Abstract

Helmholtz resonators (HRs) are widely used to damp acoustic oscillations, including in the combustors of aero-engines and power gas turbines where they damp thermoacoustic oscillations. The geometries of such combustors are often annular in shape, which means that low frequency acoustic modes exhibit both longitudinal and circumferential modeshapes, the latter across different circumferential wave numbers. For linear acoustic disturbances downstream of the flame, the presence of HRs leads to modal coupling and mode shape shifts, which makes design and placement of multiple HRs very complicated. A procedure which ensures that the design and placement of the HRs can be optimised for good acoustic damping performance would be very valuable, and such a procedure is presented in this work. A simplified linear, low-dimensional model for the acoustic behaviour in a hot annular duct sustaining a mean flow is extended to account for the attachment of multiple HRs. The HRs are assumed to sustain a cooling mean bias flow through them, towards the combustor, such that they can be modelled using linear, lumped element Rayleigh conductivity models. An optimisation method based on the gradient derived from adjoint sensitivity analysis is then applied to the low order network acoustic modelling framework for hot annular ducts incorporating HR models, for the first time. It is used to optimise over multiple HR geometry and placement parameters, to
obtain optimum acoustic damping over all acoustic modes in a given frequency range. These optimisation procedures are validated via multi-dimensional parameter sweep results. Thus a novel and efficient tool for HR optimisation for thin annular ducts is presented.

Keywords: Helmholtz resonators, annular duct, thermoacoustics, optimisation

1. Introduction

Helmholtz resonators (HRs) are widely used to damp acoustic oscillations. Of particular interest in the present work is their use in damping thermoacoustic oscillations. Modern gas turbines – both aero-engines and land-based – are often operated under lean-premixed conditions in order to reduce NOx emissions. However, this makes the combustion susceptible to acoustic disturbances [1]. The resulting two-way coupling between acoustic waves and unsteady heat release rate can lead to thermoacoustic instability (also known as combustion instability) which results in acoustic oscillations large enough to interfere with engine and gas-turbine operations, and even sometimes failure of the whole system. The combustion chambers of modern gas turbines often have thin annular geometries. The combustor circumference can be much longer than or of the same order as its longitudinal length [2, 3], meaning that thermoacoustic oscillations, which tend to occur at low frequencies, exhibit azimuthal (also known as circumferential) mode shapes [4–8]. The circumferential modes may be degenerate or coupled, resulting in the possibility for a variety of spatial instability patterns, including circumferential spinning, circumferential standing, circumferential mixed and slanted (this being a combination of longitudinal and circumferential) in limit cycle oscillations [7–9].

Helmholtz resonators (HRs) can be used as passive dampers for thermoacoustic oscillations [10–18]. A Helmholtz resonator consists of a small neck opening to a large cavity volume, as shown on the left of Fig. 1. The HR is typically much smaller than the acoustic wavelength and the neck flow acts as a mass inertia and the cavity air as a spring. Such geometries exhibit a resonance
whose frequency is given by the well-known expression
\[ f_{\text{ref}} = c \sqrt{\frac{S_n}{V l}} / (2\pi) , \]
where \( S_n, \quad l, \quad V \) are the HR neck cross-sectional area, neck length, and cavity volume respectively and \( c \) is the sound speed in the cavity. This equation does not account for acoustic damping. At resonance, small pressure perturbations at the neck mouth give rise to large flow oscillations within the neck. In practice, a cooling mean flow, known as a bias flow, is used to protect the HR from hot gas damage (shown on the right of Fig. 1). With a bias flow, incident acoustic waves cause unsteady vortices to be shed at the neck hole edge – these are then swept away by the mean flow. The presence of a bias flow significantly extends the range of acoustic amplitudes over which the damping can be modelled using linear equations – this allows the acoustic absorption to be characterized using a Rayleigh conductivity [19–21].

![Diagram of a Helmholtz resonator](image)

Figure 1: (Left) Schematic view of a typical Helmholtz resonator, showing the slug of air in the neck and the volume of air which compresses in the cavity. (Right) A typical Helmholtz resonator with a mean neck bias flow – the dotted lines represent the sheet of shed vorticity.

When HRs are used to damp thermoacoustic oscillations in annular combustors, more than a single HR is typically needed. This is for two reasons. Firstly, a given HR has a narrow sound absorption bandwidth near its resonant frequency, and annular geometries tend to exhibit a variety of acoustic modes with differing frequencies. Secondly, adding a single HR will split the circumferential mode, with one of the split modes having a pressure node at the HR.
location – there is no acoustic absorption for this mode \[11\]. As each HR need not have the same geometry, neck bias flow nor placement on the combustor, this creates many free parameters for the designer looking to achieve damping in annular combustors. Furthermore, the introduction of each HR will change the mode shapes of the system. This all points to the need for a systematic optimisation method in order to obtain the best configuration for the HRs.

The issue of incorporating HRs into annular combustors was studied by Stow and Dowling, using a low-order network model which expressed perturbations in terms of modal expansions \[11\]. This was computationally efficient (simulations on a general PC take negligible time), captured the HR effect on mode degeneracy, and could account for combustor mean flows. Note that the mode degeneracy means the clock-wise and anti clock-wise propagating components of the mode can exist independently, and thus it is impossible to identify whether the mode is spinning (only one component exists), standing (both components exist with equal strength) or mixed (both components exist with different strengths) \[22–26\]. When the mode is non-degenerate, the ratio between the strengths of the two components is fixed, and thus the spatial mode pattern is determined. HR configuration rules obtained using this method have been confirmed by subsequent studies \[13, 27\]. Low-order network models are generally very powerful when the geometry can be represented as a network of simple modules. They offer the advantage of allowing mean flow effects, including entropy \[28, 29\] and vorticity waves \[30, 31\], to be incorporated, and have been shown to be able to provide useful prediction, control and design insights \[4, 32–34\].

Studies have also been performed using Helmholtz solvers for the acoustic field. Helmholtz solvers neglect the effect of mean flow (including effects such as entropy and vorticity wave convection) but can account for complex geometry and spatial variations in the mean speed of sound \[35, 37\]. Recently, a two dimensional Helmholtz solver was applied to an annular geometry, with the locations of multiple HRs restricted to the different burner outlets azimuthally \[27\]. The HRs were modelled as simple reduced impedances. Adjoint-based pertur-
bation analysis was then performed based on both the first- and higher-order variations across multiple parameters [27, 38]. Accounting for large perturbation strengths requires higher order variations to be accounted for but the splitting of degenerate eigenvalues and the “inclination rule” of the mode move direction (in the complex plane) from the expansion point can be obtained at the first order [38].

The present work applies an optimisation procedure to a low-order network modelling tool, using it to systematically optimise the geometry and placement of multiple HRs attached to a hot annular duct with mean flow. The procedure can account for HRs with any spatial location, longitudinally and circumferentially, as well as being able to account for realistic effects including the temperature difference between the HR cavity volume and the annular duct (combustor) [29]. It uses the Rayleigh conductivity model for the HRs, this being more complex and physical than models used in many previous optimisation studies.

Firstly, a model for the propagation of acoustic waves in the hot annular duct is developed in Section 2. This assumes linear, low-dimensional perturbations which can be represented as the sum of planar and circumferential wave number variations. It follows similar lines to Stow and Dowling’s low-order network model [11]. For the sake of simplicity and focus in this work, neither the connections to other geometry modules nor the flame jump conditions are incorporated, although the present module could be connected as is to those modules by using standard methods for low order network modelling [39–42]. Section 2 also describes how the HR attachment affects the acoustic waves via flow conservation equations, and describes the Rayleigh-conductivity models used for the HR damping. Section 3 then describes how both the locations and geometric specifications of multiple HRs can be optimised via an objective functional which incorporates all the relevant acoustic modes. A quasi Newton optimisation method based on a multi-parameter gradient is presented. Section 4 demonstrates the ability of the procedure to target a variety of acoustic modal patterns across a range of combustor geometries. Conclusions are finally given
in Section 5.

2. Models

A typical annular combustor can be represented as a thin annulus whose radial gap is much smaller than the longitudinal and circumferential lengths, as shown in Fig. 2. This means that at the low frequencies at which thermoacoustic oscillations occur, high-order acoustic radial modes are highly cut-off [4]. When multiple Helmholtz resonators with differing longitudinal locations are attached to the duct, the duct is longitudinally segmented into sections with HRs located only at section interfaces, as shown in Fig. 3. Flow perturbations within each section are expanded as a sum of circumferential modes in order to obtain general solutions for oscillations in each of these sections. Conservation equations and Helmholtz resonator models are then incorporated to link oscillations across the HRs.

2.1. Linear perturbations in an annular duct

To obtain general solutions for linear oscillations inside an annular duct with a uniform mean flow, we represent all perturbations as the sum of circumferential modes. This follows similar lines to [4, 39, 43, 44]; key steps are outlined with more details provided in Section A.

Firstly, by assuming a steady, uniform mean flow with constant heat capacity and neglecting viscosity, heat input and heat conduction, the linearized equations for acoustic, entropy and vorticity perturbations are uncoupled and can be written as [4]

\[
\begin{align*}
\left( \frac{1}{c^2} \frac{\bar{D}^2}{Dt^2} - \nabla^2 \right) p' &= 0, \\
\bar{\rho}T \frac{\bar{D}s'}{Dt} &= 0, \\
\frac{\bar{D} \xi'}{Dt} &= 0,
\end{align*}
\]

where $\bar{D}/Dt = \partial/\partial t + \bar{u} \cdot \nabla$. $c$, $\rho$, $T$, $p$, $s$, $\xi$ are the sound speed, density, temperature, pressure, entropy and vorticity respectively. An overbar, $\bar{}$, denotes
Figure 2: (Left) A thin annular duct with a uniform mean flow in the longitudinal direction. Dotted lines with arrows indicate that the perturbation waves may propagate in either one or both of the longitudinal and circumferential directions. (Right) Fourier amplitudes of the waves – $\tilde{A}^{n\pm}$ are downstream and upstream propagating acoustic waves, $\tilde{V}^{(n)}$ is the vorticity wave, and $\tilde{E}^{(n)}$ the entropy wave – all with a given circumferential wave number, $n$. 
time-average and a prime, [ ... ], denotes small perturbation. From these equations, it can be seen that linear flow perturbations can be represented as the sum of three types of disturbance, 1) an acoustic disturbance which is isentropic and irrotational, 2) an entropic disturbance that is incompressible and irrotational, and 3) a vortical disturbance that is incompressible and isentropic \([4, 45]\).

Assuming the mean flow to be purely longitudinal and neglecting radial variations, perturbations due to the acoustics can be represented as an infinite expansion of modal solutions for Eq. (1a), each mode \(n\) representing a circumferential wave number which is an integer due to the circumferentially periodic boundary condition. For an angular frequency \(\omega\), and truncating at mode number \(N\), the solution can be written as

\[
p'(A)(x, \theta, t) = \text{Re}[\tilde{p}_A(x, \theta) e^{-i \omega t}],
\]

where \(\text{Re}[\cdot]\) denotes the real part, \(\tilde{p}\) denotes Fourier amplitude and \(\tilde{A}(n)\) denotes that the perturbation is due to acoustic waves. \(\tilde{A}(n)\) is the combination of downstream and upstream propagating waves, \(\tilde{A}(n)\). The acoustic component of the density perturbation and the longitudinal and circumferential velocity perturbations for a given \(n\) are similarly obtained (see Section A).

The entropy perturbation can be obtained from the modal solution of the convected wave equation Eq. (1b). Perturbations due to the entropy wave are incompressible and irrotational, and can be prescribed in the form of only density fluctuations. As oscillations are two dimensional, a vorticity disturbance in the radial direction may exist, and can be obtained from the modal solution for Eq. (1c). The vorticity does not contribute to pressure or density oscillations but contribute to the velocity oscillations.

Combining the expressions for the acoustic, entropy and vorticity waves into a wave vector

\[
W(n)(x) = (\tilde{A}^n e^{i k^n x}, \tilde{A}^n e^{i k^n - x}, \tilde{E}^n e^{i k_0 x}, \tilde{V}^n e^{i k_0 x})^T,
\]

where the Fourier wave amplitudes are defined in Fig. 2, \(k^n\) are the longitudinal acoustic wave numbers which are given by Eq. (A.3) and \(k_0\) the convective wave number defined by Eq. (A.5). The overall pressure, density and velocity perturbations can then be written as a “flow vector”

\[
F(n)(x) = (\tilde{p}^n, \tilde{\rho}^n, \tilde{u}^n, \tilde{w}^n)^T
\]

which is linked to \(W(n)(x)\) by

\[
F(n)(x) = M_{W2F}^{(n)} W(n)(x),
\]

where the wave-to-flow transfer matrix, \(M_{W2F}^{(n)}\), is given by Eq. (A.7). Wave propagation in the longitudinal
direction from a reference location to another, for example from \( x_0 \) to \( (x_0 + 160) \), can be obtained through \( W^{(m)}(x_0 + L) = M_p^{(m)} W^{(m)}(x_0) \), where \( M_p^{(m)} = \text{diag}(e^{i k_x n L}, e^{i k_x n - L}, e^{i k_y L}, e^{i k_y L}) \).

When connecting different sections, the conservation of mass flux, \( m = S \rho u \), longitudinal-momentum flux, \( f(x) = Sp + mu \), angular-momentum flux, \( f(\theta) = \bar{R} \rho w \), and energy flux, \( e = S \gamma pu / (\gamma - 1) + m(u^2/2 + w^2/2) \), are employed, where \( S \) denotes the cross sectional area and \( \bar{R} \) the mean radius of the annular duct. A “flux vector” \( \mathbf{J}^{(m)}(x) = (\tilde{m}(x), \tilde{f}(x), \tilde{f}(\theta), \tilde{e})^T \) is then defined, with the relation between \( \mathbf{F}^{(m)}(x) \) and \( \mathbf{J}^{(m)}(x) \) given by a flow-to-flux transfer matrix, \( \mathbf{J}^{(m)}(x) = M^{(m)} \mathbf{F}^{(m)}(x) \), where the flow-to-flux transfer matrix, \( M^{(m)} \), is given by Eq. (A.8).

2.2. Relations across the HRs

Helmholtz resonators are now incorporated into the model. The model can account for any number of HRs, even though in practice only a few can be installed due to space and cooling flow limitations. As shown in the top part of Fig. 3, one or more HRs with the same longitudinal location, \( x_h \), separate the annular duct into sections before and after. At low frequencies where the acoustic wavelength is much larger than the neck diameter of each HR, each HR can be represented as a discontinuity in flow perturbations located at the center of its inlet \( (x_h, \theta_h^{(i)}) \) for the \( i \)th HR at \( x_h \). Note that any two HRs need to be far away enough from each other to ensure that (i) the acoustic near field at their neck inlets do not interact, and (ii) there is enough space for them to be installed. We now consider \( H \) HRs with the same longitudinal location, as shown in Fig. 3 to illustrate how they are incorporated into the model.

The mean flow through the HRs is assumed much smaller than that inside the annular duct, with the small contribution accounted for in calculating the uniform mean flow parameters in subsequent annular sections. The HRs are assumed attached perpendicular to the duct, such that the fluctuating neck flows are radial and only directly contribute unsteady mass and energy fluxes – they do not directly contribute longitudinal or circumferential momentum fluxes. For
Figure 3: (Top) An annular duct example showing five HRs which longitudinally split the annular duct into three sections. (Bottom left) Oscillations across the HRs interfacing sections 1 and 2. (Bottom right) Cross section view of the annular duct at $x = x_h$. 
Helmholtz resonators attached to hot ducts (such as a combustors), a bias cooling flow is typically applied to protect the HR from the high duct temperature. Gas turbine combustors typically employ cooling flows up to 1000K lower than that of the combustor \[46\]. We account for this temperature difference, which has been shown to have a significant effect on the acoustic damping \[29\]. To link perturbations just before and after the HRs, as shown in Fig. 3 (bottom left), the linearised mass, momentum and energy conservation equations across the HRs can be written as

\[
\sum_{n=-N}^{N} \overline{m}^{(n)}_{h} e^{i n \theta} + \sum_{i=1}^{H} 2\pi \overline{m}^{(i)}_{h} e^{i n \theta} = \sum_{n=-N}^{N} \overline{m}^{(n)}_{h+} e^{i n \theta}, \quad (2a)
\]

\[
\sum_{n=-N}^{N} \overline{f}^{(n)}(x)_h e^{i n \theta} = \sum_{n=-N}^{N} \overline{f}^{(n)}(x)_h+ e^{i n \theta}, \quad (2b)
\]

\[
\sum_{n=-N}^{N} \overline{f}^{(n)}(\theta)_h e^{i n \theta} = \sum_{n=-N}^{N} \overline{f}^{(n)}(\theta)_h+ e^{i n \theta}, \quad (2c)
\]

\[
\sum_{n=-N}^{N} \overline{c}^{(n)}_{h} e^{i n \theta} + \sum_{i=1}^{H} 2\pi \overline{c}^{(i)}_{h} e^{i n \theta} = \sum_{n=-N}^{N} \overline{c}^{(n)}_{h+} e^{i n \theta}, \quad (2d)
\]

where \(\overline{m}^{(i)}_{h}\) and \(\overline{c}^{(i)}_{h}\) are the unsteady mass and energy fluxes from the \(i\)th HR which are given by HR models, \([\ ]_h^\pm\) denotes perturbations at \(x_h^\pm\), and \(\delta\) is the Dirac delta function. If the unsteady mass flux is known, the energy flux oscillation can be obtained from

\[
\overline{c}^{(i)}_{h} = \overline{B}^{(i)}_{h} \overline{m}^{(i)}_{h} + \overline{m}^{(i)}_{h} \overline{B}^{(i)}_{h}, \quad (3)
\]

where \(\overline{B}^{(i)}_{h} = C_p \overline{T}^{(i)}_{h} + 0.5(\overline{u}^{(i)}_{h})^2\) is the mean neck stagnation enthalpy for the \(i\)th HR, with \(\overline{B}^{(i)}_{h}\) the oscillation, \(C_p\) is the heat capacity at constant pressure, \(\overline{T}^{(i)}_{h}\) is the mean temperature and \(\overline{u}^{(i)}_{h}\) the mean velocity, all in the HR neck.

The effect of the temperature difference between the HR cavity and the duct is accounted for by using the full mass, momentum and energy equations across the HRs.

An acoustic model for the HR is then used to relate the mass flux oscillation through the neck of the HR, \(\overline{m}^{(i)}_{h}\), to the pressure oscillations at its entrance,
$\tilde{p}(x_h, \theta_h^{(i)})$. A linear model based on the Rayleigh conductivity is used, with $\tilde{m}_h^{(i)}$ related to the HR entrance pressure by

$$
\tilde{p}(x_h, \theta_h^{(i)}) = -\left( \frac{(c_v^{(i)})^2}{\omega V^{(i)}} + \frac{i\omega}{K_R^{(i)}} \right) \tilde{m}_h^{(i)},
$$

where $V^{(i)}$ is the HR cavity volume, $c_v^{(i)}$ is the sound speed within the cavity, and $K_R^{(i)}$ the revised Rayleigh conductivity defined in references [47–49]. The revised Rayleigh conductivity accounts for the length of the HR neck as a mass inertia correction to the Rayleigh conductivity for a very short hole. The cavity temperature difference on affecting this length correction could be incorporated [50]. A new model which accounts for vortex-sound interaction within the neck could also be incorporated [20, 21]. If the pressure oscillation at the HR entrance is known, the mass flux oscillation can be obtained from Eq. (4), and the energy flux oscillation follows from Eq. (3).

By multiplying $e^{-in'\theta}$ on both sides of Eqs. (2) and integrating from $\theta = 0$ to $2\pi$, they can be simplified to give

$$
\tilde{m}_{h-}^{(n')} + \sum_{i=1}^{H} \tilde{m}_h^{(i)} e^{-in'\theta_h^{(i)}} = \tilde{m}_{h+}^{(n')}, \quad (5a)
$$

$$
\tilde{f}_{(x)h-}^{(n')} = \tilde{f}_{(x)h+}^{(n')}, \quad (5b)
$$

$$
\tilde{f}_{(\theta)h-}^{(n')} = \tilde{f}_{(\theta)h+}^{(n')}, \quad (5c)
$$

$$
\tilde{e}_{h-}^{(n')} + \sum_{i=1}^{H} \tilde{e}_h^{(i)} e^{-in'\theta_h^{(i)}} = \tilde{e}_{h+}^{(n')}. \quad (5d)
$$

Equations [5] show that the difference between the $n^{th}$ component of the flux perturbation vector just ahead of the HRs, $(\tilde{m}_{h-}^{(n')}, \tilde{f}_{(x)h-}^{(n')}, \tilde{f}_{(\theta)h-}^{(n')}, \tilde{e}_{h-}^{(n')})^T$, and the $n^{th}$ component of the flux perturbation vector just after the HRs, $(\tilde{m}_{h+}^{(n')}, \tilde{f}_{(x)h+}^{(n')}, \tilde{f}_{(\theta)h+}^{(n')}, \tilde{e}_{h+}^{(n')})^T$ comes from the mass and energy perturbations from the HRs. The value of this difference (for both the mass and the energy perturbation) equals the $n^{th}$ component of the space Fourier transform of all the HRs over the circumference. As the HR number, $H$, is usually of the same order of magnitude as the maximum circumferential wave number ($N$) that we are interested in, perturbations with different circumferential wave numbers can couple
with each other even though the HR model used is linear. This is because the pressure oscillation driving the HR fluctuations contains contributions from all circumferential modes. It will be seen more clearly in the following sections.

2.3. Eigenvalue systems

Figure 4: Schematic diagram of the eigenvalue system. Waves denoted by red symbols are incident waves and those by black symbols are reflected waves at both ends of the system.

As shown in Fig. 4, in order to fully model the acoustics in the annular duct, we need to combine acoustic treatments for the:

I duct inlet,

II across each set of HRs,

III duct outlet.

At the duct inlet, for an incident upstream travelling acoustic wave, \( \tilde{A}_i^{(n)} \), with circumferential wave number \( n \) and Fourier amplitude \( \lambda^{(n)} \), an acoustic, \( \tilde{A}_i^{(n)} \), an entropy, \( \tilde{E}_i^{(n)} \), and a vorticity wave, \( \tilde{V}_i^{(n)} \), with the same \( n \) may be generated \[30\]. When the inlet acoustic boundary condition is given, an acoustic
reflection coefficient, $R^{(n)}_t$, an entropy reflection coefficient, $R^{(n)}_e$, and a vorticity reflection coefficient, $R^{(n)}_v$, can be used to prescribe the relations between the generated waves and the incident acoustics. The wave vector at the duct inlet can then be written as 

\[ \left( \tilde{A}_i^n, \tilde{A}_i^{-n}, \tilde{E}_i^n, \tilde{V}_i^n \right)^T = \lambda^{(n)} (R^{(n)}_t, 1, R^{(n)}_e, R^{(n)}_v)^T. \] (6)

Across each set of HRs, we need to convert the modal strengths upstream of the HRs to perturbations in flow variables, apply HR models and conservation equations to obtain perturbations in flow variables downstream of the HRs, and then convert these back into sum of modal contributions. Given the strengths of all the modes at the inlet, flow perturbations at the inlet of each HR can be calculated by propagating each of these modes through the sections before the considered HRs and summing their contributions at the inlet of each HR. For example, these flow perturbations at the inlet of the $i$th HRs shown in Fig. 4 are

\[ (\tilde{p}, \tilde{\rho}, \tilde{u}, \tilde{\omega})^{T}_{k-i} = M^{(i)}_{w2p} (\lambda^{-N}, \cdots \lambda^{(0)}, \cdots \lambda^{(N)})^T, \] (7)

where $i = 1, 2, 3$, $M^{(i)}_{w2p}$ is a $4 \times (2N + 1)$ matrix which is a function of only the angular frequency, system geometry, mean flow and thermodynamic parameters.

By substituting Eq. (7) into Eqs. (3) and (4), unsteady mass and energy fluxes from each HR can be obtained. The relation of modal expansions before and after the HRs can then be obtained by substituting the mass and energy results into Eqs. (5). Perturbations at the duct outlet are then obtained by propagating the waves from just downstream of the HRs to the outlet (if there are more HRs at the downstream side, the same procedure can be used again to find the relations across them).

At the duct outlet, an incident acoustic wave, $\tilde{A}_o^{n+}$, an incident entropy wave, $\tilde{E}_o^{(n)}$, and an incident vorticity wave, $\tilde{V}_o^{(n)}$, with any given specific circumferential wave number $n$, may all interact with the outlet to generate reflected acoustic waves with the same $n$ [30]. The relation between the generated acoustic wave and these incident waves can be prescribed by the acoustic reflection coefficients
corresponding to a unit incoming acoustic, entropy and vorticity waves. Thus, for each \( n = -N, \ldots, 0, \ldots, N \), boundary conditions can be written as

\[
R^{(n)}_{(A)} \tilde{A}^{n}_{\alpha} + R^{(n)}_{(E)} \tilde{E}^{n}_{\alpha} + R^{(n)}_{(V)} \tilde{V}^{n}_{\alpha} - \tilde{A}^{n}_{\alpha} = 0, \tag{8}
\]

where \( R^{(n)}_{(A)}, R^{(n)}_{(E)} \), and \( R^{(n)}_{(V)} \) are the acoustic reflection coefficients corresponding to the incident acoustic, entropy and vorticity waves respectively.

Finally, the final eigenvalue system can then be written as

\[
M_{i20} \lambda = 0, \tag{9}
\]

where \( M_{i20} \) is a \((2N + 1) \times (2N + 1)\) matrix, and \( \lambda = (\lambda^{(-N)}, \ldots, \lambda^{(0)}, \ldots, \lambda^{(N)})^T \) the wave strength vector. The multiplication of each row vector in \( M_{i20} \) with the wave strength vector \( \lambda \) gives the outlet boundary condition in Eq. (8) for a specific \( n \).

Because the HR model considered in this paper is linear, \( M_{i20} \) is a function of the angular frequency, HR parameters, boundary conditions, system geometry, mean flow and thermodynamic parameters. Thus, if the system parameters are given, the only unknown in \( M_{i20} \) is the complex angular frequency \( \omega = 2\pi f + i\sigma \) whose real part denotes the eigen-frequency and imaginary part the growth rate of the system modes. A negative \( \sigma \) means stable and a positive \( \sigma \) unstable modes. These complex angular frequencies can be calculated by solving the nonlinear eigenvalue problem, which is equivalent to solving for the determinant of \( M_{i20} \) being zero,

\[
\det(M_{i20}(\omega, g)) = 0, \tag{10}
\]

where \( g \) denotes the system parameters. This is convenient to solve for the present case as the dimension of \( M_{i20} \), \((2N + 1) \times (2N + 1)\), is low. Then each of these complex frequencies gives, with its associated eigenvector, an eigenmode of the system. Modal couplings due to the presence of HRs can be obtained by finding the eigenvectors in Eq. (9) corresponding to these eigenmodes.
3. Optimisation

The aim of this work is to provide an efficient HRs optimisation method applicable at the design stage of the combustor as well as a robust tool in the retrofitting of such devices. The objective is to optimise the acoustic damping of the least stable acoustic modes of the combustor (generally the lowest frequency eigenmodes for thermoacoustics). In order to do so, we must target multiple eigenmodes, even very stable ones, in order to guarantee that modification targeting some modes will not destabilise others. The optimisation uses gradient information, obtained with the adjoint method \[51, 52\], implemented in a quasi-Newton algorithm (commonly referred to as Limited memory -BFGS, or simply L-BFGS) \[53, 54\]. Adjoint information in the field of thermoacoustics has previously been used to provide sensitivity measures of the eigenvalues of the system to different model parameters \[55–62\]. It has been applied to cases where the thermoacoustic modes are given by a linear eigenproblem, for example when the acoustics are modelled using a Galerkin expansion, and to cases where the modes are given by a nonlinear eigenproblem \[58, 60, 62\], for example when the acoustics are modelled by using more powerful low-order network models or Helmholtz solvers. In the present work, we apply gradient based optimisation to a low-order network model for a hot annular duct with multiple HRs attached.

By targeting multiple acoustic modes, we systematically optimise the acoustic damping of these HRs.

The objective is to minimise a scalar cost functional \( J \) which depends on the growth rates of the system, obtained from Eq. \([10]\), which we group into the vector \( \text{Im}(\omega) = [\sigma_1, \sigma_2, \cdots, \sigma_M] \), and on the control variable, such as the physical parameters and locations of each HR, grouped into the vector \( g = [g_1, g_2, \ldots, g_n] \). The formulation is as follows:

\[
J(\text{Im}(\omega), g) = \sum_{i=0}^{M} e^{f(\sigma_i)} + B(g),
\]

where \( B(g) \) is any user-defined function to impose physical constraints on the control parameters (e.g. geometric limitations). The function \( f(\sigma_i) \), is another
user-defined function that can be used to give more weight to certain eigenvalues, or flatten the behaviour of the exponential in order to avoid numerical issues with large numbers (useful in case of large unstable growth rates). The eigenmodes at the initial state are specified a priori and should generally cover the lower frequency range where thermoacoustic oscillations tend to occur. This also requires including the modes that arise from the presence of the dampers.

Including $f(\sigma_i)$ in the exponential term allows us to prioritise the most unstable modes, as their contribution to the summation will be larger. It is important to note that due to our definition of the cost functional, convergence, even to a local minimum, is not guaranteed as the modes could be indefinitely damped, in this case the code needs to be augmented with a suitable stopping criterion such as a user-defined ‘small enough’ value for the growth rates. The cases considered in the present paper all converged to a local minima for which a stopping criterion based on the total derivative of the cost functional is standard practice.

The algorithm uses an L-BFGS method \cite{53, 54}, a second-order algorithm, which works particularly well with ill-conditioned problems for the cost functional and narrow valleys. The method provides an approximation of the Hessian at the $k$th iteration, which is obtained using both function values and gradient information at each iteration. Function evaluations are obtained directly by solving the system, while the gradient evaluations are obtained using the adjoint method. The updated control can be obtained from

$$g_{k+1} = \alpha_k d_k,$$

where $d_k$ is the search direction at the $k$th iteration and $\alpha_k$ is the step length obtained with an Armijo line search with strong Wolfe Powell conditions \cite{53, 54}. We would like to clarify that the step size is kept small, as this allows individual eigenvalues to be tracked with no switching, and ensures each eigenvalue is identified as a separate root even if closely approaching another (without becoming degenerate).

The gradient of the cost functional requires calculations of the sensitivities
of the vector of eigenmodes, $\omega$, to the control, $g$

$$\frac{D\mathcal{J}}{Dg} = \frac{\partial \mathcal{J}}{\partial \omega} \frac{d\omega}{dg} + \frac{\partial \mathcal{J}}{\partial g}.$$  \hspace{1cm} (13)

The adjoint formulation, derived using Lagrange multipliers, is used to obtain the required sensitivities. This has been used previously within a similar framework [58, 60, 61], with the final result being:

$$\frac{\partial \omega_i}{\partial g_j} = -\frac{(\lambda_i^+)^H \frac{\partial M_{2g}}{\partial g} \lambda_i}{(\lambda_i^+)^H \frac{\partial M_{2g}}{\partial \omega} \lambda_i},$$  \hspace{1cm} (14)

where $\lambda_i^+$ is the adjoint variable and $^H$ denotes the complex conjugate transpose.

In the present work the final sensitivities obtained with the adjoints (as shown in Eq. 14) have been validated with finite difference. Note that the derivatives of the matrix $M_{2g}$ are obtained numerically by using an $O(\epsilon)$ perturbation. The issue of large differences in the sensitivities of some search directions (e.g., sensitivities with respect to HR volumes of $\sim 10^{-3} m^3$, compared to longitudinal lengths of $\sim 1 m$ and circumferential locations of $\sim 1 rad$) is remedied by diagonally scaling the problem [63] based on the above typical order of magnitude of the parameters.

At this point some limitations of the current optimisation method are noted. Firstly, Eq. (14) is not valid for exactly degenerate modes. The optimisation therefore should not be started with degenerate modes, and the algorithm terminates if two or more eigenvalues cannot be separated by more than three decimal places in both frequency and growth rate (this can be refined). This implies that if the degenerate configuration is a local minima, the code can converge towards it, without numerically achieving degeneracy. If the degenerate case is an intermediate step, the code would stop without convergence, although we believe this would be extremely rare as the optimisation is unlikely to pass through a degeneracy with such accuracy. Extending the currently work to consider mode degeneracy is left for future work. Readers are referred to [27, 60, 64] for more discussions on considering degenerate modes. Secondly, the
current method is based on tracking of those modes present at the start of the
optimisation: any mode not in the studied region at the start and entering at a
later time would not presently be tracked. It would be possible to insert checks
for additional modes at various points in the optimisation procedure.

4. Results and discussions

An annular duct with a mean radius $\bar{R} = 0.6$ m and a cross sectional
area $S = 0.3$ m$^2$ is now considered as an example to show the capabilities
of the present optimisation method. A uniform mean flow with a mean pressure
$\bar{p} = 4$MPa, mean temperature $\bar{T} = 2000$ K and mean velocity $\bar{u} = 23.9$ m/s is
considered at the duct inlet, and HRs considered in this paper all have a cooled
cavity with 1000 K. Both the duct inlet and outlet acoustic boundary condi-
tions are assumed to be closed. In such an annular geometry, both plane and
circumferential modes can propagate. We consider two different duct lengths;
a very short duct, $L \ll 2\pi\bar{R}$, in which circumferential acoustic modes are dom-
ninant at low frequencies, and a medium length duct, $L \sim 2\pi\bar{R}$, in which both
circumferential and plane waves are important. Note that if $L \gg 2\pi\bar{R}$, the
system eigenmodes with the lowest frequencies would be purely plane modes,
which reduces the present model to the widely considered simpler longitudinal
duct case [29].

4.1. Very short duct

A very short duct length, $L = 0.2$ m, is considered, for which $L/(2\pi\bar{R}) = 0.053 \ll 1$ and so the lowest frequency eigenmodes will correspond to circum-
erential variations. As shown in Table 1 if no HR is included, below 500 Hz
there are two eigenmodes with frequencies $f_1 \approx 223$ Hz, $f_2 \approx 447$ Hz, and growth
rates $\sigma_1 = \sigma_2 = 0$. These are the first ($n = \pm 1$) and second ($n = \pm 2$) purely
circumferential propagating modes and their pressure mode shapes can be seen
in Fig. 5 – their values are constant in the longitudinal direction. Note that these
eigenfrequencies are consistent with acoustic wavelengths of a mean circum-
ference and half a mean circumference respectively by $f_1 = c/(2\pi\bar{R}) \approx 223.4$ Hz
and \( f_2 = c/(\pi \bar{R}) \approx 446.7 \text{ Hz} \) where \( c = 842.08 \text{ ms}^{-1} \). The growth rates are zero because no acoustic source or damping is included.

Table 1: Frequencies and growth rates of eigenmodes below 500 Hz without HRs and with 2 HRs where their cavity volumes are \( V_h^{(1)} = V_h^{(2)} = 1.39 \times 10^{-3} \text{ m}^3 \) and circumferential location of the second HR \( \theta_h^{(2)} = \pi/3 \) – all other HR parameters take fixed values.

<table>
<thead>
<tr>
<th>Without HRs</th>
<th>Frequency (Hz)</th>
<th>Growth rate (s(^{-1}))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>223</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>447</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>With 2 HRs</th>
<th>Mode No.</th>
<th>Frequency (Hz)</th>
<th>Growth rate (s(^{-1}))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>206</td>
<td>-77</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>222</td>
<td>-126</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>224</td>
<td>-69</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>248</td>
<td>-122</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>448</td>
<td>-1.4</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>450</td>
<td>-4.1</td>
</tr>
</tbody>
</table>

Figure 5: A standing wave pressure mode shape of the first (223 Hz, \( n = \pm 1 \) – black solid line) and second (447 Hz, \( n = \pm 2 \) – blue dash line) purely circumferential modes. Note that the modes are degenerate in the circumferential direction – using standing patterns here is only for illustration.

As discussed in [11], with only a single HR attached to the annular duct, even if its resonant frequency is tuned to one of the duct eigenfrequencies (a degenerate azimuthal mode), the HR will not provide any damping to one of the resulted non-degenerate modes – the clockwise and anti-clockwise modal components become coupled to set up a standing wave with a pressure node at the HR location.
We now use two HRs to damp the system eigenmodes. From [11], if both HR resonant frequencies are close to the targeted eigenmode and they are located properly on the duct, they can significantly damp the targeted eigenmode. In annular combustors, thermoacoustic instabilities may occur at different frequencies, and practical limitations mean it is desirable to damp all possible unstable modes with as few HRs as possible. We now apply the present optimisation algorithm to two HRs, targeting both the \( n = \pm 1 \) and \( n = \pm 2 \) modes simultaneously, to show how such optimisation can be achieved.

The two HRs have the same neck length \( l_h^{(1)} = l_h^{(2)} = 0.01 \) m, neck cross-sectional area \( S_h^{(1)} = S_h^{(2)} = 1.54 \times 10^{-4} \) m\(^2\), mean bias flow Mach number \( \bar{M}_h^{(1)} = \bar{M}_h^{(2)} = 8.2 \times 10^{-3} \), mean cavity temperature \( T_h^{(1)} = T_h^{(2)} = 1000 \) K, and longitudinal locations \( x_h^{(1)} = x_h^{(2)} = 0.1 \) m – all of these parameters are fixed. The circumferential location of the first HR is fixed at \( \theta_h^{(1)} = 0 \) while that of the second one, \( \theta_h^{(2)} \), can vary from 0 to \( 2\pi \). Note that due to the periodicity along the circumference, it is the angular difference between the two HRs, rather than the specific longitudinal locations of each of them, that matters [11]. The two HR cavity volumes, \( V_h^{(1)} \) and \( V_h^{(2)} \) can also vary, thus three control variables, \( \theta_h^{(2)} \), \( V_h^{(1)} \) and \( V_h^{(2)} \), are considered. It is worth mentioning that many parameters can be optimised simultaneously – we restrict attention to three in this example because we can then conveniently validate the optimisation via “3-D sweep” of the eigenmodes and the objective functional for different combinations of these three parameters.

As shown in Table [11], attaching 2 HRs splits the two original modes into six modes. This is because on the one hand the HRs themselves introduce extra modes, and on the other hand including the HRs makes the original modes become non-degenerate and thus brings more modes. All the six modes are damped – all the modes have negative growth rates. As both the two HRs are tuned to a resonant frequency close to 223 Hz, the four modes (modes 1–4) close to the first original mode are seen to be largely damped (all have negative growth rates with large absolute values, agreeing with [11]), while the two modes (modes 5–6) near the second original mode are slightly damped – this is mainly
because the sound absorption bandwidth of the HRs is narrow and thus their sound absorption performance near 447 Hz is poor.

4.1.1. Mode solution convergence

The mode convergence with increasing modal solution truncation number, $N$ (see e.g. Eq. (A.1)), is now checked with results shown in Fig. 6. The frequencies and growth rates of modes 1-4 are seen to be well converged for $N \geq 4$, and so $N = 4$ is used for the rest of the calculations in the present paper.

![Figure 6: Frequency and growth rate variations of modes 1-4 in Table 1 with increasing modal solution truncation number, N.](image)

4.1.2. Optimisation

We now perform our optimisation targeting the six modes with 2 HRs. The targeted mode vector in the objective functional, Eq. (11), is thus $\mathbf{ω} = [\omega_1, \omega_2, \cdots, \omega_6]$, and the control variable vector is $\mathbf{g} = [V_h^{(1)}, V_h^{(2)}, \theta_h^{(2)}]$. The optimisation algorithm requires $f$ and $B$ to be prescribed: we choose $f = x^5/50$ to weight the least stable modes and we choose $B$ to restrict the allowable range of the control variables using $B(\mathbf{g}) = \sum_{i=1}^4 e^{-\mathcal{K}(g_i - g_i^{\text{min}})} + e^{\mathcal{K}(g_i - g_i^{\text{max}})}$ where $g_i^{\text{min}}$
and \( g_i^{\text{min}} \) and \( g_i^{\text{max}} \) are the minimum and maximum limit for the \( i^{\text{th}} \) control variable (the two HR volumes are restricted to be within \([0, 10^{-2}] \) m\(^3\) and their azimuthal gap within \([0, 2\pi]\) in the present case) and \( K = 10^6 - \text{round}\left[\log\left((g_i^{\text{min}} + g_i^{\text{max}})/2\right)\right] \) where round denotes rounding to the nearest integer.

The optimisation starts from a random initial guess of the control parameters; \( \mathbf{g}_0 = [14 \times 10^{-4} \text{ m}^3, 4 \times 10^{-4} \text{ m}^3, \pi/3] \) was chosen. Evolution of the control vector \( \mathbf{g} \), the objective functional \( \mathcal{J} \), and the absolute value of the gradient \(|\mathbf{D}\mathcal{J}/\mathbf{Dg}|\) with the number of optimisation iterations are shown in Fig. 7. The objective functional, \( \mathcal{J} \), decreases from \( \sim 10^0 \) to \( \sim 10^{-16} \) over \( \sim 140 \) iterations, and then remains nearly constant. Correspondingly, the absolute value
of the gradient $|\nabla J/Dg|$ tends to decrease with iteration number, tending to zeros beyond $\sim 140$ iterations, meaning that the optimisation has converged. Both the objective functional $J$ and all the control parameters, $g$, remain approximately constant beyond this.

The evolutions of all six modes during the optimisation are shown in Fig. 8. Modes 2 and 5 are initially the two least stable modes (with the highest growth rates). As the number of iterations increases, these two modes are simultaneously damped. The growth rates of both modes 1 and 6 increase. However, the overall objective functional reduces because its form, $\sum_{i=0}^6 e^{(\sigma_i)^2/50}$, weights more strongly contributions from modes with larger growth rates.

![Figure 8: Frequency and growth rate evolutions of modes 1–6 during the optimisation in the very short duct case. The arrows indicate the number of iterations increasing. The top shows all the 6 modes and the bottom shows a zoom close to the zero growth rate axis.](image)

A 3-D sweep of all the modes and the objective functional over the compo-
nents of $g = [V_h^{(1)}, V_h^{(2)}, \theta_h^{(2)}]$ is also performed for validation. This is done by discretizing $g_i$ ($i = 1$, 2, 3), between minimum ($g_{i,\text{min}}$) and maximum ($g_{i,\text{max}}$) limits, into $X$ segments and calculating the frequencies and growth rates of the six modes across each combination of the three control parameters. This means solving Eq. (10) over the relevant frequency region $X^3$ times. By doing this full sweep, a local minimum near $[V_h^{(1)} = V_h^{(2)} = 8 \times 10^{-4} \, \text{m}^3, \theta_h^{(2)} = 0.28\pi]$ with a maximum growth rate $\sigma \approx -4.5 \, \text{s}^{-1}$ was found. These HR volumes, the circumferential spacing, and the maximum growth rate agree with optimisation findings, validating it. Note that in the present optimisation, any initial guess which is not too far away from the current local minimum or is not closer to any other local minima will give a convergence to the same local minimum. It should also be noted that this full sweep already involves heavy calculation (for a normal PC) with three control variables. In some cases, some modes approach one another and resolving them is particularly computationally heavy. The full sweep method becomes impractical for larger numbers of control parameters. The optimisation method, however, takes tens of seconds on a general PC, and could be readily extended to larger number of control parameters.

4.2. Medium length duct

We now increase the duct length from $L = 0.2 \, \text{m}$ to $L = 1.6 \, \text{m}$, while keeping all other geometry and mean flow parameters the same as for the very short duct case. Now $L/(2\pi \bar{R}) \approx 0.42$, meaning that the duct length is of the same order as its circumference which is widely encountered in a real annular combustor system [3].

Now, for the isolated duct without any HRs attached, in addition to the two purely circumferential modes (the 223 Hz and 447 Hz modes in Table 1) discussed in the previous section, two extra modes, $(0 \, \text{s}^{-1}, 263 \, \text{Hz})$ and $(0 \, \text{s}^{-1}, 345 \, \text{Hz})$, exist below 500 Hz. The 223 Hz and 447 Hz modes remain purely circumferential modes and do not involve amplitude variation in the longitudinal direction. The 263 Hz mode is the first purely longitudinal mode ($n = 0$) and involves variation only in the longitudinal direction. The 345 Hz mode is slightly more
complicated – it arises from waves which propagate both longitudinally and circumferentially, with its pressure mode shape given in Fig. 9. This mode shape exhibits a $2\pi$ phase variation circumferentially and a $\pi$ phase variation longitudinally, and is known as a mixed mode. Thus the medium length duct exhibits a much richer diversity of spatial mode patterns.

Figure 9: Pressure mode shape of the first ($n = \pm 1$) mixed mode at 345 Hz (arbitrary scale). Note that the mode is degenerate in the circumferential direction – using a standing pattern here is only for illustration.

We now consider two HRs attached to the duct and perform an optimisation to maximise acoustic damping over the first three modes, $(0 \text{ s}^{-1}, 223 \text{ Hz})$, $(0 \text{ s}^{-1}, 263 \text{ Hz})$ and $(0 \text{ s}^{-1}, 345 \text{ Hz})$. We are now targeting one purely circumferential mode at $(0 \text{ s}^{-1}, 223 \text{ Hz})$, one purely longitudinal mode at $(0 \text{ s}^{-1}, 263 \text{ Hz})$, and one mixed mode at $(0 \text{ s}^{-1}, 345 \text{ Hz})$. Since both the longitudinal and mixed mode involve a longitudinal pressure amplitude variation, the longitudinal locations of the HRs will significantly affect their damping performance. In order to balance freedom to optimise the longitudinal location with simplicity, the two HRs are assumed to have the same longitudinal location, $x_h$, which now become a control parameter in $g$. From [11], a $\pi/2$ angular spacing between two HRs gives best damping to $n = \pm 1$ circumferential modes, and so we choose $\theta_h^{(1)} = 0$ and $\theta_h^{(2)} = \pi/2$, which are fixed. Thus the 3 control parameters are now, $g = [V_h^{(1)}, V_h^{(2)}, x_h]$. 

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Figure 10: Frequency and growth rate evolutions of modes 1–7 during the optimisation in the medium length duct case. The arrows indicate the number of iterations increasing. The top shows the 5 modes close to the zero growth rate axis and the bottom shows the two most stable modes.
We start the optimisation from a random initial guess of $g$, in this case chosen to be $g_0 = [14 \times 10^{-4} \text{ m}^3, 7 \times 10^{-4} \text{ m}^3, 0.3 \text{ m}]$. The attachment of the two HRs leads to 7 modes existing below 400 Hz. Their initial frequencies and growth rates, and their evolutions during the optimisation procedure are shown in Fig. 10. The main result is that both mode 1 and mode 5 tend to be destabilised, while modes which are originally most unstable (modes 2 and 6) tend to be stabilized. The remaining three modes have such negative growth rates that they do not affect the optimisation. This means the optimisation is automatically targeting the most unstable modes and reducing their growth rates by searching for locally optimised combinations of the three control parameters.

![Graphs](image)

Figure 11: Medium length duct case. (Top) Evolution of the components of the control vector $g$, (middle) the objective functional $\mathcal{J}$, and (bottom) the absolute value of the gradient $|D\mathcal{J}/Dg|$ with the optimisation iteration steps.
This optimisation performance is illustrated via the objective functional, $J$, shown in Fig. 11. Beyond $\sim 380$ iterations, the absolute value of the gradient $|D\mathcal{J}/Dg|$ is very small, and both the control $g$ and the objective functional $\mathcal{J}$ tend to constants, meaning that a local minimum has been obtained. A full 3-D sweep confirms a local minimum near $[V_h^{(1)} = V_h^{(2)} = 1 \times 10^{-3} \text{ m}^3, x_h = 0 \text{ m}]$ with a maximum growth rate $\sigma \approx -3.4 \text{ s}^{-1}$ – agreeing with both the optimised final control parameters shown in Fig. 11 (top), and the final maximum growth rate shown in Fig. 10.

5. Conclusions

The present work has considered a linear, low order modelling framework for acoustic wave propagation in thin, hot, annular ducts. It has incorporated the attachment of multiple Helmholtz resonators (HRs) to the annular duct, representing the HRs using Rayleigh conductivity models. These are known to physically capture the resonance and damping effects for HRs sustaining a mean bias flow through the neck. Then, in order to systematically choose the geometries, mean bias flow parameters and placements of multiple HRs in order to achieve good acoustic damping, an optimisation framework is developed, based on the gradient derived from adjoint sensitivity analysis. An objective functional is used which includes the growth rates of all eigenmodes in a given frequency range, to ensure good damping across all possible modes. The HR geometries, mean bias flow parameters and their longitudinal and circumferential locations are taken as control parameters and are simultaneously optimised. A very short duct sustaining modes which propagate purely circumferentially at low-frequencies, and a medium-length duct sustaining co-existing modes which can propagate purely circumferentially, purely longitudinally and mixed are studied in detail, with optimisation of 2 attached HRs and 3 control parameters performed. Optimisation procedures are validated via a full multi-parameter sweep of the objective functional. These studies illustrate the capability of the present optimisation method. Even though they consider three optimisation pa-
rameters, the adjoint method can target many eigenmodes and optimise many HR parameters simultaneously, at little computational cost. The modelling and optimisation framework can be incorporated, without changes, as a module within a more complete thermoacoustic network model, including flame models and area changes.

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Appendices

A. Linear flow perturbations and transfer matrices

For a thin annular duct with a uniform mean flow in the longitudinal direction, as shown in Fig. 2, radial variations are neglected. Perturbations due to the acoustics can be represented as an infinite expansion of modal solutions for Eq. (1a), each mode \( n \) representing a circumferential wave number which is an integer due to the circumferentially periodic boundary condition. For an angular frequency \( \omega \), and truncating at mode number \( N \), the solution can be written as

\[
p_{A}^{\prime}(x, \theta, t) = \text{Re}[\tilde{p}_{A}(x, \theta)e^{-i\omega t}], \quad (A.1a)
\]

\[
\tilde{p}_{A}(x, \theta) = \sum_{n=-N}^{N} \tilde{p}_{A}^{(n)}(x, \theta), \quad (A.1b)
\]

where \( \text{Re}[\ ] \) denotes the real part, \( \tilde{[\ ]} \) denotes Fourier amplitude and \( [\ ]_{A} \) denotes that the perturbation is due to acoustic waves. \( \tilde{p}_{A}^{(n)} \) is the combination of
downstream and upstream propagating contributions:

\[
\tilde{p}^{(n)}_A = \tilde{A}^+ e^{in\theta + ik^nx} + \tilde{A}^- e^{in\theta + ik^-x}, \quad (A.2)
\]

where \(\tilde{A}^\pm\) are the amplitudes of the downstream and upstream propagating waves, \(k^\pm\) are the longitudinal wave numbers given by

\[
k^\pm = -\bar{M}k \pm \sqrt{(k^2 - n^2(1 - \bar{M}^2)/\bar{R}^2)} / (1 - \bar{M}^2), \quad (A.3)
\]

where \(k = \omega/\bar{c}, \bar{M} = \bar{u}/\bar{c}\) is the longitudinal mean flow Mach number and \(\bar{R}\) is the mean radius of the annular duct. The acoustic component of the density perturbation and the longitudinal and circumferential velocity perturbations can be similarly written as follows, where \(\alpha = \omega + \bar{u}k\).

\[
\tilde{\rho}^{(n)}_A = \frac{1}{c^2} \tilde{A}^+ e^{in\theta + ik^nx} + \frac{1}{c^2} \tilde{A}^- e^{in\theta + ik^-x}, \quad (A.4a)
\]

\[
\tilde{u}^{(n)}_A = -\frac{k^+}{\bar{\rho}\alpha} \tilde{A}^+ e^{in\theta + ik^nx} - \frac{k^-}{\bar{\rho}\alpha} \tilde{A}^- e^{in\theta + ik^-x}, \quad (A.4b)
\]

\[
\tilde{w}^{(n)}_A = -\frac{n}{\bar{\rho}\alpha} \tilde{A}^+ e^{in\theta + ik^nx} - \frac{n}{\bar{\rho}\alpha} \tilde{A}^- e^{in\theta + ik^-x}. \quad (A.4c)
\]

The entropy perturbation can be obtained from the modal solution of the convected wave equation Eq. (1b). Perturbations due to the entropy wave, denoted \(\tilde{[E]}\), are incompressible and irrotational, and can be prescribed in the form of only density fluctuations (\(\tilde{\rho}_E = \tilde{\rho}_V = \tilde{\tilde{E}}^E = 0\)). Representing this as a sum of components with different \(n\), the \(n\)th component can be written as

\[
\tilde{\rho}^{(n)}_E = -\frac{1}{c^2} \tilde{\tilde{E}}^{(n)} e^{in\theta + ik_0x}, \quad (A.5)
\]

with \(k_0 = \omega/\bar{u}, \tilde{\tilde{E}}^{(n)} = \bar{\rho}\bar{T} (\gamma - 1) \tilde{s}^{(n)}\) and \(\gamma\) the heat capacity ratio.

As the oscillations are two dimensional, a vorticity disturbance in the radial direction, denoted \([V]\), may exist. This can be obtained from the modal solution of Eq. (1c). This vorticity does not contribute to pressure or density oscillations (\(\tilde{\rho}_V = \tilde{\rho}_V = \tilde{\tilde{E}}^V = 0\)) and is related to the velocity oscillations via

\[
\tilde{\xi} = (\partial \tilde{\bar{w}}_V/\partial x - \partial \tilde{\bar{u}}_V/\partial \theta) \bar{e}_r, \quad \bar{e}_r\text{ is a unit vector in the radial direction.}
\]

By combining this with the mass conservation equation and representing...
the vorticity disturbance as the sum of different circumferential components, contributions of the $n^{th}$ component can then be written as

\[ \tilde{u}_{V}^{(n)} = \frac{n}{\bar{\rho}c} \tilde{V}^{(n)} e^{i k_0 x + i n \theta}, \]  
(A.6a)

\[ \tilde{w}_{V}^{(n)} = -\frac{k_0 \bar{R}}{\bar{\rho}c} \tilde{V}^{(n)} e^{i k_0 x + i n \theta}. \]  
(A.6b)

where $\tilde{V}^{(n)} = i \bar{\rho} \bar{c} \bar{R} \cdot \tilde{\xi}^{(n)}/(n^2 + k_0^2 \bar{R}^2)$.

Then by considering the overall flow perturbations, the wave-to-flow and flow-to-flux transfer matrices defined in Section 2.1 with circumferential wavenumber $n$ are

\[
M_{W2F}^{(n)} = \begin{pmatrix}
1 & 1 & 0 & 0 \\
1/\bar{c}^2 & 1/\bar{c}^2 & -1/\bar{c}^2 & 0 \\
-k^n/\bar{\rho}c^{n+} & -k^n/\bar{\rho}c^{n-} & 0 & n/\bar{\rho}c \\
-n/\bar{R}k^n/\bar{\rho}c^{n+} & -n/\bar{R}k^n/\bar{\rho}c^{n-} & 0 & -k_0 \bar{R}/(\bar{\rho}c)
\end{pmatrix} 
\]  
(A.7)

\[
M_{F2J}^{(n)} = S \begin{pmatrix}
0 & \bar{u} & \bar{\rho} & 0 \\
1 & \bar{u}^2 & 2\bar{\rho}\bar{u} & 0 \\
0 & 0 & 0 & \bar{R}\bar{\rho}\bar{u} \\
\frac{\gamma \bar{u}}{\gamma - 1} & \frac{\bar{u}^3}{2} & \left(\frac{\gamma \bar{\rho}}{\gamma - 1} + \frac{3\bar{\rho}\bar{u}^2}{2}\right) & 0
\end{pmatrix} 
\]  
(A.8)

References


