Instability and receptivity of subsonic flow in the boundary layer

A thesis presented for the degree of Doctor of Philosophy of Imperial College and the Diploma of Imperial College by

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Declaration of Originality

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To my parents, Herbert and Klaudia.
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Abstract

In this thesis, the main focus is on the interaction of an incoming Tollmien–Schlichting wave with an isolated, stationary wall roughness in subsonic flow. In Part I, this problem is analysed by means of the Triple Deck theory. The linearised sublayer equations are solved under the assumption that the horizontal extent of the roughness is of $O(L \text{Re}^{-3/8})$ and that its height $h$ is small, and an expression for the pressure perturbation is found. The transmission coefficient $T_I$, defined as the amplitude of the T–S wave downstream of the roughness divided by its initial amplitude, is then calculated, where $|T_I| > 1$ means that the wave is amplified and $|T_I| < 1$ represents an attenuation of the T–S wave. The transmission coefficient is dependent on the frequency $\omega$, the height $h$ of the roughness and on the Fourier transform of the roughness shape evaluated at zero value of the wavenumber. The same setup is investigated in Part II through numerical calculations: a DNS solver provides the base flows for 25 different gaps of varying width and height, which are then run through a PSE analysis to examine the stability of the flow. From the results of both methods it can be concluded that a surface indentation amplifies an incoming T–S wave, and that the amplification increases with the width and depth of the roughness. An additional geometry is studied in Part I by again employing the Triple Deck theory to investigate the effect small elastic vibrations of a semi-infinite plate attached to a stationary plate have on the boundary layer, and the receptivity coefficient is calculated for varying $\omega$. 
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Introduction

To make commercial air traffic more sustainable by reducing CO$_2$ emissions, aircraft manufacturers are working to develop more fuel-efficient aircraft. The key idea is to reduce the aerodynamic drag on the aircraft, which is partly comprised of the skin friction drag arising from the frictional forces between the aircraft wings and the airflow around them. One way to reduce the drag due to these frictional forces is to delay the onset of turbulence in the boundary layer, the so-called transition of laminar to turbulent flow. This topic is not only relevant for the development of aircraft, but it is currently one of the most active areas of research in fluid dynamics and can be applied to a variety of moving objects such as planes, gliders, rockets or racing cars. The main goal is to be able to predict the location where this transition happens and to identify the factors that play a significant role in triggering it. These insights make it possible to develop more advanced methods to delay transition, resulting in a decrease in drag and consequently also in fuel consumption.

Transition to turbulence is a subject of interest for both scientists and engineers. There are three main approaches to studying the topic: theoretical analysis, numerical simulations and experimental methods. The two former will be discussed in this thesis. In Part I a theoretical point of view is adopted, where various fundamental concepts in fluid dynamics are examined and later employed to determine the receptivity of the boundary layer to external disturbances. Part II tackles the subject in a more practical way, making use of numerical methods to study the behaviour of fluid flow over a range of roughness elements of various widths and heights to then draw comparisons between the transition locations for the different geometries.
Part I

Theoretical analysis
Chapter 1

Mathematical background

1.1 Navier–Stokes equations

The majority of fluid flows encountered in real life (e.g. air and water) are governed by the Navier–Stokes equations, named after Louis Navier [1823] and George G. Stokes [1845]. Drawing on previous work, most notably by Newton [1729], Euler [1757] and Laplace [1816], these form the first set of equations adequately describing fluid flow, incorporating both inviscid and viscous forces. Subsonic flows, such as the flows considered in this thesis, are treated as incompressible flows, so that their density $\rho$ is constant. For this type of flow, the Mach number $M$ is assumed to be much smaller than 1:

$$M = \frac{U}{c} \ll 1, \quad (1.1)$$

where $U$ is the speed of the object or flow and $c$ is the speed of sound, see e.g. Young et al. [2010]. There are other types of flow for higher Mach numbers, such as transonic ($0.8 < M < 1.2$), supersonic ($M > 1$) or hypersonic ($M \gg 1$) flow.

The Navier–Stokes momentum equations for incompressible flow are written as

$$\frac{\partial \hat{u}}{\partial t} + (\hat{u} \cdot \nabla) \hat{u} = -\frac{1}{\rho} \nabla \hat{p} + \nu \nabla^2 \hat{u} + \hat{f}. \quad (1.2a)$$

The Cartesian coordinate system $(\hat{x}, \hat{y}, \hat{z})$ is used, where the hat denotes dimensional variables and $\hat{t}$ is the time. The vector $\hat{u} = (\hat{u}, \hat{v}, \hat{w})$ is the velocity vector, $\hat{p}$ denotes
the pressure, $\nu$ the kinematic viscosity and $\hat{f}$ represents body forces such as gravity or centrifugal forces. Note that in the following work, these are disregarded, so $\hat{f} = 0$. In addition, the principle of conservation of mass yields the continuity equation for an incompressible flow,

$$\nabla \cdot \hat{\mathbf{u}} = 0. \quad (1.2b)$$

In this thesis, the attention is restricted to two-dimensional flows. The system of equations (1.2) should be solved with the no-slip boundary condition at the surface $S$:

$$\hat{\mathbf{u}} = \hat{\mathbf{u}}_{\text{surf}} \quad \text{on} \quad S, \quad (1.3a)$$

and the free-stream boundary condition, for which an oncoming uniform, uni-directional flow in the $\hat{x}$-direction will be considered throughout this thesis,

$$(\hat{u}, \hat{v}, \hat{p}) \rightarrow (U_\infty, 0, p_\infty) \quad \text{as} \quad \hat{x}^2 + \hat{y}^2 \rightarrow \infty, \quad (1.3b)$$

where $U_\infty$ is the modulus of the free-stream velocity vector and $p_\infty$ denotes the free-stream pressure.

1.1.1 Non-dimensionalisation

It is beneficial to simplify notation by converting (1.2) and (1.3) into dimensionless form. Introduce the characteristic length scale $L$, which will be discussed in more detail in Section 1.2. Then the non-dimensional independent and dependent variables are introduced through the relations

$$\begin{align*}
\hat{x} &= L \times, \\
\hat{y} &= L \times, \\
\hat{t} &= \frac{L}{U_\infty} t, \\
\hat{\mathbf{u}} &= U_\infty u, \\
\hat{v} &= U_\infty v, \\
\hat{p} &= p_\infty + \rho U_\infty^2 p.
\end{align*} \quad (1.4)$$
Substituting these into (1.2) yields the dimensionless Navier–Stokes equations

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right),
\]

\[(1.5a)\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{Re} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right),
\]

\[(1.5b)\]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.
\]

\[(1.5c)\]

and the corresponding boundary conditions from (1.3)

\[
u = u_{surf} \text{ on } S,
\]

\[(1.6a)\]

\[
(u, v, p) \to (1, 0, 0) \text{ as } x^2 + y^2 \to \infty,
\]

\[(1.6b)\]

where the Reynolds number Re is the dimensionless parameter defined by

\[
Re = \frac{U_\infty L}{\nu}.
\]

\[(1.7)\]

The Reynolds number represents the ratio of the inertial to viscous forces (see e.g. Batchelor [2000]). There are two special cases to consider. If the Reynolds number is small, Re \( \ll 1 \), the viscous terms are significant enough to balance changes in pressure so that the inertial terms of the momentum balance in the Navier–Stokes equations become negligible to leading order and the Stokes equations are obtained. These are used to describe creeping flows such as the swimming of microorganisms or the flow of lava, and are discussed in more detail in e.g. Kirby [2010]. If the Reynolds number is large,

\[
Re \gg 1,
\]

\[(1.8)\]

which is the key assumption from now on and is the case for aerospace flows past wings and aerofoils, the viscous terms appear to be small compared to the other terms in the momentum equations and (1.2) is reduced to the well-known Euler equations,

\[
\frac{\partial \hat{u}}{\partial t} + (\hat{u} \cdot \nabla) \hat{u} = -\frac{1}{\rho} \nabla \hat{p},
\]

\[(1.9)\]
which describe inviscid flow. It is worth noting that, due to the absence of the second-order partial derivatives, these are first-order equations whereas the Navier–Stokes equations are second-order, which means that the boundary condition should be relaxed. By taking the large-Reynolds number limit, the viscous terms are neglected, so it is the no-slip condition on the surface that can no longer be met. Therefore it can be concluded that the inviscid limit may only capture the essential physics, away from the surface, but fails to describe the entire flow field. An example to illustrate this inconsistency is fluid flow past a cylinder, for which theoretical and experimental results do not agree, as is explained in Ruban & Gajjar [2014]. While the inviscid flow theory predicts a vertically symmetric streamline pattern around the cylinder, experimental observations show a flow that detaches from the surface at the back of the cylinder leading to separated flow and a region of re-circulating flow can be observed. Furthermore, the inviscid analysis predicts zero drag force, which is in direct contradiction to experimental observations – this phenomenon is termed Alembert’s paradox, after d’Alembert [1743].

1.2 Boundary Layer theory

In 1904, Ludwig Prandtl realised that viscosity becomes important for large-Reynolds number flows in a thin region adjacent to the body surface. It is called the boundary layer, where the no-slip condition holds on the surface (see Prandtl [1904]). Since the flow sufficiently far away from the surface is inviscid and not affected by friction to leading order, the flow can be divided into two distinct regions, each characterised by different governing equations. To ensure continuity, the velocity at the outer edge of the boundary layer must match the velocity at the bottom of the inviscid region, also called the slip velocity \( u_{\text{slip}} = (u_S, 0) \). The mathematical framework describing these two distinguished limits and their subsequent matching is termed matched asymptotic expansions. This is used in a multitude of disciplines, but it was Prandtl’s boundary layer theory that inspired its development.

Having established that the Euler equations hold in the inviscid region, consider now the solution in the boundary layer. Since the slip velocity \( u_S \) at the bottom of the inviscid region is non-zero, but the no-slip condition at the surface requires the
longitudinal velocity \( u \) to be zero, there is a sharp drop in \( u \) across the boundary layer. This implies that the derivatives \( \partial u / \partial y \) and \( \partial^2 u / \partial y^2 \) must be large in this region and therefore the second viscous term on the right hand side of the \( x \)-momentum equation (1.5a) must be retained despite the small coefficient \( \text{Re}^{-1} \) in front of it. To determine how thin the boundary layer is, consider the following order-of-magnitude analysis. In the boundary layer, the horizontal coordinate is \( x = \mathcal{O}(1) \), and the longitudinal velocity component is \( u = \mathcal{O}(1) \). Assuming that the boundary layer is thin, subject to subsequent confirmation, introduce the new variables \( \delta_1, \delta_2 \) to determine the orders of magnitude for the vertical coordinate \( y \) and the transverse velocity \( v \):

\[
y \sim \delta_1, \quad v \sim \delta_2 \quad \text{such that} \quad \delta_1, \delta_2 \ll 1.
\]

Then balancing the continuity equation (1.5c) yields

\[
\frac{\partial v}{\partial y} \sim \frac{\partial u}{\partial x}, \quad \text{so} \quad \frac{\delta_2}{\delta_1} \sim \mathcal{O}(1) \quad \text{and thus} \quad \delta_1 \sim \delta_2 \sim \delta, \quad \delta \ll 1.
\]

Moreover, the first two terms in the \( x \)-momentum equation (1.5a) are both of \( \mathcal{O}(1) \),

\[
u \frac{\partial u}{\partial x} \sim v \frac{\partial u}{\partial y} \sim \mathcal{O}(1),
\]

whereas the diffusion terms on the right-hand side are of different magnitudes:

\[
\text{Re}^{-1} \frac{\partial^2 u}{\partial x^2} \sim \text{Re}^{-1} \quad \text{and} \quad \text{Re}^{-1} \frac{\partial^2 u}{\partial y^2} \sim \frac{\text{Re}^{-1}}{\delta^2}.
\]

Noting that, for \( \delta \ll 1 \), the term on the left is much smaller than the term on the right, and applying the principle of least degeneration, the latter is chosen to be of \( \mathcal{O}(1) \), too. It can then be concluded that

\[
\frac{\text{Re}^{-1}}{\delta^2} \sim \mathcal{O}(1) \quad \text{and thus} \quad \delta \sim \text{Re}^{-1/2} \quad \text{for} \quad \text{Re} \gg 1, \quad (1.10)
\]

This agrees with the initial assumption that the boundary layer is thin, \( \delta \ll 1 \). These scalings also imply that

\[
\frac{\partial u}{\partial x} / \frac{\partial u}{\partial y} \sim \text{Re}^{-1/2}, \quad (1.11a)
\]
so that the rate of change of $u$ in the $x$-direction is much slower than in the $y$-direction. This indicates that the majority of boundary layer flows, as most flows in nature, are non-parallel. (In parallel flows, $v = 0$ and $u$ is a function of the transverse coordinate only.) It should be noted however, that the lateral velocity component $v$ is much smaller than the longitudinal one, namely

$$\frac{v}{u} \sim \text{Re}^{-1/2}. \quad (1.11b)$$

A flow exhibiting these two characteristics is referred to as *nearly parallel* or *weakly non-parallel*. Furthermore, by inspecting the magnitude of the terms in the $y$-momentum equation (1.5b), it can be concluded that the pressure does not vary across the boundary layer, since

$$\frac{\partial p}{\partial y} = 0 \quad \text{to the leading order.} \quad (1.12)$$

The pressure can instead be expressed in terms of the slip velocity by introducing the new vertical variable

$$Y = \text{Re}^{1/2} y = \mathcal{O} (1) \quad (1.13)$$

and using the matching condition between the boundary layer and the inviscid flow. Then, expanding the solution in the boundary layer as

$$(u, v, p) = \left( U(x, Y), \text{Re}^{-1/2} V(x, Y), P(x) \right), \quad (1.14)$$

yields, through substitution into (1.5), the classical boundary layer equations,

$$U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial Y} = u_s(x) u'_s(x) + \frac{\partial^2 U}{\partial Y^2}, \quad (1.15a)$$

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \quad (1.15b)$$

subject to an initial condition

$$U = F(Y) \quad \text{at} \quad x = x_0, \quad Y \geq 0, \quad (1.15c)$$

the no-slip condition on the surface

$$U = V = 0 \quad \text{on} \quad S, \quad x > 0, \quad (1.15d)$$
and the matching condition with the inviscid slip velocity at the edge of the boundary layer

\[ U \to u_s(x) \quad \text{as} \quad Y \to \infty. \]  

This is a parabolic boundary-value problem which means that the flow is not affected by what happens downstream. It can therefore easily be solved numerically using a marching algorithm such as Crank & Nicolson [1947].

### 1.2.1 The Blasius boundary layer

To illustrate the importance of classical boundary layer theory and its physical significance, the special case of a uniform flow past a thin flat plate is laid out in the following. This problem is termed the Blasius boundary layer, after Prandtl’s student Paul R. H. Blasius [1908]. It forms the base flow for the different geometries considered in this part of the thesis, as it is a simplification of the flow over an aerofoil. In this case, the slip-velocity at the bottom of the inviscid layer is \( u_s(x) = 1 \), and so the pressure term in (1.15a) vanishes. For this particular problem, there exist a self-similar solution with the similarity variable

\[ \eta = \sqrt{\frac{U_\infty}{\nu}} \frac{y}{x} = \sqrt{\frac{U_\infty L}{\nu}} \frac{y}{\sqrt{x}} = \frac{Y}{\sqrt{x}}, \]  

where \( y = \text{Re}^{-1/2} Y \) from before. Moreover, the velocity components can be expressed through the function \( \varphi(\eta) \), and its derivatives \( \varphi' \), as

\[ U = \varphi' \quad \text{and} \quad V = \frac{\eta \varphi' - \varphi}{2\sqrt{x}}, \]  

where \( \varphi \) is governed by the Blasius equation

\[ \varphi''' + \frac{1}{2} \varphi \varphi'' = 0, \]  

with the boundary conditions

\[ \varphi(0) = \varphi'(0) = 0, \quad \varphi'(\infty) = 1. \]
Figure 1.1: Blasius flow velocity profile

The Blasius equation (1.18) can be solved numerically and Figure 1.1 shows that $\varphi(\eta)$ is a monotonically increasing function. The asymptotic behaviour of $\varphi$ is given by

$$
\varphi(\eta) = \frac{\lambda}{2} \eta^2 + \ldots \quad \text{as} \quad \eta \to 0, \quad (1.19a)
$$

$$
\varphi(\eta) = \eta - C + \ldots \quad \text{as} \quad \eta \to \infty, \quad (1.19b)
$$

where $\lambda = 0.332057$ is the undisturbed shear coefficient and $C = 1.7208$.

### 1.2.2 The displacement effect

It is important to note that both velocity components are not only dependent on $Y$, but also on $x$. This is expected, as boundary layer flows are non-parallel. To investigate the flow behaviour, visualise the velocity field through streamlines, tangent to the velocity vector $\mathbf{u}$. Then the presence of a vertical velocity in the boundary layer is associated with the displacement of the streamline $S$ from the plate surface. This physical phenomenon is termed the displacement effect and can be explained by the
mass conversion law, which requires that the fluid flux between the two points $A_1B_1$ must be the same as that between $A_2B_2$, as illustrated in Figure 1.2. Since the streamwise velocity $U = \varphi'$ decreases as $x$ increases (because $\varphi$ is a monotonically increasing function), the mass between the two locations can only be conserved if the distance $A_1B_1 < A_2B_2$. It can thus be concluded that the flow deceleration due to the viscous effects in the boundary layer leads to a displacement of the streamline $S$, forming the so-called effective body where the slip-velocity holds.

![Figure 1.2: Displacement of the streamline $S$](image)

To measure the height of the effective body at a point $x$, it is necessary to calculate the displacement thickness of the boundary layer,

$$
\delta^*(x) = \int_0^\infty (1 - U) \, dY.
$$

(1.20)

whose derivative determines the slope of the streamlines,

$$
\frac{d\delta^*}{dx} = \lim_{\hat{y} \to \infty} \frac{v}{u}.
$$

(1.21)

For the Blasius flow, the latter is simply equal to the transpiration velocity

$$
\frac{d\delta^*}{dx} = \text{Re}^{-1/2} \lim_{\eta \to \infty} V = \frac{C}{2} \sqrt{\frac{\nu}{U_\infty x}}, \quad \text{so} \quad \delta^* = C \sqrt{\frac{\nu x}{U_\infty}}.
$$

(1.22)

Having looked at the general concepts of boundary layer flows, in particular the Blasius boundary layer, the next step is to determine the stability of these flows.
1.3 Hydrodynamic Stability Theory

As mentioned in the Introduction, there are two different types of viscous flow over a surface such as an aeroplane wing: laminar flow, where the fluid moves in a smooth, ordered way over the surface, and turbulent flow, which is chaotic and three-dimensional. Both are illustrated in Figure 1.3 for a Hagen–Poiseuille flow (see e.g. Ruban & Gajjar [2014] for more details). Transition takes place when the Reynolds number reaches a critical value and the flow becomes unstable – this means that the perturbations that arise due to internal or external disturbances grow with time. To study the stability of the flow, the flow field is decomposed by introducing the perturbations $u'$ and $p'$ such that

$$u = U_0 + u' \quad \text{and} \quad p = P_0 + p',$$  \hspace{1cm} (1.23)

where $(U_0, P_0)$ describe the velocity and pressure components of the base flow.

![Laminar flow](image1.png)  
Laminar flow

![Turbulent flow](image2.png)  
Turbulent flow

Figure 1.3: Laminar-turbulent transition in the Hagen–Poiseuille flow in a circular tube (Van Dyke [1982])

It is clear that as the flow transitions from being laminar to turbulent, the fluid motion becomes significantly more complicated, which results in a larger shear stress and causes a significant increase in the surface drag. Therefore, the main goal in many practical applications is to maintain laminar flow as long as possible. It is thus important to study the stability of flows in more detail, in particular the process of laminar-turbulent
transition. The objective of Hydrodynamic Stability theory is to understand how and when instabilities arise and to identify the physical processes that play a role in destabilising the flow.

Osborne Reynolds [1884], after whom the Reynolds number is named, was the first to investigate the laminar-turbulent transition process experimentally by looking at the Hagen–Poiseuille flow in a circular tube. He was able to observe laminar flow for a sufficiently small fluid velocity and reported that, as the velocity was increased in small steps, the flow would suddenly develop unsteadiness. He also identified the dimensionless parameter now called the Reynolds number, for which there exists a critical value $Re_{crit}$ where this unsteadiness first arises. Following Reynolds’ discovery, various other flows were investigated. Burgers [1924] first observed transition in the boundary layer flow on a flat plate, and it was studied further by Dryden [1947], Schubauer & Skramstad [1948] and Klebanoff & Tidstrom [1959]. Their experiments revealed that near the leading edge of the plate, the flow is always laminar and may be described by the Blasius solution. However, at a certain distance from the leading edge, the flow behaviour changes and an unsteadiness starts to develop (see point $\hat{x}_c$ in Figure 1.4). This disturbance appears in the form of waves that are superimposed on the Blasius flow, termed Tollmien–Schlichting waves after Prandtl’s students Walter Tollmien and Hermann Schlichting who studied the subject theoretically (see Tollmien [1929, 1935] and Schlichting [1933a, b]). At first, the amplitude of these T–S waves is too small to provoke transition and the boundary layer stays laminar, but as they travel downstream, their amplitude increases, triggering nonlinear effects and secondary instabilities that cause the boundary layer to become turbulent (point $\hat{x}_t$).

![Image](image.png)

Figure 1.4: Laminar-turbulent transition in the boundary layer for a flow over a flat plate
Important factors to consider in the laminar-turbulent transition process are: external disturbances, such as free-stream turbulence and acoustic noise; wall roughnesses; and many others. The characteristics of these external disturbances influence how transition occurs. If the initial disturbance level is low, the process described above takes place: the perturbations penetrate the boundary layer where they excite T–S waves whose amplitude will grow in the streamwise direction until transition to turbulence occurs. However, if the initial disturbance levels are high enough, transition can happen straightaway, without the excitation and amplification of T–S waves. This process is referred to as bypass transition.

Throughout this thesis, any initial disturbances to the flow are assumed to be small. This means that some nonlinear terms can be neglected, which significantly simplifies the governing equations that need to be solved. This approach offers a deeper, albeit limited insight into the initial development of the instability modes. The focus in this research is on the early stages of transition, namely the linear and nonlinear growth of instabilities in the boundary layer.

From now on, there will be a distinction made between the global Reynolds number that has been discussed so far, and the local Reynolds number that is used in the asymptotic analysis. The previously mentioned global Reynolds number will now be denoted by

$$\text{Re}_c = \frac{U_\infty c}{\nu},$$

(1.24)

where $c$ is a fixed length scale such as the chord length of an aerofoil or the length of a plate. The abbreviation $\text{Re}$ is instead used for the local Reynolds number, which depends on the local characteristic length scale $L$ measured at each physical $\hat{x}$-location,

$$\text{Re} = \text{Re}_x = \frac{U_\infty L}{\nu},$$

(1.25)

where $L$ is often taken to be the distance from the leading edge of the plate, but can also be the radius of the curvature of the aerofoil at its nose or any other scale that is useful in studying the flow behaviour.
Furthermore, since boundary layer flows are weakly non-parallel as discussed in (1.11), the stability analysis becomes much simpler when the parallel flow approximation is applied. If the characteristic wavelength $\ell$ of the perturbations is much shorter than the local characteristic length scale $L$ measured at each point $\hat{x} = \hat{x}_0$, the base flow does not experience any noticeable change over a distance of several wavelengths. Therefore, at each location $\hat{x}_0 = L$, the base flow behaves as if it were parallel and its dependence on $\hat{x}$ can be treated as parametric. Using (1.16), (1.17) and (1.19), the horizontal velocity component then exhibits the asymptotic behaviour

$$U(Y) = \lambda Y + \ldots \quad \text{as} \quad Y \to 0,$$

$$U(Y) = 1 + \ldots \quad \text{as} \quad Y \to \infty,$$

where $\lambda = 0.332057$ is the undisturbed shear coefficient.

Several types of instability modes have been identified for boundary layers on aerofoils. For aeroplanes with a small sweep angle, in subsonic flow, such as regional aircrafts, the two-dimensional T–S waves are the dominant instabilities that need to be examined. For flows past wings with a larger sweep angle, where the spanwise component of the flow becomes non-negligible, the three-dimensional boundary layer is controlled by crossflow instabilities. This is characteristic for long-distance passenger carriers. A third type of instability mode are Görtler vortices which are of importance for flows over surfaces with curvature.

In this thesis, Tollmien–Schlichting waves are the main focus – of particular interest is how they might be amplified by a surface roughness element and whether they can be generated efficiently by a vibrating plate situated upstream. For both problems, it is interesting to establish whether these disturbances induce transition sooner. The study of this initial stage of transition is termed Receptivity theory. A distinction is made between calculating the receptivity coefficient, when investigating how efficiently the disturbance generates T–S waves on its own, and the transmission coefficient which allows to determine the amplification of the instability mode by the disturbance.
1.3.1 Receptivity theory

Receptivity theory investigates the interaction between the boundary layer and the surrounding perturbation field. Its importance stems from a multitude of experiments where it was observed that the transition locations for the same aerodynamic body tested in different wind tunnels were not in agreement with each other. This discrepancy can be attributed to seemingly unimportant imbalances between the wind tunnels such as their smoothness, the quality of the flow and the acoustic noise in the test section, the level of turbulence in the oncoming flow, or the tested model surface, among other factors. In general, the quieter the wind tunnel is, the longer the boundary layer stays laminar. Furthermore, since the level of atmospheric turbulence in real flight conditions is lower than that in wind tunnel tests, a direct experimental simulation of the laminar-turbulent transition process proves to be impossible, thus demanding that the problem be approached in a new way. In Part II of this work, a theoretical approach is chosen in which asymptotic methods for high Reynolds number flow are employed, in particular the Triple Deck theory laid out in Section 1.3.2.

Morkovin [1969] was the first to highlight the importance of receptivity as the initial stage of the laminar-turbulent transition process. Numerous experimental studies, starting with Schubauer & Skramstad [1948], revealed that some disturbances, e.g. acoustic waves, free-stream turbulence or wall roughnesses, easily penetrate into the boundary layer and excite instability modes, whereas others do not. Kachanov et al. [1982] first formulated the necessary resonance conditions for an effective transformation of external disturbances into instability modes, which state that not only the frequency, but also the wavenumber of the external perturbations need to be in tune with the natural internal oscillations of the boundary layer, specifically those of the aforementioned Tollmien–Schlichting wave. The implications of this are considered in the following.

Define a new local Reynolds number in terms of the boundary layer thickness $\delta^*$ such that

$$Re_* = \frac{U_\infty \delta^*}{\nu}.$$  \hspace{1cm} (1.27)

Lin [1945] was the first to show that the characteristic wavelength $\alpha_*$ and the charac-
teristic frequency $\omega_*$ of the T–S wave for large $Re_c$ are estimated as

$$\alpha_* \delta^* \sim Re_*^{-1/4} \quad \text{and} \quad \frac{\omega_* \delta^*}{U_\infty} \sim Re_*^{-1/2}.$$  \hfill (1.28)

Consider the horizontal scaling around $\hat{x} = \hat{x}_0 = L$ where the T–S wavelength becomes important. Note that

$$\hat{x} - L \sim \frac{2\pi}{\alpha_*} \sim 2\pi \delta^* \ Re_*^{1/4} = 2\pi \left( \frac{U_\infty}{\nu} \right)^{1/4} (\delta^*)^{5/4}.$$

For a Blasius base flow, the boundary layer thickness is given by \hbox{[1.22]}, so that the magnitude of this horizontal region is of order

$$\hat{x} - L = \mathcal{O} \left( L \ Re^{-3/8} \right). \hfill (1.29)$$

This scaling corresponds to the horizontal extent of the interaction region in the Triple Deck model. By a similar argument,

$$\hat{t} \sim \frac{2\pi}{\omega_*} \sim 2\pi \delta^* \ Re_*^{1/2} \ \frac{\delta^*}{U_\infty} = 2\pi \left( \frac{\delta^*}{\nu U_\infty} \right)^{3/2},$$

and so the period of oscillations of the T–S wave is of order

$$\hat{t} = \mathcal{O} \left( \frac{L}{U_\infty} \ Re^{-1/4} \right). \hfill (1.30)$$

This is the characteristic time scale employed in the Triple Deck model. From these scalings, it can be concluded that the characteristic frequency and wavenumber of the oscillations of the instability mode in the boundary layer, in terms of the characteristic length scale $L$ measured at each $\hat{x}$-location, are estimated as

$$\alpha_{TS} \sim Re^{3/8} \quad \text{and} \quad \omega_{TS} \sim Re^{1/4}. \hfill (1.31)$$
Lin [1945] was the first to derive these characteristic scalings as a result of investigating the stability of parallel flows by using classical Orr-Sommerfeld theory (see Orr [1907], Sommerfeld [1908]). This was achieved through a study of the upper and lower branches of the so-called stability curve: a graphic representation of the stability of a flow, with the wavenumber $\alpha$ plotted against the Reynolds number, as shown in Figure 1.5. The neutral curve is the locus of points which represent a wave of constant amplitude, so the perturbations neither grow nor decay. Inside the curve lies a region of instability where perturbations are growing, but they decay everywhere outside the curve. It is important to note that, as the Reynolds number increases, the corresponding wavenumber $\alpha$ becomes smaller. It is this limit that is investigated in this thesis.

![Figure 1.5: Neutral curve for temporal instability](image)

While Figure 1.5 is a representation of the *temporal* instability of the flow, which investigates how perturbations grow in time, perturbations in boundary layers, such as Tollmien–Schlichting waves, are observed to grow in the downstream direction. This leads to the concept of *spatial* instability, for which the frequency is assumed to be real and the wavenumber is taken to be complex. It has been observed in experiments that the frequency $\omega$ of a Tollmien–Schlichting wave remains unchanged as the wave travels
downstream, see e.g. [Schubauer & Skramstad 1948]. So, when employing the normal mode decomposition for the perturbation $u'$,

$$u' = e^{i(\alpha X + \omega T)} \bar{u}(y) = e^{-\alpha_i X} e^{i(\alpha_r X + \omega T)} \bar{u}(y),$$

(1.32)

where $\alpha = \alpha_r + i\alpha_i$, the amplitude of the mode is dependent on the imaginary part of the wavenumber $\alpha_i$. Taking $X > 0$, as the downstream behaviour of the flow is of interest, it can be concluded that the mode grows for $\alpha_i < 0$ and decays if $\alpha_i > 0$. This is visualised for Tollmien–Schlichting waves in Figure 1.6i. Below, Figure 1.6ii shows the changes in the amplitude of the highlighted mode in Figure 1.6i as it travels downstream.

When analysing the asymptotic behaviour of the lower branch in the large-Reynolds number limit for incompressible boundary layers, Lin found that the solution developed a three-tiered structure whose scalings of the perturbed quantities and length scales later became relevant for the unsteady version of the Triple Deck theory. However, due to the initial assumption of parallel flow which does not apply to boundary layers, it took over 30 years for these scalings to be confirmed by [Smith 1979a,b].

1.3.2 Triple Deck theory

The Triple Deck model describes the interaction between the boundary layer and the inviscid flow outside the boundary layer, termed viscous-inviscid interaction, and plays a key role in a wide variety of fluid dynamic phenomena; see [Sychev et al. 1998] for an exposition of its applications. The need for Triple Deck theory arose, because Prandtl’s classical boundary layer theory, where the viscous and inviscid regions are analysed independently of each other, was unequipped to deal with singularities arising in the boundary layer solutions at the point of separation and at the trailing edge of a flat plate (see [Landau & Lifshitz 1944, Goldstein 1930, 1948]). In these cases, the pressure due to the displacement of the boundary layer was found to change the flow behaviour outside the boundary layer in such a significant way that the leading order terms in the expansions for the velocity components at large Reynolds number (1.14) were no longer applicable. The Triple Deck theory for steady flows was developed independently by [Neiland 1969, Stewartson & Williams 1969, Stewartson 1969, and Messiter 1970]. Instead of using the two-tiered structure from classical boundary layer theory, the flow
Figure 1.6: Neutral curve for spatial instability and corresponding behaviour of the amplitude $A$: for a fixed $\omega$, as the observer moves downstream ($x_1 < x_2 < x_3$), the amplitude changes depending on the sign of the growth rate $\alpha_i$. 
is now divided into three distinct regions, each describing a different part of the viscous-inviscid interaction process:

- In the viscous sublayer, also termed the lower deck, the flow exhibits high sensitivity to pressure variations due to the slow motion of the fluid close to the surface. This leads to the previously discussed deformation of the streamlines called the displacement effect of the boundary layer. The velocity in this region is estimated to be small compared to the free-stream velocity.

- The main part of the boundary layer, i.e. the middle deck, acts as a continuation of the conventional boundary layer where the velocity is of the same order as $U_\infty$. Its largely passive role is to transmit the displacement effect from the lower to the upper deck, resulting in a region of parallel streamlines.

- In the inviscid layer, called the upper deck, the displacement of the streamlines is converted into a pressure perturbation. This is then transmitted through the main deck back into the viscous sublayer, promoting further velocity disturbances, thus restarting the chain of events.

This self-sustaining process is the key element of Triple Deck theory and allows to describe a large number of fluidodynamic phenomena. Contrary to the parabolic nature of the classical boundary layer equations, it permits an upstream propagation of the perturbations and removes the aforementioned singularities. Schneider [1974] extended Triple Deck theory to include unsteady flows. Whereas the viscous sublayer is sensitive to unsteady perturbations when the characteristic time scale $\hat{t}$ is an $O(L \text{Re}^{-1/4}/U_\infty)$ quantity, the middle and upper decks remain quasi-steady. Based on previous work by Lin [1945], Smith [1979a,b] was then able to demonstrate that for subsonic boundary layer flows, the Triple Deck theory describes the T–S waves at and near the lower branch of the stability curve, where the physical sizes of the three $\hat{y}$-regions, from the bottom up, are found to be of $O(L \text{Re}^{-5/8})$, $O(L \text{Re}^{-1/2})$ and $O(L \text{Re}^{-3/8})$ respectively. It was thus shown that the Triple Deck model, contrary to the previously employed Orr-Sommerfeld theory (see Schlichting [1968] for details), allows for nonlinear and nonparallel effects.

Terent’ev [1981] was the first to use Triple Deck theory to study the receptivity of the boundary layer to periodic vibrations, a theoretical model of previous experiments
by Schubauer & Skramstad [1948]. He considered an incompressible fluid flow past a flat plate whose base flow is given by the Blasius solution and assumed that a short section of the plate acts as a vibrator. His analysis showed that a T–S wave forms downstream of the vibrator whose amplitude can be determined if the amplitude and the shape of the vibrator are known, and that the frequency $\omega$ of a Tollmien–Schlichting wave remains unchanged as the wave travels downstream, and is equal to the frequency of the vibrator.

Some external disturbances are insufficient to generate a growing T–S wave on their own. In these cases, an added irregularity in the boundary layer, such as wall roughness, is necessary to produce growing perturbations. Ruban [1984] and Goldstein [1985] developed the asymptotic theory for this form of receptivity, by noting that the Triple Deck theory can be applied if the streamwise length scale of the roughness is of $O(L\text{Re}^{-3/8})$. The amplitude of the T–S wave downstream could then be found by extracting the corresponding harmonic from the spectrum of the flow perturbations and was determined to be proportional to the amplitude of the incoming acoustic wave, as well as the Fourier transform of the roughness shape evaluated at the specific wavenumber. In this thesis, a similar result is found which will be discussed in Section 3.7.

1.4 Layout for Part I

Having reviewed the historical background of the mathematical methods that are used to study fluid flow analytically, these concepts will now be employed to study two different flow geometries. The first problem examines the behaviour of the fluid flow over a small roughness element, where the incoming flow is a Blasius flow perturbed by a Tollmien–Schlichting wave. In order to do this, the flow behaviour upstream of the roughness is investigated first and expressions for the flow perturbations describing an incoming T–S wave over a flat plate are derived in Chapter 2. This is done by a local Triple Deck analysis which will take advantage of the periodicity of the T–S wave to employ the normal mode decomposition of the perturbations.

After having determined the velocity and pressure components upstream, the roughness further downstream is considered. Before looking at the interaction between the T–S wave and the roughness however, it is necessary to first analyse the roughness regime
on its own. For this purpose, another local Triple Deck analysis around the roughness will be conducted in Section 3.1 which will yield time-independent velocity and pressure perturbations in Fourier space.

Having found expressions for the flow perturbations in the two regimes, these can be used to examine what happens to the flow if both are present at once. This is done in the remainder of Chapter 3 where it can be observed that the interaction between the two regimes yields an additional term in the momentum equation which is studied in detail. Finally, the transmission coefficient for the interaction problem is calculated, and a comparison is made between the amplitudes of the T–S wave up- and downstream of the roughness.

In Chapter 4, a second, slightly different setup is considered: a semi-infinite vibrating elastic plate is joined to a stationary flat plate. The aim here is to identify how the periodic vibrations upstream influence the flow behaviour downstream, which is achieved by calculating the receptivity coefficient for this problem.
Chapter 2

Triple Deck Analysis of an incoming Blasius base flow perturbed by a Tollmien–Schlichting wave over a flat plate

In the following chapter, an investigation of the Blasius boundary layer perturbed by a Tollmien–Schlichting wave of wavenumber $\alpha_{TS}$ and frequency $\omega_{TS}$ over a flat plate is carried out. Furthermore, the scalings in each deck with the corresponding governing equations and matching conditions are described in detail here, and referred to in subsequent chapters.

2.1 Problem formulation

The global characteristic length scale $L$ is chosen to be the distance from the leading edge of the plate to the point $\hat{x}_0$ where the roughness will be located for the analyses in Chapter 3. To determine expressions for the small flow perturbations, a Triple Deck structure with non-dimensional mid-point $x_0 = 1$ is set up, as pictured in Figure 2.1, where region 3 denotes the viscous sublayer (lower deck), region 2 is the main boundary layer (main deck) and region 1 is the inviscid layer (upper deck).

The dimensionless Navier–Stokes equations (1.5) for the fluid velocity $\mathbf{u} = (u, v)$ and pressure $p$ are solved in each deck with the boundary conditions (1.6), as well as
matching conditions between the decks. Since this is the case of a flat surface, \( S \) simply becomes \( y = 0 \). Furthermore, in view of the previously established scalings (1.31), introduce a fast streamwise variable \( X^* \), as well as a fast time variable \( T^* \) such that

\[
x = 1 + \text{Re}^{-3/8} X^* \quad \text{and} \quad t = \text{Re}^{-1/4} T^*.
\] (2.1)

Finally, represent the \( \mathcal{O}(1) \) wavenumber \( \alpha^* \) and frequency \( \omega^* \) of the oscillations in the form

\[
\alpha_{TS} = \text{Re}^{3/8} \alpha^* \quad \text{and} \quad \omega_{TS} = \text{Re}^{1/4} \omega^*.
\] (2.2)

Thus, the dependence on the longitudinal coordinate \( x \) and time \( t \) of the incoming T–S wave is of the form

\[
e^{i(\alpha_{TS}(x-1)+i\omega_{TS}t)} = e^{i(\alpha^* X^* + i\omega^* T^*)} = e^{-\alpha_i^* X^*} e^{i(\alpha_r^* X^* + i\omega^* T^*)},
\] (2.3)

where \( \alpha^* = \alpha_r^* + i\alpha_i^* \) is complex and \( \omega^* \) is real.
2.2 Lower deck

In the lower deck, the vertical variable is rescaled as

\[ y = \text{Re}^{-5/8} y^*_3, \]  

(2.4a)

and the velocity components are expanded as

\[ u(x, y, t) = \text{Re}^{-1/8} U^*(X^*, y^*_3, T^*) + \ldots, \]  

(2.4b)

\[ v(x, y, t) = \text{Re}^{-3/8} V^*(X^*, y^*_3, T^*) + \ldots, \]  

(2.4c)

\[ p(x, y, t) = \text{Re}^{-1/4} P^*(X^*, y^*_3, T^*) + \ldots. \]  

(2.4d)

The scalings for the velocity and pressure were obtained as follows. First, balance the convective and the viscous terms in the x-momentum equation (1.5a) to determine the scaling for \( u \):

\[ u \frac{\partial u}{\partial x} \sim \text{Re}^{-1} \frac{\partial^2 u}{\partial y^2} \text{ or } u \frac{u}{x} \sim \text{Re}^{-1} \frac{u}{y^2}, \]  

which yields \( u \sim \text{Re}^{-1/8} \).

From this, obtain the scaling of \( v \) from the continuity equation (1.5c):

\[ \frac{\partial u}{\partial x} \sim \frac{\partial v}{\partial y} \text{ or } u \frac{v}{x} \sim \frac{v}{y}, \]  

which yields \( v \sim \text{Re}^{-3/8} \).

It remains to obtain the scaling for \( p \) by balancing the convective and pressure terms in (1.5a):

\[ u \frac{\partial u}{\partial x} \sim \frac{\partial p}{\partial x} \text{ or } u \frac{p}{x} \sim \frac{\Delta p}{x}, \]  

which yields \( \Delta p \sim \text{Re}^{-1/4} \).

Substituting (2.4) into the Navier–Stokes equations (1.5) and equating terms to leading order, the classical boundary layer equations are obtained:

\[ \frac{\partial U^*}{\partial T^*} + U^* \frac{\partial U^*}{\partial X^*} + V^* \frac{\partial U^*}{\partial y^*_3} = - \frac{\partial P^*}{\partial X^*} + \frac{\partial^2 U^*}{\partial (y^*_3)^2}, \]  

(2.5a)

\[ \frac{\partial P^*}{\partial y^*_3} = 0, \]  

(2.5b)

\[ \frac{\partial U^*}{\partial X^*} + \frac{\partial V^*}{\partial y^*_3} = 0, \]  

(2.5c)
with the no-slip condition at the surface
\[(U^*, V^*) = (0, 0) \text{ at } y_3^* = 0, \quad (2.5d)\]

and the matching condition with the main deck, where a Blasius base flow is present
\[U^* \to \lambda y_3^* + A^*(X^*, T^*) + \ldots \text{ as } y_3^* \to \infty. \quad (2.5e)\]

Here, \(\lambda\) is the undisturbed shear coefficient of the Blasius flow introduced in (1.19), and the base state is defined by
\[(U_B, V_B, P_B) = (\lambda y_3^*, 0, 0). \quad (2.6)\]

The unknown function \(A^*(X^*, T^*)\) in (2.5e) is called the displacement function. Note that the second equation shows that the pressure in the lower deck is independent of \(y_3^*, P^* = P^*(X^*, T^*).\)

### 2.2.1 Affine transformation

In order to simplify calculations, the parameter \(\lambda\) may be excluded from (2.5) by means of the affine transformation
\[
\begin{align*}
X^* &= \lambda^{-5/4} X, \\
y_3^* &= \lambda^{-3/4} y_3, \\
T^* &= \lambda^{-3/2} T, \\
A^* &= \lambda^{1/4} A,
\end{align*}
\]
\[U^* = \lambda^{1/4} U, \\
V^* = \lambda^{3/4} V, \\
P^* = \lambda^{1/2} P. \quad (2.7)\]

The governing equations and corresponding boundary conditions then become
\[
\begin{align*}
\frac{\partial U}{\partial T} + U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial y_3} &= -\frac{\partial P}{\partial X} + \frac{\partial^2 U}{\partial y_3^2}, \quad (2.8a) \\
\frac{\partial P}{\partial y_3} &= 0, \quad (2.8b) \\
\frac{\partial U}{\partial X} + \frac{\partial V}{\partial y_3} &= 0, \quad (2.8c)
\end{align*}
\]

with the no-slip condition at the surface
\[(U, V) = (0, 0) \text{ at } y_3 = 0, \quad (2.8d)\]
and the matching condition with the main deck,

\[ U \to y_3 + A(X, T) + \ldots \quad \text{as} \quad y_3 \to \infty. \quad (2.8e) \]

### 2.2.2 Linearisation

Due to the presence of the incoming T–S wave, periodic in \( T \), it can be assumed that the responding velocity and pressure perturbations exhibit the same periodicity and thus the solution can be sought in normal-mode form. Furthermore, the amplitude of the perturbation is assumed to be small, \( \varepsilon \ll 1 \). The affine transformation (2.7) also changes the wavenumber and frequency, so introduce \((\alpha, \omega)\) such that

\[ \alpha^* = \lambda^{5/4} \frac{\alpha}{\lambda}, \quad \text{and} \quad \omega^* = \lambda^{3/2} \omega. \quad (2.9) \]

Equations (2.8) can then be linearised by using the expansions

\[ U(X, y_3, T) = y_3 + \varepsilon e^{i(\omega T + \alpha X)} u^3_{TS}(y_3) + \ldots, \quad (2.10a) \]
\[ V(X, y_3, T) = \varepsilon e^{i(\omega T + \alpha X)} v^3_{TS}(y_3) + \ldots, \quad (2.10b) \]
\[ P(X, T) = \varepsilon e^{i(\omega T + \alpha X)} p^3_{TS} + \ldots, \quad (2.10c) \]
\[ A(X, T) = \varepsilon e^{i(\omega T + \alpha X)} A_{TS} + \ldots, \quad (2.10d) \]

To leading order, the linearised triple deck equations (2.8a) and (2.8c) then become:

\[ i\omega u^3_{TS} + i\alpha y_3 u^3_{TS} + v^3_{TS} = -i\alpha p^3_{TS} + \frac{d^2 u^3_{TS}}{dy_3^2}, \quad (2.11a) \]
\[ i\alpha u^3_{TS} + \frac{dv^3_{TS}}{dy_3} = 0, \quad (2.11b) \]

with the boundary conditions

\[ (u^3_{TS}, v^3_{TS}) = (0, 0) \quad \text{at} \quad y_3 = 0, \quad (2.11c) \]
\[ u^3_{TS} \to A_{TS} + \ldots \quad \text{as} \quad y_3 \to \infty, \quad (2.11d) \]
In particular, setting $y_3 = 0$ in (2.11a) gives the relation:\n\[
\frac{d^2 u_T^3}{dy_3^2} = i\alpha p_T^3 \quad \text{at} \quad y_3 = 0. \tag{2.12}
\]

### 2.3 Middle deck

In the middle deck, the vertical variable is rescaled as
\[
y = \text{Re}^{-1/2} \lambda^{-3/4} y_2, \tag{2.13a}
\]

The velocity components are rescaled and linearised as
\[
\begin{align*}
u(x, y, t) &= U_B(y_2) + \text{Re}^{-1/8} \lambda^{1/4} \tilde{u}(X, y_2, T) + \ldots, \tag{2.13b} \\
v(x, y, t) &= \text{Re}^{-1/4} \lambda^{3/4} \tilde{v}(X, y_2, T) + \ldots, \tag{2.13c} \\
p(x, y, t) &= \text{Re}^{-1/4} \lambda^{1/2} \tilde{p}(X, y_2, T) + \ldots, \tag{2.13d}
\end{align*}
\]

where $U_B$ denotes the rescaled incoming Blasius base flow with the properties
\[
\begin{align*}
U_B &\to y_2 + \ldots \quad \text{as} \quad y_2 \to 0, \tag{2.14a} \\
U_B &\to 1 \quad \text{as} \quad y_2 \to \infty. \tag{2.14b}
\end{align*}
\]

To find the scaling for $v$ in the main part of the boundary layer, note that the ratio between $u$ and $v$ needs to match between decks. From the lower deck,
\[
v = \text{Re}^{-3/8} V^* U_B \to \frac{\text{Re}^{-3/8} V^* X}{U_B},
\]
which yields $v \sim \text{Re}^{-1/4}$ since $u = \mathcal{O}(1)$ in the middle deck. The second order term for $u$ can then be found by using the continuity equation
\[
\frac{\partial u}{\partial x} \sim \frac{\partial v}{\partial y} \quad \text{or} \quad \frac{\Delta u}{\Delta x} \sim \frac{\Delta v}{\Delta y}, \quad \text{which yields} \quad \Delta u \sim \text{Re}^{-1/8}.
\]

Furthermore, to ensure continuity in the pressure, the scaling for $p$ is the same as in the lower deck, $\Delta p = \text{Re}^{-1/4}$. Substituting (2.13) into the Navier–Stokes equations (1.5)
and equating terms to leading order then yields the equations

\[ U_B \frac{\partial \tilde{u}}{\partial X} + \frac{dU_B}{dy_2} \tilde{v} = 0, \quad (2.15a) \]
\[ \frac{d\tilde{p}}{dy_2} = 0, \quad (2.15b) \]
\[ \frac{\partial \tilde{u}}{\partial X} + \frac{\partial \tilde{v}}{\partial y_2} = 0, \quad (2.15c) \]

which need to be solved with the boundary condition obtained from matching with the lower deck:

\[ \tilde{u} \to A(X, T) + \ldots \quad \text{as} \quad y_2 \to 0. \quad (2.15d) \]

From equation (2.15b), note that the pressure perturbation in the middle deck is independent of \( y_2 \): \( \tilde{p} = \tilde{p}(X, T) \), which indicates that there is no variation of the pressure perturbation in \( y \) in the lower and middle decks. Substituting (2.15c) into (2.15a) yields the differential equation

\[ -U_B \frac{\partial \tilde{v}}{\partial y_2} + \frac{dU_B}{dy_2} \tilde{v} = 0 \quad \Leftrightarrow \quad \frac{1}{\tilde{v}} \frac{\partial \tilde{v}}{\partial y_2} = \frac{1}{U_B} \frac{dU_B}{dy_2}, \quad (2.16) \]

whose solution is found to be

\[ \tilde{v}(X, y_2, T) = \frac{\partial B(X, T)}{\partial X} U_B(y_2), \]

where \( B(X, T) \) is an arbitrary function determined by the boundary condition (2.15d):

\[ B(X, T) = -A(X, T), \]

and thus

\[ \tilde{u}(X, y_2, T) = A(X, T) \frac{dU_B}{dy_2}(y_2), \quad (2.17a) \]
\[ \tilde{v}(X, y_2, T) = -\frac{\partial A(X, T)}{\partial X} U_B(y_2). \quad (2.17b) \]

Contrary to the lower deck, the middle deck equation (2.16) and its solutions (2.17) are linear. To allow for matching with the lower deck, it is necessary to expand \((\tilde{u}, \tilde{v}, \tilde{p})\)
with respect to the small parameter $\varepsilon \ll 1$, again assuming that the perturbations are periodic in $T$, and noting that $\tilde{p}$ is independent of $y_2$:

\[
\begin{align*}
\tilde{u}(X, y_2, T) &= \varepsilon \, e^{i(\omega T + \alpha X)} \, u_{TS}(y_2) + \ldots, \\
\tilde{v}(X, y_2, T) &= \varepsilon \, e^{i(\omega T + \alpha X)} \, v_{TS}(y_2) + \ldots, \\
\tilde{p}(X, T) &= \varepsilon \, e^{i(\omega T + \alpha X)} \, p_{TS} + \ldots
\end{align*}
\]

Using $A = \varepsilon \, e^{i(\omega T + \alpha X)} \, A_{TS}$, the $O(\varepsilon)$ flow perturbations in the middle deck become

\[
\begin{align*}
u_{TS}^2(y_2) &= A_{TS} \, \frac{dU_B}{dy_2}(y_2), \\
v_{TS}^2(y_2) &= -i\alpha \, A_{TS} \, U_B(y_2).
\end{align*}
\]

2.4 Upper deck

In the upper deck, the vertical variable is rescaled as

\[
y = \text{Re}^{-3/8} \, \lambda^{-3/4} \, y_1,
\]

The velocity components are represented in the form of asymptotic expansions:

\[
\begin{align*}
u(x, y, t) &= 1 + \text{Re}^{-1/4} \, \lambda^{1/4} \, \tilde{u}(X, y_1, T) + \ldots, \\
v(x, y, t) &= \text{Re}^{-1/4} \, \lambda^{3/4} \, \tilde{v}(X, y_1, T) + \ldots, \\
p(x, y, t) &= \text{Re}^{-1/4} \, \lambda^{1/2} \, \tilde{p}(X, y_1, T) + \ldots
\end{align*}
\]

The scalings for $(v, p)$ can be found by matching with the middle deck. By substituting $u = 1 + \Delta u + \ldots$ into the continuity equation, the perturbation is found to be of order

\[
\frac{\partial u}{\partial X} \sim \frac{\partial v}{\partial y} \quad \text{or} \quad \Delta u \sim \frac{\Delta v}{\Delta y} \Delta x,
\]

which yields $\Delta u \sim \text{Re}^{-1/4}$, from matching with the middle deck.
Substituting (2.20) into the Navier–Stokes equations (1.5) gives

\[ \frac{\partial \tilde{u}}{\partial X} = -\frac{\partial \tilde{p}}{\partial X}, \]  
(2.21a)

\[ \frac{\partial \tilde{v}}{\partial X} = -\frac{\partial \tilde{p}}{\partial y_1}, \]  
(2.21b)

\[ \frac{\partial \tilde{u}}{\partial X} + \frac{\partial \tilde{v}}{\partial y_1} = 0, \]  
(2.21c)

with the boundary conditions

\[ (\tilde{v}, \tilde{p}) \rightarrow (\tilde{v}, \tilde{p}) \text{ as } y_1 \rightarrow 0, \]  
(2.21d)

\[ (\tilde{u}, \tilde{v}, \tilde{p}) \rightarrow (0, 0, 0) \text{ as } X^2 + y_1^2 \rightarrow \infty. \]  
(2.21e)

Furthermore, applying (2.21d) to (2.21b) and using (2.14b), (2.17), yields the useful relation

\[ \frac{\partial \tilde{p}}{\partial y_1} \big|_{y_1=0} = -\frac{\partial \tilde{v}}{\partial X} \big|_{y_2 \rightarrow \infty} = \frac{\partial^2 A}{\partial X^2}. \]  
(2.21f)

Substituting (2.21a) and (2.21b) into (2.21c), the Laplace equation for \( \tilde{p} \) is found:

\[ \nabla^2 \tilde{p} = 0. \]  
(2.22)

The Laplace equation can easily be solved using Fourier transforms, however, in this case, the local periodicity of the flow in the X-direction renders this unnecessary. Expanding \( (\tilde{u}, \tilde{v}, \tilde{p}) \) in \( \varepsilon \ll 1 \) again, write

\[ \tilde{u}(X, y_1, T) = \varepsilon e^{i(\omega T + \alpha X)} u_{TS}^1(y_1) + \ldots, \]  
(2.23a)

\[ \tilde{v}(X, y_1, T) = \varepsilon e^{i(\omega T + \alpha X)} v_{TS}^1(y_1) + \ldots, \]  
(2.23b)

\[ \tilde{p}(X, y_1, T) = \varepsilon e^{i(\omega T + \alpha X)} p_{TS}^1(y_1) + \ldots. \]  
(2.23c)

The Laplace equation then becomes

\[ \frac{d^2 p_{TS}^1}{dy_1^2} - \alpha^2 p_{TS}^1 = 0, \]  
(2.24a)
which needs to be solved with the corresponding boundary conditions

\[ p_{TS}^1 \to 0 \quad \text{as} \quad y_1 \to \infty, \quad (2.24b) \]
\[ \frac{dp_{TS}^1}{dy_1} = -\alpha^2 A_{TS} \quad \text{at} \quad y_1 = 0. \quad (2.24c) \]

The solution to the upper deck equation \((2.24a)\) is

\[ p_{TS}^1(y_1) = B_1 e^{\alpha|y_1|} + B_2 e^{-\alpha|y_1|}, \]

where \(B_1\) and \(B_2\) are arbitrary constants to be determined. Since \(p_{TS}^1\) needs to be bounded as \(y_1 \to \infty\), set \(B_1 = 0\). In order to determine \(B_2\), use \((2.24c)\):

\[-\alpha |B_2| = -\alpha^2 A_{TS} \Rightarrow B_2 = |\alpha| A_{TS}.\]

Thus

\[ p_{TS}^1(y_1) = |\alpha| A_{TS} e^{-\alpha|y_1|}. \quad (2.25) \]

The pressure perturbation constant across the lower and middle decks is then defined by the following pressure-displacement relation

\[ p_{TS}^3 = p_{TS}^2 = p_{TS}^1|_{y_1=0} = |\alpha| A_{TS}, \quad (2.26) \]

so that relation \((2.12)\) can be written as

\[ \frac{d^2u_{TS}^3}{dy_3^2} = i\alpha|\alpha| A_{TS} \quad \text{at} \quad y_3 = 0. \quad (2.27) \]

Furthermore, the velocity components of \(\mathcal{O}(\varepsilon)\) in the upper deck become

\[ u_{TS}^1(y_1) = -|\alpha| A_{TS} e^{-\alpha|y_1|}, \quad (2.28a) \]
\[ v_{TS}^1(y_1) = -i\alpha A_{TS} e^{-\alpha|y_1|}. \quad (2.28b) \]

### 2.5 Viscous-inviscid interaction

Now it is time to return to the lower deck to calculate the remaining velocity perturbations and to establish a dispersion relation between \(\alpha\) and \(\omega\).
Differentiating (2.11a) w.r.t. $y_3$ and substituting (2.11b) gives

$$\frac{d^3 u_{TS}^3}{dy_3^3} - i(\alpha y_3 + \omega)\frac{d u_{TS}^3}{dy_3} = 0,$$

which can be transformed into an Airy equation as follows.

Introduce the new variable $z$

$$z = \gamma_{TS} y_3 + z_0 \iff y_3 = \frac{z - z_0}{\gamma_{TS}},$$

so

$$\frac{d}{dy_3} = \frac{d}{dz} \frac{dz}{dy_3} = \gamma_{TS} \frac{d}{dz}.$$

Then equation (2.29) becomes

$$\gamma_{TS}^3 \frac{d^3 u_{TS}^3}{dz^3} = i(\omega + \alpha y_3)\gamma_{TS} \frac{d u_{TS}^3}{dz}$$

$$= i\alpha \left( z - z_0 + \gamma_{TS} \frac{\omega}{\alpha} \right) \frac{d u_{TS}^3}{dz}.$$

In order to obtain the well-known Airy equation for $u_{TS}^3$,

$$\frac{d^3 u_{TS}^3}{dz^3} - z \frac{d u_{TS}^3}{dz} = 0,$$

set

$$\gamma_{TS} = (i\alpha)^{1/3} \text{ and } z_0 = \gamma_{TS} \frac{\omega}{\alpha} = \frac{i\omega}{(i\alpha)^{2/3}}.$$

Since $\gamma_{TS}$ is a multi-valued quantity in $\alpha$, it is necessary to introduce a branch cut in the complex $\alpha$-plane. Here, a branch cut along the positive imaginary $\alpha$-axis is chosen, as shown in Figure [2.2]. Writing

$$\alpha = |\alpha| e^{i\theta} \text{ where } -\frac{3\pi}{2} < \theta \leq \frac{\pi}{2},$$

$\gamma_{TS}$ can then be written as

$$\gamma_{TS} = (i\alpha)^{1/3} = (e^{i\frac{\pi}{2}} |\alpha| e^{i\theta})^{1/3} = e^{i(\frac{\pi}{6} + \frac{\pi}{3})} |\alpha|^{1/3},$$
so
\[-\frac{\pi}{3} < \arg(\gamma_{TS}) \leq \frac{\pi}{3},
\]
which means that the corresponding domain in the $\gamma_{TS}$-plane is represented by a wedge of angle $\frac{2\pi}{3}$, as shown in Figure 2.2. For $\alpha \in \mathbb{R}$, there are two cases to consider:

- If $\alpha < 0$, $\theta = -\pi$, so $\gamma_{TS} = e^{-\frac{i\pi}{3}} |\alpha|^{1/3}$.
- If $\alpha \geq 0$, $\theta = 0$, so $\gamma_{TS} = e^{\frac{i\pi}{3}} |\alpha|^{1/3}$.

This means that the real $y_3$-axis is represented by the two dashed rays emerging from the origin, each forming an angle of $\frac{\pi}{6}$ to the real $\gamma_{TS}$-axis.

Moreover, $z_0$ becomes:
\[
z_0 = \frac{i\omega}{(i\alpha)^{2/3}} = \frac{e^{\frac{i\pi}{3}} \omega}{e^{i(\frac{\pi}{3} + \frac{2\pi}{3})} |\alpha|^{2/3}} = e^{i(\frac{\pi}{3} - \frac{2\pi}{3})} \frac{\omega}{|\alpha|^{2/3}},
\]
so
\[-\frac{\pi}{6} < \arg(z_0) \leq \frac{7\pi}{6}.
\]
For the neutral T–S wave, the wavenumber is negative, $\alpha_n = -1.00049$, see for example
Zhuk & Ryzhov [1983], which means that the argument $\theta = -\pi$, so

$$z = e^{-\frac{i\pi}{6}} |\alpha|^{1/3} \left( y - \frac{\omega}{|\alpha|} \right)$$

and thus the solution for a neutral wave will lie on the ray in the lower half of the imaginary $\gamma_{TS}$-plane where $\text{arg}(z) = -\frac{\pi}{6}$.

The general solution to the Airy equation (2.31) is

$$\frac{du^3_{TS}}{dz}(z) = C_1 \text{Ai}(z) + C_2 \text{Bi}(z),$$

where Ai, Bi are the Airy functions of the first and second kind, and $C_1, C_2$ are arbitrary constants to be determined in due course, see Abramowitz & Stegun [1964]. In the chosen domain where $|\text{arg}(z)| < \frac{\pi}{3}$, as $z \to \infty$, $\text{Ai}(z) \to 0$ and $\text{Bi}(z) \to \infty$. Since $u^3_{TS}$ is bounded as $z \to \infty$, $C_2 = 0$ is required. Integrating (2.36) with respect to $z$ gives an expression for $u^3_{TS}$:

$$u^3_{TS}(z) = C_1 \int_{z_0}^{z} \text{Ai}(s) ds + C_3.$$  

To find $C_1$ and $C_3$, the previously discussed boundary conditions, (2.11c), (2.11d) and (2.27), need to be rewritten in terms of $z$. They become

$$u^3_{TS} = 0 \quad \text{at} \quad z = z_0,$$

$$u^3_{TS} \to A_{TS} \quad \text{as} \quad z \to \infty,$$

$$\frac{d^2 u^3_{TS}}{dz^2} = (i\alpha)^{1/3}|\alpha| A_{TS} \quad \text{at} \quad z = z_0.$$  

Applying (2.38a) to (2.37) gives $C_3 = 0$.

The other two boundary conditions can then be written as the matrix equation

$$\begin{pmatrix}
\int_{z_0}^{\infty} \text{Ai}(s) ds & -1 \\
\text{Ai}'(z_0) & -(i\alpha)^{1/3}|\alpha|
\end{pmatrix}
\begin{pmatrix}
C_1 \\
A_{TS}
\end{pmatrix}
= 0.$$  

(2.39)

To find a non-trivial solution to (2.39), the determinant of the matrix should be zero,
which gives the dispersion relation for lower-branch Tollmien–Schlichting modes:
\[
\Delta_{TS}(\alpha, \omega) = \operatorname{Ai}'(z_0) - (i\alpha)^{1/3} |\alpha| \int_{z_0}^{\infty} \operatorname{Ai}(s) \, ds,
\]
\[\alpha_n < 0 \Rightarrow \operatorname{Ai}'(z_0) - i(\alpha)^{4/3} \int_{z_0}^{\infty} \operatorname{Ai}(s) \, ds = 0. \quad (2.40a)\]

This expression represents the large-Reynolds-number version of the Orr-Sommerfeld equation \((6.2)\). It has an infinite (countable) number of roots \(\alpha^{(j)}\) whose positions depend on the frequency \(\omega\). Substituting \(\alpha\) for \(z_0\) by using \((2.32)\), equation \((2.40a)\) can be rewritten in the form
\[
z_0^2 \operatorname{Ai}'(z_0) + i\omega^2 \int_{z_0}^{\infty} \operatorname{Ai}(s) \, ds = 0. \quad (2.40b)
\]

This equation can be solved for \(z_0\) using the complex Newton-Raphson algorithm (see e.g. Atkinson [2008]) for varying real \(\omega\). The first five roots \(z_0^{(j)}\) are pictured in Figure \(2.3i\) with their corresponding wavenumbers \(\alpha^{(j)}\) in Figure \(2.3ii\) where \(\omega\) is increased along the trajectory of each root. For small \(\omega\), \(\alpha\) is small as well, and the solutions \(z_0\) are close to the zeros of \(\operatorname{Ai}'(z_0) = 0\).

For the neutral frequency \(\omega_n = 2.29797\), the first root is real and equal to the Tollmien–Schlichting wavenumber, \(\alpha_n = -1.00049\) which lies on the real axis in the negative half-plane. All the other roots have a positive imaginary part, so that their corresponding perturbations in the boundary layer are expected to decay downstream as \(X \to \infty\):
\[
e^{i\alpha X} = e^{i(\alpha_r + i\alpha_i) X} = e^{-\alpha_i X} e^{i\alpha_r X}. \quad (2.41)
\]

For the first root, \(\alpha_1^{(1)}\) changes sign when the neutral frequency \(\omega_n\) is reached: for \(\omega < \omega_n\), \(\alpha_1^{(1)} > 0\) and the perturbations decay, but for \(\omega > \omega_n\), \(\alpha_1^{(1)} < 0\) and the corresponding modes will start to grow downstream.
(i) Roots $z_0^{(j)}$ of the dispersion relation

(ii) Corresponding wavenumbers $\alpha^{(j)}$ for the roots of the dispersion relation

Figure 2.3: Roots of the dispersion relation
The matrix system (2.39) does not yield a unique solution. Rather, the flow perturbations in the lower deck are all dependent on the arbitrary displacement parameter $A_{TS}$ that may be treated as the amplitude of the Tollmien–Schlichting wave:

$$u_{TS}^3(z) = \frac{A_{TS}}{\int_{z_0}^{\infty} \text{Ai}(s)ds} \int_{z_0}^z \text{Ai}(s)ds,$$

$$v_{TS}^3(z) = -\left(\frac{i\alpha}{2}\right)^{2/3} A_{TS} \left[ z \int_{z_0}^z \text{Ai}(s)ds - \text{Ai}'(z) + \text{Ai}'(z_0) \right],$$

$$p_{TS}^3 = |\alpha| A_{TS},$$

similarly to their counterparts in the other decks, (2.19), (2.26) and (2.28). The velocity profiles $u_{TS}^3(z)$ and $v_{TS}^3(z)$ are visualised in Figure 2.4. This concludes the Triple Deck Analysis for an incoming Tollmien–Schlichting wave over a flat surface, as the relevant fluid dynamic quantities have been determined throughout the three decks.
Figure 2.4: Velocity profiles $u_{TS}^3(z)$ and $v_{TS}^3(z)$ for $\alpha_n = -1.00049$, $\omega_n = 2.29797$ and $A_{TS} = 1$
Chapter 3

Interaction of an incoming Tollmien–Schlichting wave over a small stationary roughness

3.1 Triple Deck Analysis of an incoming Blasius base flow over a small stationary roughness element

In the first section of this chapter, an incoming Blasius flow over a small stationary roughness element in subsonic flow is considered. In 1971, Bogolepov & Neiland [1971] first investigated how a small surface roughness element changes the flow behaviour for a supersonic viscous flow. In the same year, Hunt [1971] examined the case of a small hump such that its height is much smaller than its length and confirmed the existence of a similarity solution. Smith [1973] extended Hunt’s analysis by examining a wider range of humps and employed the Triple Deck theory to study the flow in more detail. Both localised and distributed roughnesses have been studied extensively since then by Ruban [1984], Duck [1990] or Wu [2001], amongst others.

The flow response to a wall obstacle depends on the latter’s geometry and size, so consider a roughness element of non-dimensional width $W$ and height $H$. Not every choice of these parameters leads to interesting flow responses, as is the case for a wide, shallow roughness element which would only give rise to a small linear correction to the base flow. On the other hand, if the roughness element is very short, the problem for
the viscous sublayer reduces to the condensed-flow problem, as studied by Hunt [1971] and Bogolepov & Neiland [1971]. In this chapter, the width of the roughness element is chosen such that it provokes deviations from the standard boundary-layer problem. As discussed in Section 1.3.2, this is the case if

$$W \sim \text{Re}^{-3/8},$$  \hspace{1cm} (3.1)

which is the horizontal extent of the Triple Deck region. Furthermore, since the interest lies in marginally nonlinear flow responses, the obstacle height needs to be large enough to provoke nonlinearity. Through an order-of-magnitude analysis, it can be shown that this is the case if

$$H \sim W^{5/3} \sim \text{Re}^{-5/8},$$  \hspace{1cm} (3.2)

which is the characteristic thickness of the lower deck in the Triple Deck theory.

### 3.1.1 Problem formulation

![Triple Deck structure of a Blasius flow over a stationary roughness](image)

Figure 3.1: Triple Deck structure of a Blasius flow over a stationary roughness

Consider a Blasius base flow over a flat plate with a roughness element of mid-point situated at $\hat{x}_0 = L$, where $L$ is the global characteristic length scale, defined to be the distance from the leading edge of the plate to the middle of the roughness, so
that $x_0 = 1$ in dimensionless variables. The width of the roughness is chosen to be $O(Re^{-3/8})$, so the characteristic Triple Deck scalings introduced in Chapter 2 can be applied, as shown in Figure 3.1.

Just as in the previous analysis, start with the dimensionless Navier–Stokes equations and the corresponding boundary conditions, but this time $y_{\text{wall}} = y_{\text{def}}$, the deformation function describing the wall roughness. Since the roughness is stationary, all equations and flow functions are independent of time. It is beneficial however, to reuse the variable $X$ as the fast streamwise length scale of the roughness such that

$$x = 1 + Re^{-3/8} \lambda^{-5/4} X. \quad (3.3)$$

The deformation height of the roughness is assumed to be small compared to the boundary layer thickness. The shape of the wall deformation is described by

$$y_{\text{def}} = Re^{-5/8} \lambda^{-3/4} y_0(X), \quad \text{with} \quad y_0(X) = h Y_0(X), \quad (3.4)$$

where the parameter $h \ll 1$ and $Y_0(X) = O(1)$. The assumption of a small roughness element allows the perturbations to be linearised w.r.t. $h$ in the viscous sublayer and thus simplifies the analysis.

### 3.1.2 Lower deck

Similarly to the previous analysis, the variables are rescaled as (2.4) and (2.7), so the viscous sublayer equations and their corresponding boundary conditions (2.8) are obtained. Since the roughness is stationary, the displacement function is only dependent on the horizontal variable, $A = A(X)$.

From equation (2.8b), it is clear that $P$ is independent of $y_3$. Assuming that the flow perturbations are small by using $h \ll 1$, the fluid-dynamic functions expand as

$$U(X, y_3) = y_3 + h v^3_R(X, y_3) + \ldots, \quad P(X) = h p^3_R(X) + \ldots, \quad (3.5a)$$
$$V(X, y_3) = h v^3_R(X, y_3) + \ldots, \quad A(X) = h A_R(X) + \ldots, \quad (3.5b)$$
so the corresponding equations for $u^3_R$, $v^3_R$ and $p^3_R$ are

\[
y_3 \frac{\partial u^3_R}{\partial X} + v^3_R = -\frac{dp^3_R}{dX} + \frac{\partial^2 u^3_R}{\partial y_3^2}, \quad (3.6a)
\]

\[
\frac{\partial u^3_R}{\partial X} + \frac{\partial v^3_R}{\partial y_3} = 0. \quad (3.6b)
\]

To formulate the required boundary conditions, consider the following. Since the roughness height is assumed to be $O(h)$ and $y_0(X) = h Y_0(X)$, $Y_0$ is a $O(1)$ quantity. Then, using (3.5a), in this case, (2.8d) becomes

\[
U(X, hY_0) = h Y_0(X) + h u^3_R(X, h Y_0) + \cdots = 0 \quad \text{at} \quad y_3 = h Y_0(X). \quad (3.7)
\]

Since the height of the roughness is assumed to be very small, $h \ll 1$, a Taylor expansion for $u^3_R$ can be used:

\[
u^3_R(X, h Y_0(X)) \approx u^3_R(X, 0) + h Y_0(X) \frac{\partial u^3_R}{\partial y_3}(X, 0) + \ldots
\]

Substituting this into (3.7) gives the relation

\[
u^3_R(X, 0) = -Y_0(X), \quad (3.8)
\]

to leading order. Applying the same reasoning to $v^3_R$ gives

\[
v^3_R(X, 0) = 0. \quad (3.9)
\]

Using a similar argument for equation (3.6a) at the wall yields the additional relation:

\[
\frac{\partial^2 u^3_R}{\partial y_3^2} = \frac{dp^3_R}{dX} \quad \text{at} \quad y_3 = 0. \quad (3.10)
\]
Thus the boundary conditions for equations (3.6) are

\begin{align}
(u_R^3, v_R^3) &= (-Y_0(X), 0) \quad \text{at} \quad y_3 = 0, \\
\frac{\partial^2 u_R^3}{\partial y_3^2} &= \frac{d p_R^3}{d X} \quad \text{at} \quad y_3 = 0.
\end{align}

Equations (3.6) and the corresponding boundary conditions (3.11) will be revisited in due course, after having examined the flow behaviour in the middle and upper decks.

### 3.1.3 Middle deck

Assuming an incoming Blasius flow (2.14), the middle deck scalings (2.13) remain, so the corresponding equations and boundary conditions are the same as (2.15) and the solutions are given by (2.17). Assuming the flow perturbations \((\tilde{u}, \tilde{v}, \tilde{p})\) are of \(O(h)\) to allow for matching with the lower deck, write

\begin{align}
\tilde{u}(X, y_2) &= h u_R^2(X, y_2), \\
\tilde{v}(X, y_2) &= h v_R^2(X, y_2), \\
\tilde{p}(X) &= h p_R^2(X),
\end{align}

so the \(O(h)\) perturbations in the middle deck become

\begin{align}
\frac{u_R^2(X, y_2)}{\frac{dU_B}{dy_2}(y_2),} \\
\frac{-v_R^2(X, y_2)}{\frac{dA_R}{dX} U_B(y_2)}.
\end{align}

where \(U_B(y_2)\) is the rescaled Blasius flow as defined in (2.14).
3.1.4 Upper deck

The upper deck scalings are the same as (2.20). Rewriting the flow perturbations as $\mathcal{O}(h)$ functions,

$$
\ddot{u}(X, y_1) = h \, u^1_R(X, y_1), \quad \ddot{p}(X, y_1) = h \, p^1_R(X, y_1),
\dot{v}(X, y_1) = h \, v^1_R(X, y_1),
$$

the pressure $\ddot{p}$ will again be governed by Laplace’s equation (2.22),

$$
\nabla^2 p^1_R(X, y_1) = 0, \quad (3.13)
$$

and the corresponding boundary conditions are

$$
p^1_R \rightarrow 0 \quad \text{as} \quad y_1 \rightarrow \infty, \quad (3.14a)
$$

$$
\frac{\partial p^1_R}{\partial y_1} = \frac{d^2 A_R}{dX^2} \quad \text{at} \quad y_1 = 0. \quad (3.14b)
$$

3.1.5 Viscous-inviscid interaction in Fourier space

In order to find a solution for $p^1_R$, it is easier to switch to Fourier space as this will simplify the calculations. Define $\mathcal{F}$ to be the Fourier transform of $\mathcal{F}$ such that

$$
\mathcal{F}(\xi) = \int_{-\infty}^{+\infty} \mathcal{F}(X) \, e^{-i\xi X} \, dX, \quad (3.15a)
$$

$$
\mathcal{F}(X) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathcal{F}(\xi) \, e^{i\xi X} \, d\xi, \quad (3.15b)
$$

where $\xi$ is the Fourier variable in $X$ and the derivative of $\mathcal{F}$ becomes $\mathcal{F}' = i\xi \mathcal{F}$. Before continuing the analysis, it is worth considering the validity of employing a Fourier transform, as it cannot be proven at this point that the functions in question decay as $X \rightarrow \pm \infty$. Assume for now, subsequent to later confirmation, that a function does not grow exponentially, but algebraically, $\mathcal{F} \sim X^\lambda$. After enough differentiations, the function will decay, but the differential equation will still be the same and can be solved in Fourier space. Then, before inverting $\mathcal{F}$ back to physical space, check that
the initial assumption of algebraic growth was justified, and, if needed, differentiate \( F \) enough times until the expression can be integrated over the whole domain. It then only remains to integrate w.r.t. \( X \) as many times as desired to obtain the final solution.

Rewriting equation (3.13) in Fourier Space gives

\[
\frac{\partial^2 \tilde{p}_1^R}{\partial y_1^2} - \xi^2 \tilde{p}_1^R = 0, \tag{3.16}
\]

whose solution is

\[
\tilde{p}_1^R(\xi, y_1) = D_1(\xi) \, e^{-|\xi|y_1} + D_2(\xi) \, e^{\xi|y_1}. \tag{3.17}
\]

The boundary conditions become

\[
\begin{align*}
\tilde{p}_1^R &\to 0 \text{ as } y_1 \to \infty, \tag{3.18a} \\
\frac{\partial \tilde{p}_1^R}{\partial y_1} &\equiv -\xi^2 \tilde{A}_R(\xi) \text{ at } y_1 = 0. \tag{3.18b}
\end{align*}
\]

The first boundary condition shows that \( \tilde{p}_1^R \) is bounded as \( y_1 \to \infty \), so set \( D_2(\xi) = 0 \). The second boundary condition then gives the relation

\[
D_1(\xi) = |\xi| \tilde{A}_R(\xi),
\]

so

\[
\tilde{p}_1^R(\xi, y_1) = |\xi| \tilde{A}_R(\xi) \, e^{-|\xi|y_1}, \tag{3.19}
\]

and therefore

\[
\tilde{p}_3^R(\xi) = \tilde{p}_2^R(\xi) = \tilde{p}_1^R(\xi, 0) = |\xi| \tilde{A}_R(\xi). \tag{3.20}
\]

Now that the pressure perturbations have been determined across all decks, it is time to revisit the lower deck, where equations (3.6) and the corresponding boundary conditions (3.11) have been determined. Differentiating (3.6a) with respect to \( y_3 \) and substituting (3.6b) yields the partial differential equation

\[
\frac{\partial^2 u_3^R}{\partial y_3^2} - y_3 \frac{\partial^2 u_3^R}{\partial X \partial y_3} = 0, \tag{3.21}
\]
which needs to be solved with respect to the boundary conditions. Rewriting (3.21) and (3.11) in Fourier Space gives

\[ \frac{\partial^3 \tilde{u}_R^3}{\partial y_3^4} + i\xi y_3 \frac{\partial \tilde{u}_R^3}{\partial y_3} = 0, \]  

with the corresponding boundary conditions

\[ \tilde{u}_R^3 = -Y_0(\xi) \text{ at } y_3 = 0, \]  
\[ \tilde{u}_R^3 \to \bar{A}_R(\xi) \text{ as } y_3 \to \infty, \]  
\[ \frac{\partial^2 \tilde{u}_R^3}{\partial y_3^2} = i|\xi| \bar{A}_R \text{ at } y_3 = 0. \]

Introduce the new variable

\[ w = \gamma_R y_3 \text{ where } \gamma_R = (i\xi)^{1/3}. \]  

Equation (3.22a) is transformed into the well-known Airy equation

\[ \frac{\partial^3 \tilde{u}_R^3}{\partial w^4} - w \frac{\partial \tilde{u}_R^3}{\partial w} = 0, \]  

whose solution is

\[ \frac{\partial \tilde{u}_R^3}{\partial w} = E_1(\xi) \text{Ai}(w) + E_2(\xi) \text{Bi}(w). \]  

The arbitrary functions \( E_1(\xi) \) and \( E_2(\xi) \) will be determined by the boundary conditions

\[ \tilde{u}_R^3 = -Y_0(\xi) \text{ at } w = 0, \]  
\[ \tilde{u}_R^3 \to \bar{A}_R(\xi) \text{ as } w \to \infty, \]  
\[ \frac{\partial^2 \tilde{u}_R^3}{\partial w^2} = (i\xi)^{1/3}|\xi| \bar{A}_R \text{ at } w = 0. \]
Before this analysis can be continued, it is important to define a branch cut in the complex $\xi$-plane, as $\gamma_R$ is a multi-valued quantity in $\xi$, similar to $\gamma_{TS}$. By introducing a branch cut on the positive imaginary $\xi$-axis (Figure 3.2i) and writing

$$\xi = |\xi| e^{i\phi} \text{ where } -\frac{3\pi}{2} < \phi \leq \frac{\pi}{2},$$

(3.26a)

$\gamma_R$ can be expressed as

$$\gamma_R = |\xi|^{1/3} e^{i\left(\frac{\phi}{3} + \frac{\pi}{6}\right)}.$$  

(3.26b)

For $\xi \in \mathbb{R}$, $\gamma_R = |\xi|^{1/3} e^{\pm i \frac{\pi}{6}}$, so the solution (3.25a) will lie on either of the two dashed rays on the right-hand side of the $\gamma_R$-plane (Figure 3.2ii), with $|\arg(w)| = \frac{\pi}{6}$.

![Figure 3.2: Branch cuts](image)

In the chosen domain for $w$ where $|\arg(w)| < \frac{\pi}{3}$, the Airy functions exhibit the behaviour $\text{Ai} \rightarrow 0$ and $\text{Bi} \rightarrow \infty$ as $w \rightarrow \infty$. Since $\bar{u}_R^3$ is bounded as $w \rightarrow \infty$, set $E_2(\xi) = 0$, and integrate (3.25a) to get

$$\bar{u}_R^3(\xi, w) = E_1(\xi) \int_0^w \text{Ai}(s) \, ds + E_3(\xi).$$

(3.27)

From (3.25b), it is clear that

$$E_3(\xi) = -\bar{\nabla}_0(\xi).$$
Applying (3.25c) and (3.25d) and noting that $\int_0^\infty \text{Ai}(s)\,ds = \frac{1}{3}$, $\bar{A}_R(\xi)$ and $E_1(\xi)$ are found to be:

\[
E_1(\xi) = \frac{3(i\xi)^{1/3}|\xi| Y_0(\xi)}{(i\xi)^{1/3}|\xi| - 3 \text{Ai}'(0)}, \quad (3.28a)
\]

\[
\bar{A}_R(\xi) = \frac{3 \text{Ai}'(0) Y_0(\xi)}{(i\xi)^{1/3}|\xi| - 3 \text{Ai}'(0)}, \quad (3.28b)
\]

where $\text{Ai}'(0) = \frac{-1}{3^{1/3} \Gamma \left(\frac{1}{3}\right)}$.

Using (3.20) and the no-slip condition $\bar{v}_R^3 = 0$ at $w = 0$, the Fourier transforms of the $O(h)$ flow perturbations in the lower deck for a Blasius flow over a stationary roughness are determined as

\[
\bar{u}_R^3(\xi, w) = \frac{3(i\xi)^{1/3}|\xi| Y_0(\xi)}{(i\xi)^{1/3}|\xi| - 3 \text{Ai}'(0)} \int_0^w \text{Ai}(s)\,ds - Y_0(\xi), \quad (3.29a)
\]

\[
\bar{v}_R^3(\xi, w) = -\frac{3i \text{sgn}(\xi)\xi^2 Y_0(\xi)}{(i\xi)^{1/3}|\xi| - 3 \text{Ai}'(0)} \left[ w \int_0^w \text{Ai}(s)\,ds - \text{Ai}'(w) + \text{Ai}'(0) \right] - Y_0(\xi) w, \quad (3.29b)
\]

\[
\bar{p}_R^3(\xi) = \frac{3 \text{Ai}'(0)|\xi| Y_0(\xi)}{(i\xi)^{1/3}|\xi| - 3 \text{Ai}'(0)}. \quad (3.29c)
\]

Note that these are all dependent on the Fourier transform of the roughness shape, $Y_0(\xi)$. To find $(u_R, v_R, p_R)$, it remains to take the inverse Fourier transform (3.15). However this is not necessary, as the calculations in the following chapter will also use a Fourier transform in $X$. 
3.2 Interaction: problem formulation and expansions

In the rest of this chapter, the goal is to determine the effect a small roughness element on an otherwise flat plate has on an incoming T–S wave. This is done by comparing the initial amplitude of the T–S wave to its amplitude after the roughness and calculating the transmission coefficient for this problem. A similar setup was recently investigated by Wu & Dong [2016], based on a previous paper by Wu & Hogg [2006], and the results will be compared in more detail later on.

The motivation behind this analysis is based on the numerical research presented in Part II, where the flow stability for a range of surface deformations on an aerofoil is investigated. It is important to understand how surface roughness elements influence the laminar-turbulent transition in boundary layers, especially whether and by how much they amplify or reduce existing perturbations. The application to aeroplanes in particular is discussed in Section 5.1.

Previously, the flow behaviour of two independent regimes was analysed and their perturbations were determined:

1. an incoming Tollmien–Schlichting wave over a flat plate, with $O(\varepsilon)$ flow perturbations $e^{i\alpha x + i\omega T}(u_{TS}, v_{TS}, p_{TS})$ in Chapter 2 and

2. a Blasius flow encountering a stationary roughness, $y_0(x) = Re^{-5/8}hY_0(X)$, with $O(h)$ flow perturbations $(u_R, v_R, p_R)$ in Section 3.1,

where $k = 1, 2, 3$ denotes the deck number. The aim in this chapter is to determine the perturbation terms that describe the interaction between the two regimes, i.e. what happens when an incoming Tollmien–Schlichting wave hits a roughness, as pictured in Figure 3.3.
Figure 3.3: Triple Deck Structure of the incoming Tollmien–Schlichting wave encountering a stationary roughness

In order to find expressions for the interaction terms, define the following expansions, with $k$ again denoting the deck:

\begin{align}
    u^k(X, y, T) &= U_0 + \varepsilon e^{i\alpha X + i\omega T} u_{TS}^k + h u_R^k + \varepsilon h e^{i\omega T} u_I^k \quad (3.30a) \\
    v^k(X, y, T) &= \varepsilon e^{i\alpha X + i\omega T} v_{TS}^k + h v_R^k + \varepsilon h e^{i\omega T} v_I^k \quad (3.30b) \\
    p^k(X, y, T) &= \varepsilon e^{i\alpha X + i\omega T} p_{TS}^k + h p_R^k + \varepsilon h e^{i\omega T} p_I^k \quad (3.30c)
\end{align}

The indices $TS$ and $R$ denote the previously studied $O(\varepsilon)$ Tollmien–Schlichting and $O(h)$ roughness perturbations, the index $I$ denotes the interaction terms, $U_0(y)$ is the base flow such that $U_0 = 1$ for the upper deck, $U_0 = U_B(y_2)$, the rescaled Blasius flow from (2.14) for the middle deck and $U_0 = y_3$ for the lower deck. The pair $(\alpha, \omega)$ are the dimensionless T–S wavenumber and frequency from (2.2) and (2.9). Since the roughness is stationary and independent of time, the interaction perturbations $(u_I, v_I, p_I)$ are assumed to be periodic in $T$, just like $(u_{TS}, v_{TS}, p_{TS})$: they exhibit the same periodicity as the incoming T–S wave.
3.3 Interaction: middle and upper decks

Since the middle and upper deck equations are linear, they will be the same as in the previously analysed Triple Deck cases. Using $u^2 = \tilde{u}$ in (2.13) and $u^1 = \tilde{u}$ in (2.20), the $O(\varepsilon h)$ velocity and pressure perturbations can be expressed as follows:

$$u^2_I(X, y_2) = A_I(X) \frac{dU_B}{dy_2} \quad \tilde{u}^1_I(\xi, y_2) = -|\xi| \bar{A}_I(\xi) e^{-i|\xi|y_1}, \quad (3.31a)$$

$$v^2_I(X, y_2) = -\frac{dA_I}{dX} U_B(y_2) \quad \tilde{v}^1_I(\xi, y_2) = -i\xi \bar{A}_I(\xi) e^{-i|\xi|y_1}, \quad (3.31b)$$

$$\bar{p}^2_I(\xi) = |\xi| \bar{A}_I(\xi) \quad \bar{p}^1_I(\xi, y_1) = |\xi| \bar{A}_I(\xi) e^{-i|\xi|y_1}. \quad (3.31c)$$

The pressure perturbations and the velocity perturbations in the upper deck are in Fourier space, and $A_I(X)$ is the $O(\varepsilon h)$ term of the displacement function.

3.4 Derivation of the interaction equation

Consider the viscous sublayer. Using the same balancing arguments as in (2.4), and substituting (3.30) into (2.8), $u^3 = U$, the lower deck equations become, to $O(\varepsilon h)$:

$$i\omega u_I + y_3 \frac{\partial u_I}{\partial X} + v_I + f(X, y_3) = -\frac{\partial p_I}{\partial X} + \frac{\partial^2 u_I}{\partial y_3^2} \quad (3.32a)$$

$$\frac{\partial p_I}{\partial y_3} = 0 \quad (3.32b)$$

$$\frac{\partial u_I}{\partial X} + \frac{\partial v_I}{\partial y_3} = 0, \quad (3.32c)$$

where the upper index 3 is dropped to simplify notation. Note that the factor $e^{i\omega T}$ has been scaled out of these equations which implies that the frequency of the incoming T–S wave does not change as it travels downstream over the roughness. Furthermore, the extra forcing function $f$ is defined as

$$f(X, y_3) = e^{i\alpha X} \left( u_{TS} \frac{\partial u_R}{\partial X} + i\alpha u_{TS} u_R + v_{TS} \frac{\partial u_R}{\partial y_3} + \frac{\partial u_{TS}}{\partial y_3} v_R \right), \quad (3.33)$$

$$=: e^{i\alpha X} g(X, y_3).$$
From (3.32b), it can be seen that \( p_I \) is independent of \( y_3 \), so
\[
p_I = p^3_I(X) = p^2_I(X),
\]
and thus the Fourier transform of \( p_I \) can be obtained by matching with (3.31c)
\[
\tilde{p}_I = \tilde{p}^3_I(\xi) = \tilde{p}^2_I(\xi) = |\xi|\tilde{A}_I(\xi).
\] (3.34)

### 3.4.1 Boundary conditions

This system of equations needs to be solved with the corresponding boundary conditions: the no-slip conditions on the wall and the attenuation conditions as \( y_3 \to \infty \) that allow matching with the middle deck.

First, consider the no-slip condition on the wall,
\[
(U, V) = (0, 0) \quad \text{at} \quad y_3 = y_{wall} = hY_0(X).
\] (3.35)

Using (3.30a) where \( u^3 = U \), and substituting the corresponding values for \( y_3 \) and \( u_R \):
\[
0 = hY_0 + \varepsilon u_{TS}(hY_0) e^{i\alpha X + i\omega T} - hY_0 + \varepsilon h u_I(X, hY_0) e^{i\omega T},
\]
\[
= \varepsilon e^{i\omega T} [u_{TS}(hY_0) e^{i\alpha X} + hu_I(X, hY_0)].
\] (3.36)

Taylor expanding \( u_{TS} \) at \( y_3 = hY_0(X) \) for small \( h \), and using that \( u_{TS}(0) = 0 \), gives
\[
u_{TS}(hY_0(X)) = u_{TS}(0) + hY_0(X) \frac{du_{TS}}{dy_3} \bigg|_{y_3=0} + ... = hY_0(X) \frac{(i\alpha)^{1/3}A_{TS} Ai(z_0)}{\int_{z_0}^\infty Ai(s)ds} + ... = h p_c Y_0(X) + ...,\]
where the constant \( p_c \) is defined as
\[
p_c = \frac{(i\alpha)^{1/3}A_{TS} Ai(z_0)}{\int_{z_0}^\infty Ai(s)ds}.
\] (3.37)
Similarly, to first order,
\[ u_I(X, hY_0(X)) \simeq u_I(X, 0). \]

Equating the two remaining terms in equation (3.36) then gives
\[ u_0(X) := u_I(X, 0) = p_c Y_0(X) e^{i\alpha_0 X}. \quad (3.38) \]

The same analysis can be done for \( v_{TS} \) and \( v_I \). Also, since \( p_{TS} \) and \( p_R \) are independent of \( y_3 \), \( P = 0 \) implies that \( p_I = 0 \) at \( y_3 = 0 \).

Furthermore, when matching the horizontal velocity perturbations between the lower and middle decks, it is necessary that
\[ u_3^I|_{y_3 \to \infty} = u_2^I|_{y_2 \to 0} = A_I(X). \]

Thus the boundary conditions needed to solve (3.32) are:
\[ (u_I, v_I, p_I) = (u_0(X), 0, 0) \quad \text{at} \quad y_3 = 0, \quad (3.39a) \]
\[ u_I \to A_I(X) \quad \text{as} \quad y_3 \to \infty. \quad (3.39b) \]

### 3.4.2 Fourier space

In order to solve (3.32), it is easiest to take the Fourier transform in \( X \) as defined in (3.15). The procedure is similar to that employed for the flow past a steady roughness, except for the extra term in the \( x \)-momentum equation, function \( f \). Using (1.32) and relation (3.33), it can be seen that \( f \) exhibits exponential growth if \( \alpha_i < 0 \), i.e. if the frequency of the incoming T–S wave is supercritical, \( \omega > \omega_n \). Since the use of Fourier analysis is technically only valid if the incoming perturbation is neutral or decaying, a more thorough discussion is necessary, which will be done later on in Section 3.6.5. For now, assume that the frequency and wavenumber are neutral,
\[ \omega = \omega_n = 2.29797 \quad \text{and} \quad \alpha = \alpha_n = -1.00049. \quad (3.40) \]
Then the Fourier transform of function \( f(X, y_3) \) defined by (3.33) is calculated as
\[
\tilde{f}(\xi, y_3) = \int_{-\infty}^{\infty} f(X, y_3) e^{-i\xi X} dX \\
= \int_{-\infty}^{\infty} e^{i\alpha X} g(X, y_3) e^{-i\xi X} dX \\
= \int_{-\infty}^{\infty} g(X, y_3) e^{-i(\xi-\alpha)X} dX \\
= \tilde{g}(\xi - \alpha, y_3).
\] (3.41)

From now on, to simplify nomenclature, this shift in the Fourier variable will be denoted by the index \( \alpha \), introducing the new variables
\[
\xi_\alpha = \xi - \alpha, \quad w_\alpha = w(\xi_\alpha, y_3) = (i\xi_\alpha)^{1/3} y_3,
\] (3.42a)
as well as the ‘new’ roughness functions
\[
(\bar{u}_{R\alpha}, \bar{v}_{R\alpha}) = (\bar{u}_R, \bar{v}_R) (\xi_\alpha, w_\alpha).
\] (3.42b)

Using Fourier transform properties and simplifying, \( \tilde{g} \) becomes
\[
\tilde{g}(\xi, y_3) = i\xi u_{TS} \bar{u}_{R\alpha} + v_{TS} \frac{\partial \bar{u}_{R\alpha}}{\partial y_3} + \frac{du_{TS}}{dy_3} \bar{v}_{R\alpha},
\] (3.43)
where the T–S perturbations do not depend on \( \xi \).

Switching to Fourier space, equations (3.32a), (3.32c) and their boundary conditions (3.39) become
\[
(i\omega + i\xi y_3) \bar{u}_I + \bar{v}_I + \bar{g} = -i\xi \bar{p}_I + \frac{\partial^2 \bar{u}_I}{\partial y_3^2} \\
i\xi \bar{u}_I + \frac{\partial \bar{v}_I}{\partial y_3} = 0
\] (3.44a) (3.44b)
and

\[
(\bar{u}_I, \bar{v}_I, \bar{p}_I) = (\bar{u}_0(\xi), 0, 0) \quad \text{at} \quad y_3 = 0, \quad (3.44c)
\]

\[
\bar{u}_I \rightarrow \bar{A}_I(\xi) \quad \text{as} \quad y_3 \rightarrow \infty, \quad (3.44d)
\]

where, taking the Fourier transform of \(3.38\),

\[
\bar{u}_0(\xi) = \rho_c Y_0(\xi). \quad (3.45)
\]

Differentiating \(3.44a\) with respect to \(y_3\) and substituting \(3.44b\) yields the equation

\[
\frac{\partial^3 \bar{u}_I}{\partial y_3^3} - (i\xi y_3 + i\omega) \frac{\partial \bar{u}_I}{\partial y_3} = \frac{\partial \bar{g}}{\partial y_3}, \quad (3.46)
\]

As a third boundary condition, evaluate \(3.44a\) at \(y_3 = 0\) to get the relation

\[
\frac{\partial^2 \bar{u}_I}{\partial y_3^2}(\xi, 0) = i\xi \bar{p}_I(\xi) + i\omega \bar{u}_0(\xi). \quad (3.47)
\]

**Variable change**

Introducing the new variable

\[
\eta = w + \eta_0 = (i\xi)^{1/3} y_3 + \eta_0 \quad \text{with} \quad \eta_0 = \frac{i\omega}{(i\xi)^{2/3}}, \quad (3.48)
\]

where \((i\xi)^{1/3} = \gamma_R\) was discussed previously in \(3.26b\), then yields the equation

\[
\frac{\partial^3 \bar{u}_I}{\partial \eta^3} - \eta \frac{\partial \bar{u}_I}{\partial \eta} = \frac{1}{(i\xi)^{2/3}} \frac{\partial \bar{g}}{\partial \eta}. \quad (3.49)
\]

Define the function \(G(\xi, \eta)\) such that

\[
\bar{Y}_0(\xi) G(\xi, \eta) = \frac{1}{(i\xi)^{2/3}} \frac{\partial \bar{g}}{\partial \eta} = \frac{1}{i\xi} \frac{\partial \bar{g}}{\partial y_3}
\]

\[
= \frac{1}{i\xi} \left( i\alpha \frac{du_{TS}}{dy_3} \bar{u}_{R_a} + i\xi_\alpha u_{TS} \frac{\partial \bar{u}_{R_a}}{\partial y_3} + v_{TS} \frac{\partial^2 \bar{u}_{R_a}}{\partial y_3^2} + \frac{d^2 u_{TS}}{dy_3^2} \bar{v}_{R_a} \right). \quad (3.49)
\]
It is more convenient to write this in terms of the previously used variables \((z, w_\alpha)\), as this facilitates the asymptotic analysis needed later on. For details about this, refer to Appendix A1.

\[
\mathcal{Y}_0(\xi_\alpha) \bar{G}(\xi, \eta) = \frac{(i\alpha)^{2/3}}{i\xi} u''_{TS}(z) \bar{v}_{R_\alpha}(\xi_\alpha, w_\alpha) + \frac{(i\alpha)^{4/3}}{i\xi} u'_{TS}(z) \bar{u}_{R_\alpha}(\xi_\alpha, w_\alpha) \\
+ \frac{(i\xi_\alpha)^{2/3}}{i\xi} v_{TS}(z) \frac{\partial^2 \bar{u}_{R_\alpha}}{\partial w_\alpha^2}(\xi_\alpha, w_\alpha) + \frac{(i\xi_\alpha)^{4/3}}{i\xi} u_{TS}(z) \frac{\partial \bar{u}_{R_\alpha}}{\partial w_\alpha}(\xi_\alpha, w_\alpha).
\]

(3.50)

It is possible to factor out the F.T. of the roughness function \(\mathcal{Y}_0(\xi_\alpha)\), because both \(\bar{u}_{R_\alpha}\) and \(\bar{v}_{R_\alpha}\) are multiples of it. This simplifies the interaction equation to

\[
\frac{\partial^3 \bar{u}_I}{\partial \eta^3} - \eta \frac{\partial \bar{u}_I}{\partial \eta} = \mathcal{Y}_0(\xi_\alpha) \bar{G}(\xi, \eta),
\]

(3.51a)

with its corresponding boundary conditions

\[
\bar{u}_I = \bar{u}_0(\xi_\alpha) \quad \text{at} \quad \eta = \eta_0, \quad (3.51b)
\]

\[
\bar{u}_I \to \bar{A}_I(\xi) \quad \text{as} \quad \eta \to \infty, \quad (3.51c)
\]

\[
\frac{\partial^2 \bar{u}_I}{\partial \eta^2} = (i\xi)^{1/3} \bar{p}_I(\xi) + \eta_0 \bar{u}_0(\xi_\alpha) \quad \text{at} \quad \eta = \eta_0. \quad (3.51d)
\]

Before continuing the analysis, it is important to discuss the variable change from above, as \(\eta\) is a multi-valued function of \(\xi\), just like \(w\). The Fourier variable \(\xi\) is real, but \(\eta\) is complex. However, later calculations also include a contour integration in the complex \(\xi\)-plane, so it is useful to consider the general case for complex \(\xi\) in the following. The previously introduced branch cut in the upper imaginary \(\xi\)-plane (3.26) gives the relation

\[
\xi = |\xi| e^{i\phi} \quad \text{with} \quad -\frac{3\pi}{2} < \phi \leq \frac{\pi}{2}, \quad (3.52a)
\]

so

\[
\gamma_R = e^{i\frac{\pi}{3} + i\phi} |\xi|^{1/3} \quad \text{and} \quad |\arg(\gamma_R)| \leq \frac{\pi}{3}. \quad (3.52b)
\]
This implies that
\[ \eta = \gamma_R \left( y_3 + e^{-i\phi} \frac{\omega}{|\xi|} \right), \]  
(3.52c)
as well as
\[ \eta_0 = e^{i\frac{\pi}{6} - i\frac{2\phi}{3}} \frac{\omega}{|\xi|^{2/3}}, \]  
(3.52d)and thus
\[ -\frac{5\pi}{6} < \arg(\eta_0) \leq -\frac{\pi}{6}. \]  
(3.52e)

Consider the Airy functions Ai and Bi. Their asymptotic behaviour differs in different regions of the complex plane, depending on the argument of the complex number. This is called the Stokes phenomenon. The asymptotic formulae for the Airy function of the first kind are valid for \( |\arg| < \pi \), whereas those for the Airy functions of the second kind require \( |\arg| < \frac{\pi}{3} \). As detailed on the next page, only Airy functions of the first kind will be used in the solution to the interaction equation. Since both \( |\arg(\gamma_R)| \) and \( |\arg(\eta_0)| \) are smaller than \( \pi \), it is thus justified to used the asymptotic expansions in the following analysis.

### 3.5 Solution of the interaction equation

The general solution to the interaction equation \((3.51a)\) is
\[ \frac{\partial \bar{u}_I}{\partial \eta} = K_1(\xi) \text{Ai}(\eta) + K_2(\xi) \text{Bi}(\eta) + Y_0(\xi_0) \varphi(\xi, \eta), \]  
(3.53a)where \( \varphi \) may be determined numerically by solving the following boundary-value problem for each \( \xi \):
\[ \frac{\partial^2 \varphi}{\partial \eta^2} - \eta \varphi(\xi, \eta) = \bar{G}(\xi, \eta); \]  
(3.53b)
\[ \frac{\partial \varphi}{\partial \eta}(\xi, \eta_0) = 0, \quad \varphi(\xi, \infty) = 0. \]  
(3.53c)
This is done by implementing the Thomas algorithm (see e.g. P. Niyogi [2006]) over a uniform grid, thus achieving second order accuracy by defining

\[ \eta_j = j \Delta \eta + \eta_0 \quad \text{for} \quad j = 0..N - 1, \]  
\[ \varphi_j = Q_j \varphi_{j-1} + R_j \quad \text{for} \quad j = 1..N - 2, \]  
\[ \eta_0 = \eta_j = 0, \]  
\[ \varphi_0 = \varphi_1 = \frac{Q_1}{1 - R_1}. \]  

\[ \text{with the coefficients} \]

\[ R_{N-1} = Q_{N-1} = 0 \]
\[ R_j = \left\{ \begin{array}{ll}
(\Delta \eta)^2 \eta_j + 2 - R_{j+1} \\
\frac{1}{(\Delta \eta)^2 \eta_j + 2 - R_{j+1}}
\end{array} \right. \quad \text{for} \quad j = 1..N - 2, \]
\[ Q_j = \left\{ \begin{array}{ll}
Q_{j+1} - (\Delta \eta)^2 \bar{G}_j \\
(\Delta \eta)^2 \eta_j + 2 - R_{j+1}
\end{array} \right. \quad \text{for} \quad j = 1..N - 2, \]

and the boundary condition at \( \eta = \eta_0, \)

\[ \varphi_0 = \varphi_1 = \frac{Q_1}{1 - R_1}. \]

In the following, the boundary conditions (3.51b)-(3.51d) will be applied to (3.53a) to determine \( K_1 \) and \( K_2. \)

**Boundedness as \( \eta \to \infty \)**

For large \( y_3, \eta \sim \gamma_R y_3, \) so that \( |\arg(\eta)| = |\arg(\gamma_R)| = \frac{\pi}{6} \) for real \( \xi. \) The Airy function \( \text{Bi}(\eta) \) grows exponentially along these rays in the complex \( \xi \)-plane, so, to eliminate any exponential growth, set \( K_2(\xi) = 0. \)

**Behaviour at the wall**

Integrating the solution with respect to \( \eta \) and using boundary condition (3.51c) gives

\[ \bar{u}_I(\xi, \eta) = K_1(\xi) \int_{\eta_0}^{\eta} \text{Ai}(s) ds + \nabla_0(\xi_0) \int_{\eta_0}^{\eta} \varphi(\xi, s) ds + \bar{u}_0(\xi_0). \]  

(3.55)
Attenuation condition

Matching $\bar{u}_I$ with the middle deck solution using condition (3.51c) yields

$$K_1(\xi) \int_{\eta_0}^{\infty} \text{Ai}(s) ds + \mathcal{Y}_0(\xi\alpha) \int_{\eta_0}^{\infty} \varphi(\xi, s) ds + \bar{u}_0(\xi\alpha) = \bar{A}_I(\xi).$$ \hspace{1cm} (3.56)

Pressure condition at the wall

Moreover, condition (3.51d) gives the relation

$$K_1(\xi) \text{Ai}'(\eta_0) = (i\xi)^{1/3} \bar{p}_I(\xi) + \eta_0 \bar{u}_0(\xi\alpha),$$

$$= (i\xi)^{1/3}|\xi| \bar{A}_I(\xi) + \eta_0 \bar{u}_0(\xi\alpha),$$ \hspace{1cm} (3.57)

using matching between the lower and middle decks from earlier, (3.34).

Solution in Fourier Space

Solving for the two unknowns in (3.56), (3.57), and making use of (3.45) gives the following expressions

$$K_1(\xi) = \frac{\mathcal{Y}_0(\xi\alpha)}{\Delta(\xi, \omega)} \left[ (i\xi)^{1/3}|\xi| \left( p_c + \int_{\eta_0}^{\infty} \varphi(\xi, s) ds \right) + p_c \eta_0 \right],$$ \hspace{1cm} (3.58)

$$\bar{A}_I(\xi) = \frac{\mathcal{Y}_0(\xi\alpha)}{\Delta(\xi, \omega)} \left[ p_c \left( \eta_0 \int_{\eta_0}^{\infty} \text{Ai}(s) ds + \text{Ai}'(\eta_0) \right) + \text{Ai}'(\eta_0) \int_{\eta_0}^{\infty} \varphi(\xi, s) ds \right],$$ \hspace{1cm} (3.59)

where

$$\Delta_I(\xi, \omega) = \text{Ai}'(\eta_0) - (i\xi)^{1/3}|\xi| \int_{\eta_0}^{\infty} \text{Ai}(s) ds.$$ \hspace{1cm} (3.60)

Setting $\Delta_I(\xi, \omega) = 0$ yields the dispersion equation. It represents the large-Reynolds-number version of the Orr-Sommerfeld equation, as discussed in (2.5), (2.40). In this case, $\xi \in \Re$, so both signs need to be considered:

$$\Delta_I^{\pm}(\xi, \omega) = \text{Ai}'(\eta_0) - (i\xi)^{1/3}|\xi| \int_{\eta_0}^{\infty} \text{Ai}(s) ds = 0.$$ \hspace{1cm}

For $\xi > 0$, this equation does not have any real roots. For $\xi < 0$ however, its first five roots $\eta_0^{(j)}$ are pictured in Figure 3.4i with the corresponding $\xi^{(j)}$ in Figure 3.4ii.
For the neutral frequency $\omega_n = 2.29797$, the first root is real and equal to the Tollmien–Schlichting wavenumber, $\xi^{(1)} = \alpha_n = -1.00049$. For $\omega < \omega_n$, $\xi^{(1)} > 0$ and the perturbations decay, but for $\omega > \omega_n$, $\xi^{(1)} < 0$ and the corresponding modes will start to grow downstream. All the other roots have a positive imaginary part, so that their corresponding perturbations in the boundary layer are expected to decay.

Using relation $\text{(3.34)}$, the Fourier transform of the $\mathcal{O}(\varepsilon h)$ pressure perturbation in the viscous sublayer $\text{(3.34)}$ can be written as

$$
\bar{p}_I(\xi) = \frac{|\xi|}{\Delta(\xi, \omega)} \left[ p_c \left( \eta_0 \int_{\eta_0}^{\infty} \text{Ai}(s) \, ds + \text{Ai}'(\eta_0) \right) + \text{Ai}'(\eta_0) \int_{\eta_0}^{\infty} \varphi(\xi, s) \, ds \right], \quad (3.61)
$$

where $p_c$ was defined in $\text{(3.37)}$.

Having obtained an expression for $\bar{p}_I$, it remains to take the Inverse Fourier transform to find $p_I$. 

---

(i) Roots $\eta_0^{(j)}$ of the dispersion relation

(ii) Roots $\xi^{(j)}$ of the dispersion relation

Figure 3.4: Roots of the dispersion relation
3.6 Pressure perturbation generated by the interaction

The pressure perturbation of $O(\varepsilon h)$ in the lower deck can be determined by solving the integral

$$p_I(X) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{p}_I(\xi) e^{iX\xi} d\xi$$

$$= \frac{1}{2\pi} \left[ \int_{-\infty}^{0} \bar{p}_I^-(\xi) e^{iX\xi} d\xi + \int_{0}^{\infty} \bar{p}_I^+(\xi) e^{iX\xi} d\xi \right],$$

$$= \frac{1}{2\pi} \left[ \int_{-\infty}^{0} I_-(\xi) d\xi + \int_{0}^{\infty} I_+(\xi) d\xi \right] , \quad (3.62)$$

where the index denotes the sign of $\xi$, so

$$\bar{p}_I^\pm(\xi) = \frac{\pm \xi \gamma_0(\xi)}{\Delta_I^\pm(\xi)} \left[ p_c(\eta_0) \int_{\eta_0}^{\infty} \text{Ai}(s) ds + \text{Ai}'(\eta_0) \right] + \Delta_I^\pm(\xi) \int_{\eta_0}^{\infty} \varphi(\xi, s) ds , \quad (3.63)$$

$$\Delta_I^\pm(\xi) = \text{Ai}'(\eta_0) \pm i(i\xi)^{4/3} \int_{\eta_0}^{\infty} \text{Ai}(s) ds \quad (3.64)$$

and to simplify calculations, define

$$I_\pm(\xi) = \bar{p}_I^\pm(\xi) e^{iX\xi} . \quad (3.65)$$

Now consider the analytical extensions of the integrands $I_\pm(\xi)$ into the complex upper $\xi$-plane. To look at the flow behaviour downstream, i.e. where $X > 0$, it is necessary that $\Im(\xi) > 0$, so that the modulus of the exponential decays. Instead of integrating along the real $\xi$-axis, it is easier to introduce the two contours $\gamma_\pm$ and perform two separate integrations in the complex upper half plane, as shown in Figure 3.5. Note that the angle $\phi_0$ is chosen such that only root 1 from Figure 3.4ii lies inside the wedge. Then

$$p_I(X) = \frac{1}{2\pi} \left( \int_{C^-} I_-(\xi) d\xi + \int_{C^+} I_+(\xi) d\xi \right) . \quad (3.66)$$
Furthermore, using contour properties (see e.g. Ablowitz & Fokas [2003]),

\[ \int_{\gamma_-} I_- d\xi = \int_{C^-} I_- d\xi + \int_{C_2^-} I_- d\xi + \int_{C_R^-} I_- d\xi, \quad (3.67a) \]

\[ \int_{\gamma_+} I_+ d\xi = \int_{C^+} I_+ d\xi + \int_{C_2^+} I_+ d\xi + \int_{C_R^+} I_+ d\xi, \quad (3.67b) \]

where, by Cauchy’s Theorem,

\[ \int_{\gamma_-} I_- d\xi = 2\pi i \text{Res} \{ I_-, \xi = \alpha \} \quad \text{and} \quad \int_{\gamma_+} I_+ d\xi = 0. \quad (3.68) \]

Hence

\[ \int_{C^-} I_- d\xi = 2\pi i \text{Res} \{ I_-, \alpha \} - \int_{C_2^-} I_- d\xi - \int_{C_R^-} I_- d\xi, \quad (3.69a) \]

\[ \int_{C^+} I_+ d\xi = -\int_{C_2^+} I_+ d\xi - \int_{C_R^+} I_+ d\xi. \quad (3.69b) \]

It can be shown that (see Appendix [B])

\[ \int_{C_R^-} I_- d\xi = \int_{C_R^+} I_+ d\xi = 0 \quad \text{as} \quad R \to \infty, \quad (3.70) \]

and thus

\[ p_I(X) = i \text{Res} \{ I_-(\xi), \alpha \} - \frac{1}{2\pi} \left( \int_{C_2^-} I_- d\xi + \int_{C_2^+} I_+ d\xi \right). \quad (3.71) \]
It remains to determine the residue and the two integrals on the right hand side, with
$I_{\pm}$ as defined in (3.65). To evaluate these integrals, note that they can be transformed
into Laplace-type integrals whose significant contributions will only come from the
neighbourhood where $\xi \to 0$. This concept will be discussed in more detail in the
following subsection in which the behaviour of $I_{\pm}$ for small $\xi$ will be analysed first, in
order to simplify the necessary calculations.

### 3.6.1 Behaviour of the integrand for small $\xi$

As $\xi \to 0$, $\eta_0 = \frac{i\omega}{(i\xi)^{2/3}} \to \infty$, so it is valid to use the asymptotic expansions for the Airy
functions for numbers with large moduli (Abramowitz & Stegun [1964]):

\[
\int_{\eta_0}^{\infty} \text{Ai}(s) ds \sim \frac{e^{-\zeta_0}}{2\sqrt{\pi}} \left( \eta_0^{-3/4} - \frac{165}{144} \eta_0^{-9/4} + \mathcal{O} \left( \eta_0^{-15/4} \right) \right), \tag{3.72a}
\]

\[
\text{Ai}'(\eta_0) \sim \frac{-e^{-\zeta_0}}{2\sqrt{\pi}} \left( \eta_0^{1/4} - \frac{21}{144} \eta_0^{-5/4} + \mathcal{O} \left( \eta_0^{-11/4} \right) \right), \tag{3.72b}
\]

where $\zeta_0 = \frac{2}{3}\eta_0^{3/2}$. These give the following useful relations

\[
\eta_0 \int_{\eta_0}^{\infty} \text{Ai}(s) ds + \text{Ai}'(\eta_0) = \frac{-e^{-\zeta_0}}{2\sqrt{\pi}} \eta_0^{-5/4} + \mathcal{O} \left( \eta_0^{-11/4} \right), \tag{3.73a}
\]

\[
\pm i(i\xi)^{4/3} \int_{\eta_0}^{\infty} \text{Ai}(s) ds = \pm \frac{\omega}{\zeta^2} \gg 1. \tag{3.73b}
\]

The second relation shows that the integral Airy term can be neglected for small $\xi$ in
comparison to $\text{Ai}'(\eta_0)$, so $\Delta_{\pm} \sim \text{Ai}'(\eta_0)$ for small $\xi$.

A small note on the behaviour of the Airy functions of the first kind for complex $\xi$: as discussed in (3.52e), $-\frac{5\pi}{6} < \arg(\eta_0) \leq -\frac{\pi}{6}$, which means that $\text{Ai}(\eta_0)$ and its
derivatives do not decay for every complex $\xi = |\xi|e^{i\phi}$. Consider the exponent in the
asymptotic expansion:

\[
\zeta_0 = \frac{2}{3} \eta_0^{3/2} = \frac{2}{3} \left( e^{\frac{i\phi}{6} - \frac{2\phi}{3}} \frac{\omega}{|\xi|^{2/3}} \right)^{3/2} = \frac{2\omega^{3/2}}{3|\xi|} e^{\frac{i\phi}{2} - i\phi},
\]

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\[ \Re(\zeta_0) = \frac{2\omega^{3/2}}{3|\xi|} \cos\left(\frac{\pi}{4} - \phi\right) > 0 \quad \text{if} \quad \phi \in \left(-\frac{3\pi}{2}, -\frac{5\pi}{4}\right) \cup \left(-\frac{\pi}{4}, \frac{\pi}{2}\right) \]

and therefore

\[ \text{Ai}(\eta_0) \sim e^{-\zeta_0} \eta_0^{-1/4} = e^{-\Re(\zeta_0)} e^{i\Im(\zeta_0)} \eta_0^{-1/4} \begin{cases} 
0 & \text{if} \quad \Re(\zeta_0) > 0, \\
\infty & \text{if} \quad \Re(\zeta_0) < 0.
\end{cases} \]

This implies that \( \text{Ai}(\eta_0) \) and its derivatives exhibit exponential growth for \( \phi \in \left(-\frac{5\pi}{4}, -\frac{\pi}{4}\right) \), however this can be disregarded in this case, as the exponents in the integrand cancel each other out.

In addition to using the Airy expansions, assume for now, subject to later confirmation, that

\[ \int_{\eta_0}^{\infty} \varphi(\xi, s) ds = p_\varphi \xi^\Lambda + O(\xi^{\Lambda+1}), \quad (3.74) \]

where the unknowns \( \lambda \) and \( p_\varphi(\alpha, \omega) \) can be determined by looking at relation \( (3.50) \) in the asymptotic limit as \( \xi \to 0 \). With this assumption, the Fourier transform of the pressure perturbation becomes

\[ \tilde{p}_I^\pm(\xi) = \pm \xi Y_0(\xi) \left[ \frac{p_c}{\eta_0^{3/2}} + O\left(\eta_0^{-3}\right) + \int_{\eta_0}^{\infty} \varphi(\xi, s) ds \right] \]

\[ = \pm \xi Y_0(\xi) \left[ \frac{ip_c}{\omega^{3/2}} \xi + O(\xi^2) + p_\varphi \xi^\Lambda + O(\xi^{\Lambda+1}) \right] \]

\[ = \pm Y_0(\xi) \left[ \frac{ip_c}{\omega^{3/2}} \xi^2 + p_\varphi \xi^{\Lambda+1} + O(\xi^{\Lambda}) \right] \quad \text{as} \quad \xi \to 0, \quad (3.75) \]

where \( p_c \) and \( p_\varphi \) are constants depending on \( \omega, \alpha \). The exponent is \( \Lambda = 3 \) or \( \Lambda = \lambda + 2 \) – whichever is lower. This approximation for small \( \xi \) will be used in the following two calculations of the ray integrals.
Integral $\bar{p}^+_I$ on $C^+_2$

Start with the integral $\int_{C^+_2} I_+ d\xi$ in (3.71). On $C^+_2$, $\phi = \frac{\pi}{2}$, so $\xi = |\xi| e^{i\frac{\pi}{2}} = i |\xi|$:

$$\int_{C^+_2} \bar{p}^+_I(\xi) e^{iX\xi} d\xi = \int_{0}^{\infty} \bar{p}^+_I(\xi) e^{-X|\xi|} i d|\xi| = -i \int_{0}^{\infty} \bar{p}^+_I(\xi) e^{-X|\xi|} d|\xi|.$$ 

This is a Laplace-type integral that can be approximated for large $X$ using Laplace’s method, see e.g. [Bender & Orszag 1999]. This states that the dominant contributions to the integral will come only from points in the neighbourhood of the unique global maximum of the function in the exponent which is $|\xi|_{\text{max}} = 0$ for this case. It is therefore justified to look at the integral in the neighbourhood $[0, \delta]$, where $\delta \ll 1$, such that $\delta > \frac{1}{X}$, and replace the integrand by its approximation for small $\xi$ which was derived in the above. Thus

$$\int_{C^+_2} \bar{p}^+_I(\xi) e^{iX\xi} d\xi \simeq -i \int_{0}^{\delta} \bar{p}^+_I(\xi) e^{-X|\xi|} d|\xi|$$

$$= -i \int_{0}^{\delta} Y_0(\xi_0) \left[ -i p_e \omega^{3/2} |\xi|^2 + i^{\lambda+1} p_v |\xi|^{\lambda+1} + O(|\xi|^\Lambda) \right] e^{-X|\xi|} d|\xi|$$

$$\simeq \int_{0}^{\infty} Y_0(\xi_0) \left[ -p_e \omega^{3/2} |\xi|^2 + i^{\lambda} p_v |\xi|^{\lambda+1} + O(|\xi|^\Lambda) \right] e^{-X|\xi|} d|\xi|$$

$$= Y_0(-\alpha) \left[ \frac{i^{\lambda} p_v \Gamma(\lambda + 2)}{X^{\lambda+2}} - \frac{p_e \Gamma(3)}{\omega^{3/2}X^3} \right] + O\left(X^{-\Lambda-1}\right)$$

$$\rightarrow 0 \text{ as } X \rightarrow \infty \text{ provided } \lambda > -2. \quad (3.76)$$

The final result was obtained by making use of Watson’s Lemma, see [Watson 1918], where it was assumed that the roughness function $Y_0(X)$ is integrable on the interval $X \in (-\infty, \infty)$ together with its modulus, i.e. the function, and consequently also its Fourier transform, are bounded - there exists a positive constant $B$ such that

$$|Y_0(\xi)| = \left| \int_{-\infty}^{\infty} Y_0(X) e^{-iX\xi} dX \right| \leq \int_{-\infty}^{\infty} |Y_0(X)| dX < B. \quad (3.78)$$

Note that downstream, i.e. as $X \rightarrow \infty$, the contribution from the integral is of $O(X^{-\Lambda-2})$ and becomes negligible provided $\lambda > -2$, which is also one of the conditions that need to be fulfilled to be able to apply Watson’s Lemma in the first place.
It will be shown in due course that $\lambda = 0$, so the condition holds.

**Integral $\overline{p}_I$ on $C_2^-$**

Consider the other integral in (3.71). On $C_2^-$, $\phi = -\pi - \phi_0$, so $\xi = -|\xi| e^{-i\phi_0}$:

$$
\int_{C_2^-} \bar{p}_I(\xi) e^{iX\xi} d\xi = -\int_0^\infty \bar{p}_I(\xi) e^{-iX|\xi| e^{-i\phi_0}} e^{-i\phi_0} d|\xi| = -e^{-i\phi_0} \int_0^\infty \bar{p}_I(\xi) e^{-iX|\xi| e^{-i\phi_0}} d|\xi|,
$$

using a similar justification as in the previous calculation. To evaluate this integral, an extension of Laplace’s Method could be used: the Method of steepest descent. However, since the residue contributions of the other poles $\xi_2, \xi_3, \ldots$ are transcendentally small (as their imaginary parts are positive for all $\omega$) and can therefore be neglected in comparison to the residue at $\alpha$, it is easiest to deform the contour $\gamma_-$ to simplify calculations. The new contour mirrors the shape of $\gamma_+$, where the new vertical path on the left of the imaginary axis shall be denoted by $C_{-3}^-$, but its direction is opposite to $C_2^+$’s. It can be shown that the new circular contour $C_{-R}^-$ tends to 0, just like $C_{-R}^-$. This means that

$$
\int_{C_2^-} I_-(\xi) d\xi = \int_{C_{-3}^-} I_-(\xi) d\xi.
$$

On $C_{-3}^-$, $\phi = -\frac{3\pi}{2}$, so $\xi = |\xi| e^{-i\frac{3\pi}{2}}$:

$$
\int_{C_{-3}^-} \bar{p}_I(\xi) e^{iX\xi} d\xi = i \int_0^\delta -Y_0(\xi_\alpha) \left[ -\frac{\pi}{\omega^{3/2}} |\xi|^2 + e^{-i(\lambda+1)\frac{3\pi}{2}} p_\varphi |\xi|^{\lambda+1} + O(|\xi|^\Lambda) \right] e^{-X|\xi|} d|\xi| \\
\approx \int_0^\infty Y_0(\xi_\alpha) \left[ \frac{p_\varphi}{\omega^{3/2}} |\xi|^2 + e^{-i\lambda\frac{3\pi}{2}} p_\varphi |\xi|^{\lambda+1} + O(|\xi|^\Lambda) \right] e^{-X|\xi|} d|\xi| \\
= Y_0(-\alpha) \left[ e^{-i\lambda\frac{3\pi}{2}} p_\varphi \frac{\Gamma(\lambda + 2)}{X^{\lambda+2}} - \frac{p_\varphi \Gamma(3)}{\omega^{3/2} X^3} \right] + O(X^{-\Lambda-1}) \tag{3.79}
$$

$$
\to 0 \quad \text{as} \quad X \to \infty \quad \text{provided} \quad \lambda > -2, \tag{3.80}
$$

again making use of Watson’s Lemma and the boundedness of the roughness function.
In conclusion: it has been shown that both ray integrals decay for large $X$ (downstream), provided the integral $\int_{\eta_0}^{\infty} \varphi(\xi, s) ds \sim p_\varphi \xi^\lambda$, where $\lambda > -2$. Since $\varphi$ is the solution of an Airy equation with forcing term $\bar{G}$ as defined in (3.50), asymptotic methods can be applied to verify that this is true.

**Asymptotic analysis of $\int_{\eta_0}^{\infty} \varphi(\xi, s) ds$ for small $\xi$**

The function $\varphi$ is the solution to the forced Airy equation

$$\frac{\partial^2 \varphi}{\partial \eta^2} - \eta \frac{\partial \varphi}{\partial \xi} = \frac{\partial^2 \varphi}{\partial w^2} - (w + \eta_0) \varphi = \bar{G}(\xi, \eta), \quad (3.81)$$

with the boundary conditions $\frac{\partial \varphi}{\partial \eta}(\xi, \eta_0) = \varphi(\xi, \infty) = 0$. In this paragraph, it will be shown that $\int_{\eta_0}^{\infty} \varphi(\xi, s) ds = O(1)$ in the limit as $\xi \to 0$.

First, note that, for the unstable case,

$$\eta_0 = \frac{i \omega}{(i \xi)^{2/3}} \to \infty \text{ as } \xi \to 0. \quad (3.82a)$$

Also,

$$\eta = (i \xi)^{1/3} y_3 + \eta_0 = w + \eta_0, \text{ where } \eta_0 \gg w \text{ as } \xi \to 0. \quad (3.82b)$$

With $\eta_0$ becoming very large for small $\xi$, the leading-order solution for $\varphi$ can be found by balancing the third term in the middle with the forcing term on the right in (3.81):

$$\varphi = -\frac{\bar{G}(\xi \to 0)}{\eta_0} \sim \xi^{2/3} \bar{G}(\xi \to 0). \quad (3.83)$$

To find the leading order behaviour for $\bar{G}$, consider the following. Since $\xi$ is small, the binomial series expansion can be used for any exponent $n$

$$(\xi_0)^n = (\xi - \alpha)^n = (-\alpha)^n \left(1 - \frac{\xi}{\alpha}\right)^n \sim e^{-in\pi} \alpha^n \left(1 + n \frac{\xi}{\alpha} + O(\xi^2)\right). \quad (3.84)$$
Then
\[ w_\alpha = (i\xi_\alpha)^{1/3}y_3 \simeq e^{-\frac{\pi}{3} \alpha^{1/3}} \left( 1 + \frac{\xi}{3\alpha} \right) y_3 \quad \text{as} \quad \xi \to 0, \]
so, to leading order, \( \xi_\alpha \) and \( w_\alpha \) are independent of \( \xi \), just like \( z \).
Hence
\[ \bar{G} = \frac{\hat{G}(y_3)}{\xi} \left( 1 + \mathcal{O}(\xi) \right) = \frac{\hat{G}(y_3)}{\xi} + \mathcal{O}(1) \quad \text{as} \quad \xi \to 0, \]  
(3.85)
where \( \hat{G} \) is \( \mathcal{O}(1) \) in \( \xi \), but dependent on \( y_3 \). Then, from (3.83),
\[ \varphi = -\frac{\hat{G}(y_3)}{\omega(i\xi)^{1/3}} + \mathcal{O}(\xi^{2/3}) \quad \text{as} \quad \xi \to 0. \]  
(3.86)
Integrating over \( \eta = (i\xi)^{1/3}y_3 + \eta_0 \),
\[ \int_{\eta_0}^{\infty} \varphi(\xi \to 0, \eta) \, d\eta = -\frac{1}{\omega} \int_{0}^{\infty} \hat{G}(y_3) \, dy_3 + \mathcal{O}(\xi) \quad \text{as} \quad \xi \to 0. \]  
(3.87)
Each term in \( \hat{G}(y_3) \) is dependent on one or more of the Airy functions \( \text{Ai}(z) \), \( \text{Ai}'(z) \), \( \text{Ai}(w_\alpha) \) and \( \text{Ai}'(w_\alpha) \) which are exponentially decaying as \( y_3 \) becomes large by definition of \( z \) and \( w_\alpha \). The integral of \( \varphi \) is thus convergent in \( y_3 \) and of \( \mathcal{O}(1) \) in \( \xi \), so from (3.74),
\[ \lambda = 0 > -2 \]  
(3.88)
and condition (3.80) is satisfied. This means that the contribution from the ray integrals \( C^\pm_2 \) is of \( \mathcal{O}(X^{-2}) \) and thus decays downstream.

### 3.6.2 Residue of \( p_I^-(\xi) e^{i\lambda_\xi} \) at \( \xi = \alpha \)

To calculate the residue from the pole at \( \xi = \alpha \), make use of the following formula for a complex number \( c \) ([Priestley 2003]):
\[ \text{Res} \left( \frac{g(c)}{h(c)}, c = c_0 \right) = \frac{g(c_0)}{h'(c_0)}. \]  
(3.89)
The derivative of the denominator of $\bar{p}_I$ can be calculated as follows

$$\frac{\partial}{\partial \xi} \left( Ai'(\eta_0) - i(i\xi)^{4/3} \int_{\eta_0}^{\infty} Ai(s)ds \right)$$

$$= \frac{\partial \eta_0}{\partial \xi} Ai''(\eta_0) + \frac{4}{3} (i\xi)^{1/3} \int_{\eta_0}^{\infty} Ai(s)ds + i(i\xi)^{4/3} \frac{\partial \eta_0}{\partial \xi} Ai(\eta_0)$$

$$= \frac{\partial \eta_0}{\partial \xi} Ai(\eta_0) \left[ \eta_0 + i(i\xi)^{4/3} \right] + \frac{4}{3} (i\xi)^{1/3} \int_{\eta_0}^{\infty} Ai(s)ds$$

$$= - \frac{2\eta_0}{3 \xi} Ai(\eta_0) \left[ i(i\xi)^{4/3} + \eta_0 \right] - \frac{4}{3} i(i\xi)^{4/3} \int_{\eta_0}^{\infty} Ai(s)ds$$

$$= - \frac{1}{3 \xi} \left[ 2\eta_0 Ai(\eta_0) \left[ i(i\xi)^{4/3} + \eta_0 \right] + 4i(i\xi)^{4/3} \int_{\eta_0}^{\infty} Ai(s)ds \right]$$

Hence the residue of the integrand at $\xi = \alpha$ becomes

$$\text{Res} \left( \bar{p}_I e^{i\xi X}, \xi = \alpha \right)$$

$$= \frac{3\alpha^2 Y_0(0) \left[ p_c \left( z_0 \int_{z_0}^{\infty} Ai(s)ds + Ai'(z_0) \right) + Ai'(z_0) \int_{z_0}^{\infty} \varphi(s)ds \right]}{2z_0 Ai(z_0) \left[ i(i\alpha)^{4/3} + z_0 \right] + 4i(i\alpha)^{4/3} \int_{z_0}^{\infty} Ai(s)ds} e^{i\alpha X}, \quad (3.90)$$

since $\eta_0 = z_0$ when evaluated at $\xi = \alpha$. To determine $\int_{z_0}^{\infty} \varphi(s)ds$, it remains to go back to (3.50) yet again, substitute for $\xi = \alpha$ and solve the corresponding forced Airy equation.

### 3.6.3 Analytical solution for $\int_{z_0}^{\infty} \varphi(s)ds$

When evaluating equation (3.50) at $\xi = \alpha$, $\xi_\alpha = 0$, all terms but one vanish, as can be seen by combining expressions (3.29a) and (3.29b) with (3.42b), so that

$$\bar{u}_{R_\alpha} = -\nabla \theta(0), \quad \bar{v}_{R_\alpha} = \frac{\partial \bar{u}_{R_\alpha}}{\partial y_3} = \frac{\partial^2 \bar{u}_{R_\alpha}}{\partial y_3^2} = 0 \quad \text{at} \quad \xi_\alpha = 0, \forall y_3.$$

Thus the forcing term $\bar{G}$ becomes

$$\bar{G}(\alpha, z) = \frac{-(i\alpha)^{1/3} A_{TS}}{\int_{z_0}^{\infty} Ai(s)ds} Ai(z) = G_\alpha Ai(z). \quad (3.91)$$
Then the integral
\[ \int_{\eta_0}^{\infty} \phi(\xi, \eta) d\eta \bigg|_{\xi=\alpha} = \int_{z_0}^{\infty} \phi(z) dz \]
can be determined by solving the differential equation
\[ \frac{d^2\varphi}{dz^2} - z \varphi(z) = G_\alpha \text{Ai}(z); \quad \frac{d\varphi}{dz}(z_0) = 0, \quad \varphi(\infty) = 0, \quad (3.92) \]
and integrating over the whole domain. This can be done numerically using the Thomas algorithm, as described in \[3.54\], but in this case it is also possible to solve the equation analytically using the variation of parameters method laid out in \[\text{Kamke} 1948\]. The general solution to equation \[3.92\] is
\[ \varphi = \left[ L_1 - \pi G_\alpha \int_{z_0}^{z} \text{Ai}(s) \text{Bi}(s) ds \right] \text{Ai}(z) + \left[ L_2 + \pi G_\alpha \int_{z_0}^{z} [\text{Ai}(s)]^2 ds \right] \text{Bi}(z), \]
where \( L_1, L_2 \) are arbitrary constants to be determined by the boundary conditions. To eliminate any exponential growth as \( z \to \infty \), \( L_2 \) needs to be set as
\[ L_2 = -\pi G_\alpha \int_{z_0}^{\infty} [\text{Ai}(z)]^2 dz, \]
so \( \varphi \) simplifies to
\[ \varphi(z) = \left[ L_1 - \pi G_\alpha \int_{z_0}^{z} \text{Ai}(s) \text{Bi}(s) ds \right] \text{Ai}(z) - \pi G_\alpha \text{Bi}(z) \int_{z}^{\infty} [\text{Ai}(s)]^2 ds. \]
Differentiating w.r.t. \( z \) and using the boundary condition at \( z = z_0 \) yields
\[ L_1 = \frac{\pi G_\alpha \text{Bi}'(z_0)}{\text{Ai}'(z_0)} \int_{z_0}^{\infty} [\text{Ai}(s)]^2 ds. \]
Thus
\[ \varphi(z) = \pi G_\alpha \left[ \frac{\text{Bi}'(z_0)}{\text{Ai}'(z_0)} \int_{z_0}^{\infty} [\text{Ai}(s)]^2 ds - \int_{z_0}^{z} \text{Ai}(s) \text{Bi}(s) ds \right] \text{Ai}(z) \]
\[ - \pi G_\alpha \text{Bi}(z) \int_{z}^{\infty} [\text{Ai}(s)]^2 ds. \quad (3.93) \]
Figure 3.6 visualises $\varphi(\alpha, z)$. For an initial amplitude $A_{TS} = 1$, the integral has the numerical value

$$\int_{z_0}^{\infty} \varphi(s) ds = -0.705886 - 0.529667i = 0.882509 e^{-\pi 2.49786}.$$ \hspace{1cm} (3.94)

Figure 3.6: The function $\varphi(\alpha, z)$, verified via the Thomas Method

Note that $\varphi(z)$ scales with $G_\alpha$, and can therefore be written in terms of the previously introduced constant $p_c$ from (3.37). Introduce $\tilde{\varphi}$ such that

$$\varphi = \pi G_\alpha \tilde{\varphi} = -\frac{\pi p_c}{\text{Ai}(z_0)} \tilde{\varphi}. \hspace{1cm} (3.95)$$

Then

$$\int_{z_0}^{\infty} \varphi(s) ds = -\frac{\pi p_c}{\text{Ai}(z_0)} \int_{z_0}^{\infty} \tilde{\varphi}(s) ds. \hspace{1cm} (3.96)$$
3.6.4 Final result for the pressure perturbation

From (3.71), (3.79), (3.88), (3.90) and (3.96), the final expression for the large-$X$ behaviour of the $O(\varepsilon h)$ pressure perturbation in the viscous sublayer can then be written as

$$p_l(X) = \frac{1}{2} R_I(\omega) \bar{Y}_0(0) e^{i\alpha X} + O(X^{-2}),$$

(3.97)

with

$$R_I(\omega) = \frac{3i\alpha^2 p_c}{z_0 \bar{A_i}(z_0)} \left[ z_0 \int_{z_0}^{\infty} A_i(s) ds + A_i'(z_0) - \frac{\pi A_i'(z_0)}{A_i(z_0)} \int_{z_0}^{\infty} \bar{\varphi}(s) ds \right],$$

(3.98)

where the analytic solution for $\bar{\varphi}$ can be obtained from (3.93) and $p_c$ is defined in (3.37).

Note that the coefficient $R_I$ is a function of the frequency $\omega$ – it does not depend on the wall roughness shape.

3.6.5 Validity of the solution

In Section 3.4.2, it was briefly mentioned that for incoming T–S waves of supercritical frequencies, the use of the Fourier Transform technique is technically not valid, because for the first root, $\alpha^{(1)}_i < 0$ for $\omega > \omega_n$ and thus the perturbation $e^{i\alpha X} = e^{-\alpha_c X + i\alpha_r X}$ grows downstream. Terent’ev [1981] encountered the same problem when he considered the oscillations of a vibrator in subsonic flow, and restricted his analysis to oscillations of subcritical frequency $\omega < \omega_n$. Around the same time, Bogdanova & Ryzhov [1982] studied long wavelength perturbations at the inlet of a flat semi-infinite channel and made two important observations. Firstly, they noted that none of the experimental data revealed a sudden change in the fields of perturbation when the frequency passes through the critical value $\omega_n$, in particular no upstream propagation of strong perturbations. Secondly, they stipulated the addition of an extra term to the solution in order to satisfy the observed increase in the amplitude of the perturbations downstream, eventually leading to turbulent flow. For this to make sense from a physical point of view, the solution needs to be continuous with respect to $\omega$ for any finite $x$, thus making the choice for the additional term unique. This formulation enabled a continuous evolution of the linear perturbations with respect to $\omega$, and allowed the oscillations to be studied.
for \( \omega \geq \omega_n \). In his follow-up paper, Terent’ev [1984] used this postulate to extend his previous analysis for supercritical frequencies, applying both the Fourier transform in \( x \) and the Laplace transform in time, and solving the problem of the vibrating oscillator for finite \( x \) and \( t \to \infty \). These calculations show that for these constraints, the integration contour from Figure 3.5 can be extended into the lower-half plane for \( \alpha_i < 0 \) as the main contribution from the deformed contour \( C^- \) will still be provided by the neighbourhood \( \xi \to 0 \).

This postulate, as well as Terent’ev’s subsequent analysis, make the solution (3.97) obtained in this thesis valid not only for the neutral frequency and wavenumber \((\omega_n, \alpha_n)\), but for all frequencies \( \omega > 0 \) and corresponding wavenumbers \( \alpha \).

### 3.7 Conclusion

#### 3.7.1 Downstream amplitude of the T–S wave

The solution for the pressure perturbation of \( O(\varepsilon h) \) in the viscous sublayer (3.97) shows that a stationary roughness changes the amplitude of an incoming Tollmien–Schlichting wave in such a way that the \( O(h) \) correction to its initial amplitude is dependent on the coefficient \( R_I \), as well as the Fourier transform of the wall roughness shape evaluated at \( \xi = 0 \),

\[
A_{\varepsilon h} = |R_I(\omega)| \left| Y_0(0) \right|.
\] (3.99)

This result implies that the magnitude of the amplitude of the T–S wave downstream of the roughness is influenced by the wall roughness shape, more precisely, by the integral

\[
Y_0(0) = \int_{-\infty}^{\infty} Y_0(X) \, dX.
\] (3.100)

This means that, for the same roughness shape, the higher the roughness, the bigger is the distortion of the amplitude of the Tollmien–Schlichting wave downstream, which in turn implies that nonlinear effects may be triggered sooner and thus an earlier transition to turbulence would be expected. However, this does not work for roughness shapes that are odd, as the integral vanishes in that case.
3.7.2 Comparison with incoming T–S wave amplitude

Consider the pressure term in the lower deck to leading order in $X \gg 1, h \ll 1$:

$$p^3(X, y_3, T) = \varepsilon e^{i\alpha X + i\omega T} p^3_{TS} + \varepsilon h e^{i\alpha X + i\omega T} \frac{1}{2} \mathcal{R}_I(\omega) \overline{Y}_0(0)$$

$$= \varepsilon e^{i\alpha X + i\omega T} p^3_{TS} \left( 1 + \frac{h}{2 p^3_{TS}} \mathcal{R}_I(\omega) \overline{Y}_0(0) \right) + h p^3_R(X, y_3),$$

and define the transmission coefficient as

$$\mathcal{T}_I(\omega) = 1 + \frac{h}{2 p^3_{TS}} \mathcal{R}_I(\omega) \overline{Y}_0(0). \quad (3.101)$$

In Chapter 2, the amplitude of the T–S wave was denoted by the constant $A_{TS}$ which is a scaling factor in the velocity perturbations (2.42), specifically $p^3_{TS} = |\alpha| A_{TS}$. The constant $p_c$, defined in (3.37) is also dependent on $A_{TS}$, and scales the calculated transmission coefficient $\mathcal{R}_I(\omega)$. Therefore the ratio $\mathcal{T}_I' = \mathcal{R}_I/2p^3_{TS}$ is independent of $A_{TS}$, which means that $\mathcal{T}_I(\omega)$ does not depend on the initial amplitude of the incoming T–S wave, namely

$$\mathcal{T}_I(\omega) = 1 + h \overline{Y}_0(0) \mathcal{T}_I'(\omega) \quad (3.102)$$

where

$$\mathcal{T}_I'(\omega) = \frac{3\alpha(ia)^{4/3} \text{Ai}(z_0)}{2|\alpha|} \left[ z_0 \int_{z_0}^{\infty} \text{Ai}(s) ds + \text{Ai}'(z_0) - \frac{\pi A'(z_0)}{\text{Ai}(z_0)} \int_{z_0}^{\infty} \phi(s) ds \right] \right]. \quad (3.103)$$

To determine whether the roughness dampens or amplifies the incoming T–S wave, the modulus of $\mathcal{T}_I(\omega)$ needs to be investigated, as this is the remaining factor when comparing the amplitude of the T–S wave after the roughness to its initial, upstream amplitude:

$$\frac{A_{\text{downstream}}}{A_{\text{upstream}}} = \frac{|\varepsilon p^3_{TS} \mathcal{T}_I(\omega)|}{|\varepsilon p^3_{TS}|} = |\mathcal{T}_I(\omega)|. \quad (3.104)$$

Since $\overline{Y}_0(0) \in \mathbb{R}$,

$$\mathcal{T}_I = 1 + h \overline{Y}_0(0) \Re(\mathcal{T}_I') + i h \overline{Y}_0(0) \Im(\mathcal{T}_I'), \quad (3.105)$$

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and thus, for $h \ll 1$,

$$|T_I(\omega)| = \left( 1 + h Y_0(0) \Re(T'_I) \right)^2 + \left( h Y_0(0) \Im(T'_I) \right)^2 \right)^{1/2}$$

$$= \left( 1 + 2 h Y_0(0) \Re(T'_I) + O(h^2) \right)^{1/2}$$

$$= 1 + h Y_0(0) \Re(T'_I) + O(h^2).$$

(3.106)

If $|T_I| > 1$, the T–S wave is amplified, and if $|T_I| < 1$, it is dampened by the roughness. As $Y_0(0)$ is known, it is necessary to look at the sign of the real part of $T'_I$. Figure 3.7 shows $\Re(T'_I)$ for varying $\omega$.

![Figure 3.7: The real part of $T'_I(\omega)$](image)

The real part of the coefficient $T'_I$ is negative for $\omega \in (0, 7.59)$, so for all $\omega \leq \omega_n$. This implies that a protruding roughness, $Y_0(0) > 0$, dampens the amplitude of the incoming T–S wave by a factor of $O(h)$. However, in the case of a surface dent, $Y_0(0) < 0$, and so the $O(h)$ correction term to the initial amplitude is positive, meaning that a gap always enhances the T–S wave. This conclusion was also reached by Wu & Hogg [2006].
suggesting that a surface hump can act as a wave barrier to block the incoming T–S wave. It is important to note that this is only valid for small $h$, since nonlinear terms become important as $h$ is increased, as discussed in Wu & Dong [2016].
Chapter 4

Triple Deck Analysis over a vibrating semi-infinite plate attached to a flat plate

The first time Triple Deck theory was employed to study the receptivity of the boundary layer to external disturbances was by Terent’ev [1981]. He investigated the flow over a flat plate, with a short section acting as a triangular vibrator. Ruban et al. [2013] studied the generation of T–S waves in the boundary layer due to elastic vibrations of the wing surface, as well as the interaction of the flow with a roughness element. In their paper, they assumed that the amplitude of the wing vibrations is small, and investigated the particular case where the frequency of the wing vibrations is of \( O(Re^{1/8}) \). The motivation behind the research in this chapter is the same, but the problem setup is slightly different – there is no well-defined roughness on the wing surface in this case. Just as in previous chapters, it is assumed that the frequency is of the same order as the Tollmien–Schlichting frequency \( \omega_{TS} \sim Re^{1/4} \).

4.1 Problem formulation

Consider an elastic plate whose thickness is small compared to its width and length, such as a thin aluminium shell which is only two millimetres thick, but two meters wide and long, typical for a wing surface. There are various ways it can be fixed to a solid surface, depending on the material of the plate (see e.g. Graff [1975]). Composite
materials are usually glued together, however in this chapter, it is assumed that the plate is pinned to the solid surface such that its movement about the joint is restricted. This allows for the problem to be linearised.

The difference in magnitude of the shell’s dimensions means that it is possible to zoom in on one end and treat the shell as a semi-infinite rod, moving at a small angle \( \hat{\beta} \) about the joint situated at \( \hat{x} = 0 \), where it is attached to a solid surface. Moreover, any change in or dependence on the spanwise coordinate \( \hat{z} \) will be disregarded and the focus is instead on the two-dimensional case. To model this problem, consider a solid plate with a periodically vibrating section, as pictured in Figure 4.1. The characteristic length scale \( L \) is taken to be the distance from the leading edge of the plate to the joint.

![Figure 4.1: Geometry of the problem setup](image)

To model a semi-infinite elastic vibrating plate, introduce the horizontal Triple Deck scaling from previous analyses, where the mid-point of the horizontal \( X \)-region is situated at the joint, so \( x_0 = 0 \) and

\[
x = \text{Re}^{-3/8} \lambda^{-5/4} X,
\]

where \( \lambda \) is the undisturbed Blasius shear coefficient. As the vibration frequency is assumed to be of \( \mathcal{O} \left( \text{Re}^{1/4} \right) \), introduce the new variables such that

\[
\omega_{TS} = \text{Re}^{1/4} \lambda^{3/2} \omega \quad \text{and} \quad t = \text{Re}^{-1/4} \lambda^{-3/2} T.
\]
Using these scalings, the periodicity of the vibration can be modelled as

\[ e^{i\omega T} = e^{i\omega t} = \mathcal{O}(1). \]  

(4.2)

For \( X < 0 \), the vibration of the plate surface is defined by

\[ f_{\text{vib}}(X, T) = \tan(\beta_{\text{vib}}) \times, \]

(4.3)

where the angle of the vibrations is assumed to be small, \( \beta_{\text{vib}} \ll 1 \), so

\[ \tan(\beta_{\text{vib}}) = \beta_{\text{vib}} + \mathcal{O}(\beta_{\text{vib}}^3). \]

(4.4)

Setting

\[ \beta_{\text{vib}} = \beta e^{i\omega T}, \]

(4.5)

the surface is defined as

\[ y_{\text{wall}}(X, T) = \begin{cases} \Re^{-3/8} \lambda^{-5/4} \beta e^{i\omega T} X & \text{if } X < 0 \\ 0 & \text{if } X \geq 0. \end{cases} \]

(4.6)

The aim of the following analysis is to determine the restrictions in the magnitude of \( \beta \) within the Triple Deck structure, as pictured in Figure 4.2.

The flow is again governed by the Navier–Stokes equations (1.5) and the boundary conditions for this problem are the no-slip conditions at the surface, as well as the free-stream conditions at infinity. However, since the wall is not stationary in this case, the no-slip condition for the vertical velocity \( v \) changes slightly and becomes

\[ v = \frac{\partial y}{\partial t} \text{ at } y = y_{\text{wall}}. \]

(4.7)

The corresponding boundary conditions thus become

\[ (u, v, p) \to (1, 0, 0) \text{ as } x^2 + y^2 \to \pm\infty, \]

(4.8a)

\[ (u, v) \to \left( 0, \frac{\partial y}{\partial t} \right) \text{ at } y = y_{\text{wall}}. \]

(4.8b)
As before, the flow behaviour can be analysed separately in every deck, and a matching procedure can be employed to determine any remaining unknowns.

### 4.2 Lower deck

In the viscous sublayer, \( y = O(Re^{-5/8}) \) with \( y_3 = Re^{5/8} \lambda^{3/4} y \). From the definition of the wall shape (4.6) and the new scalings (4.1),

\[
\beta \sim \frac{y}{x} \sim Re^{-1/4}, \tag{4.9}
\]

since \( e^{i\omega T}X = O(1) \). Introducing the \( O(1) \) variable \( \beta_0 \), define

\[
\beta = Re^{-1/4} \lambda^{1/2} \beta_0 \tag{4.10}
\]

and thus

\[
y_3 = \begin{cases} 
\beta_0 e^{i\omega T}X & \text{if } X < 0 \\
0 & \text{if } X \geq 0 
\end{cases} \text{ at } y_3 = y_{wall}. \tag{4.11}
\]
Using the viscous sublayer scalings from before (2.4) and the affine transformation (2.7), the lower deck equations (2.8) are once again obtained. As the amplitude of the vibrating plate is assumed to be small, $\beta_0 \ll 1$, the system can be linearised by expanding the solution in powers of $\beta_0$. Since the system is influenced by a periodic vibration in $T$, the flow response is expected to be periodic as well:

$$U(X, y_3, T) = y_3 + \beta_0 e^{i\omega T} u_3(X, y_3) + \ldots, \quad (4.12a)$$
$$V(X, y_3, T) = \beta_0 e^{i\omega T} v_3(X, y_3) + \ldots, \quad (4.12b)$$
$$P(X, T) = \beta_0 e^{i\omega T} p_3(X) + \ldots. \quad (4.12c)$$

The flow is then governed by the following set of equations

$$i\omega u_3 + y_3 \frac{\partial u_3}{\partial X} + v_3 = -\frac{\partial p_3}{\partial X} + \frac{\partial^2 u_3}{\partial y_3^2}, \quad (4.13a)$$
$$\frac{\partial u_3}{\partial X} + \frac{\partial v_3}{\partial y_3} = 0. \quad (4.13b)$$

with the corresponding boundary conditions

$$\begin{align*}
(u_3, v_3) &= \left(0, \frac{\partial y_3}{\partial T}\right) \quad \text{at} \quad y_3 = y_{wall}, \quad (4.14a) \\
u_3 &\to A_v(X) + \ldots \quad \text{as} \quad y_3 \to \infty, \quad (4.14b)
\end{align*}$$

where $A(X, T) = \beta_0 e^{i\omega T} A_v(X)$ is the displacement function.

To simplify boundary condition (4.14a), note that, at the wall, for $X < 0$, (4.12a) becomes

$$U(X, y_{3s}, T) = \beta_0 e^{i\omega T} X + \beta_0 e^{i\omega T} X u_3(X, y_{3s})$$
$$= \beta_0 e^{i\omega T} (X + u_3(X, y_{3s})) = 0.$$

Since $\beta_0$ is small, a Taylor expansion in $y_3$ for $u_3$ gives

$$u_3(X, y_{3s}) = u_3(X, 0) + \beta_0 e^{i\omega T} X \frac{\partial u_3}{\partial y_3}(X, 0) + \ldots$$
So, to first order, (4.14a) can be written as

\[ u_3(X, 0) = \begin{cases} -X & \text{if } X < 0 \\ 0 & \text{if } X \geq 0, \end{cases} \]  

(4.15a)

A similar reasoning yields the vertical velocity perturbation at the wall

\[ v_3(X, 0) = \begin{cases} i\omega X & \text{if } X < 0 \\ 0 & \text{if } X \geq 0. \end{cases} \]  

(4.15b)

Furthermore, evaluating equation (4.13a) at \( y_3 = 0 \) gives the additional boundary condition

\[ \frac{\partial^2 u_3}{\partial y_3^2} = \frac{dp_3}{dX} \quad \text{at} \quad y_3 = 0. \]  

(4.15c)

### 4.3 Main deck

In the middle tier, \( y = \mathcal{O}(Re^{-1/2}) \), so take \( y_2 = Re^{1/2}y \). As before, the base flow present in this layer is the Blasius boundary layer flow \( U_B \) with the properties defined in (2.14), and the flow perturbations can be scaled as (2.13), so that the classical boundary layer equations (2.15) are obtained. To allow for matching with the lower deck, expand as

\[ \tilde{u}(X, y_2, T) = \beta_0 e^{i\omega T} u_2(X, y_2) \quad \text{and} \quad \tilde{p}(X, T) = \beta_0 e^{i\omega T} p_2(X). \]  

(4.16)

This yields the following solutions for the \( \mathcal{O}(\beta_0) \) velocity perturbations in the middle deck:

\[ u_2(X, y_2) = A_v(X) \frac{dU_B}{dy_2}, \]  

(4.17a)

\[ v_2(X, y_2) = -\frac{\partial A_v(X)}{\partial X} U_B(y_2). \]  

(4.17b)
4.4 Upper deck

In the upper deck, \( y = \mathcal{O}(Re^{-3/8}) \), so take \( y_1 = Re^{3/8} y \). Using the previously determined scalings \((2,20)\) yields the Laplace equation for the pressure. Expanding in \( \beta_0 \) again,

\[
(\bar{u}, \bar{v}, \bar{p})(X, y_1, T) = \beta_0 e^{i\omega T} \left( u_1, v_1, p_1 \right)(X, y_1),
\]

and taking the Fourier transform in \( X \), as defined in \((3.15)\), the following solutions for the \( \mathcal{O}(\beta_0) \) flow perturbations are obtained

\[
\begin{align*}
\bar{u}_1(\xi, y_1) &= -|\xi| \bar{A}_v(\xi) e^{-|\xi|y_1}, \\
\bar{v}_1(\xi, y_1) &= -i\xi \bar{A}_v(\xi) e^{-|\xi|y_1}, \\
\bar{p}_1(\xi, y_1) &= |\xi| \bar{A}_v(\xi) e^{-|\xi|y_1}.
\end{align*}
\]

4.5 Viscous-inviscid interaction in Fourier space

From the pressure perturbation in the upper deck, it can be concluded that

\[
\bar{p}_3(\xi) = \bar{p}_2(\xi) = \bar{p}_1(\xi, 0) = |\xi| \bar{A}_v(\xi).
\]

Revisiting the lower deck, the linearised equations \((4.13)\) need to be solved subject to the boundary conditions \((4.15)\). Eliminating \( v_3 \) between \((4.13a) \) and \((4.13b) \) by differentiating \((4.13a) \) w.r.t. \( y_3 \), yields the equation

\[
i\omega \frac{\partial u_3}{\partial y_3} + y_3 \frac{\partial^2 u_3}{\partial X \partial y_3} = \frac{\partial^3 u_3}{\partial y_3^3}.
\]

After applying the Fourier transform in \( X \), this becomes

\[
\frac{\partial^3 \bar{u}_3}{\partial y_3^3} = (i\omega + i\xi y_3) \frac{\partial \bar{u}_3}{\partial y_3}.
\]

To determine the corresponding boundary conditions in Fourier Space, note the following. Let \((u_0, v_0)(X) = (u_3, v_3)(X, 0)\). Then, from \((4.15a)\),

\[
\frac{du_0}{dX}(X) = \frac{\partial u_3}{\partial X}(X, 0) = \begin{cases} 
-1 & \text{if } X < 0 \\
0 & \text{if } X \geq 0
\end{cases} = H(X) - 1,
\]
where \( H(X) \) is the Heaviside step function. Then, by definition of \( H(X) \),

\[
\frac{d^2 u_0}{dX^2} = \delta(X).
\]

Taking the Fourier transform yields

\[
-\xi^2 \bar{u}_0(\xi) = \delta(X) = 1 \quad \Rightarrow \quad \bar{u}_0(\xi) = -\frac{1}{\xi^2}.
\]

Similarly,

\[
\bar{v}_0(\xi) = \frac{i\omega}{\xi^2}.
\]

The corresponding boundary conditions to equation (4.22a) are thus

(\(4.22b\))

\[
(\bar{u}_3, \bar{v}_3) = \left( -\frac{1}{\xi^2}, \frac{i\omega}{\xi^2} \right) \quad \text{at} \quad y_3 = 0,
\]

(\(4.22c\))

\[
\bar{u}_3 \rightarrow \bar{A}_v(\xi) \quad \text{as} \quad y_3 \rightarrow \infty,
\]

(\(4.22d\))

\[
\frac{\partial^2 \bar{u}_3}{\partial y_3^2} = i\xi \bar{p}_3 \quad \text{at} \quad y_3 = 0.
\]

Re-using the variable \( \eta \) from Section 3.4.2

\[
\eta = (i\xi)^{1/3} y_3 + \eta_0 \quad \text{with} \quad \eta_0 = \frac{i\omega}{(i\xi)^{2/3}},
\]

(4.23)

again yields the Airy equation

\[
\frac{\partial^3 \bar{u}_3}{\partial \eta^3} - \eta \frac{\partial \bar{u}_3}{\partial \eta} = 0,
\]

(4.24)

with the known solution

\[
\frac{\partial \bar{u}_3}{\partial \eta} = M_1(\xi) \text{Ai}(\eta) + M_2(\xi) \text{Bi}(\eta),
\]

(4.25a)

where \( M_1 \) and \( M_2 \) are arbitrary functions dependent on \( \xi \) that can be determined by
the boundary conditions

\begin{align}
(\bar{u}_3, \bar{v}_3) &= \left( -\frac{1}{\xi^2}, \frac{i\omega}{\xi^2} \right) \quad \text{at} \quad \eta = \eta_0, \quad (4.25b) \\
\bar{u}_3 &\to \bar{A}_v(\xi) \quad \text{as} \quad \eta \to \infty, \quad (4.25c) \\
\frac{\partial^2 \bar{u}_3}{\partial \eta^2} &= (i\xi)^{1/3} \bar{b}_3(\xi) \quad \text{at} \quad \eta = \eta_0. \quad (4.25d)
\end{align}

Condition (4.25c) requires that $\bar{u}_3$ is bounded as $\eta \to \infty$, so set $M_2(\xi) = 0$. Integrating with respect to $\eta$ and applying condition (4.25b) yields

$$
\bar{u}_3(\xi, \eta) = M_1(\xi) \int_{\eta_0}^{\eta} \text{Ai}(s) ds - \frac{1}{\xi^2}.
$$

Using (4.25c), this results in

$$
M_1(\xi) \int_{\eta_0}^{\infty} \text{Ai}(s) ds - \frac{1}{\xi^2} = \bar{A}_v(\xi). \quad (4.26)
$$

Moreover, substituting (4.20) in (4.25d) gives

$$
M_1(\xi) \text{Ai}'(\eta_0) = (i\xi)^{1/3}|\xi| \bar{A}_v(\xi). \quad (4.27)
$$

The unknowns $\bar{A}_v$ and $M_1$ can then be calculated by solving the matrix equation

$$
\begin{pmatrix}
\int_{\eta_0}^{\infty} \text{Ai}(s) \, ds & -1 \\
\text{Ai}'(\eta_0) & -(i\xi)^{1/3}|\xi| \end{pmatrix}
\begin{pmatrix}
M_1 \\
\bar{A}_v
\end{pmatrix} =
\begin{pmatrix}
\frac{1}{\xi^2} \\
0
\end{pmatrix}, \quad (4.28)
$$

whose determinant again establishes the dispersion relation

$$
\Delta_vib(\xi) = \text{Ai}'(\eta_0) - (i\xi)^{1/3}|\xi| \int_{\eta_0}^{\infty} \text{Ai}(s) \, ds,
$$

which is the large-Reynolds number version of the Orr-Sommerfeld equation discussed
in Section 2.5. The unknowns are found to be

\[ M_1(\xi) = \frac{-i\xi^{1/3}}{|\xi|\Delta_{vib}(\xi)} \quad \text{and} \quad \bar{A}_v(\xi) = \frac{-Ai'(\eta_0)}{\xi^2\Delta_{vib}(\xi)}, \quad (4.30) \]

so the Fourier transform of the \( \mathcal{O}(\beta_0) \) pressure perturbation in the viscous sublayer becomes

\[ \bar{p}_3(\xi) = \frac{-Ai'(\eta_0)}{|\xi|\Delta_{vib}(\xi)}. \quad (4.31) \]

To determine \( p_3 \), it remains to invert \( \bar{p}_3 \) by taking the Inverse Fourier transform.

### 4.6 Pressure perturbation in the lower deck

Having determined an expression for \( \bar{p}_3 \), the pressure perturbation in the lower deck can be found by integrating

\[ p_3(X) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{Ai'(\eta_0)}{\xi|\Delta_{vib}(\xi)|} e^{i\xi X} d\xi. \quad (4.32) \]

Since the integrand is singular at \( \xi = 0 \), and thus makes the analysis much more difficult as it involves divergent integrals, make use of the Fourier transform property for derivatives and integrate

\[ \frac{dp_3}{dX}(X) = -\frac{i}{2\pi} \int_{-\infty}^{\infty} \text{sgn}(\xi) \frac{Ai'(\eta_0)}{\Delta_{vib}(\xi)} e^{i\xi X} d\xi \quad (4.33) \]

instead, which is non-singular at \( \xi = 0 \).

The method employed to solve this integral closely follows the method used in Section 3.6. The integration will be performed along the real axis and since the interest lies in the behaviour of the perturbations downstream of the vibrating plate, take \( X > 0 \). Introduce

\[ I_{\pm}(\xi) = \pm \frac{Ai'(\eta_0) e^{i\xi X}}{\Delta_{vib}^{\pm}(\xi)} = \frac{\pm Ai'(\eta_0) e^{i\xi X}}{Ai'(\eta_0) + i(i\xi)^{1/3} \int_{\eta_0}^{\infty} \text{Ai}(s)ds}, \quad (4.34) \]
then
\[
\frac{dp_3}{dX}(X) = -\frac{i}{2\pi} \left( \int_{-\infty}^{0} I_-(\xi) \, d\xi + \int_{0}^{\infty} I_+(\xi) \, d\xi \right).
\]  \hspace{1cm} (4.35)

Consider the analytical extensions of the integrands into the complex upper \(\xi\)-plane. Having previously introduced a branch cut on the positive imaginary \(\xi\)-axis \((3.26)\), the integration interval is again split into two parts, for each of which a complex contour integration is performed. The corresponding contours \(\gamma_{\pm}\), as pictured in Figure \ref{fig:4.3}, are the same as in Section \([3.6]\) the only difference between the two integrations is the numerator of the integrand.

![Figure 4.3: The integration contour \(\gamma = \gamma_- + \gamma_+\)](image)

Making use of the relations \((3.67)\) and \((3.68)\), as well as the fact that the contributions from \(C_R^\pm\) are negligible as the radius \(R \to \infty\) (see Appendix \([\text{B}]\)), it remains to determine
\[
\frac{dp_3}{dX}(X) = -\frac{i}{2\pi} \left( \int_{C_-} I_-(\xi) \, d\xi + \int_{C_+} I_+(\xi) \, d\xi \right)
= \text{Res} \left\{ I_-(\xi), \xi = \alpha \right\} + \frac{i}{2\pi} \left( \int_{C_-} I_-(\xi) \, d\xi + \int_{C_+} I_+(\xi) \, d\xi \right). \tag{4.36}
\]

To evaluate the integrals on the right, note that they can be transformed into Laplace-type integrals which can be approximated for \(X \to \infty\) by looking at the behaviour of the exponential. The integrand highly peaks at the point \(\xi_{\text{max}}\) where the exponential attains its maximum and thus the main contribution to the integral will
come from the vicinity around that point. Thus an approximation for the integral can be found by neglecting the subdominant contributions from outside the vicinity of \( \xi_{max} = 0 \), in this case. This means that the behaviour of \( I_\pm \) for small \( \xi \) needs to be considered.

4.6.1 Ray integrals

As \( \xi \to 0, \eta_0 \to \infty \) and thus the large-modulus expansions for the Airy functions can be used, as defined in (3.72). Then

\[
\frac{\text{Ai}'(\eta_0)}{\pm i(i\xi)^{4/3} \int_{\eta_0}^\infty \text{Ai}(s) ds} = \pm \frac{\omega}{\xi^2} \gg 1 \quad \text{as} \quad \xi \to 0,
\]

so the denominator can be approximated by \( \text{Ai}'(\eta_0) \) and therefore

\[
I_\pm(\xi) \sim \pm e^{i\xi X} \quad \text{as} \quad \xi \to 0.
\]

On \( C_2^+ \), \( \xi = i|\xi| \), so

\[
\int_{C_2^+} I_+ d\xi = -i \int_0^\infty I_+ d|\xi| \simeq -i \int_0^\delta e^{-|\xi|X} d|\xi| \simeq -i \int_0^\infty e^{-|\xi|X} d|\xi| = -\frac{i}{X}.
\]

Similarly,

\[
\int_{C_2^-} I_- d\xi = i \int_0^\infty I_- d|\xi| \simeq -i \int_0^\delta e^{-|\xi|X} d|\xi| \simeq -i \int_0^\infty e^{-|\xi|X} d|\xi| = -\frac{i}{X}.
\]

Thus

\[
\frac{i}{2\pi} \left( \int_{C_2^-} I_- d\xi + \int_{C_2^+} I_+ d\xi \right) = \frac{1}{\pi X}.
\]
4.6.2 Residue at $\xi = \alpha$

To calculate the residue of $I_-(\xi)$ at the simple pole $\xi = \alpha$, formula (3.89) can be used again and yields

$$\text{Res} \left( I_-(\xi), \xi = \alpha \right) = r(\omega) e^{i\alpha X} \tag{4.40a}$$

with

$$r(\omega) = \frac{3\alpha \text{Ai}'(z_0)}{2z_0 \text{Ai}(z_0) [i(i\alpha)^{4/3} + z_0] + 4i(i\alpha)^{4/3} \int_{z_0}^{\infty} \text{Ai}(s) ds} \tag{4.40b}$$

4.6.3 Final result

Using the above results, (4.36) becomes, for large values of $X$,

$$\frac{dp_3}{dX}(X) = \frac{1}{2} r(\omega) e^{i\alpha X} + \frac{1}{\pi X}, \tag{4.41}$$

Integrating with respect to $X$ then gives the expression for the $O(\beta_0)$ pressure perturbation in the lower deck,

$$p_3(X) = \frac{1}{2} |R_{vib}(\omega)| e^{i\alpha X+i\theta_R} + \frac{1}{\pi} \log(X) + c.c., \tag{4.42a}$$

where $R_{vib}$ is the receptivity coefficient

$$R_{vib}(\omega) = \frac{-3i \text{Ai}'(z_0)}{z_0 \text{Ai}(z_0) [i(i\alpha)^{4/3} + z_0] + 2i(i\alpha)^{4/3} \int_{z_0}^{\infty} \text{Ai}(s) ds} \tag{4.42b}$$

and $\theta_R$ denotes the argument of $R_{vib}$.

4.7 Conclusion

The physical meaning of the downstream pressure perturbation in the lower deck can be explained as follows. The first term is of the same form as the previously determined pressure perturbation in section (3.97). However, as there is not incoming wave, $R_{vib}(\omega)$ is termed the receptivity coefficient and, just like the transmission coefficient earlier, it is only dependent on the frequency $\omega$. It can be calculated for a range of real values of $\omega$ by solving the previously discussed dispersion equation (4.29) which gives the corresponding values for $\alpha$ and $z_0$. The values of the Airy function, its derivative and
integral can then be obtained by solving the initial value problem for the Airy equation. Since $R_{vib}$ is a complex number, it can also be written as $R_{vib} = |R_{vib}| e^{i\psi}$. Through the addition of the complex conjugate, it can be observed that this represents a wave of amplitude $|R_{vib}(\omega)|$. The logarithmic term on the right stems from the flow around the corner, as outlined in Appendix C and can thus also be expected at higher orders.

Hence the analysis has shown that the periodic vibrations of the plate produce a Tollmien–Schlichting wave in the boundary layer whose amplitude is specified by the modulus of the receptivity coefficient $R_{vib}$. The modulus and argument of $R_{vib}$ are shown in Figure 4.4 for varying values of $\omega$. Contrary to other geometries, this type of receptivity does not need an additional disturbance to excite Tollmien–Schlichting waves – it does not require the so-called ”double resonance” condition to be satisfied.
Part II

Numerical analysis
Chapter 5

Motivation and background

5.1 Motivation

As already discussed in the Introduction, one of the main goals in aerospace research is to figure out how to maintain laminar flow over aeroplane wings for as long as possible, thus delaying the onset of turbulence. One of the influencing factors in this process is the wing surface – ideally it should be completely smooth. However, in practice, an airplane wing possesses surface features that could amplify potential disturbances in the flow and cause a premature onset of laminar-turbulent transition: an example of these deformations would be a gap or bump at the junction of the wing leading edge and the wing-box components. Gaps may be filled, but the filler shape may deteriorate over time – this is caused by continuous variations in temperature and pressure during flight, thus causing small deformations in the wing surface. It is therefore important to understand the impact these small surface features have on the flow’s stability, as this affects decisions in the design and manufacturing of planes. Note that in this chapter, the theory will be explained by means of the incompressible Navier–Stokes equations, but the methodology used for the numerical study in the following sections employs the compressible Navier–Stokes equations, as the Mach number is 0.75.
5.2 Linearised Navier–Stokes equations

The stability of a flow can be studied computationally through direct numerical simulations (DNS). Employing the incompressible flow decomposition,

\[
\mathbf{u} = \mathbf{U}_0 + \mathbf{U}' \quad \text{and} \quad p = P_0 + P',
\]

where \((\mathbf{U}_0, P_0)\) denotes the base flow and \((\mathbf{U}', P')\) the perturbations, and substituting these expressions into the dimensionless Navier–Stokes equations (1.5) yields the nonlinear disturbance equations

\[
\frac{\partial \mathbf{U}'}{\partial t} + (\mathbf{U}' \cdot \nabla) \mathbf{U}_0 + (\mathbf{U}_0 \cdot \nabla) \mathbf{U}' - (\mathbf{U}' \cdot \nabla) \mathbf{U}' = -\nabla p' + \frac{1}{\text{Re}} \nabla^2 \mathbf{U}'.
\]

For sufficiently small disturbances, such as the external perturbations in real flight conditions,

\[(\mathbf{U}', P') = \delta (\mathbf{u}', p'), \quad \delta \ll 1, \quad (\mathbf{u}', p') = \mathcal{O}(1),\]

the last, nonlinear term on the left-hand side of (5.2) can be neglected, yielding the linearised Navier–Stokes stability equations, hereafter abbreviated by the term LNS:

\[
\frac{\partial \mathbf{u}'}{\partial t} + \mathcal{L}(\mathbf{U}_0) \mathbf{u}' = 0,
\]

where

\[
\mathcal{L}(\mathbf{U}_0) \mathbf{u}' = (\mathbf{u}' \cdot \nabla) \mathbf{U}_0 + (\mathbf{U}_0 \cdot \nabla) \mathbf{u}' + \nabla p' - \frac{1}{\text{Re}} \nabla^2 \mathbf{u}',
\]

whose coefficients are determined by the base velocity \(\mathbf{U}_0\). These describe the behaviour of the perturbations and thus give insight into the stability of the flow. This method can be extended for compressible flow by adding additional terms.

The LNS equations are used in numerical studies, since they yield nearly perfect results for small disturbances. Unfortunately, these codes are computationally expensive and thus not very practical when considering a range of geometries and frequencies.
5.3 Parabolised stability equations

Based on previous research by Gaster [1974], Herbert & Bertolotti [1987] developed a new method to analyse the behaviour of disturbances in slowly varying flows, such as boundary layers, introducing the parabolised stability equations, referred to by the abbreviation PSE. Due to their parabolic character in the streamwise direction, they can be solved directly by a marching procedure and thus allow the user to examine the stability of the flow at a fraction of the computational cost of DNS, which is why they have been used extensively ever since. In his paper, Herbert [1997] describes the derivation in detail and discusses i.a. their validity and advantages, briefly mentioned in the following. Moreover, the PSE have been extended to compressible flows by Bertolotti & Herbert [1991].

The method uses the (spatial) normal mode decomposition,

\[
\begin{align*}
\mathbf{u}'(x, y, t) &= e^{i(\bar{\alpha}(x) - \omega t)} \mathbf{u}(x, y) + (c.c.) \\
&= e^{-\bar{\alpha}(x)} e^{i(\bar{\alpha}_r(x) - \omega t)} \mathbf{u}(x, y) + (c.c.),
\end{align*}
\]

(5.5a)

where \(\bar{\alpha}\) is defined as

\[
\bar{\alpha}(x) = \int_{x_0}^{x} \alpha(s) ds,
\]

(5.5b)

and \(\bar{\alpha}_r, i\) denote the real and imaginary parts of \(\bar{\alpha}\) respectively, and are referred to as wavenumber and growth rate. The term \(\bar{\mathbf{u}}\) is the amplitude function, slowly varying in the streamwise coordinate \(x\). Careful consideration of the base flow is required, as even small deviations in the streamwise direction may accumulate during the integration.

For non-parallel flows, it is therefore helpful to use the LNS equations for comparison. Furthermore, the PSE can only be solved using a marching technique if the stability problem is governed by downstream propagating modes, i.e. if the flow is convectively unstable, meaning that the disturbances are swept away from the source. However, when marching with a sufficiently small step size, the PSE exhibit weak ellipticity, which means that there exists a slight upstream influence, similar to the boundary layer equations. This can be avoided by choosing a sufficiently large marching step, but nevertheless suggests that a validation through comparison with a LNS analysis is

\[\text{Note that the frequency here is of opposite sign as the frequency defined in Part I.}\]
advisable, especially in the case of separated flow. Besides the modest computational expense, the PSE are particularly advantageous when calculating \( N \)-factors, as they directly provide the spatial amplitude growth rates at each location along the marching path, taking the evolution of the disturbance and the streamwise variation of the base flow into account. Researchers mostly use the PSE to complement other numerical simulations, like DNS or Reynolds-averaged Navier-Stokes (RANS).

### 5.4 \( N \)-factor method

Liepmann [1943] was the first to convert linear stability results into a transition criterion for practical applications, which, after some development, is used to this day for transition prediction in aerodynamical engineering. Termed the \( N \)-factor method, or \( e^N \) method, the \( N \)-factor is the amplitude growth rate defined by

\[
N = \log \left( \frac{A}{A_0} \right) \quad \text{or} \quad A = A_0 \ e^N \tag{5.6a}
\]

for a given frequency \( \omega \), where \( A_0 \) is the initial, small amplitude of the wave under examination at the onset position of instability \( x_0 \), and \( A \) is the wave amplitude at a location \( x \) downstream. \( N \) is usually based on the chordwise velocity perturbation, \( u' \). The \( N \)-factor measures the accumulated amplification of the mode over the distance \( x - x_0 \), and can also be calculated as

\[
N(x, \omega) = - \int_{x_0}^{x} \alpha_i(s, \omega) ds, \tag{5.6b}
\]

where \( \alpha_i \) is the imaginary part of the complex wavenumber \( \alpha \), and \( \alpha_i < 0 \) denotes spatially amplified disturbances in the \( x \)-direction, as discussed in Part I, [132]. In practical applications, such as the numerical study in this thesis, \( N \)-factors are calculated for a range of frequencies, and the so-called \( N \)-factor envelope is computed, determined by the highest \( N \)-factor at each \( x \)-location:

\[
N_{env}(x) = \max_\omega N(x, \omega). \tag{5.7}
\]

When \( N_{env} \) reaches the critical \( N \)-factor \( N_{crit} \), determined by experimental observations,
transition is deemed to occur. Typical values of $N_{\text{crit}}$ range from 7 to 9, depending on the flow geometry and the external disturbances in consideration. The $N$-factor envelope allows a direct comparison of the flow stability for different geometries, and will be employed in Chapter 7 to examine the effect of different gap widths and heights on the laminar-turbulent transition process. It should be noted that the $N$-factor method only measures the relative amplification of the instability modes, and does not account for the absolute amplitude, nor for nonlinear effects, which have been shown experimentally to be the most significant factors in determining the point of transition to turbulence. However, as discussed in more detail in the review paper by [Van Ingen, 2008], the $e^N$ method remains a useful tool for predicting the position of boundary layer transition in two-dimensional flows. He notes that the linear stability theory has been estimated to cover about 75-85% of the distance between the first instability and transition, and bases this on research by Obremski et al. [1969] and experiments by Klebanoff et al. [1962].

5.5 Literature

There have been several experimental studies investigating the effect small surface deformations have on the stability of the boundary layer over an aerofoil, like Fage [1943], Holmes et al. [1985], Ohara & Holmes [1985] or Wang & Gaster [2005], amongst many others. Numerical studies exploring the stability of T–S waves over isolated deformations have been carried out by i.a. Nayfeh et al. [1987], Cebeci & Egan [1989] and Masad & Iyer [1994], making use of the $e^N$-method to measure the stability of the flow, and, more recently by Wu & Hogg [2006], Gao et al. [2011], Wu & Dong [2016] and Xu et al. [2017]. Other surface deformations have been studied as well, such as wavy walls, for which Li & Malik [1996] were the first to develop a stability criterion that needs to be met when using PSE methods to analyse the stability of separated flows. They found that the ellipticity of the disturbance equations can be overcome by choosing a stream-wise step size that is greater than the real part of the wavenumber,

$$
\Delta x > \frac{1}{\alpha_r}.
$$

(5.8)

This produces numerically stable solutions and works especially well for two-dimensional boundary layer flows, as considered here. Wie & Malik [1998] used this criterion to re-
late the amplification of the \(N\)-factor to the dimensions of the imposed surface waviness. Thomas et al. [2017] were able to draw similar conclusions for the growth of T–S waves over an unswept aerofoil by employing a different methodology, which is described in Chapter 6 and used to obtain the results discussed in this thesis.

The main conclusion to draw from previous literature is that surface deformations, even if they are small, do seem to amplify the incoming T–S wave and therefore destabilise the flow in such a way that transition is triggered sooner than for a smooth surface. Nayfeh et al. [1987] and, more recently, Ruban et al. [2016] note that laminar separation of the base flow plays a significant role in this process, and from their experiments, Obara & Holmes [1985] even concluded that separation is the major cause of transition in predominantly two-dimensional flows.

5.6 Layout for Part II

In this thesis, the numerical analysis of the flow over a variety of geometries is described, comparing the effect of depth, height and width of small surface deformations on the stability of the flow and the point of transition to turbulence. These results have been produced as part of an internship at Airbus Group Innovations\(^2\), using software developed at Imperial College London. The motivation behind this study was to obtain numerical simulations of a wide range of configurations – some have also been subject to a wind tunnel test at the Aircraft Research Association\(^3\), with results published in Ashworth et al. [2016].

In Chapter 6, the methodology of this study is described in detail, particularly focusing on the software that was used to obtain the boundary layer data and the process behind the stability analysis.

In Chapter 7, after introducing the problem setup, the PSE method is validated for the given geometries by comparing it with an LNS approach. Then, the \(N\)-factor results for a range of configurations are presented and the dominant frequencies along the aerofoil are discussed in more detail. The transition locations for the different geometries are

\(^2\)http://www.adas.com/
\(^3\)http://www.ara.co.uk/
compared to both the clean case and each other, allowing to draw conclusions about their effect on the stability of the flow.
Chapter 6

Methodology

6.1 Mesh

The model that was used for the flow simulations discussed in this thesis was originally developed by Ashworth et al. [2016] at Airbus, with the goal of drawing direct comparisons between numerical computations and results from wind-tunnel experiments. It models the flow over an aerofoil at zero angle of attack. The starting point for this project is a clean base mesh, containing at least 50 points across the boundary layer and more than 500 points along the chord. It has a large, locally refined region around the deformation (centered at 20% of the chord length) to effectively capture small scale flow mechanisms. To study different deformation types, also called fillers, the mesh in this region was deformed by applying the TauFlowSim Mesh Deformation Tool, a feature of the TAU code [DLR 2011]. In this thesis, exponential fillers are investigated, generated by using a Gaussian deformation function. An example of a filler is visualised in Figure 6.1.

6.2 Software

6.2.1 TAU

For each configuration, a flow solution was obtained by using the TAU solver [DLR 2011], which is described as a “modern software system for the prediction of viscous and inviscid flows about complex geometries from the low subsonic to the hypersonic...
flow regime, employing hybrid unstructured grids\footnote{http://tau.dlr.de/code-description/}. It solves the RANS (Reynolds Averaged Navier–Stokes) equations and has been developed by the DLR (Deutsche Luft- und Raumfahrt) for use in aerospace engineering. Using an explicit Runge-Kutta iterative procedure, TAU simulates the flow that develops past the geometry until a converged steady state solution is obtained (residuals of order $10^{-8}$ or less) and outputs a file containing the dimensional flow velocity, temperature and pressure fields in Cartesian coordinates.

For all simulations, the far field parameters that were used are

$$\begin{align*}
\text{Re}_c &= 1.232 \cdot 10^7, \\
U_\infty &= 265.2076 \text{ m/s}, \\
\text{Ma}_\infty &= 0.75, \\
T_\infty &= 311.2K.
\end{align*} \tag{6.1}$$

The chosen turbulence model is the Spalart–Allmaras Edwards model, which has been shown to give good results for boundary layers with adverse pressure gradient, see \cite{Spalart1994} and \cite{Edwards1996}. Furthermore, by imposing a transition location on the rear of the wing and thus setting the eddy viscosity to zero, laminar flow was established over the front half of the wing, which includes the gap and transition locations of interest in this thesis, and the RANS scheme was reduced to the Navier–Stokes equations. Moreover, the refined mesh around the gap region enabled TAU to capture potential regions of separation, as this phenomenon was expected to occur for deeper gaps.
6.2.2 REBL

REBL (RANS Extraction of the Boundary Layer) is a Python program developed by Dr Christian Thomas at Imperial College London. It employs the open-source program Paraview\footnote{http://www.paraview.org/}, which handles the mesh files and can also visualise the flow solutions obtained through TAU. The purpose of REBL is to extract boundary layer data from the RANS solution file to a format suitable for linear stability analysis. To do this, the TAU solutions are transformed from Cartesian to surface-fitted coordinates, non-dimensionalised by the local boundary layer thickness $\delta^*$. The velocity field is non-dimensionalised by the velocity at the edge $U_e$, and similarly, the pressure and temperature. REBL then outputs the data file PSEMEAN.PRO, which is used to conduct a stability analysis with other programs such as CoPSE or MiPSecR (see Sections 6.2.3 and 6.2.4).

The advantage of REBL to conventional boundary layer solvers is its ability to deal with pockets of separated flow. Traditional boundary layer solvers, such as the software CoBL utilised in Mughal [2001], are based on a streamwise marching procedure, a numerical method which breaks down when the flow separates from the body due to a sufficiently large adverse pressure gradient. Since REBL extracts the boundary layer data from the TAU solution, which is able to capture small-scale flow characteristics, this new method allows boundary layer profiles to be computed for separated flows.

A full description of the method is given in Thomas et al. [2016], where it was successfully verified through comparison with solutions of boundary layer equations for wing geometries with an imposed surface waviness. In the paper, as well as in Thomas et al. [2017], REBL was employed to conduct stability analyses for both the LNS and PSE methods and a good agreement was found.

6.2.3 CoPSE

CoPSE (Compressible Parabolised Stability Equations) is a boundary-layer transition analysis code developed by Dr Shahid Mughal at Imperial College London. It can be used to analyse the linear and nonlinear stability states of compressible steady bound-
ary layer flows, where the basic flow to be studied is assumed known and has been computed using RANS. Standard finite differencing techniques in both the $x$- and $y$-directions are employed to solve the stability equations. Using the non-dimensional data extracted via REBL as input file, CoPSE allows for the instability region to be mapped out, which serves to identify the most dangerous band of waves (Tollmien–Schlichting waves or crossflow instabilities), as well as their frequencies and wavenumbers. There is also an option to carry out a nonlinear analysis for a particular frequency, which allows to directly compare its initial amplitude to the amplitude downstream, and how this affects the transition location.

In this thesis, a linear compressible PSE analysis is carried out for each configuration, examining the behaviour of a Tollmien–Schlichting wave as it travels downstream for a set range of frequencies. The program first computes all the eigenvalues of the discretised system at the initial starting point $x_0$ for the input frequency by solving the Orr–Sommerfeld equation

$$\frac{1}{i \Re} \left( \frac{d^2}{dy^2} - \alpha^2 \right)^2 \phi = (\alpha U - \omega) \left( \frac{d^2}{dy^2} - \alpha^2 \right) \phi - \alpha U'' \phi.$$  \hfill (6.2)

The local non-dimensional frequency $\omega$ is calculated from the dimensional frequency $\hat{f}$ measured in Hz by the formula

$$\omega = \frac{2\pi \delta^* \hat{f}}{U_e}.$$  \hfill (6.3)

The most unstable eigenvalue is identified and taken as an initial eigenvalue guess. To obtain a more accurate result, this is recomputed by an inverse Rayleigh iteration method and the final value will serve as the initial condition for the subsequent space marching procedure to solve the PSE equations. Finally, the program outputs local spatial growth rates, wavenumbers and N-factors at each $x$-coordinate. A more detailed description of the method can be found in Mughal [2006], an overview is given in the following.

From the perturbation definition in (5.5), the linear PSE formula for the shape

---

3CoPSE manual, Dr Shahid Mughal, Imperial College London
function $\bar{u}$ is represented as

$$L \bar{u} + M \frac{\partial \bar{u}}{\partial x} = 0,$$

(6.4a)

where $L$ and $M$ are differential matrix operators in the wall-normal $y$-direction, and given in Mughal [1998]. This system of equations is closed by

$$I_c = \int_0^\infty \left( \bar{u}^\dagger \cdot \frac{\partial \bar{u}}{\partial x} \right) dy \Bigg/ \int_0^\infty \bar{u}^\dagger \cdot \bar{u} dy,$$

(6.4b)

where $\dagger$ denotes the complex conjugate. Equation (6.4a) is then solved by employing a marching procedure, where the wavenumber $\alpha$ is calculated through the relation

$$\alpha_{j+1} = \alpha_j + i I_c.$$

(6.4c)

The imaginary part of $\alpha$ is then used in the previously discussed $N$-factor analysis through (5.6b). The boundary conditions are the zero-disturbance conditions on the wall

$$u = v = T = 0 \quad \text{at} \quad y = 0,$$

(6.4d)

and the Dirichlet free-stream conditions

$$u = v = T \rightarrow 0 \quad \text{as} \quad y \rightarrow \infty.$$

(6.4e)

The free-stream pressure $p$ is also assumed to satisfy the Dirichlet condition. The method was implemented with a chordwise step size $\Delta x$ that satisfies the Lie-Malik stability criterion from (5.8).

### 6.2.4 MiPSecR

The MiPSecR (Marching in Planes, PSE Secondary instabilities and Receptivity analysis) code was also developed by Dr Shahid Mughal at Imperial College London. Amongst other features, it can be employed to conduct linear and nonlinear stability analyses by solving the LNS equations for a specified disturbance, excited by a small periodic forcing placed upstream of the region of interest, at a point $x_f$. Again, the input file contains the data extracted from the RANS solutions through the use of REBL. The equations (5.4) are discretised and solved as described in Mughal & Ashworth [2013].
The surface conditions for the small periodic forcing of the perturbations are

\[ u' = -h(x) \frac{\partial U_0}{\partial y} |_{y=0}, \quad v' = -i\omega h(x), \quad T' = -h(x) \frac{\partial T_0}{\partial y} |_{y=0}, \] (6.5a)

where the function \( h(x) \) is given by the normalised Gaussian distribution of variance \( \sigma \)

\[ h(x) = \frac{10^{-6}}{\sqrt{2\pi}\sigma} e^{\frac{(x-x_f)^2}{2\sigma^2}}. \] (6.5b)

The solutions are obtained using higher-order finite-differencing in the \( x \)-direction along the surface (7401 points) and pseudo-spectral methods (51 points) in the wall-normal \( y \)-direction. Since the PSE method technically does not allow for upstream influence, MiPSecR is needed for Section 7.4, where the results obtained through CoPSE are validated with the corresponding LNS analysis.

### 6.3 Automated stability analysis script

This Python script can be used to perform a completely automated N-factor stability analysis, a previously slow, manual process. The only input file that is required is a PSEMEAN.PRO file, which can be obtained through programs such as REBL or CoBL, and it is the main data file when using CoPSE. The required user inputs are the dimensional frequency range \( \hat{f}_R \) (in Hz), which, in this thesis, is as follows:

\[ \hat{f}_{\text{min}} = 10000, \quad \hat{f}_{\text{max}} = 30000, \quad \hat{f}_{\text{step}} = 500. \] (6.6)

In the first stage, CoPSE is run for all the frequencies, calculating the corresponding growth rates at every \( x \)-location. For each frequency, the script then finds and saves the corresponding neutral point \( x_0 \) where the growth rates become positive. In the second stage, each frequency is then run individually, choosing the previously determined neutral point as starting location and increasing this as needed, until both the eigenvalue guess as well as the amplification rate at \( x_0 \) are positive. Furthermore, the script checks whether a convergence criterion is met, and, should this not be the case, it omits every other data point in the analysis which usually results in good convergence. Finally, an envelope \( N_{env} \) is created by determining the highest N-factor and its corresponding
Figure 6.2: Automated stability analysis code description
frequency at every $x$-coordinate, and a general summary of the analysis is output as a text file. For a visualisation of this process, please refer to Figure 6.2.

6.4 Comments

As all the results presented here have been obtained numerically, it is important to address any problems that can arise due to computational errors. The most important issue to note is that TAU doesn’t take roughness elements into account when calculating the flow. The input file does not contain a deformation option, since the program can be run for any geometry, so the results need to be looked at critically. When extracting the pressure data, small, irregular peaks arise around the deformation region, as can be seen in Figure 6.3 for the deeper deformations. These irregularities stem from the numerical calculations and can be explained by the fact that the corners around the deformation region are not as smooth as would be desired for the chosen density of the mesh. The change in the geometry is quite abrupt, and complicates the correct computation of normals which in turn influences the pressure data. REBL extracts the relevant boundary layer data, including the pressure data containing the incorrect data points discussed above. A spline fitting is then used to eliminate any bad points, but unfortunately this does not work for all cases and minor irregularities are therefore carried forward. This has been rectified in newer versions of REBL.
Figure 6.3: Irregular peaks in the pressure data due to numerical error: $x$ is the chord position and the different colours indicate deformations of increasing depth, as discussed in more detail in the next chapter.
Chapter 7

Numerical study on gaps and bumps

7.1 Wing geometry

Consider the incoming flow $U_\infty$ over an unswept wing, illustrated by the aerofoil of chord length $c = 0.75$ m in Figure 7.1. In the following analysis, the body-fitted coordinates $(\hat{s}, \hat{y})$ and the chordwise position $\hat{x}$ are employed.

For each configuration, the base flow $U_0$ was obtained by using the TAU solver, and the boundary layer profiles were extracted via REBL, as described in Chapter 6.

In this thesis, the filler type geometry of interest is a bell-shaped surface indentation, generated using a Gaussian deformation function, symmetric around its mid-point.
Similar to the triple deck analyses conducted in Part I, the perturbations to the base flow are assumed to be small and periodic in time, since the interest lies in the spatial stability of the flow in the streamwise direction, as discussed in (5.3) and (5.5). For PSE and LNS calculations, the surface coordinates are non-dimensionalised with respect to the local boundary layer thickness $\delta^*$ and the non-dimensional frequency $\omega$ is defined by

$$\omega = \frac{2\pi \delta^* \hat{f}}{U_e},$$

(7.1)

where $U_e$ is the local edge velocity. To investigate the stability of the flow for a given global frequency $\hat{f}$, an $N$-factor analysis is carried out by running the script described in Section 6.3 for the clean case and each of the gap configurations.

Due to proprietary reasons, no actual $N$-factor values are stated, but instead, any results are compared to a critical value $N_{crit}$, obtained through experimental observations by Airbus. Six different depths are considered,

$$\hat{D} \in \{23, 40, 58, 83, 116, 150\} \mu m,$$

(7.2a)

and for each of these, four different deformation widths,

$$\hat{W} \in \{1, 2, 4, 8\} \ mm.$$

(7.2b)

After discussing possible pockets of separated flow in Section 7.3 and validating the PSE method through a LNS analysis in Section 7.4, the dominating $N$-factors and corresponding frequencies are shown for each case and their transition locations are compared to each other.
7.2 Exponential fillers

The exponential filler considered here is modelled by taking the clean base mesh and imposing a surface indentation of the form

\[
\hat{y}_{def}(\hat{s}) = -\hat{D} \exp \left\{ - \left( \frac{\hat{s} - \hat{s}_c}{\hat{W}} \right)^2 \right\} \quad \text{on} \quad \hat{y} = 0, \quad (7.3)
\]

where the hat again indicates dimensional variables, and \( \hat{s} \) is the body-fitted coordinate along the surface. A total of 24 deformations are examined, with the 4 widths and 6 depths given in (7.2). In the following, results will be presented in the dimensionless coordinate system

\[
\hat{x} = c \, x, \quad \hat{s} = c \, s, \quad \hat{y} = c \, y, \quad \hat{D} = c \, D, \quad \hat{W} = \sqrt{2} \, c \, W, \quad (7.4)
\]

where \( c = 0.75 \, \text{m} \) is the chord-length of the aerofoil. The gap is then defined by the Gaussian distribution function

\[
y_{def}(s) = -D \exp \left\{ - \frac{1}{2} \left( \frac{s - s_c}{W} \right)^2 \right\} \quad \text{on} \quad y = 0, \quad (7.5)
\]

where the deformation midpoint is \( s_c = 0.23 \) or \( x_c = 0.2 \), as shown in Figure 7.2. It should be noted that \( W \) does not denote the actual width of the gap, but it is rather a variance. The ‘real’ (dimensionless) gap width, denoted by \( \mathcal{W} \), is indicated for the left side of the gap by the dashed lines in Figure 7.2, and will be used to visualise the gaps in the following analysis, where

\[
\mathcal{W}_{1mm} = 0.00708, \quad \mathcal{W}_{2mm} = 0.01404, \quad (7.6)
\]

\[
\mathcal{W}_{4mm} = 0.02820, \quad \mathcal{W}_{8mm} = 0.05628. \quad (7.7)
\]
7.3 Velocity contours

First, it is necessary to investigate the behaviour of the base flow $U_0$ in the deformation region before starting the $N$-factor analysis. Separation of the boundary layer results in reversed flow, so that separation can be detected for the chosen configurations by determining where $U_0$ becomes negative. The minimum value of $U_0$ is recorded in Table 7.1. The local parameters at the point of separation are $Re_* \approx 1560$ and $Ma_* \approx 1.06$.

<table>
<thead>
<tr>
<th>$W$</th>
<th>$23\mu m$</th>
<th>$40\mu m$</th>
<th>$58\mu m$</th>
<th>$83\mu m$</th>
<th>$116\mu m$</th>
<th>$150\mu m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2mm</td>
<td></td>
<td></td>
<td></td>
<td>-0.0052</td>
<td>-0.0125</td>
<td>-0.0186</td>
</tr>
<tr>
<td>4mm</td>
<td></td>
<td></td>
<td></td>
<td>-0.0016</td>
<td>-0.0093</td>
<td>-0.0173</td>
</tr>
<tr>
<td>8mm</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>-0.0040</td>
<td>-0.0125</td>
</tr>
<tr>
<td>16mm</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>-0.0075</td>
</tr>
</tbody>
</table>

Table 7.1: $\min(U_0)$ for each configuration

It can be seen that there is no separation for the three shallowest depths of 23, 40 and 58 microns. Furthermore, it can be concluded that the wider the gap becomes, the deeper the deformation has to be for separation to occur. This is visualised for the nine relevant cases in Figures 7.3 and 7.4. There does not seem to exist a critical value of the height-to-width ratio of the deformations for which separation can be expected to occur, as described in Navfah et al. [1987], but the minimum value of $U_0$ does increase
Figure 7.3: Separation bubbles for 83 $\mu m$ and 116 $\mu m$ gaps
Figure 7.4: Separation bubbles for a 150 μm deep gap of varying width
in each row and column. Since the ratios are of the same order of magnitude, this discrepancy might be due to the difference in the shape of the roughness or its location on the wing. In Thomas et al. [2018], the relationship between the minimum value of the base flow and the depth and variance of the Gaussian gap was found to be

\[- \min(U_0) \propto 10^4 D W^{-0.25}. \quad (7.8)\]

### 7.4 Validation of PSE method

Having discussed the configurations for which localised pockets of separated flow occur, it is necessary to examine the flow stability around the gap region using an LNS approach. These results can then be compared with the results obtained through a PSE analysis, thus justifying the use of the PSE method for this study. The four different widths for the 150 μm deep configuration are presented for three frequencies in the following.

At each x-location, the frequency that causes the highest amplification rate is recorded along with the corresponding N-factor value, and termed the dominant frequency. This frequency changes depending on the roughness element, and the variation in the frequency between the different configurations is discussed in more detail in Section [7.9]. The main conclusion is that low-frequency modes play a different role to the high-frequency modes. The frequencies that are dominant in the gap region lie in the range

\[\hat{f}_{\text{gap}} = [19.5, 23.5] \text{ kHz},\]  

whereas the low-frequency modes only start growing after the gap and play an important role in triggering the transition to turbulence. Figure [7.5] visualises the dominant frequencies in the gap region for the 150 μm case.

For the clean case, as well as the \(\hat{W} = 1, 2\) mm cases, \(\hat{f} = 21.5\) kHz is the dominant frequency around the mid-point of the gap region and it also plays a big role in the 8 mm case, as it dictates the stability of the flow for the right half of the gap region. To verify the PSE results just before and after the gap, the further two frequencies \(\hat{f} = 20, 23.5\) kHz are also investigated.
Figure 7.5: Dominant frequencies in the gap region for $\hat{D} = 150 \mu m$ in kHz: the black, horizontal lines represent the clean case, and the coloured horizontal lines show how the 150 micron configurations differ. The dashed lines indicate the gap width and $x$ is the non-dimensional chord position.

To model the incoming Tollmien–Schlichting wave in the LNS analysis with MiPSecR, a small periodic forcing is placed upstream of the gap, as described in Section 6.2.4. The disturbance is excited at the forcing location $x_f = 0.11$ for the narrow gaps and $x_f = 0.13$ for the wider gaps, in order to analyse how its amplitude changes as it passes over the deformation, where the interest lies in the most unstable mode. As can be seen in Figure 7.6, immediately after $x_f$, the curve is characterised by oscillations due to the fact that the dominant mode keeps changing, but after a while, the most unstable mode emerges and dictates the long-term behaviour. A rise in the amplitude can be observed around the gap, which indicates that the deformations have a destabilising effect on the incoming T–S disturbance. In order to determine the validity of using the PSE approach, the amplitudes obtained from the PSE calculations are scaled to match the amplitude of the LNS results just after the gap region, so that the difference in the variation of the magnitude of the amplitude between the two sets of results can be measured. A good agreement was found for all cases.
Having verified that the PSE analysis returns correct results despite localised pockets of separation and upstream influence, the $N$-factors for the different configurations are discussed in the following sections and conclusions are drawn between the stability of the flow and the width and depth of the gaps.

### 7.5 $N$-factors for a fixed width

Figures (7.7) and (7.8) show the results for the $N$-factor envelopes for the four different widths, plotted against the chordwise $x$-coordinate. The horizontal, dashed, black line indicates the critical $N$-factor value $N_{crit}$ for which transition was observed experimentally in the clean configuration. For each case, it can be observed that the depth of the gap has a significant impact on the values of the $N$-factor envelope. Around the beginning of the gap region, there exists a minimum, which is lower the deeper the gap is. However, as the observer moves downstream, the $N$-factor values change in such a way that the envelope reaches a local maximum just after the gap, which triggers a bypass transition in the widest, deepest case (16mm, 150 µm). For the other cases, the $N$-factors then decrease slightly before growing again, which is due to the fact that the lower-range frequencies only start growing after the gap region. A greater depth implies a higher minimum after the gap, which has the effect that the critical value $N_{crit}$ is thus also reached sooner. This in turn means that the transition location is shifted upstream, indicated by the dashed vertical lines.

### 7.6 $N$-factors for a fixed depth

Figures (7.9), (7.10) and (7.11) visualise the influence of the gap width on the $N$-factor envelope and transition locations. It can be seen that the width of the gap, just like its depth, plays a role on the $N$-factor values around the gap and at the transition point – the wider the gap, the higher the $N$-factors, and the flow becomes unstable sooner. Moreover, it can be concluded that the width of the gap also influences the shape of the $N$-factor envelope. This is especially noticeable for the 40 µm case: although the 8mm case transitions earlier than the 4mm case and is thus a more unstable configuration, its $N$-factors after the gap region are lower than the narrower, 4mm case. Also, the
Figure 7.6: LNS compared to PSE for 150 microns: $x$ is the non-dimensional chord position, the shaded area represents the width of the gap, and the light blue line at the beginning indicates where the disturbance was excited. $u_{\text{max}}$ is the amplitude of the LNS results, and the PSE calculations are scaled to match $u_{\text{max}}$ just after the gap region, so the variation in the magnitude of the amplitude can be compared between the two methods.
Figure 7.7: Exponential $N$-factor envelopes for gaps of 2mm and 4mm widths: transition to turbulence happens when $N_{\text{crit}}$ is reached, and is visualised for each depth by the vertical dashed lines. The deeper the gap is, the sooner transition happens along the aerofoil, where $x$ denotes the chord position.
Figure 7.8: Exponential $N$-factor envelopes for gaps of 8mm and 16mm widths: transition to turbulence happens when $N_{crit}$ is reached, and is visualised for each depth by the vertical dashed lines. The deeper the gap is, the sooner transition happens along the aerofoil, where $x$ denotes the chord position.
Figure 7.9: Exponential $N$-factor envelopes for gaps of 23$\mu m$ and 40$\mu m$ depths: transition to turbulence happens when $N_{crit}$ is reached, and is visualised for each width by the vertical dashed lines. The wider the gap is, the sooner transition happens along the aerofoil, where $x = 0.2$ denotes the midpoint of the gap.
Figure 7.10: Exponential $N$-factor envelopes for gaps of 58$\mu$m and 83$\mu$m depths: transition to turbulence happens when $N_{\text{crit}}$ is reached, and is visualised for each width by the vertical dashed lines. The wider the gap is, the sooner transition happens along the aerofoil, where $x = 0.2$ denotes the midpoint of the gap.
Figure 7.11: Exponential $N$-factor envelopes for gaps of 116µm and 150µm depths: transition to turbulence happens when $N_{\text{crit}}$ is reached, and is visualised for each width by the vertical dashed lines. The wider the gap is, the sooner transition happens along the aerofoil, where $x = 0.2$ denotes the midpoint of the gap.
8mm gap envelope does not follow the 'parallel' pattern in [7.5] but instead exhibits a more pronounced minimum before reaching transition as compared to the 2mm and 4mm cases. This irregularity can be observed for the shallow depths in general, but changes as the gap becomes deeper: the 116 μm and 150 μm cases exhibit the same pattern as in [7.5].

7.7 N-factor in terms of W and D

It would be incredibly useful to establish a formula which were able to predict the maximum N-factor values and transition locations for a given width and depth. For the 24 configurations, it is possible to determine a relationship between \( \Delta N, W \) and \( D \), where \( \Delta N \) is defined to be the maximum difference between the clean and the deformed configuration within the gap region, \( I_{\text{gap}} = [0.2, 0.3] \),

\[
\Delta N = \max\left| N_{\text{env}}(\text{clean}) - N_{\text{env}}(\text{def}) \right|.
\]  

\( \Delta N \) were able to show that \( \Delta N \) is dependent on the width \( W \) and depth \( D \) of the exponential filler, such that \( \Delta N \propto D L^{3/4} Re_{\infty} \), where they considered four different values of \( Re_{\infty} \), the free-stream Reynolds number. For the slightly simpler case presented in this thesis, it can be shown that

\[
\Delta N \propto c D^{\lambda_1} W^{\lambda_2} 10^9,
\]  

where an ordinary least squares regression yields the values \( \lambda_1 = 1.08 \) and \( \lambda_2 = 0.763 \), which are close to the exponents 1 and 3/4 obtained by Thomas et al. This is visualised in Figure [7.12] and the details of the OLS can be found in Appendix D.
Figure 7.12: $\Delta N$ as defined in (7.10) in terms of the non-dimensional width $W$ and depth $D$

7.8 Transition coordinate $x_t$

A better way of checking which configuration reaches transition first is to determine the $x$-coordinate of the transition point for each case. Since this is a numerical study, the data are discrete, so that the non-dimensional transition point $x_t$ is defined to be the first data point where $N > N_{\text{crit}}$. For the clean configuration, transition is triggered at

$$x_{t,\text{clean}} = 0.4791.$$  \hfill (7.12a)

In order to measure by how much the transition point has moved upstream, the percentage

$$\%_t = \frac{x_{t,\text{def}}}{x_{t,\text{clean}}}$$  \hfill (7.12b)

can be calculated for each depth and width. The values are listed in Table 7.2.

Having quantified how much sooner transition occurs as compared to the clean case, it can be concluded that a deeper gap really does imply earlier transition: except for the narrowest, 2mm case, the numbers decrease in each row. Furthermore, again excluding...
the shallowest configuration of 23 μm, the values also decrease in each column, which implies that the greater the gap width, the sooner transition occurs.

7.9 Frequencies

When the $N$-factor envelope is created, the dominant frequencies at each $x$-coordinate are recorded, too, which allows to isolate the frequencies that induce instability and trigger the onset of transition to turbulence. In this thesis, the given frequency range is 10 to 30 kHz, taken in steps of 500 Hz. To illustrate the general variation in the frequency range, consider Figure 7.13 which shows the dominant frequency at each $x$-location for the clean case.

Figure 7.13 shows that the dominant frequency decreases downstream along the aerofoil. As mentioned in Section 7.4, the dominating frequencies around the gap region are in the 19.5-23.5 kHz range. These are high-frequency modes and they are responsible for the peak in the $N$-factor values about the deformation. At around 30%
of the chord length, the low frequency modes in the range

\[
\hat{f}_t = [10, 12.5] \text{ kHz.} \tag{7.13}
\]

start to grow until \( N_{\text{crit}} \) is reached, thus triggering the transition to turbulence. This agrees with the previous findings, see for example Cebeci & Egan [1989] or Gao et al. [2011], who also found that the low frequency modes are the most unstable and lead to transition. It is useful to know which frequencies are important around the transition region, so that trigger values can be identified and potentially avoided in practice. The dominating frequency values at the transition point \( x_t \) are recorded in Table 7.3 below.

<table>
<thead>
<tr>
<th>( \frac{d}{D} )</th>
<th>23( \mu \text{m} )</th>
<th>40( \mu \text{m} )</th>
<th>58( \mu \text{m} )</th>
<th>83( \mu \text{m} )</th>
<th>116( \mu \text{m} )</th>
<th>150( \mu \text{m} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2mm</td>
<td>10.5</td>
<td>10.5</td>
<td>10.0</td>
<td>10.0</td>
<td>11.0</td>
<td>11.0</td>
</tr>
<tr>
<td>4mm</td>
<td>10.5</td>
<td>11.0</td>
<td>11.0</td>
<td>11.5</td>
<td>11.5</td>
<td>11.5</td>
</tr>
<tr>
<td>8mm</td>
<td>11.0</td>
<td>11.0</td>
<td>11.5</td>
<td>11.5</td>
<td>12.0</td>
<td>11.5</td>
</tr>
<tr>
<td>16mm</td>
<td>11.0</td>
<td>11.5</td>
<td>12.0</td>
<td>12.5</td>
<td>12.5</td>
<td>12.5</td>
</tr>
</tbody>
</table>

Table 7.3: Dominant frequencies at the transition point in kHz: the lower the frequency, the darker the shading. It can be observed that for the wider and deeper gaps, the dominating frequency is higher than for the shallow, narrow configurations.

Whereas the depth of the gap does not seem to have a linear influence on the transition frequency, Table 7.3 indicates that a wider gap implies a higher transition
frequency. This can be expected, since the transition for wider gaps happens sooner and the dominating frequency decreases along the aerofoil.

7.10 Summary

This study on the effect that small gaps and bumps have on the stability of the flow over an aerofoil has led to the following conclusions:

- Flow separation is observed for some of the configurations, where the magnitude of the reversed flow is highest for the narrowest and deepest deformation. The wider the gap is, the deeper it has to be for separation to occur.
- For this specific aerofoil model, the PSE approach yields results that are in good agreement with LNS calculations, with only small differences in the disturbance amplitudes.
- The frequencies most affected by the deformation are the higher frequencies for which the gap significantly enhances the amplification rate of the T–S disturbance.
- For a fixed width, it was observed that a greater gap depth implies a higher amplification of the disturbance just after the gap region and transition to turbulence happens sooner.
- For a fixed depth, it can be seen that the shape of the $N$-factor envelope differs depending on the width of the gap: for a wider gap, the local extrema are more pronounced and again, transition happens sooner.
- From the 24 data points, the maximum difference in $N$-factors between the clean case and the other configurations around the gap region $\Delta N$ was found to vary proportionally to $D^{1.08} W^{0.763}$.
- Looking at the transition locations in terms of the point of transition for the clean case confirms that transition to turbulence is triggered sooner, the deeper and wider the gap is. Bypass transition happens in the deepest and widest configuration.
- The dominant frequency at the point of transition increases for larger gap depth and width. Contrary to the high frequencies dominant in the gap region, low frequencies are responsible for this.

A qualitative comparison of the results to other papers shows that, in general, it was found that flow separation has a strong positive effect on the amplification of the
incoming disturbance, just as was discovered here. For a fixed width, the deeper the gap becomes, the more likely it is for flow separation to occur and therefore the amplification rates become much higher, as discussed in Wu & Dong [2016] or Cebeci & Egan [1989]. However, whereas the findings of Nayfeh et al. [1987] and Gao et al. [2011] for flows over a flat plate with a small indentation imply that reducing the width of the deformation leads to a higher amplification rate, the results from this study in fact indicate the opposite – an increase in the width correlates with higher $N$-factors after the gap region and earlier transition to turbulence.
Conclusion
8.1 Part I

In Part I of this thesis, a theoretical approach is taken to study the receptivity of the boundary layer for two different flow geometries. The first problem analyses the stability of Blasius flow over a small, isolated roughness element, where the base flow is perturbed by a Tollmien–Schlichting wave. The method employed for this purpose is the Triple Deck Theory, applicable to a roughness of horizontal extent of $O(L \Re^{-3/8})$, which divides the flow domain in three layers to study the viscous-inviscid interaction between the boundary layer and the external inviscid flow. Having examined the simpler case of a T–S wave over a flat plate in Chapter 2 and the stationary problem of Blasius flow over an isolated roughness in Section 3.1, the interaction between the two phenomena was considered in the rest of Chapter 3, where the so-called transmission coefficient $T_I(\omega)$ was calculated. This allows to compare the initial amplitude of the incoming T–S wave to the amplitude of the T–S wave downstream of the roughness. It was found that the $O(h)$ correction to the magnitude of the downstream amplitude, where $h$ is the height of the roughness, depends on the frequency $\omega$, but also on the Fourier transform of the wall roughness shape evaluated at zero value of the wavenumber. This implies that the bigger the area of the roughness, the more the T–S wave will be affected by it. In the case of a protruding roughness, $|T_I| < 1$ and the amplitude is dampened, whereas it is amplified by an indentation for which $|T_I| > 1$.

The second geometry of interest in this thesis is the flow over a vibrating semi-infinite plate attached to a stationary plate, modelling an elastic plate whose thickness is small compared to its width and length, attached to a solid surface in such a way that the movement about the joint is restricted. Again making use of the Triple Deck Theory, the receptivity coefficient $R_{vib}(\omega)$ was calculated, showing that the vibrations of the plate produce a T–S wave of $O(\beta_0)$ in the boundary layer, where $\beta = \Re^{-1/4} \beta_0$ is the amplitude of the elastic vibrations of the plate and $\beta_0$ is assumed to be small. The modulus and argument of the receptivity coefficient are plotted for varying $\omega$ in Figure 4.4.
8.2 Part II

Contrary to the analytic analysis in Part I, a numerical study investigating the stability of the flow over small gaps and bumps on an otherwise smooth aerofoil was undertaken in Part II. In Chapter 6, the numerical techniques are described, which involve the use of the DNS solver TAU, as well as software developed at Imperial College London: REBL, a program which extracts the boundary layer data from the solution file, and CoPSE and MiPSecR, both allowing to examine the stability of the flow, the former through PSE equations and the latter via an LNS approach. In Chapter 7, the calculation results were discussed. It was found that the flow separates for the deepest gaps, and the separation is strongest for the narrowest, deepest configuration. The wider the gap becomes however, the deeper it has to be for separation to occur. The $N$-factor calculations indicated that an increase in both the width and the depth of the gap produces a higher amplification of the incoming disturbance, and thus transition to turbulence is triggered sooner along the aerofoil. A formula linking the depth and width of the gap to the difference in $N$-factors between the clean and deformed configurations was suggested based on the computation results: $\Delta N \sim D^{1.08} W^{0.763}$. Furthermore, it was discovered that the frequencies affected by the deformation region are much higher than the low frequencies responsible for triggering transition.

8.3 Comparison

In this final section, the analytical calculations from Chapter 3 are linked to the numerical results to find the similarities and differences between the two methods, and to extrapolate the implications of this research. The two main assumptions for the Triple Deck analysis are that

$$H \sim W^{5/3} \quad \text{and} \quad W \sim Re^{-3/8}.$$  \hspace{1cm} (8.1)

With the Reynolds number $Re = 1.232 \cdot 10^7$, these hold true for the configurations

- $\hat{W} = 1 \text{ mm}, \hat{D} \in \{40, 58, 83, 116\} \mu m$,
- $\hat{W} = 2 \text{ mm}, \hat{D} \in \{83, 116, 150\} \mu m$.

It should be noted however that for most asymptotic theories, results have been found to be applicable to a wider range of parameters than those given by the initial assump-
tions. A good example for this would be the predicted drag coefficient for trailing-edge flow, as discussed in Ruban [2017], for which the asymptotic theory based on the large-Reynolds-number limit agrees with experimental and computational data even for Reynolds numbers as low as 10. Since the Triple Deck theory is based on the principle of least degeneration, it is more exact and has wider applications than for example the PSE equations which are highly restrictive, as they do not allow for upstream influence.

In Part I, the amplitude of the T–S wave was taken to be the amplitude of the pressure perturbation, because the pressure does not vary vertically across the viscous sublayer, so its Fourier transform is much easier to convert back into physical variables than those of the velocity perturbations. However the numerical calculations in Part II investigated the change in the amplitude of the streamwise velocity perturbation to calculate the $N$-factors, as is standard practice in aerospace engineering. A direct comparison between the $N$-factor results and the analytical expression is thus not possible.

![Figure 8.1: Pressure perturbation functions for low frequencies: the solid lines represent the PSE pressure function $\bar{p}$ at different frequencies for the deformed configuration, the dotted lines are the same frequencies run for the clean case. The highest peak is attributed to the transition frequency.](image)

(i) 2mm, 83µm, $\hat{f}_{dom} = 10$ kHz
(ii) 4mm, 150µm, $\hat{f}_{dom} = 11.5$ kHz

Figure 8.1: Pressure perturbation functions for low frequencies: the solid lines represent the PSE pressure function $\bar{p}$ at different frequencies for the deformed configuration, the dotted lines are the same frequencies run for the clean case. The highest peak is attributed to the transition frequency.

To illustrate the qualitative agreement between the two, the pressure function $\bar{p}(x, y)$ from the PSE perturbation $p'(x, y, t) = e^{i(\alpha(x)-\omega t)} \bar{p}(x, y)$ is plotted in Figure 8.1 for the lowest frequencies considered, for two different cases. The dotted lines are the same frequencies run for the clean configuration, and the dominant frequency at the tran-
sition location is given in the subcaption. It is interesting to note that even though these low-range frequencies only start growing just before the gap region, they yield the highest change in the pressure function, where the highest peak is attributed to the transition frequency. From these examples it can be seen that the bell-shaped gap does have a destabilising effect on the amplitude of the pressure perturbation, as predicted by the asymptotic theory.

Furthermore, the integral of the exponential filler gives \( Y_0(0) = -\sqrt{2\pi} D W \), which implies that the wider/deeper the gap, the higher the change in the pressure perturbation should be. Comparing the two cases from Figure 8.1 to the 4\( \text{mm} \), 83\( \mu \text{m} \) configuration in Figure 8.2 it can be observed that this also applies to the PSE results.

\[ 
\begin{align*}
Y_0(0) &= -\sqrt{2\pi} D W, \\
2\text{mm, 83 microns} &\quad 4\text{mm, 83 microns} &\quad 4\text{mm, 150 microns}
\end{align*}
\]

Figure 8.2: Comparison between the PSE results for the pressure perturbation between different configurations: the deeper/wider the gap, the more destabilising the effect on the flow

It can thus be concluded that an isolated, stationary roughness has a significant impact on an incoming Tollmien–Schlichting wave. If the deformation is a gap, the effect is destabilising, a result obtained both from the asymptotic result in Part I, and the numerical study in Part II. In the case of a protruding roughness, the bump has a stabilising effect on the T–S wave if the height is small enough, \( h \ll 1 \).
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Appendices

A  Forcing term analysis and implications

A1  Detailed analysis of $\bar{G}$

The forcing term on the right side of the Airy equation has the analytical expression

$$\bar{Y}_0(\xi_\alpha) \bar{G} = \left( \frac{(i\alpha)^{2/3}}{i\xi} \right) u_{TS}'' \bar{v}_{R_\alpha} + \left( \frac{(i\alpha)^{4/3}}{i\xi} \right) u_{TS}' \bar{u}_{R_\alpha}$$

$$+ \left( \frac{(i\xi_\alpha)^{2/3}}{i\xi} \right) v_{TS} \bar{u}_{R_\alpha} \frac{\partial^2 \bar{u}_{R_\alpha}}{\partial w_\alpha^2} + \left( \frac{(i\xi_\alpha)^{4/3}}{i\xi} \right) u_{TS} \bar{u}_{R_\alpha} \frac{\partial \bar{u}_{R_\alpha}}{\partial w_\alpha}$$

(1.2)

where $(u_{TS}, v_{TS})(z)$ are the known Tollmien–Schlichting perturbations of $O(\varepsilon)$:

$$u_{TS} = \frac{A_{TS}}{\int_{z_0}^\infty \text{Ai}(s)ds} \int_{z_0}^z \text{Ai}(s)ds$$

$$v_{TS} = \frac{-i\alpha A_{TS}}{\int_{z_0}^\infty \text{Ai}(s)ds} \left[ z \int_{z_0}^z \text{Ai}(s)ds - \text{Ai}'(z) + \text{Ai}'(z_0) \right]$$

and $(\bar{u}_{R_\alpha}, \bar{v}_{R_\alpha})(\xi_\alpha, w_\alpha)$ are the Fourier transforms of the roughness regime perturbations of $O(h)$:

$$\bar{u}_{R_\alpha} = \frac{3(i\xi_\alpha)^{1/3}|\xi_\alpha| \bar{Y}_0(\xi_\alpha)}{(i\xi_\alpha)^{1/3}|\xi_\alpha| - 3 \text{Ai}'(0)} \int_0^{w_\alpha} \text{Ai}(s)ds - \bar{Y}_0(\xi_\alpha)$$

$$\bar{v}_{R_\alpha} = \frac{-3i \text{sgn}(\xi_\alpha)(\xi_\alpha)^2 \bar{Y}_0(\xi_\alpha)}{(i\xi_\alpha)^{1/3}|\xi_\alpha| - 3 \text{Ai}'(0)} \left[ w_\alpha \int_0^{w_\alpha} \text{Ai}(s)ds - \text{Ai}'(w_\alpha) + \text{Ai}'(0) \right] - \bar{Y}_0(\xi_\alpha)w_\alpha$$

where the lower index $\alpha$ denotes that they are evaluated at $\xi_\alpha = \xi - \alpha$. 
It is necessary to look at the behaviour of $\bar{G}$ to be able to evaluate the integrands $P_2^\pm$ over the contours $C_2^\pm$ and calculate the residue of $\varphi$ at $\xi = \alpha$. To do this, consider three different cases:

(i) $\xi \to 0$ to evaluate $\int_{C_2^\pm} P_2^\pm d\xi$, see (3.6.1),

(ii) $\xi = \alpha$ to calculate the residue at $\xi = \alpha$, i.e. $\xi_\alpha = 0$, see (3.90),

(iii) $\xi \to \infty$ to evaluate $\int_{C_R^\pm} P_2^\pm d\xi$, see (A2).

A2  Behaviour of $\bar{G}$ as $\xi \to \infty$

In order to evaluate the integral $\int_{C_R^\pm} P_2^\pm d\xi$, where $R \to \infty$, the behaviour of $\bar{G}$ as $\xi = R e^{i\theta} \to \infty$ is important. First, note that the Tollmien–Schlichting perturbations are independent of $\xi$ and of $O(1)$ in $y_3$ (as they must be bounded), so they can be treated like constants in this analysis. Furthermore,

$$\xi_\alpha \sim \xi \quad \text{and} \quad w_\alpha \sim w \quad \text{as} \quad \xi \to \infty. \quad (3)$$

Also, the coefficient of $u_R$,

$$\lim_{\xi \to \infty} \frac{3(i\xi)^{1/3} |\xi|}{(i\xi)^{1/3} |\xi| - 3 A'(0)} = 3, \quad (4)$$

and the roughness function $Y_0(\xi)$ is bounded by definition. Hence $\bar{u}_R$ and its derivatives are either bounded or exponentially decaying as $\xi \to \infty$. This implies that all the terms in (2), except for the first one, are either exponentially or algebraically decaying as $\xi \to \infty$. It remains to analyse the first term.

$$\frac{(i\alpha)^{2/3}}{i\xi} u''_{TS} \bar{u}_R \sim \frac{\xi w}{(i\xi)^{1/3} |\xi| - 3 A'(0)} - \frac{w}{\xi} \sim y_3 \quad \text{as} \quad \xi \to \infty. \quad (5)$$

Thus

$$\bar{G} \sim y_3 + O(\xi^{-2/3}) \quad \text{as} \quad \xi \to \infty. \quad (6)$$

Moreover, $\eta \sim w$ as $\xi \to \infty$. To ensure boundedness, it is necessary that $w = O(1)$, i.e. $y_3 \sim \xi^{-1/3}$. Then, using (3.53b), $\varphi(\xi, \eta)$ for large $\xi$ is the particular solution to the
system
\[ \frac{\partial^2 \varphi}{\partial w^2} - w \varphi(\xi, w) = G(\xi \to \infty, w); \quad \frac{\partial \varphi}{\partial w}(0) = 0, \quad \varphi(\infty) = 0. \] (7)

Since both \( w \) and \( \bar{G}(\xi \to \infty, w) \) are \( O(1) \) quantities, the solution \( \varphi(\xi, w) \) is also of \( O(1) \), and so
\[ \int_{0}^{\infty} \varphi(\xi, w) dw = O(1). \] (8)

\section{B Integrals along \( \gamma_{\pm} \)}

In section, it will be shown that the contributions of the integrals over the circular contours \( C_{\pm}^{\mp} \) for both the interaction problem and the vibrating plate are negligible as the radius \( R \) of the circular contour tends to infinity. In both cases, the integrand can be written as
\[ I_{\pm}(\xi) = \frac{\pm \mathcal{N}(\xi)}{\Delta_{\pm}(\xi)} e^{i\xi X}, \] (9)
where
\[ \Delta_{\pm}(\xi) = \text{Ai}'(\eta_0) \pm i(i\xi)^{4/3} \int_{\eta_0}^{\infty} \text{Ai}(s) ds, \] (10)
\[ \mathcal{N}_{\text{int}}(\xi) = \xi \nabla_0(\xi) \left[ p_c \left( \eta_0 \int_{\eta_0}^{\infty} \text{Ai}(s) ds + \text{Ai}'(\eta_0) \right) + \text{Ai}'(\eta_0) \int_{\eta_0}^{\infty} \varphi(\xi, s) ds \right], \] (11)
\[ \mathcal{N}_{\text{vib}}(\xi) = \text{Ai}'(\eta_0). \] (12)

On \( C_{R}^{\pm} \), the substitution \( \xi = R e^{i\phi} \) can be used, where \( R \) varies between 0 and infinity. For \( C_{R}^{-} \), \( \phi \in (-3\pi/2, -\pi + \phi_0) \), where \( \phi_0 \) is the angle between the ray \( C_{2}^{-} \) and the negative real semi-axis, such that only one root, \( \xi = \alpha \), is enclosed in the circular contour \( \gamma_{-} \). For \( C_{R}^{+} \), \( \phi \in (0, \pi/2) \). Then
\[ \int_{C_{R}^{\pm}} I_{\pm}(\xi) d\xi = \pm \int_{\phi_0}^{\phi_u} e^{i\phi} \frac{\mathcal{N}(R e^{i\phi})}{\Delta_{\pm}(R e^{i\phi})} e^{iX R e^{i\phi}} dR. \] (13)

Since the modulus of an integral is smaller or equal to the integral of the modulus of
its integrand,
\[ \left| \int_{c_R^\pm} I_\pm(\xi) \, d\xi \right| \leq \int_{\phi_l}^{\phi_u} \left| \frac{N(R e^{i\phi})}{\Delta_\pm(R e^{i\phi})} \right| e^{-XR \sin \phi} \, dR. \]  \hfill (14)

Consider the denominator first.
\[ |\Delta_\pm| = \left| (i\xi)^{4/3} \left( \frac{Ai'(\eta_0)}{(i\xi)^{4/3}} \pm i \int_{\eta_0}^\infty Ai(s) \, ds \right) \right| \]
\[ = \left| (iRe^{i\phi})^{4/3} \left( \frac{Ai'(\eta_0)}{(iRe^{i\phi})^{4/3}} \pm i \int_{\eta_0}^\infty Ai(s) \, ds \right) \right| \]
\[ \sim \frac{1}{3} R^{4/3} \quad \text{as} \quad R \to \infty, \]  \hfill (15)

since \( \eta_0 \sim R^{-2/3} \to 0 \) as \( R \to \infty \). Furthermore, for positive \( X \) downstream,
\[ e^{-XR \sin \phi} \to 0 \quad \text{as} \quad R \to \infty \quad \text{for} \quad \phi \in (\phi_l^\pm, \phi_u^\pm). \]  \hfill (16)

Since neither \( N_{\text{int}} \) nor \( N_{\text{vib}} \) exhibit exponential growth as \( \xi \to \infty \), it can be concluded that
\[ \lim_{R \to \infty} \int_{\phi_l}^{\phi_u} \left| \frac{N}{R^{4/3}} \right| e^{-XR \sin \phi} \, dR = 0, \]  \hfill (17)

and thus
\[ \lim_{R \to \infty} \int_{c_R^\pm} I_\pm(\xi) \, d\xi = 0, \]  \hfill (18)

q.e.d. Therefore (3.70) holds for both contour integrations.

C  Flow around a corner

To explain the second term in expression (4.42a), consider the conformal map between a flat plate and two plates forming a corner at an angle \( \theta \), measured from the horizontal axis to the plate. The conformal map between those planes is expressed as
\[ \zeta = Cz^{\frac{\pi}{\theta}}, \]
where \( C \) is an arbitrary constant. Making use of the known complex potential for the oncoming flow in the \( \zeta \)-plane, namely \( w(\zeta) = V_\infty \zeta \), the two relations can be combined to obtain the complex potential in the \( z \)-plane

\[
w(z) = V_\infty z^{\pi-\theta},
\]

which is the complex potential of the flow around a corner. The complex conjugate velocity can be obtained by taking the derivative,

\[
\overline{V} = \frac{dw}{dz} = \frac{\pi}{\pi - \theta} V_\infty z^{\pi-\theta},
\]

whose modulus is

\[
V = |\overline{V}| = \frac{\pi}{\pi - \theta} V_\infty r^{\frac{\pi}{\pi - \theta}}.
\]

Hence, making use of the Bernoulli equation [ref], the pressure is found to be

\[
p = -\rho \pi^2 V_\infty^2 \frac{2}{2(\pi - \theta)^2} r^{\frac{2\theta}{\pi - \theta}}.
\]

As the vibrating plate is assumed to vibrate at a small angle \( \theta \ll 1 \), the following approximation may be employed

\[
r^{\frac{2\theta}{\pi - \theta}} \sim r^{\frac{2\theta}{\pi}} = e^{\ln\left(r^{\frac{2\theta}{\pi}}\right)}
\]

\[
= e^{\frac{2\theta}{\pi} \ln r}
\]

\[
\sim 1 + \frac{2\theta}{\pi} \ln r + \ldots.
\]

Hence the pressure becomes

\[
p = A_p \left( 1 + \frac{2\theta}{\pi} \ln r + \ldots \right),
\]

where \( A_p \) is constant. These calculations show that flow around a corner leads to logarithmic terms in the pressure, and therefore these logarithms are also present in the pressure perturbations for the considered problem.
D  OLS analysis results

The following contains the results from an ordinary least squares analysis, which determines the coefficients $\lambda_1$ and $\lambda_2$ such that

$$\Delta N \propto W^{\lambda_1} D^{\lambda_2}. \quad (19)$$

Note that $W$ and $D$ are denoted by $W = x_1$ and $D = x_2$. The analysis was carried out by using the python package `statsmodels`.

<table>
<thead>
<tr>
<th>OLS Regression Results</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dep. Variable:</td>
</tr>
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</tr>
<tr>
<td>Method:</td>
</tr>
<tr>
<td>Date:</td>
</tr>
<tr>
<td>Prob (F-statistic):</td>
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<td>No. Observations:</td>
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<td>Df Residuals:</td>
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<td>Log-Likelihood:</td>
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<td>AIC:</td>
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<tr>
<td>BIC:</td>
</tr>
<tr>
<td>coef</td>
</tr>
<tr>
<td>---</td>
</tr>
<tr>
<td>x1</td>
</tr>
</tbody>
</table>
x2 | 1.0806 | 0.027 | 40.309 | 0.000 | 1.025 | 1.136 |
|const | 13.5614 | 0.276 | 49.081 | 0.000 | 12.987 | 14.136 |
| Omnibus: | 2.761 | Durbin-Watson: | 1.057 |
| Prob(Omnibus): | 0.251 | Jarque-Bera (JB): | 1.250 |
| Skew: | -0.306 | Prob(JB): | 0.535 |
| Kurtosis: | 3.935 | Cond. No. | 176. |

Thus

$$\Delta N \propto W^{0.763} D^{1.081}. \quad (20)$$

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