Immersion and Invariance Adaptive Control of Linear Multivariable Systems

Romeo Ortega, [a,1] Liu Hsu, [b] Alessandro Astolfi [c]

[a] *Laboratoire des Signaux et Systémes, Supelec, 91192, Gif-sur-Yvette, FRANCE.*

[b] *Electrical Engineering Department, COPPE/UFRJ, P. O. Box 68504, 21945-970 Rio de Janeiro, Brazil.*

[c] *Electrical Engineering Department, Imperial College, Exhibition Road, London SW7 2BT, UK and Dipartimento di Elettronica e Informazione, Politecnico di Milano, 20133 Milano, Italy.*

Abstract

We show in this paper that it is possible to globally adaptively stabilize linear multivariable systems with reduced prior knowledge of the high frequency gain. In particular we relax the restrictive (non–generic) symmetry condition usually required to solve this problem. Instrumental for the establishment of our result is the use of the new immersion and invariance approach to adaptive control recently proposed in the literature. The controllers obtained with this technique are not certainty equivalent—though smooth and without projections or overparameterizations—and the resulting Lyapunov functions contain cross–terms between the plant states and the parameter errors.

1 Introduction and Problem Formulation

Certainty equivalent direct adaptive control, also known in the literature as model reference adaptive control (MRAC), of single–input single–output linear time–invariant systems is a well understood problem with well identified prior knowledge requirements. In particular it is known, see e.g. [6,9,8,13], that for global stabilization (or global tracking) it is sufficient to know the sign of the first nonzero Markov parameter—the high frequency gain. [2] For multivariable

[1] Author to whom all correspondence should be addressed.
[2] This assumption can be relaxed using exhaustive search procedures in parameter space, however, it is well known that these schemes may exhibit practically unadmissible transient behaviors, see, e.g., [8].

plants, where the high frequency gain is a matrix, say K_p, MRAC typically assumes (see, e.g., Section 9.7.3 in [6]) the knowledge of a (nonsingular) matrix Γ such that

$$K_p \Gamma^\top = \Gamma K_p^\top > 0 \tag{1}$$

It is important to note that (1) implies a symmetry condition which is quite restrictive and non–generic since it involves an *equality constraint.* It has been shown with a counterexample in [3] that, contrary to claims made in the literature, e.g., [9], the weaker assumption (without the symmetry condition)

$$K_p \Gamma^\top + \Gamma K_p^\top > 0 \tag{2}$$

is not sufficient for MRAC stabilization. In [3] a new assumption, namely that the principal leading minors of K_p are nonzero and have known signs, was used to design a novel MRAC scheme that ensures stability and asymptotic tracking in the mean square sense for systems of order n with $n-1$ inputs. The scheme is a simple gradient adaptation law with a projection applied to a new parameterization that exploits the idea of hierarchy of controls (used in sliding mode theory). Still under the same assumption, convergence in the usual sense was achieved using a convenient factorization of K_p that generated an alternative parameterization [5]. The resulting adaptation law needed no projections but required some overparameterization.

The main contribution of our paper is the proof that (2) is sufficient for global stabilization (and global tracking) with a *non–certainty* equivalent adaptive scheme stemming from the application of the immersion and invariance (I&I) approach proposed in [2]. The resulting I&I adaptive controller, although non–standard, is smooth and very simple and does not require projections nor any additional prior knowledge besides the aforementioned matrix Γ. To enhance readability we consider first the simplest example that captures the central issue of prior knowledge on K_p. Namely, we consider the problems of stabilization and regulation to zero of the output of the system

$$\dot{y} = K_p u, \tag{3}$$

where $u, y \in I\!R^m$, and K_p is unknown and nonsingular. We then show that this basic result can be immediately extended to the problem of asymptotically tracking a trajectory y_*. In Section 4 we treat the case of arbitrary minimum phase systems of known order and relative degree $\{1, \ldots, 1\}$.[3] We wrap up the paper with some conclusions and concluding remarks.

[3] We recall that a system with minimal realization (C, A, B) has vector relative degree $\{1, \ldots, 1\}$ if $c_i^\top B \neq 0$ for $i = 1, \ldots, m$, where $c_i^\top$ are the rows of C.

2 Classical Approach

To understand the nature of the assumptions, and place our I&I controller in perspective, let us briefly review the solution for regulation of the classical MRAC. First, we define a regressor matrix

$$\Phi(y) := \begin{bmatrix} y^\top & 0\cdots0 & \cdots & 0\cdots0 \\ 0\cdots0 & y^\top & \cdots & 0\cdots0 \\ \vdots & \vdots & \vdots & \vdots \\ 0\cdots0 & 0\cdots0 & \cdots & y^\top \end{bmatrix} \in I\!R^{m\times m^2} \tag{4}$$

and introduce the parameterization

$$-K_p^{-1}y = \Phi(y)\theta_*$$

where $\theta_* \in I\!R^{m^2}$ contains the rows of $-K_p^{-1}$. The control law is defined as

$$\begin{aligned} u &= \Phi(y)\hat{\theta} \\ \dot{\hat{\theta}} &= -\Phi^\top(y)\Gamma^{-1}y \end{aligned}$$

where, for simplicity, we have taken the unitary adaptation gain and selected the reference model regulation dynamics as[4] $\frac{1}{s+1}I$. Then, we compute the error equations

$$\begin{aligned} \dot{y} &= -y + K_p\Phi(y)\tilde{\theta} \\ \dot{\tilde{\theta}} &= -\Phi^\top(y)\Gamma^{-1}y \end{aligned} \tag{5}$$

which are obtained immediately from the equations above and the definition of the parameter error $\tilde{\theta} := \hat{\theta} - \theta_*$. Stability is analyzed with the Lyapunov function candidate

$$V_0(y,\tilde{\theta}) = \frac{1}{2}(y^\top P^{-1}y + |\tilde{\theta}|^2)$$

where $P = P^\top > 0$ is a matrix to be determined. Taking the derivative of V_0 along the trajectories of (5) gives

$$\dot{V}_0 = -y^\top P^{-1}y + y^\top(P^{-1}K_p - \Gamma^{-T})\Phi(y)\tilde{\theta}$$

[4] As will become clear below, we can assign the regulation dynamics $\dot{y} = -\Lambda y$, with Λ a strictly Hurwitz matrix, with a suitable redefinition of the regressor matrix $\Phi(y)$ and the Lyapunov function candidate. This remark applies as well to the I&I adaptive controllers presented in Section 3.

It is then clear that, if condition (1) holds, then we can set $P = K_p\Gamma^T$ and cancel the second right term. (Obviously, the symmetry condition is necessary for this construction, hence the weaker assumption (2) is not sufficient.) The proof that $\lim_{t\to\infty} y(t) = 0$ can finally be completed with the standard signal chasing of MRAC analysis.

3 Immersion and Invariance Adaptive Control

The rationale of the adaptive I&I stabilization methodology introduced in [2] is to asymptotically immerse the dynamics of the closed–loop system that would result if we applied the known parameters controller—that we call the target system, and in the present case coincides with the reference model dynamics—into the full system dynamics (including the parameter update law). Towards this end, we first prove the existence of an invariant manifold, such that the behavior of the adaptively controlled system restricted to this manifold is described by the target dynamics. In other words, such that any trajectory of the adaptively controlled system is the image, through the mapping that defines the manifold, of a trajectory of the target system. Second, we design a control law that renders the manifold attractive and keeps the closed–loop trajectories bounded. As shown in [2], and also illustrated below, instrumental to achieve these objectives is to abandon the certainty equivalence principle.

The main distinguishing features of adaptive I&I are the following

- In contrast with most existing adaptive designs, including the MRAC approach described above, it does not rely on cancellation of terms (in the Lyapunov function derivative). The latter operation is akin to disturbance rejection that, as is well known, is intrinsically fragile and imposes certain matching conditions that restrict its application domain. In adaptive I&I, instead, the deleterious effect of the uncertain parameters is countered adopting a robustness perspective. More precisely, the disturbance term in the error equation is forced to decrease so that its effect is eventually "dominated" by the stability margin of the ideal dynamics. [5]
- Besides the classical "integral action" of the parameter estimator, we introduce in the adaptive I&I control law a "proportional" term. As thoroughly discussed in [2], and further illustrated in [1], the inclusion of this term

[5] A similar stabilization mechanism exists in indirect adaptive controllers, with the fundamental differences that in the latter case: i) special provisions should be taken to avoid singularities in the controller calculations; ii) the ability to reduce the parameter error hinges only upon the availability of persistently exciting signals. See also [12] for an ingenious design that exploits the robustness property of, a suitably generated, cascaded structure.

enhances the robustness of the design, via the incorporation of additional stable zero dynamics.

- Viewed from a Lyapunov function perspective, I&I provides a procedure to add cross terms between the parameter estimates and the plant states.

In this section we will illustrate these features of adaptive I&I through its application to the simple integrator plant (3). We consider first regulation and then the tracking problem, obtaining in both cases globally stabilizing smooth controllers that do not require projections nor invoke persistency of excitation arguments.

3.1 Regulation

Similarly to the MRAC solution above we consider the parameterization

$$-K_p^{-1} y_f = \Phi_f(y_f)\theta_* \tag{6}$$

where we have introduced the filtered output $\dot{y}_f = -y_f + y$ and the filtered regressor

$$\Phi_f(y_f) = \begin{bmatrix} y_f^\top & 0\cdots0 & \cdots & 0\cdots0 \\ 0\cdots0 & y_f^\top & \cdots & 0\cdots0 \\ \vdots & \vdots & \vdots & \vdots \\ 0\cdots0 & 0\cdots0 & \cdots & y_f^\top \end{bmatrix} \in I\!R^{m\times m^2}$$

Now, let us generate a filtered input

$$\dot{u}_f = -u_f + u \tag{7}$$

It is clear from (3) and (7) that

$$\dot{y}_f = K_p u_f + \epsilon_t,$$

where ϵ_t denotes an exponentially decaying term. [6] We will carry out our adaptive I&I design using this filtered representation of the plant. We define then the (non certainty equivalent) filtered control signal as

$$u_f = \Phi_f(y_f)[\hat{\theta} + \beta_1(y_f)] \tag{8}$$

[6] To simplify the notation we will omit in the sequel this kind of terms that, as will become clear later, do not affect the validity of our analysis, see also Subsection 6.5.4 in [6].

where $\beta_1(y_f)$ is a vector function to be determined, and formulate the following:

Adaptive I&I Control Problem Given the adaptively controlled system

$$\Sigma : \begin{cases} \dot{y}_f = K_p \Phi_f(y_f)[\hat{\theta} + \beta_1(y_f)] \\ \dot{\hat{\theta}} = \beta_2(y_f) \end{cases} \tag{9}$$

with extended state $(y_f, \hat{\theta})$ and "controls" $\beta_1(y_f)$ and $\beta_2(y_f)$, and the target (reference model) dynamics

$$\Sigma_T : \dot{\xi} = -\xi$$

with $\xi \in I\!R^n$. Find a manifold $\mathcal{M} \subset I\!R^{2n+m^2}$ which can be rendered invariant and attractive, and such that the restriction of Σ to $\mathcal{M}$ reduces to Σ_T.

In [2] it has been shown that, even for general nonlinear systems with nonlinearly parameterized controls, a suitable definition for $\mathcal{M}$ is given by [7]

$$\mathcal{M} = \left\{ (y_f, \hat{\theta}) \in I\!R^{n+m^2} \mid \begin{bmatrix} y_f \\ \hat{\theta} \end{bmatrix} := \pi(\xi) = \begin{bmatrix} \xi \\ \theta_* - \beta_1(\xi) \end{bmatrix} \right\}$$

Indeed, from (6) it is easy to see that

$$K_p \Phi_f(y_f)[\hat{\theta} + \beta_1(y_f)]|_{\mathcal{M}} = -\xi$$

while, for any β_1, we can always define

$$\beta_2(\xi) = \frac{\partial \beta_1}{\partial \xi}(\xi)\xi$$

(It is interesting to note that the manifold $\mathcal{M}$ depends on θ_*, hence it is unknown. On the other hand, it is shaped by the "proportional term" of the adaptation β_1.)

In adaptive I&I we do not apply the control that renders the manifold invariant—in this case the function β_2 defined above—instead we design a control law

[7] We recall that a mapping $x = \pi(\xi)$ defines an invariant manifold for the system $\dot{x} = f(x)$, $\dot{\xi} = \alpha(\xi)$, if $\pi(\xi)$ satisfies the partial differential equation $f(\pi(\xi)) = \frac{\partial \pi}{\partial \xi}\alpha(\xi)$.

that makes it attractive.[8] For, we express $\mathcal{M}$ in an (equivalent) implicit form

$$\mathcal{M} = \{(y_f, \hat{\theta}) \in \mathbb{R}^{n+m^2} \mid \hat{\theta} - \theta_* + \beta_1(y_f) = 0\}$$

and define the off–the–manifold coordinate

$$z := \hat{\theta} - \theta_* + \beta_1(y_f). \tag{10}$$

Clearly, our objective of rendering the manifold attractive is tantamount to ensuring $\lim_{t\to\infty} z(t) = 0$.

Now, replacing the definition of z in (9) and using (6) yields the first error equation

$$\dot{y}_f = -y_f + K_p \Phi_f(y_f) z \tag{11}$$

Comparing (11) with the first error equation of MRAC (5) we observe that the off–the–manifold coordinate z plays the same role as the parameter error $\tilde{\theta}$. The novelty of adaptive I&I resides in the way to generate the dynamics of z, which proceeds as follows. First, from (10) we evaluate

$$\begin{aligned} \dot{z} &= \dot{\hat{\theta}} + \frac{\partial \beta_1}{\partial y_f}(y_f)\dot{y}_f \\ &= \beta_2(y_f) + \frac{\partial \beta_1}{\partial y_f}(y_f)\left[-y_f + K_p\Phi_f(y_f)z\right] \end{aligned}$$

where we have replaced (9) and (11) to obtain the second line. Observation of the latter, and recalling that z is not measurable, suggests the choice

$$\beta_1(y_f) = -\Phi_f^\top(y_f)\Gamma^{-1}y_f$$

to get

$$\begin{aligned} \dot{z} &= \beta_2(y_f) - \Phi_f^\top(y_f)\Gamma^{-1}[-y_f + K_p\Phi_f(y_f)z] - \dot{\Phi}_f^\top(y_f)\Gamma^{-1}y_f \\ &= \beta_2(y_f) - \Phi_f^\top(y_f)\Gamma^{-1}K_p\Phi_f(y_f)z + [2\Phi_f(y_f) - \Phi(y)]^\top\Gamma^{-1}y_f \end{aligned}$$

where, to get the second equation, we have used $\dot{\Phi}_f(y_f) = -\Phi_f(y_f) + \Phi(y)$. The obvious choice of parameter estimator

$$\beta_2(y_f) = -[2\Phi_f(y_f) - \Phi(y)]^\top\Gamma^{-1}y_f$$

[8] The manifold attractivity condition imposes that, on the manifold, the control coincides with the one that renders $\mathcal{M}$ invariant—a nice reduction property absent in other invariant manifold based controller designs, which is similar to the ideal "infinite switching frequency" behavior of sliding mode control.

finally yields the second error equation

$$\dot{z} = -\Phi_f^\top(y_f)\Gamma^{-1}K_p\Phi_f(y_f)z \tag{12}$$

We will now prove that the origin of the error system (11), (12) is globally asymptotically stable. To this end, consider the quadratic Lyapunov function candidate

$$V(y_f, z) = \frac{1}{2}|y_f|^2 + \frac{\alpha}{2}|z|^2 \tag{13}$$

with $\alpha > 0$. The derivative of V along the trajectories of (11), (12) gives

$$\dot{V} = -|y_f|^2 + y_f^\top K_p z_\phi - \alpha z_\phi^\top M z_\phi$$

where—to simplify the notation—we defined $z_\phi := \Phi_f(y_f)z$, and the matrix

$$M := \frac{1}{2}\left(\Gamma^{-1}K_p + K_p^\top\Gamma^{-T}\right) = M^\top$$

which, in view of (2), is positive definite. Now, it is easy to show that, for all $\alpha > \lambda_{max}\{K_p M^{-1} K_p^\top\}$, we have

$$\begin{bmatrix} I & -\frac{1}{2}K_p \\ -\frac{1}{2}K_p^\top & \alpha M \end{bmatrix} > 0$$

Consequently, for all sufficiently large α, there exists $\delta > 0$ such that

$$\dot{V} \leq -\delta(|y_f|^2 + |z_\phi|^2)$$

This implies that, as $t \to \infty$, $y_f(t), z_\phi(t) \to 0$, hence (from (11)) we also have that $\dot{y}_f(t) \to 0$. Finally, from $\dot{y}_f = -y_f + y$, we conclude that $y(t) \to 0$ as desired.

To complete our design we must recover the actual control u from the expression of u_f derived above, namely

$$u_f = \Phi_f(y_f)(\hat{\theta} - \Phi_f^\top(y_f)\Gamma^{-1}y_f)$$

Replacing this expression in $u = \dot{u}_f + u_f$ and using the fact that

$$\frac{d}{dt}\left(\hat{\theta} - \Phi_f^\top(y_f)\Gamma^{-1}y_f\right) = -\Phi_f^\top(y_f)\Gamma^{-1}y$$

we immediately obtain

$$u = \Phi(y)[\hat{\theta} - \Phi_f^\top(y_f)\Gamma^{-1}y_f] - \Phi_f(y_f)\Phi_f^\top(y_f)\Gamma^{-1}y.$$

We have thus established the following.

Proposition 1 *Consider the system (3) where $u, y \in I\!R^m$ and $\det\{K_p\} \neq 0$. Assume known a (nonsingular) matrix Γ such that (2) holds. Then, the I&I adaptive stabilizer*

$$
\begin{aligned}
u &= \Phi(y)[\hat{\theta} - \Phi_f^\top(y_f)\Gamma^{-1}y_f] - \Phi_f(y_f)\Phi_f^\top(y_f)\Gamma^{-1}y && (14)\\
\dot{\hat{\theta}} &= -[2\Phi_f(y_f) - \Phi(y)]^\top\Gamma^{-1}y_f && (15)\\
\dot{y}_f &= -y_f + y && (16)
\end{aligned}
$$

where $\Phi(y)$ is given by (4) and

$$
\Phi_f(y_f) := \begin{bmatrix} y_f^\top & 0\cdots0 & \cdots & 0\cdots0 \\ 0\cdots0 & y_f^\top & \cdots & 0\cdots0 \\ \vdots & \vdots & \vdots & \vdots \\ 0\cdots0 & 0\cdots0 & \cdots & y_f^\top \end{bmatrix} \in I\!R^{m\times m^2}
$$

ensures $\lim_{t\to\infty} y(t) = 0$ with all signals bounded for all initial conditions $y(0), y_f(0) \in I\!R^m$, $\hat{\theta}(0) \in I\!R^{m^2}$.

3.2 Tracking

Proposition 2 *Consider the system (3) and a bounded reference trajectory y_* with bounded derivative $\dot{y}_*$. Assume known a (nonsingular) matrix Γ verifying (2). Then, the I&I adaptive controller*[9]

$$
\begin{aligned}
u &= \Psi(\hat{\theta} - \Psi_f^\top\Gamma^{-1}e_f) - \Psi_f\Psi_f^\top\Gamma^{-1}e\\
\dot{\hat{\theta}} &= -(2\Psi_f - \Psi)^\top\Gamma^{-1}e_f\\
e &= y - y_*\\
\dot{e}_f &= -e_f + e\\
\dot{y}_{*f} &= -y_{*f} + y_*
\end{aligned}
$$

[9] Here, and throughout the rest of the paper, we omit the arguments of the various functions for brevity.

where

$$\Psi := \begin{bmatrix} (e-\dot{y}_*)^\top & 0\cdots0 & \cdots & 0\cdots0 \\ 0\cdots0 & (e-\dot{y}_*)^\top & \cdots & 0\cdots0 \\ \vdots & \vdots & \vdots & \vdots \\ 0\cdots0 & 0\cdots0 & \cdots & (e-\dot{y}_*)^\top \end{bmatrix} \in I\!R^{m\times m^2}$$

$$\Psi_f := \begin{bmatrix} (e_f-\dot{y}_{*f})^\top & 0\cdots0 & \cdots & 0\cdots0 \\ 0\cdots0 & (e_f-\dot{y}_{*f})^\top & \cdots & 0\cdots0 \\ \vdots & \vdots & \vdots & \vdots \\ 0\cdots0 & 0\cdots0 & \cdots & (e_f-\dot{y}_{*f})^\top \end{bmatrix} \in I\!R^{m\times m^2}$$

ensures $\lim_{t\to\infty} e(t) = 0$ *with all signals bounded for all initial conditions* $y(0), e_f(0), y_{*f}(0) \in I\!R^m$, $\hat{\theta}(0) \in I\!R^{m^2}$ *and all bounded trajectories* y_* *with bounded derivative* $\dot{y}_*$. *Further, if* $\dot{y}_*$ *is persistently exciting, i.e.,* $\exists T, \delta > 0$ *such that* $\int_t^{t+T} \dot{y}_*(\tau)\dot{y}_*^\top(\tau)d\tau \geq \delta I$, *then convergence is* exponential.

Proof The proof is established with the parameterization

$$\Psi_f \theta_* = -K_p^{-1}(e_f - \dot{y}_{*f})$$

The I&I control takes now the form $u_f = \Psi_f[\hat{\theta} + \beta_1(e_f)]$, with the same definition of z given in (10), this leads to the error equation

$$\dot{e}_f = -e_f + K_p \Psi_f z.$$

The term β_1 and the parameter update law, which result in the controller given in Proposition 2, are obtained following *verbatim* the proof of Proposition 1. The exponential stability claim results from Theorem 1.5.2 of [13].

4 n–th Order Systems of Vector Relative Degree $\{1, \ldots, 1\}$

The proposition below shows that I&I allows also to relax the symmetry condition (1) for the general class of vector relative degree $\{1, \ldots, 1\}$ systems considered in MRAC. [10]

Proposition 3 *Consider a square multivariable linear time–invariant system, described by its transfer matrix* $G(s)$, *with the following assumptions:*

[10] For further details on the systems assumptions we refer the interested reader to the textbooks [13,9,6] or the more recent paper [10].

A1 $G(s)$ *is strictly proper, full rank, minimum phase, and an upperbound* $\bar{\nu}$ *on its observability index* ν *is known.*

A2 $G(s)$ *has vector relative degree* $\{1,\ldots,1\}$*, and the interactor matrix is diagonal. In particular, we can take the interactor matrix of the form* $\xi_m(s) := diag\{s+1,\ldots,s+1\}$*,*

A3 We know a (nonsingular) matrix Γ *verifying (2), where* $\lim_{s\to\infty}\xi_m(s)G(s) = K_p$*.*

A4 $y_* \in I\!R^m$ *is a reference trajectory generated as* $y_* = \xi_m^{-1} r$*, where* $r \in I\!R^m$ *is a bounded signal.*

Then, the I&I adaptive controller

$$
\begin{aligned}
u &= W(\hat{\theta} - W_f^\top \Gamma^{-1} e_f) - W_f W_f^\top \Gamma^{-1} e \\
\dot{\hat{\theta}} &= -(2W_f - W)^\top \Gamma^{-1} e_f \\
\dot{e}_f &= -e_f + e \\
\dot{w}_f &= -w_f + w \\
e &= y - y_*
\end{aligned}
$$

where

$$
W := \begin{bmatrix} w^\top & 0\cdots0 & \cdots & 0\cdots0 \\ 0\cdots0 & w^\top & \cdots & 0\cdots0 \\ \vdots & \vdots & \vdots & \vdots \\ 0\cdots0 & 0\cdots0 & \cdots & w^\top \end{bmatrix} \in I\!R^{m\times 2m^2(\bar{\nu}+1)}
$$

$$
W_f := \begin{bmatrix} w_f^\top & 0\cdots0 & \cdots & 0\cdots0 \\ 0\cdots0 & w_f^\top & \cdots & 0\cdots0 \\ \vdots & \vdots & \vdots & \vdots \\ 0\cdots0 & 0\cdots0 & \cdots & w_f^\top \end{bmatrix} \in I\!R^{m\times 2m^2(\bar{\nu}+1)}
$$

$$
w = \begin{bmatrix} w_u \\ w_y \\ y \\ r \end{bmatrix} \in I\!R^{2m(\bar{\nu}+1)},\ w_u = \frac{1}{\lambda(s)} \begin{bmatrix} u^{(\bar{\nu}-1)} \\ \vdots \\ \dot{u} \\ u \end{bmatrix} \in I\!R^{m\bar{\nu}},\ w_y = \frac{1}{\lambda(s)} \begin{bmatrix} y^{(\bar{\nu}-1)} \\ \vdots \\ \dot{y} \\ y \end{bmatrix} \in I\!R^{m\bar{\nu}}
$$

and $\lambda(s)$ *is a monic Hurwitz polynomial of degree* $\bar{\nu}$*, ensures* $\lim_{t\to\infty} e(t) = 0$ *with all signals bounded for all initial conditions* $y(0) \in I\!R^m$*,* $e_f(0) \in I\!R^m$*,* $w_f(0) \in I\!R^{2m(\bar{\nu}+1)}$*,* $\hat{\theta}(0) \in I\!R^{2m^2(\bar{\nu}+1)}$ *and all bounded signals* r*.*

Proof The proof is established using the key transfer matrix identity (equation (6.3.10) of [13]) which, given $\lambda(s), G(s)$ and $\xi_m(s)$, ensures the existence of monic polynomial matrices $C(s), D(s)$, of degrees $\bar{\nu}-1$, such that

$$I-\frac{1}{\lambda(s)}C(s)-\frac{1}{\lambda(s)}D(s)G(s)=K_p^{-1}\xi_m(s)G(s).$$

After some suitable term reordering and filtering this identity leads to the parameterization

$$W_f\theta_*=u_f-K_p^{-1}\xi_m(s)e_f$$

The I&I control takes again the usual form $u_f=W_f[\hat{\theta}+\beta_1(W_f,e_f)]$, where we notice that β_1 depends now explicitly on e_f and W_f. Mimicking the derivations of Section 3 leads to the error equation

$$\xi_m e_f=K_pW_fz$$

that, given the definition of the interactor matrix above, is exactly the error equation of Proposition 2. The proof is completed following *verbatim* the steps of the proof of Proposition 1.

5 Discussion and Concluding Remarks

i) *(I&I adaptation vs MRAC)* As seen in Section 2 the stability proof in MRAC relies on the *cancellation* in the Lyapunov function derivative of the perturbation term introduced by the uncertain parameters. It is well–known that this cancellation is the source of many of the robustness problems of MRAC (essentially due to the generation of equilibrium spaces). In the I&I formulation we do not try to cancel the perturbation term coming from z (11), but as we have seen above, only make it asymptotically vanishing (or sufficiently small). This difference becomes evident if we compare the error equations of MRAC (5) and the I&I controller (11), (12). While the first one defines a *feedback* interconnection, the latter is a *cascaded* structure. Another important difference is that, while the Lyapunov function of MRAC, $V_0(y_f,\tilde{\theta})$, is separable, the Lyapunov function of I&I (13)—expressed in terms of $(y_f,\tilde{\theta})$—contains cross terms between these arguments. Finally, from the construction of the adaptive I&I control law (8) we see that, besides the classical "integral action" of the parameter estimator, the addition of β_1 introduces in the control law a "proportional" term, i.e., the term $-\Phi(y)\Phi_f^\top(y_f)\Gamma^{-1}y_f-\Phi_f(y_f)\Phi_f^\top(y_f)\Gamma^{-1}y$ in (14).

ii) *(On the role of passivity)* It is well known [8] that passivity is the key property required for stabilization in MRAC. In the example of Section 2 the passivity condition is expressed in terms of positive realness of the transfer matrix

$$H_2(s):=\frac{1}{s+1}\Gamma^{-1}K_p$$

which defines the map $\Phi(y)\tilde{\theta} \mapsto \Gamma^{-1}y$. This operator is then placed in negative feedback with the operator $H_1 : \Gamma^{-1}y \mapsto -\Phi(y)\tilde{\theta}$ defined by the gradient estimator, which is known to be passive, see Fig. 1. If $H_2(s)$ is strictly positive real stability of the overall system follows from the passivity theorem. Premultiplying the well–known relation $PB = C$ of passive systems by $B^\top$, which yields the high–frequency gain of the transfer matrix, actually shows that the symmetry condition (1) is *necessary* for strict positive realness of $H_2(s)$. Hence, if this condition is violated stabilization cannot be established invoking passivity arguments.

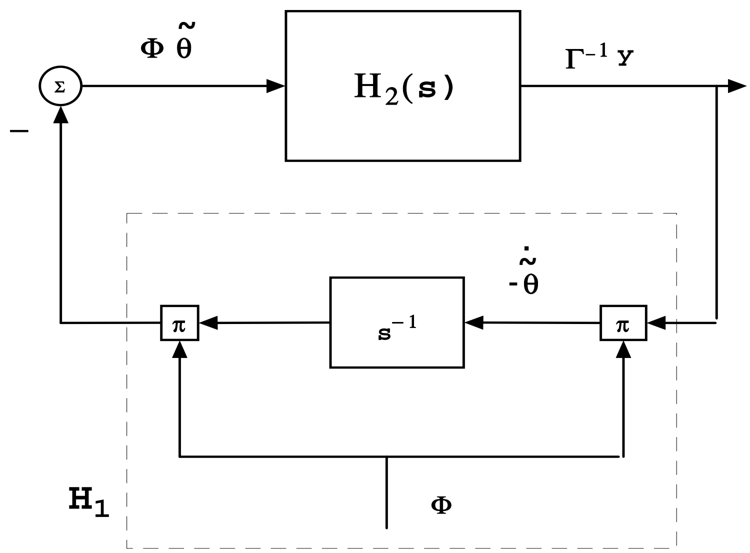

Fig. 1. Error equations of MRAC.

iii) *(Tuning gains and transient performance)* To simplify the notation, we have presented above “normalized” versions of the I&I adaptive schemes. It is well–known that to improve the transient performance some additional tuning gains should be provided. We have already pointed out in Section 2 that it is possible to select an arbitrary reference model dynamics. Also, it is clear from the proof of Proposition 1 that, for regulation of (3), we can add an adaptation gain in the estimator

$$\dot{\hat{\theta}} = -K[2\Phi_f(y_f) - \Phi(y)]^\top \Gamma^{-1} y_f, \;\; K = K^\top > 0$$

or include a time constant in the filters, e.g., $\epsilon \dot{y}_f = -y_f + y$, $\epsilon > 0$ small.

iv) *(I&I and PI adaptation)* Since, introducing a small time constant in the filter (16) we can make the filtered output y_f track y arbitrarily fast, it is interesting to study the behavior of the controller if we assume $y_f \approx y$ and

$\Phi_f(y_f) \approx \Phi(y)$. Under this approximation, equations (14), (15) of the I&I adaptive regulator reduce to

$$
\begin{aligned}
u &= \Phi(y)\hat{\theta} - \Phi(y)\Phi^\top(y)\Gamma^{-1}y \\
\dot{\hat{\theta}} &= -\Phi^\top(y)\Gamma^{-1}y
\end{aligned}
$$

which is the standard MRAC but with an estimation law consisting of integral and proportional terms—that in early references was called *PI adaptation* [7]. A schematic diagram of the resulting error system is shown in Fig. 2, where we clearly see that the proportional term of PI adaptation introduces a by–pass term to the gradient estimator. This term strengthens the passivity of H_1, and in some instances may overcome the lack of passivity of $H_2(s)$, see e.g. [11]. In the light of this discussion we are tempted to conjecture that adaptive I&I is an MRAC with PI adaptation in disguise. The stabilization mechanism of adaptive I&I is however more subtle, and the conjecture is incorrect. Indeed, a detailed derivation of the adaptive I&I controller in this case results in

$$
\begin{aligned}
u &= \Phi(y)(\hat{\theta} - \Phi_f^\top\Gamma^{-1}y_f) - \Phi_f(y_f)\Phi_f^\top(y_f)\Gamma^{-1}[(\epsilon-1)y_f + y] && (17)\\
\epsilon\dot{\hat{\theta}} &= -\left[(1+\epsilon)\Phi_f(y_f) - \Phi(y)\right]^\top\Gamma^{-1}y_f && (18)\\
\epsilon\dot{y}_f &= -y_f + y && (19)
\end{aligned}
$$

which should be compared with (14)–(16). In view of the presence of the small parameter ϵ on the update law equation (18), we conclude that MRAC with PI adaptation *is not* the slow model of the adaptive I&I (17)–(19). The stability properties of MRAC with PI adaptation in this particular problem remain to be established.

v) *(High gain control)* Clearly, if assumption (2) holds, it is trivial to design a high–gain stabilizer that achieves the regulation objective, for instance, using sliding mode control. Actually, using the recent results of [4], it is possible to further relax this assumption to diagonal stability of the matrix $\Gamma^{-1}K_p$.[11] Of course, to tackle the tracking problem or the case of systems treated in Proposition 3 requires some sort of adaptation as done in [3] or in the present paper.

vi) *(Nonlinear parameterizations)* Although not the case in the present work, I&I adaptation does not require, in general, a linear parameterization neither of the plant nor the control law. For instance, it has been used in [1] to solve

[11] A square matrix is diagonally stable if the corresponding algebraic Lyapunov equation admits a diagonal solution.

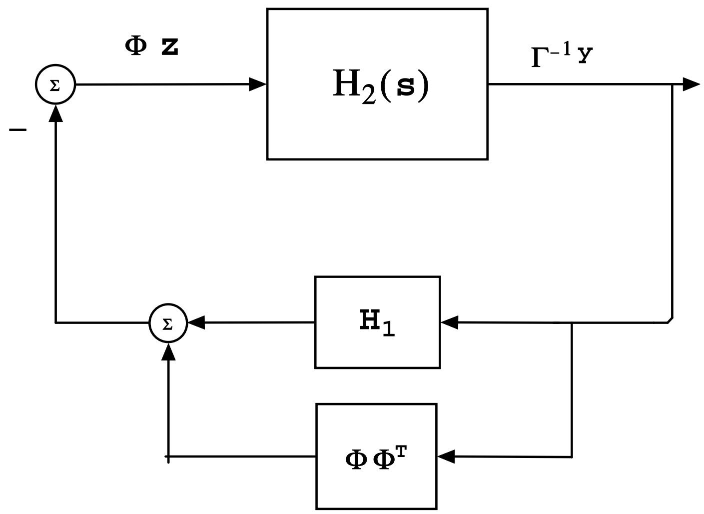

Fig. 2. MRAC with PI adaptation.

a longstanding nonlinearly parameterized problem of adaptive calibration in visual servoing.

References

[1] A. Astolfi, L. Hsu, M. Netto and R. Ortega, A solution to the adaptive visual servoing problem, *IEEE Conf. Robotics and Automation*, Seoul, Korea, May 21-26, 2001. (To appear in *IEEE Trans Robotics and Automation.*)

[2] A. Astolfi and R. Ortega, Immersion and invariance: a new tool for stabilization and adaptive control of nonlinear systems, *European Control Conf ECC 2001*, Porto, Portugal, September, 4–7, 2001.

[3] L. Hsu and R. Costa, MIMO direct adaptive control with reduced prior knowledge of the high frequency gain, *Proc. 38th IEEE CDC*, Phoenix, AZ, December 1999.

[4] L. Hsu, E. Kaszkurewicz and A. Bhaya, Matrix–theoretic conditions for the realizability of sliding manifolds, *Systems and Control Letters*, Vol. 40, No. 3, pp. 145–152, July 2000.

[5] A. Imai, R. R. Costa, L. Hsu, G. Tao and P. Kokotovic, Multivariable MRAC using high frequency gain matrix factorization, *Proc. 40th IEEE CDC*, pp. 1193–1198, Orlando, AZ, December 4–7, 2001.

[6] P. Ioannou and J. Sun, *Robust Adaptive Control*, Prentice Hall, NJ, 1996.

[7] I. Landau, *Adaptive Control: The Model Reference Approach*, Marcel Dekker, NY, 1979.

[8] I. Mareels and J. W. Polderman, *Adaptive Systems: An Introduction*, Birkhauser, Berlin, 1996.

[9] K. Narendra and A. Annaswamy, *Stable Adaptive Systems*, Prentice Hall, NJ, 1989.

[10] R. Ortega and A. Herrera, A solution to the continuous-time adaptive decoupling problem, *IEEE Trans. Aut. Cont.*, AC-39, No. 7, pp. 1639-1643, Aug. 1994.

[11] R. Ortega and T. Yu, Robustness of adaptive controllers: A survey, *Automatica*, 25(5):651–677, Sept. 1989.

[12] L. Praly. Adaptive regulation: Lyapunov design with a growth condition. *Int J Adaptive Control and Signal Proc.*, Vol. 6, No. 4, July 1992, pp. 329–352.

[13] S. Sastry and M. Bodson, *Adaptive Control: Stability, Convergence and Robustness*, Prentice–Hall, 1989.