$ behaves in a similar way (Fig.4.4d). A period of stabilization in terms of the percentage of acoustic modes and frequency
accuracy is first observed for low values of $m$. Both criteria then level to about respectively 40% and less than 1%, for larger values of $m$.

In conclusion, the method predicts a large number of acoustic modes with accuracy provided that the number of Arnoldi vectors is large enough, about 100, and that the simulation time $T = Ndt$ is long enough to let the modes converge.

4.2.4 Conclusion on Eriksson’s method

Eriksson’s method is able to predict quickly a large set of acoustic modes with accuracy. It is accurate and largely independent of the input parameters. It is therefore a good first step towards creating a fully reliable method, appropriate to the study of complex systems. But its limitations are threefold:

- Determining frequency and damping should not require the knowledge of the acoustic frequencies before hand or carrying out multiple computations. A direct method should be defined to make the method faster and more efficient.

- In Eriksson’s code, there is no possibility to identify a converged genuine acoustic mode from a spurious mode but by looking at the corresponding eigenvector and using common sense. An objective criterion would therefore be a useful addition to the method.

- The method uses block structured meshes. While it ensures rapidity and simplicity of the solver, it also limits the number of applications. An unstructured mesh would be a great addition to the tool as then the acoustic field of any geometry could be studied.

Once those drawbacks are tackled, the improved Arnoldi method will be a good alternative to the ARF method presented in Chapter 3.
Figure 4.4. Parametric study of Eriksson’s method
4.3 Our implementation

By taking into account the advantages of Eriksson’s method and considering its drawbacks, a new solver has been implemented. The improvements made on the method, addressing the three limitations highlighted above, consist in a faster way to extract frequency and damping values from the spectrum, the creation of a convergence criterion and the implementation of an unstructured linearized Euler solver.

4.3.1 Frequency and damping estimates

Eriksson uses the argument of the eigenvalue of the transformation matrix $B$ to calculate the frequency of each modes, but as seen before, this does not provide enough information to have a direct estimate of the frequency. Indeed, the argument is defined modulo $2\pi$. We found that, because the matrix $A$ is real, the real and imaginary part of its eigenvalues can be deduced directly from its eigenvectors. This will give extra information as, according to the stability system defined in (4.7), the imaginary and real part of the eigenvalues of $A$ are respectively linked to the frequency and damping of the acoustic modes.

Consider the eigensystem made of the eigenvalue $\lambda = \lambda_r + j\lambda_i$ and corresponding eigenvector $x = x_r + jx_i$:

$$A.(x_r + jx_i) = (\lambda_r + j\lambda_i).(x_r + jx_i)$$

(4.18)

Because the vector $(1, j)$ is a basis of the complex plane, (4.18) is equivalent to the system:

$$\begin{align*}
Ax_r &= \lambda_r x_r - \lambda_i x_i \\
Ax_i &= \lambda_i x_r + \lambda_r x_i
\end{align*}$$

(4.19)
This leads to:

\[
\begin{align*}
\lambda_r &= \frac{x_r^T A x_r + x_r^T A x_i}{x_r^T x_r + x_i^T x_i} \\
\lambda_i &= \frac{x_i^T A x_i - x_r^T A x_i}{x_r^T x_r + x_i^T x_i}
\end{align*}
\] (4.20)

If you now remember the definition of the eigenvalues of \( A \) (4.11), the imaginary part of the eigenvalue \( \lambda_i \) is equal to the angular frequency of the corresponding acoustic mode \( \omega_n = 2\pi f_n \). The real part of the eigenvalue \( \lambda \) is equal to the product of the damping ratio by the natural frequency. Once the frequency is known, the damping can therefore be calculated. This simple postprocessing tool allows to directly get an estimate of both the frequency and the damping of the modes. It does not require the prior knowledge of acoustic frequencies or extra computation. It is a reliable and fast addition to Eriksson’s implementation.

4.3.2 Convergence criterion

The resulting frequency, \( \omega_n^{\text{vector}} \), computed thanks to the eigenvectors of matrix \( A \), is another estimate of the frequency of the acoustic modes. Combining \( \omega_n^{\text{vector}} \) with the frequency \( \omega_n^{\text{value}} \), calculated from the eigenvalues of the preconditionned matrix \( B \), will tell us how well the eigenvector is related to its eigenvalue. In other terms, the two estimates will give an insight on how well the eigenpair has converged. A convergence criterion has thus been created looking at how close the two frequencies are. The criterion \( c_0 \) is defined as:

\[
c_0 = 100 \left| \frac{\omega_n^{\text{value}} - \omega_n^{\text{vector}}}{\omega_n^{\text{vector}}} \right|
\] (4.21)

It has been observed that on one hand, if disparities exist in the eigenvector and therefore in the acoustic modeshape, the convergence criterion will be more than
10%, on the other hand if a mode is properly converged, \( c_o \) will be small, that is, inferior to 1%. This convergence criterion gives a consistent solution to the second limitation of Eriksson’s implementation expressed earlier. It is an objective indicator of the convergence quality of a specific mode, able to dissociate acoustic modes from spurious modes.

### 4.3.3 Implementation of an unstructured solver

Now that the characteristics of the acoustic modes, i.e. frequency, damping and modeshapes are completely and accurately captured, and that a convergence criterion has been defined, the last drawback of Eriksson’s code concerning the use of structured meshes is tackled. Though large improvements have been made on modelling complex geometries with structured meshes, they often limit the number of applications. In this section, an unstructured linearized Euler solver is implemented.

Eriksson uses a second order Runge-Kutta time integration with a linearized Euler solver using a second order TVD scheme. It is likely that second-order accuracy in time and space will be necessary to fully capture the acoustic phenomenon. In the solver implemented, time integration is done thanks to the classic fourth-order Runge-Kutta scheme [65, p.460]. The space integration is carried out using a first order Roe scheme [156, p.460] or a second order TVD Roe scheme [156, p.551]. The solver’s implementation and validation are detailed in Appendix C.

### 4.3.4 Conclusion on the new implementation

The new Arnoldi solver combined the advantages of Eriksson’s method with three important improvements in response to the three limitations highlighted earlier in
A method was implemented to predict the acoustic frequencies with accuracy. This was done by directly considering the eigenvectors of the matrix $A$ and resulted in the acquisition of additional information on both frequency and damping.

The new frequency estimate is used, in combination with the frequency estimate deduced from the Hessenberg matrix eigenvalues, to create a convergence criterion. This convergence criterion is thus able to dissociate spurious modes from genuine acoustic modes and to give us valuable information on the quality of the convergence.

A second-order linearized TVD Roe scheme has been implemented in order to be able to model more complex geometries as well as ease the method’s use. The unstructured solver proved to give accurate results when modelling the propagation of two acoustic waves in the Riemann problem.

It is believed that with these improvements, the new Arnoldi method will be a fast and accurate tool for the prediction of acoustic resonances. But before studying complex systems, the method has been validated by, first, comparing its results with Eriksson’s method, and then carrying a broad parametric study.

### 4.4 Validation of the new Arnoldi method

The new Euler solver and the Arnoldi implementation are validated against both Eriksson’s method and theory. The new implementation is tested on a simple closed cylinder geometry. The latter consists in a closed circular cylinder of length $l_x = 0.693m$ and radius $a = 0.173m$. The first step of the validation of the method consists in comparing the new Arnoldi implementation with Eriksson’s results. A
4.4. VALIDATION OF THE NEW ARNOLDI METHOD

A thorough parametric study will then follow to better understand how the input parameters will impact the method’s accuracy and efficiency.

On one hand, Eriksson’s Arnoldi implementation uses a block-structured second order TVD solver with a second order Runge-Kutta time integration scheme. On the other hand, the new implementation uses an unstructured second order TVD scheme with a fourth-order Runge-Kutta time integration scheme. With different implementations, it will be interesting to see how the two methods compare. The new implementation’s results are therefore compared to Eriksson’s results presented in Section 4.2.2 using first the same hexahedral mesh and then a fully tetrahedral mesh. This study will show consistency in the results for both methods.

4.4.1 Comparison on a structured hexahedral mesh

The closed cylinder mesh, presented in Fig. 4.5, includes about 104,000 nodes. It is the unstructured equivalent to the structured mesh used for the study using Eriksson’s code. Looking at the spectra of the two systems in Fig. 4.6a, several properties of the new implementation can be observed.
On one hand, the fundamental modes such as the first axial mode (1,0,0) have the same eigenvalue: $\lambda = 0.03 + 0.999j$, for both solvers. The solvers are therefore equivalent for the fundamental, lower frequency modes. On the other hand, the eigenvalues of higher frequency modes show differences. The differences are mainly due to numerical damping errors. Whereas the two spectra are still close to the unit circle, it is seen that some eigenvalues given by the new solver have a radius superior to 1. This implies that they have a non-physical negative damping. This problem in numerics is explained by the use of a TVD scheme with an unstructured solver. The TVD scheme relies on using information at the i-1 and i+2 (see Section 4.3.3). While for structured meshes, the information is readily available, the unstructured solver uses extrapolations by gradients and Laplacians. This can introduce additional error leading to numerical damping discrepancies. The only way the problem can be solved is by refining the mesh so that the gradients are more accurate or refine the solver by considering a higher order alternative.

The acoustic frequencies and damping are then extracted from the converged eigenvalues. They are recapitulated in Fig.4.6b. The latter shows that an equivalent number of modes is found - about 40% of the Arnoldi vectors - and with the same accuracy in frequency - less than 0.3% of averaged error, compared to theory, over the range of acoustic modes. While some modes seems to have negative damping in the new solver, it stays within a $\pm 0.5\%$ band around 0% corresponding to undamped modes. In that sense, it does behave better than Eriksson’s code which reaches high damping for high frequency modes.

The modeshapes are finally computed from the set of converged eigenvectors. They are represented in Fig.4.6c. They are generally better defined than with Eriksson’s method. No discrepancies are observed which means that the modes are well con-
Figure 4.6. Acoustic prediction of the new solver on a closed circular cylinder using hexahedral mesh
verged modes. Also, in comparison to the ARF method presented in the previous, it is interesting to notice that the Arnoldi method is able to capture acoustic modes very close in frequency, such as the 2524Hz and 2525Hz modes, whose modeshapes are very different.

4.4.2 Comparison on an unstructured tetrahedral mesh

Now that the same structured mesh has been considered, lets study the accuracy of the solver on a completely unstructured mesh and compare the results to Eriksson's results. The unstuctured mesh in question is a tetrahedral mesh containing 120,000 nodes and presented in Fig.4.7. The results, presented in Fig.4.8, are analysed in terms of the spectrum, the frequency and damping.

First, the spectrum, contrary to the hexahedral mesh case above, is included in the unit circle as would be expected. It can be seen that Eriksson’s and the new solver give equivalent eigenvalues for the fundamental modes.

The first striking feature when looking at the frequencies of the acoustic modes
in Fig.4.8b, is that the new solver gives more converged physical acoustic modes than Eriksson (52% of Arnoldi vectors against 40%), while maintaining the same level of accuracy in frequency (less than 0.3% of averaged error against theory). Then, while Eriksson’s computation misses some lowest frequency modes such as the second axial mode (2,0,0), the new solver spans the whole range of lowest acoustic modes. It is then unlikely that some modes will be missed out, one of the worry of Eriksson’s code. Finally, damping is increasing steadily with frequency as you would expect from any CFD solvers. The level of damping stays nevertheless equivalent to the structured solver used by Eriksson.

In conclusion, the new solver, using a fourth order Runge-Kutta and a second order TVD Roe scheme, compares well with the solver developed by Eriksson. Optimal performances have been observed for an unstructured tetrahedral mesh, where a maximum number of acoustic modes is found. The new solver now needs to be validated by studying the impact of the different input parameters on the quality of the results.

4.5 Parametric study of the new Arnoldi solver

The run presented above constitutes a reference run on which the validation of the code will be build. It computes, using a tetrahedral mesh of about 120 000 nodes, \( m = 100 \) Arnoldi vectors with \( N = 1000 \) iterations and a time-step \( dt = 1.10^{-6} \)s. The different aspects highlighted in this section are, the relevance of the convergence criterion introduced earlier, the impact of the refinement of the mesh and the impact of the input parameters such as the number of iterations, the time-step and the number of Arnoldi vectors.
Figure 4.8. Acoustic prediction of the new solver on closed circular cylinder using tetrahedral mesh
Figure 4.8. Acoustic prediction of the new solver on a closed circular cylinder using tetrahedral mesh
4.5.1 The convergence criterion

The convergence criterion has been created to relate one eigenvector with its eigenvalue. This should give valuable information on how the eigenpair has converged. It is defined by the error between the frequency calculated with the eigenvector, following the method explained in Section 4.3.2, and the frequency calculated from the phase of the eigenvalue. It is expressed in terms of percentage:

\[ c_o = 100 \frac{|\omega_{value}^n - \omega_{vector}^n|}{\omega_{vector}^n} \] (4.22)

Fig.4.9a shows how the criterion of convergence relates to the acoustic validity of the eigenmode. It shows what has been observed in numerous cases.

- If the convergence criterion \( c_o \) is inferior to 1%, then the eigen-mode corresponds to a genuine acoustic modes.
- If \( c_o \) is superior to 1%, the eigenmode can either be spurious or not yet completely converged.
- If \( c_o \) is superior to 10%, the eigenmode is a spurious mode.

Indeed, looking at Fig.4.9b representing the evolution of the modeshape with regards to the convergence criterion, it is clear that the modeshapes for \( c_o = 3.21 \) and \( c_o = 29.47 \) are not as well defined as the ones with \( c_o = 0.02 \) and \( c_o = 0.86 \). In the case of \( c_o = 29.47 \), the modeshape is here easily recognized even with poor convergence because we are studying a very simple system. For most complex system, it is likely that such a mode could not be identified. The convergence criterion is in the end very important as it allows to capture in a glance how well the computation went, and how good the results are likely to be.

4.5.2 Impact of the mesh refinement

It is now interesting to look at the impact of the mesh refinement on the results. Three tetrahedral meshes, presented in Fig.4.10, are created. They contain respec-
4.5. PARAMETRIC STUDY OF THE NEW ARNOLDI SOLVER

Figure 4.9. Impact of the convergence criterion on the convergence of the acoustic modes.
tively 24 500, 120 000 and 944 000 nodes. For all three meshes, the percentage of converged modes stays equivalent with a percentage equal respectively to 58, 54 and 54%. The accuracy in frequency is also pretty constant with 0.66, 0.3 and 0.3% of error against theory. The main impact observed in Fig.4.10d is on the damping which dramatically decreases when the refinement of the mesh increases. This is again in agreement with second order CFD scheme theory and the way the mesh can handle wave propagation. A more refined mesh allows better extrapolation of variables and therefore gives a better estimate of the damping.

4.5.3 Parametric study

The final step in order to validate the code consists in studying how the solver behaves with regards to its input parameters: the number of iterations, the time-step, the simulation time and the number of Arnoldi vectors. When studying each parameter separately, all other input parameters are maintained constant. Fig.4.11 shows that the code follows the same trend expressed in Fig.4.4 for the Eriksson code.

- Fig.4.11a and Fig.4.11b shows that an increase in the number of iteration $N$ or the time-step $dt$ increases the number of acoustic modes predicted while decreasing the averaged error in frequency. This is logical as it implies increasing the simulation time allowing more modes to converge. Once a certain level has been reached, both the number of converged modes and the error in frequencies are seen to stay constant.

A more detailed look at the frequency and damping of each modes, in respectively Fig.4.12a and Fig.4.12b, also shows that both the number of iterations and time-step do not have a significant impact on the frequencies and damping of the converged modes.

- It seems that the main impact is made by the simulation time $T_s = N.dt$. 
4.5. PARAMETRIC STUDY OF THE NEW ARNOLDI SOLVER

Figure 4.10. Impact of the mesh refinement on the convergence of the acoustic modes
If $N$ and $dt$ are changed while maintaining $T_s$ constant, the results in Fig.4.11c stay levelled both in terms of converged modes and frequency accuracy. Actually, Fig.4.11c clearly implies that the spectra for a constant virtual time stay equivalent.

- The number of Arnoldi vectors computed will finally contribute as by increasing the number of modes, the solver is more likely to converge to proper acoustic modes. It is then possible to reach a level of converged acoustic modes of about 60%.

### 4.6 Conclusion

The Arnoldi method presented in this chapter is a solution for the prediction of acoustic resonances in core volumes. The new implementation follows the work done by Eriksson and his coworkers [133]. The method consists in transforming the Euler equations into a simple stability problem, defined by a matrix $A$. The stability characteristics of the system are then described by the spectrum of $A$, which is calculated using the Arnoldi method. The latter, an iterative method, has been chosen due to three essential properties:

- The iterative process builds a transitional Hessenberg of smaller size which is much easier to analyse. Direct eigenvalue extraction methods, such as the QR method, would indeed not be able to handle the size of CFD problems.

- The Arnoldi algorithm does not need for the matrix $A$ to be explicitly defined. It uses the image of $A$ through its multiplication with a vector. This is essential with finite volume schemes.

- The Arnoldi methods and its latest developments have good convergence properties. The eigenvalues of the transitional matrix will con-
4.6. CONCLUSION

Figure 4.11. New solver’s parameters study
Figure 4.12. Parameters impact on frequency and damping
Eriksson’s code combined the Arnoldi method with a structured solver. The method gave good results on geometries like afterburners. We aimed, in this chapter, to improve the implementation in order to make it more reliable. Three improvements have been made:

- The frequency and damping of the modes are directly computed thanks to the eigenvectors of the system. There is no need to rely on prior knowledge from analytical or experimental studies.
- A convergence criterion has been defined in order to quickly indicate the quality of convergence of the acoustic modes.
- The method has been implemented on an unstructured mesh in order to be able to model resonances in arbitrarily shaped geometries.

The new implementation has then been tested on a simple closed circular cylinder. The results compare well with Eriksson’s and theoretical results. Acoustic frequencies are usually predicted within 0.5%. Acoustic damping, small for low frequency modes, is seen to steadily increase with frequency as would be expected with any CFD solvers. The lowest frequency modes predicted have damping levels as low as Eriksson’s code. Finally, the modeshapes are well converged and clearly agree with theory.

The method is also seen to be very robust with respect to its input parameters. If the latter stay within the stability conditions of the solver, the timestep and number of iterations do not have a great impact if taken individually. As a result, the sim-
ulation time, defined as $Ndt$, is the main parameter able to improve convergence of the modes. The longer the computation is, the better the convergence. Once the virtual time has been defined, the number of Arnoldi vectors will give the number of acoustic modes predicted. It was found that, with a good set of parameters, more than 60% of computed modes corresponded to genuine acoustic modes. How then is the Arnoldi method comparing with the ARF method of Chapter 3? This is what we will find out in the next chapter.
Chapter 5

The ARF method vs. the Arnoldi method

Two acoustic resonance prediction methods, namely the ARF method and the Arnoldi method, have been presented in Chapters 3 and 4. These chapters introduced their implementation and validation on a simple enclosure. We will now focus on the comparison of the two methods. Tested on a simple geometry, the two methods will give insights on their qualities and limitations.

The first part of the chapter presents the geometry considered and the corresponding analytical results which will be used as a reference. In order to compare the two methods, we will also introduce the concept of normalization. The chapter will then present the normalization of the ARF method before looking at the normalization of the Arnoldi method. The final comparison is done in three steps. First, qualitative results are studied using the well-known response function representation.
A quantitative analysis is then undertaken. It is followed by considerations on the methods’ computational requirements.

## 5.1 Presentation of the geometry

The ARF and Arnoldi methods are compared on a simple enclosed geometry. The system has to follow three requirements:

- The system has to be simple to allow the comparison of the computational results with analytical theory.
- The system has to be small in order to allow a quick computational analysis.
- The system should be an enclosed geometry with rigid walls. In theory, such a system has undamped acoustic modes. The study would thus highlight the amount of spurious numerical damping introduced by each solver.

This section will present the geometry considered and its theoretical acoustic characteristics.

### 5.1.1 The geometry and the mesh

The geometry consists of a closed circular cylinder of length $l_x = 0.693\text{m}$ and radius $a = 0.0191\text{m}$ (Fig.5.1a). It was first used in Chapter 3 when validating the ARF method. The system was chosen as it fulfills the three criteria presented above. Furthermore, as the radius is small compared to its length, the system could be considered as one dimensional. This will give us an insight into how well both solvers behave with regards to the mesh refinement, and in particular with regard to the discretization of the acoustic wavelengths.
The CFD solver for the ARF and Arnoldi methods requires unstructured meshes. The geometry is therefore meshed using tetrahedral elements. It is composed of about 12,000 nodes. This is refined enough to make sure that the lowest frequency acoustic modes are accurately captured but coarse enough for quick computations. The mesh is shown in Fig. 5.1b. Note that the scale of the geometry is not respected, as the length of the cylinder is more than thirty times the radius. It is presented like this for clarity.

5.1.2 The analytical solution

Once the system has been meshed, a steady state is defined. The steady state flow solution corresponds to a fluid at rest with an atmospheric pressure $P_o = 101300 Pa$ and a density $\rho_o = 1.226 kg.m^{-3}$. In these conditions, the speed of sound is $c = \sqrt{\frac{P_o}{\rho_o}} = 340 m.s^{-1}$. By solving the cylindrical wave equation, we obtain the system’s acoustic modes. They are defined using the first-order Bessel function
The acoustic modeshapes are defined by the number of nodes in the axial, circumferential and radial directions: \( n_x, n_t, n_r \). In (5.1), \( k_{nx} = \frac{2\pi}{\lambda_{nx}} \) is the axial wave number associated to the wavelength \( \lambda_{nx} = \frac{2L_x}{n_x} \). The factor \( q_{nt, nr} \) are the zeroes of the function \( \frac{dJ_{nt}(r)}{dr} \). The values of \( q_{nt, nr} \) for the first circumferential and radial modes are recapitulated in Table.5.1.

The corresponding acoustic frequencies are defined as:

\[
\omega_{nx, nt, nr} = c \sqrt{\left( \frac{\pi n_x}{L_x} \right)^2 + \left( \frac{\pi q_{nt, nr}}{a} \right)^2}
\]  

(5.2)

In this geometry, the axial length is more than thirty times the radius. Thus, for a large frequency range, the acoustic modes are axial modes. Indeed, Table.5.2 shows that the first circumferential mode has a frequency of about 5200 Hertz. The system is therefore mainly one dimensional and will ease the study of the relationship between mesh discretization and wave frequency.
5.2. The ARF method normalization

Acoustic modes are characterized by their frequency, damping and modeshape. The frequency and damping estimates are inherent to the system, and therefore independent of the solver. But the modeshape amplitudes are arbitrary. In order to compare methods or systems, it is necessary to normalize these amplitudes. The first part of this section reminds the reader of the ARF method procedure. The section will then tackle the issue of normalization.

5.2.1 Building the acoustic characteristics from the time history

The ARF method, presented in Chapter 3, excites the system using a chirp excitation and collects the static pressure perturbations. Time histories are then translated into the frequency domain under the form of response functions. These allow the computation of the frequency and damping of the modes, using either the Bode plot (Fig.5.2a) or the Multivariate mode indicator function (Fig.5.2b).

Table 5.2. Theoretical frequencies of the first acoustic modes for a closed circular cylinder of length 0.693m and radius 0.0191m at atmospheric conditions, $c = 340 m/s^{-1}$.

<table>
<thead>
<tr>
<th>$n_x$</th>
<th>$n_t$</th>
<th>$n_r$</th>
<th>Frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>245</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>490</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>736</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>981</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>1226</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>1472</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>0</td>
<td>1717</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>0</td>
<td>1963</td>
</tr>
<tr>
<td>9</td>
<td>0</td>
<td>0</td>
<td>2208</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>0</td>
<td>2453</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>21</td>
<td>0</td>
<td>0</td>
<td>5152</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>5218</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>5224</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>5241</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
<td>5269</td>
</tr>
</tbody>
</table>

$\cdot$
Figure 5.2. The ARF results: the averaged response function, the multivariate mode indicator function and the acoustic node maps
In Chapter 3, the last acoustic characteristic to be determined, the modeshape, was deduced from plotting the reconstructed spatial representation of the spectrum. This was called the node map of the acoustic mode. Examples for the first four acoustic modes are represented in Fig. 5.2c to Fig. 5.2f. The acoustic node map is a quick way to represent the acoustic modeshape. But in order to achieve the normalization of the ARF method, the complete modeshape has to be computed. An additional step has to be taken. It is presented in the next section.

5.2.2 The modeshape estimate

The range of tools highlighted above gives a full picture of the acoustic behaviour of a system. Once the frequency, the damping and the node map are known, the characteristics of the system will be predictable. But if the method is to be compared with other methods, the results have to be normalized which means that the full computation of the modeshapes is required. The line-fit method presented earlier needs to be extended.

The line-fit method allows the computation of the modal model of the system. It provides an estimate of the frequency $\omega_r$, damping $\delta_r$ and modal constant $(a_r, b_r)$. With these estimates, the system’s modal response function can be computed. If $m$ modes have been captured, the individual response function resulting from the interaction of the $j^{th}$ and $k^{th}$ transducers takes the form of:

$$H_{jk}(\omega) = \sum_{r=1}^{m} \frac{a_{rk} + jb_{rk}}{\omega_r^2 - \omega^2 + j2\delta_r\omega r\omega}$$

The model is nevertheless not complete as the modeshapes are yet unknown. To deduce the modeshape using the line-fit method, it is necessary to use another defi-
nition of the response function using the modeshape vectors \( \Psi = [\psi^r]_{1 \leq r \leq m} \):

\[
H_{jk}(\omega) = \sum_{r=1}^{m} \frac{\psi_{jr} \psi_{rk}}{\omega_{r}^2 - \omega^2 + j2\delta_{r}\omega\omega}
\]  

(5.4)

(5.4) shows that any modeshape can be deduced from the set of individual response functions by considering only one column of the response function matrix \( H = [H_{jk}]_{1 \leq j \leq m, 1 \leq k \leq m} \). Indeed if only one point of excitation is used, that is to say \( k = k_e \), then the relationship between the modal constant and the modeshape is:

\[
a_{jke}^r + jb_{jke}^r = \psi_{jr} \psi_{k_e}^r
\]  

(5.5)

If the response at the excitation transducer is considered, (5.5) becomes:

\[
(\psi_{ke}^r)^2 = a_{keke}^r
\]  

(5.6)

Once \( \psi_{k_e}^r \) is known, the other values of the modeshape vector are deduced from:

\[
\psi_{j}^r = \frac{a_{jke}^r + jb_{jke}^r}{\psi_{ke}^r}
\]  

(5.7)

The acoustic modeshape is now fully determined. Before normalization, it is important to assess its quality by looking at its projection in the complex plane. The modal constant \( (a^r, b^r) \), deduced from the line-fit method, is often complex. Therefore the modeshapes are also complex. In order to rebuild the individual response functions, we are only interested in the theoretical undamped modes. These are real modes. To assess the quality of the analysis and the nature of the mode, it is possible to project the modeshape into the complex plane. Such representation is known as the Argand plot.

The modes can either have complex properties induced for example by non-linearity, or be fully rotated by a fixed angle. In the present study, the Argand plots of the four
acoustic modes, clearly show in Fig.5.3a, that the modes are rotated by a very slight angle of less than 2 degrees. We can assume that the fact that the mode is complex is mainly due to numerical errors. The theoretical modeshape is the projection of the estimated modeshape onto the real axis. The resulting acoustic modes correspond to the undamped real and theoretical acoustic modes. They are represented in Fig.5.3.

Now that the frequency, damping and modeshapes have been computed, it is time to look at how the results can be normalized.

### 5.2.3 The normalization

The modeshape estimates  \( \Psi = [\psi^r]_{1 \leq r \leq m} \) are entirely defined except for their amplitude. In order to compare different methods, the latter must be normalized. In structural dynamics, the most common normalization is the mass normalization which consist in dividing the modeshape by the modal mass:

\[
\phi^r = \frac{1}{m^r} \psi^r
\]

where the modal mass \( m^r \) is deduced from the mass matrix \( [M] \) as \( m^r = [\psi^r]^T [M] [\psi^r] \).

While in structural problems, it is possible to extract the mass matrix, there are no equivalence in the fluid domain. Using a finite volume scheme implies that the matrix of the system is not explicitly defined. Furthermore, when structural dynamics systems usually consider smaller systems allowing the use of well defined algebraic tools, CFD systems consist of a high number of nodes where five variables have to be defined: density, velocity (3 components) and pressure. The normalization we chose therefore uses the infinity norm. It imposes that the maximum amplitude of
Figure 5.3. The modeshapes of the first four fundamental acoustic modes
the modeshape is equal to one. This has been preferred to the Euclidian norm as the value of the latter will depend on the geometry’s discretization. The use of the infinity norm nevertheless suppress the relative amplitude. This is considered acceptable as our aim is to only compare results between two computational methods.

Now that the modeshapes have been normalized, it is possible to rebuild the set of individual response functions:

\[ H_{jk}(\omega) = \sum_{r=1}^{m} \frac{\phi_j^r \phi_k^r}{\omega_r^2 - \omega^2 + j2\delta_r\omega_r\omega} \]  

(5.9)

and the system’s averaged response function:

\[ |H(\omega)| = \frac{1}{n} \sum_{(j,k)} |H_{jk}(f)| \]  

(5.10)

5.3 The Arnoldi method normalization

The normalization procedure for the Arnoldi is in many ways easier than for the ARF method as the frequency, damping and modeshape are readily available. All post-processing treatment can be incorporated in the main code and therefore be transparent to the user. The first part of this section presents the Arnoldi results for the closed cylinder considered. A discussion will then follow on the post-processing for the computation of normalized modeshapes.

5.3.1 The Arnoldi results

The Arnoldi method is applied on the same geometry studied with the ARF method and presented in Fig.5.1. The fluid in the system is imposed to be at rest and at
atmospheric conditions in density and pressure. On top of the atmospheric steady state, a linear field of acoustic waves is made to propagate. The Arnoldi method extracts, from this linear perturbation, the stability characteristics of the solver. After defining the computation’s parameters, this section will focus on the study of the estimated solver’s spectrum. A more detailed analysis will then be carried out by looking at the frequency, damping and modeshapes of the predicted acoustic resonances. A short discussion will finally be presented on the influence of the mesh refinement on the results’ accuracy.

A parametric study was undertaken in Chapter 4, describing how to get the best results with the Arnoldi method in terms of convergence and accuracy. It was shown that the number of Arnoldi vectors $m$ and the virtual time $Ndt$ are the most influential parameters. In this study, the number of Arnoldi vectors is fixed to $m = 100$, the timestep is equal to $dt = 4.10^{-7}s$ and the number of iterations per vector is fixed at $N = 2500$. This setup respects the stability conditions of the solver and ensures that a maximum of vectors have converged.

### 5.3.1.1 The spectrum

The Arnoldi computation returns an estimate of the least stable modes under the form of a set of approximated eigenvalues. The approximated spectrum is presented in the complex plane in Fig.5.4.

An initial look at the system’s spectrum gives a flurry of information on the quality of the computation and the convergence of the modes. This is mainly expressed by the radius of the eigenvalues, related to the acoustic damping of the modes. In this study, a large number of eigenvalues are located close to the unit circle boundary. This shows a low value of damping, which is expected for a closed geometry with
rigid walls. But, some eigenvalues are seen outside the unit circle, indicating unstable modes. This is not physical and is explained by the approximation of the second order solver on an unstructured mesh. In order to resolve such an issue, we either need to refine the mesh or increase the order of the solver.

Another interesting feature of the spectrum presented in Fig.5.4 is its flower-like shape. This is not due to chance but more to a fortunate coincidence and leads us to now consider the argument of the eigenvalues. While the radius of the eigenvalue is related to the damping of the mode, the argument of the eigenvalue encloses information on the modes’ frequencies. Because of the specific geometry of the system, the first twenty or so acoustic modes have a frequency expressed as multiples of the fundamental frequency $f_1$. For the geometry studied, the fundamental frequency is equal to $f_1 = 245.39$ Hz. The difference between two successive frequencies can
be expressed as (See Chapter 4 for more informations):

\[ f_{n+1} - f_n = \frac{\phi_{n+1} - 2\pi k}{2\pi T_s} - \frac{\phi_n - 2\pi k}{2\pi T} \]  

(5.11)

where \( T_s \) is the virtual computational time \( T_s = N dt = 1.10^{-3} s = 1/(4 f_1) \) and \( \phi_n \) and \( \phi_{n+1} \) are the arguments of two successive eigenvalues. If we assume that the two eigenvalues are in the same \([-2\pi k, 2\pi k]\) interval, (5.11) becomes:

\[ \phi_{n+1} - \phi_n = 2\pi T_s f_1 \approx \frac{\pi}{2} \]  

(5.12)

Successive eigenvalues will therefore have a 90 degree phase angle between them. In other words, if the geometry is mainly one dimensional and if the period of the fundamental mode is multiple of the virtual time \( T_s = N dt \), the spectrum will have a specific flower-like shape.

### 5.3.1.2 The frequency and damping estimates

Once the spectrum has been studied, it is possible to extract from the eigenvalues the estimates of frequency and damping. This is done by combining information of the spectrum with the corresponding eigenvectors. The process has been described in detail in Chapter 4. For this system, out of 100 Arnoldi vectors, 70 eigenvalues, corresponding to 35 conjugate pairs, define genuine acoustic modes. The corresponding 70% rate of convergence has been observed to be the maximum achievable when changing input parameters. The information contained in these 35 conjugate pairs gives estimates of both frequency and damping:

- As can be seen in Fig.5.5a, the converged modes show very good agreement with theoretical frequencies. This is observed for a large range of frequencies and significant errors can only be observed for high frequency axial modes or circumferential modes. We will see later that this is due to mesh refinement considerations.
5.3. THE ARNOLDI METHOD NORMALIZATION

Figure 5.5. Acoustic prediction of the Arnoldi method on a closed circular cylinder using tetrahedral mesh
Figure 5.5. Acoustic prediction of the Arnoldi solver on closed circular cylinder using tetrahedral mesh
Figure 5.5. Acoustic prediction of the Arnoldi solver on closed circular cylinder using tetrahedral mesh
Fig. 5.5a also shows that low damping is achieved for a large range of frequencies. But limitations are two-fold. First, some modes have negative damping. This is due to a combination of the unstructured second order solver and the mesh refinement. It is believed that a higher order unstructured Euler solver would mitigate this effect [158]. Second, large values of damping are also observed for circumferential modes. This is due to the fact that the length of the cylinder is thirty times its radius and therefore the mesh is not refined enough in the radial direction.

In the end, a large number of acoustic modes, whose modeshapes are presented in Fig. 5.5b, have been predicted with very good accuracy in both frequency and damping. The results also highlight the strong relationship between the accuracy of the results and the mesh refinement.
5.3. THE ARNOLDI METHOD NORMALIZATION

5.3.1.3 Impact of the mesh refinement

It is known that the mesh discretization, and more importantly the number of nodes per wavelength of the acoustic modes, are important factors in the accuracy of wave propagation in unstructured CFD solvers. Because the geometry studied here is quasi one dimensional, it is an ideal candidate for a quick analysis on the influence of such a parameter.

The Arnoldi method is therefore tested on the four meshes, presented in Fig.5.6, with increasing refinements of about 300, 1700, 5000, 12 000 nodes. Plotting both the error in frequency against theory and numerical damping with respect to the

Figure 5.6. The closed cylinder geometry with increasing mesh refinements
number of nodes per wavelength clearly shows in Fig. 5.7 that, to achieve accuracy in frequency and damping, the mesh should contain more than ten nodes per wavelength. This is a good result as usually CFD solvers require to have 30 nodes or so per wavelengths in order to be accurate [65].

(a) Impact of the mesh refinement on frequency accuracy

(b) Impact of the mesh refinement on damping accuracy

**Figure 5.7. Impact of the mesh refinement on frequency and damping accuracy**
5.3.2 The Arnoldi modeshapes and normalization

The Arnoldi method successively predicted the acoustic modal model of the closed cylinder. Acoustic frequency, damping and modeshape have been accurately computed for about 35 acoustic modes. In order to compare the results with the ARF method, the modes will now be normalized. Doing so will first present the reader with an opportunity to better understand how the modeshapes are calculated. It will also help and ease the comparison with the ARF method by providing material for the creation of an Arnoldi response function.

5.3.2.1 Computation of the acoustic modeshapes

The normalization of the acoustic modes extracted with the Arnoldi method requires a better understanding of the way the modes are computed. Arnoldi transforms the matrix of fluxes of the solver $A$ into an Hessenberg submatrix $H$ of dimension $m$ via a transformation matrix $B = e^{A\cdot t}$. The eigenvalues presented above, from which modal frequency and damping are extracted, are the eigenvalues of the Hessenberg matrix. They are an approximation of the eigenvalues of the transformation matrix $B$ from which the eigenvalues of $A$ can be computed.

It is interesting to look at the composition of the modeshapes of the Hessenberg matrix in the Argand plot. Fig.5.8a shows, for example, that the so-called fundamental modeshape, i.e. the Hessenberg eigenvector, is highly complex. It does not correspond to a physical modeshape. To correct that, it is necessary to leave the mathematical domain for the physical domain.

Because $A$ is real, and because the preconidtionning matrix $B$ is defined as a polynomial function of $A$, the eigenvectors of $A$ are the eigenvectors of $B$. The matrices
Figure 5.8. The identification of the fundamental modeshape for the Arnoldi method, using the Argand plot.
B and H are real, and there exist an orthonormal matrix Q such that:

\[ \mathbf{B} = \mathbf{Q}^T \mathbf{H} \mathbf{Q} \]  
(5.13)

If \((X_i)_{i \in \mathbb{N}}\) is the family of A’s eigenvectors, they can be deduced from the eigenvectors of H, \((Y_i)_{i \in \mathbb{N}}\), by the relation:

\[ X_i = \mathbf{Q}^T Y_i \]  
(5.14)

The Argand plot for the fundamental mode of the matrix A is plotted in Fig.5.8b. The Argand plot clearly shows two orthogonal lines. The modeshape contains information on the density, velocity and pressure of the acoustic field. The orthogonal lines correspond to, respectively, the pressure/density and velocity modeshepes. This is in accordance with acoustic theory, as the pressure and velocity modeshepes are out of phase by a quarter of the wavelength (Chapter 1), while density and pressure are in phase. Isolating the pressure component of the fundamental modeshepes, represented by the green points in Fig.5.8c, clearly show the pressure mode is represented by a line in the complex plane. The theoretical acoustic mode corresponds to the projection of this line onto the real axis.

5.3.2.2 Normalization and Arnoldi’s response function

The theoretical modeshepes of the system can be computed from the Hessenberg eigenvectors by using the transformation matrix Q. Because of the way the Arnoldi vectors are calculated, the Arnoldi modeshepes are all initially normalized using the Euclidian norm. There is therefore no possibility to consider relative amplitudes and this is the main reason why, to compare the ARF and the Arnoldi methods, we introduced the infinity norm.
From the Arnoldi results, it is then possible to build the set of individual response functions. The same definition, expressed in Section 5.2.3, is used to that purpose:

$$H_{jk}(\omega) = \sum_{r=1}^{m} \frac{\phi_r^j \phi_r^k}{\omega_r^2 - \omega^2 + j2\delta_{r,\omega_r}\omega}$$  \hspace{1cm} (5.15)

The resulting averaged response function can then be computed. It is an efficient graphical tool for the comparison between the Arnoldi method and the ARF method results.

### 5.4 Comparison of the two methods

The ARF method and the Arnoldi method are two solutions achieving a similar objective: the prediction of acoustic resonances in core volumes. On one hand, the ARF method mimics real time structural experiments. On the other hand, the Arnoldi method directly studies the stability of the CFD solver. Each solution present advantages and limitations.

In previous sections, the results of the two methods have been presented on a simple enclosed cylinder. The cylinder was discretized with 12 000 nodes. To complete the study, the comparison presented now will be conducted on results for four meshes of increasing refinements (see Fig.5.6), and the results will be considered under three perspectives. First, the comparison will be carried out in terms of the number of acoustic modes predicted. Then, the difference in accuracy both in terms of frequency and damping will be highlighted. Finally, computational arguments will be studied such as the ease of implementation, computational speed and memory requirements. We will see that such a comparison, on a simple system, identifies the Arnoldi method as the best acoustic prediction alternative.
5.4. COMPARISON OF THE TWO METHODS

5.4.1 Comparison in terms of the number of modes

The comparison on the number of modes predicted, in Fig.5.9, already clearly shows the difference of philosophies between the two acoustic methods.

On one hand, the ARF method, represented in green, shows that the number of modes is independent of the mesh refinement. That is, as soon as the mesh is refined enough to capture the modes. This is to be expected as the sinusoidal chirp excitation, used to excite the system, is fixed to a specific frequency range. This was imposed in Chapter 3 to ensure a good balance between the timestep, the chirp excitation sweeping rate and the maximum frequency of excitation. One might say that, in order to excite a maximum number of modes, a white noise excitation could have been used instead. But this would have resulted in a loss of quality in the results when computing the transfer functions and would have a knock-on effect on the frequency and damping accuracy.

On the other hand, the Arnoldi method captures an increasing number of acoustic modes with increasing mesh resolution. In a sense, the Arnoldi method could be said to be evolutive. It will only be able to capture what the mesh will be able to model. And we have seen earlier that an acoustic mode will be accurately modelled if the mesh contains more than ten nodes per mode’s wavelength. Once the mesh is refined enough, the number of captured modes tends to reach a maximum level of about seventy percent of the Arnoldi vectors. This was observed for many geometries.

The Arnoldi method therefore shows better performance. For non-flow applications, the number of modes predicted seems to only be dependent on the mesh refinement. The more refined the mesh, the larger the number of modes is: the
Arnoldi method is said to be evolutive. This is true to a certain extent as it is possible that higher frequency modes will suffer from a loss of accuracy. To achieve similar results, the ARF method will require several computations with increasing frequency range and decreasing timesteps.

5.4.2 Comparison in terms of accuracy in frequency and damping

The Arnoldi method predicts more modes. But sometimes, quantity does not mean quality. The quality of the analysis is defined by the accuracy that the two methods achieve both in terms of frequency and damping. It will be considered in three ways. First, a qualitative study is undertaken using the normalized response functions defined earlier. Then, a complete quantitative study is undertaken before looking at the direct impact of the mesh refinement on the results accuracy.
5.4. COMPARISON OF THE TWO METHODS

5.4.2.1 Qualitative study

The qualitative study uses the well-known Bode plot of both individual and averaged response functions to determine the main characteristics of the two methods. The frequency range of analysis is limited between 1 and 1000 Hz. Therefore only the first four fundamental modes will be studied.

The Bode plot is widely used to identify resonant regimes, represented as peaks in the response function. In our case, four acoustic regimes are identified when looking at either individual response functions or averaged response functions. First, five individual response functions are plotted in Fig. 5.10, using the normalization introduced in Section 5.2 and Section 5.3. They correspond to response functions between the point of maximum excitation on the membrane, at \( x = 0.0 \), and five responses, captured at five transducers located on the axis of the cylinder at \( x = [0.0, 0.25l_x, 0.5l_x, 0.75l_x, l_x] \). The five Bode plots clearly show the evolution of the response functions along the cylinder’s axis with resonances and anti-resonances. The resonant peak are also cleanly identified with the ARF and Arnoldi averaged response functions, captured in Fig. 5.11.

Looking at the plots in more detail helps analysing the quality of prediction of the three acoustic properties: the modeshape, the damping and the frequency.

- The normalization presented above has leveled the two functions highlighting the similarity of the ARF and Arnoldi modeshapes.
- The sharpness of the peaks of the ARF and Arnoldi response function also shows that both methods predict low damping modes. This is in agreement with theory.
- The location of the peak in terms of frequency nevertheless shows a sharp frequency lag from the ARF predicted modes. It was seen that
Figure 5.10. The response function plotted at five different transducers
Figure 5.11. Averaged response function over the whole geometry
while the Arnoldi methods predicts acoustic frequencies within a 0.1% error margin, the ARF method underestimates the acoustic frequencies by as much as 5%. This frequency lag was attributed in Chapter 3 to the solver’s dispersive error. With the solver used, it is recommended that at least 150 timesteps are used to represent the higher frequency vibrations. In this study and in order to speed up the computation, the 1000Hz maximal vibration is represented by 100 timesteps.

5.4.2.2 Quantitative study

A quantitative study is now considered. The results for the ARF method are computed with the line-fit method (See Chapter 3). The results for the Arnoldi results are extracted from the solver’s computed eigenvalues (See Section.5.3.1). The frequencies are compared with the theoretical frequencies presented in Section.5.1.2. The damping values should be close to zero as the geometry is closed with rigid walls.

The frequency and damping values are recapitulated in Table.5.3. The averaged frequency error, for the ARF method and the Arnoldi method, are respectively 4.00 % and 0.03 %. The difference is quite important and results from different time integration schemes. As explained above, the ARF accuracy is dictated by the timestep used. The smaller the time-step is, the better the accuracy. But a smaller time-step results in longer computations. A compromise had therefore to be struck between accuracy and efficiency. In contrast, the Arnoldi method does not suffer from such a problem because instead of modelling the acoustic modes as they would develop in reality, it computes the eigenmodes of the fluxes operator directly. Provided the timestep is small enough for stability reasons, the frequency should be accurately estimated.
Table 5.3. Comparison between the ARF method and the Arnoldi method in terms of frequency and damping accuracy

<table>
<thead>
<tr>
<th>Method</th>
<th>Error in frequency (%)</th>
<th>Numerical damping (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ARF</td>
<td>Arnoldi</td>
</tr>
<tr>
<td>Mode 1 (245Hz)</td>
<td>3.6</td>
<td>0.01</td>
</tr>
<tr>
<td>Mode 2 (490Hz)</td>
<td>3.7</td>
<td>0.08</td>
</tr>
<tr>
<td>Mode 3 (736Hz)</td>
<td>4.0</td>
<td>0.07</td>
</tr>
<tr>
<td>Mode 4 (982Hz)</td>
<td>4.3</td>
<td>0.05</td>
</tr>
</tbody>
</table>

Table 5.3 also presents the damping estimates for the two methods. For the four acoustic modes detailed, the ARF method gives a better estimate of the damping when compared to the Arnoldi method. For higher frequency modes, it was seen that damping quickly creeps in in both methods. On one hand, the ARF method requires a timestep adapted to the modes modelled. On the other hand, the damping for the Arnoldi modes will be highly dependent on the mesh refinement and can sometimes be negative (Fig. 5.5a). The corresponding numerically unstable modes nevertheless have a damping close to zero. They are therefore close to the stability threshold. The problem is thought to result from the approximation made in the second order solver used. More accurate results are likely to be obtained with a higher order solver.

5.4.2.3 The impact of the mesh refinement

Reasonable level of accuracy has been achieved for frequency and damping for both the ARF method and the Arnoldi method. But in computational methods, accuracy is always in competition with efficiency and computational speed.

A parameter likely to both impact accuracy and efficiency is the mesh refinement. And it is interesting to study how each method will cope with an increasingly coarse mesh. To that purpose, four meshes of the same geometry are considered. They
correspond to tetrahedral unstructured meshes composed from about 300 nodes to 12 000 nodes. The four meshes were presented earlier in Fig.5.6 when considering the Arnoldi results. What we found is that for the lowest frequency modes, the ARF and Arnoldi results stay largely unperturbed by the mesh refinement. But as frequency increases, the mesh needs to be more refined to maintain the level of accuracy required.

Fig.5.12a and Fig.5.12b show that while the frequency error and the damping for the ARF method stay constant or so across the range of meshes, they improve for the Arnoldi method gradually. On one side, the frequency error is reasonable for the ARF method as it stays within 5%. But it does not match the accuracy achieved by Arnoldi across the mesh refinement range. The damping follows the same trend with a constant damping for the ARF method and an improved damping, with more refined meshes, for the Arnoldi method.

In the end, the Arnoldi method clearly shows its qualities in this study as it predicts more modes with better accuracy in frequency and same accuracy in damping. What needs to be considered now is the computational cost of the two methods. It will be presented in the following section.

5.4.3 Comparison in terms of computational efficiency

To have a full picture of the qualities of each method, we finally have to discuss their computational efficiency. The computational considerations are threefold: the ease of implementation, the speed of computation and the memory requirement. It is reminded here that the result looking at the computational performances are dependent on both the clusters used and the parallel efficiency of the solver. For both methods, we have used 8 to 16 nodes giving the highest parallel efficiency.
5.4. COMPARISON OF THE TWO METHODS

Figure 5.12. Impact of the mesh refinement on the frequency error and numerical damping for both methods and the first four acoustic modes.
The ease of implementation seems to favour the ARF method as it mimics a real-life structural dynamics experiment. The system is excited in a specific frequency range and the response of the system is captured thanks to transducers placed along the geometry. The concept has been around for decades and is therefore easy to grasp. Runs are easy to set up once you have an accurate CFD solver and a set of modal analysis tools. It should give you a fairly good idea of the acoustic properties of a system without much effort. On the contrary, the Arnoldi method relies on complex mathematical tools. A new linear solver must be implemented. The method can also be less accessible to first-users and results more difficult to analyse.

The speed of computation is an essential parameter able to make or break the success of a method. Fig. 5.13a shows the evolution of the computational time per number of modes captured with respect to the mesh refinement. Fig. 5.13a clearly highlight that the Arnoldi method is the fastest of the two methods. This is due to both a faster solver and more predicted modes. For the 12 000 nodes mesh, the Arnoldi method needs about 10 CPU hours per mode when the ARF methods requires 25 CPU hours. The 55 000 nodes solves the case. The Arnoldi method requires 25 CPU hours per modes while the ARF method needs 125. As a result, the latter should be restricted to coarser meshes as the accuracy in frequency and damping does not seem to suffer from differences in mesh refinements. It is also to be said here that the number of iterations in this ARF was fixed to 262 144 iterations in order to get very clean results for very undamped modes. Reducing the number of iterations would have a direct impact on the computational time required while keeping reasonably clean response functions and therefore accurate results.

Concerning the memory requirements, the Arnoldi shows again its supremacy. The problem is inherent to how the ARF method works. Indeed, the latter requires that
5.4. COMPARISON OF THE TWO METHODS

Figure 5.13. Evolution of the computational time and data size with mesh refinement for both methods.
the time histories for a large number of transducers are to be stored. Fig.5.13b clearly shows the exponential growth of the data size required with the mesh refinement. With the geometry studied, a discerning reader would be right to say that after all the same number of transducers could be used for the coarser and the more refined meshes. Our aim is nevertheless to develop the method for more complex geometries where the CFD mesh and therefore the transducers mesh needs to be adapted to the geometry’s complexity and the system’s flow patterns.

5.5 Conclusion

The objective of the thesis is to design a predicting method for acoustic resonances in core volumes. For such a method to be useful, it has to be fast, accurate, reliable and efficient. The ARF method and the Arnoldi method, presented respectively in Chapter 3 and Chapter 4, are two strong contenders. As highlighted in Table.5.4, they both present strong arguments in their favour though they also have limitations. What this chapter attempts is to compare the two methods on a simple system, highlight their pros and cons and therefore present the most obvious choice for achieving our objectives. The resulting conclusions concerning the ARF and the Arnoldi methods are clear.

On one hand, the ARF limitations are two-fold:

- The method is computationally demanding due to the modelling and storing of a large number of time histories.
- The method requires to define a frequency range in agreement with the computation’s time-step in order to ensure good accuracy. If the system is to be cleared over a large range of frequencies, numerous computations will have to be carried out.
### Table 5.4: Recapitulative comparison of the two acoustic methods

<table>
<thead>
<tr>
<th>Method</th>
<th>Determining input parameters</th>
<th>Number of predicted modes</th>
<th>Accuracy</th>
<th>Efficiency</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>ARF method</strong></td>
<td>+ + + Time-step</td>
<td>− Fixed by frequency</td>
<td>+ Good and constant accuracy, mainly function of dt</td>
<td>+ Easy implementation</td>
</tr>
<tr>
<td></td>
<td>++ Frequency window</td>
<td><em>quad</em> range of excitation</td>
<td></td>
<td>− Long and memory intensive computation</td>
</tr>
<tr>
<td></td>
<td>+ Frequency resolution</td>
<td>+ All modes predicted</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Arnoldi method</strong></td>
<td>+ + + Mesh refinement</td>
<td>+ Evolutive, function of mesh refinement</td>
<td>+ Very good accuracy for converged modes</td>
<td>+ Very fast computation</td>
</tr>
<tr>
<td></td>
<td>++ Virtual time <em>Ndt</em></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>+ Number Arnoldi vectors</td>
<td>− Possibly missing modes</td>
<td></td>
<td>+ Small memory needs</td>
</tr>
</tbody>
</table>

5.5 CONCLUSION
But the ARF method also has some strong qualities:

- The method relies on a simple concept proven and tested over the last decades. It is easy to understand and implement.
- The method gives accurate and consistent frequency and damping estimates once the time-step has been chosen. This imply that there will be not doubt over the existence or non existence of specific modes.
- The method is resilient against mesh coarsening and keeps good performance even on very coarse meshes.
- The method is defined on the full Euler equations and does not rely on linearization. This implies that it will be able to directly model the interactions between the system, the flow and the acoustic field.

On the other hand, the Arnoldi limitations are:

- The method’s concept is not as straightforward and the results could be more difficult to analyse.
- The method sometimes predicts spurious modes. This is not an issue here as the acoustic modes can be predicted by theory. This might be more complicated in more complex systems.
- The linearization implies that the acoustic field is separated from the main flow and therefore it will not be possible to model the impact of the acoustic field onto the main flow.

Nevertheless, the Arnoldi method also has strong advantages:

- The method predicts a large number of acoustic modes in a very short time.
- The frequency and damping estimates are very accurate over a wide range of frequencies.
• The method is less sensitive to the choice of the time-step, providing the solver is stable.

• The method adapts well to the mesh used and will implicitly focus on the modes the mesh is able to model.

These considerations offer an interesting insight of the abilities of both methods. They will dictate in which situation each method should be used:

• The computational limitations of the ARF method imply that the ARF method should be limited to the study of coarse mesh, simple system or pre-studies. But the consistency of the results and the fact that the full Euler equations are used imply that the method could also be used to study the development of a very specific mode which for example was highlighted as an issue during an experimental test. In other words, the ARF method could be used on either very coarse general acoustic studies or very specific detailed analysis, where flow or combustion instabilities are for example present.

• The speed and accuracy of predictions over a wide range of frequency of the Arnoldi method implies that the Arnoldi method could be used on a more general basis. It will be an ideal tool at the design stages in order to quickly assess the acoustic characteristic of a specific system.

On a simple system as studied in this chapter, the Arnoldi method seems to be the best alternative as it predicts more modes, with better accuracy in frequency and for a fraction of the computational time. Arnoldi is clearly adapted to the study of enclosures. The main drawback is the linearization of the solver which might limit the applications for more complex systems. How will Arnoldi behave on such
systems? Will the ARF method show some new strengths? This is what we will discuss in the next chapter.
Chapter 6

The end correction and Doppler effects

The thesis will now shift attention onto open systems, where the fluid is connected to the outside. Such systems will introduce the problem of boundary conditions and flow applications. The chapter’s layout is as follows. First, we will consider an open cylinder with various mouth geometries and therefore introduce the concept of end corrections. Then, we will consider a set of coaxial cylinders subject to an increasing inlet flow. It will highlight the Doppler effect on acoustic resonances.

6.1 Study of open geometries and end corrections

The first step consists of modelling a simple circular cylinder, open at one end, closed at the other. Depending on how the cylinder’s mouth is defined, the acoustic modes will change due to the different propagation properties of the acoustic waves.
This change is often modelled using the concept of end corrections introduced below. We will then look at both the ARF and Arnoldi results.

6.1.1 The theory of end corrections

The simple way to represent acoustic resonances in an open tube is to impose an acoustic pressure node at its mouth. But this does not accurately reflect the behaviour of the propagating waves and introduces an error when considering the acoustic frequencies and damping values. To better understand the phenomenon, we need to define the acoustic impedance of the semi-open cylinder in more detail.

As seen in Chapter 1, the impedance inside the cylinder, defined as the ratio between pressure and air velocity, is equal to $Z_o = \rho c/S$. If we consider the air at the open mouth and limit the study to the fundamental axial modes, we can consider that the velocity across the cylinder’s mouth is uniform. This stands as long as we can assume that the modes’ wavelengths are large compared to the radius of the cylinder. The air at the mouth can then be considered as a piston of zero mass radiating some energy in the open and reflecting some energy back in the cylinder. The impedance of the piston will be defined as $Z_m = R + jX$ where $R$ is the resistance responsible for the radiation of sound, and $X$ is the reactance responsible for the reflected energy.

In the case of a flanged piston of velocity $u'$ and radius $a$, the pressure in the open end can be expressed as [11, pp.167]:

$$p' = \frac{1}{2} j\omega \rho' u' a^2 \left( \frac{e^{-jkr}}{r} \right) \left[ \frac{2J_1(ka \sin \theta)}{ka \sin \theta} \right]$$

(6.1)

where $J_1$ is the Bessel function of first order. For large wavelengths, such that $ka \ll 1$, the ratio in the square brackets is one and therefore the impedance of the
piston, defined as \( Z_m = R + jX \), is [11, pp.180]:

\[
R = Z_o \left[ \frac{(ka)^2}{2} - \frac{(ka)^4}{2^2 \cdot 3} + \frac{(ka)^6}{2^2 \cdot 3^2 \cdot 4} - \cdots \right] \tag{6.2a}
\]

\[
X = \frac{Z_o}{\pi} \left[ \frac{(2)^3 ka}{3} - \frac{(2)^5 (ka)^3}{3^2 \cdot 5} + \frac{(2)^7 (ka)^5}{3^2 \cdot 5^2 \cdot 7} - \cdots \right] \tag{6.2b}
\]

Thus, for large wavelengths and low frequency modes, \( X \gg R \) and most of the energy is reflected back in the cylinder. With such assumptions, we obtain the relationship:

\[
Z_m = jZ_o k \left( \frac{8a}{3\pi} \right) \tag{6.3}
\]

Now that the impedance has been defined, we can consider the impedance at both ends of an opened flanged cylinder of length \( l_p \), as represented in Fig.6.1. With forward and backward propagating waves, the pressure in the cylinder is expressed as:

\[
p'(x,t) = [Ae^{-jkx} + Be^{jkx}]e^{j\omega t} \tag{6.4}
\]

Using (1.4), the resulting air velocity is:

\[
u'(x,t) = \left( \frac{1}{Z_o} \right) [Ae^{-jkx} - Be^{jkx}]e^{j\omega t} \tag{6.5}
\]
The impedance of the piston at \( x = 0 \) and \( x = l_p \), respectively defined by \( Z_{x=0} \) and \( Z_{x=l_p} \), are:

\[
Z_{x=0} = \frac{p(0, t)}{u(0, t)}, \quad Z_{x=l_p} = \frac{p(l_p, t)}{u(l_p, t)} \tag{6.6}
\]

(6.6) gives, when combined to (6.4) and (6.5):

\[
Z_{x=0} = Z_o \left[ \frac{Z_{x=l_p} \cos(kl_p) + jZ_o \sin(kl_p)}{jZ_{x=l_p} \sin(kl_p) + Z_o \cos(kl_p)} \right] \tag{6.7}
\]

If we assume we have a pressure node at the end of the cylinder, then ideally \( Z_{x=l_p} = 0 \). This gives:

\[
Z_{x=0} = jZ_o \tan(kl_p) \tag{6.8}
\]

Combining (6.3) and (6.8) implies that the impedance of the mouth of the cylinder is equivalent, when \( ka \ll 1 \), to the impedance of a purely open cylinder of length:

\[
\Delta_{flanged} = l_p = \frac{8a}{3\pi} \approx 0.85a \tag{6.9}
\]

\( \Delta_{flanged} \) is known as the end correction. The acoustic modes of a flanged cylinder of length \( l_x \) and radius \( a \), provided their wavelengths are large compared to \( a \), are equivalent to the acoustic modes of a perfectly opened cylinder of length \( l_x + \Delta_{flanged} \).

For an unflanged cylinder, the approach is similar but the method is more complicated and will not be detailed here. It is possible to calculate the unflanged end correction using a Wiener-Hopf integral equation [159]:

\[
\Delta_{unflanged} \approx 0.61a \tag{6.10}
\]

This estimate is nevertheless limited to low frequency modes. As Fig.6.2 shows, the end correction falls when the frequency of the wave increases [159]. This goes together with a damping increase. Indeed, as (6.2a) and (6.2b) point out, higher frequency modes gradually radiate more acoustic energy to the outside.
6.1. STUDY OF OPEN GEOMETRIES AND END CORRECTIONS

Figure 6.2. The end correction (ec) for an unflanged cylinder of radius a, function of $2\pi a/\lambda$ [159]

Now that the theory has been laid out, it is time to consider how the ARF and the Arnoldi methods address the problem.

6.1.2 The ARF method analysis

The impact of the location of the boundary conditions, as well as the flanged or unflanged properties of the semi-open cylinder, will first be studied with the ARF method.

Three geometries are considered. They are defined as the semi-open cylinder, the flanged cylinder and the unflanged cylinder and represented in respectively Fig.6.3a, Fig.6.3b and Fig.6.3c. In the three cases, the cylinder is open at one end only to a fluid at atmospheric conditions. The rest of the cylinder’s walls are assumed infinitely rigid. In order to save computational time, symmetry properties are used and only a quarter of the flanged and unflanged geometries are considered.
Figure 6.3. Meshes for the semi-open, flanged and unflanged cylinders
6.1. STUDY OF OPEN GEOMETRIES AND END CORRECTIONS

6.1.2.1 The simple semi-open cylinder

The initial study concerns the acoustic modes of the simple cylinder of length \( l_c = 270\, \text{mm} \) and radius \( a = 19.1\, \text{mm} \). In order to respect boundary conditions, a pressure node is imposed at the opened end of the cylinder.

The computational parameters considered are a time-step equal to \( 1.10^{-5}\, \text{s} \) and a number of iterations equal to 65536. These were chosen as a good compromise between accuracy and computational speed. The resulting averaged response function and MMIF, presented in Fig.6.4, clearly show the excitation of two acoustic modes within the 1-1000Hz frequency range. The two acoustic modes excited correspond to the first two axial modes of the system. They have an estimated frequency of respectively 300Hz and 894Hz. Their corresponding modeshapes are deduced from looking at the acoustic node maps, presented in respectively Fig.6.4c and Fig.6.4d.

The line-fit method is used to calculate accurate estimates in frequency and damping. The results are recapitulated in Table.6.1. The accuracy in both frequency and damping is similar to that achieved for enclosures. This was expected as the boundaries are purely reflective. Predicted frequency is close to theory as it is predicted within 5% of error. Damping estimates are of the order of \( 10^{-3}\% \), which corresponds to highly undamped modes.

As explained in previous chapters, accuracy in both frequency and damping can be improved at the cost of longer computations. Fig.6.5a shows that the frequency estimates will be improved if the timestep is decreased. Indeed, the smaller timestep allows a better representation of the wave propagation. The line-fit results in Table.6.2 show that with a timestep 10 times smaller, the frequency error becomes less than 1%. But to keep a good frequency resolution, dividing the time-step by ten implies
Figure 6.4. The ARF set of results: the averaged response function, the multivariate mode indicator function and the acoustic node maps
**Table 6.1.** Line-fit frequency and damping compared with theory for the semi-open cylinder

<table>
<thead>
<tr>
<th>Cylinder</th>
<th><strong>Predicted frequency (Hz)</strong></th>
<th><strong>Theoretical frequency (Hz)</strong></th>
<th><strong>Error (%)</strong></th>
<th><strong>Damping ratio (%)</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>semi-open</td>
<td>Mode 1</td>
<td>300.42</td>
<td>315.03</td>
<td>4.6</td>
</tr>
<tr>
<td></td>
<td>Mode 2</td>
<td>893.65</td>
<td>945.10</td>
<td>5.4</td>
</tr>
</tbody>
</table>

**Table 6.2.** Line-fit frequency and damping compared with theory for change in time-step and number of iterations

<table>
<thead>
<tr>
<th>Input parameter changed</th>
<th><strong>Predicted frequency (Hz)</strong></th>
<th><strong>Theoretical frequency (Hz)</strong></th>
<th><strong>Error (%)</strong></th>
<th><strong>Damping ratio (%)</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>(dt=1.10^6) s</td>
<td>Mode 1</td>
<td>314.77</td>
<td>315.03</td>
<td>0.08</td>
</tr>
<tr>
<td></td>
<td>Mode 2</td>
<td>940.23</td>
<td>945.10</td>
<td>0.52</td>
</tr>
<tr>
<td>N=262 144</td>
<td>Mode 1</td>
<td>299.03</td>
<td>315.03</td>
<td>5.0</td>
</tr>
<tr>
<td></td>
<td>Mode 2</td>
<td>892.61</td>
<td>945.10</td>
<td>5.6</td>
</tr>
</tbody>
</table>

Similarly, Fig. 6.5b and Table 6.2 show that better damping estimates will require a better frequency resolution of the response function. To achieve that, the timestep will have to stay small and the number of iterations will have to be increased. This will also result in longer computations.

In other words, the semi-open cylinder is another illustration of the compromise that has to be struck between accuracy and computational speed. In the rest of the chapter, we will be more interested in the way the acoustic properties evolve with
Figure 6.5. The ARF input parameters study on the semi-open cylinder
different boundary conditions or flow. We will therefore consider that an error of 5% in frequency and damping estimates of the order of $1.10^{-3}\%$ are sufficiently accurate. The input parameters used will therefore be a time-step equal to $1.10^{-5}\text{s}$ and a number of iterations equal to $N = 65536$.

### 6.1.2.2 The flanged and unflanged cylinders

The flanged and unflanged geometries are now considered in detail. The corresponding meshes are presented in Fig.6.3b and Fig.6.3c. These geometries will highlight the phenomenon of end corrections, as introduced earlier in this chapter. The two cylinders are opened to the exterior where atmospheric conditions are imposed.

The bode plots for the averaged response functions for the semi-open, flanged and unflanged cylinders, presented in Fig.6.6a, clearly show a shift in frequency and an increase in damping. They imply that, depending on the mouth geometry, an end correction has to be added to the physical length of the cylinder in order to take into account the change in propagation from a longitudinal wave in the cylinder to a spherical wave outside. The propagation of the modes outside the cylinder can actually be observed with the acoustic node maps presented in Fig.6.7 and the resulting loss of acoustic energy to the large domain is responsible for the increased damping of the mode.

The line-fit method applied on the three computations confirms the statement above. Table.6.3 states that the acoustic frequencies are predicted within an 8% to 5% bracket error when compared to theoretical frequencies with end corrections, while damping is seen to increase depending on the mouth geometry. Such error in frequency is typical of the time-step used $dt = 1.10^{-5}\text{s}$. But in order to confirm that
(a) The averaged response function for the semi-open, flanged and unflanged cylinder with time-step $dt = 1.10^{-5}s$

(b) The averaged response function for the semi-open, flanged and unflanged cylinder with time-step $dt = 1.10^{-6}s$

**Figure 6.6.** The averaged response function for the semi-open, flanged and unflanged cylinder
Figure 6.7. The acoustic node maps for the semi-open, flanged and unflanged cylinder
Table 6.3. Line-fit frequencies and dampings compared with theory for the semi-open, flanged and unflanged cylinders with time-step $dt = 1.10^{-5}$ s.

<table>
<thead>
<tr>
<th>Cylinder</th>
<th>Predicted frequency (Hz)</th>
<th>Theoretical frequency (Hz)</th>
<th>Error (%)</th>
<th>Damping ratio (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>semi-open</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mode 1</td>
<td>300.42</td>
<td>315.03</td>
<td>4.6</td>
<td>7.16$10^{-3}$</td>
</tr>
<tr>
<td>Mode 2</td>
<td>893.65</td>
<td>945.10</td>
<td>5.4</td>
<td>7.40$10^{-3}$</td>
</tr>
<tr>
<td>flanged</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mode 1</td>
<td>273.33</td>
<td>297.16</td>
<td>7.9</td>
<td>1.60$10^{-1}$</td>
</tr>
<tr>
<td>Mode 2</td>
<td>816.42</td>
<td>891.48</td>
<td>8.2</td>
<td>2.31$10^{-0}$</td>
</tr>
<tr>
<td>unflanged</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mode 1</td>
<td>280.88</td>
<td>301.47</td>
<td>6.8</td>
<td>8.27$10^{-1}$</td>
</tr>
<tr>
<td>Mode 2</td>
<td>837.69</td>
<td>904.41</td>
<td>7.4</td>
<td>2.03$10^{-0}$</td>
</tr>
</tbody>
</table>

the error is only due to the time-step and nothing else, computations have been carried out with a time-step ten times smaller, $dt = 1.10^{-6}$s. To limit the computation time, we used a number of time iterations equal to 163 840. This hindered the response function resolution but, as shown in Fig.6.6b, the two acoustic modes are still predicted and their acoustic frequencies can be deduced. Table 6.4 shows that with such a time-step, predicted frequencies are within 1% of the theoretical frequencies.
Table 6.4. Acoustic frequencies compared with theory for the semi-open, flanged and unflanged cylinders with time-step $dt = 1.10^{-6}$s

The end correction phenomenon has therefore been properly captured by the ARF method for the two fundamental modes of the semi-open cylinder. Will the Arnoldi method be able to do the same?

6.1.3 The Arnoldi method

6.1.3.1 The simple semi-open cylinder

The Arnoldi method is now used to study the same semi-open cylinder discretized using the same tetrahedral mesh, composed of about 5 000 nodes and presented in Fig.6.3a. The Arnoldi computation calculates 100 Arnoldi vectors with 5 000 iterations and a timestep equal to $2.10^{-7}$s.

The approximated spectrum and estimated frequency and damping are represented in respectively Fig.6.8a and Fig.6.8b. The full data defining these figures can be found in Table.D.4. The corresponding modeshapes are represented in Fig.6.8c.
modes have converged out of 50 possible vectors. The 46% convergence ratio is explained by the coarse mesh used which will only be able to model the fundamental modes of the system. What we observe though, for the converged modes, is a very accurate prediction of frequencies even for modes with a frequency higher than 8000Hz. For example, the wavelength of the fourteenth axial mode is represented by 9 mesh nodes only. This is in agreement with what was found in Chapter 5 on the performance of the Arnoldi method with regard to the nodes per wavelength discretization. The damping is also very accurately predicted for modes up to the eighth axial mode, while higher damping is observed for circumferential modes whose wavelengths are not represented by enough nodes.
Figure 6.8. Acoustic prediction of the semi-open cylinder using the Arnoldi method
Figure 6.8. Acoustic prediction of the semi-open cylinder using the Arnoldi method
6.1.3.2 The flanged and unflanged cylinders

In comparison with the ARF method and because the Arnoldi method allows a quicker and less memory intensive analysis, we have been able to model the full geometries of the flanged and unflanged cylinders. Not assuming symmetries implies that we are able to predict circumferential modes, as well as axial modes.

Both geometries, represented in Fig.6.9d and Fig.6.9e, are discretized using a tetrahedral mesh with respectively 5 000 nodes and 15 000 nodes. Because only fully reflective boundary conditions have been implemented in this thesis, the mesh is coarsened on purpose at the boundaries of the domain in order to damp spurious reflected waves.

The spectra of the two systems, presented in Fig.6.10a, show a lot of similarities. Fundamental modes actually have the same eigenvalues, as a similar virtual time $Ndt$ has been used in both computations. The spectra also highlights that only a few modes are lightly damped, leaving the majority of the modes clustered at the center of the unit circle.

Looking at the converged modes with more detail explains what is happening. Fig.6.10b and Fig.6.10c present the frequencies and damping estimates of the converged modes for respectively the flanged and unflanged cylinders (see full data in Table.D.5 and Table.D.6). Both computations, generating 200 Arnoldi vectors, show a convergence ratio of 15% which is low by the previous standards. This is explained both by the discretization of the cylinder itself, only able to capture fundamental modes, but also by the fact that Arnoldi still manages to capture modes of the outside domain. These spurious modes are non physical and are only present due to the fully reflective boundary condition used.
(d) Flanged cylinder geometry, 5,000 nodes

(c) Unflanged cylinder geometry, 15,000 nodes

Figure 6.9. The flanged and unflanged meshes for the Arnoldi study
Figure 6.10. Acoustic prediction of the flanged and unflanged cylinders using the Arnoldi method
Figure 6.10. Acoustic prediction of the flanged and unflanged cylinders using the Arnoldi method
Figure 6.10. Acoustic prediction of the flanged and unflanged cylinders using the Arnoldi method
Regarding the converged modes, the two charts Fig.6.10b and Fig.6.10c, whose data are recapitulated in Table.D.5 and Table.D.6, show a very good agreement between predicted frequencies and theory. The frequencies are indeed predicted with an averaged error of 0.5%. Theoretical frequencies have been calculated with an end correction of respectively 0.85a and 0.61a, for the flanged and unflanged cylinder. The only significant errors observed concern the unflanged case, where two frequencies in Fig.6.10c, corresponding to the 4181Hz and 5811Hz modes, seem to be undervalued. A look at the corresponding modeshapes in Fig.6.10e shows that these two modes interact with the spurious modes of the outside domain, hence the drop in frequency.

The damping estimates also show an interesting feature since the damping seems directly linked to the axial wavelength of the acoustic modes. Indeed, for equivalent axial wavelengths, both the fundamental and the first circumferential modes have the same damping level. This is in agreement with theory.

Finally the modeshapes of the modes, presented in Fig.6.10d and Fig.6.10e, show a good agreement with theory as well as the propagation of the mode to the outside domain.

6.1.3.3 Comparison with the semi-open cylinder

In order to summarize the impact of the cylinder’s mouth representation, we now focus our intention on the first eight axial modes as predicted for the three cylinders. Fig.6.11a shows the evolution of frequency and damping. The shift in frequency and increase in damping is clearly observed for the flanged and unflanged cylinders. The latter two actually see a similar level of damping of around 1% when the semi-open cylinder only has a damping of 0.1%. Acoustic energy is therefore
propagating outside the cylinder as expected.

In the unflanged cylinder case, Levine [159] predicted the evolution of the end correction with the mode’s axial wavelength. The evolution, represented in Fig.6.2, predicts the fall of the end correction with the ratio of the radius over the wavelength. Fig.6.11b, shows that Arnoldi is not able to capture the change in the end correction, as the impact of the end correction on the theoretical frequencies is less important than Arnoldi’s error margin in terms of frequencies. To study such a phenomenon, we would therefore need a geometry with a larger radius.

In the end, both implemented methods manage to model the propagation of the acoustic modes in a larger domain and predict with accuracy the corresponding shift in frequency (error when compared with theory less than 1%) and increase in damping. On one hand, the ARF method manages very accurate results provided we use a time-step small enough. On the other hand, the Arnoldi method predicts a large number of acoustic modes on a wide frequency range even with a very coarse mesh.

6.2 Impact of the Doppler effect on acoustic resonances

The organ pipe application shows that acoustic resonances can develop in a waveguide. Those resonances have frequencies defined by the speed of sound and boundary conditions. If the waveguide has a length $l_x$, the successive axial acoustic modes have frequencies of $f = \frac{nc}{2l_x}$. This stands as long as no flow is passing through the waveguide. But what happen to the acoustic frequencies when flow is introduced in the geometry?
Prediction of Acoustic Resonances in Core Volumes

Figure 6.11. The flanged and unflanged additional study

(a) Evolution of frequency and damping for the first eight modes and for the three cylinders

(b) The end correction (ec), theory and prediction
6.2. IMPACT OF THE DOPPLER EFFECT ON ACOUSTIC RESONANCES

6.2.1 Theory

To better understand the phenomenon, the fundamental axial acoustic mode is decomposed into its forward and backward propagating waves. Without flow, the time the wave would take to go back and forth in the waveguide is:

\[ T = \frac{2l_x}{c} \]  
\[ \text{(6.11)} \]

And the resulting frequency is:

\[ f = \frac{c}{2l_x} \]  
\[ \text{(6.12)} \]

If the fluid has now an axial velocity \( u \). The time for the wave to go back and forth becomes:

\[ T' = \frac{l_x}{c - u} + \frac{l_x}{c + u} = \frac{2l_x}{c(1 - M^2)} \]  
\[ \text{(6.13)} \]

Where \( M \) is the Mach number. The resulting frequency becomes:

\[ f' = f(1 - M^2) \]  
\[ \text{(6.14)} \]

The frequency therefore quadratically decreases with the Mach number until the flow is sonic. Then no acoustic resonances can develop as no backward waves will be able to propagate.

In order to test this theory and the ARF method, a simple geometry of two coaxial cylinders, will be studied.

6.2.2 The ARF method

The two coaxial cylinders, supposed infinitely rigid, consist of an outer cylinder whose length is equal to 356 mm and diameter equal to 153 mm, as well as an inner cylinder whose length is equal to 254 mm and diameter equal to 89 mm. The inner cylinder is modelled with a circular leading edge. The mesh is composed of about 50 000 nodes and is presented in Fig.6.12.
Table 6.5. Line-fit frequency and damping, compared with theory, for the set of coaxial cylinders.
6.2. IMPACT OF THE DOPPLER EFFECT ON ACOUSTIC RESONANCES

6.2.2.1 No flow study

First, the study with no general flow clearly shows three acoustic modes within the 1-1000 Hz frequency window. They are highlighted in the Bode plot Fig.6.13a. The node maps, represented in Fig.6.13b to Fig.6.13d, show that these three modes correspond to the first two axial modes of the outer cylinder and the first axial mode of the inner cylinder. The line-fit table, in Table.6.5, summarises the frequency and damping values for the three modes. With a time-step of $dt = 1.10^{-5}\text{s}$, the frequency has an error of about 5% and the damping is kept small due to fully reflective boundaries.
Figure 6.13. The averaged response function plot and the acoustic node maps for the coaxial cylinders.
6.2. IMPACT OF THE DOPPLER EFFECT ON ACOUSTIC RESONANCES

6.2.2.2 Flow study

Flow is now introduced in the geometry by imposing an inlet velocity of 10 to 250 $m.s^{-1}$, while atmospheric pressure and temperature are maintained at the boundaries. The process takes two steps.

First, a steady state is computed by imposing appropriate boundary conditions. An example of the steady state for an inlet velocity of 100 $m.s^{-1}$ is presented in Fig.6.14. The corresponding plot, representing the local Mach number, is typical of inviscid flow computations.

Second, an unsteady computation is carried out to excite the different acoustic modes. Fig.6.15a shows the evolution of the averaged response function with inlet velocity. As seen previously, the averaged response function with no flow, here represented in purple, clearly identifies the first three fundamental modes. But as soon as flow is introduced, modes are damped as part of the energy leaves the system with the flow and only part of the acoustic wave is reflected back in the domain. This goes in hand with a gradual shift in frequency.
(a) The evolution of the averaged response function with increasing inlet velocity

(b) The frequency against inlet Mach number plot

Figure 6.15. The evolution of the averaged response function and acoustic frequencies with a uniform flow
6.3. CONCLUSION

The frequency shift comes from the Doppler effect introduced earlier. Plotting the evolution of the modes frequencies against the inlet Mach number, as in Fig.6.15b, clearly confirms that the frequency follows the theoretical \( f' = f(1 - M^2) \) trend. It is also possible to note that a side effect of this frequency shift is the general increase of the response functions levels with the flow. This is explained by the fact that more and more modes are excited within the 1 to 1000 Hz frequency window. In the end, the acoustic modes are closer and closer together and more and more damped, making it difficult to dissociate them for higher Mach number flows.

6.3 Conclusion

The end correction and the Doppler phenomenon are two well known applications of sound and acoustic resonances. They are possibly the first examples of acoustic effects known to wider audiences. Applying the methods implemented for their prediction has allowed a better insight into their theory and applications. Table 6.6 sums up the results obtained throughout this chapter. It confirms what was observed in the previous chapters on enclosures. The accuracy corresponds to the frequency error when compared to theory.

The ARF method is very reliable and consistent. Input parameters directly reflect the speed and accuracy of the results. The frequency range limits the number of acoustic modes to be captured and defines which timestep to use. Indeed, the timestep is directly linked to the frequency accuracy (about 6% error with \( dt = 1.10^{-5}\text{s} \), about 1% error with \( dt = 1.10^{-6}\text{s} \)). The response function resolution and the number of iterations defines the achievable accuracy in damping. In hindsight, the flanged and unflanged geometries were over discretized and we could have used coarser meshes. When compared with the Arnoldi method, the ARF method never-
theless remains a computationally intensive method.

The Arnoldi method predicts a large number of modes with very good accuracy. As in enclosures, it is mainly the mesh refinement that will influence the frequency and damping accuracies. The accuracy stated in brackets in Table.6.6 corresponds to the average error in frequency for the first eight axial modes, that is to say where the mesh is refined enough. We observe that the average error in frequency is then less than 1% which is very good when considering the speed of the computation. Therefore, the Arnoldi method also seems to perform well for open geometries. Its application to flow systems has been left to the following chapter, which will conclude the methods validation by studying a modelled roll-post system.
### Table 6.6. Recapitulative comparison of the two acoustic methods

<table>
<thead>
<tr>
<th>Method</th>
<th>End correction</th>
<th>Coaxial cylinders</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Modes</td>
<td>Accuracy</td>
</tr>
<tr>
<td>ARF</td>
<td>s : 2/2</td>
<td>s : 5%/0.3%</td>
</tr>
<tr>
<td></td>
<td>f : 2/2</td>
<td>f : 8%/0.9%</td>
</tr>
<tr>
<td></td>
<td>u : 2/2</td>
<td>u : 7%/1.0%</td>
</tr>
<tr>
<td>Arnoldi</td>
<td>s : 23</td>
<td>s : 2.6% (0.3%)</td>
</tr>
<tr>
<td></td>
<td>f : 15</td>
<td>f : 0.9% (0.6%)</td>
</tr>
<tr>
<td></td>
<td>u : 15</td>
<td>u : 2.9% (1.0%)</td>
</tr>
</tbody>
</table>

$s$: Semi-open; $f$: Flanged; $u$: Unflanged
Chapter 7

The lock-in phenomenon

This chapter considers a modelled roll-post system. It is of interest as it has previously been studied both analytically and experimentally and can therefore be used as a reference for modelling validation. The two acoustic methods implemented are used to characterise the acoustic behaviour of the system when it is subject to flow. The results presented here will draw largely on simpler systems studied in previous chapters.

The modelled roll-post system consists of two coaxial cylinders with symmetric sidebranches. Experiments and analytical studies have shown that such a system could be prone to dramatic pressure oscillations due to the interaction of the flow with the acoustic characteristics of the branches. This phenomenon is known as the lock-in phenomenon.

The chapter’s layout will therefore be as follows: first the theory of the lock-in phenomenon will be presented; then the ARF and Arnoldi methods will be used
7.1 Theory of the lock-in phenomenon

In this section, the system studied and the theory of the acoustic phenomenon are introduced in more detail. We will see that the lock-in instability results from the vortex shedding at the branches’ mouth, a well-known viscous effect. We will therefore present how the Euler solvers implemented can model such viscous properties of the fluid.

7.1.1 Presentation of the system

The roll-posts system considered reproduces the experiment carried out by Bravo and his coworkers [26]. The model is inspired by systems used in vertical take-off and landing aircrafts to ensure the roll control during the critical take-off and landing procedures. As reminded by Bravo et al. [26], the main parameters of...
the prototype, i.e. lengths and diameters, differ from the original geometry and therefore the acoustic response of the original system will differ from the results in their paper and this thesis.

The system, presented in Fig.7.1, consists of two coaxial cylinders with symmetric sidebranches. The diameters of the outer cylinder, inner cylinder and sidebranches are respectively equal to $D_o = 153\, mm$, $D_i = 89\, mm$ and $D_b = 38\, mm$, while their lengths are equal to $L_o = 356\, mm$, $L_i = 254\, mm$ and $L_b = 270\, mm$. Finally, the inner cylinder leading edge is located 102mm from the outer cylinder front end and the sidebranches are located midway of the outer cylinder.

### 7.1.2 Sidebranches systems: the lock-in phenomenon

In the literature, systems with twin sidebranches have shown to be prone to acoustic resonances (Section 2.1.2). Indeed, vortices are shed periodically at the upstream edge of the sidebranches. They then impact the downstream edge of the mouth. If the shedding frequency approaches an acoustic frequency, then large pressure oscillations will develop. A feedback mechanism appears between the flow and the acoustic field, resulting in a self sustained phenomenon: the lock-in phenomenon.

The acoustic modeshapes of coaxial sidebranches of length $L_b$, symmetric with respect of a duct of diameter $D_o$, are defined by a quarter of the wavelength (Fig.7.2), and their frequencies are therefore equal to [24]:

$$f_2n-1 = \frac{(2n - 1)c}{4 \left(L_b + \frac{D_o}{2}\right)}$$  \hspace{1cm} (7.1)

where $c$ is the speed of sound. These modes correspond to anti-symmetric modes where a pressure node is located in the main duct. This results in very low acoustic...
radiation. Symmetric modes will be able to exist once the diameter of the axial duct is longer than half the modes wavelengths. The corresponding frequencies are then defined by:

\[ f_{2n} = \frac{(2n)c}{4 \left( L_b + \frac{D_o}{2} \right)} \quad \text{if} \quad L_b \leq D_o \left( n - \frac{1}{2} \right) \]  

(7.2)

In the end, for coaxial sidebranches, the acoustic modeshapes and frequencies are similar to the modes of a closed cylinder of length \( 2L_b + D_o \).

In terms of damping, single branch systems are often not of great concern, as a large amount of the acoustic energy will be radiated to the main duct and dissipated either by friction or by the flow. But problems occur when an acoustically closed system develops between 2 or more sidebranches. Acoustic losses are then greatly mitigated by the branch interactions. Coaxial sidebranches have actually been seen to be the most problematic system, as the acoustic field has very small interactions.
7.1. THEORY OF THE LOCK-IN PHENOMENON

with the main duct flow. This causes low damping modes and high pressure oscillations.

As an example, Bravo and his team [26] managed to successively excite the first four acoustic modes of the coupled side-branches by spanning the range of inlet velocity between 10 and 120 $m.s^{-1}$. The result shows a good agreement between the computation and the experiment, and a lock-in phenomenon is clearly identified, where vortex shedding at the mouth of the cavities excites the acoustic response of the sidebranches (Fig.7.3a). The range of Strouhal number observed, plotted in Fig.7.3b, is contained between $St = 0.27$ and $St = 0.55$. This range was previously observed in the literature and seems to be dictated by the static pressure in the main duct [24].

7.1.3 Euler solvers and vortex shedding modelling

Euler solvers were used for both acoustic methods implemented since, as seen in Section.1.2.5, acoustic phenomena are largely inviscid. Numerous numerical studies have been carried out on flows over cavities. Computationally intensive direct numerical methods have been used [160, 161, 162], but Euler solvers also proved successful in modelling the self-sustained instability [163, 164]. Indeed, though vortex shedding originates from viscous effects, the stability analysis for vortex shedding is based on inviscid arguments.

To better understand the concept of vortex shedding, we will consider flow around airfoils [165]. The phenomenon results from the generation of vorticity within the boundary layer. When an airfoil is moved from rest, the trailing edge stagnation point is originally on the suction side and a starting vortex forms around the trailing
Figure 7.3. The modelled roll-posts lock-in phenomenon [26]
edge due to large velocity gradients. Considering that, under Kelvin’s theorem, the circulation around a closed curve in an inviscid flow is constant, the starting vortex generation results in the creation of another vortex of same strength but opposite sign around the airfoil. The second vortex pushes the stagnation point towards the trailing edge. Once the value of the circulation locates the stagnation point at the trailing edge of the airfoil, the vortex separates. This is known as the Kutta condition [156], which corresponds to a minimum of the kinetic energy of the fluid around the airfoil.

With Euler solvers, the starting vortex cannot be created by viscosity, and there should be an alternative numerical mechanism for the generation of vorticity. It is the internal dissipation, inherent to any Euler solver, that provides that mechanism [156]. Once the starting vortex has been created, it defines a discontinuity also called vortex sheet. This discontinuity is a weak solution of the Euler equations [156], which satisfies the entropy condition. As a result, vortex shedding phenomena can be accurately captured by classic Euler solvers.

7.2 No flow study

The roll-posts system, as represented in Fig. 7.5a, is first studied at atmospheric conditions and with no flow. The acoustic modes of the complete system will be compared to both theoretical estimates and simpler systems results.

7.2.1 Introduction

The systems presented in previous chapters for the validation of the two methods correspond to subsystems of the complete roll-post system. The three subsystems
7.2.2 The set up

7.2.2.1 The ARF method

The set up of the ARF method computation is as follows. The mesh of the geometry is represented in Fig.7.5b. It is an unstructured mesh with tetrahedral cells and about 60,000 nodes. Symmetry is used in order to save computational time by only studying a quarter of the full geometry. The initial steady state is composed of a fluid at atmospheric conditions with no flow. The excitation is made through the excitation of the outer wall of the inner coaxial cylinder, and the frequency of the
7.2. NO FLOW STUDY

Figure 7.5. The roll-posts geometry and meshes

(a) The roll-posts full geometry

(b) The roll-posts ARF mesh, 60 000 nodes

(c) The roll-posts Arnoldi mesh, 110 000 nodes
excitation ranged from 1 to 1000Hz. The time-step used is \( dt = 1.10^{-5}s \) and the
number of iterations \( N \) is equal to 65 536. The resulting time of computation is
equal to 5270 cpuh on 16 nodes corresponding to the best parallel efficiency of the
code.

### 7.2.3 The Arnoldi method

Because the Arnoldi method is more computationally efficient, the full geometry is
modelled. This gives the ability to capture the circumferential acoustic modes as
well as the axial modes. The system is discretized with a fully unstructured mesh
(Fig.7.5c) using tetrahedral cells. The mesh contains about 110 000 nodes. In order
to capture the fundamental modes of the system, the mesh is made quite coarse for
the coaxial cylinders and more refined in the sidebranches. The time-step is equal
to \( dt = 4.10^{-7}s \), the number of iterations per vectors is \( N = 2500 \), and the number
of Arnoldi vectors is \( m = 200 \). The resulting computational time is 480 cpuh or so.

### 7.2.4 The roll-posts and its subsystems

The raw results for the ARF method and Arnoldi method will first be presented.
We will then compare the results with both experiment and theory. The question
we will ask in this study is whether a complex system can be decomposed in simpler
subsystems for faster computations or whether it should be considered in its entirety
for accuracy.

#### 7.2.4.1 The ARF method

After postprocessing the ARF computations, the averaged response function and
the MMIF, represented in Fig.7.6a and Fig.7.6b, clearly show the excitation of five
Figure 7.6. The ARF set of results: the averaged response function, the multivariate mode indicator function and the acoustic node maps.
The corresponding frequencies and damping values have been calculated using the line-fit method. They are shown in Table 7.1. Damping values are seen to be small for both axial and sidebranches modes. This is explained by the use of reflective boundaries at the inlet and outlet of the system. The corresponding acoustic nodes maps, shown in Fig. 7.6c to Fig. 7.6g, identify the modeshapes excited. They present a combination of axial and sidebranches modes.

### 7.2.4.2 The Arnoldi method

After the Arnoldi computation, Fig. 7.7 shows that all eigenvalues are within the unit circle which means that no unstable modes have been identified. Nevertheless, it is possible to see that the eigenvalues are spread within the complex domain with some heavily damped modes close to the origin, and lightly damped modes close to the unit circle.

From the 100 possible acoustic modes, 65 have converged. This is in agreement with previous convergence ratios for core geometries. The frequencies and damping estimates for the converged modes are represented in Fig. 7.8a. The data used

<table>
<thead>
<tr>
<th>Mode</th>
<th>Predicted frequency (Hz)</th>
<th>Damping ratio (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mode 1</td>
<td>269.0</td>
<td>7.5.10^{-3}</td>
</tr>
<tr>
<td>Mode 2</td>
<td>469.0</td>
<td>3.6.10^{-4}</td>
</tr>
<tr>
<td>Mode 3</td>
<td>590.3</td>
<td>2.0.10^{-3}</td>
</tr>
<tr>
<td>Mode 4</td>
<td>862.6</td>
<td>1.0.10^{-2}</td>
</tr>
<tr>
<td>Mode 5</td>
<td>893.0</td>
<td>6.3.10^{-3}</td>
</tr>
</tbody>
</table>

Table 7.1. Line-fit frequency and damping for the roll-posts system.
to generate the chart can be found in Table.D.7. We find that the lowest damping modes are also the lowest frequency modes. We also find that the estimated frequencies are in agreement with the theoretical frequencies, which are calculated by assuming that the sidebranch, the outer cylinder and the inner cylinder are similar to simple cylinders. End corrections have been taken into account for the branch and the inner cylinder.

The modeshapes corresponding to converged modes are represented in Fig.7.8b and Fig.D.1b, for respectively the lowest and highest frequency modes. They are clearly defined and show that all the modes have properly converged. The modeshapes represent complex modes composed of one or more of the subsystems. In other words, the acoustic characteristics of the individual subsystems will interact with each other and impact on frequencies, damping estimates and modeshapes.
Figure 7.8. Acoustic prediction of the roll-posts system using the Arnoldi method
Figure 7.8. Acoustic prediction of the roll-posts system using the Arnoldi method
Now that the ARF and Arnoldi results have been computed, we can compare the acoustic modes predicted with experimental results thanks to Bravo’s study [26]. Table 7.2 summarises the acoustic frequencies and compares them with analytical estimates. The analytical frequencies are split into three categories relating to the frequencies defined by: a) the theory of sidebranches ((7.1) and (7.2) as well as Fig.7.4a), b) the single branch frequencies ((Fig.7.4b)) and c) the coaxial frequencies (Fig.7.4c). The modes are named after the number of nodes for respectively a single sidebranch, the outer cylinder and the inner cylinder. The note “as” stands for anti-symmetric. Similarly, the note “s” refers to symmetric modes. The theoretical frequencies are calculated by considering simple circular cylinders. Where needed, an end correction is applied.

**Mode 1/0/0 s**

The mode corresponds to the fundamental single branch mode and therefore was not mentioned by Ziada and Buhlmann [24]. The mode is symmetric. The Arnoldi method predicts the frequency with 4.4% error when compared to the single branch analytical frequency. This is reasonable. The ARF method underpredicts the frequency by about 9%. Different parameters can explain the difference. First, The ARF computation is using a timestep equal to $10^{-5}$ s which has been seen to introduce a 5% to 7% frequency lag. Second, only a quarter of the geometry has been modelled in order to save computational time. As a result, it is possible that the symmetry imposed also impacts on the frequency. To solve the argument, computation of the full geometry with a smaller timestep would be useful.

**Mode 1/0/0 as**
Table 7.2. Estimated frequencies of the acoustic modes predicted with the ARF and Arnoldi methods. Mode defined as branch/outer cylinder/inner cylinder. The error in brackets corresponds to the error against analytical frequencies of a single branch or the coaxial cylinders.
The mode is the fundamental acoustic mode of the coaxial sidebranches system. The Arnoldi computation, which captures the mode with an error in frequency of about 0.3% also confirms the Bravo experiment. The fact that both the experiment and Arnoldi overpredicts Ziada and Buhlmann’s estimate tends to show that the presence of the inner cylinder prevents the coaxial interaction from fully developing and favours single sidebranch modes. The ARF method is not able to predict anti-symmetric modes as the symmetry condition imposes an acoustic velocity node at the center of the duct.

**Mode 1/1/0 s**

The mode shows a combination of sidebranch and axial cylinder modes. This makes it interesting as it should not be allowed to develop. Indeed, its half wavelength is equal to 1.1 times the axial cylinder’s diameter [24]. The mode is therefore only observed because its frequency is very close to the frequency of the axial cylinder’s fundamental mode. The ARF method and the Arnoldi method both predict the mode with a respectively 4.4% and 1% error when compared to the analytical frequency of the full branch system (Ziada’s estimate). This is in agreement with the set of input parameters used.

**Modes 0/0/1 and 0/2/0**

These are the two following axial modes. They are predicted with both the ARF and the Arnoldi method. They are not present in the experimental study since, as seen in Section 6.2, axial modes will be heavily damped as soon as flow is introduced in the system. The respective frequency errors are equal to 6.9% and 2.6% for the fundamental inner cylinder mode, and 6.6% and 0.2% for the second mode
of the outer cylinder. This agrees with the previous results using the same set of parameters.

**Modes 2/0/0 as, 2/0/0 s, 3/0/0 as, 3/0/0 s**

These modes correspond to the higher frequency sidebranch modes. On one hand, the ARF method only predicts mode (2/0/0s) due to symmetry conditions and a limited excitation frequency range. The error in frequency when compared to the single branch theory is equal to 3% which is similar to the accuracy achieved in previous chapters. On the other hand, the Arnoldi method predicts all four modes and agrees with both experiment and theory with averaged error against theoretical frequency of 3%.

It is also interesting to see that both the ARF and Arnoldi results, as well as the experiments, show that the sidebranche mode frequencies, recapitulated in Fig.7.9, are governed by the single branch length more than by the coupling between side-branches. This is due to the presence of the inner axial cylinder which attenuates the interaction.

As a conclusion, the ARF and Arnoldi methods show the same level of accuracy observed in previous chapters. But the Arnoldi method is seen to predict more modes with a better accuracy and less computational time. It therefore seems to be the most efficient method to quickly capture a complete acoustic map of a system.

This study also shows that looking at subsystems is useful in order to determine which frequency range needs to be focused on. But ultimately, it also highlights
that the system needs to be considered in its entirety in order to take into account acoustic mode interactions. The latter will have an impact on either the acoustic frequencies and damping estimates or the very existence of a mode.

7.3 Flow study

7.3.1 The set up

Bravo et al. showed in their paper [26] that the roll-posts system could potentially suffer from a lock-in phenomenon when vortex shedding at the mouth of the side-branches excites the cavities fundamental modes. They observed that an increase of the roll-posts inlet velocity causes the excitation of successive modes. We will now reproduce their experiment with both the ARF and Arnoldi methods by imposing an inlet velocity between 10 and 100\(m.s^{-1}\). Pressure and density at the boundaries are maintained at sea static levels. An example of the resulting steady state used for the

---

**Figure 7.9.** Comparison of ARF and Arnoldi frequencies with analytical frequencies and experiments
7.3. FLOW STUDY

ARF and Arnoldi methods are represented in respectively Fig.7.10a and Fig.7.10b for an inlet velocity of $50\text{m.s}^{-1}$. These plots show that the steady states are indeed very similar.

7.3.2 The ARF method

Regarding the ARF method, the fluid in the system is excited by vibration of the wall of the inner cylinder and the corresponding averaged response functions are computed. In order to better represent the evolution of the acoustic field with the inlet velocity, the Bode representation has been replaced by a 2D acoustic map highlighting the evolution of the averaged response function with inlet velocity. It is presented, alongside a similar plot created by Bravo et al. [26], in Fig.7.11. Comparing the two plots will give instructive informations on the ARF method abilities.

But before we compare the results, it is necessary to point out how the ARF method process differs from Bravo’s experiment. In Bravo’s experiment, the flow in the geometry is increased in steps. At each step, a limit cycle is reached where pressure oscillations are observed at the end of the sidebranches. Therefore, at each speed, there is only one frequency of excitation corresponding to the vortex shedding frequency. The ARF process differs slightly since an excitation frequency is added over the vortex shedding frequency in order to span a larger frequency range.

Once this is remembered, the comparison between the ARF results and the experiment highlights five important points:

- The lock-in resonances levels are much higher than the ones observed in the no-flow study. Here, the modulus of the averaged response function
Figure 7.10. Steady flow computation of full roll-posts system with inlet velocity of 50 m/s⁻¹
7.3. FLOW STUDY

(a) The acoustic map using the ARF method

(b) The acoustic map of the modelled roll-posts system [26]

Figure 7.11. The evolution of acoustic resonances with inlet velocity both with the ARF method and the work by Bravo
reaches about 170dB for the first mode, when the no-flow fundamental mode peaked at 120dB. This implies that another source of excitation is added to the controlled chirp excitation by the vortex shedding.

- The axial modes present in the no-flow study quickly disappear when flow is introduced in the geometry. This is due to the increased damping of those modes as part of their energy is dissipated in the flow. The axial modes are then quickly overpowered by the lock-in phenomenon.

- The geometry used for the ARF study takes advantage of symmetry properties in order to reduce computational time. But these added conditions imposed an acoustic velocity node at the center of the duct. Therefore, while Bravo observes three modes within the 1-1000Hz frequency window, we only capture two. They are the fundamental side-branches modes, symmetric in pressure, shown earlier in Fig.7.6c and Fig.7.6f.

- In Bravo’s paper, the first mode is excited at an inlet velocity of about 20\( m.s^{-1} \), while in our case, it is only fully developed at a velocity of about 30\( m.s^{-1} \). We believe the difference is due to the chirp excitation interfering with the vortex shedding. It is likely the mode would be observed at 20\( m.s^{-1} \) with the ARF method if it had been excited long enough.

- In Bravo’s paper the first acoustic mode of the sidebranch stops at a velocity of 50\( m.s^{-1} \) to let the second mode develop. As we do not represent the anti-symmetric modes in our study, the fundamental mode stays excited until 80\( m.s^{-1} \) when the second symmetric mode takes over. This also results from the chirp excitation used at each inlet velocity considered, exciting the fluid within the 1-1000Hz frequency range.

Now that we have looked at the ARF results, we will consider the Arnoldi method.
7.3.3 The Arnoldi method

With the Arnoldi method, the evolution of the computed spectrum with the mass-flow is plotted in Fig.7.12. The figure shows that the spectrum stays stable but for a small number of modes that are seen to become unstable. Looking back at previous results with regard to the Doppler effect and the lock-in phenomenon, we would expect to find that the axial modes will get damped while the sidebranches modes will successively become unstable. We therefore plot the evolution in frequency and damping of the acoustic modes included in the 1 to 2 000Hz frequency window. The two plots, Fig.7.13a and Fig.7.13b, represent the evolution of the pure sidebranch modes in colors and the axial modes, that is, the modes of the coaxial cylinders, in grey. The sidebranch modes are defined in both graphs by their number of acoustic nodes and their symmetry properties.
Fig.7.13a shows that the acoustic frequencies of the axial modes decrease when the inlet Mach number increases, while the frequencies of the sidebranch modes stay unperturbed as they are not concerned by the flow. This is in agreement with the Doppler theory highlighted in Section.6.2.
7.3. FLOW STUDY

Figure 7.13. The Arnoldi study of roll-posts system with flow.
The most striking observations nevertheless concern the damping estimates presented in Fig.7.13b. The plot shows that the sidebranch modes are the least stable modes of the system and highlights five different steps:

- **Between 0 and 20\(m.s^{-1}\), all acoustic modes are stable and therefore no acoustic resonance should develop.**
- **Between 20 and 50\(m.s^{-1}\), the symmetric and anti-symmetric fundamental sidebranch modes, represented in blue, become unstable, causing large pressure oscillations as observed by Bravo (Fig.7.11b). It is believed that Bravo only captures the anti-symmetric mode as the two fundamental modes are very close in frequency but the anti-symmetric mode has a node at the center of the duct resulting in lower radiation losses.**
- **At 50\(m.s^{-1}\), Bravo observes a switch from the first anti-symmetric to the second anti-symmetric mode. Sadly, in the runs computed with the Arnoldi method, the latter has not converged enough to be identified. This shows a limitation of the Arnoldi method and further computations would be needed to see if the mode could be predicted.**
- **Between 50 and 80\(m.s^{-1}\), the acoustic pattern changes with an increase in the damping of the sidebranch symmetric fundamental mode and the increasing instability of the second axial sidebranch modes, both symmetric and anti-symmetric. This is in agreement with Bravo’s experiment which sees the excitation of the second sidebranch mode within the same inlet velocity range.**
- **At 80\(m.s^{-1}\), Bravo witnesses the development of the second symmetric mode. The Arnoldi method confirms that, at that speed, the second symmetric mode is more unstable than the second anti-symmetric mode.**
Figure 7.13. The Arnoldi study of roll-posts system with flow
These successive changes in damping estimates are an alternative way to represent the lock-in phenomenon, and we have seen that most of the acoustic patterns, within the $10$ to $100 m.s^{-1}$ velocity range, were captured. Looking back at the system’s spectra, Fig. 7.12 shows that the evolution of the modes could actually be observed, as the same input parameters have been used throughout the study and because the frequencies of the sidebranches modes are unperturbed by the flow. The three fundamental sidebranch modes are located around respectively the $(-0.25,0.96)$, $(0.7,0.6)$ and $(-0.95,0.2)$ eigenvalues.

This Arnoldi study therefore presents features confirming the experiment. It also shows two current limitations of the code implemented. First, important modes are sometimes not predicted as they have not converged enough. Second, we have seen throughout this thesis that the damping estimates will be highly dependent on the mesh and on the convergence of the mode. In such complex studies, this damping sensitivity might result in approximate or erroneous results.

### 7.4 Conclusion

The study of the modelled roll-post system presented in this chapter concludes the thesis by comparing the ARF and Arnoldi results with theory and experimental data. The higher complexity of the system and the study of flow applications further challenged the methods, increasing our understanding of their behaviours.

The ARF method has been consistent throughout the thesis and this chapter confirms its qualities as well as its limitations. The method is reliant and consistent. A set of input parameters will correspond to a constant error margin in terms of
frequency and damping. The method is computationally intensive but it is able to predict vortex shedding and the lock-in phenomenon accurately.

The Arnoldi method is fast and efficient, predicting a large number of acoustic modes in a very short computational time when no flow is considered. Nevertheless, the flow applications showed that important modes could be missed and damping estimates are sometimes not reliable enough to have a clean understanding of the phenomenon. It is thought that these two problems could be solved by implementing a higher order solver.
Chapter 8

Conclusions and recommendations

The objective of the thesis was to develop a computational method able to predict acoustic resonances in core volumes. Though the occurrence of acoustic resonances can be found in many applications, a particular emphasis has been put on turbomachinery systems for two specific reasons. On one hand, spurious acoustic resonances are source of unwanted noise that every turbomachine manufacturer wishes to eliminate. On the other hand, acoustic resonances have also been found to be the source of spurious and sometimes dangerous vibrations, leading to structural failure of key engine components.

Throughout the thesis, two acoustic methods, known as the ARF method and the Arnoldi method have been put to the test. While the concept of the methods were already known, numerous improvements have been developed in order to make each method more accurate, reliable and efficient. Both methods are now able to predict the three components of resonances: the frequency, damping and modeshape, with great accuracy.
To the author’s knowledge, this is the first time that two acoustic prediction methods have been implemented, validated and compared on a wide range of testcases comprising enclosures, open systems and flow systems. The results are a deeper knowledge of acoustic phenomena and a better understanding of the methods’ origins, implementations and abilities.

We will now summarize the computational methods’ principles, results and recommended uses. The different points highlighted are also presented in Table.8.1. We will then consider possible research developments.

8.1 The ARF method

8.1.1 Implementation

The ARF method, presented in Chapter 3, is a combined time and frequency domain approach where a Favre averaged Navier-Stokes solver is used along with traditional modal analysis tools. The method has been tested on a closed circular cylinder (Chapter 3), a semi-open cylinder, a set of coaxial cylinders (Chapter 6) and finally a modelled roll-post geometry (Chapter 7).

8.1.2 Results

For all four testcases enumerated above, the ARF method has given reliable and very consistent results when compared to theory. Modes were predicted with good accuracy in frequency and very low damping throughout the thesis. Furthermore, complex flow interactions, such as the lock-in phenomenon, were captured.

The most important parameter, when considering the accuracy of the result, is the
time-step $dt$. If $dt$ is set so that the highest frequency mode is represented by 100 computational steps, then we can expect the error in frequency to be approximately 5%. If the timestep is set so that 1000 time iterations represent the highest frequency mode, then the predicted frequencies were found to be within 0.5% of theory. In further studies, the author would recommend the use of 200 steps to represent the highest excited modes.

$d t$ is the main input parameter, as it defines how accurate the frequency will be, but it also impacts on the resolution of the response function. Indeed, the resolution is equal to the inverse of the product $N dt$, where $N$ is the number of iterations. This highlights the compromise that has to be struck. On one hand, an accurate study requires a very small time step $dt$ which has to be compensated by a large number of iterations in order to be able to extract any valuable information from the time histories. On the other hand, a larger timestep would allow the use of a smaller $N$ for an equivalent response function resolution.

8.1.3 Recommended use

The above considerations imply that the method can be computationally intensive in terms of speed and memory. They impose how the ARF method should be used:

- The ARF method has been seen to behave very well even on coarser meshes. Therefore, it could be used at the design stage for a fast general acoustic study. Indeed, coarse meshes and large timestep, i.e. small number of iterations, would result in quick computations to generate the acoustic map of the system. At such stage, pinpoint accuracy in frequency is not essential.
• The ARF has also been seen to model complex phenomena such as the end correction, the Doppler effect and the lock-in phenomenon very accurately. It could be used on a small frequency range with a more refined mesh and small time-step in order to better understand and model the development of one specific and potentially dangerous mode. Because it uses the full Euler equations, it can also be used to directly study flow and acoustic interactions.

8.2 The Arnoldi method

8.2.1 Implementation

The Arnoldi method combines a CFD solver with an eigenvalue extraction method (Chapter 4). It reduces the system to a stability problem defined by the matrix $A$:

$$Ax = \lambda x$$

Because we are studying fluid applications, the eigenpairs of the matrix $A$ correspond to the acoustic modes of the system.

The method’s improvements include the direct computation of frequencies and dampings, the definition of a convergence criterion differentiating between clean acoustic modes and spurious numerical modes, and last but not least, the use of an unstructured solver able to model complex geometries.

8.2.2 Results

The method has been tested on simple systems such as enclosed and semi-open cylinders as well as more complex geometries like the roll-posts. These testcases
showed the potential as well as the current limitations of the method.

Concerning simple systems, the Arnoldi method showed, in Chapter 4 and Chapter 5, very interesting results, giving a high number of converged acoustic modes with very good accuracy in frequency (within 0.5% of theory) and low damping (typically $1 \times 10^{-3}$%). The algorithm is very quick and computationally efficient when compared with the ARF method. This performance is all the more impressive that the method predicted modes over a wide range of frequencies with no important influence from the time step used. In these cases, the limitation was actually identified to be the mesh refinement (Chapter 5). It is recommended that the mesh contains 10 nodes to represent the wavelength of the highest frequency mode of interest.

The Arnoldi method also held its own when studying more complex geometries. It accurately captured the end corrections and the lock-in phenomenon. But flow applications also showed some current limitations as convergence and damping accuracy were not consistently achieved.

### 8.2.3 Recommended use

Today, the Arnoldi method has shown a great potential in identifying a large number of acoustic modes in a very short time. The method is indeed able to span a wide range of frequencies, and give very accurate results in frequency, damping and mode shape. Because it has proved faster than the ARF method, the ideal use of the Arnoldi method would therefore be to predict the general acoustic map of a specific system at the design stage. Applications with no-flow conditions have given entire satisfaction but we have seen that special care should be taken if flow is introduced in the system.
<table>
<thead>
<tr>
<th>Method</th>
<th>Qualities</th>
<th>Developments</th>
<th>Limitations</th>
<th>Recommended use</th>
</tr>
</thead>
<tbody>
<tr>
<td>ARF method</td>
<td>+ + + Consistent and accurate</td>
<td>Weighted averaged response function</td>
<td>• Coarse mesh, quick low frequency study</td>
<td>• Coarse mesh, quick low frequency study</td>
</tr>
<tr>
<td></td>
<td>+ + Easy implementation</td>
<td>Full Navier-Stokes solver</td>
<td>• Refined mesh, complex acoustic interactions</td>
<td>• Refined mesh, complex acoustic interactions</td>
</tr>
<tr>
<td></td>
<td>+ + coarse mesh results</td>
<td>Hot flow studies</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Arnoldi method</td>
<td>+ + Speed</td>
<td>Boundary conditions</td>
<td>• General acoustic studies of no flow applications</td>
<td>• General acoustic studies of no flow applications</td>
</tr>
<tr>
<td></td>
<td>+ + Number of modes</td>
<td>Higher order Euler solver</td>
<td>• Convergence and missing modes</td>
<td>• Convergence and missing modes</td>
</tr>
<tr>
<td></td>
<td>+ + Accuracy</td>
<td>Latest Arnoldi algorithms</td>
<td>• Dependence to mesh refinement</td>
<td>• Dependence to mesh refinement</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Navier-Stokes solver</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 8.1. Recapitulative comparison of the two acoustic methods
8.3  Further work

To conclude, we believe the understanding of acoustic resonances would benefit from further research in three different domains: the development of the computational methods, the study of computational testcases and the experimental and analytical studies.

8.3.1  Computational methods

8.3.1.1  The ARF method

The ARF method is now mature but further work could be undertaken in order to improve the results and widen the abilities of the method:

• For studies of complex systems with highly non-uniform meshes, it would be useful to create a weighted average of the individual response functions. The applied weight would be representative of the volume cell around the corresponding transducer. In non-uniform meshes, this would prevent over-estimating the influence of one highly refined area of the mesh over a coarser area.

• Because the method is based on a Navier-Stokes solver, it would be interesting to study the impact of viscosity on the acoustic resonances in contrast with the inviscid computations carried out in this thesis. The use of the full Navier-Stokes equations could have an important impact on the damping of the modes and could also better capture the role of boundary layers for example.

• The study of hot flows would also be of interest as thermo-acoustic instabilities are one of the primary concerns in jet engines. The first step would consist of studying the propagation of acoustic waves over
a hot flow steady state. The change in temperature across the system will induce changes in frequencies, which are bound to impact on the development of acoustic modes. The second step would consist of the implementation of a combustion model in order to accurately model the interaction between combustion and acoustics.

8.3.1.2 The Arnoldi method

Further developments of the Arnoldi method could also resolve its remaining limitations:

- Boundary conditions should be further improved. So far we have implemented Dirichlet conditions for fully reflective boundaries and Neumann conditions for strictly rigid walls. The method could be developed by implementing Robin conditions in order to take into account specific boundary impedances as well as non reflective boundary conditions in order to study open systems such as jet intakes. The technique of perfectly matched layer absorbing boundaries [166, 127] has, for example, been widely used in aero-acoustics studies.

- The linearized solver could be improved to, first, attenuate its dependency to the mesh refinement, and second, improve its numerical damping characteristics. This should be achieved by increasing the order of the solver. Higher-order unstructured solver are still under development. A full review can be found in a paper by Wang [158].

- The convergence of the Arnoldi method could be enhanced by implementing the latest Arnoldi developments. The work of Sorensen [153] on an implicitly restarted Arnoldi looks particularly promising as it would allow focusing on a specific part of the spectrum.
8.3. FURTHER WORK

- A full Euler and later Navier-Stokes solver could be considered in order to model flow applications with full interactions between flow or combustion and the acoustic field.

8.3.2 Computational testcases

The testcases presented for the validation of the two methods have introduced three well known acoustic phenomena: the end-correction, the Doppler effect and the lock-in phenomenon. The developments highlighted above would widen the possibilities of studies of the acoustic resonances.

Indeed, we could consider viscous flows, turbulent flows and combustion flows. The study of Parker and Rossiter modes would be an ideal example of fluid-acoustic interactions. We would also be able to model complex thermo-acoustic instabilities as in the Rijke tube, combustion chambers or afterburners. Finally, the methods developed here could be used to study the cut-on/cut-off properties of acoustic modes in jet intakes, the interaction between acoustic modes and compressor bladerows and the role of acoustic dampers and porous walls in mitigating the resonances.

8.3.3 Experimental and analytical studies

From the simplest of geometries to complex systems, the computational studies will need to be validated against both experimental and analytical results. We nevertheless observed that, while the frequency and modeshapes have always been mentioned in previous research, the third characteristic of resonances, namely the damping, is often omitted. In computational studies, the latter is critical and therefore the work achieved in this thesis would benefit from a more thorough investigation of acoustic damping.
Finally, in order to predict all possible acoustic modes with accurate frequencies and damping, we have seen that the two methods presented here required the study of the system in its entirety. Inspired by current structural methods, the development of a method able to decompose a complex system into simple subsystems and use their acoustic characteristics to recreate the complete acoustic map, would both ease and fasten the study of complex acoustic applications.
Appendices
Appendix A

From the Navier-Stokes equations to the wave equation

In acoustics, the Navier-Stokes equations [65], ruling the behaviour of fluid, can be simplified by a set of assumptions in order to model simple systems. This appendix will describe in detail both the assumptions used and their impact on the equations. This will lead form the full Navier-Stokes equations to the very simple wave equation.

First, it was proved in Section 1.2.5 that acoustic phenomena are largely inviscid. If we also assume that heat conduction is negligible, the Navier-Stokes equations are simplified into Euler’s equations:

\[
\begin{align*}
\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) &= 0 \quad (A.1a) \\
\partial_t (\rho \mathbf{v}) + \nabla \cdot (p\mathbf{I} + \rho \mathbf{v} \times \mathbf{v}) &= \mathbf{f} \quad (A.1b)
\end{align*}
\]
The variables $\rho$, $v$, $E$ and $H$ are respectively the density, the velocity, the total energy and total enthalpy of the fluid, while $f$ corresponds to the external forces applied. The equations (A.1a), (A.1b) and (A.1c) express the conservation of respectively the mass, the momentum and the energy of the fluid. Current acoustic studies use the Euler equations, often linearised around a steady state.

Earlier acoustic studies though focused on enclosed volumes of fluid. In such systems it is possible to further simplify the Euler equations, using the following assumptions:

- No external forces are applied on the fluid. This implies that the fluid is isentropic.
- The acoustic perturbations are very small compared to the mean variable, which implies that the convective terms of the Euler equations are negligible.
- The fluid is stagnant.

The Euler equations then become:

\begin{align*}
2.2a \Rightarrow & \quad \partial_t \rho + \rho \nabla \cdot v = 0 \\
2.2b \Rightarrow & \quad \rho \partial_t v + \nabla p = 0
\end{align*}

A quantity interesting to study in such a flow is the vorticity, defined as:

$$\Omega = \nabla \times v$$
Taking the rotational of (A.2b) leads to:

\[ \partial_t \Omega = \left( -\frac{1}{\rho} \right) \nabla \times \nabla p = 0 \quad (A.4) \]

The acoustic flow is then irrotational and a velocity potential \( \phi \) can be defined as

\[ \mathbf{v} = \nabla \phi \quad (A.5) \]

The equation (A.2b) then gives:

\[ -\nabla p = \nabla \left( \rho \partial_t \phi \right) \quad (A.6) \]

This gives, after integration:

\[ p = -\rho \partial_t \phi + Ct \quad (A.7) \]

Coming back to the mass conservation (A.2a) and substituting the relationship (A.7) leads to:

\[ \rho \nabla \cdot \mathbf{v} = - \left( \frac{\partial \rho}{\partial p} \right)_s \frac{\partial p}{\partial t} = \left( \frac{\rho}{c^2} \right)_s \frac{\partial^2 \phi}{\partial t^2} \quad (A.8) \]

Substituting in (A.8) the definition of the velocity potential definition (A.5) leads to the wave equation for velocity potential:

\[ \nabla^2 \phi - \frac{1}{c^2} \cdot \frac{\partial^2 \phi}{\partial t^2} = 0 \quad (A.9) \]

Classical wave equations in pressure (A.10a) or velocity (A.10b) can be deduced using the definitions (A.7) and (A.5) respectively:

\[ \nabla^2 p - \frac{1}{c^2} \cdot \frac{\partial^2 p}{\partial t^2} = 0 \quad (A.10a) \]

\[ \nabla^2 \mathbf{v} - \frac{1}{c^2} \cdot \frac{\partial^2 \mathbf{v}}{\partial t^2} = 0 \quad (A.10b) \]
Finally, if we assume an harmonic behaviour of the variables, the pressure is then expressed as $p(x,t) = P(x)e^{j\omega t}$, the wave equation becomes the well known Helmholtz equation:

$$\nabla^2 p + \frac{\omega^2}{c^2} \cdot \frac{\partial^2 p}{\partial t^2} = 0$$  \hspace{1cm} (A.11)

Depending on the system studied, different assumptions will be made leading to the use of complex or simple equations. Early work used the Helmholtz or wave equations while recent CFD studies focus on the Euler equations.
Appendix B

The Arnoldi eigenvalue method

Eigenvalue problems, defined by the equation $A.x = \lambda.x$, can be found in a wide range of applications, from pure mathematics to structural dynamics. As soon as a system is modelled by a series of equations, it can be reduced to a matrix whose eigenpairs, including the eigenvalues and eigenvectors, will describe the system’s behaviour in a clearer basis. In this thesis, the behaviour of the system, for example, will be described by its acoustic frequencies and modeshapes.

The last decades have seen the development of two families of methods able to extract the eigenpairs from a known matrix $A$. The first methods, including for example the well known QR method, are called the direct methods. They consist in a series of matrix manipulations to express the matrix $A$ in a more suitable vector basis. The fact that they are direct means that all the eigenpairs are extracted. But because of the exponential increase in computational power in recent years, more and more complex systems have been studied, resulting in having to solve larger
and larger matrices. And such large eigen-problems reach the limit of what current direct methods can do. As a result, other methods have gathered pace. They are known as the iterative methods and consist in using advantageous convergence properties to only extract the eigenpairs of interest. The latest developments of such methods will be presented in this appendix.

### B.1 The Lanczos algorithm

Among the iterative methods, a particular method has grasped our attention and has been used in the thesis. It is known as the Arnoldi method and is the extension of the Lanczos method. While the Lanczos method is limited to symmetric real matrices, Arnoldi extends to the study of unsymmetric real matrices. This section will present the theory behind the method as well as its convergence properties. It is largely inspired by the classic book written by Golub and Van Loan [138].

#### B.1.1 The theory

Lanczos [143] implemented a method to extract the eigenvalues of a large, sparse and symmetric matrix \( A \) in \( \mathbb{R}^{n \times n} \). His aim was to iteratively build a basis of orthonormal vectors \( (q_i)_{i \in \{1, n\}} \) to transform \( A \) in a simpler tridiagonal matrix \( T \). That way, the eigenvalues of \( T \) will be easier to extract. But it was later found that the extremal eigenvalues of the intermediate matrix \( T_k \), taken at the iteration \( k \), and defined by \( Q_k^T A Q_k = T_k \), would progressively converge to the extremal eigenvalues of \( A \).

For a better insight into the mathematics, let’s consider the Rayleigh quotient defined
for $x$ a real vector non equal to zero:

$$r(x) = \frac{x^T Ax}{x^T x} \quad (B.1)$$

If the eigenvalues of $A$ are sorted decreasingly, such that $\lambda_1 \geq \cdots \geq \lambda_n$, the maximum and minimum of the Rayleigh quotient are equal to respectively $\lambda_1$ and $\lambda_n$.

The aim is to build a series of orthonormal matrices $(Q_k)$ such that $M_k$ defined as the largest eigenvalue of $Q_k^T AQ_k$ is getting closer and closer to the largest eigenvalue of $A$, that is to say $M_{k+1} > M_k$.

$$M_k = \lambda_1 \left( Q_k^T AQ_k \right) = \max_{\|y\|=1} r(Q_k y) \leq \lambda_1(A) \quad (B.2)$$

By considering the gradient of the Rayleigh coefficient, it is possible to get $M_{k+1} > M_k$ if, for a vector $u_k \in \text{span}\{q_1, \cdots, q_k\}$ such that $M_k = r(u_k)$, the vector $q_{k+1}$ is chosen such that the gradient $\nabla r(u_k)$ is within the domain defined by the basis $(q_i)_{i \in J_1, k+1K}$, or:

$$\nabla r(u_k) \in \text{span}\{q_1, \cdots, q_{k+1}\} \quad (B.3)$$

The same is true for the lowest eigenvalue of $Q_k^T AQ_k$, $m_k$. For a vector $v_k \in \text{span}\{q_1, \cdots, q_k\}$ such that $m_k = r(v_k)$, the gradient $\nabla r(v_k)$ should be within the domain defined by the basis $(q_i)_{i \in J_1, k+1K}$:

$$\nabla r(v_k) \in \text{span}\{q_1, \cdots, q_{k+1}\} \quad (B.4)$$

The gradient of the Rayleigh quotient is:

$$\nabla r(x) = \frac{2}{x^T x} (Ax - r(x)x) \quad (B.5)$$

and is therefore a linear combination of the vectors $\{x, Ax\}$. For two non identical vectors $u_k$ and $v_k$, equations (B.3) and (B.4) are true if the basis $(q_i)_{i \in J_1, kK}$ is build recursively with $q_{k+1}$ such that :

$$\text{span}\{q_1, \cdots, q_{k+1}\} = \text{span}\{q_1, \cdots, A^k q_1\} \quad (B.6)$$
The subspace \( \text{span}\{q_1, \cdots, A^k q_1\} \) is known as the Krylov subspace for the matrix \( A \) and the vector \( q_1 \).

In the end, it is possible to compute the orthonormal matrix \( Q = [q_i]_{i \in J_1, nK} \) using the definition of the symmetric tridiagonal matrix \( Q^T A Q = T \). If \((\alpha_i)_{i \in J_1, nK}\) and \((\beta_i)_{i \in J_1, n-1K}\) are respectively the diagonal and sub-diagonal elements of \( T \), then:

\[
\beta_k q_{k+1} = (A - \alpha_k I)q_k - \beta_{k-1} q_{k-1} = r_k
\]

The orthonormality of the basis \( (q_i)_{i \in J_1, nK} \) implies that \( \alpha_k = q_k^T A q_k \) and \( q_{k+1} = \frac{r_k}{\beta_k} \) with \( \beta_k = \|r_k\|_2 \). The Lanczos iterations at a stage \( k \) can then be written in a more general expression as:

\[
A Q_k = Q_k T_k + r_k e_k^T
\]

**B.1.2 The Lanczos convergence, implementation and limitations**

The convergence properties of the eigenvalues of \( T_k (\theta_i)_{i \in J_1, nK} \), also known as Ritz values, towards the eigenvalues of the matrix \( A (\alpha_i)_{i \in J_1, nK} \) were defined in two very important papers \([144, 145]\) under the name of the Kaniel-Paige theory. It states that the maximum Ritz value \( \theta_1 \), with the associated Ritz vector \( z_1 \), is bounded by:

\[
\lambda_1 \geq \theta_1 \geq \lambda_1 - \frac{(\lambda_1 - \lambda_n) \tan(\phi_1)^2}{(c_{k-1}(1 + 2\rho_1))^2}
\]

where \( \cos(\phi_1) = |q_1^T z_1| \), \( \rho_1 = (\lambda_1 - \lambda_2)/(\lambda_2 - \lambda_n) \) and \( c_{k-1} \) is the Chebyshev polynomial of degree \( k - 1 \). Because the Chebyshev polynomials are bounded by unity on the \([-1, 1]\) interval and grows rapidly outside it, Lanczos will converge better if \( \lambda_1 \) is far from the rest of the eigenvalues. This has also been proved for eigenvalues inside \( A \)'s spectrum. In other terms, Lanczos will first converge towards the eigenvalues that are the furthest apart from each other. Its convergence will therefore be improved if \( A \)'s spectrum is unclustered.
The advantage of the Lanczos method is that the matrix $A$ does not need to be expressed and the whole procedure can be achieved by only storing two $n$-vectors. Following the relationship (B.8), the original algorithm takes the form of:

**Algorithm B.1 The Lanczos Algorithm**

\[
\begin{align*}
v &= 0 \\
\beta_0 &= 0 \\
k &= 0 \\
\textbf{while } \beta_k \neq 0 \textbf{ do} \\
\quad \textbf{if } k \neq 0 \textbf{ then} \\
\quad \quad t &= w, w = v/\beta_k, v = -\beta_k t \\
\quad \textbf{end if} \\
\quad v &= v + A \cdot w \\
\quad k &= k + 1, \alpha_k = w^T v, v = v - \alpha_k w, \beta_k = \| c \|_2 \\
\textbf{end while}
\end{align*}
\]

The fact that the algorithm is used on computers introduces the problem of round-off errors. While the general expression $AQ_k = Q_k T_k + r_k c_k^T$ is satisfied to working precision that is to say that they are only dependent on the norm of $A$. The same can not be said of orthogonality of the vectors. If the computed values are denoted by a hat, such as $\hat{Q}_k$ for the matrix of Lanczos vectors and $\hat{r}_k$ for the Lanczos residual, it has been shown that the scalar product between two Lanczos vectors is approximatively equal to:

\[
|\hat{q}_{k+1}^T \hat{q}_i| \approx \frac{|\hat{r}_k^T \hat{q}_i| + u \| A \|_2}{|\beta_k|} \quad (B.10)
\]

The orthogonality is therefore highly dependent of $\beta_k$ and is threatened if $\beta_k$ is small or even is equal to zero. Because loss of orthogonality is then directly linked to the convergence of the algorithm, steps needs to be taken to reorthogonalize the vectors.
at each steps. This leads to the first practical algorithms where, after calculation, the new Lanczos vector $q_{k+1}$ is re-orthogonalized against the set of previous vectors $(q_i)_{i \in J_1, nK}$.

### B.2 The Arnoldi method

Sadly, the linearized Euler equations, considered in this thesis, do not result in a symmetric matrix $A$. It is therefore necessary to consider a non-symmetric counterpart to Lanczos iterations. The most developed and widely used method is nowadays known as the Arnoldi algorithm. It has therefore been preferred here to its competitors known as the non-symmetric Lanczos algorithm [138], Davidson’s algorithm [140] or the Jacobi-Davidson algorithm [141].

#### B.2.1 The theory

The Arnoldi algorithm has first been defined by Arnoldi himself [142] in 1951. It uses the same method than the Lanczos but the non-symmetric matrix $A$ is now converted into an Hessenberg matrix $H$:

$$ Q^T A Q = H \quad \text{(B.11)} $$

The Hessenberg matrix is defined as an upper triangular matrix with an additional subdiagonal. The orthonormal basis defining $Q$ is created iteratively by breaking down the relationship (B.11) giving:

$$ h_{k+1,k} q_{k+1} = A q_k - \sum_{i=1}^{k} h_{i,k} q_i = r_k \quad \text{(B.12)} $$

The orthonormality of the $(q_i)_{i \in J_1, nK}$ implies that the Hessenberg coefficients correspond to:

$$ h_{i,k} = q_i^T A q_k \quad \text{(B.13)} $$
And if the residual $r_k$ is not equal to zero, the new unit-2 norm vector is defined as:

$$q_{k+1} = \frac{r_k}{h_{k+1,k}}; \quad \text{with} \quad h_{k+1,k} = \| r_k \|_2 \quad (B.14)$$

In the end, the method can be expressed as previously under a matrix form:

$$AQ_k = Q_k H_k + r_k e_k^T \quad (B.15)$$

Though the Arnoldi method follows the steps of the Lanczos algorithm, it does not benefit from the same performances in terms of convergence and efficiency. Indeed the creation of the Hessenberg matrix is more computationally demanding and will require more power for larger systems. For a problem of size $n$, the number of operations involved at the $k$th-stage, is of the order of $k.n$. In addition, the starting vector will have a very important impact on the convergence of the eigenvalues. These properties lead to the development of better suited and more efficient algorithms as presented in the next section.

### B.2.2 The implementation

#### B.2.2.1 Re-orthogonalization

The first step towards a more effective algorithm is to ensure that the Arnoldi vectors are orthogonal to each other. This is done by imposing the orthogonality of the newly defined $k$th vector with the family of previous Arnoldi vectors. A well known method is called the Gramm-Schmidt modified Arnoldi method [155]. It has been used in this thesis. The resulting algorithm is given in Algorithm.B.2.

Combined with the techniques of preconditionning and restarting either explicitly [147, 148, 149] or impicitly [152, 153, 154, 167], the Arnoldi algorithm seems totally adapted to its use in CFD computations because we only need to look at a
Algorithm B.2 The Gramm-Schmidt modified Arnoldi Algorithm

\[ r_0 = q_1 \]
\[ h_{1,0} = 1.0 \]
\[ k = 0 \]

while \( h_{k+1,k} \neq 0 \) do //Loop of the number of Arnoldi vectors
\[ q_{k+1} = \frac{r_k}{h_{k+1,k}} \]
\[ k = k + 1 \]
\[ r_k = A.q_k \]

for \( i = 1 \) to \( k \) do //Building the Hessenberg matrix
\[ h_{i,k} = q_i^T.r_k \]
\[ r_k = r_k - h_{i,k}.q_i \]
end for

for \( i = 1 \) to \( k \) do //Gramm-Schmidt re-orthogonalization
\[ s_i = q_i^T.r_k \]
\[ r_k = r_k - s_i.q_i \]
end for

\[ h_{k+1,k} = \| r_k \|_2 \]
end while
small part of the spectrum and because it only defines the matrix of the differential operator implicitly. Structural dynamics computation favoured direct methods such as the QR method to iterative method because of the smaller scale of the systems. But the development of the Arnoldi method has now gone a long way. As a consequence, while earlier computational models, focusing on enclosure resonances, were using QR methods, the current state of the art, in terms of prediction of acoustic resonances, tends to use Arnoldi algorithms.
Appendix C

The unstructured second order TVD Roe scheme

C.1 The theory

The solver implemented consists of the classic fourth-order accurate Runge-Kutta scheme [65, p.460], combined with a first order Roe scheme [156, p.460] or a second order TVD Roe scheme [156, p.551].

C.1.1 The first order Roe Scheme

The first-order Roe scheme is a standard method that can be found in many textbooks. It consists in correcting the numerical fluxes between two adjacent cells by introducing a matrix $A_R$, representing the variation of the conservative variables $U$ as simple waves. The general form of the numerical flux $f_{i+1/2}^s$ between the $i$ and
\[ f_{i+1/2}^* = \frac{1}{2} (f_i + f_{i+1}) - \frac{1}{2} |A_R| (U_{i+1} - U_i) \]  
\hspace{1cm} \text{(C.1)} 

### C.1.2 The second order TVD Roe Scheme

The second order TVD solver builds up on this by introducing additional flux corrections using the information available at cells \( i-1 \) and \( i+2 \). The linearization of the Euler equations consists in discretizing the equations by considering the first order terms only. Assuming that all unknowns, such as the pressure \( p = p_0 + p' \), can be linearized, the linear fluxes are expressed as:

\[ f_{i+1/2}^{**} = \frac{1}{2} (f_i' + f_{i+1}' - \frac{1}{2} f_{i+1/2}^{Roe} + \frac{1}{2} \Psi_{i-1/2} f_{i-1/2}^{Roe*} - \frac{1}{2} \Psi_{i+3/2} f_{i+3/2}^{Roe*} \] 
\hspace{1cm} \text{(C.2)}

The \( \frac{1}{2} (f_i' + f_{i+1}') \) term corresponds to the first-order linear fluxes between two adjacent cells. \( \frac{1}{2} f_{i+1/2}^{Roe} \) corresponds to the first-order Roe scheme corrective fluxes. Finally, \( \frac{1}{2} \Psi_{i-1/2} f_{i-1/2}^{Roe*} - \frac{1}{2} \Psi_{i+3/2} f_{i+3/2}^{Roe*} \) adds TVD limiters to obtain a second-order scheme.

Extrapolating the variables at nodes \( i-1 \) and \( i+2 \), in order to have a four nodes stencil able to represent second order fluxes can be tricky on an unstructured mesh. The gradient extrapolation will not be refined enough to handle second order accuracy due to its linearity. The extrapolation is therefore done by considering the Laplacian at each nodes. The approach used is the one presented in [1]. The Laplacian \( L_j \) is averaged over the median-dual volume surrounding the node.

\[ L_i(U') = \frac{1}{N_{\text{cells}}} \sum_{j:j \neq i} U'_i - U'_j \]  
\hspace{1cm} \text{(C.3)} 

In order to ensure second order accuracy, the Laplacian of a variable vector \( U' \) should be such that:

\[ L_i(U') \approx O(h^2)U'_{|x=x_i} \]  
\hspace{1cm} \text{(C.4)}
A simple Taylor expansion shows that this can not hold. Indeed:

\[ L_i = L_i(x) \nabla Q \big|_{x=x_i} + O(h^2) \]  

(C.5)

A pseudo-Laplacian operator is therefore defined to ensure it respects linearity and improves the accuracy. It does so by considering the gradient:

\[ L_i^p(U') = L_j(U') - \nabla U_i^p \cdot L_i(x) \]  

(C.6)

Estimates of the variables at nodes i-1 and i+2 can then be deduced using the central difference Galerkin approximation:

\[ L_i^p(U^0) = \left( \frac{1}{2} U_{i-1}^0 - U_i^0 + \frac{1}{2} U_{i+1}^0 \right), \quad L_{i+1}^p(U^0) = \left( \frac{1}{2} U_i^0 - U_{i+1}^0 + \frac{1}{2} U_{i+2}^0 \right) \]  

(C.7)

Once the variable are known at the four nodes, the numerical fluxes can be defined, starting with the first order linear fluxes:

\[ f_i^0 = \begin{cases} 
(\rho o u_n')_i + (\rho' u_n)_i \\
(\rho o u_n')_i (v')_i + (\rho' u_n + \rho o u'_n)_i (v_o)_i + (p')_i \\
(\rho o u_n')_i (e')_i + (\rho' u_n + \rho o u'_n)_i (e_o)_i + \frac{c}{\gamma - 1} (p' u_n + p o u'_n)_i 
\end{cases} \]  

(C.8)

The first-order Roe fluxes are then defined. The Roe matrix \( A_R \) is calculated from the steady values only [156, p.465]. To solve the entropy condition, Harten’s correction is applied on the Roe eigenvalues, \( v', v' + c \) and \( v' - c \), where \( c \) is the speed of sound. The Roe flux is then expressed in terms of [156, p.466]:

\[ \frac{1}{2} f_{i+1/2}^{Roe} = \frac{1}{2} |A_R| (U_{i+1}^0 - U_i^0) \]  

(C.9)

The second-order fluxes extends Roe’s methodology to the i-1 and i+2 nodes [156, p.551]:

\[ f_{i-1/2}^{Roe} = A_{Ro}^- (U_i^0 - U_{i-1}^0) \]

\[ f_{i+1/2}^{Roe} = A_{R_o}^+ (U_{i+2}^0 - U_{i+1}^0) \]  

(C.10)
$A^+_{Ro}$ and $A^-_{Ro}$ translate as considering only the positive and negative eigenvalues of the Roe matrix. The delimiters are finally defined using the classic Roe’s Superbee function [156, p.544], only dependent on the flux ratio:

$$
\Psi_i^{t-1/2} = \max \left( 0.0, \min \left( 2 \frac{f_{i+1/2}^{Roe+}}{f_{i-1/2}^{Roe+}}, 1 \right), \min \left( 2 \frac{f_{i+1/2}^{Roe-}}{f_{i-1/2}^{Roe-}}, 2 \right) \right)
$$

$$
\Psi_i^{t+3/2} = \max \left( 0.0, \min \left( 2 \frac{f_{i+3/2}^{Roe+}}{f_{i+3/2}^{Roe-}}, 1 \right), \min \left( 2 \frac{f_{i+1/2}^{Roe-}}{f_{i+3/2}^{Roe-}}, 2 \right) \right)
$$

(C.11)

The second order solver, thus defined, will now be tested on a simple testcase: the shock-tube problem, also known as the Riemann problem.

### C.2 The validation on a shock-tube application

The Riemann problem is a classic testcase to study how a CFD solver handles the propagation of waves [156]. It consists in a waveguide geometry divided at time $t = 0s$ into two domains by a diaphragm. In this study, the waveguide is a rectangular cylinder with a length of 5m, meshed with either hexahedral or tetrahedral cells as represented in Fig.C.1a and Fig.C.1b. The left-hand side domain contains a fluid at rest with pressure and density equal to $0.01bars$ and $1.23 \times 10^{-2} kg.m^{-3}$ respectively, while the right-hand side domain contains the same fluid at pressure and density equal to $0.001bars$ and $1.53 \times 10^{-3} kg.m^{-3}$.

The separating diaphragm is broken at $t = 0s$ giving way to the propagation of a shock wave and an expansion wave within the geometry. In our case, in order to respect the acoustic approximation and the linearity of the solver, the pressure difference between the two domains is small and the two propagating waves actually correspond to weak acoustic waves propagating in opposite directions at the speed of sound $c$. 

(a) The shock-tube case with an hexahedral mesh

(b) The shock-tube case with a tetrahedral mesh

Figure C.1. Hexahedral and tetrahedral meshes of the waveguide used in the Riemann problem
304 Prediction of Acoustic Resonances in Core Volumes

Figure C.2. Propagation of a wave as modelled by a first and second order TVD Roe scheme
C.2. THE VALIDATION ON A SHOCK-TUBE APPLICATION

Figure C.2. Propagation of a wave as modelled by a first and second order TVD Roe scheme

(e) Pressure repartition along the waveguide axis at $t=10\text{ms}$

(f) Pressure repartition along the waveguide axis at $t=15\text{ms}$

(g) Pressure repartition along the waveguide axis at $t=20\text{ms}$
Fig.C.2 presents the evolution of the wave at different times. The theoretical propagation, along with the first order solver and the second order on hexahedral and tetrahedral meshes estimates, can be seen in Fig.C.2a, Fig.C.2b, Fig.C.2c and Fig.C.2d. Depending on the order of the numerical scheme, two observations can be made. On one hand, the first order Roe scheme quickly introduces a large amount of damping, sign of diffusion in the solver. On the other hand, the second order TVD Roe scheme implemented gives an accurate representation of the wave propagation on both hexahedral and tetrahedral meshes. Second order oscillations are observed on the hexahedral mesh while the wave front is perfectly captured for the tetrahedral mesh.

After about 8ms, the waves are made to reflect on the endwalls of the waveguide. This gives us the opportunity to check the boundary conditions implementation for rigid walls. Fig.C.2e, Fig.C.2f and Fig.C.2g, taken respectively at a time t=10ms, t=15ms and t=20ms, clearly show that wave reflection is correctly modelled, with no damping or phase lag, thanks to an appropriate boundary condition implementation.
Appendix D

Raw Arnoldi results

The Arnoldi method generates a large amount of data by extracting a large number of acoustic modes in one computation. To ease the read of the thesis, it has been decided that the results would be presented in the thesis chapters under the form of pictures and charts. The data used to generate these figures is presented here under its raw form for reference.
## Chapter 4

### D.1  Chapter 4

#### Table D.1. Data for the closed cylinder study with hexahedral mesh in Chapter 4

<table>
<thead>
<tr>
<th>Mode number</th>
<th>1</th>
<th>1</th>
<th>1.7</th>
<th>1.9</th>
<th>2</th>
<th>2</th>
<th>4</th>
<th>5</th>
<th>5</th>
<th>7</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency (Hz)</td>
<td>1620.0</td>
<td>2430.0</td>
<td>3240.0</td>
<td>4050.0</td>
<td>4860.0</td>
<td>5670.0</td>
<td>6480.0</td>
<td>7290.0</td>
<td>8100.0</td>
<td>8910.0</td>
<td>9720.0</td>
<td>10530.0</td>
</tr>
<tr>
<td>Frequency (Hz)</td>
<td>2430.0</td>
<td>3240.0</td>
<td>4050.0</td>
<td>4860.0</td>
<td>5670.0</td>
<td>6480.0</td>
<td>7290.0</td>
<td>8100.0</td>
<td>8910.0</td>
<td>9720.0</td>
<td>10530.0</td>
<td>11340.0</td>
</tr>
<tr>
<td>Frequency (Hz)</td>
<td>3240.0</td>
<td>4050.0</td>
<td>4860.0</td>
<td>5670.0</td>
<td>6480.0</td>
<td>7290.0</td>
<td>8100.0</td>
<td>8910.0</td>
<td>9720.0</td>
<td>10530.0</td>
<td>11340.0</td>
<td>12150.0</td>
</tr>
</tbody>
</table>

#### Table D.2. Data for the closed cylinder study with tetrahedral mesh in Chapter 4

<table>
<thead>
<tr>
<th>Mode number</th>
<th>1</th>
<th>1</th>
<th>1.7</th>
<th>1.9</th>
<th>2</th>
<th>2</th>
<th>4</th>
<th>5</th>
<th>5</th>
<th>7</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency (Hz)</td>
<td>1620.0</td>
<td>2430.0</td>
<td>3240.0</td>
<td>4050.0</td>
<td>4860.0</td>
<td>5670.0</td>
<td>6480.0</td>
<td>7290.0</td>
<td>8100.0</td>
<td>8910.0</td>
<td>9720.0</td>
<td>10530.0</td>
</tr>
<tr>
<td>Frequency (Hz)</td>
<td>2430.0</td>
<td>3240.0</td>
<td>4050.0</td>
<td>4860.0</td>
<td>5670.0</td>
<td>6480.0</td>
<td>7290.0</td>
<td>8100.0</td>
<td>8910.0</td>
<td>9720.0</td>
<td>10530.0</td>
<td>11340.0</td>
</tr>
<tr>
<td>Frequency (Hz)</td>
<td>3240.0</td>
<td>4050.0</td>
<td>4860.0</td>
<td>5670.0</td>
<td>6480.0</td>
<td>7290.0</td>
<td>8100.0</td>
<td>8910.0</td>
<td>9720.0</td>
<td>10530.0</td>
<td>11340.0</td>
<td>12150.0</td>
</tr>
</tbody>
</table>
### Table D.3. Data for the closed cylinder study in Chapter 5

<table>
<thead>
<tr>
<th>Mode number</th>
<th>Frequency vector (Hz)</th>
<th>Frequency vector (Hz)</th>
<th>Frequency vector (Hz)</th>
<th>Error (Hz)</th>
<th>Convergence criterion</th>
<th>Sampling ratio</th>
<th>Mode number</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 0 0</td>
<td>7</td>
<td>584.3</td>
<td>584.3</td>
<td>0.012</td>
<td>0.05</td>
<td>3.00E-03</td>
<td>1 0 0</td>
</tr>
<tr>
<td>2 0 0</td>
<td>25</td>
<td>-1.000</td>
<td>984.3</td>
<td>490.78</td>
<td>0.004</td>
<td>0.28</td>
<td>2 0 0</td>
</tr>
<tr>
<td>3 0 0</td>
<td>19</td>
<td>-2.000</td>
<td>768.8</td>
<td>576.71</td>
<td>0.075</td>
<td>4.30</td>
<td>1.86E-03</td>
</tr>
<tr>
<td>4 0 0</td>
<td>29</td>
<td>-1.000</td>
<td>964.7</td>
<td>664.11</td>
<td>0.006</td>
<td>0.25</td>
<td>1.70E-02</td>
</tr>
<tr>
<td>5 0 0</td>
<td>5</td>
<td>0.144</td>
<td>1333.75</td>
<td>1220.55</td>
<td>0.045</td>
<td>0.85</td>
<td>2.00E-02</td>
</tr>
<tr>
<td>6 0 0</td>
<td>23</td>
<td>-2.000</td>
<td>1472.29</td>
<td>1472.29</td>
<td>0.005</td>
<td>0.11</td>
<td>6.10E-02</td>
</tr>
<tr>
<td>7 0 0</td>
<td>14</td>
<td>-0.200</td>
<td>1683.51</td>
<td>1717.73</td>
<td>0.004</td>
<td>2.10</td>
<td>6.10E-02</td>
</tr>
<tr>
<td>8 0 0</td>
<td>24</td>
<td>0.200</td>
<td>2687.25</td>
<td>1963.14</td>
<td>0.004</td>
<td>0.30</td>
<td>6.10E-02</td>
</tr>
<tr>
<td>9 0 0</td>
<td>3</td>
<td>0.250</td>
<td>2712.44</td>
<td>2212.82</td>
<td>0.006</td>
<td>1.70</td>
<td>6.10E-02</td>
</tr>
<tr>
<td>10 0 0</td>
<td>21</td>
<td>-1.000</td>
<td>2485.81</td>
<td>2453.81</td>
<td>0.056</td>
<td>0.43</td>
<td>6.10E-02</td>
</tr>
<tr>
<td>11 0 0</td>
<td>29</td>
<td>-2.000</td>
<td>2765.42</td>
<td>2765.48</td>
<td>0.059</td>
<td>6.05E-02</td>
<td></td>
</tr>
<tr>
<td>12 0 0</td>
<td>32</td>
<td>0.200</td>
<td>2684.50</td>
<td>2583.84</td>
<td>0.065</td>
<td>1.70</td>
<td></td>
</tr>
<tr>
<td>13 0 0</td>
<td>1</td>
<td>0.250</td>
<td>3162.86</td>
<td>3162.86</td>
<td>0.054</td>
<td>6.05E-02</td>
<td></td>
</tr>
<tr>
<td>14 0 0</td>
<td>15</td>
<td>-0.200</td>
<td>3154.98</td>
<td>3154.98</td>
<td>0.054</td>
<td>6.05E-02</td>
<td></td>
</tr>
<tr>
<td>15 0 0</td>
<td>12</td>
<td>0.200</td>
<td>3162.86</td>
<td>3162.86</td>
<td>0.054</td>
<td>6.05E-02</td>
<td></td>
</tr>
<tr>
<td>16 0 0</td>
<td>29</td>
<td>0.250</td>
<td>3661.77</td>
<td>3661.77</td>
<td>0.054</td>
<td>6.05E-02</td>
<td></td>
</tr>
<tr>
<td>17 0 0</td>
<td>17</td>
<td>0.270</td>
<td>3657.94</td>
<td>400.85</td>
<td>0.035</td>
<td>2.10</td>
<td></td>
</tr>
<tr>
<td>18 0 0</td>
<td>35</td>
<td>-2.000</td>
<td>4355.65</td>
<td>4427.81</td>
<td>0.040</td>
<td>6.05E-02</td>
<td></td>
</tr>
<tr>
<td>19 0 0</td>
<td>42</td>
<td>-0.400</td>
<td>4473.25</td>
<td>4473.25</td>
<td>0.045</td>
<td>6.05E-02</td>
<td></td>
</tr>
<tr>
<td>20 0 0</td>
<td>49</td>
<td>0.500</td>
<td>4743.16</td>
<td>4743.16</td>
<td>0.055</td>
<td>6.05E-02</td>
<td></td>
</tr>
<tr>
<td>21 0 0</td>
<td>27</td>
<td>0.500</td>
<td>4921.27</td>
<td>4921.27</td>
<td>0.055</td>
<td>6.05E-02</td>
<td></td>
</tr>
<tr>
<td>22 0 0</td>
<td>75</td>
<td>0.500</td>
<td>4743.16</td>
<td>4743.16</td>
<td>0.055</td>
<td>6.05E-02</td>
<td></td>
</tr>
<tr>
<td>23 0 0</td>
<td>79</td>
<td>0.500</td>
<td>4921.27</td>
<td>4921.27</td>
<td>0.055</td>
<td>6.05E-02</td>
<td></td>
</tr>
<tr>
<td>24 0 0</td>
<td>73</td>
<td>0.500</td>
<td>4921.27</td>
<td>4921.27</td>
<td>0.055</td>
<td>6.05E-02</td>
<td></td>
</tr>
<tr>
<td>25 0 0</td>
<td>48</td>
<td>0.500</td>
<td>4921.27</td>
<td>4921.27</td>
<td>0.055</td>
<td>6.05E-02</td>
<td></td>
</tr>
<tr>
<td>26 0 0</td>
<td>67</td>
<td>0.500</td>
<td>4921.27</td>
<td>4921.27</td>
<td>0.055</td>
<td>6.05E-02</td>
<td></td>
</tr>
<tr>
<td>27 0 0</td>
<td>73</td>
<td>0.500</td>
<td>4921.27</td>
<td>4921.27</td>
<td>0.055</td>
<td>6.05E-02</td>
<td></td>
</tr>
<tr>
<td>28 0 0</td>
<td>59</td>
<td>0.500</td>
<td>4921.27</td>
<td>4921.27</td>
<td>0.055</td>
<td>6.05E-02</td>
<td></td>
</tr>
<tr>
<td>29 0 0</td>
<td>59</td>
<td>0.500</td>
<td>4921.27</td>
<td>4921.27</td>
<td>0.055</td>
<td>6.05E-02</td>
<td></td>
</tr>
<tr>
<td>30 0 0</td>
<td>59</td>
<td>0.500</td>
<td>4921.27</td>
<td>4921.27</td>
<td>0.055</td>
<td>6.05E-02</td>
<td></td>
</tr>
<tr>
<td>31 0 0</td>
<td>59</td>
<td>0.500</td>
<td>4921.27</td>
<td>4921.27</td>
<td>0.055</td>
<td>6.05E-02</td>
<td></td>
</tr>
<tr>
<td>32 0 0</td>
<td>59</td>
<td>0.500</td>
<td>4921.27</td>
<td>4921.27</td>
<td>0.055</td>
<td>6.05E-02</td>
<td></td>
</tr>
<tr>
<td>33 0 0</td>
<td>59</td>
<td>0.500</td>
<td>4921.27</td>
<td>4921.27</td>
<td>0.055</td>
<td>6.05E-02</td>
<td></td>
</tr>
<tr>
<td>34 0 0</td>
<td>59</td>
<td>0.500</td>
<td>4921.27</td>
<td>4921.27</td>
<td>0.055</td>
<td>6.05E-02</td>
<td></td>
</tr>
<tr>
<td>35 0 0</td>
<td>59</td>
<td>0.500</td>
<td>4921.27</td>
<td>4921.27</td>
<td>0.055</td>
<td>6.05E-02</td>
<td></td>
</tr>
<tr>
<td>36 0 0</td>
<td>59</td>
<td>0.500</td>
<td>4921.27</td>
<td>4921.27</td>
<td>0.055</td>
<td>6.05E-02</td>
<td></td>
</tr>
<tr>
<td>37 0 0</td>
<td>59</td>
<td>0.500</td>
<td>4921.27</td>
<td>4921.27</td>
<td>0.055</td>
<td>6.05E-02</td>
<td></td>
</tr>
<tr>
<td>38 0 0</td>
<td>59</td>
<td>0.500</td>
<td>4921.27</td>
<td>4921.27</td>
<td>0.055</td>
<td>6.05E-02</td>
<td></td>
</tr>
<tr>
<td>39 0 0</td>
<td>59</td>
<td>0.500</td>
<td>4921.27</td>
<td>4921.27</td>
<td>0.055</td>
<td>6.05E-02</td>
<td></td>
</tr>
</tbody>
</table>

Table D.3. Data for the closed cylinder study in Chapter 5
### Table D.4. Data for the semi-opened cylinder study in Chapter 7

<table>
<thead>
<tr>
<th>Mode ( n )</th>
<th>Error frequency &lt; 10%</th>
<th>Vector</th>
<th>Frequency (Hz)</th>
<th>Frequency (Hz)</th>
<th>Frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 0</td>
<td>0.040</td>
<td>5</td>
<td>316.29</td>
<td>315.03</td>
<td>314.57</td>
</tr>
<tr>
<td>2 0</td>
<td>0.240</td>
<td>24</td>
<td>943.03</td>
<td>945.49</td>
<td>943.70</td>
</tr>
<tr>
<td>3 0</td>
<td>0.210</td>
<td>2</td>
<td>1884.37</td>
<td>1756.17</td>
<td>1872.04</td>
</tr>
<tr>
<td>4 0</td>
<td>0.300</td>
<td>9</td>
<td>2187.78</td>
<td>2207.98</td>
<td>2291.97</td>
</tr>
<tr>
<td>5 0</td>
<td>0.530</td>
<td>14</td>
<td>2241.82</td>
<td>2239.13</td>
<td>2331.10</td>
</tr>
<tr>
<td>6 0</td>
<td>0.070</td>
<td>3</td>
<td>952.05</td>
<td>947.32</td>
<td>948.24</td>
</tr>
<tr>
<td>7 0</td>
<td>0.730</td>
<td>15</td>
<td>4112.50</td>
<td>4112.88</td>
<td>4089.37</td>
</tr>
<tr>
<td>8 0</td>
<td>0.080</td>
<td>12</td>
<td>4732.68</td>
<td>4718.51</td>
<td>4718.61</td>
</tr>
<tr>
<td>1 1</td>
<td>0.020</td>
<td>29</td>
<td>4721.00</td>
<td>4278.87</td>
<td>5277.76</td>
</tr>
<tr>
<td>2 1</td>
<td>0.280</td>
<td>21</td>
<td>5322.96</td>
<td>5350.48</td>
<td>5392.83</td>
</tr>
<tr>
<td>3 1</td>
<td>0.210</td>
<td>19</td>
<td>5319.92</td>
<td>5306.92</td>
<td>5347.64</td>
</tr>
<tr>
<td>4 1</td>
<td>0.630</td>
<td>34</td>
<td>5673.47</td>
<td>5613.16</td>
<td>5683.05</td>
</tr>
<tr>
<td>5 1</td>
<td>0.070</td>
<td>50</td>
<td>6149.91</td>
<td>6074.89</td>
<td>5976.76</td>
</tr>
<tr>
<td>6 1</td>
<td>0.340</td>
<td>27</td>
<td>6246.08</td>
<td>6358.95</td>
<td>6281.29</td>
</tr>
<tr>
<td>7 1</td>
<td>0.060</td>
<td>46</td>
<td>6915.06</td>
<td>6731.98</td>
<td>6828.74</td>
</tr>
<tr>
<td>8 1</td>
<td>0.310</td>
<td>80</td>
<td>7108.47</td>
<td>7155.38</td>
<td>7235.25</td>
</tr>
<tr>
<td>9 1</td>
<td>0.290</td>
<td>41</td>
<td>7260.96</td>
<td>7277.86</td>
<td>7235.04</td>
</tr>
<tr>
<td>10 1</td>
<td>0.220</td>
<td>57</td>
<td>7533.41</td>
<td>7618.38</td>
<td>7471.80</td>
</tr>
<tr>
<td>11 1</td>
<td>0.180</td>
<td>60</td>
<td>8493.06</td>
<td>8057.73</td>
<td>7884.19</td>
</tr>
<tr>
<td>12 1</td>
<td>0.010</td>
<td>73</td>
<td>7647.16</td>
<td>8006.13</td>
<td>7934.25</td>
</tr>
<tr>
<td>13 1</td>
<td>0.010</td>
<td>66</td>
<td>9278.52</td>
<td>9007.71</td>
<td>9418.35</td>
</tr>
<tr>
<td>14 1</td>
<td>0.120</td>
<td>63</td>
<td>8282.11</td>
<td>8703.94</td>
<td>8493.31</td>
</tr>
</tbody>
</table>

### Table D.5. Data for the flanged cylinder study in Chapter 7

<table>
<thead>
<tr>
<th>Mode ( n )</th>
<th>Error frequency &lt; 10%</th>
<th>Vector</th>
<th>Frequency (Hz)</th>
<th>Frequency (Hz)</th>
<th>Frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 0</td>
<td>0.020</td>
<td>1</td>
<td>290.79</td>
<td>292.91</td>
<td>290.74</td>
</tr>
<tr>
<td>2 0</td>
<td>0.190</td>
<td>10</td>
<td>686.28</td>
<td>690.28</td>
<td>690.23</td>
</tr>
<tr>
<td>3 0</td>
<td>0.130</td>
<td>3</td>
<td>1469.18</td>
<td>1496.41</td>
<td>1483.72</td>
</tr>
<tr>
<td>4 0</td>
<td>0.030</td>
<td>35</td>
<td>2301.46</td>
<td>2306.03</td>
<td>2377.21</td>
</tr>
<tr>
<td>5 0</td>
<td>0.020</td>
<td>6</td>
<td>2690.43</td>
<td>2693.43</td>
<td>2709.69</td>
</tr>
<tr>
<td>6 0</td>
<td>0.140</td>
<td>23</td>
<td>3388.00</td>
<td>3329.95</td>
<td>3284.18</td>
</tr>
<tr>
<td>7 0</td>
<td>0.370</td>
<td>99</td>
<td>3537.39</td>
<td>3594.50</td>
<td>3597.57</td>
</tr>
<tr>
<td>8 0</td>
<td>0.400</td>
<td>35</td>
<td>4223.75</td>
<td>4409.08</td>
<td>4451.16</td>
</tr>
<tr>
<td>9 0</td>
<td>0.100</td>
<td>25</td>
<td>4266.90</td>
<td>4270.98</td>
<td>4230.71</td>
</tr>
<tr>
<td>10 0</td>
<td>0.020</td>
<td>2</td>
<td>5223.31</td>
<td>5246.01</td>
<td>5293.69</td>
</tr>
<tr>
<td>11 0</td>
<td>0.240</td>
<td>15</td>
<td>5370.26</td>
<td>5376.58</td>
<td>5425.12</td>
</tr>
<tr>
<td>12 0</td>
<td>0.320</td>
<td>16</td>
<td>5539.58</td>
<td>5564.09</td>
<td>5616.52</td>
</tr>
<tr>
<td>13 0</td>
<td>0.200</td>
<td>81</td>
<td>4989.03</td>
<td>5062.00</td>
<td>5092.03</td>
</tr>
<tr>
<td>14 0</td>
<td>0.320</td>
<td>110</td>
<td>5341.99</td>
<td>6008.00</td>
<td>6155.11</td>
</tr>
<tr>
<td>15 0</td>
<td>0.010</td>
<td>64</td>
<td>6022.35</td>
<td>6433.34</td>
<td>6499.33</td>
</tr>
</tbody>
</table>
### Table D.6. Data for the unflanged cylinder study in Chapter 7

<table>
<thead>
<tr>
<th>Mode number</th>
<th>Vector</th>
<th>Eigenvalue</th>
<th>Frequency vector [Hz]</th>
<th>Frequency value [Hz]</th>
<th>Frequency ratio [Hz]</th>
<th>Error (%)</th>
<th>Convergence criterion</th>
<th>Damping ratio (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 0 0</td>
<td>1</td>
<td>-0.200</td>
<td>296.33</td>
<td>297.91</td>
<td>331.57</td>
<td>1.214</td>
<td>0.38</td>
<td>2.20E+09</td>
</tr>
<tr>
<td>2 0 0</td>
<td>8</td>
<td>0.730</td>
<td>922.34</td>
<td>895.23</td>
<td>934.71</td>
<td>1.048</td>
<td>3.00</td>
<td>1.40E+09</td>
</tr>
<tr>
<td>3 0 0</td>
<td>3</td>
<td>-0.870</td>
<td>1477.72</td>
<td>1482.04</td>
<td>1507.64</td>
<td>1.048</td>
<td>0.98</td>
<td>1.40E+09</td>
</tr>
<tr>
<td>4 0 0</td>
<td>22</td>
<td>-0.670</td>
<td>2100.25</td>
<td>2101.35</td>
<td>2110.08</td>
<td>0.280</td>
<td>0.40</td>
<td>1.70E+09</td>
</tr>
<tr>
<td>5 0 0</td>
<td>6</td>
<td>-0.270</td>
<td>2720.40</td>
<td>2688.15</td>
<td>2714.11</td>
<td>0.956</td>
<td>1.29</td>
<td>2.00E+09</td>
</tr>
<tr>
<td>6 0 0</td>
<td>19</td>
<td>-0.120</td>
<td>3000.53</td>
<td>3202.06</td>
<td>3317.26</td>
<td>1.061</td>
<td>0.94</td>
<td>2.40E+09</td>
</tr>
<tr>
<td>7 0 0</td>
<td>99</td>
<td>0.400</td>
<td>3556.50</td>
<td>3887.09</td>
<td>3820.39</td>
<td>0.849</td>
<td>3.40</td>
<td>2.60E+09</td>
</tr>
<tr>
<td>8 0 0</td>
<td>36</td>
<td>-0.430</td>
<td>4236.42</td>
<td>4680.37</td>
<td>4523.52</td>
<td>0.954</td>
<td>5.40</td>
<td>3.00E+09</td>
</tr>
<tr>
<td>1 1 0</td>
<td>54</td>
<td>0.270</td>
<td>4345.33</td>
<td>4181.78</td>
<td>5126.66</td>
<td>18.431</td>
<td>3.93</td>
<td>1.60E+09</td>
</tr>
<tr>
<td>2 1 0</td>
<td>23</td>
<td>0.050</td>
<td>5176.04</td>
<td>5245.62</td>
<td>5226.66</td>
<td>0.356</td>
<td>1.30</td>
<td>1.30E+09</td>
</tr>
<tr>
<td>3 1 0</td>
<td>13</td>
<td>-0.430</td>
<td>5230.26</td>
<td>5378.10</td>
<td>5431.77</td>
<td>1.025</td>
<td>0.05</td>
<td>1.50E+09</td>
</tr>
<tr>
<td>4 1 0</td>
<td>18</td>
<td>-0.510</td>
<td>5519.95</td>
<td>5665.52</td>
<td>5626.10</td>
<td>1.129</td>
<td>0.32</td>
<td>1.70E+09</td>
</tr>
<tr>
<td>5 1 0</td>
<td>81</td>
<td>0.200</td>
<td>6552.11</td>
<td>6185.74</td>
<td>5729.60</td>
<td>1.500</td>
<td>4.50</td>
<td>9.00E+09</td>
</tr>
<tr>
<td>6 1 0</td>
<td>127</td>
<td>0.310</td>
<td>6844.74</td>
<td>6108.20</td>
<td>6183.42</td>
<td>1.216</td>
<td>3.70</td>
<td>2.40E+09</td>
</tr>
<tr>
<td>7 1 0</td>
<td>69</td>
<td>0.190</td>
<td>5600.75</td>
<td>5811.54</td>
<td>6526.66</td>
<td>10.980</td>
<td>3.00</td>
<td>1.80E+09</td>
</tr>
</tbody>
</table>
### Table D.7. Data for the roll-posts system study in Chapter 7

<table>
<thead>
<tr>
<th>Vector</th>
<th>Error frequency (% of theoretical frequency)</th>
<th>Error frequency (%)</th>
<th>Error frequency (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Branch</td>
<td>Outboard</td>
<td>Inboard</td>
</tr>
<tr>
<td>1</td>
<td>0.21</td>
<td>0.97</td>
<td>3.26</td>
</tr>
<tr>
<td>3</td>
<td>0.28</td>
<td>0.95</td>
<td>3.03</td>
</tr>
<tr>
<td>7</td>
<td>1.00</td>
<td>0.02</td>
<td>4.01</td>
</tr>
<tr>
<td>5</td>
<td>0.69</td>
<td>0.78</td>
<td>3.26</td>
</tr>
<tr>
<td>6</td>
<td>0.24</td>
<td>0.67</td>
<td>3.26</td>
</tr>
<tr>
<td>8</td>
<td>0.86</td>
<td>0.52</td>
<td>3.26</td>
</tr>
<tr>
<td>9</td>
<td>0.07</td>
<td>0.48</td>
<td>3.26</td>
</tr>
<tr>
<td>10</td>
<td>0.74</td>
<td>0.66</td>
<td>3.26</td>
</tr>
<tr>
<td>12</td>
<td>0.26</td>
<td>0.87</td>
<td>3.26</td>
</tr>
<tr>
<td>13</td>
<td>0.32</td>
<td>0.87</td>
<td>3.26</td>
</tr>
<tr>
<td>14</td>
<td>0.27</td>
<td>0.78</td>
<td>3.26</td>
</tr>
<tr>
<td>15</td>
<td>0.06</td>
<td>0.78</td>
<td>3.26</td>
</tr>
<tr>
<td>16</td>
<td>0.32</td>
<td>0.87</td>
<td>3.26</td>
</tr>
<tr>
<td>17</td>
<td>0.57</td>
<td>0.87</td>
<td>3.26</td>
</tr>
<tr>
<td>18</td>
<td>0.96</td>
<td>0.87</td>
<td>3.26</td>
</tr>
<tr>
<td>19</td>
<td>0.24</td>
<td>0.87</td>
<td>3.26</td>
</tr>
<tr>
<td>20</td>
<td>0.26</td>
<td>0.87</td>
<td>3.26</td>
</tr>
<tr>
<td>21</td>
<td>0.26</td>
<td>0.87</td>
<td>3.26</td>
</tr>
<tr>
<td>22</td>
<td>0.26</td>
<td>0.87</td>
<td>3.26</td>
</tr>
<tr>
<td>23</td>
<td>0.26</td>
<td>0.87</td>
<td>3.26</td>
</tr>
<tr>
<td>24</td>
<td>0.26</td>
<td>0.87</td>
<td>3.26</td>
</tr>
<tr>
<td>25</td>
<td>0.26</td>
<td>0.87</td>
<td>3.26</td>
</tr>
<tr>
<td>26</td>
<td>0.26</td>
<td>0.87</td>
<td>3.26</td>
</tr>
<tr>
<td>27</td>
<td>0.26</td>
<td>0.87</td>
<td>3.26</td>
</tr>
<tr>
<td>28</td>
<td>0.26</td>
<td>0.87</td>
<td>3.26</td>
</tr>
<tr>
<td>29</td>
<td>0.26</td>
<td>0.87</td>
<td>3.26</td>
</tr>
<tr>
<td>30</td>
<td>0.26</td>
<td>0.87</td>
<td>3.26</td>
</tr>
<tr>
<td>31</td>
<td>0.26</td>
<td>0.87</td>
<td>3.26</td>
</tr>
<tr>
<td>32</td>
<td>0.26</td>
<td>0.87</td>
<td>3.26</td>
</tr>
<tr>
<td>33</td>
<td>0.26</td>
<td>0.87</td>
<td>3.26</td>
</tr>
<tr>
<td>34</td>
<td>0.26</td>
<td>0.87</td>
<td>3.26</td>
</tr>
<tr>
<td>35</td>
<td>0.26</td>
<td>0.87</td>
<td>3.26</td>
</tr>
<tr>
<td>36</td>
<td>0.26</td>
<td>0.87</td>
<td>3.26</td>
</tr>
</tbody>
</table>

Table D.7. Data for the roll-posts system study in Chapter 7
Figure D.1. Acoustic modeshapes of the roll-posts system using the Arnoldi method
### Figure D.1

**Acoustic modeshapes of the roll-posts system using the Arnoldi method**
Bibliography


[125] S. Ahmad and P.K. Banerjee. Free vibration analysis using BEM particular

the acoustic eigenfrequency analysis. *International Journal for Numerical


the control of thermoacoustic instabilities, The acoustic modes in a combus-


[131] L. Selle, L. Benoit, T. Poinsot, and W. Krebs. Joint use of compressible large-
eddy simulation and Helmholtz solvers for the analysis of rotating modes in

bustors with complex impedances and multidimensional active flames. *AIAA

[133] L.E. Eriksson, L. Andersson, and K. Lindblad. Development of a cooled ra-
dial flameholder for the F404/RM12 afterburner, Part III: Afterburner rumble


