Heavy tails and copulas: 
Limits of diversification revisited*

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Abstract

We show that diversification does not reduce Value-at-Risk for a large class of dependent heavy tailed risks. The class is characterized by power law marginals with tail exponent no greater than one and by a general dependence structure which includes some of the most commonly used copulas.

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1 Introduction

Level-$q$ Value-at-Risk $VaR_q$ ($q > 0$), also known as the level-$q$ quantile of a distribution of losses, is a commonly used risk measure, whose popularity in a wide range of areas in finance is attributed to the recommendations of the Basel Committee on Banking Supervision. A series of recent papers studied the problem of portfolio optimization in the VaR framework, mostly focusing on the situation when the portfolio components are independent and have a heavy tailed distribution (see, e.g., Embrechts et al. 1997, 2009b; Ibragimov and Walden 2011). An important conclusion from that work is that if the tails of return distributions are extremely heavy then diversification increases portfolio riskiness in terms of VaR.

This property of VaR known as non-subadditivity has been studied in i.i.d. settings by many authors. For example, Garcia et al. (2007), Ibragimov and Walden (2007) and Ibragimov (2009b) focus on i.i.d. stable random variables (r.v.’s) with infinite variance and show that VaR is subadditive provided that the mean of the distribution is finite. Similar results are obtained for asymptotically large losses of portfolios of i.i.d. risks with general power law distributions.

There are a few extensions to non-independence. For example, Ibragimov and Walden (2007, 2011) consider dependence arising from common multiplicative and additive shocks, Embrechts et al. (2009b) and Chen et al. (2012) consider Archimedian copulas, Asmussen and Rojas-Nandayapa (2008) consider the normal copula, Albrecher et al. (2006) consider Archimedian copulas. Barbe et al. (2006) use a spectral measure of the tail dependence. Embrechts et al. (2009a) and Jessen and Mikosch (2006) use multivariate regular variation. The studies find that the subadditivity property of VaR is generally affected by both the strength of dependence and the tail behavior of the marginals, however in some cases only heavy tails of the marginals matter (see Ibragimov and Prokhorov 2017 for a survey of these and related results).

The purpose of this paper is to provide new results on subadditivity of VaR in non-iid settings.

2 Diversification under independence

2.1 Heavy tails and power law family

A tail index of a univariate distribution characterizes the heaviness, or the rate of decay, in the tails of the distribution, assuming it obeys a power law. Let $X$ denote a loss. Then, $X$ belongs to the power law family of distributions if

$$P(|X| > x) \sim x^{-\alpha},$$

$$\alpha > 0.$$
where $\alpha$ is the tail index, or tail exponent, and “$\sim$” means that the left hand side is asymptotically equivalent to a nonzero constant times the right hand side (and asymptotics is with respect to $x \to \infty$). The power law family is commonly used in financial econometrics to model heavy tailed distributions (see, e.g., [Gabaix 2009 Ibragimov 2009b]).

Power law distributions are attractive because they permit modelling rates of tail decay that are slower than the exponential decay of a Gaussian distribution. Such distributions often form the basis of a wider class obtained by introducing a slight disturbance to the tail behavior in the form of a slowly varying function (see, e.g., [Embrechts et al. 1997 Ibragimov and Walden 2008]). Many distributions can be viewed as special cases of power laws, at least for asymptotically large losses. This includes Pareto and Student-$t$ distributions as well as Cauchy, Levy and other stable distributions with the index of stability $\alpha < 2$.

The tail index $\alpha$ governs the likelihood of observing outliers or large fluctuations of risks or returns in consideration: a smaller tail index means slower rate of decay of tails of risk distributions, which means that the above likelihood is higher. When the tail index is less than two, the tail decay is so slow that the second moment of the underlying risk or return distribution is infinite; when the tail index is less than one, the first moment is infinite. More generally, the power law distributions have the property that absolute moments of $X$ are finite if and only if their order is less than tail index $\alpha$. That is,

$$\mathbb{E}|X|^p < \infty \quad \text{if} \quad p < \alpha; \quad \mathbb{E}|X|^p = \infty \quad \text{if} \quad p \geq \alpha.$$

A large number of studies in economics, finance and insurance have documented that financial returns and other important financial and economic variables have heavy-tailed distributions with values of $\alpha$ ranging from significantly lower than one to above four ([Jansen and Vries 1991 Loretan and Phillips 1994 McCulloch 1997 Rachev and Mittnik 2000 Gabaix et al. 2006 Chavez-Demoulin et al. 2006 Silverberg and Verspagen 2007] and references therein).

We will say that a risk has extremely heavy tails if $\alpha < 1$, and moderately heavy tails if $\alpha > 1$.

### 2.2 Limits of diversification under heavy tails and independence

Consider a simple problem of optimal portfolio allocation in the VaR framework with possibly extremely heavy tailed losses $X_j > 0$, $j = 1, 2$. Let $w = (w_1, w_2) \in \mathbb{R}^2$ be the portfolio weights such that $w_1, w_2 \geq 0$, $w_1 + w_2 = 1$. Consider the tail of the aggregate loss distribution $\mathbb{P}(w_1X_1 + w_2X_2 > x)$, where the weighted average loss $w_1X_1 + w_2X_2$ corresponds to a portfolio with weights $w_1$ and $w_2$. Unless one of the weights is zero, the portfolio is diversified.
A $q\%$ Value-at-Risk of a portfolio risk $Z$ is $\text{VaR}_q(z) = \inf\{z \in \mathbb{R} : \mathbb{P}(Z > z) \leq q\}$, or the $(1-q)$-th quantile of the loss distribution. The problem of interest is to minimize $\text{VaR}_q(w_1X_1 + w_2X_2)$ over the weights $w$ for a given $q \in (0, 1/2)$.

When $X_1$ and $X_2$ are i.i.d. with a stable distribution, it is now well understood that, for all non-zero $w$’s, $\mathbb{P}(w_1X_1 + w_2X_2 > x) \geq \mathbb{P}(X_1 > x)$ if $\alpha_j > 1$, $j = 1, 2$. In other words, the VaR of a diversified portfolio of moderately (but not extremely) heavy tailed risks is no greater than that of a not diversified.

If $X_j$’s are i.i.d. with $\alpha_j < 1$ then $\mathbb{P}(w_1X_1 + w_2X_2 > x) \leq \mathbb{P}(X_1 > x)$; that is, for extremely heavy-tailed risks the benefits of diversification disappear and the least risky portfolio is one that has a single risk. For example, if $X_j$’s are i.i.d. stable with $\alpha = 1/2$, that is, if $X_j$’s are Levy distributed, the aggregate loss of an equally weighted portfolio $\frac{X_1 + X_2}{2}$ has the same distribution as $2X_1$ and thus $\text{VaR}_q\left(\frac{X_1 + X_2}{2}\right) = 2\text{VaR}_q(X_1) > \text{VaR}_q(X_1)$.

Ibragimov (2009b) showed that analogous statements hold for portfolios of any size, using majorization theory. Similar results are available for bounded risks concentrated on a sufficiently large interval: for such cases, VaR-based diversification is suboptimal up to a certain number of risks and then becomes optimal (Ibragimov and Walden, 2007).

There is a growing range of applications of these seemingly counterintuitive results in finance, economics and insurance. Ibragimov et al. (2009) demonstrate how this analysis can be used to explain abnormally low levels of reinsurance among insurance providers in markets for catastrophic insurance. Ibragimov et al. (2011) show how to analyze the recent financial crisis as a case of excessive risk sharing between banks when risks are extremely heavy-tailed. Gabaix (2009) provides a review of applications of the above conclusions in different areas of economics and finance.

Let $(\xi_1(\alpha), \xi_2(\alpha))$ denote independent random variables from a power-law distribution with a common tail index $\alpha$. It follows from the two-risks example above that the limits of diversification results hold for i.i.d. losses regardless of the weights $w_j$. In what follows we consider an equally weighted portfolio $w_1 = w_2 = 1/2$, for simplicity. All our results can be easily extended to a portfolio of size $n$ with any weights, in which case the aggregate loss $\left(\frac{\xi_1(\alpha) + \xi_2(\alpha)}{2}\right)$ is replaced with $\sum_{i=1}^{n} w_i \xi_i(\alpha)$.

**Theorem 1** (Theorems 4.1 and 4.2 of Ibragimov (2009b)) For sufficiently small loss probability $q$,

$$\text{VaR}_q\left(\frac{\xi_1(\alpha) + \xi_2(\alpha)}{2}\right) < \text{VaR}_q(\xi_1(\alpha)), \quad \text{if} \quad \alpha > 1$$

(2)

$$\text{VaR}_q\left(\frac{\xi_1(\alpha) + \xi_2(\alpha)}{2}\right) > \text{VaR}_q(\xi_1(\alpha)), \quad \text{if} \quad \alpha < 1$$

(3)

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An interesting boundary case corresponds to $\alpha = 1$. This is when diversification has no effect, i.e. it neither increases nor reduces VaR. For example, if $\xi$’s are i.i.d. stable with $\alpha = 1$, which means they have a Cauchy distribution, one has that $\frac{\xi_1(\alpha) + \xi_2(\alpha)}{2}$ has the same distribution as $\xi_1(\alpha)$, so a diversified and a non-diversified portfolios have identical VaRs.

It is not obvious what happens if we relax the independence assumptions. The two extreme cases, corresponding to a comonotone (extreme positive) and countermonotone (extreme negative) dependence do not present a consistent picture. For example, if we consider extreme positive dependence with $\xi_1 = \xi_2$ (a.s.) then, obviously, $\text{VaR}_q (w_1 \xi_1(\alpha) + w_2 \xi_2(\alpha)) = \text{VaR}_q (\xi_1(\alpha))$ and so diversification has no effect regardless of the tails; while if we have extreme negative dependence with $\xi_1 = -\xi_2$ (a.s.) then $\text{VaR}_q (w_1 \xi_1(\alpha) + w_2 \xi_2(\alpha)) = (w_1 - w_2) \text{VaR}_q (\xi(\alpha))$ and it is optimal to fully diversify regardless of the tails.

3 Diversification under dependence

3.1 Dependence and copulas

Copulas are joint distributions with uniform marginals. They are useful because given the marginal distributions, they represent the dependence in the joint distribution. Specifically, let $H(x_1, \ldots, x_n)$ and $h(x_1, \ldots, x_n)$ denote the joint cdf and density, respectively, of $n$ random variables $(X_1, \ldots, X_n)$ and suppose that the marginal density and cdf of $X_j$ are $f_j(x_j)$ and $F_j(x_j)$ respectively, $j = 1, \ldots, n$. Then, an $n$-dimensional copula of $(X_1, \ldots, X_n)$ is a function $C : [0, 1]^n \to [0, 1]$ such that

(a) $C(u_1, \ldots, u_n)$ is increasing in each $u_i$, $i = 1, \ldots, n$.

(b) $C(u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_n) = 0$, $i = 1, \ldots, n$.

(c) $C(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i$, $i = 1, \ldots, n$.

(d) for any $a_j \leq b_j$, $j = 1, \ldots, n$,

$$\sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{i_1+\cdots+i_n} C(u_{i_1}, \ldots, u_{i_n}) \geq 0,$$

where $u_{j1} = a_j$ and $u_{j2} = b_j$ for all $j = 1, \ldots, n$.

(e) $H(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n))$, or, for absolutely continuous copulas with density $c(u_1, \ldots, u_n)$, $h(x_1, \ldots, x_n) = c(F_1(x_1), \ldots, F_n(x_n)) \prod_{i=1}^{n} f(x_i)$.
It is well known that the copula $C$ (the copula density $c$) is uniquely determined if the univariate cdf’s $F_j$ are continuous. The probability integral transforms $u_j = F_j(X_j), j = 1, 2,$ are the uniform random variables that form the marginals of $C$ and $c$. So, equivalently $C$ can be defined as a joint cdf of $n$ random variables, each of which is uniform on $[0, 1]$. The fact that we can model $F_j$ separately from modelling the dependence between $F_j$’s is what makes copulas natural in the analysis of dependent risks with heavy-tailed power-law marginals.

A well known property of the copula function is that it is bounded by the Frechet-Hoeffding bounds, which correspond to joint cdf’s with extreme positive and, in the case $n = 2$, extreme negative dependence. For two risks $X_1, X_2$, let $X_1$ be a fixed increasing function of $X_2$, then the copula of $(X_1, X_2)$ can be written as $\min(u_1, u_2)$ and this is the upper bound for bivariate copulas. Now let $X_1$ be a fixed decreasing function of $X_2$; then the copula of $(X_1, X_2)$ can be written as $\max(u_1 + u_2 - 1, 0)$. So the two extreme cases when diversification does not have any effect (comonotonicity) and when it is always beneficial (countermonotonicity) regardless of the heavy-tailedness are nested within the copula framework. Joe (1997) and Nelsen (2006) provide excellent introductions to copulas.

If we return to the two-risk example above, we are interested in how the aggregate loss probability for a diversified portfolio compares to that of a single risk. That is, we are interested in evaluating

$$P\left(\frac{X_1 + X_2}{2} > x\right) = \int \int_{z_1 + z_2 > x} f(z_1; \alpha)f(z_2; \alpha)c(F(z_1; \alpha), F(z_2; \alpha); \gamma) \, dz_1 dz_2$$

$$= \mathbb{E}\left\{c(F(\xi_1; \alpha), F(\xi_2; \alpha); \gamma) \mathbb{I}\left[\frac{\xi_1 + \xi_2}{2} > x\right]\right\}$$

where $c(u_1, u_2; \gamma)$ is a copula density parameterized by $\gamma,$ $f(\cdot; \alpha)$’s are power-law marginal densities, $\mathbb{I}[\cdot]$ is the indicator function and $\xi_j$’s are independent copies of $X_j$’s.

There is no general way of expressing this probability in terms of $P(X_1 > x)$ and whether diversification helps or hurts depends on the copula family as well as on the interaction between $\alpha$ and $\gamma$. However, there exist classes of copulas for which we can make explicit comparisons.

### 3.2 Power-type copulas

The class of copulas we propose contains copula families that are multiplicative or additive in powers of the margins, or can be approximated using such copulas. We call this class power-type. It is similar but more general than the power copula family and than the polynomial copula family which we discuss below.
The most common family in this class is the Eyraud-Farlie-Gumbel-Morgenstern (EFGM) copula family and its generalizations. The bivariate EFGM copula family can be written as follows:

\[ C(u_1, u_2) = u_1 u_2 [1 + \gamma (1 - u_1)(1 - u_2)], \]

where \( \gamma \in [-1, 1] \), and its density has the form \( c(u_1, u_2) = 1 + g(u_1, u_2) \), where \( g(u_1, u_2) \) is an expansion by linear functions \( 1 - 2u_j, j = 1, 2 \). This is a non-comprehensive copula in the sense that it has a limited range of dependence it can accommodate. For example, Kendall’s \( \tau \) of an EFGM copula is restricted to \( [-\frac{2}{3}, \frac{2}{3}] \).

The multivariate version of the EFGM copula introduced by Cambanis [1977] has the following form:

\[ C(u_1, u_2, \ldots, u_n) = u_1 u_2 \cdots u_n \left[ 1 + \sum_{c=2}^{n} \sum_{1 \leq i_1 < i_2 < \ldots < i_c \leq n} \gamma_{i_1, i_2, \ldots, i_c} (1 - u_{i_1})(1 - u_{i_2}) \cdots (1 - u_{i_c}) \right], \]

where \( -\infty < \gamma_{i_1, i_2, \ldots, i_c} < \infty \) are such that \( \sum_{c=2}^{n} \sum_{1 \leq i_1 < i_2 < \ldots < i_c \leq n} \gamma_{i_1, i_2, \ldots, i_c} \delta_{i_1} \cdots \delta_{i_c} \geq -1 \) for all \( \delta_i \in [-1, 1], i = 1, \ldots, n \). This copula family can be viewed as a special case of a wider family of n-dimensional power copulas introduced by Ibragimov [2009a].

The power copula family can be written as follows

\[ C(u_1, \ldots, u_n) = u_1 u_2 \cdots u_n \left[ 1 + \sum_{c=2}^{n} \sum_{1 \leq i_1 < i_2 < \ldots < i_c \leq n} \gamma_{i_1, i_2, \ldots, i_c} (u_{i_1}^l - u_{i_1}^l)(u_{i_2}^l - u_{i_2}^l) \cdots (u_{i_c}^l - u_{i_c}^l) \right], \]

where \( \gamma_{i_1, i_2, \ldots, i_c} \in (-\infty, \infty) \) are such that

\[ \sum_{c=2}^{n} \sum_{1 \leq i_1 < i_2 < \ldots < i_c \leq n} |\gamma_{i_1, i_2, \ldots, i_c}| \leq 1. \]

This corresponds to using nonlinear rather than linear functions in the expansion of the copula density function.

Another relevant copula family, of which the EFGM copula in (4) is a special case, is known as a polynomial copula family (see, e.g., Drouet Mari and Kotz [2001] p. 74). An order \( m (m \geq 4) \) polynomial copula can be written as follows:

\[ C(u, v) = uv \left[ 1 + \sum_{k+q \leq m-2} \gamma_{kq} (u^k - 1)(v^q - 1) \right], \]

where \( \gamma_{kq} = \frac{\theta_{kq}}{(k+1)(q+1)}, \theta_{kq} \in \mathbb{R}, \) and \( 0 \leq \min \left( \sum_{k+q \leq m-2} k^{q}, \sum_{k+q \leq m-2} q^{k} \right) \leq 1. \)

One example of this copula family is Nelsen et al.’s [1997] copula with cubic section, which is written as follows

\[ C(u, v) = uv + 2\gamma uv(1 - u)(1 - v)(1 + u + v - 2uv), \]
where $\gamma \in [0, \frac{1}{4}]$.

Several other copula families can be written as approximations of the EFGM copula. For example, it is well known that the EFGM copula is a first-order approximation to the Ali-Mikhail-Haq (AMH) copula family. The AMH copula can be written as follows:

$$C(u_1, \ldots, u_n) = (1 - \gamma) \left[ \prod_{i=1}^{n} \left( \frac{1 - \gamma}{u_i} + \gamma \right) - \gamma \right]^{-1},$$

where $\gamma \in [-1, 1]$.

A less known result is that the Plackett and the Frank copula families are first order Taylor approximations of the EFGM copula at independence (see, e.g., Nelsen, 2006, p. 100, 133). The $n$-variate Frank copula, which is comprehensive, radially symmetric and Archimedian, can be written as follows

$$C(u_1, \ldots, u_n) = \log \gamma \left[ 1 + \frac{\prod_{i=1}^{n} (\gamma u_i - 1)}{(\gamma - 1)^{n-1}} \right],$$

where $\gamma \geq 0$.

The $n$-variate Plackett copula, which is also comprehensive, is rarely discussed in the literature unless $n = 2$, in which case it has the following form:

$$C(u_1, u_2) = \frac{1}{2(\gamma - 1)} \left[ 1 + (\gamma - 1)(u_1 + u_2) - \sqrt{1 + (\theta - 1)(u_1 + u_2)^2 - 4\gamma(\gamma - 1)u_1u_2} \right],$$

where $1 \neq \gamma > 0$. However, a way to generalize to $n > 2$ is presented by Molenberghs and Lesaffre (1994). It is also worth mentioning that for all the three copula families, there exist improved second-order approximations (see, e.g., Nelsen, 2006, p. 83).

An interesting set of approximation results are given by Nelsen et al. (1997), Cuadras (2009) and Cuadras and Diaz (2012). Nelsen et al. (1997) provide a generalization of the bivariate EFGM copula using cubic terms as in (8) and show that it can be used to approximate some well-known families of copulas, both symmetric and not, such as the copulas of Kimeldorf and Sampson (1975) and Lin (1987), as well as the Sarmanov copula. They also show that copulas in (8) are second-degree Maclaurin approximations to members of the Frank and Plackett copula families.

Cuadras (2009) studies the power series class of copulas, obtained as weighted geometric means of the EFGM and AMH copulas, and shows that the Gumbel-Barnett and Cuadras-Auge copulas can be expressed as first-order approximations to that class. Cuadras and Diaz (2012) provide approximations of the tail-dependent Clayton-Oakes copula, which also have the form of a power-type generalization of the EFGM copula.
3.3 Diversification under dependence

We start with the bivariate EFGM copula. Let \((X_1, X_2)\) be random variables with the EFGM copula and power-law marginals with the tail index \(\alpha > 0\). Then, for \(j = 1, 2\),

\[
1 - F_j(x) \sim x^{-\alpha},
\]

\[
f_j(x) \sim \alpha x^{-\alpha - 1}, \text{ as } x \to \infty
\]

\[
H(x_1, x_2) = F_1(x_1)F_2(x_2)[1 + \gamma(1 - F_1(x_1))(1 - F_2(x_2))],
\]

\[
h(x_1, x_2) = f_1(x_1)f_2(x_2)[1 + \gamma(1 - 2F_1(x_1))(1 - 2F_2(x_2))].
\]

As before, let \((\xi_1(\alpha), \xi_2(\alpha))\) be independent copies of \((X_1, X_2)\), that is, independent random variables that have the same power-law distributions with tail index \(\alpha\) as \(X_1, X_2\). Our key insight is that in the tail, we can characterize the behavior the joint distribution of \(X_j, j = 1, 2\), in terms of the distribution of the independent copies. This makes it possible to provide asymptotic (with respect to the loss) comparisons between the VaR of an aggregated loss and that of a single risk. More specifically, the crucial component of \(P(X_1 + X_2 > x)\) under the EFGM copula can be written as follows

\[
\int_{\frac{s+t}{2} > x} \alpha^2 s^{-\alpha - 1} t^{-\alpha - 1}(2s^{-\alpha} - 1)(2t^{-\alpha} - 1)dsdt
\]

\[
\sim 4\alpha^2 P\left(\frac{\xi_1(2\alpha) + \xi_2(2\alpha)}{2} > x\right) - 2\alpha^2 P\left(\frac{\xi_1(2\alpha) + \xi_2(\alpha)}{2} > x\right)
\]

\[
-2\alpha^2 P\left(\frac{\xi_1(\alpha) + \xi_2(2\alpha)}{2} > x\right) + \alpha^2 P\left(\frac{\xi_1(\alpha) + \xi_2(\alpha)}{2} > x\right),
\]

where the behavior of the individual summands for large \(x\) is driven by the lowest tail index of \(\xi_j\) in the portfolio.

We formalize this result in the following theorem, which generalizes to \(n\) dependent heavy-tailed random variables \(X_1, X_2, ..., X_n\) with multivariate EFGM copula given in (5) and power-law marginals.

**Theorem 2** For an asymptotically large \(x > 0\), and any \(n, \alpha > 0\),

\[
P\left(\sum_{i=1}^{n} X_i > xn\right) \sim P\left(\sum_{i=1}^{n} \xi_i(\alpha) > xn\right). \tag{9}
\]

The result suggests that suboptimality of diversification in the VaR framework for extremely heavy tailed losses carries over from independence to the EFGM copula. That is, diversification increases VaR of dependent extremely heavy tailed risks within this copula family. Specifically, combining the results of Theorems 1 and 2, it is easy to see that the following corollary holds.
Corollary 1 For dependent losses with the EFGM copula and sufficiently small loss probability $q$,

$$\text{VaR}_q\left(\frac{X_1 + X_2}{2}\right) < \text{VaR}_q(X_1), \quad \text{if} \quad \alpha > 1$$  \hspace{1cm} (10)

$$\text{VaR}_q\left(\frac{X_1 + X_2}{2}\right) > \text{VaR}_q(X_1), \quad \text{if} \quad \alpha < 1$$  \hspace{1cm} (11)

Another interesting corollary of Theorem 2 can be obtained by combining this result with Theorem 1 of Sharakhmetov and Ibragimov (2002). The EFGM copula family has restrictive dependence, for example, it is not comprehensive in the sense that it cannot accommodate all possible values of Kendall’s $\tau$. Yet, as shown by Sharakhmetov and Ibragimov (2002), it can be used to represent any joint distribution of two-valued random variables (see also de la Pena et al., 2006, p. 190). Therefore, for two-valued random variables, our Theorem 2 applies to all dependence patterns.

Important generalizations of Theorem 2 arise if we consider the wider class of power-type copulas discussed in Section 3.2. Most popular members of this class such as the polynomial copula of Drouet Mari and Kotz (2001), the copula with cubic section of Nelsen et al. (1997) and the power copula of Ibragimov (2009b) can be written in the following general form

$$C(u_1, \ldots, u_n) = \sum_{i_1, \ldots, i_n = 0, 1, \ldots} \gamma_{i_1, i_2, \ldots, i_n} \cdot u_1^{i_1} \cdot u_2^{i_2} \cdot \ldots \cdot u_n^{i_n},$$  \hspace{1cm} (12)

for a multiple index $i = (i_1, i_2, \ldots, i_n)$ and a set of corresponding parameters $\gamma_i$ with appropriate restrictions that make $C(u_1, \ldots, u_n)$ a copula. For example, Drouet Mari and Kotz (2001, Section 4.5.2) show how to obtain the polynomial copula in (7) from function $f = u^k v^q$. The key feature of such copulas is that they and their densities can be expressed as powers of $u_j$’s. This allows to apply similar arguments as for EFGM.

Theorem 3 For dependent losses with a power-type copula in (12) and for an asymptotically large $z > 0$, and any $n, \alpha > 0$, the conclusions of Theorem 2 hold.

One may argue that the class of copulas in (12) is not sufficiently general. For example, it is not clear whether it can incorporate tail dependence or comprehensive copulas. However, the power-type copulas also include copulas which can approximate or be approximated by the class in (12). And, as discussed in Section 3.2, there are comprehensive and tail-dependent copulas among these copulas families. A corollary from the above theorem is that for dependent losses with copulas whose Taylor or Maclaurin expansions can be written as (12), the results of Theorem 2 should hold locally at the point of approximation.
This covers all the copula families discussed in Section \ref{sec:copula-families} including the AMH, Plackett, Frank, Clayton-Oakes, Kimeldorf and Sampson, Lin, Gumbel-Barnett and many others, but only to the extent the approximations are valid. That is, the results of Theorem \ref{thm:main-theorem} hold for expansions at the point at which we expand, which often coincides with independence. Clearly, they may fail when the approximation error is large and the precise meaning of “holding locally” needs to be clarified. Therefore, applicability of Theorem \ref{thm:main-theorem} to a specific copula family needs to be checked on a case-by-case basis but the class of copulas that can be covered is very rich – it includes comprehensive copulas (Plackett, Frank), asymmetric copulas (Nelsen et al.'s copulas with cubic sections) and tail-dependent copulas (Clayton-Oakes).

4 Concluding remarks

We have revisited the limits of diversification for dependent risks. The revisit focused on a wide class of copulas that are additive in powers of margins. This class covers several well known families such as EFGM, power and polynomial families but also contains a number of other copula classes which do not have this form but can be approximated using Taylor-type expansions. So the resulting class we consider is very wide – comprehensive, tail-dependent and asymmetric copula families can be considered within this class.

The main result of the paper is that within the class, diversification increases riskiness in a VaR framework if the tail index of the individual risks falls below one. This makes dependent risks within this class no different from independent in the sense that the same threshold value of the tail index delineates the benefits of diversification.

We have looked at equally weighted portfolios with components having the same tail index. The restriction of equal tail indices can easily be relaxed because the tail behavior of the aggregate loss is dominated by the component with the lowest tail index. The limits of diversification are determined by whether the lowest index in the portfolio is above or below one. A similar result can be obtained for unequally weighted portfolios but we leave both these extensions for future work.

A Proofs

Proof of Theorem \ref{thm:main-theorem}

We provide the proof for the case $n = 2$. A working paper version of this paper (available on the authors’ webpages) contains the proof for any $n$. 

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Due to independence between $\xi_1$ and $\xi_2$, we have that
\[
\mathbb{P}\left(\frac{\xi_1(\beta_1) + \xi_2(\beta_2)}{2} > z\right) \sim \beta_1 \beta_2 \int_{\frac{z}{\beta_1} > z} s^{-\beta_1-1} t^{-\beta_2-1} dsdt.
\] (13)

Now for non-independent $(X_1, X_2)$ under the EFGM copula, we can write using (13):
\[
\mathbb{P}\left(\frac{X_1 + X_2}{2} > z\right) = \int_{\frac{z}{\xi_1} > z} f_1(s) f_2(t) \left[1 + \gamma(1 - 2F_1(s))(1 - 2F_2(t))\right] dsdt
= \mathbb{P}\left(\frac{\xi_1(\alpha) + \xi_2(\alpha)}{2} > z\right) + \gamma \int_{\frac{z}{\xi_1} > z} f_1(s) f_2(t)(1 - 2F_1(s))(1 - 2F_2(t)) dsdt
= \mathbb{P}\left(\frac{\xi_1(\alpha) + \xi_2(\alpha)}{2} > z\right) + \gamma E(1 - 2F_1(\xi))(1 - 2F_2(\eta)) I\left(\frac{\xi_1 + \xi_2}{2} > z\right),
\]
where $I(\cdot)$ denotes the indicator function.

Now consider the last term:
\[
\int_{\frac{z}{\xi_1} > z} f_1(s) f_2(t)(1 - 2F_1(s))(1 - 2F_2(t)) dsdt \sim \int_{\frac{z}{\xi_1} > z} \alpha^2 s^{-\alpha-1} t^{-\alpha-1} (2s^{-\alpha} - 1)(2t^{-\alpha} - 1) dsdt
= 4\alpha^2 \int_{\frac{z}{\xi_1} > z} s^{-2\alpha-1} t^{-2\alpha-1} dsdt
- 2\alpha^2 \int_{\frac{z}{\xi_1} > z} s^{-2\alpha-1} t^{-\alpha-1} dsdt
- 2\alpha^2 \int_{\frac{z}{\xi_1} > z} s^{-\alpha-1} t^{-2\alpha-1} dsdt
+ \alpha^2 \int_{\frac{z}{\xi_1} > z} s^{-\alpha-1} t^{-\alpha-1} dsdt
= 4\alpha^2 \mathcal{I}_1 - 2\alpha^2 \mathcal{I}_2 - 2\alpha^2 \mathcal{I}_3 + \alpha^2 \mathcal{I}_4,
\]
where $\mathcal{I}_1 = \mathbb{P}\left(\frac{\xi_1(2\alpha) + \xi_2(2\alpha)}{2} > z\right)$, $\mathcal{I}_2 = \mathbb{P}\left(\frac{\xi_1(2\alpha) + \xi_2(\alpha)}{2} > z\right)$, $\mathcal{I}_3 = \mathbb{P}\left(\frac{\xi_1(\alpha) + \xi_2(2\alpha)}{2} > z\right)$ and $\mathcal{I}_4 = \mathbb{P}\left(\frac{\xi_1(\alpha) + \xi_2(\alpha)}{2} > z\right)$.

Thus we obtain
\[
\mathbb{P}\left(\frac{X + Y}{2} > z\right) \sim (1 + \gamma \alpha^2) \mathbb{P}\left(\frac{\xi_1(\alpha) + \xi_2(\alpha)}{2} > z\right)
- 2\gamma \alpha^2 \mathbb{P}\left(\frac{\xi_1(\alpha) + \xi_2(2\alpha)}{2} > z\right)
- 2\gamma \alpha^2 \mathbb{P}\left(\frac{\xi_1(2\alpha) + \xi_2(\alpha)}{2} > z\right)
+ 4\gamma \alpha^2 \mathbb{P}\left(\frac{\xi_1(2\alpha) + \xi_2(2\alpha)}{2} > z\right).
\]
It is a well-known result in the power law literature (see, among others, Corollary 1.3.2 in Embrechts et al. [1997]) that, asymptotically as $z \to \infty$,

$$
P\left(\frac{\xi_1(\beta) + \xi_2(\beta)}{2} > z\right) \sim 2P(\xi_1(\beta) > 2z) \sim 2^{1-\beta}z^{-\beta}
$$

for all $\beta > 0$. In addition, if $\beta_1 < \beta_2$, then

$$
P\left(\frac{\xi_1(\beta_1) + \xi_2(\beta_2)}{2} > z\right) \sim P(\xi_1(\beta_1) > 2z) \sim 2^{-\beta_1}z^{-\beta_1}
$$

It follows from (14)-(15) that, as $z \to \infty$,

$$
P\left(\frac{X + Y}{2} > z\right) \sim (1 + \gamma \alpha^2)2^{1-\alpha}z^{-\alpha} - 2\gamma \alpha^22^{1-\alpha}z^{-\alpha} + 4\gamma \alpha^22^{1-2\alpha}z^{-2\alpha}
$$

$$
\sim (1 - \gamma \alpha^2)2^{1-\alpha}z^{-\alpha}
$$

$$
\sim P\left(\frac{\xi_1(\alpha) + \xi_2(\alpha)}{2} > z\right).
$$

**Proof of Corollary 1**

The result follows trivially by inversion of the cdfs in Theorem 2 for $n = 2$ and applying equations (2)-(3).

**Proof of Theorem 3**

The density corresponding to (12) is a polynomial of a lower order, which we write in the following generic form:

$$
c(u_1, \ldots, u_n) = \sum_{k_1, \ldots, k_n = 0, 1, \ldots} \phi_{k_1, k_2, \ldots, k_n} \cdot u_1^{k_1} \cdot u_2^{k_2} \cdot \ldots \cdot u_n^{k_n},
$$

Then, using arguments similar to Theorem 2,

$$
P\left(\sum_{i=1}^{n} X_i > zn\right) = \mathbb{E}\left[\sum_{k_i \in \{0, 1, \ldots\}} \phi_{k_1, k_2, \ldots, k_n} \cdot F_{k_1}^\xi(\xi_1(\alpha))F_{k_2}^\xi(\xi_2(\alpha))\ldots F_{k_n}^\xi(\xi_n(\alpha))\mathbb{I}\left(\sum_{i=1}^{n} \xi_i(\alpha) > zn\right)\right]
$$

$$
= P\left(\sum_{i=1}^{n} \xi_i(\alpha) > zn\right) + \mathbb{E}\left[\sum_{k_i \in \{0, 1, \ldots\} \setminus \{k_i = 0\forall i\}} \phi_{k_1, k_2, \ldots, k_n} \cdot F_{k_1}^\xi(\xi_1(\alpha))F_{k_2}^\xi(\xi_2(\alpha))\ldots F_{k_n}^\xi(\xi_n(\alpha))\mathbb{I}\left(\sum_{i=1}^{n} \xi_i(\alpha) > zn\right)\right].
$$
Now consider the last term.

\[
\mathbb{E}\left[ \sum_{k_i \in \{0,1,\ldots\} \setminus \{k_i=0\}} \phi_{k_1,k_2,\ldots,k_n} F_1^{k_1}(\xi_1(\alpha))F_2^{k_2}(\xi_2(\alpha))\ldots F_n^{k_n}(\xi_n(\alpha))\mathbb{I}\left( \sum_{i=1}^n \xi_i(\alpha) > zn \right) \right]
\]

(20)

\[
\sim \int_{\sum_{i=1}^n s_i > nz} \sum_{k_i \in \{0,1,\ldots\}} \psi_{k_1,k_2,\ldots,k_n} s_1^{-\alpha(k_1+1)} s_2^{-\alpha(k_2+1)} \ldots s_n^{-\alpha(k_n+1)} ds_1 \ldots ds_n
\]

\[
= \sum_{k_i \in \{0,1,\ldots\}} \psi_{k_1,k_2,\ldots,k_n} \int_{\sum_{i=1}^n s_i > nz} s_1^{-\alpha(k_1+1)} s_2^{-\alpha(k_2+1)} \ldots s_n^{-\alpha(k_n+1)} ds_1 \ldots ds_n
\]

\[
= \sum_{k_i \in \{0,1,\ldots\}} \psi_{k_1,k_2,\ldots,k_n} \mathbb{P}\left( \frac{\xi_1(\alpha(k_1+1)) + \ldots + \xi_n(\alpha(k_n+1))}{n} > z \right),
\]

where the new coefficients \(\psi\)'s are different from \(\phi\)'s because we have expressed \((1 - s_i^{-\alpha})^{k_i}\) in terms of powers of \(s_i^\alpha\). Now, using the same arguments as for (14)-(15),

\[
\mathbb{P}\left( \frac{\xi_1(\alpha) + \ldots + \xi_n(\alpha)}{n} > z \right) \sim n\mathbb{P}(\xi_1(\alpha) > nz) \sim n^{1-\alpha}z^{-\alpha}
\]

\[
\mathbb{P}\left( \frac{\xi_1(\alpha(k_1+1)) + \ldots + \xi_n(\alpha(k_n+1))}{n} > z \right) \sim \mathbb{P}(\xi_1(\alpha) > nz) \sim n^{-\alpha}z^{-\alpha},
\]

for all \(k_i \geq 0\). It thus follows that \(\mathbb{P}(\sum_{i=1}^n X_i > zn) \sim \mathbb{P}(\sum_{i=1}^n \xi_i(\alpha) > zn)\).

References


