A unified theory for brittle and ductile shear mode fracture

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Abstract A unified theory captures both brittle and ductile fracture. The fracture toughness is proportional to the applied stress squared and the length of the crack. For purely brittle solids this criterion is equivalent to Griffith's theory. In other cases it provides a theoretical basis for the Irwin-Orowan formula. For purely ductile solids the theory makes direct contact with the Bilby-Cottrell-Swinden model. The toughness is highest in ductile materials because the shielding dislocations in the plastic zone provide additional resistance to crack growth. This resistance is the force opposing dislocation motion, and the Peach-Koehler force overcomes it. A dislocation-free zone separates the plastic zone from and the tip of the crack. The dislocation-free zone is finite because molecular forces responsible for the cohesion of the surfaces near the crack tip are not negligible. At the point of crack growth the length of the dislocation-free zone is constant and the shielding dislocations advance in concert. As in Griffith's theory the crack is in unstable equilibrium. The theory shows that a dimensionless variable controls the elastoplastic behaviour. A relationship for the size of the dislocation-free zone is derived in terms of the macroscopic and microscopic parameters that govern the fracture.

Keywords Mechanical properties; fracture toughness; cracks; Griffith's theory; Bilby-Cottrell-Swinden model; dislocation-free zone

1 Introduction

A brittle solid such as glass and Si at low temperatures is fragile and when a sharp crack grows, it suddenly cleaves. When we rejoin the macroscopic parts of the broken solid, it nearly completely regains its initial shape [1]. In Griffith's theory fracture is assumed to be fully reversible and quasi-static. It describes perfectly brittle solids. However, these solids do not exist because even in very brittle materials there are dislocations. In general, dislocations collect in a plastic zone ahead of the crack and shield the tip of the crack. The fracture toughness of these materials is orders of magnitude higher than that predicted by Griffith's theory. For instance, strong ductile materials such as precipitation-hardened alloys, have high fracture toughnesses. In an attempt to account for these higher toughnesses, Irwin and Orowan [1, 2] suggested an extension to Griffith's theory. In this extension an empirical term accounts for the work associated with the plastic zone. This paper presents a continuum theory for the fracture toughness of crystalline materials. We will use the small-scale yielding approximation, where the applied stress is much smaller than the yield stress. In this approximation the size of the plastic zone is much smaller than the length of the crack.

We assume that an elastic enclave separates the crack tip from the plastic zone. Thomson [3] and Weertman [4] were the first to introduce this elastic region. We also assume that the slip plane is coplanar with the crack and that the external loading is either by plane shear (mode II) or by anti-plane shear (mode III). In this case the elastic enclave is called a dislocation-free zone (DFZ) [5–9]. Ohr [10] showed direct observations of the DFZ and also reviewed the DFZ models. These models showed that as the size of the DFZ increases, the shielding by the plastic zone diminishes. Taking the limit as the DFZ is maximum, the models tend to the elastic crack, e.g. [11, 12]. This limit describes the completely brittle solids. In contrast, taking the limit as the DFZ goes to zero, the models tend to the Dugdale-Bilby-Cottrell-Swinden model [13, 14]. This limit describes the completely ductile solids. When the size of the DFZ varies between these two limiting cases the behaviour of the solid changes from purely brittle to purely ductile.

These studies did not calculate the stress field everywhere in the medium. We derive this stress field in terms of mathematical functions. We also calculate the J-integrals. Then, we find a relationship for the length of the DFZ in terms of the elastic constants, the yield stress, the total Burgers vector in the plastic zone, the applied shear stress and the crack length. Majumdar and Burns [8] derived the ratio between the DFZ length and the extension of the plastic zone. But they did not obtain the DFZ length because they needed an additional relationship or physical argument. In the present model the path-independence of the J-integrals provides this additional relationship.

When in molybdenum a crack grows the DFZ length is unaltered and the dislocations in the plastic zone move ahead of the crack as a group [10]. We assume that at the point of crack growth the DFZ length is constant. Using this assumption and the path independence of the J-integral, we derive a fracture criterion. The fracture criterion provides a theoretical basis for the relationship between the macroscopic parameters characterizing the fracture and the state of stress and deformation very near the crack tip. In the purely brittle limit the fracture criterion is equivalent to Griffith's equation. Otherwise, it proves the Irwin-Orowan formula and shows that the plastic zone moves ahead of the crack.

The outline of the paper is as follows. In the next section we describe the geometry of the model. Then, we calculate the stress field and the J-integrals. A dimensionless variable, which we call λ, becomes evident. This variable controls the mechanical properties of solids. It comprises the elastic constants, the yield stress, the total Burges vector in the plastic zone, the applied shear stress, and the crack length. We predict the DFZ length in terms of λ. We derive the fracture criterion. Finally, we discuss the implications of the model.

2 Geometry

We consider an infinite, homogeneous, isotropic linear elastic medium where a stationary crack of length c extends in the plane $y = 0$ from $x = -c$ to $x = 0$ and is infinitely extended along the z-axis. For mode II loading a constant shear stress $\sigma_{xy} = \sigma$ is applied at $y = \pm\infty$. For mode III loading a constant shear stress $\sigma_{yz} = \sigma$ is applied at $y = \pm\infty$. The crack tip has an inverse square root singularity and intensifies the applied stress. The intensity of this singularity is $k_{\mathrm{a}} = \sigma\sqrt{c}$, which is called the applied stress intensity factor. In the stress intensity factor π is omitted without compromising the accuracy of the model [15]. Ahead of the crack, there is a DFZ between $x = 0$ and $x = d \geq 0$. We assume a plastic zone between $x = d$ and $x = a \geq d$, and infinitely extended along the z-axis. In the plastic zone work hardening is neglected and the shear stress σ_{xy} (mode II), σ_{yz} (mode III), is assumed to be constant and equal to the yield stress, σ_1. For small-scale yielding we assume $\sigma \ll \sigma_1$. The system is shown in Figure 1.

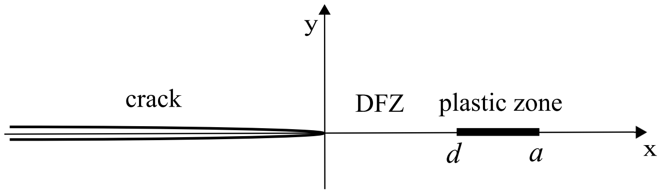

Figure 1: Geometry comprising a long stationary crack along the negative x-axis, a dislocation-free zone (DFZ) and a plastic zone. For a mode II crack the medium is subjected to a uniform applied shear stress $\sigma_{xy} = \sigma$, and the elastic field is one of plane strain. For a mode III crack the uniform applied shear stress is $\sigma_{yz} = \sigma$, and the elastic field is one of antiplane strain. The geometry is invariant normal to the page.

3 Results

The plastic zone and the crack are modelled using a continuous distribution of dislocations with infinitesimal Burgers vector, e.g. [16, 17]. For a mode II crack the glide edge dislocations have

Burgers vectors along the negative x-axis and lines parallel to the z-axis, consistent with the FS/RH convention. In the plastic zone the xy-component of the stress tensor equals σ_1. Similarly, for a mode III crack the screw dislocations have Burgers vectors along the negative z-axis and lines parallel to the z-axis, consistent with the FS/RH convention. In the plastic zone the yz-component of the stress tensor equals σ_1. The tractions at any point on the faces of these cracks must be zero. For crack much larger than the plastic zone we can extend the domain of the crack in the negative (semi-infinite) axis without compromising the accuracy of the model. At any point x in the crack or plastic zone the Burgers vector density $b(x)$ must satisfy the following singular integral equation:

$$A \int_L \mathrm{d}x' \, \frac{b(x')}{x - x'} = \sigma_1 \left(H(x-d) - H(x-a) \right), \tag{1}$$

where L extends from $x' = -\infty$ to $x' = 0$ and from $x' = d$ to $x' = a$. The variable $x \in L$, and the integral is a Cauchy principal value integral because a straight dislocation does not exert a stress upon itself. Let μ and ν be the shear modulus and Poisson's ratio. The constant A equals:

$$A = \begin{cases} \dfrac{\mu}{2\pi(1-\nu)} & \text{mode II;} \\ \dfrac{\mu}{2\pi} & \text{mode III.} \end{cases} \tag{2}$$

$H(x)$ denotes the Heaviside step function.

The length of the plastic zone is $p \equiv a - d$. Throughout the plastic zone σ_1 is finite and, hence, the Burgers vector density must not diverge at $x = d$ and $x = a$. Let $\eta = d/a$. It implies that a solution of Equation (1) exists only when the following condition is satisfied:

$$E\left(\sqrt{1-\eta}\right) = \frac{\pi}{2} \frac{\sigma}{\sigma_1} \sqrt{\frac{c}{2a}} \, . \tag{3}$$

$E\,(m\,)$ is the complete elliptic integral of the second kind, where m is the parameter. The properties of the elliptic integrals may be found in [18]. Majumdar and Burns [8] first derived Equation (3). This equation does not define d or a. But it is an implicit relationship for the ratio d/a in terms of the variable $(\sigma/\sigma_1)\sqrt{c/(2a)}$. $1 \leq E\left(\sqrt{1-\eta}\right) \leq \pi/2$. Taking the limit $\eta \to 1$, $p = 0$ and the plastic zone vanishes. Taking the limit $\eta \to 0$, $p = \pi^2\sigma^2 c/(8\sigma_1^2)$ and the length of the plastic zone is the maximum.

To find the solution of Equation (1) we introduce the complex function $j(z) = i\sqrt{[a(z-a)]\,/\,[z(z-d)]}$, where $z = x + iy$. Then, the Burgers vecotr density is as follows:

$$b(x) = \frac{2\sigma_1}{A\pi^2}\, j(x+i0)\, \eta\, \Pi\left(\frac{(1-\eta)x}{x-d} \middle| \sqrt{1-\eta} \right). \tag{4}$$

$\Pi\,(n|m)$ is the complete elliptic integral of the third kind, where m is the parameter, n is the characteristic. It is plotted in Figure 2. For $d \leq x \leq a$ the density was first calculated in [8].

We may use the density $b(x)$ (4) to calculate the total Burgers vector in the plastic zone Δu:

$$\Delta u = \int_x^a \mathrm{d}x' \, b(x') = \frac{2a\sigma_1}{A\pi^2} \left\{ \left[E\left(\sqrt{1-\eta}\right) \right]^2 - \eta \left[K\left(\sqrt{1-\eta}\right) \right]^2 \right\}. \tag{5}$$

$K\,(m)$ is the complete elliptic integral of the first kind, where m is the parameter. This equation is also the crack tip sliding displacement. It was first calculated in [8].

3.1 The stress field

The dislocations with density $b(x)$ (4) generate the stress field in the medium. The stress field is of interest because it governs the fracture of the solid. This stress tensor $\sigma_{\alpha\beta}(\mathbf{x})$ is:

$$\sigma_{\alpha\beta}(\mathbf{x}) = A \int_L \mathrm{d}x' \, b(x') \zeta_{\alpha\beta}\left(x - x', y\right), \tag{6}$$

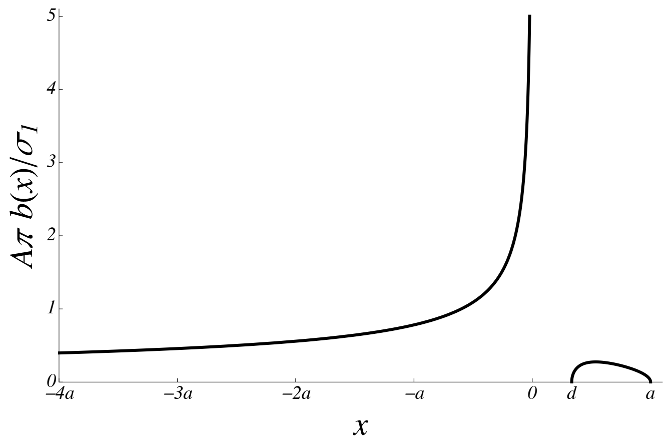

Figure 2: The Burgers vector density $b(x)$, normalised to $\sigma_1/(A\pi)$, plotted for $d = a/3$.

where $\mathbf{x} = (x, y)$. For a mode II crack the functions $\zeta_{\alpha\beta}(\mathbf{x})$ are determined by the stress field of glide edge dislocations:

$$\zeta_{xx}(\mathbf{x}) = -\frac{y(3x^2+y^2)}{(x^2+y^2)^2}, \tag{7a}$$

$$\zeta_{yy}(\mathbf{x}) = \frac{y(x^2-y^2)}{(x^2+y^2)^2}, \tag{7b}$$

$$\zeta_{zz}(\mathbf{x}) = \nu\left(\zeta_{xx}(\mathbf{x}) + \zeta_{yy}(\mathbf{x})\right), \tag{7c}$$

$$\zeta_{xy}(\mathbf{x}) = \zeta_{yx}(\mathbf{x}) = \frac{x(x^2-y^2)}{(x^2+y^2)^2}, \tag{7d}$$

$$\zeta_{xz}(\mathbf{x}) = \zeta_{zx}(\mathbf{x}) = 0, \tag{7e}$$

$$\zeta_{yz}(\mathbf{x}) = \zeta_{zy}(\mathbf{x}) = 0. \tag{7f}$$

For a mode III crack the functions $\zeta_{\alpha\beta}(\mathbf{x})$ are determined by the stress field of screw dislocations:

$$\zeta_{xz}(\mathbf{x}) = \zeta_{zx}(\mathbf{x}) = -\frac{y}{x^2+y^2}, \tag{8a}$$

$$\zeta_{yz}(\mathbf{x}) = \zeta_{zy}(\mathbf{x}) = \frac{x}{x^2+y^2}, \tag{8b}$$

$$\zeta_{xy}(\mathbf{x}) = \zeta_{yx}(\mathbf{x}) = 0, \tag{8c}$$

$$\zeta_{xx}(\mathbf{x}) = \zeta_{yy}(\mathbf{x}) = \zeta_{zz}(\mathbf{x}) = 0. \tag{8d}$$

To calculate the stress field (6) we may use the contour integration in the complex plane [19]. The function which generates this stress field $\psi(z) = \psi_1(\mathbf{x}) + i\,\psi_2(\mathbf{x})$ is:

$$\psi(z) = \eta\sqrt{\frac{a(z-a)}{z(z-d)}}\,\Pi\left(\frac{(1-\eta)z}{z-d}\middle|\sqrt{1-\eta}\right). \tag{9}$$

It is interesting to note that this generator is the analytic continuation of the density $b(x)$ (4) in the complex plane. It covenient to introduce a second function $\chi(z) = \chi_1(\mathbf{x}) + i\,\chi_2(\mathbf{x})$:

$$\chi(z) = iy\frac{d\psi}{dz}(z) = \frac{iy}{2}\sqrt{\frac{a}{z(z-a)(z-d)}}\left[\frac{d}{z}K\left(\sqrt{1-\eta}\right) - E\left(\sqrt{1-\eta}\right)\right]. \tag{10}$$

For a mode II crack we obtain the following expression for the components of the stress tensor:

$$\sigma_{xx} = \frac{2\sigma_1}{\pi}\left(\chi_2 + 2\psi_2\right), \tag{11a}$$

$$\sigma_{yy} = -\frac{2\sigma_1}{\pi}\chi_2, \tag{11b}$$

$$\sigma_{xy} = \sigma_{yx} = \frac{2\sigma_1}{\pi}\left(\chi_1 + \psi_1\right). \tag{11c}$$

In these equations the dependence on the position $\mathbf{x}$ is omitted for simplicity. For a mode III crack the non-zero components of the stress tensor are:

$$\sigma_{xz} = \sigma_{zx} = \frac{2\sigma_1}{\pi}\psi_2, \tag{12a}$$

$$\sigma_{yz} = \sigma_{zy} = \frac{2\sigma_1}{\pi}\psi_1. \tag{12b}$$

In these stress fields the region near the crack tip has a $1/\sqrt{2z}$ singularity. The intensity of this singularity is:

$$k = \frac{2\sigma_1}{\pi}\sqrt{2d}\,K\left(\sqrt{1-\eta}\right). \tag{13}$$

For a mode II and a mode III crack this equation is the local stress intensity factor. It was first calculated in [8] using a different method. In this work the local stress intensity factor is calculated from the sum of the stress intensity factor of the elastic crack plus the stress intensity factor at the crack tip due to the plastic zone only.

3.2 The J-integrals

Eshelby [20, 21] derived the force on elastic singularities and inhomogeneities in terms of a line integral of the stress energy-momentum tensor enclosing the singularity. Rice [22] discovered the integral independently and called it the J-integral. The J-integral is minus the ratio between the infinitesimal variation of the total energy, which equals the sum of the elastic energy of the medium and the potential energy of the external mechanism connected with it, and the infinitesimal variation of the position of the singularity. Because the medium has either plane or antiplane strain condition, the J-integral has only two components, namely J_x and J_y. We calculate these forces as follows.

Let u_α be the displacement associated with the stress field $\sigma_{\alpha\beta}$. Using Equation (11), the non-zero derivative of the displacement $u_{\alpha,\beta}$ are:

$$u_{x,x} = \frac{\sigma_1}{\mu\pi}\left(\chi_2 + 2(1-\nu)\,\psi_2\right), \tag{14a}$$

$$u_{x,y} = \frac{\sigma_1}{\mu\pi}\left(\chi_1 + (3-2\nu)\,\psi_1\right), \tag{14b}$$

$$u_{y,x} = \frac{\sigma_1}{\mu\pi}\left(\chi_1 + (2\nu-1)\,\psi_1\right), \tag{14c}$$

$$u_{y,y} = -\frac{\sigma_1}{\mu\pi}\left(\chi_2 + 2\nu\,\psi_2\right). \tag{14d}$$

Using Equation (12), the non-zero derivatives of the displacement are:

$$u_{z,x} = \frac{2\sigma_1}{\mu\pi}\psi_2, \tag{15a}$$

$$u_{z,y} = \frac{2\sigma_1}{\mu\pi}\psi_1. \tag{15b}$$

For a linear elastic medium the stress energy-momentum tensor is:

$$P_{\alpha\beta} = \frac{1}{2}\sigma_{\gamma\kappa}u_{\gamma,\kappa}\delta_{\alpha\beta} - \sigma_{\gamma\beta}u_{\gamma,\alpha}, \tag{16}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta. We use the suffix notation for convenience. Using Equations (11) and (14), the components of the stress energy-momentum tensor in the $x-y$ plane are:

$$P_{xx} = -P_{yy} = \frac{4(1-\nu)\sigma_1^2}{\mu\pi^2}\left(\psi_1^2 - \psi_2^2 + \chi_1\psi_1 - \chi_2\psi_2\right), \tag{17a}$$

$$P_{xy} = -\frac{4(1-\nu)\sigma_1^2}{\mu\pi^2}\left(\chi_1\psi_2 + \psi_1\psi_2 + \chi_2\psi_1\right), \tag{17b}$$

$$P_{yx} = -\frac{4(1-\nu)\sigma_1^2}{\mu\pi^2}\left(\chi_1\psi_2 + 3\psi_1\psi_2 + \chi_2\psi_1\right), \tag{17c}$$

Using Equations (12) and (15), the components of the stress energy-momentum tensor are:

$$P_{xx} = -P_{yy} = \frac{2\sigma_1^2}{\mu\pi^2}\left(\psi_1^2 - \psi_2^2\right), \tag{18a}$$

$$P_{xy} = P_{yx} = -\frac{4\sigma_1^2}{\mu\pi^2}\psi_1\psi_2. \tag{18b}$$

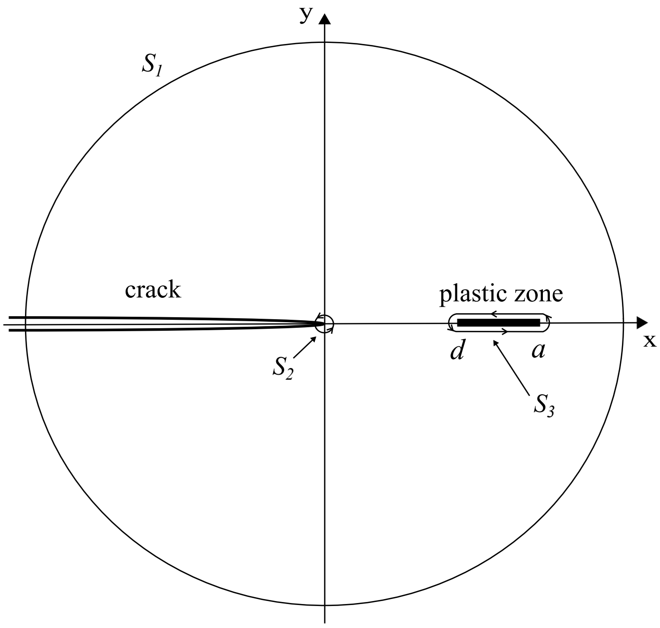

Figure 3: The contours S_1, S_2, and S_3 include the crack-tip and the plastic zone, i.e. the singularities of the medium.

Let S be a contour in the $x - y$ plane. The J-integral is as follows:

$$J_i(S) = \oint_S \mathrm{d}s\, P_{ik} n_k, \tag{19}$$

where the Latin indices i and k refer to the cartesian coordinates x and y, and n_i is the outward unit normal to the contour S. For a mode II crack the stress energy-momentum tensor is given in Equation (17). For a mode III crack the stress energy-momentum tensor is given in Equation (18). We consider three contours, S_1, S_2, and S_3, illustrated in Figure 3. S_1 encloses the singularity at the crack tip and the plastic zone. S_2 encloses only the crack tip. S_3 encloses only the plastic zone.

To calculate the J-integral (20) we may use the complex contour integration [19, 23]. For a mode II and a mode III crack the y component of the J-integral is always zero. The x component of the J-integral is:

$$J_x(S_1) = \frac{k_{\mathrm{a}}^2}{4A}, \tag{20a}$$

$$J_x(S_2) = \frac{k^2}{4A}, \tag{20b}$$

$$J_x(S_3) = \sigma_1 \Delta u. \tag{20c}$$

There are no other forces in the system. Using the condition (3) we have:

$$J_x(S_1) = J_x(S_2) + J_x(S_3). \tag{21}$$

The J-integral depends only on the singularity within the contours as expected, and hence it can be deformed as I have indicated. The relationship (21) provides a theoretical basis for the relation between the applied stress intensity factor and the fracture mechanism operating at the microscale. We will see below that this equation determines the variable η. But η must also satisfy the condition (3). Because Equations (3) and (21) are independent, the theory predicts the DFZ length d in terms of the macroscopic and microscopic parameters which govern the fracture.

The force on the entire system is $J_x(S_1)$ (20a), while $J_x(S_2)$ (20b) is the force on the crack tip. Similar forces were calculated in [23, 24]. In these works the distribution of the dislocations ahead of the crack was arbitrary. This distribution was not self-consistent because the net force acting on each singularity was different from zero. In contrast, in the present model the condition (3) ensures the static equilibrium. The J-integral $J_x(S_3)$ (20c) is the Peach-Koehler force on the plastic zone.

3.3 The elastic-plastic behaviour

The identity (21) can be written as follows:

$$\frac{J_x(S_2)}{J_x(S_1)} = 1 - \frac{J_x(S_3)}{J_x(S_1)}. \tag{22}$$

Using the J-integrals (20) this relation becomes:

$$\frac{k}{k_\mathrm{a}} = \sqrt{1 - \frac{4A\sigma_1\Delta u}{\sigma^2 c}}. \tag{23}$$

The right-hand side of this equation is always positive, and less than or equal to 1. The ratio k/k_a can be rewritten using the local stress intensity factor (13) and the condition (3). In this case Equation (23) becomes:

$$\sqrt{\eta}\,\frac{K\left(\sqrt{1-\eta}\right)}{E\left(\sqrt{1-\eta}\right)} = \sqrt{1 - \frac{4A\sigma_1\Delta u}{\sigma^2 c}}. \tag{24}$$

This equation is an implicit relationship for the variable η in terms of the dimensionless variable $\lambda \equiv 4A\sigma_1\Delta u/(\sigma^2 c)$. *We will show that the dimensionless variable λ controls the elastic vs plastic nature of the fracture.*

The implicit relationship (24) does not have an analytic solution for the variable η in terms of λ. It can be calculated numerically. Let $f(\eta)$ be the function on the left-hand side of Equation (24). Then, we have $f(\eta) = \sqrt{1-\lambda}$. The inverse of this function equals η and is as follows:

$$\eta = f^{-1}\left(\sqrt{1-\lambda}\right). \tag{25}$$

It is illustrated in Figure 4. For a purely brittle solid there is no plastic zone, $\Delta u = 0$, $\lambda = 0$, and $\eta = 1$. For a purely ductile solid $\Delta u = \sigma^2 c/(4A\sigma_1)$ is maximum, $\lambda = 1$, and $\eta = 0$. This is consistent with Figure 4. For intermediate solids $\Delta u > 0$, $0 < \lambda < 1$ and $0 < \eta < 1$.

The variable η (25) and the condition (3) determine the DFZ length:

$$d = \frac{\pi^2\sigma^2}{8\,\sigma_1^2}\,\frac{c\,f^{-1}\left(\sqrt{1-\lambda}\right)}{\left[E\left(\sqrt{1-f^{-1}\left(\sqrt{1-\lambda}\right)}\right)\right]^2}. \tag{26}$$

It is plotted in Figure 5, where it is normalised to $\sigma^2 c/(2\sigma_1^2)$. For a purely brittle solid $\lambda = 0$ and $d = c\,\sigma^2/(2\sigma_1^2)$. For a purely ductile solid $\lambda = 1$ and $d = 0$. In general, $0 < \lambda < 1$ and d is always smaller than that in the purely brittle limit.

The length of the plastic zone p in terms of λ follows from Equations (3), (25) and (26). It is shown in Figure 6. For a purely brittle solid $\lambda = 0$ and the length of the plastic zone is zero. In contrast, for a purely ductile solid $\lambda = 1$ and the length of the plastic zone is maximum. In general, $0 < \lambda < 1$ and the length of the plastic zone has an intermediate value.

Taking the limit $\lambda \to 0$, the stress generator (9) tends to $(\pi/2)\sqrt{a/z}$. The fields $\sigma_{\alpha\beta}$ (11-12) are those of the elastic crack, e.g. [11, 12]. The local stress intensity factor equals the applied stress intensity factor. Taking the limit $\lambda \to 1$, the stress generator (9) tends to $\arcsin\sqrt{a/z}$. The

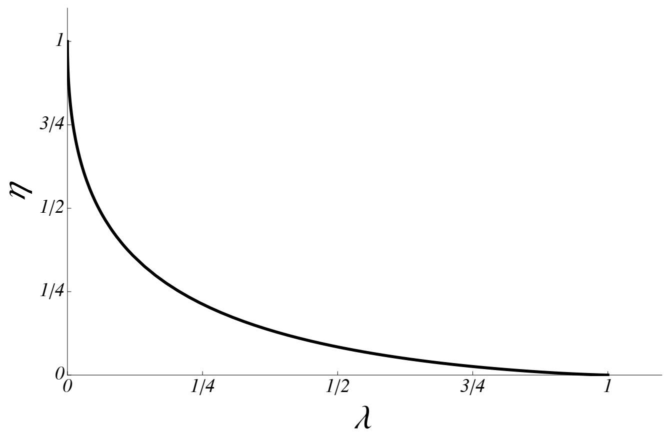

Figure 4: The variable $\eta = d/a$ as a function of the dimensionless variable λ.

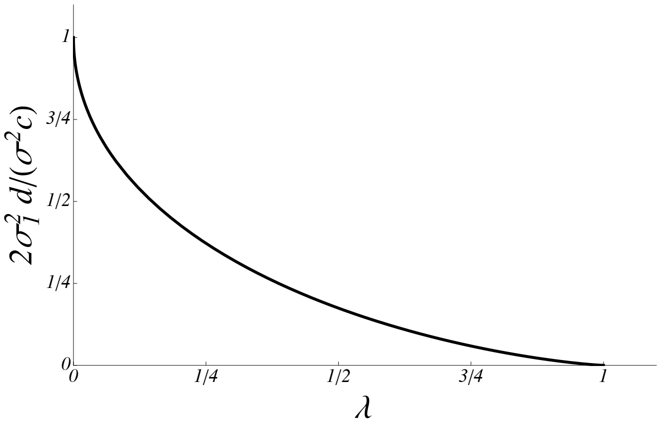

Figure 5: The length of the dislocation-free zone d normalised to $\sigma^2 c/(2\sigma_1^2)$, as a function of the dimensionless variable λ.

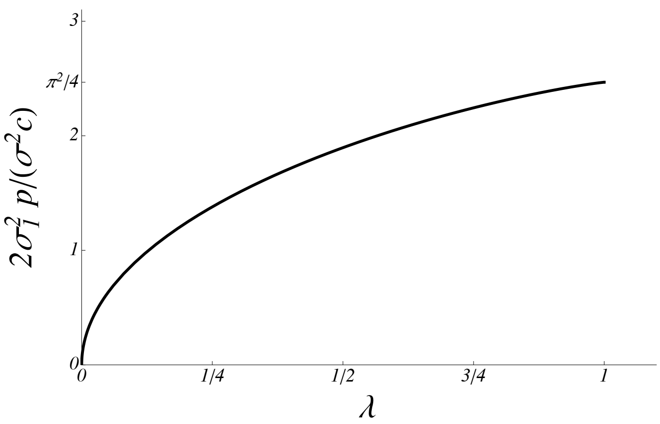

Figure 6: The length of the plastic zone $p = a - d$ normalised to $\sigma^2 c/(2\sigma_1^2)$, as a function of the dimensionless variable λ.

fields $\sigma_{\alpha\beta}$ (11-12) are those of the BCS model [14, 25]. The local stress intensity factor is zero and the applied stress intensity factor is positive. When $0 < \lambda < 1$, the plastic zone shields the crack tip, and the stress singularity at the crack tip does not vanish, so that $k < k_{\mathrm{a}}$.

Taking the limit $\lambda \to 0$, $J_x(S_3) = 0$. $J_x(S_2) = k_{\mathrm{a}}^2/(4A)$ is the J-integral of the elastic crack, e.g. [11, 12]. Taking the limit $\lambda \to 1$, $J_x(S_2) = 0$. $J_x(S_3) = 2a\sigma_1^2/(A\pi^2)$ is the J-integral of the

BCS model [11, 25, 26], and $J_x(S_1) = 2a\sigma_1^2/(A\pi^2)$. Rice [27] derived an equation for the emission of the dislocation from a crack tip. His work assumes a periodic relationship between the shear stress and the atomic shear displacement on the most stressed slip plane ahead of the crack tip. The energy barrier to complete the block-like shear replaces the above term $2a\sigma_1^2/(A\pi^2)$, so that the J-integral can be understood in terms of the atomic mechanisms for dislocation emission. In the present model $J_x(S_1) = \sigma_1\Delta u$ is the Peach-Koehler force on the microscopic plastic zone.

3.4 The toughness

We assume that the toughness τ is the critical force acting on all the singularities $J_x(S_1)$ (20a) at the point of crack growth. The fracture criterion is as follows:

$$\frac{\sigma^2 c}{4A} = \tau. \tag{27}$$

This toughness is the resistive force per unit length along the z-axis that a material exerts against fracture. For a mode II crack Equation (27) implies that the critical stress at the point of crack growth is $\sigma = \sqrt{E\tau/[(1-\nu^2)\pi c]}$, where E is Young's modulus. For a mode III crack $\sigma = \sqrt{2\mu\tau/(\pi c)}$. This critical stress is smaller than that at the general yielding (i.e. σ_1) because the size of the plastic zone is much smaller than the crack length.

For a purely brittle solid the plastic zone is negligible, $\lambda = 0$ and $J_x(S_1) = J_x(S_2)$ (20). We identify the toughness with the work required to create two unit area surfaces $\tau = 2\gamma$. In this case the criterion (27) is Griffith's equation. In other cases the plastic zone is not negligible, $\lambda > 0$ and $J_x(S_1) = J_x(S_2) + J_x(S_3)$ (20). τ comprises the surface energy of the crack faces and the force needed to overcome the friction force opposing dislocation motion. This force is the Peach-Koehler force $\sigma_1\Delta u$. The criterion (27) provides a theoretical basis for the Irwin-Orowan formula. For a purely ductile solid Δu is maximum. $\lambda = 1$ and $J_x(S_1) = J_x(S_3)$ (20). In this case $\tau = \sigma_1\Delta u$.

4 Discussion

The dislocations in the plastic zone reduce the elastic singularity at the crack tip. The local stress intensity factor is the strength of this singularity. Rice and Thomson [28] suggested that when the local stress intensity factor is sufficiently high to create a repulsive force within the core of the dislocation, the crack tip emits the dislocation. This criterion enabled them to distinguish between intrinsically brittle and ductile solids. The present model describes brittle (i.e. $\Delta u = 0$), intermediate (i.e. $\Delta u > 0$) and ductile solids (i.e. Δu maximum). When $\Delta u \approx 0$, it describes the realistic case of a brittle solid with a small number of shielding dislocations (i.e. $\sigma_1 < \infty$).

Chang and Ohr [5] and Ohr and Chang [6] argued that the DFZ controls the nucleation of the dislocations at the crack tip because it affects the local stress intensity factor. Using [28] they found a critical value of the local stress intensity factor which activates the dislocation nucleation. Experiments show that the DFZ length is a function of the applied stress and the crack length [10]. Figure 5 shows that the DFZ length d is a function of the elastic constants, σ_1, Δu, σ, and c.

The DFZ may originate because the stress singularity at the crack tip repels a shielding dislocation. For instance, Hirsch et al. [29] support this process. They developed a model to predict the brittle-ductile transition (BDT) of precracked crystals. They found that in bulk Si, although the dislocation density was very low, the existing dislocations controlled the BDT. These dislocations moved to the crack tip and generated new sources. When the source at or close to the crack tip, nucleates a dislocation loop, the shielding dislocation moves away and shields the crack tip. In contrast, the anti-shielding dislocation is absorbed into the crack tip and shifts the crack faces. This mechanism leaves behind a DFZ.

It is usually assumed that when the local stress intensity factor reaches a critical value, the crack grows [5, 6, 8, 9, 12, 24]. For purely brittle solids this criterion corresponds to Griffith's theory because the local stress intensity factor equals the applied stress intensity factor. However, when the dislocations in the plastic zone shield the crack tip the criterion lacks in a theoretical basis. For $\sigma \ll \sigma_1$ we prove that the toughness τ (27) is always proportional to the square of the applied stress intensity factor whether there is plastic zone or not. The Griffith's equation and the Irwin-Orowan formula are particular cases of this criterion.

Rice [30] showed that in any continuum elastic-plastic medium if the stress field saturates to a finite value at large strain, then there is no elastic singularity at the crack tip. Also, there is no energy surplus to create the new surfaces of the crack. The plastic zone absorbs all the energy supplied by the external loading machine. For instance, when Δu is maximum the criterion (27) satisfies Rice's theorem. A consequence of this theorem is that a boundary condition must account for the separation process within the fracture zone. When a crack grows it creates new free surfaces. The surface tension in a solid is a resistance force against the opening surfaces at the crack tip. The toughness includes this resistance so that $\tau > \sigma_1 \Delta u$.

In the BCS model the local stress intensity factor is always zero. Thomson [12] argued that for a cleavage crack this solution is inconsistent because the local stress intensity factor must reflect the strengths of the bonds at the crack tip. We provide a theoretical basis for Thomsons's argument as follows. At the point of crack growth $\Delta u \geq 0$ reaches a critical value, which corresponds to a critical size of the plastic zone. $\lambda < 1$ because $\tau > \sigma_1 \Delta u$. Figure 5 shows that the DFZ length is always finite (i.e. $d > 0$) and, hence, the local stress intensity factor is positive.

To describe the cohesion forces at the crack tip, Barenblatt [31] assumed that in an elastic medium the faces of the crack join smoothly near the crack tip. When the surfaces are very close, the molecular attraction between them is not negligible. Because the stress must be finite at the edges of the crack, he derived Griffith's theory [32]. The DFZ includes the small region where the molecular forces dominate.

When $\sigma^2 c/(4A) < \tau$, the crack is in stable equilibrium. We assume that during crack growth d is constant. The experimental observation in molybdenum [10] supports this assumption. When $\sigma^2 c/(4A) \geq \tau$ the crack grows and the equilibrium is unstable. The plastic zone moves ahead of the crack. The Peach-Koehler force enables this motion. It follows that the high toughness of the materials originates from the friction force we must overcome to move the shielding dislocations as a group ahead of the crack.

5 Conclusions

A unified theory was developed to capture both brittle and ductile fracture for mode II or III loadings. It shows that the dimensionless variable $\lambda = 4A\sigma_1 \Delta u/(\sigma^2 c)$ controls the elastoplastic behaviour of solids, where A is a geometrical factor which depends on the elastic constants and the mode of loading; σ_1 is the yield stress; Δu is the total Burgers vector in the plastic zone; σ is the applied shear stress; and c is the crack length. The fracture toughness τ is the critical force acting on all singularities in the medium at the point of crack growth, and it equals $\tau = \sigma^2 c/(4A)$. For purely brittle solids $\lambda = 0$ and the fracture criterion equals Griffith's theory. Otherwise, $\lambda > 0$ and the fracture criterion provides a theoretical basis for the Irwin-Orowan formula. When $\lambda \approx 0$, the size of the plastic zone is negligible and the solid is quite brittle. BCC metals at low temperatures often satisfy this limit. For purely ductile solids $\lambda = 1$ and the plastic zone is maximum. This limit equals the Bilby-Cottrell-Swinden model for $\sigma \ll \sigma_1$. The Peach-Koehler force overcomes the friction resistance opposing dislocation motion and the plastic zone moves ahead of the crack. The toughness of ductile solids is higher than that of brittle solids because the plastic zone absorbs the energy provided by the external loading. There is a dislocation-free zone between the plastic zone and the tip of the crack. The theory derives a relationship for the length of the dislocation-free zone d in terms of λ. For quasi-brittle solids $d \approx \sigma^2 c/(2\sigma_1^2)$ is maximum. In general, $d > 0$ because the cohesive forces between the bonds at the crack tip are not negligible. When the crack grows d is constant. The extension of the crack is unstable.

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