Asymptotic consensus on the average of a field for time-varying nonlinear networks under almost periodic connectivity

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***Abstract*—The paper presents new results on asymptotic consensus for continuous time non-autonomous nonlinear networks under almost-periodic interactions. We introduce such consensus algorithms in order to estimate the average of a measured field, despite the presence of limited agents' interaction (herein represented by almost periodic connectivity). To this end, a suitable notion of integral connectivity is exploited, frozen in state variables, and of simple verification, thanks to ergodicity of the underlying agents' spatial dynamics. In the considered set up, consensus variables are different than those affecting network's connectivity unlike most of the existing literature on asymptotic agreement. The application of the proposed results is illustrated considering two representative examples in the scenario of autonomous sampling by mobile sensor agents.**

***Index Terms*—: autonomous agents, nonlinear networks, Consensus, Multi agent systems, almost periodic function.**

I. Introduction

During recent years the scientific community has devoted considerable attention to the consensus problem (see [7], [6], [15] [9] and references therein). The distinguishing feature of the above approaches is to allow every agent to autonomously converge towards a common *agreement* value (or trajectory about some variables of interest) by only using local information available at the node and/or received from neighboring agents. In the literature, several criteria have been adopted to assess consensus, both in discrete and continuous time ([11], [17], [18], [19], [10], [14], just to cite a few) and under different classes of both nonlinear time-invariant and switching/time-varying networks [25], [16], [31], [32], [28], [30]. In particular, by adopting averaged or integrated notions of connectivity as in the following references: [17], [25], [21]. Moreover some classes of non-autonomous linear time-varying consensus algorithms are studied in the presence of bounded measurement errors and vanishing weights [43]. This property is usually guaranteed by requiring the sign definiteness of off diagonal entries of the Jacobian matrix $F(x)$ and absence of negative cycles when regarding the Jacobian as the adjacency matrix of a graph, ([33]). Recently, emerging application paradigms such as surveillance networks, formation flight, clusters of satellites or automated highway systems, have led to the need for agreement protocols within distributed algorithms for fault detection or decision making ([36], [37]), optimization ([38]), control and monitoring of some measures of interest ([42], [41], [35], [39], [40]). In such scenarios, it seems appropriate to model agents' interactions and connectivity strength as dependent on configuration variables distinct from those involved in the asymptotic consensus protocol. For instance, the connectivity of mobile sensor nodes may depend on the agents' relative position (or even orientation, in the case of directional communications) while the agreement value might concern the estimation of either agent features (i.e. reliability, reputation) or aggregate measure of a field (for instance pollutant concentrations, temperatures and so on, i.e. the global average or variance of local node measurements). At the same time, convergence towards the average of the field can be enforced by designing the mobility of individual agents so as to guarantee that the whole area of interest is spanned, not just individually by agents, but also jointly, when regarding their collective motion as a single point in the cartesian product of individual configuration spaces. This is achieved, from a technical point of view, through almost periodic and ergodic mobility of agents, and will result in sufficient uniformity across time and interaction strength among agents visiting nearby or overlapping regions.

A. Paper Contribution

In the light of the above considerations the paper contributions are both of a theoretical and practical nature. Specifically:

a) we propose, in a nonlinear network set-up, new criteria for asymptotic agreement of consensus variables distinct from those affecting the network's connectivity. This generalizes existing studies on linear non-autonomous consensus protocols ([39], [40], [43]) in two respects: by allowing nonlinear agents interactions and, most remarkably, by introducing a cascaded structure into the network whereby spatial dynamics is explicitly taken into account and affects both agents' interaction strength of the downstream consensus protocol and exogenous input measurements. The agreement is guaranteed under weak connectivity properties (just existence of a spanning tree for a suitable averaged graph is required) for a large class of nonlinear time-varying non monotone networks. This encompasses most of the agents models normally adopted in the literature in the linear and non

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linear time varying setting. Moreover, both the dynamics at the node (self-feedback) and the coupling can be time varying and state dependent with the notable feature that the strength of attraction between two agents may vanish as the distance between their "opinion" becomes larger. Finally, the considered type of almost periodic-dependent interactions may effectively model the case of autonomous agents that are characterized by partial or limited interactions among them due to energy (i.e. reduced number of wireless communications), physical (reduced space to allocate agents) or cost (reduced number of available agents) constraints;

b) in recent papers [22], [24], we introduced a condition for asymptotic agreement (*state frozen integral connectivity*), suitable for nonlinear time-varying monotone and non-monotone autonomous networks. The condition extended the notion of integral connectivity introduced by Moreau for linear networks [20], with the additional merit to be frozen in state variables and therefore of simple verification. While the above state-frozen connectivity conditions are formulated in [22], [23] for monotone networks and [24] for non-monotone ones, herein we undertake a non trivial further step of allowing exogenous inputs and almost periodic interactions;

c) we consider two illustrative applications of the proposed results including the above features. Specifically, we show how in the case of asymmetric and almost periodic interactions the criteria may be used to guarantee exponential convergence of autonomous agents state to a common consensus equilibrium about some variable of interest (e.g. level of network/node reliability or reputation). Moreover, in the case of symmetric almost periodic interactions, the result may be used to guarantee asymptotic convergence of individual agents estimates towards the average value of a field over the region spanned by the agents. These find direct application to autonomous sampling scenarios.

II. Notation and problem statement

Throughout the paper all vectors are assumed to be column vectors. To denote vectors we write $x = [x_1, \ldots, x_n]'$ for the column vector $x \in \mathbb{R}^n$. $|x|$ denotes the Euclidean norm of x. $\mathbf{1}$ is the vector of all ones and e_j is the j-th element of the canonical basis of $\mathbb{R}^n$, where n should normally be clear from the context. The integer interval $N = \{1, 2, \ldots, n\}$ will be identified with the set of interacting agents. We adopt the following definition of *almost periodic function* ([1]): a continuous function $g : \mathbb{R} \to \mathbb{R}$ is almost periodic if $\forall \epsilon_a > 0$, $\exists\ l(\epsilon_a) > 0$ such that all intervals of length $l(\epsilon_a)$ contain at least one τ satisfying: $|g(t+\tau) - g(t)| \leq \epsilon_a$, $\forall t \in \mathbb{R}$. $[a]^+ \doteq \max\{0, a\}$, (resp. $[a]^- \doteq \min\{0, a\}$). Finally, given a signal $s(t)$, the asymptotic time average is defined as $\lim_{t\to\infty} \frac{\int_0^t s(\tau)d\tau}{t}$.

Consider an ensemble of agents moving in k-dimensional Euclidean space, within a region that, for the sake of simplicity, we assume to be a box $B \subset \mathbb{R}^k$. In typical applications $k = 2$ but more general scenarios can be envisioned. Each agent has a configuration space which is a q-dimensional torus, $\mathbb{S}^q$, with $q \geq k$. The inequality $q \geq k$ holds true because coordinates of the agent i within the box B are part of the configuration variable $\theta_i \in \mathbb{S}^q$, which characterizes agent i. Adopting coordinates on a torus, rather than standard Euclidean coordinates, is useful to avoid discontinuities in the speed of agents when they hit the boundary of the box and need to bounce backwards as balls subject to an elastic collision in a billiard like fashion. Of course, coordinates in the torus need to be projected down to Euclidean coordinates when assessing relative positions of agents in the plane. This is a well defined procedure [2] usually considered for $\mathbb{R}^2$ also in sensor and mobile applications ([3]). More configuration variables could be included in θ_i so as to model situations where availability of a communication channel between agents is not a function of reciprocal position alone, for instance if agents communicate through directional antennas.

The evolution of θ_i in $\mathbb{S}^q$, for each agent i, follows the equation below:

$$\dot{\theta}_i(t) = \omega_i$$

for some constant vector of angular speeds $\omega_i \in \mathbb{R}^q$. Notice that angular variables are regarded as making sense up to integer multiples of 2π or, equivalently, by embedding $\mathbb{S}^q$ in $\mathbb{R}^{2q}$ through the map $\theta_i \mapsto [\cos(\theta_i)', \sin(\theta_i)']'$ (where $\sin$ and $\cos$ are meant componentwise). Each agent i continuously measures, by means of an onboard sensor, a field of interest $\bar{z} : B \subset \mathbb{R}$ as a function of its own position. To model this, it is convenient to stack configuration variables in $\sigma := (\theta_1, \theta_2, \ldots, \theta_n) \in \mathbb{S}^{qn} := \Sigma$ and then define individual read-out functions $z_i(\sigma) : \Sigma \to \mathbb{R}$, just by composing $\bar{z}(\cdot)$ with the projection of σ over its i-th angular coordinate (and a suitable map to convert it to cartesian coordinates in Euclidean space), for instance: $z_i(\sigma) := \bar{z}(\arcsin(\sin(\theta_i))$ (this is appropriate if the motion of agent i happens on the box $[-\pi/2, \pi/2]^k$; slightly more complicated expressions could work if suitable translations or rescaling factors are applied). We also define, for later use, the stack of $z_i(\sigma)$ functions as $z(\sigma) := [z_1(\sigma), z_2(\sigma), \ldots, z_n(\sigma)]'$. Usually, in sampling applications, mobile agents need to estimate in a distributed way a value of interest (i.e. average of the field) under partial or limited interactions (e.g. due to energy, cost or physical constraints) and when the estimation variables are different than those affecting network's connectivity (i.e. position). In order to achieve this goal, we design agents mobility so as to guarantee almost periodic functions $\sigma(t) : [0, +\infty] \to \mathbb{S}^{qm}$ and ergodicity of σ dynamics[1]. Thanks to such assumptions, each agent i moves according to an almost periodic trajectory spanning a box $B_i \subseteq B$ which, in the considered set-up, can be expressed as $B_i = [\underline{b}_i, \bar{b}_i]$.

Coupled to the former equations is the distributed consensus

[1]Ergodicity of a dynamical system on a probability space (Σ, m) is defined as the property that the flow on Σ preserves the measure m, and moreover, invariant sets of Σ only have measure zero or 1. A consequence of ergodicity, which is used in the paper, is the Ergodic Theorem, the fact that asymptotic time averages of any output function h of the system can be computed taking averages on phase-space of Σ, [26], thus simplifying verification of condition (5).

protocol, as described by the following system of nonlinear differential equations:

$$\dot{x}(t) = -\frac{x(t)}{t} + \frac{z(\sigma(t))}{t} + f(t, x(t), \sigma(t)) \qquad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, representing the current estimate of the average held by individual agents, $t \in [0, +\infty]$ denotes time and $f : [0, +\infty) \times \mathbb{R}^n \times \Sigma \to \mathbb{R}^n$ is a t and σ dependent vector-field describing the dynamics of the interaction between agents. Notice that in this framework the assumption of $t \geq 1$ is made to simplify notation: it could be replaced by $t \geq 0$ by using $t + 1$ at the denominator in (1).

Remark 1 *Most of the formulations existing in the literature assume (implicitly or explicitly) the property of monotonicity [34] of the interactions represented by the vector field f. Given a function $f(t, x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, piecewise continuous in t and locally Lipschitz continuous with respect to x, the associated system of differential equations $\dot{x}(t) = f(t, x(t))$, is called monotone if for any $i \in \{1, 2, \ldots, n\}$, $f_i(t, x)$ is non-decreasing with respect to x_j for all $j \neq i$. Notice that this condition implies monotonicity of the flow $\phi(t; t_0, x_0)$ with respect to initial conditions, namely, for all t_0 and all $t \geq t_0$, it holds $\phi(t; t_0, x_1) \geq \phi(t; t_0, x_2)$ if $x_1 \geq x_2$ (where "$\geq$" is meant componentwise), [34]. The assumption of monotonicity is widely and implicitly assumed in the literature, both in linear and nonlinear networks scenarios, as it appears natural because it models coupling influence growing with distance, thus allowing reasonable convergence speed to the consensus equilibria. However, many networks of theoretical and practical interest (i.e. opinion dynamics, swarm of robots, sensor networks) might be characterized or designed so as to implement limited or vanishing influence as the distance among consensus variables goes to infinity. Therefore, given the considered set-up, we remove the assumption of monotonicity of the vector field f.*

The main aim of algorithm (1) is to allow agents to asymptotically agree on an estimated value of interest $\bar{e}$. Specifically.

Definition We say that agents asymptotically agree to $\bar{e}$ if $\lim_{t\to\infty} x_i(t) = \bar{e}$ for all i.

Additionally, algorithm (1) may be also used to guarantee agreement on a value $\bar{e}$ independent of measured exogenous inputs, thus extending the standard consensus protocols to this set up. This autonomous case is still of practical interest when each agent implements a consensus algorithm (15) for fault detection or decision making purposes ([36], [37]). We denote with $F(x, \sigma) = [F_{ij}(x, \sigma)]$ the Jacobian matrix, when this can be defined and with $F_{ij}^{+}(x, \sigma)$ (resp. $F_{ij}^{-}(x, \sigma)$) the right (resp. left) partial derivative of $f_i(x, \sigma)$ with respect to x_j at (x, σ). We assume: f is locally Lipschitz continuous with respect to x uniformly in time and σ, namely for all compacts $\mathcal{K} \in \mathbb{R}^n$ there exists $L_{\mathcal{K}} > 0$, such that, for all $x_a, x_b \in \mathcal{K}$ and all $t \geq 0$ and $\sigma \in \Sigma$ it holds $|f(t, x_a, \sigma) - f(t, x_b, \sigma)| \leq L_{\mathcal{K}}|x_a - x_b|$. Moreover f is piecewise continuous in t. The assumptions on f, imply the local existence and the uniqueness of the system's solution on some maximally extended open time interval. f admits an agreement equilibrium set, that is:

$$\mathcal{E} := \text{span}_{\mathbb{R}}\{\mathbf{1}\} \subseteq \{x \in \mathbb{R}^n : f(t, x, \sigma) = 0 \; \forall\, t \in \mathbb{R}_+, \forall\, \sigma \in \Sigma\}. \qquad (2)$$

Let $x(t)$ denote a solution of (1). At any time instant t the following quantities are of interest:

$$x_{\max}(t) = \max_{k\in N}\{x_k(t)\}; \quad x_{\min}(t) = \min_{k\in N}\{x_k(t)\},$$

$\delta_k(t) = |x_{max}(t) - x_k(t)|$ for all $k \in N$ (or symmetrically $\tilde{\delta}_k(t) = |x_{\min}(t) - x_k(t)|$), and the agent diameter $V(x(t)) = x_{\max}(t) - x_{\min}(t)$. Notice that being $z(\sigma(t))$ bounded, there exists $\bar{z}$ so that $|z(\sigma(t))| \leq \bar{z}$, for all $t \geq 1$. Fixed an arbitrary solution $x(\cdot)$ and an arbitrary time t we define a time-dependent permutation $p_j(t)$ of indices $j \in N$ such that it fulfils

$$x_{p_1(t)}(t) \leq x_{p_2(t)}(t) \leq x_{p_3(t)}(t) \leq \ldots \leq x_{p_n(t)}(t).$$

Notice that, if two or more entries of x take some given value, then the permutation is not uniquely defined. Nevertheless the permutation always exists and the value $x_{p_i(t)}(t)$ is independent of how it is selected. Therefore, for any solution $x(t)$ of (1) we can define the corresponding re-ordered solution as $[x_{p_i(t)}(t)]$, $i \in N$.

III. Main Results

Next we state our connectivity assumptions which are crucial to attain asymptotic consensus.

Assumption 1 (*Sign-definite interactions*) *For all $x \in \mathbb{R}^n$ and all $i \neq j \in N^2$, it holds:*

$$sign(x_j - x_i)[f_i(t, x, \sigma) - f_i(t, x + (x_i - x_j)e_j, \sigma)] \geq 0. \qquad (3)$$

Assumption 1 is a condition stating that influence of agent j on i is never repelling. The above assumption alone is not enough to guarantee agreement. A positive average interaction strength is needed to guarantee contraction and attain consensus, at least along a tree spanning the agents formation. Therefore we state the following additional requirements:

Assumption 2 (*State-Frozen, σ-dependent connectivity*) *Given network (1), we say that state-frozen σ-dependent connectivity holds, provided for all compact intervals $\mathcal{K} \subseteq \mathbb{R}$ there exist a root node $r \in N$, a rooted spanning tree $\mathcal{T}_r \subset N \times N$, $\varepsilon_{\mathcal{K}} > 0$ such that for all $x \in \mathcal{K}^n$, and for all $(i, j) \in \mathcal{T}_r$ and any $\sigma \in \Sigma$*

$$\begin{aligned} &sign(x_j - x_i)[f_i(t, x, \sigma) - f_i(t, x + (x_i - x_j)e_j, \sigma)] \qquad (4) \\ &\geq \Psi_{ij}(\sigma)\varepsilon_{\mathcal{K}}|x_i - x_j|, \end{aligned}$$

where each $\Psi_{ij} : \Sigma \to \mathbb{R}$ is a nonnegative continous, vector valued function.

Assumption 3 (*Positive average link strength*) *The signal σ is almost periodic and fulfills for all $(i, j) \in \mathcal{T}_r$:*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \Psi_{ij}(\sigma(\tau))\, d\tau > 0, \qquad (5)$$

Remark 2 *Notice that:*

1 *unlike most of existing literature, we deal with the scenario where the connectivity property depends on variables $\sigma_i(t)$ distinct from the consensus variables $x_i(t)$;*

2 *in the light of equation (5) this is an assumption of asymptotic averaged link strength (on a spanning tree), while, by condition* (4)*, the node interaction property is defined on frozen state variables, making its verification straightforward.*

It is somewhat convenient for the following developments to let the function Ψ_{ij} be defined for all $i \neq j \in N^2$, and assuming an inequality as (4) to hold. This can be done, without loss of generality, by letting $\Psi_{ij} = 0$ if $(i,j) \notin \mathcal{T}_r$. In the following we will present some Lemmas that are instrumental to prove asymptotic consensus.

The first Lemma shows that condition (5) actually implies integral connectivity across uniform time intervals, namely that there exists a sufficient large $T > 0$, $\varepsilon_T > 0$ so that $\int_t^{t+T} \Psi_{ij}(\sigma(\tau))\, d\tau \geq \varepsilon_T$ for any $t \geq 1$.

Lemma 1 *Let $\Psi_{ij} : \Sigma \to \mathbb{R}$ be a nonnegative continous vector-valued function of $\sigma(t) = [\sigma_i(t)]$, with $\sigma_i(t)$ almost periodic function. If $\lim_{t\to\infty} \frac{1}{t}\int_0^t \Psi_{ij}(\sigma(\tau))\, d\tau > 0$ then there exist $T > 0$ and $\varepsilon_T > 0$ so that $\int_t^{t+T} \Psi_{ij}(\sigma(\tau))\, d\tau \geq \varepsilon_T$ for any $t \geq 1$.*

Proof: Let $\sigma : \mathbb{R} \to \Sigma$ be component-wise an almost periodic function. Since Ψ_{ij} is a continous function of σ, then the composition $\Psi_{ij}(\sigma(t))$ is also almost periodic (see Proposition 3.3. in [4]). In the following, for the sake of notation, we highlight only the dependence of Ψ_{ij} on t in place of $\Psi_{ij}(\sigma(t))$.

By assumption $\lim_{t\to\infty} \frac{1}{t}\int_0^t \Psi_{ij}(\tau)\, d\tau = \bar{\Psi}_{ij} > 0$. Let $\hat{T} > 1$, be such that:

$$\frac{\int_0^{\hat{T}} \Psi_{ij}(\tau)\, d\tau}{\hat{T}} \geq \bar{\Psi}_{ij}/2.$$

Let $T = \max\{2\hat{T}, 2l(\bar{\Psi}_{ij}/4)\}$, where $l(\cdot)$ is the function l as in the definition of almost periodic function associated to Ψ_{ij}. For any $t \in \mathbb{R}$, pick the "almost period" τ in the interval $[t, t+l(\bar{\Psi}_{ij}/4)]$ as from the definition of almost periodic function, associated to $\varepsilon = \bar{\Psi}_{ij}/4$. Then:

$$\int_t^{t+T} \Psi_{ij}(s)ds \geq \int_\tau^{\tau+\hat{T}} \Psi_{ij}(s)\, ds,$$

where the inequality follows from non-negativity of Ψ_{ij} and because $[\tau, \tau+\hat{T}] \subset [t, t+T]$. Moreover,

$$\int_\tau^{\tau+\hat{T}} \Psi_{ij}(s)\, ds =$$

$$= \int_\tau^{\tau+\hat{T}} \Psi_{ij}(s-\tau) + [\Psi_{ij}(s) - \Psi_{ij}(s-\tau)]\, ds$$

$$\geq \int_\tau^{\tau+\hat{T}} (\Psi_{ij}(s-\tau) - \bar{\Psi}_{ij}/4)\, ds$$

$$= \int_0^{\hat{T}} \Psi_{ij}(s)ds - \hat{T}\bar{\Psi}_{ij}/4 \geq \hat{T}\bar{\Psi}_{ij}/4 \geq \bar{\Psi}_{ij}/4.$$

This proves the claim for T as defined above and $\varepsilon_T := \bar{\Psi}_{ij}/4$. ∎

Remark 3 *Notice that almost periodicity of signals $\sigma(t)$ is used as a sufficient condition to conclude uniform time-averaged strength of interconnection between pair of agents corresponding to arcs in the spanning tree $\mathcal{T}_r$. All our consensus results could be proved under such weaker and more widespread assumption.*

The next Lemma extends to the case of networks with exogenous inputs the classical results on monotonicity of $x_{\max}$ and $x_{\min}$.

Lemma 2 *The functions $\max\{x_{\max}(t), \bar{z}\}$ and $\min\{x_{\min}(t), -\bar{z}\}$ are (respectively) monotonically non-increasing and non-decreasing with respect to t.*

Proof: The claim is equivalent to the set:

$$\mathcal{M}_c := \{x : \max_i\{x_i, \bar{z}\} \leq c\},$$

being forward invariant for all $c \in \mathbb{R}$. Let x in $\mathcal{M}_c$ be arbitrary. Since $\mathcal{M}_c$ is convex, its tangent cone at x is simply given by $TC_x\mathcal{M}_c = \{w : \forall\, i \text{ with} : x_i = c, w_i \leq 0\}$ (see Proposition 5.5, [12]). For all i such that $x_i = c = \max\{x_{max}(t), \bar{z}\} = x_{max}(t)$ and any t it holds:

$$-\frac{x_i(t)}{t} + \frac{z_i(\sigma(t))}{t} + f_i(t, x(t), \sigma(t)) \leq$$
$$-\frac{x_i(t)}{t} + \frac{\bar{z}}{t} + f_i(t, x(t), \sigma(t))$$

Being $f_i(t, x_i\mathbf{1}, \sigma) = 0$ and taking into account condition (3) yields:

$$\begin{aligned}
&f_i(t,x,\sigma) = [f_i(t,x,\sigma) - f_i(t, x + (x_i - x_1)e_1, \sigma)]\\
&+ [f_i(t, x + (x_i - x_1)e_1, \sigma)\\
&- f_i(t, x + (x_i - x_1)e_1 + (x_i - x_2)e_2, \sigma)]\\
&+ [f_i(t, x + (x_i - x_1)e_1 + (x_i - x_2)e_2, \sigma) -\\
&f_i(t, x + (x_i - x_1)e_1 + (x_i - x_2)e_2 + (x_i - x_3)e_3, \sigma)]\\
&+ \ldots + [f_i(t, x + (x_i - x_1)e_1 + (x_i - x_2)e_2\\
&+ \ldots + (x_i - x_{n-1})e_{n-1}, \sigma)\\
&- f_i(t, x + (x_i - x_1)e_1 + (x_i - x_2)e_2\\
&+ \ldots + (x_i - x_n)e_n, \sigma)] + f_i(t, x_i\mathbf{1}, \sigma) \leq 0
\end{aligned}$$

Additionally being in this case $x_i \geq \bar{z}$ it results $-\frac{x_i}{t} + \frac{\bar{z}}{t} \leq 0$ and hence: $-\frac{x_i}{t} + \frac{\bar{z}}{t} + f_i(t,x,\sigma) \leq 0$. Hence $-\frac{x_i(t)}{t} + \frac{\bar{z}}{t} + f_i(t, x(t), \sigma(t)) \in TC_x\mathcal{M}_c$. The same conclusion trivially holds for all i such that $x_i = c$ when $c = \max\{x_{max}(t), \bar{z}\} = \bar{z}$. As this holds for all $x \in \mathcal{M}_c$ forward invariance of $\mathcal{M}_c$ follows by Nagumo's Theorem - [13], and therefore monotonicity of $\max\{x_{max}(t), \bar{z}\}$ holds.

A symmetric argument can be used to prove monotonicity of $\min\{x_{min}(t), -\bar{z}\}$ by showing forward invariance of $\mathcal{N}_c = \{x : \min_{i\in N}\{x_i, -\bar{z}\} \geq c\}$. ∎

When $x_{\max}(t) \leq \bar{z}$ (and symmetrically $x_{\min}(t) \geq -\bar{z}$), monotonicity of such functions cannot be inferred from Lemma 2. Indeed $x_{\max}(t)$ (resp. $x_{\min}(t)$) may increase (decrease). It will be useful in the following derivations to have an estimate of how quickly this might occur. This is stated in the next Lemma.

Lemma 3 *The following inequality holds for almost all $\theta \geq t \geq 1$:*

$$x_{\max}(\theta) \leq \frac{t}{\theta} x_{\max}(t) + \frac{\theta - t}{\theta} \bar{z}. \tag{6}$$

Proof: The claim follows by remarking that: $\dot{x}_{\max}(\theta) \leq -\frac{(x_{\max}(\theta)-\bar{z})}{\theta}$ for all θ. Hence, by a standard comparison principle ([44]) we see: $x_{\max}(\theta) - \bar{z} \leq e^{-\int_t^\theta 1/s ds}(x_{\max}(t) - \bar{z})$. By explicit integration of the right-hand side of the previous inequality, we get:

$$x_{\max}(\theta) \leq \bar{z} + \frac{t}{\theta}(x_{\max}(t) - \bar{z}) = \frac{t}{\theta} x_{\max}(t) + \frac{\theta - t}{\theta} \bar{z}$$

■

Notice that the above Lemma could be extended by considering the Dini derivative in place of derivative as made in some approaches in the literature (i.e. [29]). In what follows we will present a key lemma which will allow us to later prove asymptotic consensus in the considered non autonomous case. This is a far from trivial adaptation of the main technical result in [24] to accomodate for exogenous input signals.

Lemma 4 *Let $r \in N$ be the root of the spanning tree as from Assumption 2. For all initial conditions $x(1) \in \mathbb{R}^n$, there exists a finite positive integer $\bar{k}$ and $\mu > 0$ (uniform in time) and constant $\hat{K}$ such that, for all $t \geq 1$, the following holds along the solutions of (1):*

$$x_{\max}(t + \bar{k}T) \leq x_{\max}(t) - \mu|x_{\max}(t) - x_r(t)| + \frac{\hat{K}}{t} \tag{7}$$

and:

$$x_{\min}(t + \bar{k}T) \geq x_{\min}(t) + \mu|x_{\min}(t) - x_r(t)| - \frac{\hat{K}}{t}. \tag{8}$$

While the kind of robustness portrayed by this Lemma is desirable and, to a certain extent, expected, we emphasize that, due to nonlinearity of the considered system, superposition principles cannot be invoked in this context. Likewise, adopting existing converse Lyapunov arguments for nominal exponential stability in order to claim tolerance of asymptotically vanishing disturbances appears to be unsuitable given the complex nature of the considered network and the non-compact nature of its equilibrium set. For the above reasons we adopt a direct trajectory based proof that explicitly takes into account the effect of exogenous disturbances. Our main induction step is based on the technical Lemma stated below. Let

$$d(q) : N \to \mathbb{N}$$

be the distance in the spanning tree $\mathcal{T}_r$ of node q from the root r.

Lemma 5 *For any node $k \in N$ at distance $d(k)$ there exists positive constants $\mu(d(k))$ and $K(d(k))$ such that:*

$$x_k(t+2d(k)T) \leq x_{\max}(t) - \mu(d(k))|x_{\max}(t) - x_r(t)| + \frac{K(d(k))}{t}, \tag{9}$$

and:

$$x_k(t+2d(k)T) \geq x_{\min}(t) + \mu(d(k))|x_{\min}(t) - x_r(t)| + \frac{K(d(k))}{t}, \tag{10}$$

where r is the root node and T is as in Lemma 1.

The proof of Lemma 5 is deferred to the appendix for the sake of readability. Below we show how Lemma 4 can be derived exploiting Lemma 5.

Proof: Notice that inequality (9) is of the form needed for Lemma 4, except for the distance dependent number of time-steps ($2d(k)$) needed for agent k to 'feel' the effect of the pulling influence from the root. By "integrating" inequality (9) for agent k at distance $d(k)$ over a finite number of steps S_k, one gets:

$$x_k(t+2d(k)S_kT) \leq x_{\max}(t) - \mu(d(k))|x_{\max}(t) - x_r(t)| + S_k \frac{K(d(k))}{t}. \tag{11}$$

Hence, by letting $S_k = n!/d(k)$ we see from (11) that:

$$x_k(t+2n!T) \leq x_{\max}(t) - \mu(d(k))|x_{\max}(t) - x_r(t)| + \frac{n!}{d(k)} \frac{K(d(k))}{t}. \tag{12}$$

Being $K(d(k)) = d(k)(K_a + \bar{K})$ and $\mu(n-1) \leq \mu(d(k))$, the latter inequality proves the Lemma 4 with $\bar{k} = 2n!$, $\mu = \mu(n-1)$ and $\hat{K} = n!(K_a + \bar{K})$. ■

We are now ready to state the Main Result of the Section stating asymptotic agreement for non autonomous nonlinear time varying networks in the presence of almost periodic connectivity and input signals z_i.

Theorem 1 *Consider the network modeled by equations* (1)*, if Assumptions 1 hold, then the equilibrium set is uniformly asymptotically stable and, for any initial condition $x(1)$, $x(t)$ converges to an agreement equilibrium state.*

Proof: Consider the function earlier introduced:

$$V(x) = \max_{k \in N} x_k - \min_{k \in N} x_k.$$

This is positive definite with respect to the equilibrium set $\mathcal{E}$ and radially unbounded. The network's solutions asymptotically converge to the equilibrium set. Indeed, by the results (42)-(43) of Lemma 4 it results:

$$V(x(t + \bar{k}T)) \leq (1 - \mu)V(x(t)) + \frac{2\hat{K}}{t}.$$

By a standard comparison principle, a linear difference inequality forced by a converging exogenous signal only admits converging solutions. In fact:

$$\lim_{s \to \infty} V(x(t + s\bar{k}T)) \leq \frac{2\hat{K}}{\mu} \lim_{s \to \infty} \frac{1}{t + \bar{k}sT} = 0.$$

By Lipschitz continuity of the considered equations, then:

$$\lim_{t\to+\infty} V(x(t)) = 0,$$

where the limit is taken with respect to $t \in \mathbb{R}$. This completes the proof of our claim. ■

Now we present a Corollary addressing the problem of asymptotic agreement on the estimate the average value of input signals z_i.

Corollary 1 *Consider a network* (1) *as in Theorem 1. If in addition* $f(t, x, \sigma)$ *satisfies* $\mathbf{1}^T f(t, x, \sigma) = 0$ *for all* $t \geq 1$, $x \in \mathbb{R}^n$, $\sigma \in \Sigma$ *then each agent state* $x_i(t)$ *asymptotically agrees to the mean of the asymptotic time averages of input signals* z_i.

Proof: If Assumptions 1 hold from Theorem 1 asymptotic consensus follows. Additionally being $\mathbf{1}^T f(t, x(t), \sigma(t)) = 0$, from (1) it results:

$$\sum_{i=1}^{n} \dot{x}_i(t) = \frac{\sum_{i=1}^{n} z_i(\sigma(t))}{t} - \frac{\sum_{i=1}^{n} x_i(t)}{t}.$$

Define the following quantity:

$$z_s(t) := \sum_{i=1}^{n} \frac{\int_0^t z_i(\sigma(\tau))d\tau}{t}.$$

It is straightforward to verify that:

$$\dot{z}_s(t) = -\frac{z_s(t) - \sum_{i=1}^{n} z_i(\sigma(\theta(t)))}{t}.$$

Hence, letting $\tilde{z}_s(t) = z_s(t) - \sum_{i=1}^{n} x_i(t)$ we can verify that

$$\dot{\tilde{z}}_s = -\frac{\tilde{z}_s}{t}$$

which is the equation of a time-varying linear system globally asymptotically stable at the origin. Therefore, for any initial condition, it holds:

$$\lim_{t\to+\infty} z_s(t) - \sum_{i=1}^{n} x_i(t) = \lim_{t\to+\infty} \tilde{z}(t) = 0,$$

which concludes the proof of our claim. ■

Finally we give a Corollary assessing exponential convergence to the equilibria for the autonomous systems $\dot{x} = f(t, x(t), \sigma(\theta(t)))$ under almost periodic integral interactions.

Corollary 2 *Let a network* $\dot{x} = f(t, x(t), \sigma(\theta(t)))$ *under almost periodic-dependent connectivity, then the equilibrium set is uniformly exponentially stable and, for any initial condition* $x(1)$, $x(t)$ *converges to an agreement equilibrium state. Additionally, if* $f(t, x(t), \sigma(t))$ *satisfies* $\mathbf{1}^T f(t, x(t), \sigma(t)) = 0$ *for all* t *then each agent state* $x_i(t)$ *asymptotically agrees to the mean of the initial state* $x(1)$.

Proof: The result trivially follows by Theorem 1 that assures agreement equilibrium ($x_1 = x_2 = \ldots = x_n$), and assumption $\mathbf{1}^T f(t, x(t), \sigma(t)) = 0$ that guarantees flow conservativeness, $\sum_{i=1}^{n} x_i(t) = \sum_{i=1}^{n} x_i(1)$ for all $t \geq 1$. ■

IV. Application to sampling by mobile sensors

In this Section we consider a representative application where each agent is an autonomous robot or sensor node moving with constant speed in a rectangular region. When an agent reaches the boundary of the region its velocity component orthogonal to the boundary swaps its sign and, as a result, the agent bounces back towards the interior preserving its speed. Each agent (say the i-th one) could be visiting a different region $[-a_i, a_i] \times [-b_i, b_i]$, and these could be overlapped or partially overlapped in arbitrarily complex ways. In order to design an algorithm for uniformly weighted field's averages a suitable time-dependent weighting function $w_i(t)$ should be adopted so as to take into account of how many agents are visiting each point that agent i visits along its trajectory. In the following description, for the sake of simplicity, we will assume that all regions are identical and that, as a consequence, unitary weighting functions are appropriate. This is also the case if regions are non-overlapping (as in the second example considered).

Notice that, by a standard embedding used in the study of biliards, we could regard agents moving with constant speed and bouncing elastically at the boundary as material points rotating at constant speed inside a 2-dimensional torus.

The torus, in turn, can be embedded in $\mathbb{R}^2$, by identifying points in $\mathbb{R}^2$ whose difference is an integer multiple of 2π (coordinatewise). Accordingly the rotation inside the torus of agent i can be described by a simple differential equation for the vector $\theta_i = [\theta_{i_x}, \theta_{i_y}]$:

$$\begin{aligned} \dot{\theta}_{i_x} &= \omega_{i_x} \\ \dot{\theta}_{i_y} &= \omega_{i_y} \end{aligned} \tag{13}$$

with ω_{i_x}, ω_{i_y}, $i = 1 \ldots n$, uncommensurable real numbers. Incommensurability guarantees that rotation in the torus is not periodic and each point in the torus is approached in the limit by the solution.

The coordinates θ_i defined in $\mathbb{R}^{2N}$ are projected to planar coordinates in the box $[-a, a] \times [-b, b]$ according to $[2a\tilde{\sigma}(\theta_{i_x})/\pi, 2b\tilde{\sigma}(\theta_{i_y})/\pi]$ with $\tilde{\sigma}(\theta) = \arcsin(\sin(\theta))$. Overall we let $\sigma(\theta)$ denote the vector $[\ldots \tilde{\sigma}(\theta_{i_x}), \tilde{\sigma}(\theta_{i_y}) \ldots]'$.

Each agent i carries out a measure of a field of interest as a function of its position $z_i(\sigma(t))$ (e.g. local temperature, local ambient pollution concentration) by an onboard sensor and implements algorithm (1) to estimate the spatial average of the field (which we assume to be constant in time) both by integrating in time his local information and by communicating the current estimated average value to its neighboors. As an example, we assume that moving agents can communicate information of their own consensus variable x_i just with agent at distance less than R, while the strength of interaction vanishes for higher distances R. Therefore we may assume function Ψ_{ij} of the following form:

$$\Psi_{ij} = B_{\gamma,R}\left(\sqrt{[(\sigma(\theta_{i_x}) - \sigma(\theta_{j_x})]^2 + [\sigma(\theta_{i_y}) - \sigma(\theta_{j_y})]^2}\right) \tag{14}$$

with $B_{\gamma,R}(w)$ is a bump function so that $B_{\gamma,R}(0) = \gamma$ and $B_{\gamma,R}(w) = 0$ for $|w| > R$.

In the following we will give two Corollaries by tailoring the Main Results presented in Section III to the considered scenario. The first one copes with the aim to estimate the average value of a measured field $\bar{z}$, while the second one has the objective to agree on a value depending on the initial state variables $x_i(1)$ (e.g. their mean value).

Corollary 3 *Let a network* (1) *be composed of agents moving in a region according to* (13) *with a law of interaction* (14) *and sampling local measurements $z_i(\sigma(\theta(t)))$, if $f(t, x(t), \sigma(t))$ satisfies* (4) *then $x(t)$ asymptotically converges to an agreement equilibrium state. If, in addition, $f(t, x(t), \sigma(t))$ fulfils $\mathbf{1}^T f(t, x(t), \sigma(t)) = 0$ for all t then each agent asymptotically agrees to an estimate of the average of the field $\bar{z}$ over the box $[-a, a] \times [-b, b]$.*

Proof: Being ω_{i_x} and ω_{i_y} incommensurable, the functions $[\sin(\omega_{i_x} t + \theta_{i_x}(0)), \sin(\omega_{i_y} t + \theta_{i_y}(0))]$ is almost periodic and its image visits a dense set in $[-1, 1]^2$. Similarly $\sigma(\theta(t))$ as well as $\Psi(\sigma(\theta(t)))$ are almost periodic functions, (as composition of a continuous function and an almost periodic one) ([4]). Additionally, from the Ergodic Theorem [5], condition (5) holds being: $\lim_{t\to\infty} \frac{1}{t}\int_0^t \Psi_{ij}(\sigma(\theta(\tau)))\,d\tau = \int_{\mathbb{T}^n} \Psi_{ij}(\theta)d\theta = \frac{\int_{[0,2\pi]^{2n}} \Psi_{ij}(\theta)\,d\theta}{(2\pi)^{2n}} > 0$. From assumption $f(t, x(t), \sigma(t))$ satisfies the state frozen interaction condition (4). Therefore conditions of Assumptions 1 are satisfied and from Theorem 1 asymptotic consensus follows. Additionally if $\mathbf{1}^T f(t, x(t), \sigma(\theta(t))) = 0$, from Corollary 2 we have the convergence of the estimations $\sum_{i=1}^n x_i$ towards the sum of the field's time averages. Finally, letting $\bar{x} = \lim_{t\to+\infty} x_i(t)$ we see that:

$$\bar{x} = \lim_{t\to+\infty} \frac{\sum_{i=1}^n x_i(t)}{n} = \lim_{t\to+\infty} \frac{1}{n}\sum_{i=1}^n \frac{\int_0^t z_i(\sigma(\theta(\tau)))dt}{t}$$
$$= \frac{1}{n}\sum_{i=1}^n \frac{\int_{[-a_i,a_i]\times[-b_i,b_i]} \bar{z}(x, y)dxdy}{4a_ib_i},$$

where the last equality follows from ergodicity of system (13). ■

Corollary 4 *Consider the autonomous network $\dot{x} = f(t, x(t), \sigma(t))$, composed of agents moving in a region according to* (13) *and with a strength of interaction fulfilling* (14)*. If $f(t, x(t), \sigma(t))$ satisfies* (4) *then $x(t)$ exponentially converges to an agreement equilibrium state. If, additionally, $f(t, x(t), \sigma(t))$ fulfills $\mathbf{1}^T f(t, x(t), \sigma(t)) = 0$ for all $t \geq 0$ then each agent asymptotically agrees to an estimate of the mean of the initial state $x_i(1)$.*

Proof: The result follows from Corollary 2 and Corollary 3. ■

We remark that in the considered set up, the proposed results allow to assess consensus both in the absence and presence of inputs z under limited agents' interactions (due to bounded range of action R and almost periodic connectivity). In this respect in the following we will present two examples showing the application of the results of Corollary 3 and Corollary 4: a.) the autonomous and asymmetric case in the absence of inputs, where the agents need to reach consensus on a variable of interest and b.) the algorithm is used to estimate the average value of a field.

A. Consensus on a variable of interest: asymmetric case

This example is representative of the autonomous sampling scenario where each agent has a different function and hardware (e.g. sampling or storing different kind of measures, different radius of action). Nevertheless each agent implements a consensus algorithm (15) (once the vanishing term $[z_i(t) - x_i(t)]/t$ is dropped) for fault detection or decision making purposes ([36], [37]). In this asymmetric scenario it is therefore required agreement on a variable x different from σ, which instead affects the connectivity. To give an example, let us consider the following nonlinear non-monotone and autonomous network composed of 3 agents:

$$\dot{x}_i = \sum_{j=1}^3 \Psi_{ij}(\sigma(t)) \frac{(x_j - x_i)}{1 + (x_j - x_i)^2} \qquad (15)$$

$i = \{1, 2, 3\}$, with $\sigma(\theta(t))$ defined above. $\Psi_{ij}(\sigma(t))$ is defined according to (14) with $\gamma = 5$ for all i and with different radius of action for each agent, namely: $R_1 = 1$, $R_2 = 5$, $R_3 = 10$: this implies asymmetric interactions. Verification of condition (4) in Assumption 2 for any i, and j yields:

$$\text{sign}(x_j - x_i)\Psi_{ij} \frac{(x_j - x_i)}{1 + (x_j - x_i)^2} \geq \Psi_{ij}\varepsilon_{\mathcal{K}}|x_j - x_i|,$$

provided $\varepsilon_{\mathcal{K}} := 1$. The agents move in a region ($a = 100$, $b = 200$) as shown in Fig. 1 while connecting to agents in the own range of action. From Corollary 4 consensus is assessed as shown by the dynamic of agents' state variable in Fig. 2. Taking into account the level of coverage of the region by the motion of each agent (Fig. 1), notice that as agent (marked in the Figures by red colour) with larger radius ($R = 10$) more frequently can observe other agents, it acts as a "bridge agent" to facilitate consensus between the agents (marked by green colour) with minimum range ($R = 1$) and agent (marked by green colour) with medium range ($R = 5$) of action.

B. Estimation of the average value of a field: symmetric case

Now we consider the scenario of mobile agents with limited range of action (i.e. robots) carrying out a measure as a function of its position $z_i(\sigma(\theta(t)))$ (e.g. local temperature, local ambient pollution concentration) by an onboard sensor and implementing algorithm (1) to estimate the spatial average of the field. Let us consider a network composed of 3 agents:

$$\dot{x}_i = -\frac{x_i(t)}{t} + \frac{z_i(\sigma(\theta(t)))}{t} + \sum_{j=1}^3 \Psi_{ij}(\sigma(t)) \frac{(x_j - x_i)}{1 + (x_j - x_i)^2} \qquad (16)$$

$i = 1..3$, with $\sigma(\theta(t))$ and $\Psi_{ij}(\sigma(t))$ defined in (14) with $\gamma = 5$ and $R = 45$ for all i The agents move on a rectangular field in Fig. 3(a) with $a = 300$, $b = 200$. Each agent measures variable z_i depending on the node positions $(\theta_{i_x}, \theta_{i_y})$ and assuming the value represented by the colour in Fig. 3(a). From the observation carried out in the previous Section,

$f_i(t, x(t), \sigma(t))$ satisfies (4) in Assumption 1 for any i, and j. Additionally, having all mobile sensors the same distance of action R, Ψ_{ij} are symmetric. This condition along with the form of $f_i(t, x(t), \sigma(t))$ in (16) implies $\mathbf{1}^T f(t, x(t), \sigma(t)) = 0$. Therefore from Corollary 3 the asymptotic convergence of x_i to the mean value of the field is assessed.

In order to highlight the minimal sensors set up and agents' interaction needs to guarantee the convergence result and therefore the desired monitoring functionality, we consider each agent moving in just one subregion (i.e. blue, green, or red) of the monitoring field as depicted in Fig. 3(b) with $R = 45$ (i.e. one agent of a subregion may connect at most with agent of the adjacent subregion). The initial state condition is $x(1) = [138\ 220\ 336]^{'}$. Notice that although both the limited agents' interactions, communication range and low number of agents, the swarm asymptotically may coverage all the area (Fig. 3(b)) and each agent can estimate the average measure of the region (i.e. 50) as shown in Fig. 4(a). The example shows as the proposed conditions may find application to autonomous sampling of large areas by using a low number of mobile agents, reduced energy consumption (due to reduced wireless communications) and cost (reduced number of agents). Nevertheless, the algorithm may be tuned in a different way (e.i. increasing the range of action R or the interaction gain γ) or increasing the number of agents if a higher responsiveness in getting the average value is a requirement. For instance, in Fig. 4(b) is shown as the convergence rate is increased than the case of Fig. 4(a) when the range of action, strength of interaction and number of agents per subregion are increased (i.e. $R = 55$, $\gamma = 10$ and four agents).

By large, each agent may implement both of the kind of algorithm presented in Subsections IV-A-IV-B to simultaneously assess both monitoring and decision making/fault detection functions.

Remark 4 *From the above verifications it appears that the introduced condition is fulfilled (assumed the existence of a node to be taken as the root of the graph of interactions). It is worth pointing out that the assumption verification is performed with "frozen" state variables, greatly simplifying the a priori verification of the conditions. Indeed, it allows to avoid the circular argument by which solutions depend on the connectivity and the latter is in turn influenced by state evolutions: this type of circular argument normally makes up for conditions that can hardly be tested in the case of time-varying nonlinear agent dynamics without explicit apriori knowledge of solutions. Therefore the proposed results may be used to formulate (resp. test) a (resp. given) nonlinear, time varying monotone algorithm to assess consensus under limited agents' interactions.*

V. CONCLUSIONS

In this paper we introduced condition to guarantee asymptotic consensus of a class of non-monotone, non-autonomous networks in the presence of inputs and almost periodic interactions. The state agreement is assessed for state variables different than those affecting the network connectivity and the measured inputs. This is representative of many monitoring and surveillance scenarios where the connectivity of mobile sensor nodes depends on the relative position while the agreement value concerns either different estimation variables (i.e. reliability, reputation) or sampled measures (i.e. temperature, chemical concentrations). The agreement is guaranteed under weak connectivity properties (just existence of a spanning tree for a suitable averaged graph is required). The adopted notion of integral connectivity has the additional merit to be very easy to test due to almost periodic and ergodic agents dynamics. As as additional feature, the monotonicity property of the vector-field is not required, unlike most of existing literature on the subject. The proposed framework is of practical interest when consensus is required between autonomous agents characterized by partial or limited interactions, herein represented by the almost periodic function-dependent connectivity. A direct application to practical scenarios of interest like autonomous sampling is highlighted by two representative examples. The first models mobile agents interested in achieving consensus on a variable different from the one affecting connectivity for decision making or fault detection purposes. The simulation results validate the effectiveness of the result in the case of asymmetric interactions. The second example is representative of agents taking homogeneous measures and moving in a region with the consensus algorithm used to estimate the average value of the field measured. The simulation results validate the effectiveness of the result in the case of symmetric interactions and also in the presence of cost (use of reduced number of agents), energy consumption (reduced number of wireless communications) or region accessibility constraints. The possibility of considering more general dynamics at the nodes and or higher order filters in processing the measured data was not addressed in the present manuscript and is a promising direction for future research.

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APPENDIX A
PROOF OF LEMMA 5

We prove the Lemma for $x_{\max}(t)$, a similar argument holds for $x_{\min}(t)$. Assume $x(1)$ belongs to $\mathcal{K}^n$ for some compact interval $\mathcal{K}$ and denote by ε the product $\varepsilon_T \cdot \varepsilon_{\mathcal{K}}$ where T and the corresponding ε_T are as in Lemma 1 and $\varepsilon_{\mathcal{K}}$ is as from Assumption 1. We prove the result by induction.

*1) **STEP 1**:* Let us deal first with any node $q \in \mathcal{D}_1 := \{q \in N : d(q) = 1\}$.

We consider different cases:
Case $a)$ $x_r(\tau) \le x_q(\tau)$, for all $\tau \in [t, t+2T]$.
Define $\bar{q}(\tau) \in \{1, \ldots, n\}$ so as to fulfil $p_{\bar{q}(\tau)}(\tau) = q$. In the following expressions, the time dependence of $\bar{q}$ will be omitted for the sake of simplicity of notation. Then, for all $\tau \in [t, t+2T]$ the following holds:

$$
\begin{aligned}
&x_q(\tau) - x_q(t) = \int_t^\tau f_q(\theta, x(\theta), \sigma(\theta))\, d\theta \\
&+ \int_t^\tau \frac{z_q(\sigma(\theta)) - x_q(\theta)}{\theta}\, d\theta = \int_t^\tau \Big([f_q(\theta, x(\theta), \sigma(\theta)) \\
&- f_q(\theta, x(\theta) + (x_q(\theta) - x_{p_{\bar{q}-1}(\theta)}(\theta)) e_{p_{\bar{q}-1}(\theta)}, \sigma(\theta))] \\
&+ [f_q(\theta, x(\theta) + (x_q(\theta) - x_{p_{\bar{q}-1}(\theta)}(\theta)) e_{p_{\bar{q}-1}(\theta)}, \sigma(\theta))
\end{aligned}
$$

$$
\begin{aligned}
&- f_q(\theta, x(\theta) + (x_q(\theta) - x_{p_{\bar{q}-1}(\theta)}(\theta)) e_{p_{\bar{q}-1}(\theta)} \\
&\quad + (x_q(\theta) - x_{p_{\bar{q}-2}(\theta)}(\theta)) e_{p_{\bar{q}-2}(\theta)}, \sigma(\theta))] \\
&+ [f_q(\theta, x(\theta) + (x_q(\theta) - x_{p_{\bar{q}-1}(\theta)}(\theta)) e_{p_{\bar{q}-1}(\theta)} \\
&\quad + (x_q(\theta) - x_{p_{\bar{q}-2}(\theta)}(\theta)) e_{p_{\bar{q}-2}(\theta)}, \sigma(\theta)) \\
&- f_q(\theta, x(\theta) + (x_q(\theta) - x_{p_{\bar{q}-1}(\theta)}(\theta)) e_{p_{\bar{q}-1}(\theta)} \\
&\quad + (x_q(\theta) - x_{p_{\bar{q}-2}(\theta)}(\theta)) e_{p_{\bar{q}-2}(\theta)} \\
&\quad + (x_q(\theta) - x_{p_{\bar{q}-3}(\theta)}(\theta)) e_{p_{\bar{q}-3}(\theta)}, \sigma(\theta))] + \ldots + \\
&\quad + [f_q(\theta, x(\theta) + (x_q(\theta) - x_{p_{\bar{q}-1}(\theta)}(\theta)) e_{p_{\bar{q}-1}(\theta)} + \ldots \\
&+ (x_q(\theta) - x_{p_2(\theta)}(\theta)) e_{p_2(\theta)}, \sigma(\theta)) \\
&- f_q(\theta, x(\theta) + (x_q(\theta) - x_{p_{\bar{q}-1}(\theta)}(\theta)) e_{p_{\bar{q}-1}(\theta)} \\
&+ \ldots + (x_q(\theta) - x_{p_1(\theta)}(\theta)) e_{p_1(\theta)}, \sigma(\theta))] \\
&+ f_q(\theta, x(\theta) + (x_q(\theta) - x_{p_{\bar{q}-1}(\theta)}(\theta)) e_{p_{\bar{q}-1}(\theta)} \\
&+ \ldots + (x_q(\theta) - x_{p_1(\theta)}(\theta)) e_{p_1(\theta)}, \sigma(\theta)) \Big) d\theta \\
&+ \int_t^\tau \frac{z_q(\sigma(\theta)) - x_q(\theta)}{\theta} d\theta
\end{aligned}
$$

The application of Assumption 1 to each of the terms in the first integrand of the previous expression (except for the last one) leads to:

$$
\begin{aligned}
&x_q(\tau) - x_q(t) \le \\
&- \int_t^\tau \sum_{j : x_j(\theta) < x_q(\theta)} \Psi_{qj}(\theta) \varepsilon_{\mathcal{K}} |x_q(\theta) - x_j(\theta)| d\theta \\
&+ \int_t^\tau f_q(\theta, x(\theta) + (x_q(\theta) - x_{p_{\bar{q}-1}(\theta)}(\theta)) e_{p_{\bar{q}-1}(\theta)} \\
&+ \ldots + (x_q(\theta) - x_{p_1(\theta)}(\theta)) e_{p_1(\theta)}, \sigma(\theta))\ d\theta \\
&+ \int_t^\tau \frac{z_q(\sigma(\theta)) - x_q(\theta)}{\theta}\ d\theta
\end{aligned}
$$

The former calculations are instrumental for the subsequent exploitation of uniform Lipschitz continuity of f as detailed below:

$$
\begin{aligned}
&x_q(\tau) - x_q(t) \le \\
&- \int_t^\tau \sum_{j : x_j(\theta) < x_q(\theta)} \Psi_{qj}(\theta) \varepsilon_{\mathcal{K}} |x_q(\theta) - x_j(\theta)| d\theta \\
&+ \int_t^\tau \Big([f_q(\theta, x(\theta) + (x_q(\theta) - x_{p_{\bar{q}-1}}) e_{p_{\bar{q}-1}} + \\
&\ldots + (x_q(\theta) - x_{p_1(\theta)}(\theta)) e_{p_1(\theta)}) - f_q(\theta, x_q(\theta)\mathbf{1}, \sigma(\theta))] \\
&+ f_q(\theta, x_q(\theta)\mathbf{1}, \sigma(\theta)) \Big)\ d\theta + \int_t^\tau \frac{z_q(\sigma(\theta)) - x_q(\theta)}{\theta}\ d\theta.
\end{aligned}
$$

Being $f_q(\theta, x_q(\theta)\mathbf{1}, \sigma(\theta)) = 0$, it results:

$$
\begin{aligned}
&x_q(\tau) - x_q(t) \le \\
&- \int_t^\tau \sum_{j : x_j(\theta) < x_q(\theta)} \Psi_{qj}(\theta) \varepsilon_{\mathcal{K}} |x_q(\theta) - x_j(\theta)| d\theta \\
&- L \int_t^\tau \sum_{j : x_j(\theta) \ge x_q(\theta)} [x_q(\theta) - x_j(\theta)]\ d\theta \\
&+ \int_t^\tau \frac{z_q(\sigma(\theta)) - x_q(\theta)}{\theta}\ d\theta
\end{aligned}
\tag{17}
$$

with L denoting the (time-independent) Lipschitz constant of f_q. Moreover let a constant $\bar{V}$ (later defined), Lemma 3 yields for $\theta \in [t, \tau]$:

$$
\begin{aligned}
x_j(\theta) &\le x_{max}(\theta) \le \frac{t}{\theta}[x_{\max}(t) + \bar{V}] - \frac{t}{\theta}\bar{V} + \frac{\theta - t}{\theta}\bar{z} \\
&\le x_{\max}(t) + \bar{V} - \frac{t}{\theta}\bar{V} + \frac{\theta - t}{\theta}\bar{z} \\
&= x_{\max}(t) + \frac{\theta - t}{\theta}(\bar{z} + \bar{V}) \\
&\le x_{\max}(t) + \frac{2T}{t}(\bar{z} + \bar{V}) = x_{\max}(t) + \frac{\bar{K}}{t},
\end{aligned}
\tag{18}
$$

where $\bar{K} \doteq 2T(\bar{z} + \bar{V})$. Therefore, combining the inequalities in (18) and (17), we have:

$$
\begin{aligned}
&x_q(\tau) - x_q(t) \le \\
&\le - \int_t^\tau \sum_{j : x_j(\theta) < x_q(\theta)} \Psi_{qj}(\theta) \varepsilon_{\mathcal{K}} |x_q(\theta) - x_j(\theta)| d\theta \\
&- L \int_t^\tau \sum_{j : x_j(\theta) \ge x_q(\theta)} [x_q(\theta) - (x_{\max}(t) + \frac{\bar{K}}{t})]\ d\theta \\
&+ \int_t^\tau \frac{z_q(\sigma(\theta)) - x_q(\theta)}{\theta}\ d\theta \\
&\le - \int_t^\tau \sum_{j : x_j(\theta) < x_q(\theta)} \Psi_{qj}(\theta) \varepsilon_{\mathcal{K}} |x_q(\theta) - x_j(\theta)| d\theta \\
&- (n-1)L \int_t^\tau [x_q(\theta) - (x_{\max}(t) + \frac{\bar{K}}{t})]\ d\theta \\
&+ \int_t^\tau \frac{z_q(\sigma(\theta)) - x_q(\theta)}{\theta}\ d\theta.
\end{aligned}
$$

In particular, further manipulating the previous inequality to highlight the influence of the root node r yields for all $\tau \in$

$[t+T, t+2T]$:

$$
\begin{aligned}
& x_q(\tau) - (x_{\max}(t) + \frac{\bar{K}}{t}) \le x_q(\tau) - x_{\max}(t) \\
& \le x_q(\tau) - x_q(t) \\
& \le - \int_t^\tau \Psi_{qr}(\theta)\, \varepsilon_{\mathcal{K}} |x_q(\theta) - x_r(\theta)| d\theta \\
& - (n-1)L \int_t^\tau [x_q(\theta) - (x_{\max}(t) + \frac{\bar{K}}{t})]\ d\theta \\
& + \int_t^\tau \frac{z_q(\sigma(\theta)) - x_q(\theta)}{\theta}\ d\theta. \qquad (19)
\end{aligned}
$$

By the triangular inequality it holds:

$$
\begin{aligned}
& - |x_q(\theta) - x_r(\theta)| \le -|x_{\max}(t) + \frac{\bar{K}}{t} - x_r(\theta)| \qquad (20) \\
& + |x_{\max}(t) + \frac{\bar{K}}{t} - x_q(\theta)|,
\end{aligned}
$$

moreover, by Lipschitz continuity of f, convergence of $x_r(\theta)$ towards $x_{\max}(t) + \frac{\bar{K}}{t}$ is at most exponential in time and therefore we may infer:

$$
\begin{aligned}
& \left| x_{\max}(t) + \frac{\bar{K}}{t} - x_r(\theta) \right| \\
& \ge e^{-(L+1/t)(\theta - t)} \left| x_{\max}(t) + \frac{\bar{K}}{t} - x_r(t) \right| \\
& \ge e^{-(L+1)(\theta - t)} \left| x_{\max}(t) - x_r(t) \right| \qquad (21)
\end{aligned}
$$

So, it holds:

$$
-|x_{\max}(t) + \frac{\bar{K}}{t} - x_r(\theta)| \le -e^{-(L+1)(\theta-t)} |x_{\max}(t) - x_r(t)|. \qquad (22)
$$

Combining the above inequalities (22) and (20), we may restate the bound expressed in (19) as detailed below:

$$
\begin{aligned}
& x_q(\tau) - (x_{\max}(t) + \frac{\bar{K}}{t}) \le \\
& - \int_t^\tau \Psi_{qr}(\theta) \varepsilon_{\mathcal{K}} e^{-(L+1)(\theta-t)} |x_{\max}(t) - x_r(t)| d\theta \\
& - \int_t^\tau \Psi_{qr}(\theta) \varepsilon_{\mathcal{K}} [x_q(\theta) - (x_{\max}(t) + \frac{\bar{K}}{t})]\ d\theta \\
& - (n-1)L \int_t^\tau [x_q(\theta) - (x_{\max}(t) + \frac{\bar{K}}{t})]\ d\theta \\
& + \int_t^\tau \frac{z_q(\sigma(\theta)) - x_q(\theta)}{\theta}\ d\theta
\end{aligned}
$$

Notice that Ψ_{ij} are uniformly bounded functions. In the following, without loss of generality, we will assume $\Psi_{ij} \le 1$. Additionally, taking into account the result of Lemma 1 (i.e. there exists a sufficient large $T > 0$, $\varepsilon_T > 0$ so that $\int_t^{t+T} \Psi_{ij}(\sigma(\tau))\, d\tau \ge \varepsilon_T$ for any t), it results:

$$
\begin{aligned}
& x_q(\tau) - [x_{\max}(t) + \frac{\bar{K}}{t}] \le \\
& - \varepsilon_{\mathcal{K}} e^{-2(L+1)T} |x_{\max}(t) - x_r(t)| \int_t^\tau \Psi_{qr}(\theta)\, d\theta \\
& - (\varepsilon_{\mathcal{K}} + (n-1)L) \int_t^\tau [x_q(\theta) - (x_{\max}(t) + \frac{\bar{K}}{t})]\ d\theta \\
& + \int_t^\tau \frac{z_q(\sigma(\theta)) - x_q(\theta)}{\theta}\ d\theta \\
& \le -\varepsilon e^{-2(L+1)T} |x_{\max}(t) - x_r(t)| \\
& - (\varepsilon_{\mathcal{K}} - (n-1)L) \int_t^\tau [x_q(\theta) - (x_{\max}(t) + \frac{\bar{K}}{t})]\ d\theta \\
& + \int_t^\tau \frac{z_q(\sigma(\theta)) - x_q(\theta)}{\theta}\ d\theta \qquad (23)
\end{aligned}
$$

with $\varepsilon = \varepsilon_{\mathcal{K}} \varepsilon_T$.
By defining $\Delta(\tau) = \int_t^\tau [x_q(\theta) - (x_{\max}(t) + \frac{\bar{K}}{t})]\, d\theta$, we can recast for all $\tau \in [t+T, t+2T]$ equation (23) as:

$$
\begin{aligned}
& \frac{d}{d\tau} \Delta(\tau) \le -\varepsilon e^{-2(L+1)T} |x_{\max}(t) - x_r(t)| \qquad (24) \\
& - ((n-1)L + \varepsilon_{\mathcal{K}}) \Delta(\tau) + \int_t^\tau \frac{z_q(\sigma(\theta)) - x_q(\theta)}{\theta}\ d\theta
\end{aligned}
$$

Defined the constant $\bar{V}$ as $\bar{V} \doteq \max\{x_{\min}(1), \bar{z}\}$, it results $x_q(\theta) \ge -\bar{V}$. Indeed from Lemma 2 we may infer: $x_q(\theta) \ge x_{\min}(\theta) \ge \min\{x_{\min}(\theta), -\bar{z}\} \ge \min\{x_{\min}(1), -\bar{z}\} \ge -\max\{x_{\min}(1), \bar{z}\} = -\bar{V}$. This implies:

$$
\begin{aligned}
& \int_t^\tau \frac{z_q(\sigma(\theta)) - x_q(\theta)}{\theta}\ d\theta \le (\tau - t) \frac{\bar{z} + \bar{V}}{t} \\
& \le 2T \frac{\bar{z} + \bar{V}}{t} = \frac{\bar{K}}{t}
\end{aligned}
$$

Hence:

$$
\begin{aligned}
& \frac{d}{d\tau} \Delta(\tau) \le -\varepsilon e^{-2(L+1)T} |x_{\max}(t) - x_r(t)| \\
& - ((n-1)L + \varepsilon_{\mathcal{K}}) \Delta(\tau) + \frac{\bar{K}}{t}
\end{aligned}
$$

Since $\Delta(t+T) \le 0$, by a standard comparison principle ([44]) we see that:

$$
\begin{aligned}
& \Delta(\tau) \le -\mu_\Delta(\tau) |x_{\max}(t) - x_r(t)| + \qquad (25) \\
& \frac{\bar{K}}{t} \int_{t+T}^\tau e^{-((n-1)L + \varepsilon_{\mathcal{K}})(\theta - T - t)}\ d\theta,
\end{aligned}
$$

with $\mu_\Delta(\tau) = e^{-2(L+1)T} \frac{\varepsilon[1 - e^{-((n-1)L+\varepsilon_{\mathcal{K}})(\tau - T - t)}]}{((n-1)L + \varepsilon_{\mathcal{K}})}$, which holds for all $\tau \in [t+T, t+2T]$. For $\tau = t+2T$ equation (25) yields:

$$
\begin{aligned}
& \Delta(t+2T) \le -\mu_\Delta |x_{\max}(t) - x_r(t)| \qquad (26) \\
& + \frac{\bar{K}}{t} \frac{[1 - e^{-((n-1)L+\varepsilon_{\mathcal{K}})T}]}{((n-1)L + \varepsilon_{\mathcal{K}})},
\end{aligned}
$$

with $\mu_\Delta = \mu_\Delta(t+2T)$. From the mean value theorem it results: $\exists t^* \in [t, t+2T]$:

$$
x_q(t^*) - [x_{\max}(t) + \frac{\bar{K}}{t}] = \frac{\Delta(t+2T)}{2T}. \qquad (27)
$$

By Lipschitz continuity of f, convergence of x_q towards $x_{\max}(t) + \frac{\bar{K}}{t}$ is at most exponential in time and therefore we may infer:

$$\begin{aligned}&\left|x_{\max}(t)+\frac{\bar K}{t}-x_q(t+2T)\right|\\&\geq e^{-(L+1)(t+2T-t^*)}\left|x_{\max}(t)+\frac{\bar K}{t}-x_q(t^*)\right|\\&\geq e^{-2(L+1)T)}\left|x_{\max}(t)+\frac{\bar K}{t}-x_q(t^*)\right|.\end{aligned}$$

So, it holds:

$$\begin{aligned}&x_q(t+2T)-[x_{\max}(t)+\frac{\bar K}{t}]\\&\leq[x_q(t^*)-(x_{\max}(t)+\frac{\bar K}{t})]e^{-2(L+1)T}\end{aligned}\tag{28}$$

From (27) and (28) it results:

$$x_q(t+2T)-[x_{\max}(t)+\frac{\bar K}{t}]\leq\frac{\Delta(t+2T)}{2T}e^{-2(L+1)T}.\tag{29}$$

Finally, in order to derive an estimate of how decreasing is $x_q(t)$ which is uniform in time we combine (26) and (29) to obtain:

$$x_q(t+2T)-[x_{\max}(t)+\frac{\bar K}{t}]\leq-\mu_a\delta_r(t)+\frac{K_a}{t},\tag{30}$$

with

$$K_a=\bar K\frac{e^{-2(L+1)T}[1-e^{-((n-1)L+\varepsilon_{\mathcal K})T}]}{2((n-1)L+\varepsilon_{\mathcal K})T},$$

$$\mu_a=e^{-4(L+1)T}\frac{\varepsilon[1-e^{-((n-1)L+\varepsilon_{\mathcal K})T}]}{2((n-1)L+\varepsilon_{\mathcal K})T},$$

and $\delta_r(t)=|x_{\max}(t)-x_r(t)|$.

Case b): $x_r(\tau)\geq x_q(\tau)$ for all $\tau\in[t,t+2T]$
In this case considering that $\delta_q(t)\geq\delta_r(t)$ and exploiting Lipschitz continuity of f, we may infer again:

$$\begin{aligned}&x_q(t+2T)-[x_{\max}(t)+\frac{\bar K}{t}]\leq-e^{-2(L+1)T}|x_{\max}(t)-x_q(t)|\\&=-\mu_b\delta_q(t)\leq-\mu_b\delta_r(t),\end{aligned}$$

with $\mu_b=e^{-2(L+1)T}$.

Case c): $\exists\bar\tau\in(0,2T]$ such that $x_q(t+\bar\tau)=x_r(t+\bar\tau)$.
By Lipschitz continuity of f, convergence of x_r and x_q towards the value $x_{\max}(t)+\frac{\bar K}{t}$ is at most exponential. This, along with assumption $x_q(t+\bar\tau)=x_r(t+\bar\tau)$, yields:

$$\begin{aligned}&|\frac{\bar K}{t}+x_{\max}(t)-x_q(t+2T)|\geq\\&e^{-(L+1)(2T-\bar\tau)}|\frac{\bar K}{t}+x_{\max}(t)-x_q(t+\bar\tau)|\\&=e^{-(L+1)(2T-\bar\tau)}|\frac{\bar K}{t}+x_{\max}(t)-x_r(t+\bar\tau)|\\&\geq e^{-2(L+1)T}|\frac{\bar K}{t}+x_{\max}(t)-x_r(t)|\\&\geq e^{-2(L+1)T}|x_{\max}(t)-x_r(t)|,\end{aligned}$$

and therefore $x_q(t+2T)-[x_{\max}(t)+\frac{\bar K}{t}]\leq-\mu_c\delta_r(t)$ with $\mu_c=e^{-2(L+1)T}$.
Therefore, in any of cases a, b and c it results:

$$x_q(t+2T)-[x_{\max}(t)+\frac{\bar K}{t}]\leq-\mu_1\delta_r(t)+\frac{K_1}{t}\tag{31}$$

or in other terms:

$$|\frac{\bar K}{t}+x_{\max}(t)-x_q(t+2T)|\geq\mu_1\delta_r(t)-\frac{K_1}{t}\tag{32}$$

with $\mu_1=\min\{\mu_a,\mu_b,\mu_c\}$, $K_1=K_a$.

2) ***STEP 2:*** Next we deal with nodes $k\in N$ with $d(k)=2$. Let q be such that $d(q)=1$ and $(q,k)\in\mathcal{T}_r$. We consider different cases.
Case a): $x_k(t+\tau)\geq x_q(t+\tau)$, for all $\tau\in[2T,4T]$
For node k, let $\bar k(\tau)$ be such that $p_{\bar k(\tau)}(\tau)=k$. In the following we omit the time dependence of $\bar k$ for the sake of simplicity of notation. Then, the following holds for all $\tau\in[t+2T,t+4T]$:

$$\begin{aligned}&x_k(\tau)-x_k(t+2T)=\int_{t+2T}^{\tau}f_k(\theta,x(\theta),\sigma(\theta))\,d\theta\\&+\int_{t+2T}^{\tau}\frac{z_k(\sigma(\theta))-x_k(\theta)}{\theta}\,d\theta\\&=\int_{t+2T}^{\tau}\Big([f_k(\theta,x(\theta),\sigma(\theta))\\&-f_k(\theta,x(\theta)+(x_k(\theta)-x_{p_{\bar k-1}(\theta)}(\theta))e_{p_{\bar k-1}(\theta)},\sigma(\theta))]\\&+[f_k(\theta,x(\theta)+(x_k(\theta)-x_{p_{\bar k-1}(\theta)}(\theta))e_{p_{\bar k-1}(\theta)},\sigma(\theta))\\&-f_k(\theta,x(\theta)+(x_k(\theta)-x_{p_{\bar k-1}(\theta)}(\theta))e_{p_{\bar k-1}(\theta)}\\&\quad+(x_k(\theta)-x_{p_{\bar k-2}(\theta)}(\theta))e_{p_{\bar k-2}(\theta)},\sigma(\theta))]\\&+[f_k(\theta,x(\theta)+(x_k(\theta)-x_{p_{\bar k-1}(\theta)}(\theta))e_{p_{\bar k-1}(\theta)}\\&\quad+(x_k(\theta)-x_{p_{\bar k-2}(\theta)}(\theta))e_{p_{\bar k-2}(\theta)},\sigma(\theta))\\&-f_k(\theta,x(\theta)+(x_k(\theta)-x_{p_{\bar k-1}(\theta)}(\theta))e_{p_{\bar k-1}(\theta)}\\&\quad+(x_k(\theta)-x_{p_{\bar k-2}(\theta)}(\theta))e_{p_{\bar k-2}(\theta)}\\&+(x_k(\theta)-x_{p_{\bar k-3}(\theta)}(\theta))e_{p_{\bar k-3}(\theta)}),\sigma(\theta))]\\&+\ldots+[f_k(\theta,x(\theta)+(x_k(\theta)-x_{p_{\bar k-1}(\theta)}(\theta))e_{p_{\bar k-1}(\theta)}+\\&+\ldots+(x_k(\theta)-x_{p_2(\theta)}(\theta))e_{p_2(\theta)},\sigma(\theta))\\&-f_k(\theta,x(\theta)+(x_k(\theta)-x_{p_{\bar k-1}(\theta)}(\theta))e_{p_{\bar k-1}(\theta)}+\\&\ldots+(x_k(\theta)-x_{p_1(\theta)}(\theta))e_{p_1(\theta)},\sigma(\theta))]\\&+f_k(\theta,x(\theta)+(x_k(\theta)-x_{p_{\bar k-1}(\theta)}(\theta))e_{p_{\bar k-1}(\theta)}+\\&\ldots+(x_k(\theta)-x_{p_1(\theta)}(\theta))e_{p_1(\theta)},\sigma(\theta))\Big)d\theta\\&+\int_{t+2T}^{\tau}\frac{z_k(\sigma(\theta))-x_k(\theta)}{\theta}\,d\theta\\&\leq-\int_{t+2T}^{\tau}\sum_{j:x_j(\theta)<x_k(\theta)}\Psi_{kj}(\theta)\varepsilon_{\mathcal K}|x_k(\theta)-x_j(\theta)|d\theta\\&+\int_{t+2T}^{\tau}f_k(\theta,x(\theta)+(x_k(\theta)-x_{p_{\bar k-1}(\theta)}(\theta))e_{p_{\bar k-1}(\theta)}\\&+\ldots+(x_k(\theta)-x_{p_1(\theta)}(\theta))e_{p_1(\theta)},\sigma(\theta))\;d\theta\\&+\int_{t+2T}^{\tau}\frac{z_k(\sigma(\theta))-x_k(\theta)}{\theta}\,d\theta\end{aligned}$$

$$\leq -\int_{t+2T}^{\tau} \sum_{j:x_j(\theta)<x_k(\theta)} \Psi_{kj}(\theta)\varepsilon_{\mathcal{K}}|x_k(\theta)-x_j(\theta)|d\theta +$$
$$\int_{t+2T}^{\tau}\Big([f_k(\theta,x(\theta)+(x_k-x_{p_{\bar{k}-1}(\theta)}(\theta))e_{p_{\bar{k}-1}(\theta)}+$$
$$\ldots+(x_k(\theta)-x_{p_1(\theta)}(\theta))e_{p_1(\theta)})-f_k(\theta,x_k(\theta)\mathbf{1},\sigma(\theta))]$$
$$+f_k(\theta,x_k(\theta)\mathbf{1},\sigma(\theta))\Big)\,d\theta$$
$$+\int_{t+2T}^{\tau}\frac{z_k(\sigma(\theta))-x_k(\theta)}{\theta}\,d\theta.$$

Being $f_k(\theta,x_k(\theta)\mathbf{1},\sigma(\theta))=0$, it results:

$$x_k(\tau)-x_k(t+2T) \tag{33}$$
$$\leq -\int_{t+2T}^{\tau} \sum_{j:x_j(\theta)<x_k(\theta)} \Psi_{kj}(\theta)\varepsilon_{\mathcal{K}}|x_k(\theta)-x_j(\theta)|d\theta$$
$$-L\int_{t+2T}^{\tau} \sum_{j:x_j(\theta)\geq x_k(\theta)} [x_k(\theta)-x_j(\theta)]\,d\theta \tag{34}$$
$$+\int_{t+2T}^{\tau}\frac{z_k(\sigma(\theta))-x_k(\theta)}{\theta}\,d\theta$$

By Lemma 3 for all $\theta\in[t+2T,\tau]$ and taken any $j\in N$ it results:

$$x_j(\theta)\leq x_{\max}(\theta)\leq x_{\max}(t)+\frac{2\bar{K}}{t}, \tag{35}$$

then we have:

$$x_k(\tau)-x_k(t+2T)\leq \tag{36}$$
$$-\int_{t+2T}^{\tau} \sum_{j:x_j(\theta)<x_k(\theta)} \Psi_{kj}(\theta)\varepsilon_{\mathcal{K}}|x_k(\theta)-x_j(\theta)|d\theta$$
$$-L\int_{t+2T}^{\tau} \sum_{j:x_j(\theta)\geq x_k(\theta)} [x_k(\theta)-(x_{\max}(t)+\frac{2\bar{K}}{t})]\,d\theta$$
$$+\int_{t+2T}^{\tau}\frac{z_k(\sigma(\theta))-x_k(\theta)}{\theta}\,d\theta$$
$$\leq -\int_{t+2T}^{\tau} \sum_{j:x_j(\theta)<x_k(\theta)} \Psi_{kj}(\theta)\varepsilon_{\mathcal{K}}|x_k(\theta)-x_j(\theta)|d\theta$$
$$-(n-1)L\int_{t+2T}^{\tau}[x_k(\theta)-(x_{\max}(t)+\frac{2\bar{K}}{t})]\,d\theta$$
$$+\int_{t+2T}^{\tau}\frac{z_k(\sigma(\theta))-x_k(\theta)}{\theta}\,d\theta$$

where again L denotes the (time-independent) Lipschitz constant of f_k. In particular, from observation (35), we have:

$$x_k(t+2T)\leq x_{\max}(t)+\frac{\bar{K}}{t}\leq x_{\max}(t)+\frac{2\bar{K}}{t},$$

and resuming the series of inequalities in (33), we see that for all $\tau\in[t+3T,t+4T]$ it holds:

$$x_k(\tau)-[x_{\max}(t)+\frac{2\bar{K}}{t}]\leq x_k(\tau)-x_k(t+2T) \tag{37}$$
$$\leq -\int_{t+2T}^{\tau} \sum_{j:x_j(\theta)<x_k(\theta)} \Psi_{kj}(\theta)\varepsilon_{\mathcal{K}}|x_k(\theta)-x_j(\theta)|d\theta$$
$$-(n-1)L\int_{t+2T}^{\tau}[x_k(\theta)-[x_{\max}(t)+\frac{2\bar{K}}{t}]\,d\theta \tag{38}$$
$$+\int_{t+2T}^{\tau}\frac{z_k(\sigma(\theta))-x_k(\theta)}{\theta}\,d\theta$$
$$\leq -\int_{t+2T}^{\tau}\Psi_{kq}(\theta)\varepsilon_{\mathcal{K}}|x_k(\theta)-x_q(\theta)|d\theta$$
$$-(n-1)L\int_{t+2T}^{\tau}[x_k(\theta)-[x_{\max}(t)+\frac{2\bar{K}}{t}]\,d\theta$$
$$+\int_{t+2T}^{\tau}\frac{z_k(\sigma(\theta))-x_k(\theta)}{\theta}\,d\theta.$$

By the triangular inequality we may infer:

$$-|x_k(\theta)-x_q(\theta)|\leq -|x_{\max}(t)+\frac{\bar{K}}{t}-x_q(\theta)|$$
$$+|x_{\max}(t)+\frac{\bar{K}}{t}-x_k(\theta)|\leq$$
$$\leq -|x_{\max}(t)+\frac{\bar{K}}{t}-x_q(\theta)|+|x_{\max}(t)+\frac{2\bar{K}}{t}-x_k(\theta)|.$$

Moreover, by Lipschitz continuity of f, we may infer: $|x_{\max}(t)+\frac{\bar{K}}{t}-x_q(\theta)|\geq e^{-(L+1/t)(\theta-(t+2T))}|x_{\max}(t)+\frac{\bar{K}}{t}-x_q(t+2T)|\geq e^{-(L+1)(2T)}|x_{\max}(t)+\frac{\bar{K}}{t}-x_q(t+2T)|$. Therefore, substituting the previous inequalities in (37), we have:

$$x_k(\tau)-[x_{\max}(t)+\frac{2\bar{K}}{t}]\leq$$
$$-\int_{t+2T}^{\tau}\Psi_{kq}(\theta)\varepsilon_{\mathcal{K}}e^{-2(L+1)T}|x_{\max}(t)+\frac{\bar{K}}{t}$$
$$-x_q(t+2T)|d\theta-\int_{t+2T}^{\tau}\Psi_{kq}(\theta)\varepsilon_{\mathcal{K}}[x_k(\theta)-[x_{\max}(t)$$
$$+\frac{2\bar{K}}{t}]\,d\theta\;-(n-1)L\int_{t+2T}^{\tau}[x_k(\theta)-[x_{\max}(t)$$
$$+\frac{2\bar{K}}{t}]\,d\theta+\int_{t+2T}^{\tau}\frac{z_k(\sigma(\theta))-x_k(\theta)}{\theta}\,d\theta.$$
$$\leq -\varepsilon_{\mathcal{K}}e^{-2LT}|x_{\max}(t)+\frac{\bar{K}}{t}-x_q(t+2T)|\int_{t+2T}^{\tau}\Psi_{kq}(\theta)d\theta$$
$$-\varepsilon_{\mathcal{K}}\int_{t+2T}^{\tau}[x_k(\theta)-[x_{\max}(t)+\frac{2\bar{K}}{t}]\,d\theta$$
$$-(n-1)L\int_{t+2T}^{\tau}[x_k(\theta)-[x_{\max}(t)+\frac{2\bar{K}}{t}]\,d\theta$$
$$+\int_{t+2T}^{\tau}\frac{z_k(\sigma(\theta))-x_k(\theta)}{\theta}\,d\theta$$
$$\leq -\varepsilon e^{-2LT}|x_{\max}(t)+\frac{\bar{K}}{t}-x_q(t+2T)|$$
$$-\varepsilon_{\mathcal{K}}\int_{t+2T}^{\tau}[x_k(\theta)-[x_{\max}(t)+\frac{2\bar{K}}{t}]\,d\theta$$
$$-(n-1)L\int_{t+2T}^{\tau}[x_k(\theta)-[x_{\max}(t)+\frac{2\bar{K}}{t}]\,d\theta$$
$$+\int_{t+2T}^{\tau}\frac{z_k(\sigma(\theta))-x_k(\theta)}{\theta}\,d\theta.$$

By defining $\Delta(\tau)=\int_{t+2T}^{\tau}[x_k(\theta)-(x_{\max}(t)+\frac{2\bar{K}}{t})]\,d\theta$ we can recast the above equation as:

$$\frac{d}{d\tau}\Delta(\tau)\leq-((n-1)L+\varepsilon_x)\Delta(\tau) \tag{39}$$
$$-\varepsilon e^{-2LT}|x_{\max}(t)+\frac{\bar{K}}{t}-x_q(t+2T)|$$
$$+\int_{t+2T}^{\tau}\frac{z_k(\sigma(\theta))-x_k(\theta)}{\theta}\,d\theta,$$

which holds for all $\tau\in[t+3T,t+4T]$. Since $\Delta(t+3T)\leq 0$, from the standard comparison principle and exploiting for (39) the same derivations given at $Step$ 1 for (24), finally we get:

$$\Delta(t+4T)\leq-\mu_\Delta|x_{\max}(t)+\frac{\bar{K}}{t}-x_q(t+2T)|+ \tag{40}$$
$$\frac{\bar{K}}{t}\frac{[1-e^{-((n-1)L+\varepsilon_{\mathcal{K}})T}]}{((n-1)L+\varepsilon_{\mathcal{K}})},$$

with $\mu_\Delta=\frac{\varepsilon}{((n-1)L+\varepsilon_{\mathcal{K}})}e^{-2(L+1)T}[1-e^{-((n-1)(L+1)+\varepsilon_{\mathcal{K}})T}]$. By applying the mean value theorem and exploiting Lipschitz continuity as at $Step$ 1, it results:

$$x_k(t+4T)-[x_{\max}(t)+\frac{2\bar{K}}{t}]\leq\frac{\Delta(t+4T)}{2T}e^{-2(L+1)T}$$
$$\leq-\mu_a|x_{\max}(t)+\frac{\bar{K}}{t}-x_q(t+2T)|$$
$$+\frac{\bar{K}}{t}\frac{[1-e^{-((n-1)L+\varepsilon_{\mathcal{K}})T}]}{((n-1)L+\varepsilon_{\mathcal{K}})}$$

or in other terms:

$$x_k(t+4T)-[x_{\max}(t)+\frac{2\bar{K}}{t}]\leq-\mu_a|x_{\max}(t)+\frac{\bar{K}}{t}$$
$$-x_q(t+2T)|+\frac{K_a}{t},$$

with

$$K_a=\bar{K}\frac{e^{-2(L+1)T}[1-e^{-((n-1)L+\varepsilon_{\mathcal{K}})T}]}{2((n-1)L+\varepsilon_{\mathcal{K}})T},$$
$$\mu_a=e^{-4(L+1)T}\frac{\varepsilon[1-e^{-((n-1)L+\varepsilon_{\mathcal{K}})T}]}{2((n-1)L+\varepsilon_{\mathcal{K}})T}.$$

Case b): $x_q(t+\tau)\geq x_k(t+\tau)$ $\tau\in[2T,4T]$

In this case, we may infer by Lipschitz continuity of f that:

$$|\frac{2\bar{K}}{t}+x_{\max}(t)-x_k(t+4T)|\geq$$
$$e^{-2(L+1)T}|\frac{2\bar{K}}{t}+x_{\max}(t)-x_k(t+2T)|\geq$$
$$e^{-2(L+1)T}|\frac{\bar{K}}{t}+x_{\max}(t)-x_k(t+2T)|\geq$$
$$e^{-2(L+1)T}|\frac{\bar{K}}{t}+x_{\max}(t)-x_q(t+2T)|$$

and therefore: $x_k(t+4T)-[x_{\max}(t)+\frac{2\bar{K}}{t}]\leq$ $-\mu_b[\frac{\bar{K}}{t}+x_{\max}(t)-x_q(t+2T)]$ with $\mu_b=e^{-2(L+1)T}$.

Case c): $x_q(t+\bar{\tau})=x_k(t+\bar{\tau})$ for some $\bar{\tau}\in(2T,4T]$.

By Lipschitz continuity of f, it results:

$$|\frac{2\bar{K}}{t}+x_{\max}(t)-x_k(t+4T)|\geq$$
$$e^{-(L+1)(4T-\bar{\tau})}|\frac{2\bar{K}}{t}+x_{\max}(t)-x_k(t+\bar{\tau})|$$
$$=e^{-(L+1)(4T-\bar{\tau})}|\frac{2\bar{K}}{t}+x_{\max}(t)-x_q(t+\bar{\tau})|$$
$$\geq e^{-2(L+1)T}|\frac{2\bar{K}}{t}t+x_{\max}(t)-x_q(t+2T)|$$
$$\geq e^{-2(L+1)T}|\frac{\bar{K}}{t}+x_{\max}(t)-x_q(t+2T)|$$

and therefore: $x_k(t+4T)-[x_{\max}(t)+\frac{2\bar{K}}{t}]\leq$ $-\mu_c[\frac{\bar{K}}{t}+x_{\max}(t)-x_q(t+2T)]$ with $\mu_c=e^{-2(L+1)T}$. Therefore, in any of cases a, b and c it results:

$$x_k(t+4T)-[x_{\max}(t)+\frac{2\bar{K}}{t}]\leq \tag{41}$$
$$-\mu_2[\frac{\bar{K}}{t}+x_{\max}(t)-x_q(t+2T)]+\frac{K_2}{t}$$

with $\mu_2=\min\{\mu_a,\mu_b,\mu_c\}$, $K_2=K_a$. Consequently, in order to derive an estimate of how decreasing is $x_k(t)$ which is uniform in time by combining (41) and (32) we obtain:

$$x_k(t+4T)-[x_{\max}(t)+\frac{2\bar{K}}{t}]$$
$$\leq-\mu_1\mu_2|x_{\max}(t)-x_r(t)|+\mu_2\frac{K_1}{t}+\frac{K_2}{t}$$
$$\leq-\mu_1\mu_2|x_{\max}(t)-x_r(t)|+\frac{K_1}{t}+\frac{K_2}{t}.$$

3) ***STEP d(k)+1****:* A similar procedure can be used to construct an estimate of the convergence rate for an arbitrary node at distance $d(k)$ based on the estimate for nodes at distance $d(k)-1$. By induction, for any node k at distance $d(k)$ from the root, and being $\mu_i\in(0,1)$ for all i, the following inequality holds:

$$x_k(t+2d(k)T)-[x_{\max}(t)+d(k)\tfrac{\bar{K}}{t}]$$
$$\leq-\left(\prod_{i=1}^{d(k)}\mu_i\right)|x_{\max}(t)-x_r(t)|+\tfrac{1}{t}\sum_{i=1}^{d(k)}K_i$$
$$=-\mu(d(k))|x_{\max}(t)-x_r(t)|+\tfrac{1}{t}\sum_{i=1}^{d(k)}K_i,$$

and hence:

$$x_k(t+2d(k)T)-x_{\max}(t) \tag{42}$$
$$\leq-\mu(d(k))|x_{\max}(t)-x_r(t)|+\frac{1}{t}K(d(k)),$$

with $\mu(d(k))=\prod_{i=1}^{d(k)}\mu_i$ and $K(d(k))=\sum_{i=1}^{d(k)}K_i+d(k)\bar{K}$ being positive constants for any fixed $d(k)$. As by construction $K_i=K_a$ for all i, it results: $K(d(k))=d(k)(K_a+\bar{K})$.

Similar arguments yield to the following inequality:

$$x_k(t+2d(k)T)-x_{\min}(t)$$
$$\geq\mu(d(k))\,|x_{\min}(t)-x_r(t)|-\tfrac{1}{t}K(d(k)). \tag{43}$$

This concludes the proof.

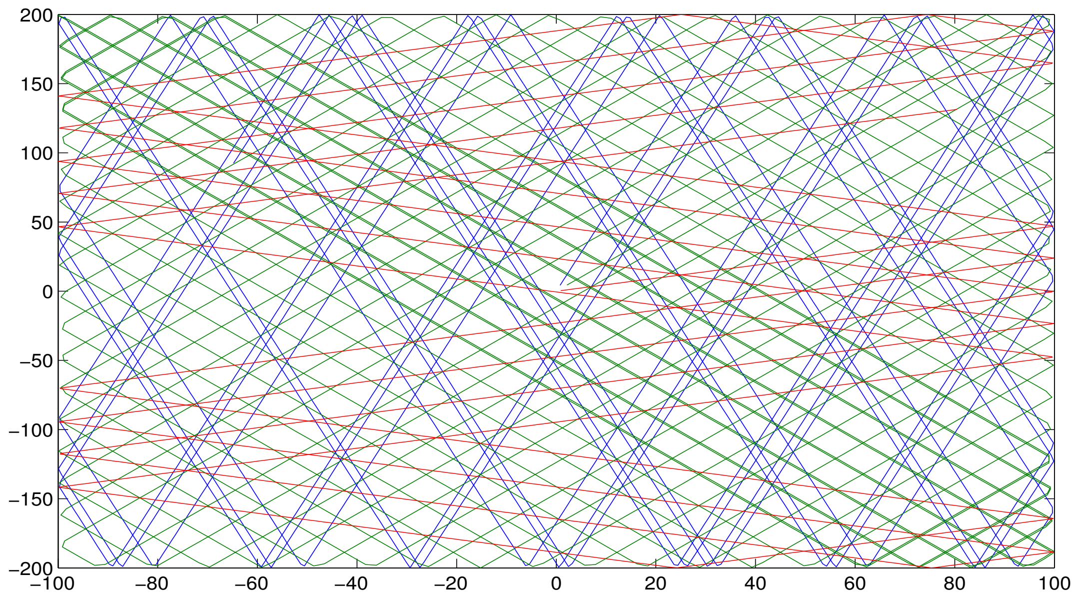


Fig. 1. Mobile sensors trajectory after 4000 simulation steps. Each sensor has different range of action: $R_1 = 1$ (blue), $R_2 = 5$ (green), $R_3 = 10$ (red)

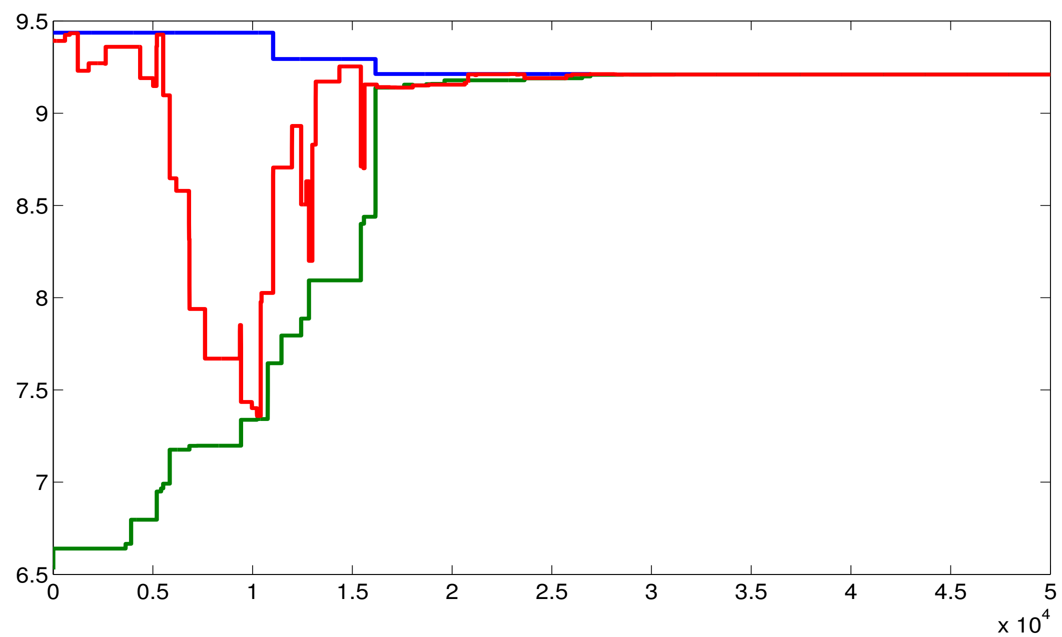
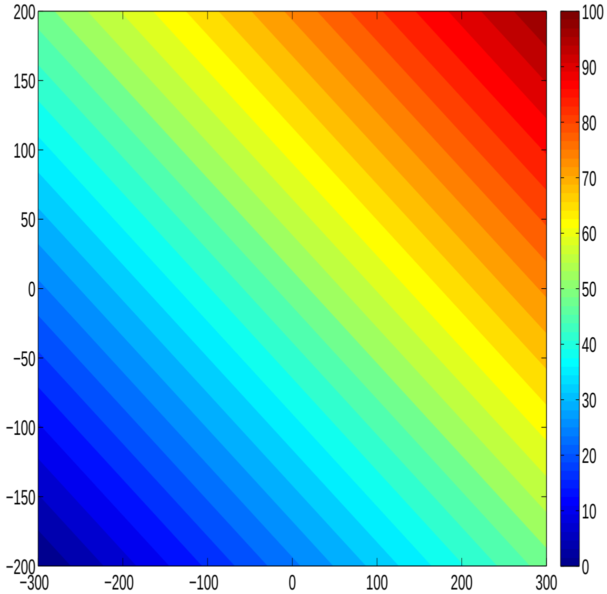
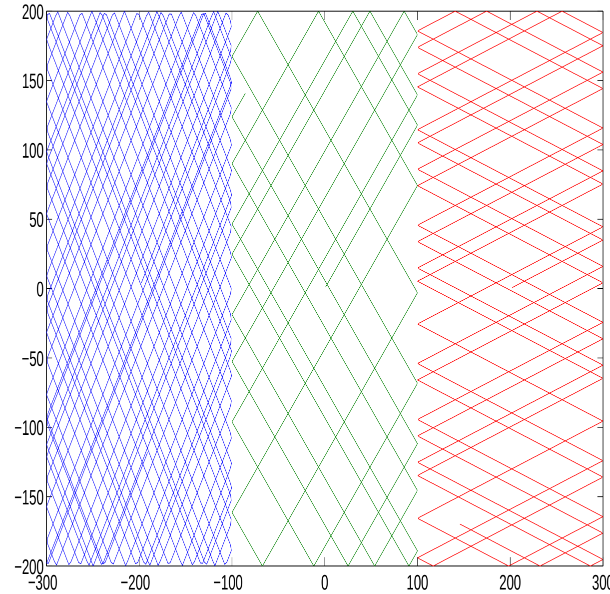


Fig. 2. Asymptotic convergence to the consensus state: agent with $R_1 = 1$ (blue line), agent with $R_2 = 5$ (green), agent with $R_3 = 10$ (red line)

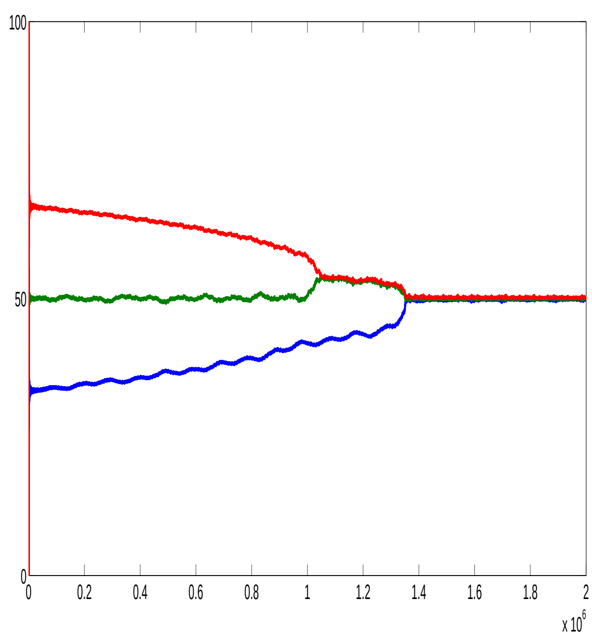


(a)

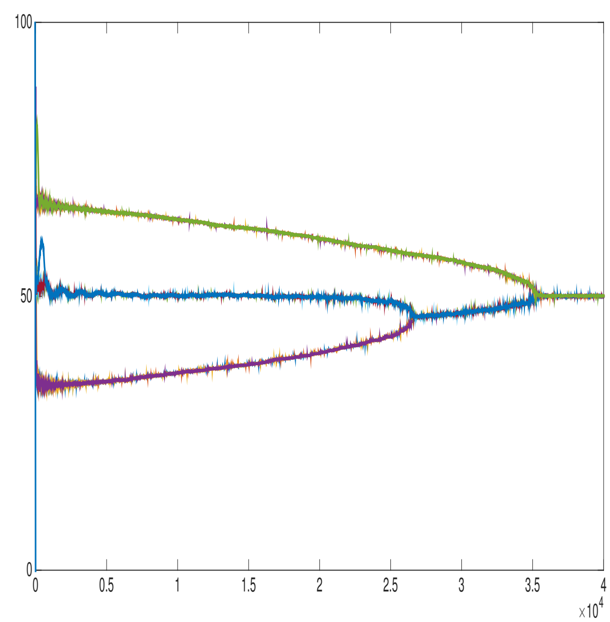


(b)

Fig. 3. Scenario of network composed of 3 mobile sensors: **(a)** measure field; **(b)** mobile sensors trajectory after 800 simulation steps. Each sensor just operates in one subreagion (blue, green or red) of the field



(a)



(b)

Fig. 4. Asymptotic convergence to the consensus state: dynamic state evolution: **(a)** $R = 45$, $\gamma = 5$, one agent per subregion; **(b)** $R = 55$, $\gamma = 10$, four agents per subregion;