PROCEEDINGS OF THE ROYAL SOCIETY A

rspa.royalsocietypublishing.org

Research

Article submitted to journal

Subject Areas:
xxxxx, xxxxx, xxxx

Keywords:
Variational principles, slice models, Kelvin circulation laws

Author for correspondence:
C. J. Cotter
e-mail: colin.cotter@imperial.ac.uk

A variational formulation of vertical slice models

C. J. Cotter[1] and D. D. Holm[2]

[1]Department of Aeronautics, Imperial College London SW7 2AZ, UK.
[2]Department of Mathematics, Imperial College London SW7 2AZ, UK.

A variational framework is defined for vertical slice models with three dimensional velocity depending only on x and z. The models that result from this framework are Hamiltonian, and have a Kelvin-Noether circulation theorem that results in a conserved potential vorticity in the slice geometry. These results are demonstrated for the incompressible Euler–Boussinesq equations with a constant temperature gradient in the y-direction (the Eady–Boussinesq model), which is an idealised problem used to study the formation and subsequent evolution of weather fronts. We then introduce a new compressible extension of this model. Unlike the incompressible model, the compressible model does not produce solutions that are also solutions of the three-dimensional equations, but it does reduce to the Eady–Boussinesq model in the low Mach number limit. Hence, the new model could be used in asymptotic limit error testing for compressible weather models running in a vertical slice configuration.

Royal Society Publishing
Informing the science of the future

Contents

1. Introduction

This paper introduces a variational framework for deriving geophysical fluid dynamics models in a vertical slice geometry (*i.e.* the x-z plane). The work is motivated by the asymptotic limit solutions framework advocated in [Cul07], in which model error in dynamical cores for numerical weather prediction models can be quantified by comparing limits of numerical solutions with solutions from semigeostrophic (SG) models. In particular, the SG solutions of the Eady frontogenesis problem specified in a vertical slice geometry prove very useful since they can be solved in a two-dimensional domain, which means that they can be run quickly on a single workstation. In the incompressible hydrostatic and nonhydrostatic cases these solutions are equivalent to exact solutions of the full three dimensional equations. As described in [Cul07], this proves to be a challenging test problem. Using a Lagrangian numerical discretisation that utilises the optimal transport formulation, converged numerical integrations of the SG model indicate an almost periodic cycle in which fronts form, change shape, and then relax again to a smooth solution. However, primitive equation solutions obtained by [GNH92] are rather dissipative due to the need for eddy viscosity to stabilise the numerics, and the periodic behaviour is not observed; this leads to a loss of predictability after the formation of the front. In [Cul07], it is suggested that greater predictability in this limit might be possible if the numerical solution exhibits energy and potential vorticity conservation over long time periods; it is also suggested that a form of Lagrangian averaging may be required to obtain accurate predictions of the subsequent front evolution. Since energy conservation can be derived from a variational framework and potential vorticity arises from the particle relabelling symmetry, this has motivated us to develop such a framework in the case of "slice geometries" in which there are three components of velocity, but they are functions of x and z only.

Another motivation for our work is that efforts to compare compressible models with the two dimensional SG solutions have been thwarted by the fact that it is not possible to construct a

compressible vertical slice model with solutions that are consistent with the full three dimensional model, with conserved energy and potential vorticity. This is because of the nonlinear dependence in the equation of state on the y-dependent component of the temperature. Hence, so far asymptotic limit studies of compressible models have only been performed over short time intervals corresponding to the initial stages of front formation [Cul08]. In this paper we introduce a new compressible slice model that can be used in asymptotic limit studies, since it has a conserved energy and potential vorticity. The price to pay is that the solutions are not consistent with the full three dimensional equations. However, the model should still be very useful in studying the behaviour of discretisation methods and averaging procedures for numerical weather prediction in the presence of fronts.

Our approach is to derive models in the Euler-Poincaré framework [HMR98]. This framework is a way of obtaining variational models without resorting to Lagrangian coordinates, by providing formulas that express how infinitesimal variations in the Lagrangian flow map correspond to variations in the Eulerian prognostic variables. The present paper specialises to the case where all the Eulerian fields are independent of y. This corresponds to a subgroup of the group of diffeomorphisms in three dimensions, which can be expressed as a semi-direct product of two dimensional diffeomorphisms in the vertical slice and rigid displacements in the y-direction. Having selected this group, the Euler-Poincaré theory immediately tells us how to perform Hamilton's principle. In this framework, the problem of developing slice models reduces to the problem of choosing which Lagrangian to substitute into the action.

The structure of this paper is as follows. In Section (a), we identify the slice subgroup, and set up the geometric framework. In Section (b), we then obtain the general equations of motion corresponding to the Euler-Poincaré equation with advected density and tracer variables (temperature). In Section (c) we reformulate the equations in a more geometric notation, and show that the equations conserve energy in the case of Lagrangians without explicit time-dependence; this is shown by recasting the equations in Lie-Poisson form. We also show that the equations have a Kelvin-Noether circulation theorem. This circulation theorem differs from the usual circulation theorem for baroclinic fluids which have a baroclinic circulation production term on the right-hand side that only vanishes if the circulation loop lies on an isentropic surface. In the slice geometry, this baroclinic term can be rewritten as the time-derivative of another circulation term, and we obtain conservation of the total circulation on arbitrary curves within the slice. This circulation theorem leads to a conserved potential vorticity that turns out to correspond to the usual three-dimensional Ertel potential vorticity. We then use this framework to present a number of models in the slice geometry. In Section 3 we show how to obtain the Euler-Boussinesq Eady model. We present the corresponding Lagrangian-averaged Eady model in Section 4 and introduce our new compressible slice model in Section 5, comparing it with the model used in [Cul08]. Finally we provide a summary and outlook in Section 6. The appendices provide proofs and show how this framework relates to known Lie-Poisson formulations of superfluid models. This relationship is significant since it shows how to build conservative numerical schemes in the slice geometry. This last point is also discussed in Section 6.

2. Vertical slice models

(a) Definition

Physically, slice models are used to describe the formation of fronts in the atmosphere and ocean. These fronts arise when there is a strong North-South temperature gradient (maintained by heating at the Equator and cooling at the Pole), which maintains a vertical shear flow in the East-West direction through geostrophic balance. In an idealised situation, neglecting the Earth's curvature and assuming a constant Coriolis parameter f, this basic steady state can be modelled with a three-dimensional flow in which there is a constant temperature gradient in the y-direction, and the velocity points in the x-direction with a linear shear in the z-direction. This basic flow is unstable to y-independent perturbations in all three components of velocity and temperature

which rapidly lead to the formation of fronts that vary sharply in the x direction but do not vary in structure in the y direction. The presence of the constant gradient of the temperature in the y direction means that the y-component of velocity is coupled to other variables since it can lead to a source or sink of temperature in each vertical slice. Since all of the perturbations are y-independent, we can consider the dynamics in a single vertical slice without loss of generality.

To build a variational vertical slice model of this type, it is assumed that the forward Lagrangian map takes the form

$$\phi(X,Y,Z,t) = (x(X,Z,t), y(X,Z,t) + Y, z(X,Z,t)) \,, \tag{2.1}$$

where (X,Y,Z) are Lagrangian labels, (x,y,z) are particle locations and t is time, *i.e.*

$$\frac{\partial \phi}{\partial Y} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} .$$

Such maps form a subgroup of the diffeomorphisms[1] $\mathrm{Diff}(\Omega \times \mathbb{R})$ (where $\Omega \in \mathbb{R}^2$ is the domain in the x-z plane, and $\mathbb{R}$ represents an infinite line in the y-direction). This subgroup is isomorphic to $\mathrm{Diff}(\Omega) \circledS \mathcal{F}(\Omega)$ where $\circledS$ denotes the semidirect product, and $\mathcal{F}(\Omega)$ denotes an appropriate space of smooth functions on Ω that specify the displacement of Lagrangian particles in the y-direction at each point in Ω. Multiplication in the semidirect product group is given by a standard formula [HMR98],

$$(\phi_1, f_1) \cdot (\phi_2, f_2) = (\phi_1 \circ \phi_2, \phi_1 \circ f_2 + f_1). \tag{2.2}$$

The corresponding Lie algebra is isomorphic to $\mathfrak{X}(\Omega) \circledS \mathcal{F}(\Omega)$ where $\mathfrak{X}(\Omega)$ denotes the vector fields on Ω, representing the two components of the velocity $u_S \in \mathfrak{X}(\Omega)$ in the x-z plane, and the smooth function $u_T \in \mathcal{F}(\Omega)$ represents the y-component of the velocity. We write elements of $\mathfrak{X}(\Omega) \circledS \mathcal{F}(\Omega)$ as (u_S, u_T) where u_S is the "slice" component in the x-z plane, and u_T is the "transverse" component in the y direction. In component notation, the *Lie bracket* for the Lie algebra $\mathfrak{X}(\Omega) \circledS \mathcal{F}(\Omega)$ of the semidirect product group $\mathrm{Diff}(\Omega) \circledS \mathcal{F}(\Omega)$ takes the form

$$[(u_S, u_T), (w_S, w_T)] = ([u_S, w_S],\ u_S \cdot \nabla w_T - w_S \cdot \nabla u_T) \,, \tag{2.3}$$

where $[u_S, w_S] = u_S \cdot \nabla w_S - w_S \cdot \nabla u_S$ is the Lie bracket for the time-dependent vector fields $(u_S, w_S) \in \mathfrak{X}(\Omega)$, and ∇ denotes the gradient in the x-z plane.

We introduce two types of advected quantities in this framework.

First, mass is conserved locally, so the mass element $D\,\mathrm{d}^3x$ is advected in three-dimensional space. That is, the mass density $D(x,y,z,t)$ satisfies

$$(\partial_t + \mathcal{L}_{(u_S,u_T)})(D\,\mathrm{d}^3x) = \big(\partial_t D + \nabla \cdot (u_S D) + \partial_y (u_T D)\big)\,\mathrm{d}^3x = 0 \,,$$

with partial time derivative $\partial_t = \partial/\partial t$ and partial space derivative $\partial_y = \partial/\partial y$ in the y-direction normal to the x-z plane. If u_T and D are specified to be y-independent consistently with the slice motion assumption, then the last term vanishes and the equation for conservation of mass reduces to advection of an *areal density* $D\,\mathrm{d}S \in \Lambda^2(\Omega)$, in which $D(x,z,t)$ satisfies the continuity equation,

$$\partial_t D + \nabla \cdot (u_S D) = 0 \,. \tag{2.4}$$

Second, in order to represent potential temperature that has a constant gradient in the y-direction, $s = \partial\bar{\theta}/\partial y =$ constant, we shall require advected scalars $\theta(x,y,z,t)$ that may be decomposed into

[1] Diffeomorphisms are smooth invertible maps with smooth inverses.

dynamic and static parts, as

$$\theta(x,y,z,t) = \theta_S(x,z,t) + (y-y_0)s. \tag{2.5}$$

Consequently, the three-dimensional scalar tracer equation

$$\partial_t \theta_S + u_S \cdot \nabla \theta_S + u_T \partial_y \theta = 0$$

becomes a dynamic equation for $\theta_S(x,z,t) \in \mathcal{F}(\Omega)$, which satisfies,

$$\partial_t \theta_S + u_S \cdot \nabla \theta_S + u_T s = 0\,, \tag{2.6}$$

in which we keep in mind that s is a constant and $u_T(x,z,t)$ has been specified to be y-independent. The space of advected scalars of this type is isomorphic to $\mathcal{F}(\Omega) \times \mathbb{R}$, represented as pairs (θ_S, s), with infinitesimal Lie algebra action

$$\mathcal{L}_{(u_S,u_T)}(\theta_S,\, s) = (u_S \cdot \nabla \theta_S + u_T s, 0)\,.$$

(b) Variational formulation via Hamilton's principle

In this section we show how to perform variational calculus in the slice geometry. Vector fields of infinitesimal variations (w_S, w_T) in the Lie algebra $\mathfrak{X}(\Omega) \circledS \mathcal{F}(\Omega)$ of the semidirect product group $\mathrm{Diff}(\Omega) \circledS \mathcal{F}(\Omega)$ induce infinitesimal variations in (u_S, u_T), D, and (θ_S, s) as follows:[2]

$$\begin{aligned}
\delta\,(u_S, u_T) &= (\partial_t w_S + [u_S, w_S], \partial_t w_T + u_S \cdot \nabla w_T - w_S \cdot \nabla u_T)\,,\\
\delta D &= -\nabla \cdot (w_S D)\,,\\
\delta\,(\theta_S, s) &= (-w_S \cdot \nabla \theta_S - w_T s, 0)\,.
\end{aligned} \tag{2.7}$$

For a Lagrangian functional $l[(u_S,u_T),(\theta_S,s),D] : (\mathfrak{X} \circledS \mathcal{F}(\Omega)) \circledS ((\mathcal{F}(\Omega)\times\mathbb{R}) \times \Lambda^2(\Omega)) \to \mathbb{R}$, we apply Hamilton's principle and obtain

$$\begin{aligned}
0 &= \delta S\\
&= \delta \int_0^T l\big[(u_S,u_T),(\theta_S,s),D\big]\,\mathrm{d}t\\
&= \int_0^T \left\langle \frac{\delta l}{\delta(u_S,u_T)}, \delta(u_S,u_T) \right\rangle + \left\langle \frac{\delta l}{\delta(\theta_S,s)}, \delta(\theta_S,s) \right\rangle + \left\langle \frac{\delta l}{\delta D}, \delta D \right\rangle \mathrm{d}t\\
&= \int_0^T \left\langle \frac{\delta l}{\delta u_S},\, \partial_t w_S + (u_S\cdot\nabla)w_S - (w_S\cdot\nabla)u_S \right\rangle + \left\langle \frac{\delta l}{\delta u_T},\, \partial_t w_T + u_S\cdot\nabla w_T - w_S\cdot\nabla u_T \right\rangle\\
&\quad + \left\langle \frac{\delta l}{\delta D},\, -\nabla\cdot(w_S D) \right\rangle + \left\langle \frac{\delta l}{\delta \theta_S}\,,\, -(w_S\cdot\nabla)\theta_S - w_T s \right\rangle \mathrm{d}t\\
&= \int_0^T \left\langle -\frac{\partial}{\partial t}\frac{\delta l}{\delta u_S} - \nabla\cdot\left(u_S \otimes \frac{\delta l}{\delta u_S}\right) - (\nabla u_S)^T \frac{\delta l}{\delta u_S} - \frac{\delta l}{\delta u_T}\nabla u_T + D\nabla\frac{\delta l}{\delta D} - \frac{\delta l}{\delta \theta_S}\nabla\theta_S,\, w_S \right\rangle\\
&\quad + \left\langle -\frac{\partial}{\partial t}\frac{\delta l}{\delta u_T} - \nabla\cdot\left(u_s \frac{\delta l}{\delta u_T}\right) - \frac{\delta l}{\delta \theta_S} s,\, w_T \right\rangle \mathrm{d}t\\
&\quad + \left[\left\langle \frac{\delta l}{\delta u_S},\, w_S \right\rangle + \left\langle \frac{\delta l}{\delta u_T},\, w_T \right\rangle \right]_0^T,
\end{aligned} \tag{2.8}$$

where the angle brackets indicate L_2 inner products with integration over $\mathbb{R}^2$. The last term makes no contribution for velocity variations (w_S, w_T) that vanish at the endpoints in time.

[2] These are standard formulas for defining the variations in Hamilton's principle. See [HMR98] and Appendix A for details.

Hence, we obtain the Euler-Poincaré equations on the slice semidirect product with advected density D and scalar θ:

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\delta l}{\delta u_S}+\nabla\cdot\left(u_S\otimes\frac{\delta l}{\delta u_S}\right)+(\nabla u_S)^T\cdot\frac{\delta l}{\delta u_S}+\frac{\delta l}{\delta u_T}\nabla u_T=D\nabla\frac{\delta l}{\delta D}-\frac{\delta l}{\delta\theta_S}\nabla\theta_S,\\ \frac{\partial}{\partial t}\frac{\delta l}{\delta u_T}+\nabla\cdot\left(u_S\frac{\delta l}{\delta u_T}\right)=-\frac{\delta l}{\delta\theta_S}s\,.\end{aligned}\tag{2.9}$$

The system (2.9) is completed by including the advection equations (2.4) and (2.6) for D and θ_S, respectively.

(c) Geometric reformulation and Kelvin-Noether circulation theorem

Theorem 2.1 (Energy conservation)**.** *If the Lagrangian l has no explicit time-dependence, the energy functional defined by the Legendre transformation*

$$h[(m_S,\,m_T),(\theta_S,s),D]=\langle(m_S,\,m_T),(u_S,\,u_T)\rangle-l[(u_S,\,u_T),(\theta_S,s),D]\,.\tag{2.10}$$

is conserved for solutions of Equations (2.4), (2.6) and (2.9).

Proof. In Appendix B, we show that Equations (2.4), (2.6) and (2.9) are Hamiltonian, with Hamiltonian given by h in Equation (2.10). If l has no explicit time-dependence, then h has no explicit time-dependence and is therefore an invariant of the Hamiltonian system. □

Theorem 2.2 (Kelvin-Noether circulation theorem)**.** *Equations (2.4), (2.6) and (2.9) imply a conservation law for circulation,*

$$\frac{d}{dt}\oint_{c(u_S)}\left(s\left(\frac{1}{D}\frac{\delta l}{\delta u_S}\right)-\left(\frac{1}{D}\frac{\delta l}{\delta u_T}\right)\nabla\theta_S\right)\cdot\mathrm{dx}=0\,,\tag{2.11}$$

in which $c(u_S)$ is a circuit in the vertical slice moving with velocity u_S and $s=s$ is a constant parameter.

Proof. The proof of the theorem is facilitated by rewriting the system of equations (2.4), (2.6) and (2.9) equivalently in the following geometric form,

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\mathcal{L}_{u_S}\right)\left(\frac{1}{D}\frac{\delta l}{\delta u_S}\cdot\mathrm{dx}\right)+\left(\frac{1}{D}\frac{\delta l}{\delta u_T}\right)\mathrm{d}u_T=\mathrm{d}\left(\frac{\delta l}{\delta D}\right)-\left(\frac{1}{D}\frac{\delta l}{\delta\theta_S}\right)\mathrm{d}\theta_S\,,\\ \left(\frac{\partial}{\partial t}+\mathcal{L}_{u_S}\right)\left(\frac{1}{D}\frac{\delta l}{\delta u_T}\right)=-\left(\frac{1}{D}\frac{\delta l}{\delta\theta_S}\right)s\,,\\ \left(\frac{\partial}{\partial t}+\mathcal{L}_{u_S}\right)\theta_S+u_Ts=0\,,\\ \left(\frac{\partial}{\partial t}+\mathcal{L}_{u_S}\right)(D\,\mathrm{d}S)=0\,,\end{aligned}\tag{2.12}$$

where $\mathcal{L}_{u_S}$ denotes *Lie derivative* along the vector field u_S. One may then verify the circulation theorem (2.11) for slice models by applying the relation

$$\frac{d}{dt}\oint_{c(u_S)}v(x,t)\cdot\mathrm{dx}=\oint_{c(u_S)}\left(\frac{\partial}{\partial t}+\mathcal{L}_{u_S}\right)(v(x,t)\cdot\mathrm{dx})\,,$$

for any vector $v(x,t)$ in the slice.

Corollary 1. *The system of equations (2.12) implies that the following potential vorticity (PV, denoted as q) is conserved along flow lines of the fluid velocity u_S,*

$$\partial_t q+u_S\cdot\nabla q=0\quad\textit{for potential vorticity}\quad q:=\frac{1}{D}\left(\mathrm{curl}\left(s\frac{1}{D}\frac{\delta l}{\delta u_S}\right)+\nabla\theta_S\times\nabla\left(\frac{1}{D}\frac{\delta l}{\delta u_T}\right)\right)\cdot\hat{y}\,.\tag{2.13}$$

Proof. Applying the differential operation d to the first equation in the system (2.12) yields

$$(\partial_t + \mathcal{L}_{u_S})\left(\left(\mathrm{curl}\left(\frac{1}{D}\frac{\delta l}{\delta u_S}\right) + s^{-1}\nabla\theta_S \times \nabla\left(\frac{1}{D}\frac{\delta l}{\delta u_T}\right)\right)\cdot\hat{y}\,\mathrm{d}S\right) = 0\,, \tag{2.14}$$

where $\mathrm{d}S$ is the surface element in the vertical slice, whose normal vector is $\hat{y}$. Applying the Lie derivative and using the continuity equation for D then yields the local conservation law (2.13). □

Upon introducing the new notation,

$$v_S := \frac{1}{D}\frac{\delta l}{\delta u_S}\,, \quad v_T := \frac{1}{D}\frac{\delta l}{\delta u_T}\,, \quad \pi := \frac{\delta l}{\delta D}\,, \quad \gamma_S := \frac{1}{D}\frac{\delta l}{\delta \theta_S}\,, \tag{2.15}$$

the system (2.12) takes a slightly more transparent form

$$\begin{aligned}
(\partial_t + \mathcal{L}_{u_S})\,(v_S\cdot\mathrm{d}\mathrm{x}) &= \mathrm{d}\pi - v_T\,\mathrm{d}u_T - \gamma_S\,\mathrm{d}\theta_S\,,\\
(\partial_t + \mathcal{L}_{u_S})\,v_T &= -\,s\,\gamma_S\,,\\
(\partial_t + \mathcal{L}_{u_S})\,\mathrm{d}\theta_S &= -\,s\,\mathrm{d}u_T\,,\\
(\partial_t + \mathcal{L}_{u_S})\,(D\,\mathrm{d}S) &= 0\,,
\end{aligned} \tag{2.16}$$

in which the differential of the third equation has also been taken. Hence, combining the middle two equations in (2.16) results in

$$(\partial_t + \mathcal{L}_{u_S})\,(v_T\,\mathrm{d}\theta_S) = -s(v_T\,\mathrm{d}u_T + \gamma_S\,\mathrm{d}\theta_S)\,. \tag{2.17}$$

Inserting this formula into the first equation in (2.16) implies that

$$(\partial_t + \mathcal{L}_{u_S})\,(sv_S\cdot\mathrm{d}\mathrm{x} - v_T\,\mathrm{d}\theta_S) = \mathrm{d}\pi\,. \tag{2.18}$$

This relation then yields the Kelvin-Noether circulation theorem as stated above in (2.11),

$$\frac{d}{dt}\oint_{c(u_S)}(sv_S - v_T\nabla\theta_S)\cdot\mathrm{d}\mathrm{x} = \oint_{c(u_S)}(\partial_t + \mathcal{L}_{u_S})\,(sv_S\cdot\mathrm{d}\mathrm{x} - v_T\,\mathrm{d}\theta_S) = \oint_{c(u_S)} d\pi = 0\,, \tag{2.19}$$

and potential vorticity conservation as in (2.13),

$$\partial_t q + u_S\cdot\nabla q = 0 \quad \text{for potential vorticity} \quad q := \frac{1}{D}\,(s\,\mathrm{curl}\,v_S + \nabla\theta_S\times\nabla v_T)\cdot\hat{y}\,. \tag{2.20}$$

□

Condition 2.1. *Note that this circulation theorem is different from the case of general 3D motions, in which the circulation is only preserved if the loop integral is restricted to lie on a temperature isosurface. In the special case of slice motions, the baroclinic generation term can itself be written as the total derivative of a loop integral. The physical interpretation is that q is in fact the usual three-dimensional potential vorticity. Due to the existence of the linear y-variation in θ, it is always possible to find an equivalent three-dimensional loop on a temperature isosurface that projects onto any given two-dimensional loop in the vertical slice plane.*

Condition 2.2. *In Appendix (c) we will discuss the geometric meaning of the Kelvin-Noether circulation theorem (2.11) and the potential vorticity conservation law (2.13) from the viewpoint of the Lie-Poisson brackets in the Hamiltonian formulation of these equations.*

3. The Euler–Boussinesq Eady model

(a) Specialising the Euler–Poincaré equations to deal with the Eady model

The Euler–Boussinesq Eady model in a periodic channel of width L and height H, has Lagrangian

$$l[u_S, u_T, D, \theta, p] = \int_\Omega \frac{D}{2}\left(|u_S|^2 + u_T^2\right) + D f u_T x + \frac{g}{\theta_0} D\left(z - \frac{H}{2}\right)\theta_S + p(1-D)\,\mathrm{d}V, \tag{3.1}$$

where g is the acceleration due to gravity, θ_0 is the reference temperature, f is the Coriolis parameter, and we have introduced the Lagrange multiplier p to enforce constant density. We obtain the following variational derivatives of this Lagrangian,

$$\begin{aligned} v_S &= \frac{1}{D}\frac{\delta l}{\delta u_S} = u_S\,, \qquad v_T = \frac{1}{D}\frac{\delta l}{\delta u_T} = u_T + fx\,, \\ \pi &= \frac{\delta l}{\delta D} = \frac{1}{2}\left(|u_S|^2 + u_T^2\right) + f u_T x - p + \frac{g}{\theta_0}\theta_S\left(z - \frac{H}{2}\right), \\ \gamma_S &= \frac{1}{D}\frac{\delta l}{\delta \theta_S} = \frac{g}{\theta_0}\left(z - \frac{H}{2}\right), \qquad \frac{\delta l}{\delta p} = 1 - D\,. \end{aligned} \tag{3.2}$$

Substitution of these variational derivatives into the Euler-Poincaré equations in (2.9) gives

$$\begin{aligned} &\partial_t u_S + u_S\cdot\nabla u_S + (\nabla u_S)^T\cdot u_S + (u_T + fx)\nabla u_T \\ &\quad = \nabla\left(\frac{1}{2}\left(|u_S|^2 + u_T^2\right) + u_T f x - p + \frac{g}{\theta_0}\theta_S\left(z - \frac{H}{2}\right)\right) - \frac{g}{\theta_0}\left(z - \frac{H}{2}\right)\nabla\theta_S, \\ &\partial_t u_T + u_S\cdot\nabla(u_T + fx) = -\frac{g}{\theta_0}\left(z - \frac{H}{2}\right)s. \end{aligned} \tag{3.3}$$

Upon substituting $D = 1$, $\nabla\cdot u_S = 0$ and combining with equations (2.4) and (2.6) for the advected quantities D and θ, the system of equations (3.3) becomes

$$\begin{aligned} \partial_t u_S + u_S\cdot\nabla u_S - f u_T\hat{x} &= -\nabla p + \frac{g}{\theta_0}\theta_S\hat{z}, \\ \partial_t u_T + u_S\cdot\nabla u_T + f u_S\cdot\hat{x} &= -\frac{g}{\theta_0}\left(z - \frac{H}{2}\right)s, \\ \nabla\cdot u_S &= 0, \\ \partial_t\theta_S + u_S\cdot\nabla\theta_S + u_T s &= 0, \end{aligned} \tag{3.4}$$

where $\hat{x}$ is the unit normal in the x-direction.

Condition 3.1. *The system (3.4) is the* standard *Euler-Boussinesq Eady slice model.*

(b) Geometric reformulation and circulation theorem for the Eady model

Substitution of the variational derivatives in (3.2) into the geometric form of the system of Euler-Poincaré equations in (2.16) gives the following equivalent form of this system,

$$\begin{aligned} (\partial_t + \mathcal{L}_{u_S})\,(u_S\cdot\mathrm{d}x) &= -\,\mathrm{d}p - (u_T + fx)du_T - \theta_S\,\mathrm{d}\gamma_S\,, \\ (\partial_t + \mathcal{L}_{u_S})\,(u_T + fx) &= -\,s\,\gamma_S\,, \\ D = 1 \quad\Longrightarrow\quad \nabla\cdot u_S &= 0\,, \\ (\partial_t + \mathcal{L}_{u_S})\,\mathrm{d}\theta_S &= -\,s\,\mathrm{d}u_T\,. \end{aligned} \tag{3.5}$$

Consequently, we recover the Kelvin circulation conservation law (2.19) for the Eady model in the form

$$\frac{d}{dt}\oint_{c(u_S)}(s u_S - (u_T + fx)\nabla\theta_S)\cdot\mathrm{d}x = \oint_{c(u_S)}\mathrm{d}\left(\frac{1}{2}|u_S|^2 - p + \gamma_S\theta_S\right) = 0\,. \tag{3.6}$$

Corollary 2. *Equation (3.6) and incompressibility imply that potential vorticity (PV, denoted as q) is conserved along flow lines of the fluid velocity u_S in the Eady model,*

$$\partial_t q + u_S \cdot \nabla q = 0 \quad \textit{for potential vorticity} \quad q := (\mathrm{scurl}\, u_S + \nabla\theta_S \times \nabla(u_T + fx)) \cdot \hat{y}\,. \tag{3.7}$$

On denoting $u_S = (u, w)$, $u_T = v$, this potential vorticity may be written as

$$q = -\frac{\partial\bar{\theta}}{\partial y}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) + \frac{\partial(v + fx, \theta')}{\partial(x, z)}.$$

Applying the Legendre transform to the Lagrangian (3.1) yields the energy

$$h[u_S, u_T, D, \theta, p] = \int_\Omega \frac{D}{2}\left(|u_S|^2 + u_T^2\right) - \frac{g}{\theta_0} D\left(z - \frac{H}{2}\right)\theta_S \,\mathrm{d}V. \tag{3.8}$$

Corollary 3. *The energy (3.8) is conserved for the Eady Boussinesq slice model.*

4. Lagrangian-averaged Boussinesq model

Numerical forecast models are restricted in grid resolution due to the stringent time requirements of operational forecasting, and hence it is necessary to perform some form of averaging on the equations in order to prevent energy and enstrophy accumulating at the gridscale, either explicitly by introducing extra terms (*i.e.* eddy viscosities, or Large Eddy Simulation), or implicitly by numerical stabilisation in advection schemes. All of these examples amount to some form of Eulerian averaging that leads to dissipation, which is thought to be detrimental to evolution of fronts. To avoid this, [Cul07] suggested that some form of Lagrangian averaging may be required, also suggesting that it is important for averaging to retain energy and potential vorticity conservation if agreement with the SG limiting solution is to be obtained.

In this section we obtain a Lagrangian averaged Boussinesq model from a variational principle, and so energy and potential vorticity conservation will follow immediately. Here, we shall interpret Lagrangian averaging as a regularisation of the equations that is consistent with the Lagrangian flow map for slice models in Equation (2.1). This regularisation is obtained by replacing Equation (3.1) with

$$\begin{aligned} l[u_S, u_T, D, \theta, p] = \int_\Omega \Bigg[\frac{D}{2}\left(|u_S|^2 + \alpha^2|\nabla u_s|^2 + u_T^2 + \alpha^2|\nabla u_T|^2\right) + Dfu_T x \\ + \frac{g}{\theta_0} D\left(z - \frac{H}{2}\right)\theta_S + p(1 - D)\Bigg] \mathrm{d}V, \end{aligned} \tag{4.1}$$

where α is a regularisation lengthscale. We obtain the following variational derivatives of this Lagrangian,

$$\begin{aligned} \tilde{u}_S &= \frac{1}{D}\frac{\delta l}{\delta u_S} = (1 - \alpha^2\nabla_D^2)u_S\,, \qquad \tilde{u}_T = \frac{1}{D}\frac{\delta l}{\delta u_T} = (1 - \alpha^2\nabla_D^2)u_T + fx\,, \\ \pi &= \frac{\delta l}{\delta D} = \frac{1}{2}\left(|u_S|^2 + u_T^2\right) + fu_T x - p + \frac{g}{\theta_0}\theta_S\left(z - \frac{H}{2}\right), \\ \gamma_S &= \frac{1}{D}\frac{\delta l}{\delta\theta_S} = \frac{g}{\theta_0}\left(z - \frac{H}{2}\right), \qquad \frac{\delta l}{\delta p} = 1 - D\,, \end{aligned} \tag{4.2}$$

where

$$\nabla_D^2 = \frac{1}{D}\nabla \cdot D\nabla.$$

Substitution into the Euler-Poincaré equations and applying $D=1$ gives

$$\begin{aligned}
\partial_t \tilde{u}_S + u_S \cdot \nabla \tilde{u}_S + \nabla u_S^T \tilde{u}_S - f u_T \hat{x} &= -\nabla p + \frac{g}{\theta_0}\theta_S \hat{z}, \\
\partial_t \tilde{u}_T + u_S \cdot \nabla \tilde{u}_T + f u_S \cdot \hat{x} &= -\frac{g}{\theta_0}\left(z - \frac{H}{2}\right) s, \\
\nabla \cdot u_S &= 0, \\
\partial_t \theta_S + u_S \cdot \nabla \theta_S + u_T s &= 0, \\
\tilde{u}_S &= (1 - \alpha^2 \nabla^2) u_S, \\
\tilde{u}_T &= (1 - \alpha^2 \nabla^2) u_T.
\end{aligned} \tag{4.3}$$

This is the Lagrangian averaged Boussinesq Eady slice model.

Corollary 4. *Equations* (4.3) *have conserved energy*

$$h = \int_\Omega \frac{D}{2}\left(|u_S|^2 + \alpha^2 |\nabla u_s|^2 + u_T^2 + \alpha^2 |\nabla u_T|^2\right) - \frac{g}{\theta_0} D \left(z - \frac{H}{2}\right) \theta_S \, \mathrm{d}V.$$

Corollary 5. *Equations* (4.3) *have Lagrangian potential vorticity conservation*

$$\partial_t q + u_S \cdot \nabla q = 0 \quad \textit{for potential vorticity} \quad q := \frac{1}{D}\left(s \operatorname{curl} \tilde{u}_S + \nabla \theta_S \times \nabla \tilde{u}_T\right) \cdot \hat{y}\,. \tag{4.4}$$

5. Sliced Compressible Model (SCM)

In this section we present a model that is a compressible extension of the Boussinesq Eady model described in the previous section. The aim of the model is to provide a framework where nonhydrostatic compressible dynamical cores can be benchmarked in a slice geometry. Due to the nonlinear equation of state, it is not possible to write down a compressible slice model with solutions that correspond to solutions of the full three dimensional equations, and we need to proceed by replacing the full potential temperature θ in the internal energy by the slice component θ_S. This approximation would be valid if the potential temperature were slowly varying in the y-direction. We derive a model that has conserved energy, potential vorticity, and supports baroclinic instability leading to front formation, so that dynamical cores in this configuration can be compared with the corresponding model in the SG limit.

In the present notation, the Lagrangian for the Sliced Compressible Model (SCM) in Eulerian (x, y, z) coordinates is,

$$l\left[u_S, u_T, D, \theta_S\right] = \int_\Omega \frac{D}{2}\left(|u_S|^2 + u_T^2\right) + f D u_T x + g D z - D c_v \theta_S \Pi \, \mathrm{d}V, \tag{5.1}$$

where Π is the Exner function given by

$$\Pi = \left(\frac{p}{p_0}\right)^{R/c_p},$$

where p_0 is a reference pressure level and c_p and R are gas constants. The equation for an ideal gas becomes

$$p_0 \Pi^{c_p/R} = D R \theta_S \Pi,$$

and differentiating with respect to θ_S and D gives

$$\begin{aligned}\frac{\partial \Pi}{\partial \theta_S} &= \frac{\partial}{\partial \theta_S}\left(\frac{DR\theta_S}{p_0}\right)^{\gamma-1},\\ &= (\gamma-1)\frac{DR\theta_S}{p_0}\left(\frac{DR\theta_S}{p_0}\right)^{\gamma-2},\\ &= \frac{\gamma-1}{\theta_S}\Pi = \frac{c_P-c_v}{c_v\theta_S}\Pi = \frac{R}{c_v\theta_S}\Pi.\end{aligned}$$

Similarly we obtain

$$\frac{\partial \Pi}{\partial D} = \frac{R}{c_v D}\Pi.$$

Note that we use θ_S in both the internal energy term in the Lagrangian, and in the equation of state. This removes all y-dependence from the Lagrangian, making a slice model possible.

We obtain the following variational derivatives of this Lagrangian,

$$\begin{aligned} v_S &= \frac{1}{D}\frac{\delta l}{\delta u_S} = u_S\,, \qquad v_T = \frac{1}{D}\frac{\delta l}{\delta u_T} = u_T + fx\,,\\ \frac{\delta l}{\delta D} &= \frac{1}{2}\left(|u_S|^2 + u_T^2\right) + fu_T x + gz - c_p\Pi\theta_S\,,\\ \gamma_S &= \frac{1}{D}\frac{\delta l}{\delta \theta_S} = \frac{1}{D}\frac{\delta l}{\delta \theta_S} = -\,c_p\Pi\,,\end{aligned} \tag{5.2}$$

where we have used the decomposition (2.5) in the last line.

Substitution of the variational derivatives (5.2) of the SCM Lagrangian (5.1) into the Euler-Poincaré equations in (2.9) gives the system

$$\begin{aligned} (\partial_t + \mathcal{L}_{u_S})\,(u_S\cdot \mathrm{d}\mathbf{x}) &= -c_p\theta_S\, d\Pi + \mathrm{d}\left(\frac{1}{2}|u_S|^2 - gz\right) + fu_T\,\mathrm{d}x\,,\\ (\partial_t + \mathcal{L}_{u_S})\,(u_T + fx) &= sc_p\Pi\,,\\ (\partial_t + \mathcal{L}_{u_S})\,\theta_S &= -\,su_T\,,\\ (\partial_t + \mathcal{L}_{u_S})(D\,\mathrm{d}S) &= 0\,.\end{aligned} \tag{5.3}$$

Consequently, we recover the expected Kelvin circulation conservation law (2.19) for the SCM in the form

$$\frac{d}{dt}\oint_{c(u_S)}\left(u_S - s^{-1}(u_T + fx)\nabla\theta_S\right)\cdot \mathrm{d}\mathbf{x} = 0\,. \tag{5.4}$$

Corollary 6. *The system of SCM equations in (5.3) implies that potential vorticity q is conserved along flow lines of the fluid velocity u_S,*

$$\partial_t q + u_S\cdot\nabla q = 0 \quad \textit{with potential vorticity} \quad q := \frac{1}{D}\left(s\,\mathrm{curl}\,u_S + \nabla\theta\times\nabla(u_T + fx)\right)\cdot\hat{y}\,. \tag{5.5}$$

Corollary 7. *These equations are Hamiltonian, with conserved energy*

$$E = \int_\Omega \frac{D}{2}\left(|u_S|^2 + u_T^2\right) - gDz + c_v D\Pi\theta'\,\mathrm{d}V.$$

Condition 5.1. *The system of SCM equations in (2.18) may also be written equivalently in* standard *fluid dynamics notation as*

$$\begin{aligned}
\partial_t u_S + u_S\cdot\nabla u_S - f u_T\hat{x} &= -\,c_p\theta\nabla\Pi - g\hat{z},\\
\partial_t u_T + u_S\cdot\nabla u_T + f u_S\cdot\hat{x} &= s c_p\Pi,\\
\partial_t\theta_S + u_S\cdot\nabla\theta_S &= -\,s\,u_T\,,\\
\frac{\partial D}{\partial t} + \nabla\cdot(D u_S) &= 0\,.
\end{aligned} \tag{5.6}$$

Next we check that the basic state of these equations supports a shear profile (and hence allows baroclinic instability and frontogenesis). Reverting to more standard notation $u_s = (u, w)$, $u_T = v$, the balance equations are

$$-fv = -c_p\theta'\frac{\partial\pi}{\partial x}, \tag{5.7}$$

$$fu = \frac{\partial\bar{\theta}}{\partial y}c_p\Pi, \tag{5.8}$$

$$0 = -c_p\theta\frac{\partial\pi}{\partial z} - g. \tag{5.9}$$

Assuming a x-independent temperature field, then Equation (5.7) implies that $v = 0$. For positive θ, equation (5.9) implies that Π will increase with height, and equation (5.8) then implies that u decreases with height, leading to a shear profile in the basic state.

We now compare our SCM with the slice compressible model in [Cul08] and identify the differences. On defining velocity $\mathbf{u} = (u_S, u_T)$ with u_S in the vertical slice, and u_T transverse to it, the model in [Cul08] in Eulerian (x, y, z) coordinates becomes, in the present notation,

$$\begin{aligned}
\partial_t u_S + u_S\cdot\nabla u_S - f u_T\hat{x} &= -\,c_p\theta\,\nabla\Pi - g\hat{z}\,,\\
\partial_t u_T + u_S\cdot\nabla u_T + f u_S\cdot\hat{x} &= -\,c_p\theta\,\Pi'_0\,,\\
\partial_t\theta_S + u_S\cdot\nabla\theta_S &= -\,s\,u_T\,,\\
\frac{\partial D}{\partial t} + \nabla\cdot(D u_S) &= 0\,.
\end{aligned} \tag{5.10}$$

Writing the [Cul08] equations in Lie-derivative form yields, cf. equation (2.18),

$$\begin{aligned}
(\partial_t + \mathcal{L}_{u_S})(u_S\cdot\mathrm{d}x) &= -\,c_p\theta\,d\Pi + \mathrm{d}\left(\frac{1}{2}|u_S|^2 - gz\right) + f u_T\,\mathrm{d}x\,,\\
(\partial_t + \mathcal{L}_{u_S})(u_T + fx) &= -\,c_p\theta\,\Pi'_0\,,\\
(\partial_t + \mathcal{L}_{u_S})\theta &= -\,s\,u_T\,,\\
(\partial_t + \mathcal{L}_{u_S})(D\,\mathrm{d}S) &= 0\,.
\end{aligned} \tag{5.11}$$

These equations differ from the SCM equations in (2.18), by only one term. Namely, the right hand sides of the second equation in each set differ, with $(-c_p\Pi'_0\,\theta)$ in these equations and $(sc_p\Pi)$ in (2.18). It turns out that this single difference has important consequences for their respective circulation laws.

The circulation law for the compressible slice models in [Cul08] is similar to that for the SCM in the previous section, but with one important difference. Namely,

Theorem 5.1. Circulation for the compressible slice models in [Cul08] is not conserved. Instead, we find

$$\frac{d}{dt}\oint_{c(u_S)}\left(u_S - s^{-1}(u_T + fx)\nabla\theta\right)\cdot\mathrm{d}x = -\oint_{c(u_S)} c_p\theta\,\nabla\Pi\cdot\mathrm{d}x\,. \tag{5.12}$$

Proof. The proof uses the first three equations in the system (5.11). The middle two equations yield

$$(\partial_t + \mathcal{L}_{u_S})(-s^{-1}(u_T + fx)\,\mathrm{d}\theta) = \frac{1}{2}\,\mathrm{d}\left(s^{-1}c_p\Pi'_0\theta^2 + u_T^2\right) + fx\,\mathrm{d}u_T.$$

Combining this formula with the first equation in the system (5.11) then yields the circulation law, (5.12). □

Corollary 8. Equation (5.12) implies that potential vorticity (PV, still denoted as q) is *created* along flow lines of the fluid velocity u_S, as

$$\partial_t q + u_S\cdot\nabla q = c_p D^{-1}\nabla\Pi\times\nabla\theta\cdot\hat{y} \quad \text{with PV given by} \quad q := D^{-1}\left(s\,\mathrm{curl}\,u_S + \nabla\theta\times\nabla(u_T+fx)\right)\cdot\hat{y} \tag{5.13}$$

Proof. Applying Stokes theorem to the circulation equation in (5.12) yields

$$\frac{d}{dt}\iint_{\partial S(u_S)}\Big(\mathrm{curl}\,u_S + s^{-1}\nabla\theta\times\nabla(u_T+fx)\Big)\cdot\hat{y}\,\mathrm{d}S = \iint_{\partial S(u_S)} c_p\nabla\Pi\times\nabla\theta\cdot\hat{y}\,\mathrm{d}S\,, \tag{5.14}$$

where $\hat{y}\,\mathrm{d}S$ is the surface element in the vertical slice, whose normal vector is $\hat{y}$. Expanding the time derivative in (5.14) and applying the Lie derivative relation for D in the last equation of the system (5.11), which is the continuity equation for D, then yields the local PV evolution equation in (5.13). □

Condition 5.2. This is the main difference between the SCM here and in [Cul08]. According to Corollary 6, the potential vorticity in the SCM is conserved and this conservation is a general property of this class of Euler-Poincaré equations, as given by Corollary 1. In contrast, according to Corollary 8, the potential vorticity in the model of [Cul08] (when viewed as a slice model) is *created* whenever the gradients of θ and Π are not aligned. This, combined with the lack of a conserved energy meant that it was not possible to obtain long time asymptotic convergence results in a compressible model, because these quantities *are* conserved in the equivalent SG model; [Cul08] restricted to looking at the asymptotic magnitude of the geostrophic imbalance in the solution. Our new model addresses this problem, allowing asymptotic limit tests to be performed with compressible models in a slice configuration.

6. Summary and Outlook

In this paper we have shown how to construct variational models for geophysical fluid dynamics problems in a vertical slice configuration in which there is motion transverse to the slice, but the velocity field is independent of the transverse coordinate. (The vertical slice configuration may be taken as the x-z plane. Then the transverse coordinate is y.) Any model developed in this framework has a conserved energy, and corresponding conserved potential vorticity. The formulation has a number of interesting geometric features, arising from the semidirect product structure of the slice subgroup of the group of three-dimensional diffeomorphisms. Firstly, the formulation leads to a Kelvin-Noether circulation theorem in which circulation is preserved on arbitrary loops in the slice, unlike the usual circulation theorem in which circulation is only preserved on isentropic surfaces. Secondly, as shown in Appendix B, the equations can always be rewritten in terms of a pair of two dimensional momenta, one comprising the x- and z-components of linear momentum, and one formed from the temperature and the y-component of linear momentum, plus the density. This formulation involving only two-dimensional momenta and density means that potential vorticity conserving numerical schemes for the shallow-water equations can be adapted for vertical slice problems. In the shallow-water case, the equations can

be written in the form

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_u\right) v \cdot \mathrm{d}x + \mathrm{d}\pi = 0,$$

where u is the velocity, v is the total momentum divided by the layer thickness, and π is a pressure. It is possible using mimetic/discrete exterior calculus methods [TC12] to use u as a prognostic variable, but to also apply d to the above equation, use some chosen stable conservative advection scheme for potential vorticity, and then to obtain a discrete form of $\mathcal{L}_u v \cdot \mathrm{d}x$ which is consistent with that scheme (so that potential vorticity advection is stabilised even though it is a diagnostic variable). This programme cannot be easily extended to three dimensions when advected temperature is present, since we gain an extra term of the form $G\,\mathrm{d}\theta$ (for some scalar function G), and it is not currently clear how to obtain a discrete form of $\mathcal{L}_u v$ that is consistent with a stable advection scheme for the Ertel potential vorticity. However, for the slice model equation set (2.12), it should be possible to use conservative advection schemes for the last three equations, then apply $\mathrm{d}$to equation (2.18), apply a stable conservative advection scheme for potential vorticity, and obtain a discrete form of $\mathcal{L}_{u_s} v_s \cdot \mathrm{d}x$ that is consistent with that scheme. This becomes possible for slice models, the extra term can be moved inside the Lie derivative to obtain equation (2.18).

This work has led to the development of new model equations: a Lagrangian-averaged form of the Eady model of frontogenesis and a new compressible model. We plan to use both of these models to investigate how to improve prediction of front evolution, following the programme set out in [Cul07]. Whilst solutions of the slice compressible model do not recover solutions of the full three dimensional equations, this model approximates the slice Boussinesq model in the Boussinesq limit, and is easily obtained by very minor modifications to standard dynamical core slice configurations, so will allow asymptotic limit analysis to be performed with compressible codes, addressing a problem highlighted in [Cul08].

The authors are grateful to Mike Cullen and Abeed Visram for very useful and interesting discussions about slice models. The work by DDH was partially supported by Advanced Grant 267382 from the European Research Council and the Royal Society of London Wolfson Award Scheme. The work by CJC was partially supported by the Natural Environment Research Council Next Generation Weather and Climate programme.

References

Cul07 M.J.P. Cullen.
Modelling atmospheric flows.
Acta Numerica, 2007.

Cul08 M. J. P. Cullen.
A comparison of numerical solutions to the eady frontogenesis problem.
Q. J. R. Meteorol. Soc., 134:2143–2155, 2008.

DV80 I. E. Dzyaloshinskii and G. E. Volovik.
Poisson brackets in condensed matter physics.
Ann. Phys., 125:67–97, 1980.

GNH92 S.T. Garner, N. Nakamura, and I.M. Held.
Nonlinear equilibration of two-dimensional eady waves : a new perspective.
Journal of the Atmospheric Sciences, 49(21):1984–1996, 1992.

HK82 D. D. Holm and B. A. Kupershmidt.
Poisson structures of superfluids.
Phys. Lett. A, 91:425–430, 1982.

HM04 D. D. Holm and J. E. Marsden.
Momentum maps and measure valued solutions (peakons, filaments, and sheets) of the Euler-Poincaré equations for the diffeomorphism group.
In J.E. Marsden and T.S. Ratiu, editors, *In The Breadth of Symplectic and Poisson Geometry, A Festshrift for Alan Weinstein*, volume 232 of *Progr. Math.*, pages 203–235. Birkhäuser Boston, 2004.

HMR98 D. D. Holm, J. E. Marsden, and T. S. Ratiu.

The Euler–Poincaré equations and semidirect products with applications to continuum theories.
Adv. in Math., 137:1–81, 1998.

TC12 J. Thuburn and C.J. Cotter.
A framework for mimetic discretization of the rotating shallow-water equations on arbitrary polygonal grids.
SIAM J. Sci. Comp., 2012.

Appendices

A. Euler–Poincaré semidirect-product formulation

The advection equations (2.4)–(2.6) for (θ_S, s) and D may be rewritten in Lie-derivative notation as

$$\begin{aligned} \partial_t\,(\theta_S, s) &= -\mathcal{L}_{(u_S,u_T)}\,(\theta_S, s) = (-\,u_S\cdot\nabla\theta_S - u_T s, 0)\,, \\ \partial_t(D\ \mathrm{d}S) &= -\mathcal{L}_{u_S}(D\ \mathrm{d}S) = -\,\mathrm{div}(u_S D)\ \mathrm{d}S\,. \end{aligned} \tag{A 1}$$

The corresponding infinitesimal variations in (θ_S, s) and D, in (2.7) induced by the Lie-derivative actions of the Lie algebra of vector fields $\mathfrak{X}(\Omega)\circledS\mathcal{F}(\Omega)$ are given by:

$$\begin{aligned} \delta\,(\theta_S, s) &= -\mathcal{L}_{(w_S,w_T)}\,(\theta_S, s) = (-w_S\cdot\nabla\theta_S - w_T s, 0)\,, \\ \delta D\ \mathrm{d}S &= -\mathcal{L}_{w_S}(D\ \mathrm{d}S) = -\,\mathrm{div}(w_S D)\ \mathrm{d}S\,. \end{aligned} \tag{A 2}$$

The infinitesimal variations in (u_S, u_T) in (2.7) may be expressed in terms of the adjoint action in the Lie algebra $\mathfrak{X}(\Omega)\circledS\mathcal{F}(\Omega)$ of the semidirect-product group $\mathrm{Diff}(\Omega)\circledS\mathcal{F}(\Omega)$. Namely,

$$\begin{aligned} \delta\,(u_S, u_T) &= (\partial_t w_S, \partial_t w_T) - \mathrm{ad}_{(u_S,u_T)}\,(w_S, w_T) \\ &= (\partial_t w_S + [u_S, w_S],\ \partial_t w_T + u_S\cdot\nabla w_T - w_S\cdot\nabla u_T)\ . \end{aligned} \tag{A 3}$$

For a Lagrangian functional $l[(u_S, u_T), (\theta_S, s), D] : (\mathfrak{X}\circledS\mathcal{F}(\Omega))\circledS((\Lambda^0(\Omega)\times\mathbb{R})\times\Lambda^2(\Omega))\to\mathbb{R}$, one defines Hamilton's principle using the L^2 pairing, which is denoted as $\langle\cdot,\cdot\rangle$. Hence, inserting the infinitesimal variational formulas in (2.7) for (u_S, u_T), (θ_S, s) and D yields, in semidirect-product notation,

$$\begin{aligned} 0 &= \delta S \\ &= \delta\int_0^T l[(u_S, u_T), (\theta_S, s), D]\,\mathrm{d}t \\ &= \int_0^T \left\langle \frac{\delta l}{\delta(u_S, u_T)}, \delta(u_S, u_T)\right\rangle + \left\langle \frac{\delta l}{\delta(\theta_S, s)}, \delta(\theta_S, s)\right\rangle + \left\langle \frac{\delta l}{\delta D}, \delta D\right\rangle \mathrm{d}t \\ &= \int_0^T \left\langle \frac{\delta l}{\delta(u_S, u_T)}, \frac{\partial}{\partial t}(w_S, w_T) - \mathrm{ad}_{(u_S,u_T)}(w_S, w_T)\right\rangle \\ &\quad + \left\langle \frac{\delta l}{\delta(\theta_S, s)}, -\mathcal{L}_{(w_S,w_T)}\,(\theta_S, s)\right\rangle + \left\langle \frac{\delta l}{\delta D}, -\,\mathrm{div}(w_S D)\right\rangle \mathrm{d}t \\ &= \int_0^T \left\langle -\frac{\partial}{\partial t}\frac{\delta l}{\delta(u_S, u_T)} - \mathrm{ad}^*_{(u_S,u_T)}\frac{\delta l}{\delta(u_S, u_T)} + \frac{\delta l}{\delta(\theta_S, s)}\diamond(\theta_S, s) + \left(\frac{\delta l}{\delta D}\diamond D,\, 0\right),\, (w_S, w_T)\right\rangle \mathrm{d}t \\ &\quad + \left[\left\langle \frac{\delta l}{\delta(u_S, u_T)},\, (w_S, w_T)\right\rangle\right]_0^T . \end{aligned} \tag{A 4}$$

In comparison, see equation (2.8) for the *same* Hamilton's principle in vector notation. As before, the last term in the previous equation vanishes because (w_S, w_T) vanishes at the endpoints. The

ad* notation in (A 4) denotes the dual of the ad operation with respect to the L^2 pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ [HMR98]. Explicitly, the L^2 dual of the ad operation is defined by

$$\left\langle \mathrm{ad}^*_{(u_S,u_T)}\frac{\delta l}{\delta(u_S,u_T)},(w_S,w_T)\right\rangle=\left\langle \frac{\delta l}{\delta(u_S,u_T)},\mathrm{ad}_{(u_S,u_T)}\,(w_S,w_T)\right\rangle. \tag{A 5}$$

Likewise, the diamond ($\diamond$) operation is defined in the present notation by the L^2 pairings,

$$\begin{aligned}\left\langle \frac{\delta l}{\delta(\theta_S,s)}\diamond(\theta_S,s),\,(w_S,w_T)\right\rangle&:=\left\langle \frac{\delta l}{\delta(\theta_S,s)},\,-\mathcal{L}_{(w_S,w_T)}\,(\theta_S,s)\right\rangle,\\ \left\langle\left(\frac{\delta l}{\delta D}\diamond D,\,0\right),\,(w_S,w_T)\right\rangle&:=\left\langle \frac{\delta l}{\delta D},\,-\mathcal{L}_{w_S}D\right\rangle=\left\langle \frac{\delta l}{\delta D},\,-\mathrm{div}(w_S D)\right\rangle.\end{aligned} \tag{A 6}$$

Hence, the last equality of (A 4) yields the Euler-Poincaré equations on the dual Lie algebra $(\mathfrak{X}(\Omega)\circledS\mathcal{F}(\Omega))^*$ with the advected areal density $D\in\Lambda^2$ and advected scalars $(\theta_S,s)\in\Lambda^0\times\mathbb{R}$ in semidirect-product form, as

$$\frac{\partial}{\partial t}\frac{\delta l}{\delta(u_S,u_T)}+\mathrm{ad}^*_{(u_S,u_T)}\frac{\delta l}{\delta(u_S,u_T)}=\frac{\delta l}{\delta(\theta_S,s)}\diamond(\theta_S,s)+\left(\frac{\delta l}{\delta D}\diamond D,\,0\right). \tag{A 7}$$

The system (A 7) is completed by including the advection equations (A 1) for D and (θ_S,s).

B. Lie-Poisson Hamiltonian formulation

(a) Equations on the dual of $(\mathfrak{X}\circledS\mathcal{F}(\Omega))\circledS((\Lambda^0(\Omega)\times\mathbb{R})\times\Lambda^2(\Omega))$

The Legendre transformation to the Hamiltonian is defined by,

$$h[(m_S,\,m_T),(\theta_S,s),D]=\langle(m_S,\,m_T),(u_S,\,u_T)\rangle-l[(u_S,\,u_T),(\theta_S,s),D]\,. \tag{A 1}$$

Therefore, we find the variational relations

$$(m_S,\,m_T)=\frac{\delta l}{\delta(u_S,u_T)}\,,\quad (u_S,\,u_T)=\frac{\delta h}{\delta(m_S,m_T)}\,,\quad \frac{\delta h}{\delta(\theta_S,s)}=-\frac{\delta l}{\delta(\theta_S,s)}\,,\quad \frac{\delta h}{\delta D}=-\frac{\delta l}{\delta D}\,. \tag{A 2}$$

Consequently, the system (A 7) may be written in terms of the Hamiltonian as

$$\frac{\partial}{\partial t}(m_S,m_T)=-\,\mathrm{ad}^*_{\delta h/\delta(m_S,m_T)}(m_S,m_T)-\frac{\delta h}{\delta(\theta_S,s)}\diamond(\theta_S,s)-(\frac{\delta h}{\delta D}\diamond D,\,0)\,. \tag{A 3}$$

The advection equations (A 1) for (θ_S,s) and D are then written as

$$\begin{aligned}\frac{\partial}{\partial t}(\theta_S,s)&=-\mathcal{L}_{\delta h/\delta(m_S,m_T)}\,(\theta_S,s)\,,\\ \frac{\partial}{\partial t}(D,\,0)&=-\mathcal{L}_{\delta h/\delta(m_S,m_T)}(D,\,0)\,.\end{aligned} \tag{A 4}$$

Hence, the entire system (A 3)–(A 4) may be written in Hamiltonian form as

$$\frac{\partial}{\partial t}\begin{bmatrix}(m_S,m_T)\\(\theta_S,s)\\(D,\,0)\end{bmatrix}=-\begin{bmatrix}\mathrm{ad}^*_{\square}(m_S,m_T) & \square\diamond(\theta_S,s) & \square\diamond(D,\,0)\\ \mathcal{L}_{\square}(\theta_S,s) & 0 & 0\\ \mathcal{L}_{\square}(D,\,0) & 0 & 0\end{bmatrix}\begin{bmatrix}\delta h/\delta(m_S,m_T)\\ \delta h/\delta(\theta_S,s)\\ \delta h/\delta(D,\,0)\end{bmatrix}, \tag{A 5}$$

in which the box $\square$ indicates the appropriate substitutions. The matrix operator in (A 5) defines a Lie-Poisson bracket dual to the semidirect product action $(\mathfrak{X}\circledS\mathcal{F}(\Omega))\circledS((\Lambda^0(\Omega)\times\mathbb{R})\times\Lambda^2(\Omega))$ with coordinates $(u_S,u_T)\in\mathfrak{X}\circledS\mathcal{F}(\Omega)$, $(\theta_S,s)\in\Lambda^0(\Omega)\times\mathbb{R}$ and $D\in\Lambda^2(\Omega)$. This identification of the Lie-Poisson bracket with the dual of a Lie algebra action guarantees that it satisfies the Jacobi

identity. Explicitly, the Lie-Poisson bracket is the following

$$\Big\{f,\,h\Big\} = -\left\langle \begin{bmatrix} \delta f/\delta(m_S, m_T) \\ \delta f/\delta(\theta_S, s) \\ \delta f/\delta(D,\,0) \end{bmatrix}^T, \begin{bmatrix} \mathrm{ad}^*_\square(m_S, m_T) & \square\diamond(\theta_S, s) & \square\diamond(D,\,0) \\ \mathcal{L}_\square(\theta_S, s) & 0 & 0 \\ \mathcal{L}_\square(D,\,0) & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta h/\delta(m_S, m_T) \\ \delta h/\delta(\theta_S, s) \\ \delta h/\delta(D,\,0) \end{bmatrix} \right\rangle, \tag{A 6}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the L^2 pairing.

Expanding out the operations in (A 6) makes it clear that this Lie-Poisson bracket has the required property of being antisymmetric under exchange of f and h. That is, $\{h,\,f\} = -\{f,\,h\}$, which is evident upon expanding out the operations to express the bracket in (A 6) equivalently as

$$\begin{aligned} \Big\{f,\,h\Big\} = &-\left\langle (m_S, m_T),\, \left[\frac{\delta f}{\delta(m_S, m_T)},\, \frac{\delta h}{\delta(m_S, m_T)}\right]\right\rangle \\ &+\left\langle (\theta_S, s),\, \mathcal{L}^+_{\delta f/\delta(m_S,m_T)}\frac{\delta h}{\delta(\theta_S, s)} - \mathcal{L}^+_{\delta h/\delta(m_S,m_T)}\frac{\delta f}{\delta(\theta_S, s)}\right\rangle \\ &+\left\langle (D,\,0),\, \mathcal{L}^+_{\delta f/\delta(m_S,m_T)}\frac{\delta h}{\delta(D,\,0)} - \mathcal{L}^+_{\delta h/\delta(m_S,m_T)}\frac{\delta f}{\delta(D,\,0)}\right\rangle. \end{aligned} \tag{A 7}$$

Here, $\mathcal{L}^+$ denotes the L^2 adjoint of the Lie derivative $\mathcal{L}$. In particular, upon denoting $\delta f/\delta(m_S, m_T) = (w_S, w_T)$, we find the following relations among the operations $\mathcal{L}^+$, $\mathcal{L}$ and $\diamond$,

$$\left\langle (\theta_S, s)\,,\, \mathcal{L}^+_{(w_S,w_T)}\frac{\delta h}{\delta(\theta_S, s)}\right\rangle := \left\langle \frac{\delta h}{\delta(\theta_S, s)},\, \mathcal{L}_{(w_S,w_T)}\,(\theta_S, s)\right\rangle =: \left\langle -\frac{\delta h}{\delta(\theta_S, s)}\diamond(\theta_S, s),\, (w_S, w_T)\right\rangle. \tag{A 8}$$

Condition B.1. If desired, one may now substitute the expressions for Lie derivative (A 1), ad^* (A 5) and diamond ($\diamond$) (A 6) into the L^2 pairings (A 6) or (A 7) to find the Lie-Poisson bracket $\{f,\,h\}$ as an integral over the slice domain, Ω, involving ordinary vector calculus operations. However, the present forms (A 6) and (A 7) readily reveal its semidirect-product nature and suggest further rearrangements, which we pursue next.

(b) Equations on the dual of $\mathfrak{X}\circledS(\Lambda^0 \oplus \Lambda^2 \oplus \Lambda^0)$

To explore the particular case at hand further, one may rewrite the system of equations (2.4), (2.6) and (2.9) equivalently as,

$$\begin{aligned} \frac{\partial}{\partial t}\frac{\delta l}{\delta u_S} &= -\,\mathrm{ad}^*_{u_S}\frac{\delta l}{\delta u_S} - \frac{\delta l}{\delta \theta_S}\nabla\theta_S - \frac{\delta l}{\delta u_T}\nabla u_T + D\nabla\frac{\delta l}{\delta D}\,, \\ \frac{\partial}{\partial t}\frac{\delta l}{\delta u_T} &= -\,\mathcal{L}_{u_S}\frac{\delta l}{\delta u_T} - \frac{\delta l}{\delta \theta_S}s, \\ \frac{\partial}{\partial t}\theta_S &= -\mathcal{L}_{u_S}\theta_S - u_T s, \\ \frac{\partial}{\partial t}D &= -\,\mathcal{L}_{u_S}D\,, \end{aligned} \tag{A 9}$$

where $\mathcal{L}_{u_S}$ denotes Lie derivative along the vector field u_S and we have identified $\mathcal{L}_{u_S}$ and $\mathrm{ad}^*_{u_S}$ when acting on the 1-form density $\delta l/\delta u_S$ in the first equation. For more details in this matter, see [HMR98].

We define the Legendre transformation to the Hamiltonian in this case by

$$h[m_S,\, m_T\,, \theta_S, D;\, s] = \langle m_S,\, u_S\rangle + \langle m_T,\, u_T\rangle - l[u_S,\, u_T,\, \theta_S,\, D;\, s]\,, \tag{A 10}$$

where the semicolon $[\dots;s]$ denotes parametric dependence on the constant $s\in\mathbb{R}$. The Legendre transformation (A 10) yields the variational relations

$$m_S=\frac{\delta l}{\delta u_S}\,,\quad u_S=\frac{\delta h}{\delta m_S}\,,\quad m_T=\frac{\delta l}{\delta u_T}\,,\quad u_T=\frac{\delta h}{\delta m_T}\,,\quad \frac{\delta h}{\delta\theta_S}=-\frac{\delta l}{\delta\theta_S}\,,\quad \frac{\delta h}{\delta D}=-\frac{\delta l}{\delta D}\,. \tag{A 11}$$

Consequently, the system (A 7) may be written in terms of the Hamiltonian as

$$\begin{aligned}
\frac{\partial}{\partial t}m_S&=-\operatorname{ad}^*_{\delta h/\delta m_S}m_S-m_T\nabla\frac{\delta h}{\delta m_T}+\frac{\delta h}{\delta\theta_S}\nabla\theta_S-D\nabla\frac{\delta h}{\delta D}\,,\\
\frac{\partial}{\partial t}m_T&=-\mathcal{L}_{\delta h/\delta m_S}m_T+\frac{\delta h}{\delta\theta_S}s\,,\\
\frac{\partial}{\partial t}\theta_S&=-\mathcal{L}_{\delta h/\delta m_S}\theta_S-\frac{\delta h}{\delta m_T}s\,,\\
\frac{\partial}{\partial t}D&=-\mathcal{L}_{\delta h/\delta m_S}D\,,
\end{aligned} \tag{A 12}$$

The corresponding Hamiltonian matrix is

$$\frac{\partial}{\partial t}\begin{bmatrix}m_S\\ m_T\\ \theta_S\\ D\end{bmatrix}=-\begin{bmatrix}\operatorname{ad}^*_{\square}m_S & \square\diamond m_T & \square\diamond\theta_S & \square\diamond D\\ \mathcal{L}_{\square}m_T & 0 & -s & 0\\ \mathcal{L}_{\square}\theta_S & s & 0 & 0\\ \mathcal{L}_{\square}D & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\delta h/\delta m_S\\ \delta h/\delta m_T\\ \delta h/\delta\theta_S\\ \delta h/\delta D\end{bmatrix}, \tag{A 13}$$

in which the box $\square$ indicates the appropriate substitutions.

After this rearrangement, one recognises (A 13) as the Hamiltonian matrix for the Lie-Poisson bracket on the dual of the semidirect-product Lie algebra $\mathfrak{X}\circledS(\Lambda^0\oplus\Lambda^2\oplus\Lambda^0)$ with a *symplectic two-cocycle* between m_T and θ_S. The Lie bracket for this semidirect-product algebra is

$$\big[(X,f,\omega,g),\,(\tilde X,\tilde f,\tilde\omega,\tilde g)\big]=\big([X,\tilde X],\,X(\tilde f)-\tilde X(f),\,X(\tilde\omega)-\tilde X(\omega),\,X(\tilde g)-\tilde X(g)\big)\,, \tag{A 14}$$

where, *e.g.*, $X(\tilde f)=\mathcal{L}_X\tilde f$ denotes Lie derivative of $\tilde f$ by vector field X. The dual coordinates are: m_S dual to $X\in\mathfrak{X}$; m_T to $f\in\Lambda^0$; θ_S to $\omega\in\Lambda^2$; and D to $g\in\Lambda^0$. The spaces in which the coordinates themselves are defined are $(m_S,m_T,\theta_S,D)\in(\Lambda^1\otimes\Lambda^2,\Lambda^2,\Lambda^0,\Lambda^2)$ and $s\in\mathbb{R}$ is a parameter. The second part of the bracket (A 13) is the standard two-cocycle (symplectic form) on $\Lambda^0\oplus\Lambda^2$ arising from the natural projection $\mathfrak{X}\circledS(\Lambda^0\oplus\Lambda^2\oplus\Lambda^0)\to\Lambda^0\oplus\Lambda^2$.

Condition B.2. The Hamiltonian matrix with the two-cocycle in (A 13) has been seen before. Namely, it is the same as that for ^4He superfluids [DV80,HK82] in the spatially two-dimensional case. For ^4He superfluids, the function θ_S here plays the role of the phase of the Bose-condensate wave function, whose gradient $\nabla\theta_S$ is the superfluid velocity. The other variables m_S, m_T and D correspond respectively, to total momentum density, mass density and entropy density of the superfluid.

(c) Equations on the dual of $\mathfrak{X}_1\circledS(\mathfrak{X}_2\oplus\Lambda^0)$

[HK82] showed that the two-cycle in (A 13) may be removed by transforming to new variables

$$(m_S,m_T,\theta_S,D)\to(m_S,m_R,D)\quad\text{where}\quad m_R:=(s)^{-1}m_T\nabla\theta_S\,. \tag{A 15}$$

The quantity m_R is the momentum map for right action of the diffeomorphisms on the buoyancy θ_S in two spatial dimensions, see [HM04] for more details. The resulting Lie-Poisson bracket has the standard form dual to the Lie algebra $\mathfrak{X}_1\circledS(\mathfrak{X}_2\oplus\Lambda^0)$, whose Lie bracket is

$$\begin{aligned}
&\big[(X_1,X_2,f),\,(\tilde X_1,\tilde X_2,\tilde f)\big]=\\
&\qquad\big([X_1,\tilde X_1],\,[X_2,\tilde X_2]+[X_1,\tilde X_2]-[\tilde X_1,X_2],\,X_1(\tilde f)-\tilde X_1(f)\big)\,.
\end{aligned} \tag{A 16}$$

Dual coordinates in this case are: m_S dual to $X_1\in\mathfrak{X}_1$; m_R to $X_2\in\mathfrak{X}_2$; and D to $f\in\Lambda^0$.

Transformation of the Hamiltonian matrix (A 13) into these variables yields the following Lie-Poisson Hamiltonian system

$$\frac{\partial}{\partial t}\begin{bmatrix} m_S \\ m_R \\ D \end{bmatrix} = -\begin{bmatrix} \mathrm{ad}^*_\square m_S & \mathrm{ad}^*_\square m_R & \square \diamond D \\ \mathrm{ad}^*_\square m_R & \mathrm{ad}^*_\square m_R & 0 \\ \mathcal{L}_\square D & 0 & 0 \end{bmatrix}\begin{bmatrix} \delta h/\delta m_S =: u_S \\ \delta h/\delta m_R =: u_R \\ \delta h/\delta D =: p \end{bmatrix}. \tag{A 17}$$

This system produces a system of equations for relative momentum $(m_S - m_R)$, momentum map $m_R = (s)^{-1} m_T \nabla\theta_S$ and mass density D, given by

$$\begin{aligned} \partial_t(m_S - m_R) &= -\mathrm{ad}^*_{u_S}(m_S - m_R) - p \diamond D\,, \\ \partial_t m_R &= -\mathrm{ad}^*_{(u_S+u_R)} m_R\,, \\ \partial_t D &= -\mathcal{L}_{u_S} D\,. \end{aligned} \tag{A 18}$$

Upon evaluating $p \diamond D = D\nabla p$, the first of these equations explains the geometric origin of the Kelvin-Noether circulation theorem (2.11) that was found by direct manipulation in Section (c). Together, the three equations in (A 18) show that the slice dynamics may be expressed in terms of (m_S, m_R, D) as a Lie-Poisson Hamiltonian system on the semidirect product

$$\mathrm{Diff}_1(\Omega) \circledS \left(\mathrm{Diff}_2(\Omega) \times \Lambda^2(\Omega)\right),$$

in the slice domain Ω. When $D = 1$ is imposed, we have $\nabla \cdot u_S = 0$ and this simplifies to

$$\mathrm{SDiff}_1(\Omega) \circledS \mathrm{Diff}_2(\Omega)\,.$$