

# A Primal-Dual Lifting Scheme for Two-Stage Robust Optimization

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May 17, 2018

## Abstract

Two-stage robust optimization problems, in which decisions are taken both in anticipation of and in response to the observation of an unknown parameter vector from within an uncertainty set, are notoriously challenging. In this paper, we develop convergent hierarchies of primal (conservative) and dual (progressive) bounds for these problems that trade off the competing goals of tractability and optimality: While the coarsest bounds recover a tractable but suboptimal affine decision rule approximation of the two-stage robust optimization problem, the refined bounds lift extreme points of the uncertainty set until an exact but intractable extreme point reformulation of the problem is obtained. Based on these bounds, we propose a primal-dual lifting scheme for the solution of two-stage robust optimization problems that accommodates for discrete here-and-now decisions, infeasible problem instances as well as the absence of a relatively complete recourse. The incumbent solutions in each step of our algorithm afford rigorous error bounds, and they can be interpreted as piecewise affine decision rules. We illustrate the performance of our algorithm on illustrative examples and on an inventory management problem.

**Keywords:** robust optimization; two-stage problems; decision rules; error bounds.

## 1 Introduction

In the last two decades, robust optimization has emerged as a powerful methodology for immunizing mathematical programs against uncertainty in the problem data. Many dynamic optimization

problems can be naturally formulated as *two-stage robust optimization problems* of the form

$$\begin{aligned}
& \text{minimize} && \mathbf{q}^\top \mathbf{x} \\
& \text{subject to} && \mathbf{T}(\boldsymbol{\xi}) \mathbf{x} + \mathbf{W} \mathbf{y}(\boldsymbol{\xi}) \geq \mathbf{h}(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \Xi \\
& && \mathbf{x} \in \mathbb{R}^{n_1}, \quad \mathbf{y} : \Xi \mapsto \mathbb{R}^{n_2},
\end{aligned} \tag{\mathcal{P}}$$

where  $\mathbf{q} \in \mathbb{R}^{n_1}$  and  $\mathbf{W} \in \mathbb{R}^{m \times n_2}$ , while  $\mathbf{T} : \Xi \xrightarrow{A} \mathbb{R}^{m \times n_1}$  and  $\mathbf{h} : \Xi \xrightarrow{A} \mathbb{R}^m$  are affine functions of the uncertain parameter vector  $\boldsymbol{\xi}$ , which is only known to reside in the nonempty and bounded uncertainty set  $\Xi = \{\boldsymbol{\xi} \in \mathbb{R}^k : \mathbf{F}\boldsymbol{\xi} \leq \mathbf{g}\}$ ,  $\mathbf{F} \in \mathbb{R}^{l \times k}$  and  $\mathbf{g} \in \mathbb{R}^l$ . Problem  $\mathcal{P}$  determines a first-stage decision  $\mathbf{x}$ , which does not depend on the realization of  $\boldsymbol{\xi}$ , as well as a second-stage policy  $\mathbf{y}(\boldsymbol{\xi})$ , which can adapt to the realization of  $\boldsymbol{\xi}$ , that are immunized against all parameter realizations  $\boldsymbol{\xi} \in \Xi$ . Without loss of generality, we may assume that the objective function only involves the first-stage decision  $\mathbf{x}$  as we can move the second-stage cost to the constraints via an epigraph reformulation.

The two-stage robust optimization problem  $\mathcal{P}$  has been used in diverse application domains, ranging from network design and operations, such as network flow problems [5] and vehicle routing [35, 54], to railway shunting and timetabling [45], energy systems [46, 47, 50, 57], the strategic [1] and operative [4, 44] aspects of operations management as well as healthcare [42]. It is also frequently used to determine approximately optimal solutions to more generic (but at the same time computationally more challenging) multi-stage robust optimization problems [26]. For a detailed review of the applications of problem  $\mathcal{P}$ , we refer the reader to [10, 27, 60].

The two-stage robust optimization problem  $\mathcal{P}$  is convex, but it contains infinitely many decision variables and constraints. In fact, problem  $\mathcal{P}$  is NP-hard [36], and the instances that can be solved in polynomial time are both rare and restrictive [2, 18, 35]. As a result, much of the existing research has focused on developing tractable conservative approximations to problem  $\mathcal{P}$ . Early attempts in this direction have proposed to restrict the second-stage decision  $\mathbf{y}$  to affine [36, 43], segregated affine [24, 25, 32], piecewise affine [31] and algebraic as well as trigonometric polynomial functions [19] (so-called *decision rules*) of the parameters  $\boldsymbol{\xi}$ . Decision rules have recently been extended to incorporate both continuous and discrete second-stage decisions, either by partitioning the uncertainty set  $\Xi$  into hyperrectangles [34, 56] or by resorting to a semi-infinite solution scheme [14]. By themselves, decision rule approximations only provide a conservative bound on the optimal value of the two-stage robust optimization problem  $\mathcal{P}$ . To estimate the incurred sub-

optimality, decision rules are often combined with progressive bounds that emerge from replacing the uncertainty set  $\Xi$  in  $\mathcal{P}$  with a finite subset of the parameter realizations. Scenario subsets that lead to good progressive bounds can be obtained from the Lagrange multipliers associated with the optimal solution of the decision rule problem [12, 37]. The decision rule approaches naturally extend to robust optimization problems with more than two stages. The suboptimality of decision rules has been investigated in [15, 18, 43]. For a survey of the decision rule literature, see [26].

Instead of relying on decision rules, the two-stage robust optimization problem  $\mathcal{P}$  can also be conservatively approximated by its  $K$ -adaptability problem or via its copositive reformulation. The  $K$ -adaptability problem selects  $K$  candidate second-stage decisions here-and-now (that is, before observing the realization of  $\xi$ ) and implements the best of these decisions after the realization of  $\xi$  is known. Different  $K$ -adaptability solution schemes have been proposed in [11, 39, 53], and their suboptimality has been analyzed in [17]. The copositive reformulation of problem  $\mathcal{P}$ , on the other hand, is exact and convex, but it is typically difficult to solve. Tractable conservative approximations to this reformulation can be obtained via semidefinite programming [38, 59].

All of the solution schemes reviewed so far have in common that they provide a conservative approximation to the two-stage robust optimization problem  $\mathcal{P}$ . Problem  $\mathcal{P}$  can be solved *exactly* through an iterative approximation of its worst-case second-stage cost function or its uncertainty set. An iterative approximation of the second-stage cost function can be obtained through a variant of Benders' decomposition [20, 41, 55, 62]. To this end, problem  $\mathcal{P}$  is decomposed into a convex master problem involving the first-stage decisions and an outer (progressive) approximation of the worst-case second-stage cost, as well as a non-convex subproblem that provides cuts for the cost function. The Benders' decomposition scheme has been extended to the multi-stage version of problem  $\mathcal{P}$  in [30]. Alternatively, the papers [6] and [61] use semi-infinite programming techniques to iteratively approximate the uncertainty set in problem  $\mathcal{P}$ . Here, the convex master problem is a relaxation of problem  $\mathcal{P}$  that involves finitely many realizations  $\xi \in \Xi$ , and the non-convex subproblem identifies parameter realizations  $\xi \in \Xi$  to be added to the master problem. Both the Benders' decomposition approaches and the semi-infinite programming schemes rely on sequences of progressive approximations to determine an optimal solution to problem  $\mathcal{P}$  in finite time. In contrast, the iterative solution schemes presented in [13, 49] extend the uncertainty set partitioning approaches of [34, 56] to construct a sequence of conservative approximations to problem  $\mathcal{P}$ . Here,

the convex master problem determines constant or affine decision rules for each set of the partition, and the subproblem identifies a refined partition for the master problem. Both approaches extend to integer decisions and more than two stages. It has been shown, however, that even asymptotic convergence to an optimal solution of problem  $\mathcal{P}$  cannot be guaranteed in general [13].

Instead of approximating the second-stage decisions, the second-stage cost function or the uncertainty set of the two-stage robust optimization problem  $\mathcal{P}$ , it has been proposed in [63] to solve  $\mathcal{P}$  through an iterative reformulation of the problem itself. The authors use Fourier-Motzkin elimination to reduce the number of second-stage decisions in problem  $\mathcal{P}$  at the expense of additional constraints. This results in a hierarchy of increasingly accurate conservative approximations of  $\mathcal{P}$  which converge to a static robust optimization problem that is equivalent to  $\mathcal{P}$ .

In this paper, we develop an alternative solution scheme for the two-stage robust optimization problem  $\mathcal{P}$  that aims to provide an attractive trade-off between the conflicting objectives of optimality and tractability. We summarize our key contributions as follows.

1. We develop convergent hierarchies of primal (conservative) and dual (progressive) bounds to the two-stage robust optimization problem  $\mathcal{P}$ . Our bounds combine affine decision rules with an extreme point enumeration to trade off the conflicting goals of tractability and optimality. While the primal bounds apply to any bounded polyhedral uncertainty set, the dual bounds require information about the sum of outer products of the extreme points of the uncertainty set, which can be computed in closed form for several classes of common uncertainty sets.
2. We propose a primal-dual lifting scheme that is inspired by polyhedral combinatorics. Our solution approach accommodates for discrete here-and-now decisions, infeasible problem instances as well as the absence of a relatively complete recourse. The initial bounds are based on affine decision rules and can thus be computed efficiently.
3. We highlight the intimate relationship between our bounds and piecewise affine decision rules over simplicial decompositions of the uncertainty set  $\Xi$  in problem  $\mathcal{P}$ .

We believe that the proposed approach fills a gap in the literature: Both the Benders' decomposition schemes and the semi-infinite programming approaches reviewed above typically consider instances of problem  $\mathcal{P}$  with right-hand uncertainty and a relatively complete recourse. Moreover, while the master problems in these approaches provide tractable progressive bounds, the subprob-

lems providing the conservative bounds constitute bi-affine or mixed-integer optimization problems that are difficult to solve. Although the adaptive uncertainty partitioning approaches can provide tractable conservative and progressive bounds, they do not offer a convergence guarantee. To our best knowledge, the Fourier-Motzkin elimination scheme presented in [63] is the only approach that satisfies all of the properties outlined above. We will show in our numerical experiments that our algorithm is competitive with the Fourier-Motzkin elimination, and that in fact both schemes can be combined beneficially.

The remainder of the paper is structured as follows. Section 2 presents our hierarchies of primal and dual bounds to problem  $\mathcal{P}$ , and Section 3 employs these bounds to develop a primal-dual lifting scheme for the solution of problem  $\mathcal{P}$ . We discuss the relationship between our bounds and piecewise affine decision rules in Section 4, and we report on numerical experiments in Section 5. The appendix shows how to compute the sum of outer products of the extreme points of an uncertainty set, which is required for our dual bound, for several classes of common uncertainty sets.

**Remark 1** (Discrete Here-and-Now Decisions). *Throughout the paper we will discuss how our algorithm extends to a variant of problem  $\mathcal{P}$  that contains continuous and discrete first-stage decisions:*

$$\begin{aligned}
& \text{minimize} && \mathbf{q}_c^\top \mathbf{x}_c + \mathbf{q}_d^\top \mathbf{x}_d \\
& \text{subject to} && \mathbf{T}_c(\boldsymbol{\xi}) \mathbf{x}_c + \mathbf{T}_d(\boldsymbol{\xi}) \mathbf{x}_d + \mathbf{W} \mathbf{y}(\boldsymbol{\xi}) \geq \mathbf{h}(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \Xi \\
& && \mathbf{x}_c \in \mathbb{R}^{nc_1}, \quad \mathbf{x}_d \in \mathbb{Z}^{nd_1}, \quad \mathbf{y} : \Xi \mapsto \mathbb{R}^{n_2}
\end{aligned} \tag{P_d}$$

*We will see that our bounds and the lifting scheme naturally extend to this more generic problem.*

**Notation.** For a finite-dimensional set  $\Omega$ , we denote by  $\text{ext } \Omega$ ,  $\text{conv } \Omega$  and  $\text{cl } \Omega$  the set of extreme points, the convex hull and the closure of  $\Omega$ , respectively. We define by  $\{f : \mathbb{R}^n \xrightarrow{A} \mathbb{R}^m\}$  the set of all affine functions from  $\mathbb{R}^n$  to  $\mathbb{R}^m$ . The vector of ones and the identity matrix are denoted by  $\mathbf{e}$  and  $\mathbf{I}$ , respectively, and their dimensions will be clear from the context. We use  $\langle \mathbf{A}, \mathbf{B} \rangle = \text{tr}(\mathbf{A}^\top \mathbf{B})$  to denote the trace inner product between two symmetric matrices  $\mathbf{A}$  and  $\mathbf{B}$ .

## 2 A Duality Scheme for Two-Stage Robust Optimization

Problem  $\mathcal{P}$  is a convex but challenging optimization problem as it involves infinitely many decision variables and constraints. This can be partially resolved if we replace the uncertainty set  $\Xi$  in  $\mathcal{P}$

with its extreme points  $\text{ext } \Xi$ :

$$\begin{aligned}
& \text{minimize} && \mathbf{q}^\top \mathbf{x} \\
& \text{subject to} && \mathbf{T}(\boldsymbol{\xi}) \mathbf{x} + \mathbf{W} \mathbf{y}(\boldsymbol{\xi}) \geq \mathbf{h}(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \text{ext } \Xi \\
& && \mathbf{x} \in \mathbb{R}^{n_1}, \quad \mathbf{y} : \text{ext } \Xi \mapsto \mathbb{R}^{n_2},
\end{aligned} \tag{P'}$$

In the following, we refer to this problem as the *extreme point reformulation*  $\mathcal{P}'$ .

**Proposition 1** (Extreme Point Reformulation). *Problems  $\mathcal{P}$  and  $\mathcal{P}'$  are equivalent in the following sense: Any feasible solution to one problem can be transformed into a feasible solution to the other problem that attains the same objective value.*

**Proof of Proposition 1.** One readily verifies that for every feasible solution  $\mathbf{x} \in \mathbb{R}^{n_1}$  and  $\mathbf{y} : \Xi \mapsto \mathbb{R}^{n_2}$  to the robust optimization problem  $\mathcal{P}$ , the restriction of  $\mathbf{y}$  to  $\text{ext } \Xi$  provides a feasible solution to the extreme point reformulation  $\mathcal{P}'$  that attains the same objective value.

Let  $\mathbf{x} \in \mathbb{R}^{n_1}$  and  $\mathbf{y}' : \text{ext } \Xi \mapsto \mathbb{R}^{n_2}$  be a feasible solution to the extreme point reformulation  $\mathcal{P}'$ . By construction, there is a function  $\lambda : \Xi \times \text{ext } \Xi \mapsto \mathbb{R}_+$  that satisfies

$$\sum_{\boldsymbol{\xi}' \in \text{ext } \Xi} \lambda(\boldsymbol{\xi}, \boldsymbol{\xi}') = 1 \quad \text{and} \quad \sum_{\boldsymbol{\xi}' \in \text{ext } \Xi} \lambda(\boldsymbol{\xi}, \boldsymbol{\xi}') \cdot \boldsymbol{\xi}' = \boldsymbol{\xi} \quad \forall \boldsymbol{\xi} \in \Xi.$$

Hence,  $(\mathbf{x}, \mathbf{y})$  with  $\mathbf{y}(\boldsymbol{\xi}) := \sum_{\boldsymbol{\xi}' \in \text{ext } \Xi} \lambda(\boldsymbol{\xi}, \boldsymbol{\xi}') \cdot \mathbf{y}'(\boldsymbol{\xi}')$ ,  $\boldsymbol{\xi} \in \Xi$ , is feasible in problem  $\mathcal{P}$  since

$$\begin{aligned}
\mathbf{T}(\boldsymbol{\xi}) \mathbf{x} + \mathbf{W} \mathbf{y}(\boldsymbol{\xi}) &= \sum_{\boldsymbol{\xi}' \in \text{ext } \Xi} \lambda(\boldsymbol{\xi}, \boldsymbol{\xi}') \cdot [\mathbf{T}(\boldsymbol{\xi}') \mathbf{x} + \mathbf{W} \mathbf{y}'(\boldsymbol{\xi}')] \\
&\geq \sum_{\boldsymbol{\xi}' \in \text{ext } \Xi} \lambda(\boldsymbol{\xi}, \boldsymbol{\xi}') \cdot \mathbf{h}(\boldsymbol{\xi}') = \mathbf{h}(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \Xi,
\end{aligned}$$

where the first identity follows from the definition of  $(\mathbf{x}, \mathbf{y})$  and the fact that the mapping  $\mathbf{T}$  is affine, the inequality follows from the feasibility of  $(\mathbf{x}, \mathbf{y}')$  in the extreme point reformulation  $\mathcal{P}'$ , and the second identity holds since  $\sum_{\boldsymbol{\xi}' \in \text{ext } \Xi} \lambda(\boldsymbol{\xi}, \boldsymbol{\xi}') \cdot \boldsymbol{\xi}' = \boldsymbol{\xi}$ . The statement now follows from the fact that  $(\mathbf{x}, \mathbf{y})$  attains the same objective value in  $\mathcal{P}$  as  $(\mathbf{x}, \mathbf{y}')$  does in  $\mathcal{P}'$ .  $\square$

Note that the proof of Proposition 1 does not rely on the characteristics of the first-stage decision  $\mathbf{x}$ . Hence, the statement immediately extends to the generalized two-stage robust optimization problem  $\mathcal{P}_d$  from Remark 1 that contains both continuous and discrete here-and-now decisions.

The extreme point reformulation  $\mathcal{P}'$  is finite-dimensional, but it remains computationally burdensome since the number of decision variables and constraints scales with the number of extreme points of  $\Xi$ . By the dual upper bound theorem [51, § 26.3], this number satisfies

$$|\text{ext } \Xi| \leq \binom{l - \lceil k/2 \rceil}{\lfloor k/2 \rfloor} + \binom{l - 1 - \lceil (k-1)/2 \rceil}{\lfloor (k-1)/2 \rfloor},$$

and the bound is attained by dual cyclic polytopes. This superpolynomial growth is to be expected as problem  $\mathcal{P}$  is NP-hard [36, Theorem 3.4]. We note that for a fixed number  $k$  of uncertain problem parameters, the dual upper bound theorem implies that  $|\text{ext } \Xi| \in \mathcal{O}(l^{\lfloor k/2 \rfloor})$ . Moreover, the *expected* number of vertices of uniformly sampled polyhedra is much lower than this upper bound [22].

We emphasize that the statement of Proposition 1 does not hold if problem  $\mathcal{P}$  exhibits random recourse, that is, if  $\mathbf{W}$  in  $\mathcal{P}$  depends on  $\xi$ . This has also been observed in [6].

**Example 1 (Random Recourse).** *Consider the following instance of  $\mathcal{P}$  with random recourse:*

$$\begin{aligned} & \text{minimize} && 0 \\ & \text{subject to} && \xi \cdot y(\xi) \geq 1 && \forall \xi \in [-1, 1] \\ & && y : [-1, 1] \mapsto \mathbb{R} \end{aligned}$$

*This problem is infeasible since any second-stage decision  $y$  violates the constraint  $0 \cdot y(0) \geq 1$ , but its extreme point formulation is solved by any  $y$  satisfying  $y(-1) \leq -1$  and  $y(1) \geq 1$ .  $\square$*

The next two subsections develop families of primal and dual bounds on the optimal value of the two-stage robust optimization problem  $\mathcal{P}$  that combine the extreme point reformulation  $\mathcal{P}'$  with affine decision rules.

## 2.1 Hierarchy of Primal Bounds

A popular conservative approximation to the two-stage robust optimization problem  $\mathcal{P}$  replaces the second-stage decision  $\mathbf{y} : \Xi \mapsto \mathbb{R}^{n_2}$  with an affine decision rule  $\mathbf{y} : \Xi \xrightarrow{A} \mathbb{R}^{n_2}$ . The resulting problem has finitely many decision variables (the intercept and the slopes of  $\mathbf{y}$ ) but infinitely many constraints. Classical robust optimization techniques then employ linear programming duality to reformulate this semi-infinite problem as a linear program that scales polynomially in the size  $(n_1, n_2, m, k, l)$  of the input data [36]. The affine decision rule approximation performs surprisingly

well on practical problems [43], and it is even optimal in some problem classes [2, 18, 35]. In general, however, affine decision rules are suboptimal even in seemingly benign problems [31, Example 4.5].

In the following we present a family of conservative approximations to  $\mathcal{P}$  that includes the highly tractable but usually suboptimal affine decision rules and the optimal but typically intractable extreme point reformulation  $\mathcal{P}'$  as special cases. We parameterize our approximation by the *scenario set*  $\Xi_S \subseteq \text{ext } \Xi$ , which gives rise to a complementary *affine set*  $\Xi_A \subseteq \Xi$  defined by  $\Xi_A = \text{conv}([\text{ext } \Xi] \setminus \Xi_S)$ :

$$\begin{aligned}
& \text{minimize} && \mathbf{q}^\top \mathbf{x} \\
& \text{subject to} && \mathbf{T}(\boldsymbol{\xi}) \mathbf{x} + \mathbf{W} \mathbf{y}_S(\boldsymbol{\xi}) \geq \mathbf{h}(\boldsymbol{\xi}) && \forall \boldsymbol{\xi} \in \Xi_S \\
& && \mathbf{T}(\boldsymbol{\xi}) \mathbf{x} + \mathbf{W} \mathbf{y}_A(\boldsymbol{\xi}) \geq \mathbf{h}(\boldsymbol{\xi}) && \forall \boldsymbol{\xi} \in \Xi_A \\
& && \mathbf{x} \in \mathbb{R}^{n_1}, \mathbf{y}_S : \Xi_S \mapsto \mathbb{R}^{n_2}, \mathbf{y}_A : \Xi_A \xrightarrow{A} \mathbb{R}^{n_2}
\end{aligned} \tag{\bar{\mathcal{P}}(\Xi_S)}$$

Problem  $\bar{\mathcal{P}}(\Xi_S)$  optimizes over individual second-stage decisions  $\mathbf{y}_S : \Xi_S \mapsto \mathbb{R}^{n_2}$  for all realizations  $\boldsymbol{\xi}$  in the scenario set  $\Xi_S$  and over an affine decision rule  $\mathbf{y}_A : \Xi_A \xrightarrow{A} \mathbb{R}^{n_2}$  for the affine set  $\Xi_A$ . The scenario set  $\Xi_S$  is a finite subset of the extreme points of  $\Xi$ , whereas the affine set  $\Xi_A$  is a polyhedral subset of  $\Xi$ . Robust optimization techniques allow us to reformulate problem  $\bar{\mathcal{P}}(\Xi_S)$  as a linear program that scales polynomially in the parameters  $(n_1, n_2, m, |\Xi_S|, k, \ell)$ , where  $\ell$  denotes the number of inequalities required to describe  $\Xi_A$  (see page 9 below). The choices  $\Xi_S = \emptyset$  (and thus  $\Xi_A = \Xi$ ) and  $\Xi_S = \text{ext } \Xi$  (and thus  $\Xi_A = \emptyset$ ) recover the affine decision rule approximation and the extreme point reformulation  $\mathcal{P}'$ , respectively. The requirement that  $\Xi_A = \text{conv}([\text{ext } \Xi] \setminus \Xi_S)$  is illustrated in Figure 1, and it ensures that

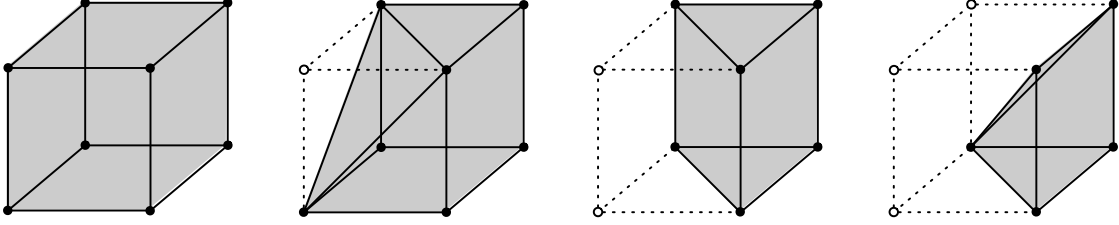
$$\text{conv}(\Xi_S \cup \Xi_A) = \text{conv}(\Xi_S \cup \text{conv}([\text{ext } \Xi] \setminus \Xi_S)) = \text{conv}(\Xi_S \cup ([\text{ext } \Xi] \setminus \Xi_S)) = \text{conv}(\text{ext } \Xi) = \Xi.$$

In the following, we use  $\bar{\mathcal{P}}(\Xi_S)$  both to refer to the bounding problem and to its optimal objective value. We now show that  $\bar{\mathcal{P}}(\Xi_S)$  bounds problem  $\mathcal{P}$  from above.

**Proposition 2** (Primal Bound). *Problem  $\bar{\mathcal{P}}(\Xi_S)$  satisfies the following two properties.*

- (i) *Any feasible solution to  $\bar{\mathcal{P}}(\Xi_S)$  can be transformed into a feasible solution to  $\bar{\mathcal{P}}(\Xi'_S)$ ,  $\Xi_S \subseteq \Xi'_S$ , that attains the same objective value.*
- (ii) *Any feasible solution to  $\bar{\mathcal{P}}(\text{ext } \Xi)$  can be transformed into a feasible solution to  $\mathcal{P}$  that attains*





**Figure 1.** Different decompositions of a three-dimensional hypercube into scenario sets  $\Xi_S$  (hollow circles) and implied affine sets  $\Xi_A$  (shaded regions). In all cases, the convex hull of  $\Xi_S$  and  $\Xi_A$  recovers the original hypercube.

*the same objective value, and vice versa.*

**Proof of Proposition 2.** As for the first statement, let  $(\mathbf{x}, \mathbf{y}_S, \mathbf{y}_A)$  be a feasible solution to  $\overline{\mathcal{P}}(\Xi_S)$ . In that case,  $(\mathbf{x}, \mathbf{y}'_S, \mathbf{y}'_A)$  with  $\mathbf{y}'_S(\boldsymbol{\xi}) := \mathbf{y}_S(\boldsymbol{\xi})$  for  $\boldsymbol{\xi} \in \Xi_S$ ;  $:= \mathbf{y}_A(\boldsymbol{\xi})$  for  $\boldsymbol{\xi} \in \Xi'_S \setminus \Xi_S$  and  $\mathbf{y}'_A(\boldsymbol{\xi}) := \mathbf{y}_A(\boldsymbol{\xi})$ ,  $\boldsymbol{\xi} \in \Xi'_A$ , is a feasible solution to  $\mathcal{P}(\Xi'_S)$  that attains the same objective value.

The second statement directly follows from Proposition 1.  $\square$

The second property states that  $\overline{\mathcal{P}}(\text{ext } \Xi)$  is equivalent to problem  $\mathcal{P}$ . The first property implies that  $\overline{\mathcal{P}}(\Xi_S)$  provides a conservative approximation to  $\overline{\mathcal{P}}(\Xi'_S)$  if  $\Xi_S \subseteq \Xi'_S$ , and in particular—by virtue of the second statement—to problem  $\mathcal{P}$ . We remark that the second statement also holds true for  $\Xi_S \neq \text{ext } \Xi$  if the implied affine set  $\Xi_A$  is a simplex [16, Theorem 1].

The proof of Proposition 2 does not exploit any properties of the first-stage decision  $\mathbf{x}$ , and hence the result immediately extends to the mixed-integer two-stage robust optimization problem  $\mathcal{P}_d$ .

The size of problem  $\overline{\mathcal{P}}(\Xi_S)$  depends on the number of scenarios  $|\Xi_S|$  and the number of inequalities  $\ell$  required to describe  $\Xi_A$ , both of which depend on the choice of the uncertainty set  $\Xi$ . If  $\Xi$  is the 1-norm ball in  $\mathbb{R}^k$ ,  $\Xi = \{\boldsymbol{\xi} \in \mathbb{R}^k : \|\boldsymbol{\xi}\|_1 \leq 1\}$ , for example, then  $\Xi_S$  contains at most  $|\text{ext } \Xi| = 2k$  scenarios, and any set  $\Xi_A$  can be described efficiently by (subsets of) the projection of  $\Xi' = \{(\boldsymbol{\xi}, \boldsymbol{\chi}) \in \mathbb{R}^k \times \mathbb{R}^k : \mathbf{e}^\top \boldsymbol{\chi} \leq 1, \boldsymbol{\chi} \geq \pm \boldsymbol{\xi}\}$  onto its first  $k$  components. If the uncertainty set is the  $\infty$ -norm ball in  $\mathbb{R}^k$ ,  $\Xi = \{\boldsymbol{\xi} \in \mathbb{R}^k : \|\boldsymbol{\xi}\|_\infty \leq 1\}$ , on the other hand, then  $\Xi_S$  can contain up to  $|\text{ext } \Xi| = 2^k$  scenarios, and the sets  $\Xi_A$  form the class of 0/1 polytopes which can have exponentially many facets [64]. The hope is that in practice, instances of  $\overline{\mathcal{P}}(\Xi_S)$  with compact descriptions of  $\Xi_A$  and  $\Xi_S$  produce tight bounds on the optimal value of problem  $\mathcal{P}$ . Section 3 presents an iterative procedure that aims to determine such instances, and Section 5 investigates the performance of this procedure in an inventory management problem.

## 2.2 Hierarchy of Dual Bounds

The paper [37] proposes a progressive bound for the two-stage robust optimization problem  $\mathcal{P}$  that replaces the uncertainty set  $\Xi$  with a subset of its extreme points  $\text{ext } \Xi$ . This bound has found widespread use in both two-stage [6, 61] and multi-stage [13, 14, 49] robust optimization. In this section we derive a family of dual bounds that tighten the bound of [37]. Our bounds can be interpreted as an application of the primal approximation from the previous section to a dual extreme point formulation of  $\mathcal{P}$ . To this end, we consider the linear programming dual of the extreme point reformulation  $\mathcal{P}'$ :

$$\begin{aligned}
& \text{maximize} && \sum_{\xi \in \text{ext } \Xi} \mathbf{h}(\xi)^\top \boldsymbol{\lambda}(\xi) \\
& \text{subject to} && \sum_{\xi \in \text{ext } \Xi} \mathbf{T}(\xi)^\top \boldsymbol{\lambda}(\xi) = \mathbf{q} \\
& && \mathbf{W}^\top \boldsymbol{\lambda}(\xi) = \mathbf{0} \quad \forall \xi \in \text{ext } \Xi \\
& && \boldsymbol{\lambda} : \text{ext } \Xi \mapsto \mathbb{R}_+^m
\end{aligned} \tag{1}$$

Contrary to the primal extreme point reformulation  $\mathcal{P}'$ , problem (1) is a maximization problem. Therefore, a conservative approximation to problem (1) will amount to a progressive approximation to problem  $\mathcal{P}'$ , and hence to problem  $\mathcal{P}$ . In order to conservatively approximate problem (1), we proceed along the lines of the previous section: We partition the extreme points  $\text{ext } \Xi$  into a scenario set  $\Xi_S$ , for which we select individual decisions  $\boldsymbol{\lambda}_S : \Xi_S \mapsto \mathbb{R}_+^m$ , and an implied affine set  $\Xi_A = \text{conv}([\text{ext } \Xi] \setminus \Xi_S)$ , for which the decisions  $\boldsymbol{\lambda}_A : \Xi_A \xrightarrow{A} \mathbb{R}_+^m$  are restricted to an affine function of  $\boldsymbol{\xi}$ . Contrary to the primal bounding problem  $\bar{\mathcal{P}}(\Xi_S)$ , however, the uncertain problem parameters  $\boldsymbol{\xi}$  enter the dual extreme point reformulation (1) quadratically, and they thus require some additional care. One can then derive the following family of dual bounds to problem  $\mathcal{P}$ :

$$\begin{aligned}
& \text{maximize} && \left\langle \mathbf{H}^\top \boldsymbol{\Lambda}, \boldsymbol{\Sigma}_A \right\rangle + \sum_{\xi \in \Xi_S} \mathbf{h}(\xi)^\top \boldsymbol{\lambda}_S(\xi) \\
& \text{subject to} && \left\langle \mathbf{T}_i^\top \boldsymbol{\Lambda}, \boldsymbol{\Sigma}_A \right\rangle + \sum_{\xi \in \Xi_S} \mathbf{T}_i(\xi)^\top \boldsymbol{\lambda}_S(\xi) = q_i \quad \forall i = 1, \dots, n_1 \\
& && \mathbf{W}^\top \boldsymbol{\lambda}_S(\xi) = \mathbf{0} \quad \forall \xi \in \Xi_S \\
& && \mathbf{W}^\top \boldsymbol{\lambda}_A(\xi) = \mathbf{0} \quad \forall \xi \in \Xi_A \\
& && \boldsymbol{\lambda}_S : \Xi_S \mapsto \mathbb{R}_+^m, \quad \boldsymbol{\lambda}_A : \Xi_A \xrightarrow{A} \mathbb{R}_+^m
\end{aligned} \tag{\mathcal{P}(\Xi_S)}$$

In this problem, we use the notational shorthands  $\mathbf{H} = (\mathbf{h}_0, \mathbf{H}_J)$ ,  $\mathbf{T}_i = (\mathbf{t}_{i,0}, \mathbf{T}_{i,J})$  and  $\mathbf{\Lambda} = (\boldsymbol{\lambda}_0, \boldsymbol{\Lambda}_J)$ , where  $\mathbf{h}_0, \mathbf{t}_{i,0}, \boldsymbol{\lambda}_0 \in \mathbb{R}^m$  denote the intercepts and  $\mathbf{H}_J, \mathbf{T}_{i,J}, \boldsymbol{\Lambda}_J \in \mathbb{R}^{m \times k}$  the slopes (Jacobians) of the affine functions  $\mathbf{h}(\boldsymbol{\xi})$ ,  $\mathbf{T}_i(\boldsymbol{\xi}) = (T_{1,i}(\boldsymbol{\xi}), \dots, T_{m,i}(\boldsymbol{\xi}))^\top$  and  $\boldsymbol{\lambda}_A(\boldsymbol{\xi})$ , and the *moment matrix*

$$\boldsymbol{\Sigma}_A = \sum_{\boldsymbol{\xi} \in \text{ext } \Xi_A} \begin{pmatrix} 1 \\ \boldsymbol{\xi} \end{pmatrix} \begin{pmatrix} 1 \\ \boldsymbol{\xi} \end{pmatrix}^\top = \begin{pmatrix} |\text{ext } \Xi_A| & \sum_{\boldsymbol{\xi} \in \text{ext } \Xi_A} \boldsymbol{\xi}^\top \\ \sum_{\boldsymbol{\xi} \in \text{ext } \Xi_A} \boldsymbol{\xi} & \sum_{\boldsymbol{\xi} \in \text{ext } \Xi_A} \boldsymbol{\xi} \boldsymbol{\xi}^\top \end{pmatrix}$$

records the sum of the extreme point scenarios in  $\Xi_A$ , as well as their outer products. Standard robust optimization techniques allow us to reformulate problem  $\underline{\mathcal{P}}(\Xi_S)$  as a linear program that scales polynomially in the parameters  $(n_1, n_2, m, |\Xi_S|, k, \ell)$ , where  $\ell$  denotes the number of inequalities in the description of  $\Xi_A$  (see page 14 below). The choice  $\Xi_S = \emptyset$  (and thus  $\Xi_A = \Xi$ ) corresponds to a highly tractable but usually suboptimal dual affine decision rule formulation, whereas  $\Xi_S = \text{ext } \Xi$  (and thus  $\Xi_A = \emptyset$ ) recovers the optimal but typically intractable dual extreme point reformulation (1). We use  $\underline{\mathcal{P}}(\Xi_S)$  both to refer to the bounding problem and to its optimal objective value.

We now formalize our reasoning that  $\underline{\mathcal{P}}(\Xi_S)$  bounds problem  $\mathcal{P}$  from below.

**Proposition 3** (Dual Bound). *Problem  $\underline{\mathcal{P}}(\Xi_S)$  satisfies the following two properties.*

- (i) *Any feasible solution to  $\underline{\mathcal{P}}(\Xi_S)$  can be transformed into a feasible solution to  $\underline{\mathcal{P}}(\Xi'_S)$ ,  $\Xi_S \subseteq \Xi'_S$ , that attains the same objective value.*
- (ii)  *$\underline{\mathcal{P}}(\text{ext } \Xi)$  is infeasible if  $\mathcal{P}$  is unbounded and vice versa. If both problems are feasible, then their optimal values coincide.*

**Proof of Proposition 3.** By weak linear programming duality, the dual extreme point reformulation (1) bounds the extreme point reformulation  $\mathcal{P}'$ —and hence, by virtue of Proposition 1, the two-stage robust optimization problem  $\mathcal{P}$ —from below. In particular, (1) is infeasible if  $\mathcal{P}$  is unbounded and vice versa. Moreover, strong linear programming duality implies that both problems attain the same optimal value if they are both feasible. The second statement of the proposition now follows from the fact that the dual extreme point reformulation (1) is equivalent to  $\underline{\mathcal{P}}(\text{ext } \Xi)$ .

As for the first statement, fix a feasible solution  $(\boldsymbol{\lambda}_S, \boldsymbol{\lambda}_A)$  for the problem  $\underline{\mathcal{P}}(\Xi_S)$ . We claim that the solution  $(\boldsymbol{\lambda}'_S, \boldsymbol{\lambda}'_A)$  defined through  $\boldsymbol{\lambda}'_S(\boldsymbol{\xi}) := \boldsymbol{\lambda}_S(\boldsymbol{\xi})$  for  $\boldsymbol{\xi} \in \Xi_S$ ;  $:= \boldsymbol{\lambda}_A(\boldsymbol{\xi})$  for  $\boldsymbol{\xi} \in \Xi'_S \setminus \Xi_S$  and  $\boldsymbol{\lambda}'_A(\boldsymbol{\xi}) := \boldsymbol{\lambda}_A(\boldsymbol{\xi})$ ,  $\boldsymbol{\xi} \in \Xi'_A$ , is feasible in  $\underline{\mathcal{P}}(\Xi'_S)$  and attains the same objective value as  $(\boldsymbol{\lambda}_S, \boldsymbol{\lambda}_A)$

does in  $\underline{\mathcal{P}}(\Xi_S)$ . Indeed, the objective function value of  $(\lambda'_S, \lambda'_A)$  in  $\underline{\mathcal{P}}(\Xi'_S)$  evaluates to

$$\begin{aligned}
\langle \mathbf{H}^\top \boldsymbol{\Lambda}, \boldsymbol{\Sigma}'_A \rangle + \sum_{\boldsymbol{\xi} \in \Xi'_S} \mathbf{h}(\boldsymbol{\xi})^\top \lambda'_S(\boldsymbol{\xi}) &= \langle \mathbf{H}^\top \boldsymbol{\Lambda}, \boldsymbol{\Sigma}'_A \rangle + \sum_{\boldsymbol{\xi} \in \Xi'_S \setminus \Xi_S} \mathbf{h}(\boldsymbol{\xi})^\top \lambda'_S(\boldsymbol{\xi}) + \sum_{\boldsymbol{\xi} \in \Xi_S} \mathbf{h}(\boldsymbol{\xi})^\top \lambda'_S(\boldsymbol{\xi}) \\
&= \langle \mathbf{H}^\top \boldsymbol{\Lambda}, \boldsymbol{\Sigma}'_A \rangle + \sum_{\substack{\boldsymbol{\xi} \in \text{ext } \Xi_A: \\ \boldsymbol{\xi} \notin \text{ext } \Xi'_A}} \mathbf{h}(\boldsymbol{\xi})^\top \lambda_A(\boldsymbol{\xi}) + \sum_{\boldsymbol{\xi} \in \Xi_S} \mathbf{h}(\boldsymbol{\xi})^\top \lambda_S(\boldsymbol{\xi}) \\
&= \langle \mathbf{H}^\top \boldsymbol{\Lambda}, \boldsymbol{\Sigma}'_A \rangle + \langle \mathbf{H}^\top \boldsymbol{\Lambda}, \boldsymbol{\Sigma}_A - \boldsymbol{\Sigma}'_A \rangle + \sum_{\boldsymbol{\xi} \in \Xi_S} \mathbf{h}(\boldsymbol{\xi})^\top \lambda_S(\boldsymbol{\xi}) \\
&= \langle \mathbf{H}^\top \boldsymbol{\Lambda}, \boldsymbol{\Sigma}_A \rangle + \sum_{\boldsymbol{\xi} \in \Xi_S} \mathbf{h}(\boldsymbol{\xi})^\top \lambda_S(\boldsymbol{\xi}),
\end{aligned}$$

where  $\boldsymbol{\Lambda} = (\boldsymbol{\lambda}_0, \boldsymbol{\Lambda}_J)$  and  $\boldsymbol{\Lambda}_J \in \mathbb{R}^{m \times k}$  and  $\boldsymbol{\lambda}_0 \in \mathbb{R}^m$  denote the slopes and the intercept of  $\lambda_A$ , respectively, while  $\boldsymbol{\Sigma}_A$  and  $\boldsymbol{\Sigma}'_A$  are the moment matrices corresponding to the affine sets  $\Xi_A$  and  $\Xi'_A$ . Here, the second identity follows from the definition of  $\lambda'_S$  and the fact that

$$\text{ext } \Xi_A = \text{ext}(\text{conv}([\text{ext } \Xi] \setminus \Xi_S)) = (\text{ext } \Xi) \setminus \Xi_S$$

and hence,

$$(\text{ext } \Xi_A) \setminus (\text{ext } \Xi'_A) = ([\text{ext } \Xi] \setminus \Xi_S) \setminus ([\text{ext } \Xi] \setminus \Xi'_S) = \Xi'_S \setminus \Xi_S.$$

The third identity holds because  $\mathbf{h}(\boldsymbol{\xi}) = \mathbf{H}_J \boldsymbol{\xi} + \mathbf{h}_0$  and  $\lambda_A(\boldsymbol{\xi}) = \boldsymbol{\Lambda}_J \boldsymbol{\xi} + \boldsymbol{\lambda}_0$ , and the last identity follows from the linearity of the inner product operator. An analogous argument shows that  $(\lambda'_S, \lambda'_A)$  satisfies the first constraint in  $\underline{\mathcal{P}}(\Xi'_S)$ , and one readily verifies that the remaining constraints are satisfied as well. Thus, the first statement follows.  $\square$

The second property states that the (extended real-valued) optimal values of  $\underline{\mathcal{P}}(\text{ext } \Xi)$  and the two-stage robust optimization problem  $\mathcal{P}$  coincide if at least one of the problems is feasible. It is possible, however, that both problems are infeasible: This is the case, for example, if  $\mathbf{T}(\boldsymbol{\xi}) = \mathbf{0}$  and  $\mathbf{W} = \mathbf{0}$  but  $\mathbf{q} \neq \mathbf{0}$  and  $\mathbf{h}(\boldsymbol{\xi}) \not\leq \mathbf{0}$  for some  $\boldsymbol{\xi} \in \Xi$ . Our incremental lifting scheme in Section 3 avoids this pathological case by solely operating on feasible instances of the bounding problems. For such instances, the second statement guarantees that the optimal values of  $\underline{\mathcal{P}}(\text{ext } \Xi)$  and  $\mathcal{P}$  coincide. The first statement implies that  $\underline{\mathcal{P}}(\Xi_S)$  provides a progressive approximation to  $\underline{\mathcal{P}}(\Xi'_S)$  if  $\Xi_S \subseteq \Xi'_S$ , and in particular—by virtue of the second statement—to problem  $\mathcal{P}$ .

**Remark 2** (Discrete Here-and-Now Decisions). *Perhaps surprisingly, we can derive a convergent hierarchy of dual bounds to the mixed-integer two-stage robust optimization problem  $\mathcal{P}_d$  as well. To this end, consider the following extreme point reformulation  $\mathcal{P}'_d$  of problem  $\mathcal{P}_d$ , where we separately optimize over the discrete and continuous here-and-now decisions:*

$$\begin{aligned} & \text{minimize} && \mathbf{q}_d^\top \mathbf{x}_d + \left[ \begin{array}{l} \text{minimize} \quad \mathbf{q}_c^\top \mathbf{x}_c \\ \text{subject to} \quad \mathbf{T}_c(\boldsymbol{\xi}) \mathbf{x}_c + \mathbf{W} \mathbf{y}(\boldsymbol{\xi}) \geq [\mathbf{h}(\boldsymbol{\xi}) - \mathbf{T}_d(\boldsymbol{\xi}) \mathbf{x}_d] \quad \forall \boldsymbol{\xi} \in \text{ext } \Xi \\ \mathbf{x}_c \in \mathbb{R}^{n_{c1}}, \quad \mathbf{y} : \text{ext } \Xi \mapsto \mathbb{R}^{n_2} \end{array} \right] \\ & \text{subject to} && \mathbf{x}_d \in \mathbb{Z}^{nd_1} \end{aligned}$$

For any fixed discrete here-and-now decision  $\mathbf{x}_d \in \mathbb{Z}^{nd_1}$ , we can replace the embedded minimization problem over the continuous here-and-now decision  $\mathbf{x}_c \in \mathbb{R}^{n_{c1}}$  with its linear programming dual:

$$\begin{aligned} & \text{minimize} && \left[ \begin{array}{l} \text{maximize} \quad \mathbf{q}_d^\top \mathbf{x}_d + \sum_{\boldsymbol{\xi} \in \text{ext } \Xi} [\mathbf{h}(\boldsymbol{\xi}) - \mathbf{T}_d(\boldsymbol{\xi}) \mathbf{x}_d]^\top \boldsymbol{\lambda}(\boldsymbol{\xi}) \\ \text{subject to} \quad \sum_{\boldsymbol{\xi} \in \text{ext } \Xi} \mathbf{T}_c(\boldsymbol{\xi})^\top \boldsymbol{\lambda}(\boldsymbol{\xi}) = \mathbf{q}_c \\ \mathbf{W}^\top \boldsymbol{\lambda}(\boldsymbol{\xi}) = \mathbf{0} \quad \forall \boldsymbol{\xi} \in \text{ext } \Xi \\ \boldsymbol{\lambda} : \text{ext } \Xi \mapsto \mathbb{R}_+^m \end{array} \right] \\ & \text{subject to} && \mathbf{x}_d \in \mathbb{Z}^{nd_1} \end{aligned}$$

Similar arguments as before show that the inner maximization problem can be bounded from below:

$$\begin{aligned} & \text{minimize} && \left[ \begin{array}{l} \text{maximize} \quad \mathbf{q}_d^\top \mathbf{x}_d + \left\langle \left[ \mathbf{H} - \sum_{i=1}^{nd_1} \mathbf{T}_{d,i} \mathbf{x}_{d,i} \right]^\top \boldsymbol{\Lambda}, \boldsymbol{\Sigma}_A \right\rangle + \sum_{\boldsymbol{\xi} \in \Xi_S} [\mathbf{h}(\boldsymbol{\xi}) - \mathbf{T}_d(\boldsymbol{\xi}) \mathbf{x}_d]^\top \boldsymbol{\lambda}_S(\boldsymbol{\xi}) \\ \text{subject to} \quad \left\langle \mathbf{T}_{c,i}^\top \boldsymbol{\Lambda}, \boldsymbol{\Sigma}_A \right\rangle + \sum_{\boldsymbol{\xi} \in \Xi_S} \mathbf{T}_{c,i}(\boldsymbol{\xi})^\top \boldsymbol{\lambda}_S(\boldsymbol{\xi}) = q_i \quad \forall i = 1, \dots, n_{1c} \\ \mathbf{W}^\top \boldsymbol{\lambda}_S(\boldsymbol{\xi}) = \mathbf{0} \quad \forall \boldsymbol{\xi} \in \Xi_S \\ \mathbf{W}^\top \boldsymbol{\lambda}_A(\boldsymbol{\xi}) = \mathbf{0} \quad \forall \boldsymbol{\xi} \in \Xi_A \\ \boldsymbol{\lambda}_S : \Xi_S \mapsto \mathbb{R}_+^m, \quad \boldsymbol{\lambda}_A : \Xi_A \xrightarrow{A} \mathbb{R}_+^m \end{array} \right] \\ & \text{subject to} && \mathbf{x}_d \in \mathbb{Z}^{nd_1} \end{aligned}$$

Here,  $\mathbf{T}_{c,i} = (\mathbf{t}_{c,i,0}, \mathbf{T}_{c,i,J})$ ,  $\mathbf{T}_{d,i} = (\mathbf{t}_{d,i,0}, \mathbf{T}_{d,i,J})$  with  $\mathbf{t}_{c,i,0}, \mathbf{t}_{d,i,0} \in \mathbb{R}^m$  denote the intercepts and  $\mathbf{T}_{c,i,J}, \mathbf{T}_{d,i,J} \in \mathbb{R}^{m \times k}$  the slopes of the affine functions  $\mathbf{T}_{c,i}(\boldsymbol{\xi}) = (T_{c,1,i}(\boldsymbol{\xi}), \dots, T_{c,m,i}(\boldsymbol{\xi}))^\top$  and

$\mathbf{T}_{d,i}(\boldsymbol{\xi}) = (T_{d,1,i}(\boldsymbol{\xi}), \dots, T_{d,m,i}(\boldsymbol{\xi}))^\top$ . For any fixed discrete here-and-now decision  $\mathbf{x}_d \in \mathbb{Z}^{nd_1}$ , the embedded maximization problem is a robust optimization problem that is linear in the decision variables  $\boldsymbol{\lambda}_S$  and  $\boldsymbol{\lambda}_A$ . Using robust optimization techniques to reformulate this semi-infinite problem as a linear program and subsequently invoking strong linear programming duality results in an embedded minimization problem that attains the same optimal value. The resulting min-min problem collapses to a single-stage mixed-integer linear program that can be solved with standard software.

As in the previous section, the size of problem  $\underline{\mathcal{P}}(\Xi_S)$  depends on the number of scenarios  $|\Xi_S|$  and the number of inequalities  $\ell$  required to describe  $\Xi_A$ , both of which depend on the choice of the uncertainty set  $\Xi$ . In addition, the dual approximation  $\underline{\mathcal{P}}(\Xi_S)$  contains the moment matrix  $\boldsymbol{\Sigma}_A$  which appears difficult to compute as it involves sums of the (outer products of the) extreme points in the affine set  $\Xi_A$ . For several class of commonly used uncertainty sets, we show in the appendix how  $\boldsymbol{\Sigma}_A$  can be derived analytically when  $\Xi_A = \Xi$ . In the next section we present a solution scheme for the two-stage robust optimization problem  $\mathcal{P}$  which computes the dual affine decision rule bound  $\underline{\mathcal{P}}(\emptyset)$  and subsequently removes individual vertices of  $\text{ext } \Xi_A$ . Thus, if the initial moment matrix  $\boldsymbol{\Sigma}_A$  for  $\Xi_A = \Xi$  can be computed efficiently, then the subsequent updates of  $\Xi_A$  merely require the subtraction of individual extreme points. On the other hand, it remains unclear whether the moment matrices  $\boldsymbol{\Sigma}_A$  corresponding to generic polyhedral uncertainty sets can be calculated efficiently, that is, without explicitly enumerating them.

We note that our dual bound  $\underline{\mathcal{P}}(\Xi_S)$  is closely related to the lower bound proposed in [37].

**Remark 3 (Relation to the Sampling Bound of [37]).** *The progressive bound of [37] can be interpreted as a relaxation of the extreme point reformulation  $\mathcal{P}'$  that only considers the scenarios  $\boldsymbol{\xi} \in \Xi_S$  in a single, fixed scenario set  $\Xi_S \subseteq \text{ext } \Xi$ :*

$$\begin{aligned} & \text{minimize} && \mathbf{q}^\top \mathbf{x} \\ & \text{subject to} && \mathbf{T}(\boldsymbol{\xi}) \mathbf{x} + \mathbf{W} \mathbf{y}(\boldsymbol{\xi}) \geq \mathbf{h}(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \Xi_S \\ & && \mathbf{x} \in \mathbb{R}^{n_1}, \quad \mathbf{y} : \Xi_S \mapsto \mathbb{R}^{n_2} \end{aligned} \tag{SB}$$

The dual associated with this problem corresponds to an instance of our dual bound  $\underline{\mathcal{P}}(\Xi_S)$  where we fix  $\boldsymbol{\lambda}_A(\boldsymbol{\xi}) = \mathbf{0}$ ,  $\boldsymbol{\xi} \in \Xi_A$ , and only optimize over  $\boldsymbol{\lambda}_S$ . Thus, our bound  $\underline{\mathcal{P}}(\Xi_S)$  is at least as tight as the bound of [37]. □

We close this section with a summary of the findings of Propositions 2 and 3:

**Theorem 1** (Duality). *The primal and dual bounds  $\overline{\mathcal{P}}(\Xi_S)$  and  $\underline{\mathcal{P}}(\Xi_S)$  satisfy:*

(i) **Weak Duality.**  $\underline{\mathcal{P}}(\Xi_S) \leq \mathcal{P} \leq \overline{\mathcal{P}}(\Xi_S)$  for all  $\Xi_S \subseteq \text{ext } \Xi$ .

(ii) **Strong Duality.**  $\underline{\mathcal{P}}(\Xi_S) = \mathcal{P} = \overline{\mathcal{P}}(\Xi_S)$  for  $\Xi_S = \text{ext } \Xi$  if  $\mathcal{P}$  is feasible.

### 3 Primal-Dual Lifting Scheme

Our solution scheme for problem  $\mathcal{P}$  starts with a *feasibility phase*, which determines a feasible solution (or recognizes that no such solution exists), and then proceeds with an *optimality phase*, which computes an optimal solution (or identifies that the problem is unbounded). The algorithm starts with the efficiently computable affine decision rule bounds  $\overline{\mathcal{P}}(\emptyset)$  and  $\underline{\mathcal{P}}(\emptyset)$  and iteratively transfers extreme points  $\boldsymbol{\xi}^* \in \text{ext } \Xi_A$  of the affine set to the scenario set  $\Xi_S$ .

Our algorithm can be summarized as follows.

1. **Initialization.** Set  $\Xi_A = \Xi$  and  $\Xi_S = \emptyset$ .

2. **Feasibility Phase.** Consider the *feasibility problem*

$$\begin{aligned} & \text{minimize} && v \\ & \text{subject to} && \mathbf{T}(\boldsymbol{\xi}) \mathbf{x} + \mathbf{W} \mathbf{y}(\boldsymbol{\xi}) \geq \mathbf{h}(\boldsymbol{\xi}) - v \mathbf{e} && \forall \boldsymbol{\xi} \in \Xi \\ & && \mathbf{x} \in \mathbb{R}^{n_1}, \mathbf{y} : \Xi \mapsto \mathbb{R}^{n_2}, v \in \mathbb{R}_+. \end{aligned}$$

(i) Solve  $\overline{\mathcal{P}}(\Xi_S)$  and  $\underline{\mathcal{P}}(\Xi_S)$  associated with this problem. If the optimal value of  $\overline{\mathcal{P}}(\Xi_S)$  is zero, go to Step 3: the current solution to  $\overline{\mathcal{P}}(\Xi_S)$  is feasible in  $\mathcal{P}$ . If the optimal value of  $\underline{\mathcal{P}}(\Xi_S)$  is strictly positive, terminate: problem  $\mathcal{P}$  is infeasible.

(ii) Select any  $\boldsymbol{\xi}^* \in \Xi_A^*$  (defined below), update  $\Xi_S \leftarrow \Xi_S \cup \{\boldsymbol{\xi}^*\}$  as well as  $\Xi_A \leftarrow \text{conv}([\text{ext } \Xi_A] \setminus \{\boldsymbol{\xi}^*\})$ , and go to back to Step 2(i).

3. **Optimality Phase.** Consider the *optimality problem*

$$\begin{aligned} & \text{minimize} && \mathbf{q}^\top \mathbf{x} \\ & \text{subject to} && \mathbf{T}(\boldsymbol{\xi}) \mathbf{x} + \mathbf{W} \mathbf{y}(\boldsymbol{\xi}) \geq \mathbf{h}(\boldsymbol{\xi}) && \forall \boldsymbol{\xi} \in \Xi \\ & && \mathbf{x} \in \mathbb{R}^{n_1}, \mathbf{y} : \Xi \mapsto \mathbb{R}^{n_2}. \end{aligned}$$

- (i) Solve  $\overline{\mathcal{P}}(\Xi_S)$  and  $\underline{\mathcal{P}}(\Xi_S)$  associated with this problem. If  $\overline{\mathcal{P}}(\Xi_S)$  is unbounded, terminate: problem  $\mathcal{P}$  is unbounded. Otherwise, if the optimal values of  $\overline{\mathcal{P}}(\Xi_S)$  and  $\underline{\mathcal{P}}(\Xi_S)$  coincide, terminate: the current solution to  $\overline{\mathcal{P}}(\Xi_S)$  is optimal in  $\mathcal{P}$ .
- (ii) Select any  $\xi^* \in \Xi_A^*$  (defined below), update  $\Xi_S \leftarrow \Xi_S \cup \{\xi^*\}$  as well as  $\Xi_A \leftarrow \text{conv}([\text{ext } \Xi_A] \setminus \{\xi^*\})$ , and go to back to Step 3(i).

The *feasibility phase* operates on a feasibility variant of problem  $\mathcal{P}$  that minimizes the maximum constraint violation. Both bounding problems  $\overline{\mathcal{P}}(\Xi_S)$  and  $\underline{\mathcal{P}}(\Xi_S)$  associated with this feasibility problem are always feasible since  $\Xi$  is bounded. Thus, Propositions 2 and 3 guarantee the validity of the conservative and progressive bounds, and the update step 2(ii) ensures that both bounds converge monotonically. After finitely many iterations, either the conservative bound evaluates to zero (indicating that a feasible solution has been found), or the progressive bound becomes strictly positive (indicating that the problem is infeasible).

If a feasible solution has been found, the algorithm proceeds with the *optimality phase* and computes conservative and progressive bounds on the problem  $\mathcal{P}$  itself. Since a feasible solution has already been determined, these bounds are valid and converge monotonically. After finitely many iterations, either the bounds coincide (indicating that an optimal solution has been found), or the conservative bound evaluates to  $-\infty$  (indicating that the problem is unbounded).

The update steps 2(ii) and 3(ii) move one extreme point of the polyhedron  $\Xi_A$  to the scenario set  $\Xi_S$ . Ideally, we would transfer the extreme point that leads to the greatest improvement of the conservative and progressive bounds. To this end, we define the *binding scenario set* as  $\Xi_A^* = \{\xi \in \text{ext } \Xi_A : \mathbf{T}(\xi)\mathbf{x} + \mathbf{W}\mathbf{y}(\xi) \not\preceq \mathbf{h}(\xi) - \mathbf{v}\mathbf{e}\}$  in the feasibility phase and as  $\Xi_A^* = \{\xi \in \text{ext } \Xi_A : \mathbf{T}(\xi)\mathbf{x} + \mathbf{W}\mathbf{y}(\xi) \not\preceq \mathbf{h}(\xi)\}$  in the optimality phase. We now show that we can restrict ourselves to these binding scenarios in the update steps.

**Observation 1** (Binding Scenarios). *For any scenario set  $\Xi_S \subseteq \text{ext } \Xi$ , the binding scenario set  $\Xi_A^*$  satisfies the following two properties.*

- (i) *If  $\xi^* \notin \Xi_A^*$ , then the update  $\Xi_S \leftarrow \Xi_S \cup \{\xi^*\}$  and  $\Xi_A \leftarrow \text{conv}([\text{ext } \Xi_A] \setminus \{\xi^*\})$  does not improve the objective value of the conservative approximation  $\overline{\mathcal{P}}(\Xi_S)$ .*
- (ii) *If  $\Xi_A^* = \emptyset$ , then  $\overline{\mathcal{P}}(\Xi_S) = \underline{\mathcal{P}}(\Xi_S)$ .*



**Proof of Observation 1.** We prove both statements for the optimality problem in Step 3 of our primal-dual lifting scheme; similar arguments apply to the feasibility problem in Step 2.

As for the first statement, let  $(\mathbf{x}, \mathbf{y}_A, \mathbf{y}_S)$  be an optimal solution to the conservative approximation  $\bar{\mathcal{P}}(\Xi_S)$  in Step 3(i), and assume that  $\mathbf{y}_A(\boldsymbol{\xi}) = \mathbf{Y}_A \boldsymbol{\xi} + \mathbf{y}_A$  for all  $\boldsymbol{\xi} \in \Xi_A$ . Fix a non-binding scenario  $\boldsymbol{\xi}^* \in \text{ext } \Xi_A \setminus \Xi_A^*$ , and assume to the contrary that  $\bar{\mathcal{P}}(\Xi'_S) < \bar{\mathcal{P}}(\Xi_S)$  for the bounding problem  $\bar{\mathcal{P}}(\Xi'_S)$ ,  $\Xi'_S = \Xi_S \cup \{\boldsymbol{\xi}^*\}$ , that results from lifting the scenario  $\boldsymbol{\xi}^*$ . Let  $(\mathbf{x}', \mathbf{y}'_A, \mathbf{y}'_S)$  be an optimal solution to the lifted problem  $\bar{\mathcal{P}}(\Xi'_S)$ , and assume that  $\mathbf{y}'_A(\boldsymbol{\xi}) = \mathbf{Y}'_A \boldsymbol{\xi} + \mathbf{y}'_A$  for all  $\boldsymbol{\xi} \in \Xi'_A$ . Consider the convex combinations  $\hat{\mathbf{x}} = \lambda \mathbf{x} + (1 - \lambda) \mathbf{x}'$ ,  $\hat{\mathbf{y}}_A(\boldsymbol{\xi}) = [\lambda \mathbf{Y}_A + (1 - \lambda) \mathbf{Y}'_A] \boldsymbol{\xi} + [\lambda \mathbf{y}_A + (1 - \lambda) \mathbf{y}'_A]$ ,  $\boldsymbol{\xi} \in \Xi_A$ , and  $\hat{\mathbf{y}}_S(\boldsymbol{\xi}) = \lambda \mathbf{y}_S(\boldsymbol{\xi}) + (1 - \lambda) \mathbf{y}'_S(\boldsymbol{\xi})$ ,  $\boldsymbol{\xi} \in \Xi_S$ . We show that for  $\lambda \uparrow 1$ ,  $(\hat{\mathbf{x}}, \hat{\mathbf{y}}_A, \hat{\mathbf{y}}_S)$  is feasible in the conservative approximation  $\bar{\mathcal{P}}(\Xi_S)$ . Since  $\mathbf{q}^\top \hat{\mathbf{x}} < \mathbf{q}^\top \mathbf{x}$ , this would contradict the optimality of  $(\mathbf{x}, \mathbf{y}_A, \mathbf{y}_S)$  in  $\bar{\mathcal{P}}(\Xi_S)$ .

We first note that for all  $\boldsymbol{\xi} \in \Xi_S$ , we have

$$\begin{aligned} \mathbf{T}(\boldsymbol{\xi}) \hat{\mathbf{x}} + \mathbf{W} \hat{\mathbf{y}}_S(\boldsymbol{\xi}) \geq \mathbf{h}(\boldsymbol{\xi}) &\iff \mathbf{T}(\boldsymbol{\xi}) [\lambda \mathbf{x} + (1 - \lambda) \mathbf{x}'] + \mathbf{W} [\lambda \mathbf{y}_S(\boldsymbol{\xi}) + (1 - \lambda) \mathbf{y}'_S(\boldsymbol{\xi})] \geq \mathbf{h}(\boldsymbol{\xi}) \\ &\iff \lambda [\mathbf{T}(\boldsymbol{\xi}) \mathbf{x} + \mathbf{W} \mathbf{y}_S(\boldsymbol{\xi})] + (1 - \lambda) [\mathbf{T}(\boldsymbol{\xi}) \mathbf{x}' + \mathbf{W} \mathbf{y}'_S(\boldsymbol{\xi})] \geq \mathbf{h}(\boldsymbol{\xi}), \end{aligned}$$

and the last inequality holds since  $\mathbf{T}(\boldsymbol{\xi}) \mathbf{x} + \mathbf{W} \mathbf{y}_S(\boldsymbol{\xi}) \geq \mathbf{h}(\boldsymbol{\xi})$  and  $\mathbf{T}(\boldsymbol{\xi}) \mathbf{x}' + \mathbf{W} \mathbf{y}'_S(\boldsymbol{\xi}) \geq \mathbf{h}(\boldsymbol{\xi})$  for all  $\boldsymbol{\xi} \in \Xi_S$  by construction. Similarly, we observe that for all  $\boldsymbol{\xi} \in \Xi_A$ , we have

$$\begin{aligned} \mathbf{T}(\boldsymbol{\xi}) \hat{\mathbf{x}} + \mathbf{W} \hat{\mathbf{y}}_A(\boldsymbol{\xi}) &\geq \mathbf{h}(\boldsymbol{\xi}) \\ \iff \mathbf{T}(\boldsymbol{\xi}) [\lambda \mathbf{x} + (1 - \lambda) \mathbf{x}'] + \mathbf{W} [\lambda (\mathbf{Y}_A \boldsymbol{\xi} + \mathbf{y}_A) + (1 - \lambda) (\mathbf{Y}'_A \boldsymbol{\xi} + \mathbf{y}'_A)] &\geq \mathbf{h}(\boldsymbol{\xi}) \\ \iff \lambda [\mathbf{T}(\boldsymbol{\xi}) \mathbf{x} + \mathbf{W} (\mathbf{Y}_A \boldsymbol{\xi} + \mathbf{y}_A)] + (1 - \lambda) [\mathbf{T}(\boldsymbol{\xi}) \mathbf{x}' + \mathbf{W} (\mathbf{Y}'_A \boldsymbol{\xi} + \mathbf{y}'_A)] &\geq \mathbf{h}(\boldsymbol{\xi}). \end{aligned}$$

The last inequality holds for all  $\boldsymbol{\xi} \in \text{ext } \Xi'_A$  since  $\mathbf{T}(\boldsymbol{\xi}) \mathbf{x} + \mathbf{W} \mathbf{y}_A(\boldsymbol{\xi}) \geq \mathbf{h}(\boldsymbol{\xi})$  and  $\mathbf{T}(\boldsymbol{\xi}) \mathbf{x}' + \mathbf{W} \mathbf{y}'_A(\boldsymbol{\xi}) \geq \mathbf{h}(\boldsymbol{\xi})$  for all  $\boldsymbol{\xi} \in \text{ext } \Xi'_A$  by construction. We furthermore observe that  $\mathbf{T}(\boldsymbol{\xi}^*) \hat{\mathbf{x}} + \mathbf{W} \hat{\mathbf{y}}_A(\boldsymbol{\xi}^*) \geq \mathbf{h}(\boldsymbol{\xi}^*)$  for all  $\lambda$  sufficiently close to 1 since  $\mathbf{T}(\boldsymbol{\xi}^*) \mathbf{x} + \mathbf{W} (\mathbf{Y}_A \boldsymbol{\xi}^* + \mathbf{y}_A) > \mathbf{h}(\boldsymbol{\xi}^*)$  by assumption. We thus conclude that  $\mathbf{T}(\boldsymbol{\xi}) \hat{\mathbf{x}} + \mathbf{W} \hat{\mathbf{y}}_A(\boldsymbol{\xi}) \geq \mathbf{h}(\boldsymbol{\xi})$  for all  $\boldsymbol{\xi} \in \text{ext } \Xi_A$  as long as  $\lambda$  sufficiently close to 1. The linearity of  $\mathbf{T}$ ,  $\hat{\mathbf{y}}_A$  and  $\mathbf{h}$  then implies that  $\mathbf{T}(\boldsymbol{\xi}) \hat{\mathbf{x}} + \mathbf{W} \hat{\mathbf{y}}_A(\boldsymbol{\xi}) \geq \mathbf{h}(\boldsymbol{\xi})$  for all  $\boldsymbol{\xi} \in \Xi_A$  as desired.

In view of the second statement, we note that if  $\Xi_A^* = \emptyset$ , then we can remove from the conservative approximation  $\bar{\mathcal{P}}(\Xi_S)$  all constraints involving the realizations  $\boldsymbol{\xi} \in \Xi_A$  without changing the

optimal value of the problem. The dual of this reduced problem can be interpreted as an instance of the progressive approximation  $\underline{\mathcal{P}}(\Xi_S)$  where  $\lambda_A(\xi) = \mathbf{0}$  for all  $\xi \in \Xi_A$ . Strong linear programming duality holds since  $\overline{\mathcal{P}}(\Xi_S)$  is feasible by construction. The statement then follows since the dual of the reduced problem is a restriction of the progressive approximation  $\underline{\mathcal{P}}(\Xi_S)$  that attains the same optimal value as the conservative approximation  $\overline{\mathcal{P}}(\Xi_S)$ .  $\square$

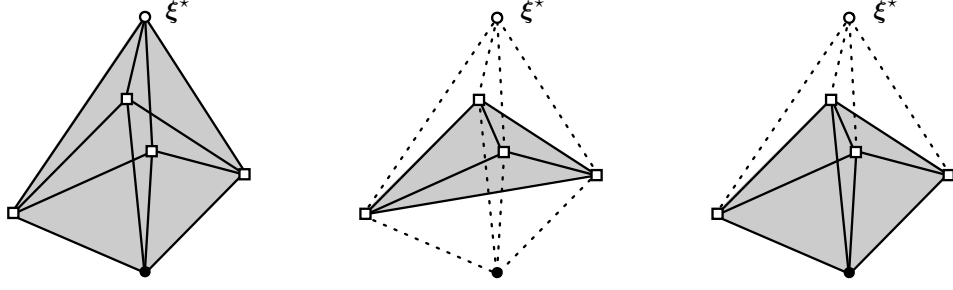
The first property of Observation 1 implies that the update steps 2(ii) and 3(ii) only need to consider the scenarios  $\xi^*$  of the binding scenario set  $\Xi_A^*$ . The second property guarantees that the binding scenario set is never empty: If  $\Xi_A^*$  was empty in Step 2(ii), then  $\overline{\mathcal{P}}(\Xi_S) = \underline{\mathcal{P}}(\Xi_S)$  in Step 2(i), which implies that the algorithm would have either proceeded with Step 3 (if both bounds are zero) or terminated (if both bounds are positive). A similar reasoning applies if  $\Xi_A^*$  was empty in Step 3(ii). We note that the crucial role of binding scenarios has previously been recognized in the context of uncertainty set partitioning approaches, see [13], [49] and [52].

Observation 1 does not specify which scenario  $\xi^* \in \Xi_A^*$  to lift. A natural approach to determine candidate scenarios  $\xi^* \in \Xi_A^*$  to lift is to fix the optimal solution to the current bounding problem  $\overline{\mathcal{P}}(\Xi_S)$  and determine one binding scenario for each constraint by minimizing the constraint's slack over all  $\xi \in \Xi_A$ . This requires the solution of a linear program for each constraint. Any of the binding scenarios thus identified is a candidate for the lifting in Steps 2(ii) and 3(ii) of the algorithm. We describe a more elaborate selection heuristic in our numerical example in Section 5.2.

The updates of the affine set  $\Xi_A$  in Steps 2(ii) and 3(ii) guarantee that  $\Xi_A = \text{conv}([\text{ext } \Xi] \setminus \Xi_S)$  throughout the algorithm. The update steps are intimately related to the ‘Forbidden Vertices Problem’ [3], which optimizes a linear function over all but a few designated vertices of a polyhedron. The update steps can be implemented without enumerating all vertices of  $\Xi_A$ . To this end, we determine the neighbouring extreme points of  $\xi^*$  in  $\Xi_A$ ,  $N(\xi^*)$ , for example through simplex pivoting steps [28]. We then determine the halfspaces defining  $\text{conv } N(\xi^*)$  via facet enumeration [51], and we add those halfspaces of  $\text{conv } N(\xi^*)$  to the description of  $\Xi_A$  that do not contain the extreme point  $\xi^*$  to be removed. Figure 2 illustrates our update procedure.

We are now ready to prove the correctness of our iterative solution scheme.

**Theorem 2** (Finite Convergence). *The algorithm terminates in finite time, and it either determines an optimal solution to  $\mathcal{P}$  or it correctly identifies infeasibility or unboundedness of the problem.*



**Figure 2.** Illustration of the update step in  $\mathbb{R}^3$ . On the left, the circular vertex  $\xi^*$  to be removed from the distorted diamond has the four square vertices as neighbours. The middle polyhedron illustrates  $\text{conv } N(\xi^*)$ , the convex hull of these neighbours. Out of the four halfspaces defining this convex hull, the two upper do not contain  $\xi^*$  and are thus added to the updated description of  $\Xi_A$  on the right.

**Proof of Theorem 2.** We first show that the algorithm terminates after at most  $|\text{ext } \Xi|$  executions of the solution steps 2(i) and 3(i). Indeed, assume to the contrary that the algorithm would execute Steps 2(i) and 3(i) more than  $|\text{ext } \Xi|$  times. During each execution of the update steps 2(ii) and 3(ii), one extreme point of the affine set  $\Xi_A$  is transferred to the scenario set  $\Xi_S$ . Thus, after  $|\text{ext } \Xi|$  executions of the update steps, the bounding problems in the solution steps 2(i) and 3(i) become  $\underline{\mathcal{P}}(\text{ext } \Xi)$  and  $\overline{\mathcal{P}}(\text{ext } \Xi)$ . Since the feasibility and optimality bounding problems are feasible by construction, we must have  $\underline{\mathcal{P}}(\text{ext } \Xi) = \overline{\mathcal{P}}(\text{ext } \Xi)$ . In that case, however, either  $\underline{\mathcal{P}}(\text{ext } \Xi) > 0$  in Step 2(i) or one of the two termination criteria in Step 3(i) is met.

If the algorithm terminates because  $\underline{\mathcal{P}}(\Xi_S) > 0$  in Step 2(i), then there is no solution  $(\mathbf{x}, \mathbf{y}, v)$  to the progressive bound  $\underline{\mathcal{P}}(\Xi_S)$  such that  $\mathbf{x}$  and  $\mathbf{y}$  satisfy  $\mathbf{T}(\boldsymbol{\xi}) \mathbf{x} + \mathbf{W}\mathbf{y}(\boldsymbol{\xi}) \geq \mathbf{h}(\boldsymbol{\xi})$  for all  $\boldsymbol{\xi} \in \Xi$ . Proposition 3 then implies that there is no solution  $(\mathbf{x}, \mathbf{y})$  to problem  $\mathcal{P}$  that satisfies  $\mathbf{T}(\boldsymbol{\xi}) \mathbf{x} + \mathbf{W}\mathbf{y}(\boldsymbol{\xi}) \geq \mathbf{h}(\boldsymbol{\xi})$  for all  $\boldsymbol{\xi} \in \Xi$ , that is, problem  $\mathcal{P}$  is indeed infeasible.

Assume now that the algorithm terminates because  $\overline{\mathcal{P}}(\Xi_S)$  unbounded in Step 3(i). By Proposition 2, any feasible solution to  $\overline{\mathcal{P}}(\Xi_S)$  can be transformed into a feasible solution to  $\mathcal{P}$  that achieves the same objective value. We thus conclude that problem  $\mathcal{P}$  is unbounded as well.

Finally, assume that the algorithm terminates because  $\underline{\mathcal{P}}(\Xi_S) = \overline{\mathcal{P}}(\Xi_S)$  in Step 3(i). Since the bounding problems are feasible by construction, Propositions 2 and 3 imply that

$$\overline{\mathcal{P}}(\Xi_S) \geq \overline{\mathcal{P}}(\text{ext } \Xi) = \mathcal{P} = \underline{\mathcal{P}}(\text{ext } \Xi) \geq \underline{\mathcal{P}}(\Xi_S),$$

where we denote by  $\mathcal{P}$  the optimal value of problem  $\mathcal{P}$ . Since  $\underline{\mathcal{P}}(\Xi_S) = \overline{\mathcal{P}}(\Xi_S)$ , however, we know that the optimal values of all these problems coincide. Proposition 2 then allows us to transform any optimal solution to  $\overline{\mathcal{P}}(\Xi_S)$  into an optimal solution to problem  $\mathcal{P}$ .  $\square$

The runtime of the algorithm is determined by (i) the number of iterations (*i.e.*, the number of times that Steps 2 and 3 are executed), (ii) the size of the bounding problems  $\overline{\mathcal{P}}(\Xi_S)$  and  $\underline{\mathcal{P}}(\Xi_S)$  in each iteration, (iii) the complexity of selecting the scenario  $\xi^* \in \Xi_A^*$  to be removed, (iv) the complexity of the update  $\Xi_A \leftarrow \text{conv}([\text{ext } \Xi_A] \setminus \{\xi^*\})$  and (v) the computation of the moment matrices  $\Sigma_A$ . The algorithm performs up to  $|\text{ext } \Xi|$  iterations in the worst case, and an upper bound for this number is provided in Section 2. The complexity of the bounding problems, as well as the selection of  $\xi^* \in \Xi_A^*$ , primarily depends on the description of the intermediate sets  $\Xi_A$ , which has been discussed in Section 2.1. For our lifting method, the time required for the update  $\Xi_A \leftarrow \text{conv}([\text{ext } \Xi_A] \setminus \{\xi^*\})$  is determined by:

- (i) *The size of  $N(\xi^*)$* : This quantity depends on the degeneracy of  $\xi^*$  in  $\Xi_A$ . If  $k + \sigma$  constraints of  $\Xi_A$  are binding at  $\xi^*$ , then  $\xi^*$  has at most  $\binom{k + \sigma}{k - 1}(\ell - k - \sigma)$  neighbours, where  $\ell$  denotes the number of constraints that describe  $\Xi_A$ . In the worst case, each vertex of  $\Xi_A$  is a neighbour of  $\xi^*$ , which is the case if  $\Xi_A$  is 2-neighbourly. The primal upper bound theorem implies that  $\text{conv } N(\xi^*)$  can consist of up to  $\binom{|N(\xi^*)| - \lceil k/2 \rceil}{\lfloor k/2 \rfloor} + \binom{|N(\xi^*)| - 1 - \lceil (k - 1)/2 \rceil}{\lfloor (k - 1)/2 \rfloor}$  halfspaces, and this bound is attained by primal cyclic polytopes [51].
- (ii) *The complexity of computing the convex hull  $\text{conv } N(\xi^*)$* : This convex hull can be computed in time  $\mathcal{O}(|N(\xi^*)|^{\lfloor k/2 \rfloor})$ .

It can be shown that even when a single scenario  $\xi^*$  is lifted, our description of  $\Xi_A = \text{conv}([\text{ext } \Xi_A] \setminus \{\xi^*\})$  can grow exponentially [3, Proposition 5]. While there are *lifted* formulations of  $\Xi_A$  that only grow *quadratically* if a single scenario is lifted [3, Proposition 6], it is known that optimizing a linear function over  $\Xi_A$  after a flexible (*i.e.*, not a priori fixed) number of scenarios has been lifted is NP-hard [3, Theorem 11] for generic polyhedral uncertainty sets. On the other hand, for special classes of uncertainty sets, such as uncertainty sets with 0-1 vertices, the same problem is solvable in polynomial time if a suitable lifted formulation of  $\Xi_A$  is used [3, §3]. The initial moment matrix  $\Sigma_A$ , finally, can often be computed efficiently, see the appendix, and updating  $\Sigma_A$  only requires to subtract the (outer product of the) scenario  $\xi^* \in \Xi_A^*$  that is removed in each iteration.

We close this section by outlining possible extensions of our iterative solution scheme.

**Remark 4** (Discrete Here-and-Now Decisions). *Theorem 2 does not exploit any specific properties of the here-and-now decisions  $\mathbf{x}$ , and our primal-dual lifting scheme therefore immediately carries over to the mixed-integer two-stage robust optimization problem  $\mathcal{P}_d$ . Observation 1, on the other hand, crucially relies on the convexity in  $\mathbf{x}$ , and one can construct counterexamples where lifting a non-binding scenario results in an improved bound. Thus, in the presence of discrete here-and-now decisions we can no longer restrict ourselves to lifting scenarios  $\boldsymbol{\xi}^* \in \Xi_A^*$ .*

**Remark 5** (Other Extensions). *The algorithm can be altered in several ways. For example, the progressive and conservative bounding problems in Steps 2(i) and 3(i) could involve multiple affine sets  $\Xi_{A,1}, \dots, \Xi_{A,s}$ , or they could assign piecewise affine decision rules to the realizations  $\boldsymbol{\xi} \in \Xi_A$ . If convergence to an optimal solution is not required, then the affine set  $\Xi_A$  could be replaced with outer approximations of  $\text{conv}([\text{ext } \Xi] \setminus \Xi_S)$  without affecting the validity of the bounds. This is advantageous if these outer approximations have compact descriptions, as is the case for Löwner-John ellipsoids [40], for example. One could also transfer multiple extreme points to  $\Xi_S$  in Steps 2(ii) and 3(ii), and it might be advantageous to transfer extreme points back to  $\Xi_A$ . Finally, one could envision lifting different extreme points for the primal and the dual bounds.  $\square$*

## 4 Relation to Piecewise Affine Decision Rules

While a feasible solution  $(\mathbf{x}, \mathbf{y}_A, \mathbf{y}_S)$  to the conservative approximation  $\overline{\mathcal{P}}(\Xi_S)$  provides an implementable first-stage decision, it only provides implementable recourse decisions for the parameter realizations  $\boldsymbol{\xi} \in \Xi_A \cup \Xi_S$ . This is of no concern for most applications, where only the first-stage decision will be implemented. In some situations, however, an implementable recourse decision  $\mathbf{y}(\boldsymbol{\xi})$  for the two-stage robust optimization problem  $\mathcal{P}$  might be required here-and-now for every  $\boldsymbol{\xi} \in \Xi$ . This is frequently the case in real-time control applications, where there is not enough time to solve optimization problems to determine the recourse actions, as well as in embedded systems that lack the processing power or energy supply to solve optimization problems. In this section, we therefore elaborate how the second-stage decision  $(\mathbf{y}_A, \mathbf{y}_S)$  of a feasible solution  $(\mathbf{x}, \mathbf{y}_A, \mathbf{y}_S)$  to  $\overline{\mathcal{P}}(\Xi_S)$  can be transformed into a decision rule  $\mathbf{y} : \Xi \mapsto \mathbb{R}^{n_2}$  that prescribes implementable recourse

decisions to  $\mathcal{P}$  for all  $\xi \in \Xi$ . Along the way, we will discover some insightful connections between our lifting scheme and the piecewise affine decision rules studied in [24, 25, 31, 32] and others.

We define the *graph* of a decision rule  $\mathbf{y} : \Xi \mapsto \mathbb{R}^{n_2}$  as

$$\text{gr } \mathbf{y} = \left\{ \begin{pmatrix} \xi \\ \mathbf{y}(\xi) \end{pmatrix} : \xi \in \Xi \right\}.$$

By construction,  $\text{gr } \mathbf{y}$  is uniquely specified through  $\mathbf{y}$  and vice versa. We also define the *recourse set* of a second-stage decision  $(\mathbf{y}_A, \mathbf{y}_S)$  in problem  $\overline{\mathcal{P}}(\Xi_S)$  as

$$\text{rec}(\mathbf{y}_A, \mathbf{y}_S) = \text{conv} \left( \left\{ \begin{pmatrix} \xi \\ \mathbf{y}_A(\xi) \end{pmatrix} : \xi \in \Xi_A \right\} \cup \left\{ \begin{pmatrix} \xi \\ \mathbf{y}_S(\xi) \end{pmatrix} : \xi \in \Xi_S \right\} \right).$$

To economize on notation, we omit the dependence of  $\text{rec}(\mathbf{y}_A, \mathbf{y}_S)$  on  $\Xi_A$  and  $\Xi_S$ . Intuitively speaking,  $\text{rec}(\mathbf{y}_A, \mathbf{y}_S)$  constitutes the convex hull of the graphs  $\text{gr } \mathbf{y}_A$  and  $\text{gr } \mathbf{y}_S$ , restricted to their respective domains  $\Xi_A$  and  $\Xi_S$ . We now show that  $\text{rec}(\mathbf{y}_A, \mathbf{y}_S)$  contains those recourse decisions  $\mathbf{y}$  to the two-stage problem  $\mathcal{P}$  that correspond to the second-stage decision  $(\mathbf{y}_A, \mathbf{y}_S)$  in  $\overline{\mathcal{P}}(\Xi_S)$ .

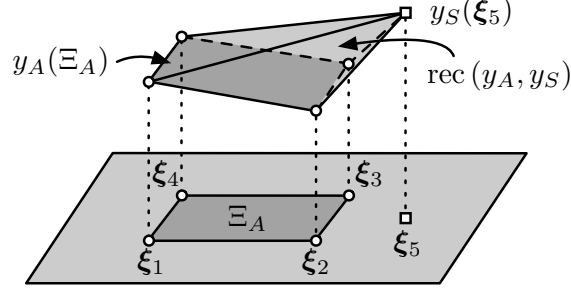
**Proposition 4.** *If  $(\mathbf{x}, \mathbf{y}_A, \mathbf{y}_S)$  is feasible in  $\overline{\mathcal{P}}(\Xi_S)$ , then for any decision rule  $\mathbf{y} : \Xi \mapsto \mathbb{R}^{n_2}$  satisfying  $\text{gr } \mathbf{y} \subseteq \text{rec}(\mathbf{y}_A, \mathbf{y}_S)$ ,  $(\mathbf{x}, \mathbf{y})$  is feasible in  $\mathcal{P}$  and attains the same objective value as  $(\mathbf{x}, \mathbf{y}_A, \mathbf{y}_S)$  in  $\overline{\mathcal{P}}(\Xi_S)$ .*

**Proof of Proposition 4.** It is clear that  $(\mathbf{x}, \mathbf{y})$  attains the same objective value in  $\mathcal{P}$  as  $(\mathbf{x}, \mathbf{y}_A, \mathbf{y}_S)$  does in  $\overline{\mathcal{P}}(\Xi_S)$ . We now show that  $(\mathbf{x}, \mathbf{y})$  is feasible in  $\mathcal{P}$ . One readily verifies that

$$\text{ext } \text{rec}(\mathbf{y}_A, \mathbf{y}_S) = \left\{ \begin{pmatrix} \xi' \\ \mathbf{y}_A(\xi') \end{pmatrix} : \xi' \in \text{ext } \Xi_A \right\} \cup \left\{ \begin{pmatrix} \xi' \\ \mathbf{y}_S(\xi') \end{pmatrix} : \xi' \in \Xi_S \right\},$$

and since  $\text{gr } \mathbf{y} \subseteq \text{rec}(\mathbf{y}_A, \mathbf{y}_S)$ , by construction, for every  $\xi \in \Xi$  there is a function  $\lambda : \text{ext } \Xi \mapsto \mathbb{R}_+$  that satisfies  $\sum_{\xi' \in \text{ext } \Xi} \lambda(\xi') = 1$  and

$$\begin{pmatrix} \xi \\ \mathbf{y}(\xi) \end{pmatrix} = \sum_{\xi' \in \text{ext } \Xi_A} \lambda(\xi') \begin{pmatrix} \xi' \\ \mathbf{y}_A(\xi') \end{pmatrix} + \sum_{\xi' \in \Xi_S} \lambda(\xi') \begin{pmatrix} \xi' \\ \mathbf{y}_S(\xi') \end{pmatrix}. \quad (2)$$



**Figure 3.** For  $\Xi_A = \text{conv}\{\xi_1, \dots, \xi_4\}$  and  $\Xi_S = \{\xi_5\}$  in the shaded plane, the recourse set  $\text{rec}(y_A, y_S)$  is shown as the light-shaded pyramid. Its darker shaded bottom represents  $y_A(\Xi_A) = \{y_A(\xi) : \xi \in \Xi_A\}$ .

We thus obtain that

$$\begin{aligned}
& \mathbf{T}(\xi) \mathbf{x} + \mathbf{W} \mathbf{y}(\xi) \\
&= \mathbf{T} \left( \sum_{\xi' \in \text{ext } \Xi_A} \lambda(\xi') \xi' + \sum_{\xi' \in \Xi_S} \lambda(\xi') \xi' \right) \mathbf{x} + \mathbf{W} \mathbf{y} \left( \sum_{\xi' \in \text{ext } \Xi_A} \lambda(\xi') \xi' + \sum_{\xi' \in \Xi_S} \lambda(\xi') \xi' \right) \\
&= \sum_{\xi' \in \text{ext } \Xi_A} \lambda(\xi') \mathbf{T}(\xi') \mathbf{x} + \sum_{\xi' \in \Xi_S} \lambda(\xi') \mathbf{T}(\xi') \mathbf{x} + \mathbf{W} \left( \sum_{\xi' \in \text{ext } \Xi_A} \lambda(\xi') \mathbf{y}_A(\xi') + \sum_{\xi' \in \Xi_S} \lambda(\xi') \mathbf{y}_S(\xi') \right) \\
&= \sum_{\xi' \in \text{ext } \Xi_A} \lambda(\xi') [\mathbf{T}(\xi') \mathbf{x} + \mathbf{W} \mathbf{y}_A(\xi')] + \sum_{\xi' \in \Xi_S} \lambda(\xi') [\mathbf{T}(\xi') \mathbf{x} + \mathbf{W} \mathbf{y}_S(\xi')] \\
&\geq \sum_{\xi' \in \text{ext } \Xi_A} \lambda(\xi') \mathbf{h}(\xi') + \sum_{\xi' \in \Xi_S} \lambda(\xi') \mathbf{h}(\xi') = \mathbf{h} \left( \sum_{\xi' \in \text{ext } \Xi_A} \lambda(\xi') \xi' + \sum_{\xi' \in \Xi_S} \lambda(\xi') \xi' \right) = \mathbf{h}(\xi),
\end{aligned}$$

where the first, the second and the last identity are due to (2) and the fact that  $\mathbf{T}$  and  $\mathbf{h}$  are affine, and the inequality holds because of the feasibility of  $(\mathbf{x}, \mathbf{y}_A, \mathbf{y}_S)$  in  $\bar{\mathcal{P}}(\Xi_S)$ .  $\square$

Figure 3 illustrates the recourse set of a one-dimensional recourse decision. Interestingly, the reverse implication of Proposition 4 does not hold in general. In fact, the next example shows that there can be feasible solutions  $(\mathbf{x}, \mathbf{y})$  to problem  $\mathcal{P}$  for which  $\text{gr } \mathbf{y} \not\subseteq \text{rec}(y_A, y_S)$  for every feasible solution  $(\mathbf{x}, \mathbf{y}_A, \mathbf{y}_S)$  to every conservative approximation  $\bar{\mathcal{P}}(\Xi_S)$  of  $\mathcal{P}$ .

**Example 2.** Consider the two-stage robust optimization problem

$$\begin{aligned} & \text{minimize} && x \\ & \text{subject to} && \left. \begin{aligned} x &\geq y(\xi) \\ y(\xi) &\geq \xi, \quad y(\xi) \geq -\xi \end{aligned} \right\} \forall \xi \in [-1, 1]. \end{aligned} \quad (3)$$

For every scenario set  $\Xi_S \subseteq \{-1, 1\}$ , the unique optimal solution  $(x, y_A, y_S)$  to  $\overline{\mathcal{P}}(\Xi_S)$  satisfies  $x = 1$  and  $\text{rec}(y_A, y_S) = [-1, 1] \times \{1\}$ . Problem (3) is optimized, however, by  $x = 1$  and every decision rule  $y : [-1, 1] \mapsto \mathbb{R}_+$  satisfying  $y(\xi) \in [|\xi|, 1]$ .  $\square$

The question naturally arises which decision rule from within  $\text{rec}(\mathbf{y}_A, \mathbf{y}_S)$  we should select. A subclass of decision rules that are of special interest are *piecewise affine decision rules*  $\mathbf{y} : \Xi \mapsto \mathbb{R}^{n_2}$  for which there is a partition of  $\Xi$  into finitely many polyhedra  $\Xi_1, \dots, \Xi_L$  such that  $\mathbf{y}$  is affine on each  $\Xi_\ell$ ,  $\ell = 1, \dots, L$ . It turns out that piecewise affine decision rules are intimately related to *simplicial decompositions* of  $\Xi$ , which are subdivisions of  $\Xi$  into finitely many  $k$ -dimensional simplices that only intersect at their boundaries and whose union recovers  $\Xi$ .

**Proposition 5.** Consider a scenario set  $\Xi_S \subseteq \text{ext } \Xi$  as well as the second-stage decisions  $\mathbf{y}_A : \Xi_A \mapsto \mathbb{R}^{n_2}$  and  $\mathbf{y}_S : \Xi_S \mapsto \mathbb{R}^{n_2}$ . Then:

- (i) For every piecewise affine decision rule  $\mathbf{y} : \Xi \mapsto \mathbb{R}^{n_2}$  with  $\text{gr } \mathbf{y} \in \text{rec}(\mathbf{y}_A, \mathbf{y}_S)$ , there is a simplicial decomposition  $\{\Xi_\ell\}_\ell$  of  $\Xi$  such that  $\mathbf{y}$  is affine on every simplex  $\Xi_\ell$ .
- (ii) For every simplicial decomposition  $\{\Xi_\ell\}_\ell$  of  $\Xi$ , there is a decision rule  $\mathbf{y} : \Xi \mapsto \mathbb{R}^{n_2}$  with  $\text{gr } \mathbf{y} \in \text{rec}(\mathbf{y}_A, \mathbf{y}_S)$  such that  $\mathbf{y}$  is affine on every simplex  $\Xi_\ell$ .

**Proof of Proposition 5.** As for the first statement, there is a partition  $\{\hat{\Xi}_\ell\}_\ell$  of  $\Xi$  into polyhedra such that  $\mathbf{y}$  is affine over each  $\hat{\Xi}_\ell$ . Fix a simplicial decomposition  $\{\Xi_{\ell,\ell'}\}_{\ell,\ell'}$  for each polyhedron  $\hat{\Xi}_\ell$ . Then  $\mathbf{y}$  is also affine over the simplicial decomposition  $\{\Xi_{\ell,\ell'}\}_{\ell,\ell'}$  of  $\Xi$ .

In view of the second statement, fix any simplicial decomposition  $\{\Xi_\ell\}_\ell$  of  $\Xi$ , as well as

$$\mathbf{y}(\boldsymbol{\xi}') \in \left\{ \mathbf{y} \in \mathbb{R}^{n_2} : \begin{pmatrix} \boldsymbol{\xi}' \\ \mathbf{y} \end{pmatrix} \in \text{rec}(\mathbf{y}_A, \mathbf{y}_S) \right\} \quad \text{for } \boldsymbol{\xi}' \in \bigcup_\ell \text{ext } \Xi_\ell, \quad (4)$$

where  $\bigcup_\ell \text{ext } \Xi_\ell$  constitutes the set of all corner points of the simplices  $\Xi_\ell$  in the simplicial decomposition. For each simplex  $\Xi_\ell$ , we set  $\mathbf{y}(\boldsymbol{\xi}) = \sum_{\boldsymbol{\xi}' \in \text{ext } \Xi_\ell} \lambda(\boldsymbol{\xi}; \boldsymbol{\xi}') \cdot \mathbf{y}(\boldsymbol{\xi}')$  for all remaining points



$\boldsymbol{\xi} \in \Xi_\ell \setminus (\text{ext } \Xi_\ell)$ , where  $\lambda : \Xi_\ell \times \text{ext } \Xi_\ell \mapsto \mathbb{R}_+$  is the unique weighting function that satisfies

$$\sum_{\boldsymbol{\xi}' \in \text{ext } \Xi_\ell} \lambda(\boldsymbol{\xi}; \boldsymbol{\xi}') = 1 \quad \text{and} \quad \sum_{\boldsymbol{\xi}' \in \text{ext } \Xi_\ell} \lambda(\boldsymbol{\xi}; \boldsymbol{\xi}') \cdot \boldsymbol{\xi}' = \boldsymbol{\xi} \quad \forall \boldsymbol{\xi} \in \Xi_\ell,$$

that is,  $\lambda(\boldsymbol{\xi}; \cdot)$  are the barycentric coordinates of  $\boldsymbol{\xi}$  in  $\Xi_\ell$ . The statement now follows if

$$\begin{pmatrix} \boldsymbol{\xi} \\ \mathbf{y}(\boldsymbol{\xi}) \end{pmatrix} \in \text{rec}(\mathbf{y}_A, \mathbf{y}_S) \quad \forall \boldsymbol{\xi} \in \Xi_\ell.$$

Due to (4), this holds for  $\boldsymbol{\xi} \in \text{ext } \Xi_\ell$ . Moreover, for  $\boldsymbol{\xi} \in \Xi_\ell \setminus (\text{ext } \Xi_\ell)$  we have

$$\begin{pmatrix} \boldsymbol{\xi} \\ \mathbf{y}(\boldsymbol{\xi}) \end{pmatrix} = \sum_{\boldsymbol{\xi}' \in \text{ext } \Xi_\ell} \lambda(\boldsymbol{\xi}; \boldsymbol{\xi}') \cdot \begin{pmatrix} \boldsymbol{\xi}' \\ \mathbf{y}(\boldsymbol{\xi}') \end{pmatrix} \in \text{rec}(\mathbf{y}_A, \mathbf{y}_S),$$

where the membership follows from (4) and the convexity of  $\text{rec}(\mathbf{y}_A, \mathbf{y}_S)$ . Since the simplex  $\Xi_\ell$  was chosen arbitrarily, we thus conclude that  $\text{gr } \mathbf{y} \subseteq \text{rec}(\mathbf{y}_A, \mathbf{y}_S)$ .  $\square$

Propositions 4 and 5 allow us to complete any feasible solution  $(\mathbf{x}, \mathbf{y}_A, \mathbf{y}_S)$  in problem  $\overline{\mathcal{P}}(\Xi_S)$  to a feasible solution  $(\mathbf{x}, \mathbf{y})$  in problem  $\mathcal{P}$  that attains the same objective value and that is piecewise affine over any fixed simplicial decomposition  $\{\Xi_\ell\}_\ell$  of  $\Xi$ . To construct a piecewise affine decision rule with a compact description, we propose to combine the affine decision rule  $\mathbf{y}_A$  over  $\Xi_A$  with a piecewise affine decision rule over  $\Xi \setminus \Xi_A$  that is affine on every simplex of a simplicial decomposition of  $\text{cl}(\Xi \setminus \Xi_A)$ . Note that  $\text{cl}(\Xi \setminus \Xi_A) = \text{conv}(\Xi_S \cup \bigcup\{N(\boldsymbol{\xi}) : \boldsymbol{\xi} \in \Xi_S\})$ , where  $N(\boldsymbol{\xi})$  denotes the neighbouring extreme points of  $\boldsymbol{\xi}$ . A simplicial decomposition of  $\text{cl}(\Xi \setminus \Xi_A)$  can be found with standard triangulation schemes [23].

## 5 Numerical Experiments

We now analyze the computational performance of our primal-dual lifting scheme from Section 3 in the context of two illustrative examples (Section 5.1) as well as an inventory management problem (Section 5.2). In our experiments, we will assess the scalability of our algorithm in terms of the problem size (measured by the number of decision variables and constraints) and the number of uncertain problem parameters. We will also investigate to which degree the different components

	Technique	Convergence	Recourse	Uncertainty Sets	Subproblems
<b>Primal-Dual</b>	lifting of uncertainty set	finite	generic	specific classes	convex hulls & LPs
<b>FME &amp; SB</b>	elimination of second-stage decisions	finite	generic <sup>†</sup>	generic polyhedra	elimination & LPs
<b>Zeng &amp; Zhao</b>	column-and-constraint generation	finite	rel. compl.	generic polyhedra	MILPs
<b>ADR &amp; SB</b>	(piecewise) affine decision rules	none	generic <sup>†</sup>	generic polyhedra	LPs
<b>SB-Heuristic</b>	scenario sampling	none	rel. compl.	generic polyhedra	MILPs

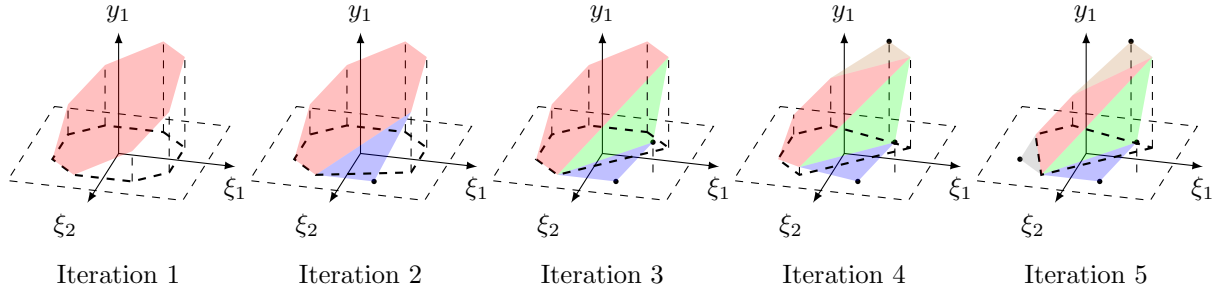
<sup>†</sup> with the exception of [12] which requires relatively complete recourse.

**Table 1.** Comparison of our method (‘Primal-Dual’) with the Fourier-Motzkin elimination approach proposed by Zhen et al. [63] (‘FME & SB’), the column-and-constraint generation scheme due to Zeng and Zhao [61] (‘Zeng & Zhao’), the piecewise linear decision rules [9, 31, 43] combined with the progressive bounds provided by Hadjiyiannis et al. [37] or Bertsimas and de Ruiter [12] (‘ADR & SB’), as well as a sampling-based heuristic (‘SB-Heuristic’).

of our algorithm add to the algorithm’s runtime, and we will explore how the conservative bounds from Section 2.1 and the progressive bounds from Section 2.2 contribute to the optimality gap.

A secondary objective of this section is to compare our algorithm with some of the state-of-the-art solution approaches for two-stage robust optimization problems. To this end, we will compare our solution scheme with the conservative bounds offered by linear and piecewise linear decision rules [9, 31, 43], the progressive bounds provided by Hadjiyiannis et al. [37] and Bertsimas and de Ruiter [12], the solution of the (exact) extreme point reformulation  $\mathcal{P}'$ , the column-and-constraint generation scheme due to Zeng and Zhao [61] as well as the Fourier-Motzkin elimination approach proposed by Zhen et al. [63]. We also compare our algorithm against a sampling-based heuristic derived from the progressive bounds of Hadjiyiannis et al. [37], which we describe further below. Table 1 compares the guarantees offered by these approaches, as well as their underlying assumptions. While the pitfalls of drawing conclusions from a small test set are well-known, we hope to generate some insights into the intricate relationship between the characteristics of problem  $\mathcal{P}$  and the suitability of the different solution schemes.

All optimization problems in this section were solved in single-threaded mode with the Gurobi 7.5 optimization package (see <http://www.gurobi.com>) on a 2.9GHz computer with 8GB RAM. Our C++ implementation of the primal-dual lifting scheme uses the LRS package for vertex and



**Figure 4.** Illustration of the piecewise affine decision rules  $y_1(\xi)$  corresponding to the optimal primal solutions  $(x, \mathbf{y}_A, \mathbf{y}_S)$  in every iteration of our lifting scheme. From left to right, the optimal objective values of  $\bar{\mathcal{P}}(\theta)$  are 6, 6, 5, 4.86 and 4.

facet enumeration (see <http://cgm.cs.mcgill.ca/~avis/C/lrs.html>).

## 5.1 Illustrative Examples

We adapt two examples from the literature to illustrate the structure of the piecewise affine decision rules of our lifting approach (Section 5.1.1), and we show how the affine decision rules  $\lambda_A$  in the dual problem  $\underline{\mathcal{P}}(\Xi_S)$  can contribute to the tightness of the lower bounds (Section 5.1.2).

### 5.1.1 Worst-Case Value of a Sum-of-Max Function

We consider the following adaptation of the example (TOY2) from [33]:

$$\begin{array}{ll}
 \text{minimize} & \tau \\
 \text{subject to} & \tau \geq \mathbf{e}^\top \mathbf{y}(\xi) \\
 & \left. \begin{array}{l}
 y_1(\xi) \geq \max\{x, x + \xi_1 + \xi_2\} \\
 y_2(\xi) \geq \max\{x, x + \xi_1 - \xi_2\} \\
 y_3(\xi) \geq \max\{x, x - \xi_1 + \xi_2\} \\
 y_4(\xi) \geq \max\{x, x - \xi_1 - \xi_2\}
 \end{array} \right\} \forall \xi \in \Xi \\
 & \tau, x \in \mathbb{R}_+, \quad \mathbf{y} : \Xi \mapsto \mathbb{R}^4
 \end{array}$$

In this problem, we set the uncertainty set to  $\Xi = \{\xi \in [-2, 2]^2 : \|\xi\|_1 \leq 3\}$ . The problem is optimized by the here-and-now decisions  $(\tau^*, x^*) = (4, 0)$ .

The uncertainty set  $\Xi$  satisfies  $|\text{ext } \Xi| = 8$ . If we lift one of the binding scenarios  $\xi^* \in \Xi_A^*$  randomly in each iteration, then our lifting scheme converges after 5 iterations. Figure 4 visualizes

the piecewise affine decision rules  $y_1(\boldsymbol{\xi})$  corresponding to the optimal primal solutions  $(\mathbf{x}, \mathbf{y}_A, \mathbf{y}_S)$  in every iteration of our lifting scheme. The initial dual bound  $\underline{\mathcal{P}}(\emptyset)$  has an optimal objective value of 3.33, and the dual bound attains the optimal objective value 4 of the two-stage robust optimization problem after lifting any single extreme point  $\boldsymbol{\xi} \in \text{ext } \Xi_A$ .

### 5.1.2 Worst-Case Makespan of a Temporal Network

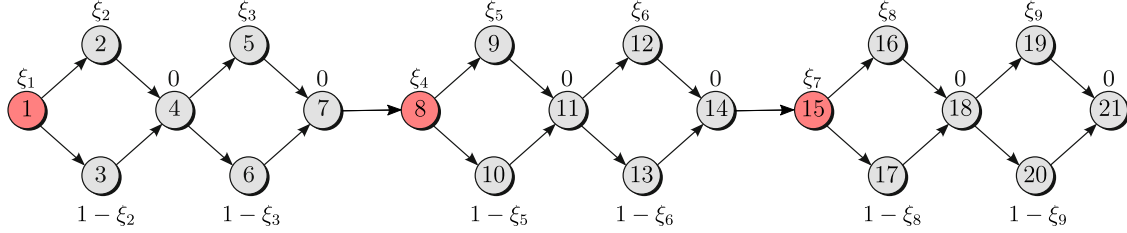
We now employ the two-stage robust optimization problem  $\mathcal{P}$  to estimate the worst-case makespan of a temporal network (*e.g.*, a project, a digital circuit or a production process). To this end, we define a temporal network as a directed, acyclic graph  $G = (V, E)$  whose nodes  $V = \{1, \dots, n\}$  represent the tasks and whose arcs  $E \subset V \times V$  denote the temporal precedences between the tasks. We assume that the duration  $d_i(\boldsymbol{\xi})$  of each task  $i \in V$  depends on the uncertain problem parameters  $\boldsymbol{\xi} \in \Xi$ . Moreover, we assume that the start times of the tasks  $i \in V_0 \subseteq V$  have to be chosen here-and-now, that is, before the realizations of the uncertain parameters  $\boldsymbol{\xi}$  are observed. This could be required to synchronize the network with other projects, circuits or production processes.

The problem is a variant of model (1) in [58], and it can be formulated as

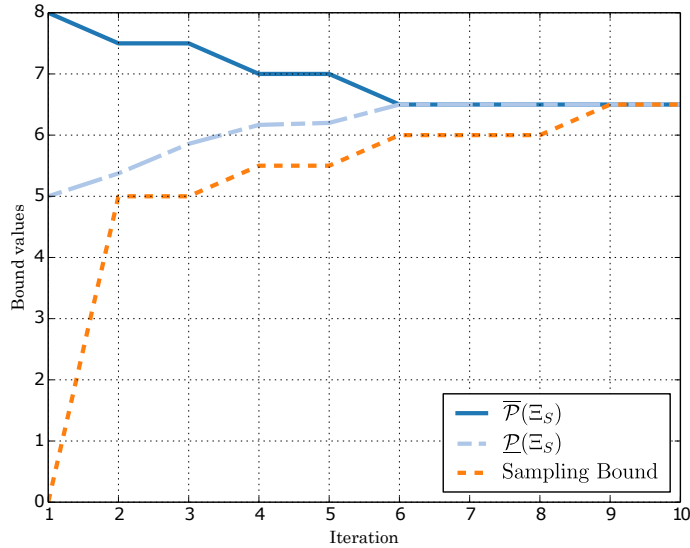
$$\begin{array}{ll}
 \text{minimize} & \tau \\
 \text{subject to} & \tau \geq y_n(\boldsymbol{\xi}) + d_n(\boldsymbol{\xi}) \\
 & y_j(\boldsymbol{\xi}) \geq y_i(\boldsymbol{\xi}) + d_i(\boldsymbol{\xi}) \quad \forall (i, j) \in E \\
 & y_i(\boldsymbol{\xi}) = x_i \quad \forall i \in V_0 \\
 & \tau \in \mathbb{R}_+, \quad x_i \in \mathbb{R}_+, \quad i \in V_0, \quad \mathbf{y} : \Xi \mapsto \mathbb{R}_+^n.
 \end{array} \left. \vphantom{\begin{array}{l} \\ \\ \\ \\ \end{array}} \right\} \forall \boldsymbol{\xi} \in \Xi$$

In this formulation, the wait-and-see decisions  $y_i(\boldsymbol{\xi})$  capture the start times of the tasks  $i \in V$ , the epigraphical here-and-now decision  $\tau$  records the worst-case makespan of the network, and the here-and-now decisions  $x_i$  correspond to the static start times of the restricted tasks  $i \in V_0$ . The presence of the restricted tasks  $i \in V_0$  implies that the problem has no relatively complete recourse.

We apply our primal-dual lifting scheme to the temporal network in Figure 5. We also present a variant of our dual bound  $\underline{\mathcal{P}}(\Xi_S)$  where we fix all affine decisions to  $\boldsymbol{\lambda}_A(\boldsymbol{\xi}) = \mathbf{0}$  for all  $\boldsymbol{\xi} \in \Xi_A$ . This variant, which we refer to as ‘sampling bound’, can be interpreted as an extension of the sampling bound in [37] that iteratively grows the scenario set  $\Xi_S$  (see Remark 3). The results of both methods are shown in Figure 6. Our approach converges to the optimal objective value of the



**Figure 5.** Temporal network with 21 tasks and 26 precedences. The durations  $d_i(\boldsymbol{\xi})$  are displayed above the tasks  $i \in V$ , and the uncertainty set is  $\Xi = \{\boldsymbol{\xi} \in \mathbb{R}^9 : \|\boldsymbol{\xi} - \mathbf{e}/2\|_1 \leq 1/2\}$ . The start times of the tasks 1, 8 and 15 have to be selected here-and-now.



**Figure 6.** Results of our primal-dual lifting scheme (top two blue lines) and the sampling bound (bottom orange line) applied to the temporal network in Figure 5.

two-stage robust optimization problem  $\mathcal{P}$  after 6 out of  $|\text{ext } \Xi| = 18$  possible iterations. The less flexible sampling bound requires 9 iterations to converge in this example.

## 5.2 Case Study: Inventory Management

We consider an inventory management problem with  $n$  products whose demands  $D_i(\boldsymbol{\xi})$ ,  $i = 1, \dots, n$ , are governed by the uncertain risk factors  $\boldsymbol{\xi} \in \mathbb{R}^k$ . The demand for product  $i$  can be served through a standard order  $x_i$  (with unit cost  $c_x$ ), which has to be placed before the demand is known, or through an express order  $y_i$  (with unit cost  $c_y > c_x$ ), which can be submitted after the demand has been observed. Any excess inventory  $h_i(\boldsymbol{\xi})$  and any backlogged demand  $b_i(\boldsymbol{\xi})$  in the second period incurs unit costs of  $c_h$  and  $c_b$ , respectively, and the express orders across all products must not exceed  $B$  units. The objective is to determine an ordering policy that minimizes the worst-case sum

of the ordering, inventory holding and backlogging costs over all anticipated demand realizations.

The problem can be formulated as the following instance of problem  $\mathcal{P}$ :

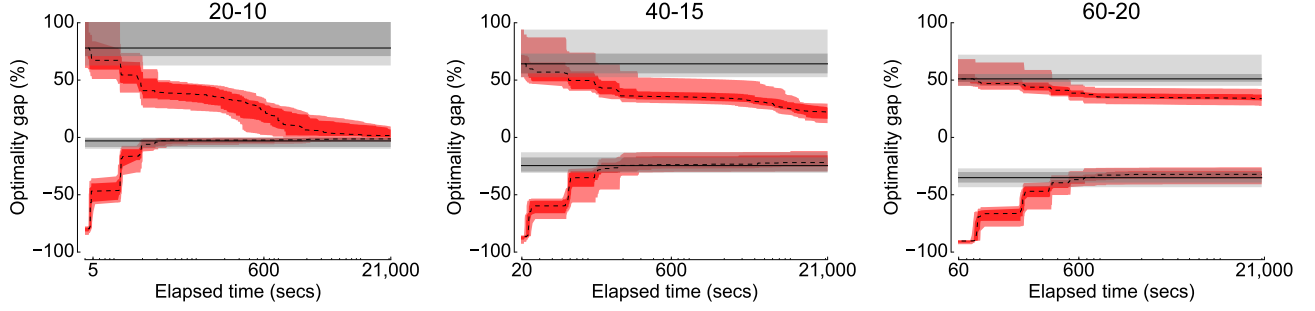
$$\begin{array}{ll}
\text{minimize} & \tau \\
\text{subject to} & \tau \geq \mathbf{e}^\top [c_x \cdot \mathbf{x} + c_y \cdot \mathbf{y}(\boldsymbol{\xi}) + c_h \cdot \mathbf{h}(\boldsymbol{\xi}) + c_b \cdot \mathbf{b}(\boldsymbol{\xi})] \\
& \mathbf{h}(\boldsymbol{\xi}) \geq \mathbf{x} + \mathbf{y}(\boldsymbol{\xi}) - \mathbf{D}(\boldsymbol{\xi}) \\
& \mathbf{b}(\boldsymbol{\xi}) \geq \mathbf{D}(\boldsymbol{\xi}) - \mathbf{x} - \mathbf{y}(\boldsymbol{\xi}) \\
& \mathbf{e}^\top \mathbf{y}(\boldsymbol{\xi}) \leq B \\
& \mathbf{x} \in \mathbb{R}_+^n, \quad \mathbf{y}, \mathbf{h}, \mathbf{b} : \Xi \mapsto \mathbb{R}_+^n
\end{array} \quad \left. \vphantom{\begin{array}{l} \\ \\ \\ \\ \\ \end{array}} \right\} \forall \boldsymbol{\xi} \in \Xi \quad (5)$$

Here, the epigraphical variable  $\tau$  records the worst-case costs over all demand realizations. Note that we model the second-stage decisions  $\mathbf{y}$ ,  $\mathbf{h}$  and  $\mathbf{b}$  as functions of the (typically unobservable) risk factors, as opposed to functions of the (eventually observable) product demands  $\mathbf{D}(\boldsymbol{\xi})$ . This simplification is justified since by construction, no optimal solution to problem (5) will take different second-stage decisions for different realizations of the risk factors  $\boldsymbol{\xi}$  that give rise to the same product demands  $\mathbf{D}(\boldsymbol{\xi})$ . Thus, we can always convert the optimal second-stage decisions in problem (5) to equivalent implementable decisions that only depend on the product demands  $\mathbf{D}(\boldsymbol{\xi})$ .

We assume that the  $n$  products are grouped into  $\lfloor \sqrt{n} \rfloor$  different product categories such that each category contains between  $\lfloor \sqrt{n} \rfloor$  and  $\lfloor \sqrt{n} \rfloor + 1$  products. The product demands are governed by a factor model of the form

$$D_i(\boldsymbol{\xi}) = \boldsymbol{\phi}_i^\top \boldsymbol{\xi} + \varphi_i, \quad i = 1, \dots, n, \quad \text{with} \quad \boldsymbol{\xi} \in \Xi = [-1, 1]^k.$$

For each product  $i = 1, \dots, n$ , we choose the factor loading vector  $\boldsymbol{\phi}_i \in \mathbb{R}^k$  uniformly at random from  $[-1, 1]^k$  and subsequently scale it so that  $\{\boldsymbol{\phi}_i^\top \boldsymbol{\xi} + \varphi_i : \boldsymbol{\xi} \in \Xi\} = [0, 2]$  for a suitably chosen scalar  $\varphi_i \in \mathbb{R}$ . Moreover, we ensure that the factor loading vectors  $\boldsymbol{\phi}_i, \boldsymbol{\phi}_j$  associated with two products  $i, j$  of the same category have the same signs (but typically not the same values) for each component, that is,  $\text{sgn}(\phi_{il}) = \text{sgn}(\phi_{jl})$  for all  $l = 1, \dots, k$ . This expresses the assumption that the demands for products of the same category are positively correlated. We fix  $c_x = 0$  and select  $c_y$  uniformly at random from  $[0, 2]$ . Thus, we interpret the standard orders as sunk costs, and we aim to minimize the amount of express deliveries, which carry a per-unit premium of  $c_y$  over the standard orders. We select the inventory holding and backlogging costs uniformly at random from



**Figure 7. Inventory Management:** Comparison of our solution scheme and the affine decision rule approximation combined with the sampling bound of [37] for small (left), medium (middle) and large (right) instances. The dashed (bold) line represents the median, the dark red (dark gray) region the range spanned by the first and third quartile, and the light red (light gray) region the range spanned by the 10% and 90% quantile of the relative distances of the upper and lower bounds of our solution scheme (the affine decision rules combined with the sampling bound).

[3, 5], and we set the upper bound on the express deliveries to  $B = n$ . We found this parameter choice to be particularly challenging for affine decision rules, which can otherwise be close to optimal in inventory management problems [9].

To select the binding scenario  $\xi^* \in \Xi_A^*$  to lift in each iteration of our primal-dual lifting scheme, we fix the optimal solution to the bounding problem  $\bar{\mathcal{P}}(\Xi_S)$  and determine one binding scenario for each constraint by minimizing the constraint’s slack over all  $\xi \in \Xi_A$ . For each such scenario  $\xi^*$ , we solve the lifted lower bound problem  $\underline{\mathcal{P}}(\Xi'_S)$ ,  $\Xi'_S = \Xi_S \cup \{\xi^*\}$ , under the additional restriction that  $\lambda_A(\xi) = \mathbf{0}$  for all  $\xi \in \Xi'_A$ . This restriction ensures that the lifted lower bound problem can be solved without actually calculating  $\Xi'_A$ . If the lifted lower bound problem  $\underline{\mathcal{P}}(\Xi'_S)$  improves upon  $\underline{\mathcal{P}}(\Xi_S)$  for any of the binding scenarios, then the binding scenario corresponding to the largest improvement is lifted. Otherwise, for each of the identified binding scenarios, we solve the lifted upper bound problem  $\bar{\mathcal{P}}(\Xi'_S)$ ,  $\Xi'_S = \Xi_S \cup \{\xi^*\}$ , and we lift the first scenario that improves the upper bound. If no such scenario is found, then we lift one of the identified binding scenarios randomly.

We first investigate the scalability of our primal-dual lifting scheme as the number of products and the number of uncertain parameters vary. The results are presented in Figure 7 and the first part of Table 2. Each instance class is identified by the label ‘ $n$ - $k$ ’, where  $n$  denotes the number of products and  $k$  refers to the number of uncertain parameters, respectively. The figure reports the optimality gaps of (i) the upper and lower bounds of our solution scheme and (ii) the affine decision rule approximation combined with the sampling bound of [37], both relative to the optimal solution

Instance	Primal-Dual			FME & SB			Zeng & Zhao			ADR & SB	SB-Heuristic
	10 min	1 hour	6 hours	10 min	1 hour	6 hours	10 min	1 hour	6 hours		
10-10	2.6%	0.4%	0%	25.8%	25.8%	25.8%	0%	0%	0%	72.8%	16.3%
15-10	15.6%	9.1%	1.0%	37.6%	37.2%	37.2%	31.9%	0.1%	0%	79.3%	13.0%
15-15	44.5%	28.2%	21.6%	41.3%	40.1%	40.1%	108.4%	4.6%	0%	78.9%	15.3%
20-10	29.9%	13.7%	6.1%	45.6%	44.9%	43.8%	-	52.3%	21.3%	82.3%	12.0% <sup>†</sup>
20-15	55.2%	39.6%	25.6%	51.7%	50.6%	48.8%	-	-	74.2%	90.4%	44.7% <sup>†</sup>
20-20	58.0%	55.9%	54.4%	52.5%	50.7%	50.7%	-	-	132.4%	73.6%	63.0% <sup>†</sup>
40-10	40.1%	26.2%	11.1%	56.8%	55.7%	55.7%	-	-	-	85.6%	85.6% <sup>†</sup>
40-15	57.1%	53.3%	42.7%	64.9%	62.4%	62.4%	-	-	-	90.1%	90.1% <sup>†</sup>
40-20	63.4%	61.8%	59.6%	70.6%	68.1%	67.4%	-	-	-	80.0%	80.0% <sup>†</sup>
60-10	41.7%	33.3%	21.3%	67.9%	65.8%	65.8%	-	-	-	94.0%	94.0% <sup>†</sup>
60-15	63.6%	57.9%	51.5%	69.5%	65.3%	65.3%	-	-	-	84.1%	84.1% <sup>†</sup>
60-20	72.4%	67.1%	66.5%	74.5%	72.5%	72.5%	-	-	-	86.5%	86.5% <sup>†</sup>

**Table 2. Inventory Management:** Overall optimality gaps of our method, the Fourier-Motzkin elimination proposed by Zhen et al. [63], the column-and-constraint generation of Zeng and Zhao [61], the affine decision rule approximation as well as the sampling-based heuristic for various time limits and instance classes. Missing elements indicate that no feasible solution has been obtained. ‘†’ indicates that the MILP for the worst-case cost estimation has not been solved within six hours. We do not show the runtimes for the affine decision rule approximation as the results were obtained within 1 minute for all instances.

of the inventory management problem. When the optimal solution has not been determined by either method, we use the mean value of (i) the smaller of the two upper bounds and (ii) the larger of the two lower bounds instead. The table reports the corresponding overall gaps, which we define as the sums of the optimality gaps of the respective upper and lower bounds. All values are averages over 25 randomly generated instances.

We observe that the affine decision rule approximation combined with the sampling bound of [37] incurs a large optimality gap in all problem instances. We suspect that in this case study, the sampling bound is consistently close to optimal, and that the optimality gap is caused by the affine decision rule approximation. Our primal-dual lifting scheme can close the gap of the affine decision rule approximation for small instances and significantly reduce the gap for larger instances. While the performance of our algorithm deteriorates with both the number of products and the number of uncertain parameters, the latter has a more pronounced impact on the efficacy of our algorithm. Nevertheless, our method closes between 99% (15-10) and 23% (60-20) of the optimality gap incurred by the affine decision rule approximation and the sampling bound.

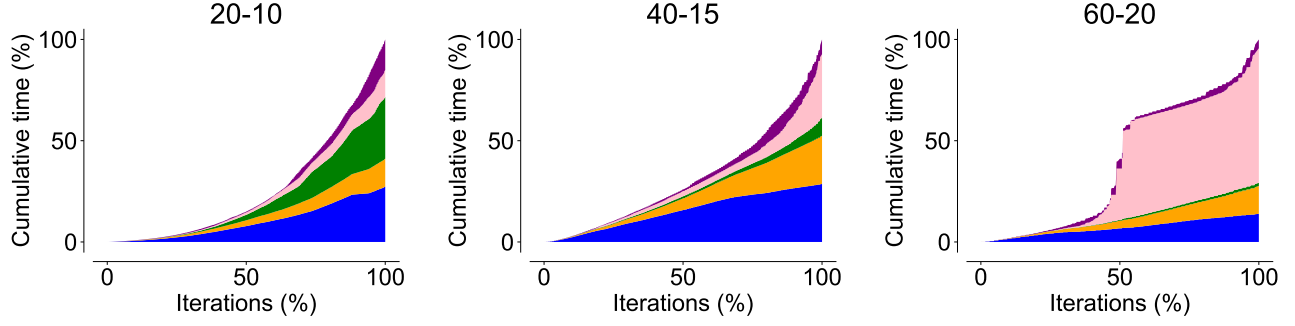
Table 2 compares our method (‘Primal-Dual’) with the Fourier-Motzkin elimination approach proposed by Zhen et al. [63] combined with the sampling bound of [37] (‘FME & SB’), the column-



and-constraint generation scheme of Zeng and Zhao [61] ('Zeng & Zhao'), as well as the affine decision rule approximation combined with the sampling bound of [37] ('ADR & SB'). As a benchmark, we also report the results of a sampling-based heuristic ('SB-Heuristic') that solves the affine decision rule problem, followed by the progressive approximation of [37] on all binding scenarios. The here-and-now decision of the progressive approximation is feasible for the two-stage robust optimization problem due to the relatively complete recourse. We subsequently evaluate the worst-case cost of this decision using the subproblem of Zeng and Zhao's approach [61], either exactly (if it can be solved within six hours) or approximately (using the best bound obtained after six hours). While our method is strongly affected by the number of uncertain parameters, the approach of Zeng and Zhao is impacted primarily by the number of products. This is expected since each iteration of their method requires the solution of a subproblem, which is a mixed-integer linear program whose size scales with the number of products. The Fourier-Motzkin elimination, on the other hand, is moderately affected by both the number of uncertain parameters and the number of products. We have also solved the inventory management problem with the piecewise affine decision rules proposed in [31]. We have not been able to obtain any noticeable improvements relative to the results of the affine decision rules, which we attribute to the fact that it appears difficult to choose appropriate breakpoints for the decision rules *a priori*. Moreover, we have attempted to replace the sampling bound of [37] with the refined bound proposed in [12]. This has not led to significant improvements either. This is in line with our earlier conjecture that the sampling bound of [37] appears to be close to optimal in this problem. Finally, we have tried to solve the instances with the extreme point reformulation  $\mathcal{P}'$ . For the given time limit of six hours, this reformulation could only be solved for instances with 10 random variables.

We conclude that our primal-dual lifting scheme, the Fourier-Motzkin elimination approach and the column-and-constraint generation scheme are complementary in the considered problem, and each method can outperform the other two in some of the instances. It is noteworthy that the methods can also be combined: We could, for example, combine the variable elimination steps of the Fourier-Motzkin elimination with the parameter lifting steps of our scheme. Initial attempts to do so have produced promising results in our numerical tests (not reported here).

Figure 8 shows the fractions of the runtime that are spent on the different steps of our primal-dual lifting scheme. We observe that for instances with 10 and 15 random variables, most of the



**Figure 8. Inventory Management:** Percentages of the overall runtime of our algorithm (ordinate) that are spent on the different steps in each iteration (abscissa; normalized to percentages). From bottom to top, the shaded areas represent the times spent on solving the upper bound problems  $\overline{\mathcal{P}}(\Xi_S)$ , solving the lower bound problems  $\underline{\mathcal{P}}(\Xi_S)$ , finding the neighbours of the extreme points  $\xi^* \in \Xi_A^*$  to be lifted, calculating the convex hulls  $\text{conv}([\text{ext } \Xi_A] \setminus \{\xi^*\})$  and the remaining overhead.

runtime is spent on the solution of the upper and lower bound problems  $\overline{\mathcal{P}}(\Xi_S)$  and  $\underline{\mathcal{P}}(\Xi_S)$ . This is due to the fact that our scenario selection scheme solves both of these problems for a large number of scenarios in each iteration in order to determine the scenario  $\xi^* \in \Xi_A^*$  to lift. For instances with 20 random variables, on the other hand, the calculation of the convex hull  $\text{conv}([\text{ext } \Xi_A] \setminus \{\xi^*\})$  that is used to update the affine set  $\Xi_A$  dominates the runtime, especially in the later stages of the algorithm. It therefore appears justified to carefully select the most promising scenario to lift in each iteration of the algorithm. The identification of the neighbourhood  $N(\xi^*)$ , finally, consumes a negligible amount of the overall time once the problem size grows.

### 5.3 Case Study: Inventory Management with Pre-Commitment

We now consider two variants of the inventory management problem from Section 5.2 that require different types of pre-commitment to the express orders  $\mathbf{y}(\xi)$  taken in the second stage. Both pre-commitments are modeled as binary here-and-now decisions, and they thus result in two-stage robust optimization problems with mixed-binary here-and-now decisions (see Remarks 1, 2 and 4).

In the first variant, the decision maker can choose the express order quantities  $\mathbf{y}(\xi)$  freely in the second stage, but she has to decide here-and-now which  $B = \lfloor n/4 \rfloor$  of the  $n$  products will carry reduced express order unit costs of  $c_y/2$  (instead of  $c_y$ , as before). This variant of our inventory

management problem can be formulated as the following instance of the model  $\mathcal{P}_d$ :

$$\begin{aligned}
& \text{minimize} && \tau \\
& \text{subject to} && \left. \begin{aligned}
& \tau \geq \mathbf{e}^\top [c_x \cdot \mathbf{x} + \mathbf{o}(\boldsymbol{\xi}) + c_h \cdot \mathbf{h}(\boldsymbol{\xi}) + c_b \cdot \mathbf{b}(\boldsymbol{\xi})] \\
& \mathbf{h}(\boldsymbol{\xi}) \geq \mathbf{x} + \mathbf{y}(\boldsymbol{\xi}) - \mathbf{D}(\boldsymbol{\xi}) \\
& \mathbf{b}(\boldsymbol{\xi}) \geq \mathbf{D}(\boldsymbol{\xi}) - \mathbf{x} - \mathbf{y}(\boldsymbol{\xi}) \\
& \mathbf{o}(\boldsymbol{\xi}) \geq \frac{c_y}{2} \cdot \mathbf{y}(\boldsymbol{\xi}), \quad \mathbf{o}(\boldsymbol{\xi}) \geq c_y \cdot \mathbf{y}(\boldsymbol{\xi}) - M \cdot \mathbf{z} \\
& \mathbf{e}^\top \mathbf{z} \leq B \\
& \mathbf{x} \in \mathbb{R}_+^n, \quad \mathbf{y}, \mathbf{h}, \mathbf{b}, \mathbf{o} : \Xi \mapsto \mathbb{R}_+^n, \quad \mathbf{z} \in \{0, 1\}^n
\end{aligned} \right\} \forall \boldsymbol{\xi} \in \Xi \tag{6}
\end{aligned}$$

In this problem, the here-and-now decisions  $z_i$  indicate whether the express order costs of product  $i = 1, \dots, n$  are reduced to  $c_y/2$  (if  $z_i = 1$ ) or not (if  $z_i = 0$ ). At optimality, the auxiliary express order cost variables  $\mathbf{o}(\boldsymbol{\xi})$  will take the values  $o_i(\boldsymbol{\xi}) = \frac{c_y}{2} \cdot \mathbf{y}(\boldsymbol{\xi})$  if  $z_i = 1$  and  $o_i(\boldsymbol{\xi}) = c_y \cdot \mathbf{y}(\boldsymbol{\xi})$  otherwise. Since  $c_y \in [0, 2]$  and  $D_i(\boldsymbol{\xi}) \in [0, 2]$  for all  $\boldsymbol{\xi} \in \Xi$  and  $i = 1, \dots, n$ , we can set  $M = 4$ .

Instance	Primal-Dual			FME & SB			Zeng & Zhao			ADR & SB	SB-Heuristic
	10 min	1 hour	6 hours	10 min	1 hour	6 hours	10 min	1 hour	6 hours		
10-10	0.03%	0%	0%	44%	41%	41%	-	0.08%	0.08%	83%	36.7%
15-10	0.02%	0%	0%	43%	40%	40%	-	-	-	56%	44.4% <sup>†</sup>
15-15	67%	52%	15%	60%	55%	55%	-	-	-	84%	73.3% <sup>†</sup>
20-10	37%	22%	1.2%	72%	66%	53%	-	-	-	77%	77% <sup>†</sup>
20-15	-	72%	66%	102%	96%	71%	-	-	-	102%	102% <sup>†</sup>
20-20	-	61%	50%	72%	66%	51%	-	-	-	71%	71% <sup>†</sup>

**Table 3. Inventory Management with Pre-Commitment (1):** The table has the same interpretation as Table 2. We do not show the runtimes for the affine decision rule approximation as the results were obtained within 10 minutes for all instances.

Table 3 compares our primal-dual lifting scheme with the alternative solution approaches on the revised problem (6). Again, the extreme point reformulation  $\mathcal{P}'$  could only be solved for instances with 10 random variables within the given time limit of six hours. Apart from a general increase of the optimality gaps, which is owed to the increased runtimes caused by the presence of the binary variables, the results for the primal-dual lifting scheme and the Fourier-Motzkin elimination are qualitatively similar to those of the previous section. The column-and-constraint generation scheme, on the other hand, fails to provide useful bounds within the set time limit. A detailed inspection of the algorithm's progress revealed that the lack of progress is caused by the subproblems, whose increased size (both in terms of the number of variables and the number of constraints) have made

their (approximate) solution significantly more challenging for all but the smallest instances.

We finally consider a variant of the inventory management problem from Section 5.2 where the decision maker has to decide here-and-now for which  $B = \lfloor n/2 \rfloor$  of the  $n$  products she can submit express orders in the second stage. This variant of our inventory management problem can be formulated as the following instance of the model  $\mathcal{P}_d$ :

$$\begin{aligned}
 & \text{minimize} && \tau \\
 & \text{subject to} && \tau \geq \mathbf{e}^\top [c_x \cdot \mathbf{x} + c_y \cdot \mathbf{y}(\boldsymbol{\xi}) + c_h \cdot \mathbf{h}(\boldsymbol{\xi}) + c_b \cdot \mathbf{b}(\boldsymbol{\xi})] \\
 & && \mathbf{h}(\boldsymbol{\xi}) \geq \mathbf{x} + \mathbf{y}(\boldsymbol{\xi}) - \mathbf{D}(\boldsymbol{\xi}) \\
 & && \mathbf{b}(\boldsymbol{\xi}) \geq \mathbf{D}(\boldsymbol{\xi}) - \mathbf{x} - \mathbf{y}(\boldsymbol{\xi}) \\
 & && \mathbf{y}(\boldsymbol{\xi}) \leq M \cdot \mathbf{z} \\
 & && \mathbf{e}^\top \mathbf{z} \leq B \\
 & && \mathbf{x} \in \mathbb{R}_+^n, \mathbf{y}, \mathbf{h}, \mathbf{b} : \Xi \mapsto \mathbb{R}_+^n, \mathbf{z} \in \{0, 1\}^n
 \end{aligned} \quad \left. \vphantom{\begin{aligned} \text{subject to} \end{aligned}} \right\} \forall \boldsymbol{\xi} \in \Xi \quad (7)$$

In this problem, there here-and-now decisions  $z_i$  indicate whether express orders for product  $i = 1, \dots, n$  can be submitted (if  $z_i = 1$ ) or not (if  $z_i = 0$ ) in the second stage. Since  $D_i(\boldsymbol{\xi}) \in [0, 2]$  for all  $\boldsymbol{\xi} \in \Xi$  and  $i = 1, \dots, n$ , we can set  $M = 2$ .

Instance	Primal-Dual			FME & SB			Zeng & Zhao			ADR & SB	SB-Heuristic
	10 min	1 hour	6 hours	10 min	1 hour	6 hours	10 min	1 hour	6 hours		
10-10	27%	18%	0%	63%	63%	63%	0%	0%	0%	92%	16.9%
15-10	49%	22%	18%	61%	61%	59%	0.4%	0%	0%	79%	19.9%
15-15	76%	69%	52%	80%	80%	79%	19%	0%	0%	90%	21.8%
20-10	-	65%	56%	73%	73%	72%	86%	13%	0%	88%	31.3%
20-15	-	-	71%	79%	79%	79%	85%	20%	3%	85%	36.6%
20-20	-	-	71%	77%	77%	77%	-	72%	7%	80%	34.4%

**Table 4. Inventory Management with Pre-Commitment (2):** The table has the same interpretation as Table 2. We do not show the runtimes for the affine decision rule approximation as the results were obtained within 10 minutes for all instances.

Table 4 compares our primal-dual lifting scheme with the alternative solution approaches on the revised problem (7). As before, the extreme point reformulation  $\mathcal{P}'$  could only be solved for instances with 10 random variables within the given time limit of six hours, and the results for the primal-dual lifting scheme and the Fourier-Motzkin elimination are qualitatively similar to the previous ones. Interestingly, the column-and-constraint generation scheme performs much better in this problem. Further investigation revealed that this is due to the subproblems, which simplify

dramatically since (i) there are no additional second-stage decisions  $o_i(\boldsymbol{\xi})$  and (ii) the binary here-and-now decisions  $z_i$  set  $n - B = \lfloor n/2 \rfloor$  of the  $n$  express order decisions  $y_i(\boldsymbol{\xi})$  to zero, which essentially removes these decisions (as well as the associated constraints) from the subproblems.

## Acknowledgments

The authors gratefully acknowledge funding from the Imperial Business Analytics Centre and the EPSRC grants EP/M028240/1, EP/M027856/1 and EP/N020030/1. We are also indebted to David Avis and Komei Fukuda for valuable advice on various aspects of polyhedral combinatorics, as well as the constructive suggestions of the three anonymous referees.

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## A Appendix: Moment Matrices for Common Uncertainty Sets

This appendix elaborates on the calculation of the moment matrix  $\Sigma_A$  in the dual problem  $\underline{\mathcal{P}}(\Xi_S)$ . We first present the moment matrices  $\Sigma_A$  of several commonly used primitive uncertainty sets in Section A.1. Afterwards, Section A.2 shows how a moment matrix changes if particular classes of transformations are applied to an uncertainty set.

### A.1 Primitive Uncertainty Sets

In the following, we list the moment matrices  $\Sigma_A$  of several commonly used primitive uncertainty sets. Since the derivations are basic but tedious, we omit them for the sake of brevity. In the following, we denote by  $\lfloor x \rfloor$  and  $\lceil x \rceil$  the largest (smallest) integer less than or equal to (larger than or equal to)  $x \in \mathbb{R}$ .

#### A.1.1 1-Norm Ball Uncertainty Sets

For a 1-norm ball uncertainty set of the form

$$\Xi = \left\{ \boldsymbol{\xi} \in \mathbb{R}^k : \|\boldsymbol{\xi}\|_1 \leq 1 \right\},$$

the set of extreme points satisfies

$$\text{ext} \left\{ \boldsymbol{\xi} \in \mathbb{R}^k : \|\boldsymbol{\xi}\|_1 \leq 1 \right\} = \text{ext} \left\{ \boldsymbol{\xi} \in \mathbb{R}^k : \sum_{i=1}^k |\xi_i| \leq 1 \right\} = \left\{ \pm \mathbf{e}_i \in \mathbb{R}^k : i = 1, \dots, k \right\},$$

and we therefore obtain the moment matrix

$$\Sigma_A = \begin{pmatrix} 2k & \mathbf{0}^\top \\ \mathbf{0} & 2 \cdot \mathbf{I} \end{pmatrix}.$$

#### A.1.2 $\infty$ -Norm Ball Uncertainty Sets

We now consider an  $\infty$ -norm ball uncertainty set of the form

$$\Xi = \left\{ \boldsymbol{\xi} \in \mathbb{R}^k : \|\boldsymbol{\xi}\|_\infty \leq 1 \right\}.$$

Uncertainty sets of this type are commonly employed to describe the demand in operations management applications [8, 34] or when the uncertainty underlying the parameters  $\boldsymbol{\xi}$  is described by a factor model [35]. The set of extreme points satisfies

$$\text{ext} \left\{ \boldsymbol{\xi} \in \mathbb{R}^k : \|\boldsymbol{\xi}\|_\infty \leq 1 \right\} = \text{ext} \left\{ \boldsymbol{\xi} \in \mathbb{R}^k : \max_{i=1,\dots,k} |\xi_i| \leq 1 \right\} = \{-1, 1\}^k,$$

and we therefore obtain the moment matrix

$$\boldsymbol{\Sigma}_A = \begin{pmatrix} 2^k & \mathbf{0}^\top \\ \mathbf{0} & 2^k \cdot \mathbf{I} \end{pmatrix}.$$

### A.1.3 $(1 \cap \infty)$ -Norm Ball Uncertainty Sets

We now consider an uncertainty set that emerges from the intersection of a scaled 1-norm ball and an  $\infty$ -norm ball:

$$\Xi = \left\{ \boldsymbol{\xi} \in \mathbb{R}^k : \|\boldsymbol{\xi}\|_1 \leq \kappa, \|\boldsymbol{\xi}\|_\infty \leq 1 \right\}$$

For  $\kappa \leq 1$  and  $\kappa \geq k$ , the uncertainty set reduces to a scaled 1-norm ball and an  $\infty$ -norm ball, respectively. We therefore assume that  $\kappa \in (1, k)$ . Uncertainty sets of this type are commonly used as polyhedral outer approximations of ellipsoidal uncertainty sets [29, 35].

If  $\kappa \in \mathbb{N}$ , we obtain the moment matrix

$$\boldsymbol{\Sigma}_A = \begin{pmatrix} 2^\kappa \binom{k}{\kappa} & \mathbf{0}^\top \\ \mathbf{0} & 2^\kappa \binom{k-1}{\kappa-1} \cdot \mathbf{I} \end{pmatrix}.$$

For fractional  $\kappa$ , on the other hand, the moment matrix is

$$\boldsymbol{\Sigma}_A = \begin{pmatrix} 2^{\lceil \kappa \rceil} \binom{k}{\lceil \kappa \rceil} & \mathbf{0}^\top \\ \mathbf{0} & 2^{\lceil \kappa \rceil} \binom{k-1}{\lceil \kappa \rceil} (\lfloor \kappa \rfloor + (\kappa - \lfloor \kappa \rfloor)^2) \cdot \mathbf{I} \end{pmatrix}.$$

### A.1.4 Budget Uncertainty Sets

We next consider a budget uncertainty set of the form

$$\Xi = \left\{ \boldsymbol{\xi} \in [0, 1]^k : \mathbf{e}^\top \boldsymbol{\xi} \leq B \right\},$$

where  $B \in \mathbb{N}_0$ . Note that  $B = 0$  and  $B \geq k$  correspond to the cases  $\Xi = \{\mathbf{0}\}$  and  $\Xi = [0, 1]^k$  (the latter being a translation of the  $\infty$ -norm ball, which can be calculated using the results of Section A.1.2 and the transformations from Section A.2) and can therefore be omitted. Budget uncertainty sets have been popularized by [21] and have since been applied widely across domains.

For  $B \in \{1, \dots, k-1\}$ , the moment matrix is

$$\Sigma_A = \begin{pmatrix} \sum_{i=0}^B \binom{k}{i} & \left[ \sum_{i=0}^{B-1} \binom{k-1}{i} \right] \cdot \mathbf{e}^\top \\ \left[ \sum_{i=0}^{B-1} \binom{k-1}{i} \right] \cdot \mathbf{e} & \left[ \sum_{i=0}^{B-1} \binom{k-1}{i} - \sum_{i=0}^{B-2} \binom{k-2}{i} \right] \cdot \mathbf{I} + \left[ \sum_{i=0}^{B-2} \binom{k-2}{i} \right] \cdot \mathbf{e}\mathbf{e}^\top \end{pmatrix}.$$

### A.1.5 Central Limit Theorem-Type Uncertainty Sets

We finally consider a central limit theorem-type uncertainty set of the form

$$\Xi = \left\{ \boldsymbol{\xi} \in [-1, 1]^k : -\Gamma \leq |\mathbf{e}^\top \boldsymbol{\xi}| \leq +\Gamma \right\},$$

where  $\Gamma \in (0, k)$  [7]. In this case, we have  $|\text{ext } \Xi| = \eta_1 + \eta_2 + \eta_3$ , where

$$\eta_1 = \begin{cases} \binom{k}{\frac{k}{2}} + 2 \sum_{i=1}^{\lfloor \frac{\Gamma}{2} \rfloor} \binom{k}{\frac{k}{2} + i} & \text{if } k \text{ is even,} \\ 2 \sum_{i=0}^{\lfloor \frac{\Gamma-1}{2} \rfloor} \binom{k}{\frac{k+1}{2} + i} & \text{if } k \text{ is odd;} \end{cases}$$

$$\eta_2 = \begin{cases} 2k \binom{k-1}{\frac{k+\Gamma-1}{2}} & \text{if } k \text{ is even and } \Gamma \text{ is odd, or vice versa,} \\ 0 & \text{otherwise;} \end{cases}$$

$$\eta_3 = \begin{cases} 2 \binom{k}{\frac{k}{2} + \lfloor \frac{\Gamma}{2} \rfloor} \cdot \left( \frac{k}{2} - \left\lfloor \frac{\Gamma}{2} \right\rfloor \right) & \text{if } k \text{ is even and } \Gamma \text{ is fractional,} \\ 2 \binom{k}{\frac{k+1}{2} + \lfloor \frac{\Gamma-1}{2} \rfloor} \cdot \left( \frac{k-1}{2} - \left\lfloor \frac{\Gamma-1}{2} \right\rfloor \right) & \text{if } k \text{ is odd and } \Gamma \text{ is fractional,} \\ 0 & \text{otherwise.} \end{cases}$$

Due to the symmetry of  $\Xi$ , the sum of first moments satisfies  $\sum_{\xi \in \text{ext } \Xi} \xi = \mathbf{0}$ . The sum of second moments, finally, satisfies  $\sum_{\xi \in \text{ext } \Xi} \xi \xi^\top = \omega \cdot (\mathbf{e} \mathbf{e}^\top) + (\delta - \omega) \cdot \mathbf{I}$ , where  $\delta = \delta_1 + \delta_2 + \delta_3$  with

$$\delta_1 = \begin{cases} \binom{k}{\frac{k}{2}} + 2 \sum_{i=1}^{\lfloor \frac{\Gamma}{2} \rfloor} \binom{k}{\frac{k}{2} + i} & \text{if } k \text{ is even,} \\ 2 \sum_{i=0}^{\lfloor \frac{\Gamma-1}{2} \rfloor} \binom{k}{\frac{k+1}{2} + i} & \text{if } k \text{ is odd;} \end{cases}$$

$$\delta_2 = \begin{cases} 2(k-1) \binom{k-1}{\frac{k+\Gamma-1}{2}} & \text{if } k \text{ is even and } \Gamma \text{ is odd, or vice versa,} \\ 0 & \text{otherwise;} \end{cases}$$

$$\delta_3 = \begin{cases} 2 \binom{k}{\frac{k}{2} + \lfloor \frac{\Gamma}{2} \rfloor} \binom{k}{\frac{k}{2} - \lfloor \frac{\Gamma}{2} \rfloor} \left[ \frac{k-1}{k} + \frac{1}{k} \cdot \left( 1 - \left[ \Gamma - 2 \lfloor \frac{\Gamma}{2} \rfloor \right] \right)^2 \right] & \text{if } k \text{ is even and} \\ & \Gamma \text{ is fractional,} \\ 2 \binom{k}{\frac{k+1}{2} + \lfloor \frac{\Gamma-1}{2} \rfloor} \binom{k}{\frac{k-1}{2} - \lfloor \frac{\Gamma-1}{2} \rfloor} \left[ \frac{k-1}{k} + \frac{1}{k} \cdot \left( 2 - \Gamma + 2 \lfloor \frac{\Gamma-1}{2} \rfloor \right)^2 \right] & \text{if } k \text{ is odd and} \\ & \Gamma \text{ is fractional,} \\ 0 & \text{otherwise,} \end{cases}$$

as well as  $\omega = \omega_1 + \omega_2 + \omega_3$  with

$$\omega_1 = \begin{cases} 2 \sum_{i=\max\{2-\frac{k}{2}, -\lfloor \frac{\Gamma}{2} \rfloor\}}^{\min\{\frac{k}{2}, \lfloor \frac{\Gamma}{2} \rfloor\}} \binom{k-2}{\frac{k}{2} + i - 2} - 2 \sum_{i=\max\{1-\frac{k}{2}, -\lfloor \frac{\Gamma}{2} \rfloor\}}^{\min\{\frac{k}{2}-1, \lfloor \frac{\Gamma}{2} \rfloor\}} \binom{k-2}{\frac{k}{2} + i - 1} & \text{if } k \text{ is even,} \\ 2 \sum_{i=\max\{\frac{1}{2}-\frac{k}{2}, -\lfloor \frac{\Gamma-1}{2} \rfloor-2\}}^{\min\{\frac{k}{2}-\frac{3}{2}, \lfloor \frac{\Gamma-1}{2} \rfloor-1\}} \binom{k-2}{\frac{k-1}{2} + i} - 2 \sum_{i=\max\{\frac{1}{2}-\frac{k}{2}, -\lfloor \frac{\Gamma-1}{2} \rfloor-1\}}^{\min\{\frac{k}{2}-\frac{3}{2}, \lfloor \frac{\Gamma-1}{2} \rfloor\}} \binom{k-2}{\frac{k-1}{2} + i} & \text{if } k \text{ is odd;} \end{cases}$$

$\omega_2 = \omega_{21} + \omega_{22} + \omega_{23}$  if  $k$  is even and  $\Gamma$  is odd, or vice versa, and  $\omega_2 = 0$  otherwise, where

$$\omega_{21} = \begin{cases} 2(k-2) \binom{k-3}{\frac{k+\Gamma-5}{2}} & \text{if } k + \Gamma \geq 5, k \geq \Gamma + 1, \\ 0 & \text{otherwise;} \end{cases}$$

$$\omega_{22} = \begin{cases} 2(k-2) \binom{k-3}{\frac{k-\Gamma-5}{2}} & \text{if } k \geq \Gamma + 5, \\ 0 & \text{otherwise;} \end{cases} \quad \omega_{23} = \begin{cases} -4 \cdot (k-2) \binom{k-3}{\frac{k+\Gamma-3}{2}} & \text{if } k \geq \Gamma + 3, \\ 0 & \text{otherwise;} \end{cases}$$

and finally  $\omega_3 = \omega_{31} - \omega_{32} + \omega_{33} - \omega_{34}$  if  $k$  is even and  $\Gamma$  is fractional, where

$$\omega_{31} = \begin{cases} 2 \binom{k-2}{\frac{k}{2} + \lfloor \frac{\Gamma}{2} \rfloor - 2} \cdot \left( \frac{k}{2} - \lfloor \frac{\Gamma}{2} \rfloor \right) + 2 \binom{k-2}{\frac{k}{2} - \lfloor \frac{\Gamma}{2} \rfloor - 2} \cdot \left( \frac{k}{2} - \lfloor \frac{\Gamma}{2} \rfloor - 2 \right) & \text{if } \frac{k}{2} \geq \lfloor \frac{\Gamma}{2} \rfloor + 2, \\ 0 & \text{otherwise;} \end{cases}$$

$$\omega_{32} = 4 \binom{k-2}{\frac{k}{2} + \lfloor \frac{\Gamma}{2} \rfloor - 1} \cdot \left( \frac{k}{2} - \lfloor \frac{\Gamma}{2} \rfloor - 1 \right);$$

$$\omega_{33} = \begin{cases} 4 \binom{k-2}{\frac{k}{2} - \lfloor \frac{\Gamma}{2} \rfloor - 2} & \text{if } \frac{k}{2} \geq \lfloor \frac{\Gamma}{2} \rfloor + 2, \\ 0 & \text{otherwise;} \end{cases} \quad \omega_{34} = \begin{cases} 4 \binom{k-2}{\frac{k}{2} - \lfloor \frac{\Gamma}{2} \rfloor - 1} & \text{if } \frac{k}{2} \geq \lfloor \frac{\Gamma}{2} \rfloor + 1, \\ 0 & \text{otherwise;} \end{cases}$$

as well as  $\omega_3 = \omega_{35} + \omega_{36} - \omega_{37} + \omega_{38} - \omega_{39}$  if  $k$  is odd and  $\Gamma$  is fractional, where

$$\omega_{35} = \begin{cases} 2 \binom{k-2}{\frac{k-3}{2} + \lfloor \frac{\Gamma-1}{2} \rfloor} \cdot \left( \frac{k-1}{2} - \lfloor \frac{\Gamma-1}{2} \rfloor \right) & \text{if } \frac{k-1}{2} \geq \lfloor \frac{\Gamma-1}{2} \rfloor, \\ 0 & \text{otherwise;} \end{cases}$$

$$\omega_{36} = \begin{cases} 2 \binom{k-2}{\frac{k-5}{2} - \lfloor \frac{\Gamma-1}{2} \rfloor} \cdot \left( \frac{k-5}{2} - \lfloor \frac{\Gamma-1}{2} \rfloor \right) & \text{if } \frac{k-5}{2} \geq \lfloor \frac{\Gamma-1}{2} \rfloor, \\ 0 & \text{otherwise;} \end{cases}$$

$$\omega_{37} = \begin{cases} 4 \binom{k-2}{\frac{k-1}{2} + \lfloor \frac{\Gamma-1}{2} \rfloor} \cdot \left( \frac{k-3}{2} - \lfloor \frac{\Gamma-1}{2} \rfloor \right) & \text{if } \frac{k-3}{2} \geq \lfloor \frac{\Gamma-1}{2} \rfloor, \\ 0 & \text{otherwise;} \end{cases}$$

$$\omega_{38} = \begin{cases} 4 \binom{k-2}{\frac{k-5}{2} - \lfloor \frac{\Gamma-1}{2} \rfloor} & \text{if } \frac{k-5}{2} \geq \lfloor \frac{\Gamma-1}{2} \rfloor, \\ 0 & \text{otherwise;} \end{cases} \quad \omega_{39} = \begin{cases} 4 \binom{k-2}{\frac{k-3}{2} - \lfloor \frac{\Gamma-1}{2} \rfloor} & \text{if } \frac{k-3}{2} \geq \lfloor \frac{\Gamma-1}{2} \rfloor, \\ 0 & \text{otherwise.} \end{cases}$$

## A.2 Transformations of Primitive Uncertainty Sets

In the following, we show how a moment matrix  $\Sigma_A$  for an uncertainty set  $\Xi$  changes if  $\Xi$  is transformed by an injective affine map, or if  $\Xi$  is composed of the cross product of primitive uncertainty sets  $\Xi_i$  with known moment matrices  $\Sigma_{A,i}$ .

### A.2.1 Injective Affine Maps

It follows from Theorem 9.2.3 in [48] that extreme points are preserved under injective affine maps.

Let us assume that  $\Xi' = f(\Xi)$  for an injective affine map  $f : \mathbb{R}^k \xrightarrow{A} \mathbb{R}^{k'}$ . We then obtain that

$$\Sigma'_A = \begin{pmatrix} |\text{ext } \Xi| & \sum_{\xi \in \text{ext } \Xi} f(\xi)^\top \\ \sum_{\xi \in \text{ext } \Xi} f(\xi) & \sum_{\xi \in \text{ext } \Xi} f(\xi)f(\xi)^\top \end{pmatrix}.$$

Let us now additionally assume that  $f(\xi) = \mathbf{F}\xi + \mathbf{f}$ . From the previous assumption that  $f$  is injective, we conclude that the matrix  $\mathbf{F}$  has full column rank. We then have

$$\sum_{\xi \in \text{ext } \Xi} f(\xi) = \sum_{\xi \in \text{ext } \Xi} (\mathbf{F}\xi + \mathbf{f}) = \mathbf{F} \left[ \sum_{\xi \in \text{ext } \Xi} \xi \right] + \mathbf{f} + |\text{ext } \Xi| \mathbf{f}.$$

In other words, we can calculate  $\sum_{\xi \in \text{ext } \Xi} f(\xi)$  efficiently from the quantities  $|\text{ext } \Xi|$  and  $\sum_{\xi \in \text{ext } \Xi} \xi$ .

In a similar way, we obtain that

$$\begin{aligned} \sum_{\xi \in \text{ext } \Xi} f(\xi)f(\xi)^\top &= \sum_{\xi \in \text{ext } \Xi} (\mathbf{F}\xi + \mathbf{f})(\mathbf{F}\xi + \mathbf{f})^\top = \left( \mathbf{F} \left[ \sum_{\xi \in \text{ext } \Xi} \xi \right] \mathbf{f}^\top + \left[ \mathbf{F} \left[ \sum_{\xi \in \text{ext } \Xi} \xi \right] \mathbf{f}^\top \right]^\top \right) \\ &\quad + \mathbf{F} \left[ \sum_{\xi \in \text{ext } \Xi} \xi \xi^\top \right] \mathbf{F}^\top + |\text{ext } \Xi| \mathbf{f} \mathbf{f}^\top. \end{aligned}$$

In other words, we can calculate  $\sum_{\xi \in \text{ext } \Xi} f(\xi)f(\xi)^\top$  efficiently from the quantities  $|\text{ext } \Xi|$ ,  $\sum_{\xi \in \text{ext } \Xi} \xi$  and  $\sum_{\xi \in \text{ext } \Xi} \xi \xi^\top$ .

### A.2.2 Cross Products

Assume that  $\Xi = \Xi_1 \times \Xi_2$  with  $\Xi_1 \subseteq \mathbb{R}^{k_1}$  and  $\Xi_2 \subseteq \mathbb{R}^{k_2}$  such that  $k_1 + k_2 = k$ . We then have

$$|\text{ext } \Xi| = |\text{ext } \Xi_1| \cdot |\text{ext } \Xi_2|,$$

$$\sum_{\xi \in \text{ext } \Xi} \xi = \begin{pmatrix} |\text{ext } \Xi_2| \cdot \sum_{\xi_1 \in \text{ext } \Xi_1} \xi_1 \\ |\text{ext } \Xi_1| \cdot \sum_{\xi_2 \in \text{ext } \Xi_2} \xi_2 \end{pmatrix},$$

$$\sum_{\xi \in \text{ext } \Xi} \xi \xi^\top = \begin{pmatrix} |\text{ext } \Xi_2| \cdot \sum_{\xi_1 \in \text{ext } \Xi_1} \xi_1 \xi_1^\top & \begin{pmatrix} \sum_{\xi_1 \in \text{ext } \Xi_1} \xi_1 \\ \sum_{\xi_2 \in \text{ext } \Xi_2} \xi_2 \end{pmatrix} \begin{pmatrix} \sum_{\xi_2 \in \text{ext } \Xi_2} \xi_2 \\ \sum_{\xi_1 \in \text{ext } \Xi_1} \xi_1 \end{pmatrix}^\top \\ \begin{pmatrix} \sum_{\xi_2 \in \text{ext } \Xi_2} \xi_2 \\ \sum_{\xi_1 \in \text{ext } \Xi_1} \xi_1 \end{pmatrix} \begin{pmatrix} \sum_{\xi_1 \in \text{ext } \Xi_1} \xi_1 \\ \sum_{\xi_2 \in \text{ext } \Xi_2} \xi_2 \end{pmatrix}^\top & |\text{ext } \Xi_1| \cdot \sum_{\xi_2 \in \text{ext } \Xi_2} \xi_2 \xi_2^\top \end{pmatrix}.$$

In other words, we can calculate  $|\text{ext } \Xi|$ ,  $\sum_{\xi \in \text{ext } \Xi} \xi$  and  $\sum_{\xi \in \text{ext } \Xi} \xi \xi^\top$  efficiently from the quantities  $|\text{ext } \Xi_i|$ ,  $\sum_{\xi \in \text{ext } \Xi_i} \xi$  and  $\sum_{\xi \in \text{ext } \Xi_i} \xi \xi^\top$ ,  $i = 1, 2$ .