Length effects in elastic imperfect cylindrical shells under uniform bending

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Abstract

Recent computational investigations into the buckling behaviour of perfect elastic thin-walled cylindrical shells under uniform bending, a ubiquitous reference structural system that enjoys wide practical applications, have demonstrated that the stability behaviour of this shell system depends largely on the length. Consequently, four distinct length domains – short, medium, transitional and long – were introduced to categorise the response of the system based on the relative influence of end boundary restraint and cross-sectional ovalisation. However, most investigations on this particular shell system have only focused on near-perfect geometric cases despite the vast research efforts that were made on the subject of imperfection sensitivity in a related reference system of cylindrical shells under uniform axial compression. Furthermore, the potential coupling between length and imperfection sensitivity has never been studied for any shell system.

This research thus seeks to understand and characterise the effect of length on the elastic stability of imperfect cylindrical shells under uniform bending, considering diverse forms of geometric imperfections. The stability investigations were performed over a wide parametric variation of length, radius-to-thickness ratio, end boundary condition, form and amplitude of geometric imperfection, using a combination of modern finite element analysis software and programming languages.

It was confirmed that there exists a relationship between the length of the shell system and imperfection sensitivity and this relationship was characterised into realistic, but conservative closed-form, algebraic expressions as a proxy to undertaking further computational investigations by analysts. The study also offers an efficient computational strategy that may be adopted in managing large computational analyses through most modern finite element suites and it is envisaged that this strategy will appeal to computational analysts who are encouraged to adopt the automation methodology described herein to explore other structural systems.
Declaration of Originality

I hereby declare that the work presented in the current thesis is original research, undertaken by me and that no part of it has been submitted for consideration towards another degree. Any other work, which is not my own work, has been properly referenced.

Oluwole Kunle Fajuyitan
September 2018
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Publications

The publications based on this thesis are as follows:


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Nomenclature

The definitions of all symbols and abbreviations used in this thesis are given below, except otherwise defined:

**Abbreviations**

- **ECCS** – European Convention for Constructional Steelwork
- **EDR** – European Design Recommendation
- **FE** – Finite Element
- **FEA** – Finite Element Analysis
- **GMNA** – Geometrically and Materially Nonlinear Analysis
- **GMNIA** – Geometrically and Materially Nonlinear Analysis with Imperfections
- **GNA** – Geometrically Nonlinear Analysis
- **GNIA** – Geometrically Nonlinear Analysis with Imperfections
- **KC** – Kill Condition
- **LBA** – Linear Bifurcation Analysis
- **LS** – Limit State
- **MNA** – Materially Nonlinear Analysis
- **MRF** – Mesh Refinement Factor
- **PDE** – Partial Differential Equation
- **RRD** – Reference Resistance Design

**Symbols**

- **A** – area of cross-section
- **D** – flexural stiffness for plates and shells
- **E, E_h, G** – modulus of elasticity, modulus of strain hardening, shear modulus
- **f_y** or **σ_y** – material yield stress
- **I** – second moment of area of cross-section
- **k_{GMNIA}** – calibration factor for validating GMNIA analysis
- **l, L** – length of structural member
- **l_{gs}, l_{gθ}, l_w** – stick length measurement across axial, circumferential dimple and welds
\( M \) – applied moment on cylinders under bending
\( m, n \) – buckling wave number in the axial, circumferential direction respectively
\( M_{Braz} \) – Brazier moment
\( M_{cl} \) – classical elastic critical buckling moment
\( M_{cr} \) – linear bifurcation moment in cylinders under bending
\( M_k \) – nonlinear buckling moment in cylinders under bending, with or without imperfections
\( N \) – applied stress resultant in plates and shells
\( N_{cl} \) – elastic critical buckling stress resultant in cylindrical shells under compressive loading
\( P \) – applied compressive loading
\( P_{cl} \) – critical buckling load
\( r \) – radius to shell midsurface
\( R_{cr} \) – linear elastic critical buckling load factor
\( R_d \) – design buckling resistance
\( R_{GMNIA} \) – imperfect elastic-plastic critical buckling resistance
\( R_{pl} \) – plastic collapse load factor
\( t \) – thickness of shell wall
\( u, v, w \) – meridional, circumferential and radial deformations
\( UR \) – cross-sectional rotation at each cylinder end
\( W_{el} \) – elastic section modulus of cylinder
\( w_0 \) or \( w \) – deflection of the column’s neutral axis
\( x_1, x_2 \) – pair of scaling parameters for nonlinear curve fitting
\( Z \) – Batdorf’s dimensionless length parameter
\( \alpha \) – elastic buckling reduction factor
\( \alpha_G \) – geometrical reduction factor
\( \alpha_I \) – imperfection reduction factor
\( \beta \) – plastic range factor
$\gamma_M$ – partial factor for resistance to buckling

$\gamma_{\theta}$ or $\gamma_z$ – shear strain in shells

$\delta$ or $\delta_o$ – nominal amplitude of geometric imperfection

$\delta_e$ – equivalent geometric deviation

$\Delta w_o$, $U_o$ – depth of initial dimple, dimple parameter

$\varepsilon$ – normal strain in shell

$\eta$ – interaction exponent for elastic-plastic behaviour in shells

$\lambda$ – linear axial bending half-wavelength

$\lambda_{axi}$ or $\lambda_{cl}$ – axisymmetric buckle half-wavelength

$\lambda_{ov}$ – global slenderness of the shell structure

$\lambda_p$ – plastic limit relative slenderness

$\nu$ – Poisson’s ratio

$\rho$, $\theta$, $z$ – radial, circumferential and meridional axes

$\sigma_{cl}$ – classical elastic critical buckling stress in cylindrical shells under compressive loading

$\sigma_m$, $\sigma_b$ – membrane, bending stresses

$\sigma_z$, $\sigma_\theta$ – normal stresses in the shell along axial, circumferential directions

$\tau_{\theta z}$ or $\tau_{\theta z}$ – shear stress in the shell along axial and circumferential directions

$\varphi$ – mean curvature in cylinders under bending

$\varphi_{Braz}$ – Brazier curvature in cylinders under bending

$\varphi_{cl}$ – classical critical curvature in cylinders under bending

$\chi$ – relative buckling strength

$\Omega$ – dimensionless length group governing circumferential bending in perfect cylinders under bending, where $\Omega = \frac{L}{r} \sqrt{\frac{r}{t}}$

$\omega$ – dimensionless length parameter governing local buckling with $\omega = L/\sqrt{rt}$
Chapter 1 - Introduction

1.1 General background

A thin-walled cylindrical shell is an optimised structural form that is able to support high loads with minimal material use (Thompson & Hunt, 1984; Rotter, 2004; Chapelle & Bathe, 2010). As a result of this structural efficiency and in addition to the unique aesthetical virtues they possess, cylindrical shells, hereafter referred to as cylinders, enjoy wide practical applications such as in tubular piles, circular hollow sections, wind turbine support towers and slender silos, as well as aerospace vehicles, where the load case is of a general bending type (Fig. 1.1).

![Fig. 1.1](image1.png)

a) Tubular piles in offshore applications.  b) Wind turbine support towers.  c) A corrugated, internally ring-stiffened plane fuselage

Fig. 1.1 – Structural applications of cylinders under global bending loads (from trinityproducts.com, renewable-technology.com and boeingimages.com respectively, all accessed on 4th January 2018).

The practical forms of these cylinder structures are usually very thin with a slenderness ratio, defined in terms of the radius-to-thickness ratio, \( r/t \geq 100 \), and as a result, they fail mostly by elastic buckling instability at a stress level that may be significantly below the material yield stress. This detrimental instability has prompted extensive research studies into the phenomenon of buckling in cylinders and till date, it still constitutes an area of active research. Many of the findings from these studies have demonstrated that the cylinder length plays a key role in the elastic stability of the shell system (e.g. Seide & Weingarten, 1961; Brush & Almroth, 1975; Yamaki, 1984; Rotter, 2004; Schmidt & Winterstetter, 2004; Greiner, 2004; Rotter et al., 2014), although the nature of the loading condition influences the stability of the system too (Brazier, 1927; von Kármán, 1941; Wadee et al., 2006; Sadowski & Rotter, 2012). Furthermore, because these structural systems are highly optimised to resist membrane actions, slight deviations in the original geometry create perturbations in the smooth membrane actions.
assumed in the optimisation and can therefore translate into disproportionate change in the predicted buckling resistance (Koiter, 1945; Rotter & Teng, 1989). Altogether, this demonstrates how important it is to fully understand the elastic buckling phenomenon often found in the fundamental response of cylinders and the several factors that may trigger it before attempting to establish a suitable design guide for the structure.

1.2 Instability consideration for the design of cylinders

Shell structures are widely used in various applications. They occur naturally in egg shells, crustacean shells, bone structures etc and in civil engineering applications, they come in different forms such as tanks, silos, bio-digesters, wind turbine towers, chimneys, masts and tubular piles to mention but a few. One of the simplest geometric forms is the cylinder, which, despite its geometric simplicity, exhibits a myriad of complex instability phenomena even under nominally axisymmetric condition (Calladine, 1983). A very well-known early civil engineering application of cylinders may be found in the massive tubular compression members of the Forth Bridge (Fig. 1.2) in Edinburgh, built in the late 19th century following a major advancement in the production of steel through the Bessemer process invented in 1855 and named after the inventor – Sir Henry Bessemer.

Failure of these structural members by buckling under compressive loads has equally been an important subject even before the arrival of the Bessemer process (see Fairbairn, 1849; Clark & Stephenson, 1850). Algebraic analyses to understand this buckling phenomenon was never accomplished until the simultaneous and independent works of Lorenz (1908), Timoshenko (1910) and Southwell (1914). Meanwhile, the algebraic solution that was offered by these authors was later found to be an approximation to the true behaviour under ideal conditions because many other factors, mostly geometrical, were found to influence the true buckling behaviour of axially compressed cylinders. These factors include the length and end boundary condition of the cylinder (e.g. Brush & Almroth, 1975; Yamaki, 1984; Rotter, 2004), influence of pre-buckling rotations at the edges and the shape and amplitude of geometric imperfections (von Kármán & Tsien, 1941; Koiter, 1945; Donnell & Wan, 1950; Koiter, 1963; Budiansky & Hutchinson, 1966; Yamaki, 1984; Rotter & Teng, 1989; Rotter, 1997).
In a closely related structural system of elastic cylinders under uniform bending, numerous studies have investigated the influence of detrimental pre-buckling geometric changes (cross-section ovalisation) and local buckling on the elastic stability of the perfect cylinder (e.g. Brazier, 1927; Wood, 1958; Seide & Weingarten, 1961; Stephens et al., 1975; Tatting et al., 1997; Karamanos, 2002). Although it was also known for long that the cylinder length influences the elastic stability of perfect cylinders under uniform bending (since Axelrad, 1965), the effect on their linear and nonlinear buckling behaviour was only recently characterised and documented fully by Rotter et al. (2014). Surprisingly, unlike for axially compressed cylinders, the only known imperfection sensitivity study on cylinders under uniform bending is the work of Chen et al. (2008), who conducted a limited parametric study on elastic-plastic clamped cylinders that are not susceptible to the cross-sectional ovalisation phenomenon with the aid of realistic weld depression as a suitable imperfection form.

This raises an important question, which the current research attempts to address among other things, regarding the research gaps in imperfection sensitivity of elastic cylinders under uniform bending since a systematic investigation and documentation of the sensitivity of cylinders under bending to multiple imperfection forms across a wide range of parameters above all, the cylinder length, has never been performed.

1.3 Current design rule on shell buckling limit state

The advanced European design standard on strength and stability of metal shells EN 1993-1-6 (2007) deals with the strength and stability of shells and offers flexibility in the choice of design
concept to employ. This flexibility however depends on the level of conservatism anticipated and the computational skills possessed by the analyst. Furthermore, the European Convention for Constructional Steelwork (ECCS) European Design Recommendations on the Buckling of Metal Shells (ECCS EDR5, 2013) provides extensive commentary to guide on the use of this novel design standard. On the design guide, a lot is devoted on how the different design philosophies contained in EN 1993-1-6 (2007) may be implemented in structural design giving adequate warnings of the potential pitfalls that can affect a strength verification badly. The latest amendment to the design standard (EN 1993-1-6+A1, 2017) introduced a new Reference Resistance Design (RRD) concept (Rotter, 2016a; 2016b), which in addition to being a method of design, here offers a powerful exploration lens through which all the nonlinearities governing the behaviour of any structural system may be established and understood in isolation.

For the structural system of elastic cylinders under uniform bending, which is the focal point of the current research, these nonlinearities are characterised in terms of two dimensionless algebraic parameters, each accounting for a distinct physical phenomenon, namely: geometric nonlinearity ($\alpha_G$) and imperfection sensitivity ($\alpha_I$). The study of Rotter et al. (2014) characterised $\alpha_G$ for clamped elastic cylinders under bending into sets of algebraic expressions depending on the cylinder length. However, the only expression currently available for $\alpha_I$ is from the limited parametric study of Chen et al. (2008), which did not accommodate the potential length-dependency or coupling between cross-sectional ovalisation and imperfection sensitivity as this effect was not yet understood at the time.

1.4 Research approach

To achieve a full understanding of any shell structure, one very important pre-requisite is an in-depth investigation of several conditions that individually play a vital role in the behaviour of the shell structure, ranging from the choice of deformation theory, definition of material property law, imperfections in material and geometry to the plastic collapse mechanism. Algebraic investigations into any of the above scenarios can be particularly complex since the simplest and natural geometrical properties of the shell alone can render the governing differential equations too difficult to be solved in exact closed-form (Calladine, 1983). As a result, most of the existing closed-form analytical solutions were obtained for a reference shell system where the applied loading generates a completely uniform state of stress e.g. cylinders
under uniform axial compression, external pressure or torsion (Rotter, 2016b), although even here many simplifying assumptions must be employed. Despite the fact that rigorous analyses are performed in obtaining the solutions for these reference shell systems, the simplifying assumptions come with the drawback of misrepresenting the true complex state of stress that the structure undergoes in service. Consequently, the analytical solutions offer valuable ‘anchor points’, to which the ‘real’ behaviour can tend to asymptotically, under idealised conditions.

The choice of performing laboratory experiments in understanding shell behaviour is further thwarted by the fact that it is naturally unfeasible to achieve an exact replica of practical shell structure condition during service as in the laboratory (Michel et al., 2000; Mathon & Limam, 2006; Rotter, 2016a). This is in addition to the huge cost of repetitively building massive laboratory test models in order to investigate the many hypothesised elastic systems with many varying parameters experimentally, a cost that is above the current research budget. Nonetheless, with the advancement in numerical methods first in obtaining solutions to the governing stability equations and ultimately in finite element analysis (FEA), the option of numerical investigation through generic finite element computer software is considered the most cost effective for the current research endeavour.

A number of validation studies exists that have compared predictions from numerical analysis with either test data or standard formulae. These include Kobayashi et al., 2012 who demonstrated that the arc-length path tracing algorithms in FEA software ABAQUS can indeed model a post-buckling path that is similar, both in the initial and advanced stage, to the path traced by the high-speed photography measurements of Eslinger (1969) and the experiment investigations of Yamaki (1984). Another validation was offered by Wang & Sadowski (2018) who validated the finite element analysis of pressurised short cylinders under global transverse shear by comparing the numerical predictions with test data from Yamaki (1984); Rotter et al. (2014) who compared the ovalisation phenomenon in long cylinders under bending using FEA with standard formulae; and Sadowski and Rotter (2013a) who verified the buckling predictions of cylinders under uniform bending using FEA against selected test data from Kyriakides and Ju (1992). In all cases, a reasonably close agreement was observed in the predictions. Consequently, all of the buckling investigations in this research were performed through modern computational methods using the validated commercial finite element analysis software ABAQUS v. 6.14-2 (2014). However, recourse had to be made to additional
specialised ADAPTIC software (Izzuddin, 1991) to validate the complex behaviour demonstrated by some specific shell structures that were investigated.

1.5 Research objectives

As of the time of writing the current thesis, the European design standard for metal shells (EN 1993-1-6, 2007) is ‘under revision’ as part of the Mandate for Amending Existing Eurocodes and Extending the Scope of Structural Eurocodes (Mandate M/515) and an actualisation of the rules for cylinders under uniform bending is one of the mandated tasks of the Technical Committee (TC250/SC3/PT5) working on it. There is thus a ‘pressing need’ to fully understand and characterise the buckling behaviour of elastic cylinders under uniform bending, a ubiquitous reference structural system that enjoys wide structural application and the current research aims to address this ‘need’ directly, in addition to the following:

- A full investigation and characterisation of the length-dependent linear and nonlinear buckling behaviour of perfect elastic cylinders under uniform bending and different end boundary conditions as a follow-up to the work of Rotter et al. (2014);
- A detailed investigation of the sensitivity of elastic cylinders under uniform bending, unrestrained against cross-sectional ovalisation, to three different forms of geometric imperfections (eigenmode, imposed ovalisation and axisymmetric weld depression) under the clamped and simply-supported end boundary conditions and across a wide range of other parameters (e.g. length, radius-to-thickness ratio, amplitude of geometric imperfections)
- A conservative and full characterisation of the imperfection reduction factor $\alpha_I$ over the parametric variations of imperfection amplitudes, lengths, radius-to-thickness ratios and end support conditions within the framework of the recently developed Reference Resistance Design (RRD) method.

Since the parametric studies to be conducted require so many computational analyses to be performed and processed, this becomes unfeasible within the constraint of the current PhD if there’s no recourse to the use of automation. Hence, part of the aims of the current research is to develop an efficient computational management strategy, which exploits the modern finite element (FE) suite interfacing with programming languages (e.g. Python, FORTRAN,
MATLAB) to automate all the processes (model creation, submission, termination and output processing) involved in each computational buckling investigation.

1.6 Review of thesis contents

The focus of this thesis is on the stability, with the imperfection sensitivity in particular, of elastic cylinders under uniform bending, studied with the aid of advanced finite element methods. The elastic behaviour under linear and nonlinear geometric condition is comprehensively investigated over a wide parametric variation of slenderness, length, end support conditions, forms and amplitudes of geometric imperfections. The contents of this report are presented below:

In Chapter 2, a historical overview of buckling analysis in columns, plates and shells is introduced before a review of the buckling analysis of cylinder, considering the most common shell bending theories, is presented. In line with modern trend of research, the significance of utilising computational methods for the buckling investigation of this and many other structural systems is narrated. Furthermore, the role of computer software packages and programming languages in managing large amount of numerical investigations that are required to cover parameter hyperspace to a high resolution using computational tools as a means of stability investigation is assessed. A comprehensive summary of the buckling behaviour of perfect cylinders under the reference load case of uniform axial compression is narrated with emphasis on the crucial influence of length and end boundary condition on their elastic stability, to set the tone for a detailed review of the behaviour of elastic cylinders under a closely-related load case of uniform bending. The literature on the highly detrimental imperfection sensitivity demonstrated by axially compressed cylinders and the important role of forms and amplitudes of geometric imperfections are also reviewed. The potential coupling between imperfection sensitivity and length effects is also introduced. Existing knowledge, together with the historical trend in research into the linear and nonlinear buckling behaviour of elastic perfect cylinders under bending loads, is then carefully reviewed. The role of the cylinder length on the elastic stability of elastic cylinders under bending is also assessed. Finally, the place of this research study in filling an existing gap in the most recent Reference Resistance Design (RRD) concept now in EN 1993-1-6+A1 (2017) is elaborated upon after a careful review of the trend in structural design of metal shells over the years.
In Chapter 3, the result of an initial investigation into the general applicability of exploiting double symmetry conditions for the finite element analyses of cylinders under uniform bending is first narrated. Following the recommendations from this initial investigation, the methodology adopted for this research work is presented. A preliminary mesh convergence analysis to guide on the optimum mesh density to employ for all the computational analyses is also reported. Following these, the details of all the different parameters considered are narrated as well as a novel automation strategy, which was developed to manage large amount of computational analyses in ABAQUS. Finally, effective ‘kill conditions’, which were introduced to help detect the exact point of numerical instability during a computational analysis are then described. The methodology adopted herein contributes to the journal article published in a peer-review journal with details below:


Chapter 4 presents the results of a computational investigation into the buckling behaviour of very short elastic cylinders under uniform bending, whose behaviours are dominated by compatibility bending actions due to the kinematic requirement with the end boundary condition. First, the individual parameter of cylinder length and how it affects the elastic stability of cylinders under the fundamental loads and uniform bending is introduced. Then, two different end boundary conditions are employed to investigate the potential dependency of the linear and nonlinear buckling strengths of the cylinder on the condition of the rotational degree of freedom at the edges, through a linear elastic bifurcation analysis (LBA) and a geometrically nonlinear analysis (GNA) respectively. Cylinders with different radius-to-thickness ratios ranging between $r/t = 100 – 1000$ are modelled, where the critical eigenmode and weld depression imperfections are adopted to explore the imperfection sensitivity of these cylinders under varying amplitudes through a geometrically nonlinear analysis with imperfection (GNIA). Finally, the relationship between the linear or nonlinear buckling strengths with lengths are characterised as algebraic expressions to aid design. On the basis of the findings from this chapter, two peer-reviewed journal articles together with one conference article were published with details given below:


In Chapter 5, an investigation into the imperfection sensitivity of medium-length and long cylinders under uniform bending is presented and how this is influenced by cross-section ovalisation. Cylinders with a single instance of \( r/t = 100 \) are first studied while the lengths are varied to cover the most practical geometric cases (medium, transitional and long length domains). Two different end boundary conditions are also considered. The role of length on the nonlinear buckling behaviour of the structure is investigated through geometrically nonlinear analyses with or without imperfections (GNIA or GNA respectively). The imperfection sensitivity of these structures is investigated by means of three unique forms of geometric imperfections, namely: critical eigenmode, imposed ovalisation and weld depression imperfections. Finally, the influence of varying slenderness ratio \( (r/t) \) on the imperfection sensitivity relationship is investigated with the aid of elastic ‘modified’ capacity curves.

Based on the results of the length-dependent imperfection sensitivity conducted employing only the critical eigenmode imperfection in the current Chapter, one conference article was published as part of the Proceedings of the 8th International Conference on Advances in Steel Structures:


Chapter 6 outlines the algebraic characterisation of the length-dependent imperfection sensitivity in elastic cylinders under uniform bending within the framework of RRD. The previously established buckling moment versus length relationship dataset from Chapter 5, for all the forms of imperfection considered, is first reduced into a single lower bound relationship for each end support condition employed. Thereafter, a synthetic imperfection sensitivity relationship is constructed to enforce a conservative design rule where a buckling resistance at a larger imperfection amplitude is constrained to never exceed that at the preceding amplitude.
Using this synthetic data set, a power law functional relationship is then adopted to capture the dependency of the buckling resistance of the cylinder on the imperfection amplitude. The resulting scaling factors are thereafter made to be individual functions of the cylinder length. The place of this novel ‘fit-to-a-fit’ procedure in ensuring that the ensuing algebraic expressions capture the underlying physics and the length-dependency in the imperfection sensitivity of cylinders under uniform bending is also highlighted.

The research outcomes from Chapters 5 & 6 of this thesis may be found in two separate journal articles, one has been accepted for publication with the journal of Advances in Structural Engineering while the other one is currently in preparation for another journal, viz:


In Chapter 7, a summary of the conclusions drawn from preceding chapters is presented. In addition, important studies that have already followed up on the context or methodology of the current research study are highlighted. A key example is the study of Rotter & Al-Lawati (2016) that provided a first glimpse at the role of cylinder length on the imperfection sensitivity of axially compressed cylinders following an overview of length-dependency presented for imperfect cylinders under uniform bending by Fajuyitan *et al.*, (2015). In addition, Wang & Sadowski (2018) employed the imperfection amplitude definition from Fajuyitan *et al.* (2018) and Fajuyitan & Sadowski (2018) to investigate the sensitivity of clamped cantilever cylinders under global transverse shear to different forms of geometric imperfections. The adoption of the current systemic investigation for detailed exploration of other structural systems is also recommended.
Chapter 2 - Review of literature on the buckling of cylindrical metal shells

2.1 Introduction

This chapter contextualises the research by first providing background information on the phenomenon of buckling in columns, plates and shells. With the focus now shifted on shells, literatures on shell buckling analyses and the common shell bending theories are reviewed and the place of finite element analysis in shell buckling analysis, through computational modelling, is assessed. A comprehensive review of the buckling behaviour of cylinders under uniform axial compression is narrated together with a critical assessment of the numerous factors that affect the stability of this ‘fundamental’ reference shell system. Thereafter, the behaviour of cylinder under non-uniform axial compression induced by uniform bending is reviewed, with emphasis on the current research gaps particularly in the area of sensitivity to different forms and amplitudes of geometric imperfection. Finally, the current European design rules on the buckling limit state (LS3) of general shells is reviewed and the place of this research outcome in aiding structural design is highlighted.

Metal structures have properties that demonstrate great efficiency in transmitting stresses from one point to another with failure under tension occurring in various stages: first by yielding at a stress value corresponding to the yield stress $\sigma_y$ of the material, followed by necking and then a rupture (or fracture) occurring at the least cross-sectional area. On the other hand, whenever compressive stresses are present, the elastic stability of the shell structure may become controlled by buckling – the phenomenon of a disproportionate increase in displacement arising from only a small increase in load (Brush & Almroth, 1975) – although this is dependent on the non-dimensional slenderness of the structure (EN 1993-1-1, 2005). Buckling in a compression member is a stiffness-related phenomenon where a loss of stability occurs as a result of loss of stiffness, primarily geometric. Buckling typically occurs at stresses significantly below $\sigma_y$ and can take any of four unique forms, dependent on the equilibrium path followed by the member, which is defined as a plot of the applied loading against the corresponding degree of freedom. One of these forms is a limit-point bifurcation, where the behaviour of the structure is characterised by a smooth transition from a stable to an unstable response on reaching a local maximum on the equilibrium path (Fig. 2.1a). In symmetric bifurcation systems, the behaviour of a structure may change from a stable fundamental response to either another stable post-bifurcation response (Fig. 2.1b) but at reduced stiffness,
i.e. slope of the equilibrium path or to a completely unstable post-buckling response (Fig. 2.1c). The behaviours of these symmetric bifurcation systems are characterised on the equilibrium path by positive or negative stiffness respectively immediately after bifurcation. The last form is the asymmetric system, where the same structure may demonstrate both stable and unstable post-bifurcations (Fig. 2.1d).

![Diagram showing various types of elastic instabilities](image)

**Fig. 2.1** – Various types of elastic instabilities exhibited by metal structures under compressive loading (from Thompson & Hunt, 1973; 1984). Here, the critical buckling point is shown in circles, stable and unstable post-bifurcation behaviours are represented by continuous and dash lines respectively while the dotted lines depict the behaviour of imperfect systems.

In this chapter, the concept of buckling in metal structures is first reviewed by considering Euler column buckling phenomenon to set the tone for a comprehensive understanding of buckling in plates before offering a first glimpse at buckling in shells under uniform axial compression. Algebraic analyses of buckling in shells through the common shell theories, are then presented with in-depth descriptions of the fundamental assumptions in each theory. The modern analysis of shell buckling problems through computational modelling is then outlined and the different types of computational analyses currently available in the EN 1993-1-6 (2007) are reviewed. Intensive review of the reference shell system of elastic cylinders under uniform axial compression is then presented, covering all the different parameters that affect their elastic stability, ranging from small or large displacement theory, length, boundary conditions
and geometric imperfections. A similar review is subsequently presented for elastic cylinders under uniform bending before the structural design methods for metal shells are outlined.

2.1.1 An overview of buckling in columns under axial compression

The search for a comprehensive understanding of buckling phenomenon dates back by almost three centuries, with perhaps the first recorded buckling analysis being the well-known Euler (1759) analysis of a pin-ended column subjected to axial compression (Fig. 2.2). By considering the column in a slightly deflected form (Fig. 2.2b), Euler generated the now well-known 4th order homogenous differential equation (Eq. (2.1)), governing the equilibrium of the column:

\[ EIw'' + Pw = 0 \]  

(2.1)

where \( w \) is the lateral displacement, \( EI \) is the flexural rigidity of the column (composed of the Young’s modulus \( E \) and the second moment of area \( I \) of the cross-section), \( P \) is the applied compressive loading and all derivatives are with respect to the axial coordinate \( x \) (Brush & Almroth, 1975). The solution of this governing differential equation produced the critical (i.e. minimum) buckling load \( P_{cl} \) (first term of Eq. (2.2)) of the column, otherwise referred to as the Euler load.

\[ P_{cl} = \frac{\pi^2EI}{l^2} \left[ F \right] \quad \text{such that} \quad \sigma_{cl} = \frac{P_{cl}}{A} = \frac{\pi^2E}{(l/r)^2} \left[ F/L^2 \right] \quad \text{since} \quad I = Ar^2 \]  

(2.2)
Here \( l \) is the effective length of the pin-ended column. The implication of this buckling load prediction on the mechanics of axially loaded columns is that it varies inversely with the square of the column length and consequently, the load-carrying capacity reduces in geometrical progression with increasing effective length. Furthermore, under ideal elastic and geometrically perfect conditions, columns can sustain compressive loads with no out-of-plane deformation \( w \) up to \( P_{cl} \) and slightly beyond this value, although at increasingly large deformations. For an imperfect column in the shape of a sine curve, shown in Fig. 2.2b, Timoshenko & Gere (1961) established that the relationship between the lateral displacement \( w \) (Fig. 2.2c) at the centre of the column and the applied loading \( P \) may be approximated with sufficient accuracy by means of Eq. (2.3):

\[
w = \frac{w_0}{1 - P/P_{cl}}
\]

in which \( w_0 \) represents the maximum initial deflection of the column in its slightly deflected form. The equilibrium paths followed by these perfect and imperfect columns under axial compression are illustrated in Fig. 2.3 below:

![Fig. 2.3 – A weakly stable or neutral post-buckling behaviour of columns under axial compression.](image)

The behaviours of perfect and imperfect pin-ended columns under axial compression constitute a weakly stable or neutral post-bifurcation system where, although the structure can sustain additional loads beyond the critical value, excessive deformations usually prohibit the exploitation of this post-buckling strength. In structural design of a column, a conservative approach is thus employed on the assumption that a column fails by buckling once the
Compressive loading reaches $P_{cl}$. Although the original formulation of the critical Euler load $P_{cl}$ assumes a pin-ended boundary condition at each end of the column, the applicability of this buckling load prediction is still maintained for other types of end boundary conditions by the definition of the effective length of the column $l$ as the distance between points of contraflexure along the length of the column.

### 2.1.2 Buckling of flat plates under uniaxial compression

For a simply-supported flat plate under the action of uniaxial compression, described in terms of a distributed membrane stress resultant per unit width $N_x [F.L^{-1}]$ shown in Fig. 2.4, the governing differential equation and its solution were originally provided by Bryan (1891) and confirmed by other authors (e.g. Timoshenko & Woinowsky-Krieger, 1959; Timoshenko & Gere, 1961; Brush & Almroth, 1975).

![Fig. 2.4 – Uniaxially-loaded, simply-supported flat plate in compression](image)

The general solution of the governing differential equation was simply given as in Eq. (2.4), where $m$ and $n$ are the integer number of half-waves that are admissible along the direction of compression and perpendicular to it respectively, $t$ represents the thickness of the plate, while $E$ and $v$ are the Young’s modulus and Poisson ratio respectively.

$$N = \frac{\pi^2 a^2 D}{m^2} \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2$$

where $D = \frac{E t^3}{12(1-v^2)}$ is the bending rigidity of the plate.

$$\text{(2.4)}$$
On simplifying this solution (Eq. (2.4)) to obtain a form of the critical buckling stress resultant $N_{cl}$ of the plate, first by minimising the wave number perpendicular to the axis of loading to a value of unity ($n = 1$), Timoshenko & Gere (1961) showed that the first term of the derived critical load expression ($\pi^2 D/a^2$) is similar to the Euler buckling load in column (Eq. (2.2)). This becomes valid since the flat plate may now be considered as a strip of unit width and length $a$, with the bending rigidity $D$ of the flat plate now also representing the flexural rigidity ($EI$) of the column in the Euler load prediction. Finally, another form of the critical stress resultant $N_{cl}$ shown in Eq. (2.5) is obtained by minimising both wave numbers, i.e. $m = n = 1$ in the general solution of the governing differential equation (Eq. (2.4)), together with the corresponding critical stress $\sigma_{cl}$.

$$
N_{cl} = \begin{cases} 
\frac{\pi^2 D}{a^2} \left( m + \frac{a^2}{m b^2} \right)^2 & \text{for } n = 1 \\
\frac{k \pi^2 D}{b^2} & \text{for } m = n = 1 
\end{cases}
$$

$$
\sigma_{cl} = \frac{N_{cl}}{t} = \frac{k \pi^2 E}{12(1-\nu^2)} \left( \frac{t}{b} \right)^2 
$$

where $k = \left( \frac{b}{a} + \frac{a}{b} \right)^2 \geq 4$

Unlike in columns, the behaviour of perfect and imperfect plates constitutes a stable-symmetric bifurcation such that flat plates are naturally able to sustain additional loading significantly beyond $N_{cl}$, but at a reduced stiffness (Fig. 2.5).

![Fig. 2.5 – Stable post-buckling behaviour in axially compressed flat plates.](image)

The behaviour of columns and plates described above reveals some key points. The first point is that buckling requires only the bending rigidity of the structure ($EI$ for columns and $D$ for plates), informing that buckling must be a bending phenomenon. Another point revealed is that
buckling does not necessarily signify the end of the life of the structure, especially plated structures, whose post-buckling strengths are usually exploited in structural design. The stable symmetric bifurcation behaviour demonstrated by these structures originates from the fact that buckling in these structures triggers the conversion of membrane strain energy into additional membrane strain energy and with a relatively higher membrane stiffness than bending stiffness, these structures are able to sustain further loads beyond bifurcation. Lastly, both columns and plates exhibit mild or neutral sensitivity to geometric imperfections since the strength predictions of the imperfect structures are reasonably close to those of the perfect structures.

2.1.3 Introduction to buckling in cylinders under axial compression

By contrast with the buckling behaviours of columns and plates, the behaviour of shell structures dominated by axial compression may reveal an entirely different buckling response, where the structure loses capacity to carry loads anything close to the critical load immediately after attaining it. The elastic stability of this shell system may be understood by studying a typical load path of the shell system, which is illustrated in Fig. 2.6. The shell system follows a very linear fundamental curve up to the critical buckling load $N_{cl}$, and thereafter a sharply descending post-buckling path follows, an evidence of unstable post-buckling behaviour since the initial portion of the secondary path has a negative slope (Koiter, 1945; 1963).

![Fig. 2.6 – Unstable post-buckling behaviour in axially compressed cylinder structures (from Rotter, 2004)](image)

Here, buckling signifies an engineering failure of the structure because the shell structure is never able to sustain loads close to $N_{cl}$ upon attaining this value, a consequence of buckling triggering the conversion of membrane strain energy into bending strain energy in shells
ABAQUS, 2014) and with limited bending stiffness in shells to sustain such an amount of energy. However, it should be understood that the typical buckling behaviour narrated above is not representative of all cases of shell structures since there are diverse factors that influence the buckling phenomenon, and which may affect the type of equilibrium path followed by the shell structure under compressive loading. A more in-depth review of all these factors and how they affect the response of a reference shell system of cylinders under uniform axial compression is presented shortly.

2.2 Algebraic analysis of buckling in cylinders using membrane and bending theories

The governing differential equations and solution for structural systems of columns and plates have been presented above and a hint on the typical buckling behaviour of cylinder structures was offered. However, information on the procedures employed in deriving these governing equations or the assumptions made before obtaining the solutions have been excluded. Since buckling phenomenon has been the subject of many research investigation in shells and other civil engineering structures, this important background information is reviewed in detail herein and presented below, specifically for elastic cylinder systems.

The phenomenon of buckling was simply described by Brush & Almroth (1975) as a disproportionate increase in displacement in response to a small increase in load, where the member or structure converts its stretching (or membrane) strain energy into inextensional (or bending) strain energy (Bushnell, 1985; ABAQUS, 2014). Meanwhile, the mechanical properties of a cylinder describe its resistance to deformation in terms of two unique mechanisms: stretching (membrane) and inextensional (bending) effects and transmits loads by means of a combination of these two actions which vary over the surface of the cylinder (Calladine, 1983). Consequently, in analysing the stresses in thin cylinders under loading, two methods are commonly used, namely: membrane and bending theories. The shell membrane theory is a simple treatment that considers only in-plane ‘membrane’ stresses, assumed to be constant through the thickness, in deriving the static equilibrium equations along the normal (ρ), circumferential (θ) or meridional (z) directions (Fig. 2.7). In this theory, the final governing equations are expressed only in terms of stress resultants and are statically determinate.

By contrast, a more complex but complete treatment is offered in the shell bending theory, where all bending or twisting moments, material stiffness or changes in geometry are
considered together with transverse shears (as the ones equilibrating the bending moments). As a result, the final governing equations are expressed in terms of the deformation of the shell mid-surface and are statically indeterminate, following the procedure of analysis summarised in Fig. 2.8 below.

![Diagram](image)

Fig. 2.7: The cylinder geometry showing the local deformation and the corresponding axis in parenthesis.

Local static equilibrium relates stress resultants to each other

- Stress resultants:
  - Membrane stress resultants $N$'s $[FL^{-1}]$, transverse shears $Q$'s $[FL^{-1}]$ and twisting moments $M$'s $[FLL^{-2}]$

- Through-thickness equilibrium

- Stresses:
  - Normal stresses $\sigma$'s $[FL^{-2}]$ and shear stresses $\tau$'s $[FL^{-2}]$

- Constitutive relations

- Strains:
  - Normal strains $\varepsilon$'s, shear strains $\gamma$'s, and bending curvature $\kappa$'s

- Kinematic relations

- Deformations:
  - Meridional deformation $u$'s $[L]$, circumferential deformation $v$'s $[L]$, and radial deformation $w$'s $[L]$

Fig. 2.8: Procedure for deriving the governing bending theory equations

The relationship between the applied forces and stress resultants may be established by means of local static force or moment equilibrium while an integration of the membrane or bending stress distribution through the thickness of the shell wall yields the membrane stress resultant.
or bending moment stress resultant respectively. As mentioned earlier, the distribution of membrane stress $\sigma_m$ is assumed to be constant through the thickness of the shell wall, hence, the membrane stress resultants may be estimated as in Eq.(2.6):

$$N = \int_{r-t/2}^{r+t/2} \sigma_m(\rho)d\rho = \sigma_m \int_{r-t/2}^{r+t/2} d\rho = \sigma_m t \rightarrow \sigma_m = \frac{N}{t}$$

where $\rho$ represents the radial or normal axis defined in Fig. 2.7.

For the bending moment stress resultant, a linear variation of the bending stress $\sigma_b$ through the thickness is assumed under elastic condition and the moment resultant may simply be approximated as in Eq. (2.7):

$$\sigma_b(\rho) = 2\sigma_b (r - \rho)/t$$

$$M = \int_{r-t/2}^{r+t/2} \sigma_b(\rho)(r-\rho)d\rho = \frac{2}{t} \sigma_b \int_{r-t/2}^{r+t/2} (r-\rho)^2 d\rho = \frac{\sigma_b t^2}{6} \rightarrow \sigma_b = \frac{6M}{t^2}$$

where $r$ and $t$ are the radius to middle surface and the cylinder wall thickness respectively. The relationship between stresses and strains is governed by the material law assumed. For a thin shell, a simple linear-elastic constitutive relation based on generalised Hooke’s Law is widely applied:

$$\sigma_z = \frac{E}{1-\nu^2} \left(\varepsilon_z + \nu \varepsilon_\theta\right), \quad \sigma_\theta = \frac{E}{1-\nu^2} \left(\varepsilon_\theta + \nu \varepsilon_z\right)$$

$$\tau_{z\theta} = G \gamma_{z\theta} = \tau_{\theta z} = G \gamma_{\theta z} \text{ where } G = \frac{E}{2(1+\nu)}$$

In Eq. (2.8) above, $E$ is the Young’s modulus, $G$ is the shear modulus and $\nu$ is the Poisson’s ratio. An important part of this procedure for deriving the governing bending theory equations is the kinematic relations. This relates in-plane deformations and changes in curvature, where bending occurs, to strains and it is also the point where different shell theories begin to branch out with the most common ones being those of Donnell, Timoshenko, Koiter-Sanders and Flügge.
2.2.1 Donnell shell theory

Possibly the first shell theory is that of Donnell (1933), which was initially developed for a torsional buckling analysis of thin-walled cylindrical tubes. Donnell shell theory simplified the kinematic relations by assuming that the curvature changes are small and may be represented by linear functions of the radial deformation $w$ only such that the influence of meridional and circumferential deformations $u$ and $v$ respectively is negligible (Yamaki, 1984). This approximation otherwise known as the ‘shallow shell approximation’, is perhaps more representative of flat plates but can also predict reasonably accurate results for cylinders having a very high circumferential radius of curvature, akin to flat plates. In addition, the following assumptions are incorporated to generate the full kinematic relations for circular cylinder described in Eq. (2.9):

- The cylinder is sufficiently thin i.e. having a radius-to-thickness ratio $r/t$ in excess of 100 and dimensionless length $L/r >> 1$;
- The structure undergoes small strains $\varepsilon$ and Hooke’s law of elasticity is applicable;
- Kirchhoff-Love hypothesis, a 2D equivalent to the Euler-Bernoulli theory for beams, which assumes that plane cross-sections normal to the shell midsurface remain plane and normal after deformation. In the cylinder, this theory assumes that the shell will never distort through its thickness but only along the midsurface, since the transverse displacements due to transverse shear must be negligible when compared with those due to bending as any transverse shear stress resultants that do exist only act as reactions to equilibrate a moment gradient. Furthermore, the transverse normal stresses through the thickness of the cylinder wall are assumed to be negligible, i.e. $\sigma_\theta = 0$;

\[
\varepsilon_z = \varepsilon_{z0} + \rho \cdot \kappa_z \\
\varepsilon_\theta = \varepsilon_{\theta0} + \rho \cdot \kappa_\theta \\
\gamma_{z\theta} = \gamma_{z\theta0} + \rho \cdot \kappa_{z\theta}
\]

\[
\varepsilon_{z0} = \frac{\partial u}{\partial z} + \frac{1}{2} \left( \frac{\partial w}{\partial z} \right)^2 \\
\varepsilon_{\theta0} = \frac{\partial v}{r \partial \theta} - \frac{w}{r} + \frac{1}{2} \left( \frac{\partial w}{r \partial \theta} \right)^2 \\
\gamma_{z\theta0} = \frac{\partial u}{r \partial \theta} + \frac{\partial v}{\partial z} + \left( \frac{\partial w}{\partial z} \cdot \frac{\partial w}{r \partial \theta} \right) \\
\kappa_z = -\frac{\partial^2 w}{\partial z^2} \\
\kappa_\theta = -\frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \\
\kappa_{z\theta} = -\frac{2}{r} \frac{\partial^2 w}{\partial z \partial \theta}
\]

Here, the displacements in Eq. (2.9) are as defined in Fig. 2.7. Donnell theory enjoys wide usage for both buckling and post-buckling analyses, owing to its relative simplicity and reasonable accuracy. However, because of the ‘shallow shell assumptions’ that cannot
accurately model the deformations of a cylinder in which the magnitude of the in-plane displacement ($u$ or $v$) is of a similar order as that of the deflection ($w$), several criticisms on its applicability exist (Yamaki, 1984). Nevertheless, the limitations from these shallow shell approximations were addressed by Flügge (1932), who developed the most complete basic equations for the buckling of cylinders under typical loading.

### 2.2.2 Timoshenko shell theory

In Timoshenko (1932; 1959) shell theory, the influence of circumferential deformation $v$ in predicting changes in circumferential bending curvature $\kappa_\theta$ and twisting curvature $\kappa_{z\theta}$ is considered. Although this theory also employs all of the remaining assumptions contained in Donnell theory, the modification to the former theory is seen in the computation of changes in curvature, as described in Eq. (2.10):

$$
\kappa_z = -\frac{\partial^2 w}{\partial z^2}, \quad \kappa_\theta = -\frac{1}{r^3} \left( \frac{\partial v}{\partial \theta} + \frac{\partial^2 w}{\partial \theta^2} \right) \quad \text{and} \quad \kappa_{z\theta} = -\frac{1}{r} \left( \frac{\partial v}{\partial z} + \frac{\partial^2 w}{\partial z \partial \theta} \right)
$$

### 2.2.3 Koiter-Sanders shell theory

The Koiter-Sanders nonlinear theory is a more rationally accurate theory for finite deformation of thin shells (Yamaki, 1984), simultaneously developed by Sanders (1963) and Koiter (1966). This theory supersedes Donnell theory by utilising all the displacement fields i.e. $u$, $v$ and $w$ in predicting the changes in curvature arising from bending, although transverse shear strains are still ignored. The corresponding strain-displacement and curvature-displacement relationships following Koiter-Sanders theory are:

$$
\varepsilon_{z\theta} = \frac{\partial u}{\partial z} + \frac{1}{2} \left( \frac{\partial w}{\partial z} \right)^2 + \frac{1}{8} \left( \frac{\partial v}{\partial \theta} - \frac{\partial u}{r \partial \theta} \right)^2 \quad \kappa_z = -\frac{\partial^2 w}{\partial z^2} \\
\varepsilon_{\theta\theta} = \frac{\partial v}{r \partial \theta} - \frac{w}{r} + \frac{1}{2} \left( \frac{\partial w}{r \partial \theta} + \frac{v}{r} \right)^2 + \frac{1}{8} \left( \frac{\partial u}{r \partial \theta} - \frac{\partial v}{\partial z} \right)^2 \quad \kappa_\theta = -\frac{1}{r^3} \left( \frac{\partial^2 w}{\partial \theta^2} + \frac{\partial v}{\partial \theta} \right) \\
\gamma_{z\theta\theta} = \frac{\partial u}{r \partial \theta} + \frac{\partial v}{r \partial \theta} + \frac{\partial w}{r \partial \theta} \left( \frac{\partial w}{r \partial \theta} + \frac{v}{r} \right) \quad \kappa_{z\theta} = -\frac{2}{r} \left[ \frac{\partial^2 w}{\partial z \partial \theta} + \frac{1}{4} \left( \frac{\partial v}{\partial z} - \frac{\partial u}{r \partial \theta} \right) \right]
$$

Koiter-Sanders shell theory is perhaps the favourite for computational analysis through finite element method (Yamaki, 1984) and it is unsurprising that the bending strain measures used in
all the shell elements within the element library of the commercial finite element suite ABAQUS (2014) follow the approximations in this theory.

2.2.4 Conclusion

Although all the afore-mentioned theories offer important basis and simplification on which basic shell analysis may be performed, algebraic manipulation following these theories can only be possible under axisymmetry and pure membrane conditions with anything else inevitably requiring numerical solution, usually by finite element analysis. Consequently, the classical solutions to the governing differential equations derived through these theories all assumed a pre-buckling stress state that is uniform (i.e. axisymmetric) and dominated by membrane actions. Nevertheless, the era of powerful personal computers and advanced computational tools has created the possibilities of incorporating as much complexity as may be needed to obtain accurate solutions to the shell buckling problem, including pre-buckling stress states that are non-uniform (i.e. non-axisymmetric) and dominated by bending actions. Therefore, more complete kinematic relations can now be adopted in deriving the governing bending theory equations, as may be found in the formulation of Combescure (1986) or Rotter & Jumikis (1988). The simplification associated with the Kirchhoff-Love hypotheses can also now be improved for thicker cylinders, i.e. those with \( r/t \ll 100 \), by the adoption of the Mindlin-Reissner theory, similar to the Timoshenko theory for beams, which also approximately accounts for significant transverse shear deformations.

2.3 Computational modelling of buckling in general shells

2.3.1 Introduction

The analysis of shell buckling may be complex, even for simple cylinder geometries, because there are many different mechanisms such as local buckling, circumferential bending, material plasticity, sensitivity to diverse forms of geometric imperfections, that potentially control the behaviour of shell structures (Brazier, 1927; von Kármán & Tsien, 1941; Wood, 1958; Stein, 1962; Hoff & Rehfield, 1965; Brush & Almroth, 1975; Yamaki, 1984; Bushnell, 1985; Rotter & Teng, 1989; Rotter, 2002; 2004; Chen et al., 2008; Rotter et al., 2014). A first glimpse of this complexity may be noticed in the different shell theories, outlined above, that may be adopted in deriving different kinematic relations for these cylinders. However, other stages in
the procedures for deriving the governing bending equations for cylinders (Fig. 2.8) come with their peculiar complexities as well.

Prior to the modern computer age, attempts to solve shell buckling problems were mostly undertaken through algebraic analyses (see Flügge, 1932; Timoshenko & Woinowsky-Krieger, 1940; Koiter, 1945; 1963; Brush & Almroth, 1975) but due to the complex nature of the governing equations describing shell stability, only limited and idealised shell systems could be solved with reasonable accuracy by this method. As more complex shell geometries and loading conditions emerged, there arose the need to adopt numerical methods in obtaining the solution of the governing equations and which eventually led to the development of the finite element analysis (FEA). The FEA works by transforming a structural system into individual smaller finite elements connected at the nodes and approximating a field variable using simple interpolation function that is defined over each of the finite elements (Cook et al., 2002). Consequently, all restrictions and limitations to shell buckling analysis drawn from the type of geometry, end boundary conditions, material properties or potential variation in any of these properties along any axis may now be considered to be non-existent (Felippa & Clough, 1970).

Early buckling analyses for shells using FEA may be traced to the works of numerous authors (e.g. Hsu et al., 1964; Kalnins, 1964; Marcal & Pilgrim, 1966; Cohen, 1968; Popov & Sharifi, 1971; Bushnell, 1974), with each method formulated to address a particular type of shell buckling problems. However, a number of generic finite element computer software (e.g. ABAQUS, NASTRAN, ANSYS) has since been developed that can be installed on personal computers, therefore permitting an analyst to perform buckling calculations with ease, although this undoubtedly requires some level of computational modelling skills from the analyst (Rotter, 2017a). While the use of FEA offers a robust platform to perform any shell buckling calculation, the current European standard for the design of metal shells (EN 1993-1-6, 2007) and ECCS EDR5 (2008; 2013) prescribed different types of computational analyses that may be performed on shells with the intent of separating out and classifying the nonlinearities governing the buckling behaviour of that shell structure.

2.3.2 Linear Analysis (LA)

This analysis may be performed on the perfect elastic shell structure to obtain primary and secondary stresses with the assumption of linear constitutive and kinematic relations.
Compatibility with kinematic boundary condition is ensured by the membrane and bending deformations that arise in the shell structure although no buckling or yielding calculation is permitted as this is basically a stress analysis.

2.3.3 Linear Bifurcation Analysis (LBA)

The LBA is a linear eigenvalue analysis, based on a full linear shell bending theory and so incorporates all local bending deformation characteristic of kinematic compatibility requirement with the end boundary condition that affects the pre-buckling stress state. It is employed to compute the critical bifurcation buckling load, corresponding to the lowest eigenvalue of the structural system and the associated critical buckling mode (eigenmode) of the perfect structure, assuming small deformation theory and a generalised Hooke’s law. This critical buckling load is one of the two important reference resistances \( R_{cr} \) used in evaluating the overall slenderness of a shell structure in design (EN 1993-1-6, 2007) and is distinct from the classical theoretical buckling load that is derived on the assumption of a meridionally uniform pre-buckling membrane stress state.

2.3.4 Materially Nonlinear Analysis (MNA)

The second important reference resistance used in dimensioning the global slenderness of a shell structure in design is obtained from a materially nonlinear analysis. This analysis computes the plastic collapse load \( R_{pl} \) and the corresponding collapse mechanism of the perfect structure, assuming small deformation theory but a nonlinear ideal elastic-plastic constitutive relation. The equilibrium path followed by the shell structure during a materially nonlinear analysis is characterised by an initially linear portion up to the load level that corresponds to the yield stress of the material followed by the development of a clear yield plateau of indefinite deformation without any further increase in load. The resistance level at this plateau corresponds to the plastic collapse load \( R_{pl} \) of the shell structure.

2.3.5 Geometrically Nonlinear Analysis (GNA)

The GNA is an analysis of the perfect structure, assuming nonlinear strain-to-displacement relations (large deformation theory) but linear stress-to-strain relations, to compute the nonlinear characteristic buckling load of the perfect shell structure. In this analysis, nonlinear pre-buckling bending deformations are additionally considered, therefore the initial nonlinear
equilibrium path are correctly traced with the subsequent possibility of capturing a limit-point or, with some care, bifurcation instabilities. The amplification of pre-buckling rotations near the ends during this analysis usually accounts for some loss of stiffness close to the point of buckling.

2.3.6 Geometrically and Materially Nonlinear Analysis (GMNA)

The GMNA is essentially a GNA but with a nonlinear constitutive relation. One fundamental difference between this computational analysis and the MNA is that there is no limitation in the definition of material law, with the possible inclusion of strain hardening in the material law provided it can be justified. The equilibrium path of the shell structure during this analysis follows a similar initial path like the GNA, until the initiation of plasticity which may precipitate early buckling.

2.3.7 Geometrically Nonlinear Analysis with Imperfections (GNIA)

This is one of the most important computational analyses for buckling investigation of thin-walled shell structures due to the historical precedent of imperfections having an important influence on the buckling behaviour of the widely studied elastic cylinders under uniform axial compression. It is a geometrically nonlinear analysis but with a particular pattern of geometric imperfections explicitly defined in the shell model. Imperfections may occur in a shell structure as a geometric deviation of the shell midsurface, irregularities in the boundary conditions or as a form of material imperfections such as residual stresses or a combination of these geometric and material imperfections. The equilibrium path followed by GNIA usually reveals a lower stiffness than that of the perfect structure with the possibility of buckling at a much reduced load level, depending on the post-buckling behaviour of the initially perfect shell model obtained through GNA (von Kármán, 1941; Donnell & Wan, 1950).

2.3.8 Geometrically and Materially Nonlinear Analysis with Imperfections (GMNIA)

The GMNIA is the most sophisticated computational analysis for shells, an advancement that may perhaps be considered a growing achievement of computational methods. GMNIA considers all the possible effects, with justification, in the shell model. This includes large deformation theory, nonlinear material law that may include strain hardening, together with geometric or material imperfections or a combination of these. As a way of predicting more
accurate and economical buckling resistances in shells (ECCS EDR5, 2008; 2013), this method of analysis is prescribed in EN 1993-1-6 (2007) as a method of design through global numerical analysis. A summary of these computational analyses is presented in Table 2.1.

Table 2.1 – Computational analyses in shell buckling (from EN 1993-1-6, 2007; ECCS EDR5, 2008; 2013)

<table>
<thead>
<tr>
<th>Acronym</th>
<th>Type of shell analysis</th>
<th>Kinematic relations</th>
<th>Constitutive relations</th>
<th>Shell geometry</th>
<th>Objective of analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>n/a</td>
<td>Membrane analysis</td>
<td>n/a</td>
<td>n/a</td>
<td>Perfect</td>
<td>Primary reference stresses</td>
</tr>
<tr>
<td>LA</td>
<td>Linear Elastic</td>
<td>Linear</td>
<td>Linear</td>
<td>Perfect</td>
<td>Critical buckling eigenvalue + mode</td>
</tr>
<tr>
<td>LBA</td>
<td>Linear Bifurcation</td>
<td>Linear</td>
<td>Linear</td>
<td>Perfect</td>
<td>Plastic collapse load + mode</td>
</tr>
<tr>
<td>MNA</td>
<td>Materially Nonlinear</td>
<td>Linear</td>
<td>Nonlinear (ideal elastic-plastic)</td>
<td>Perfect</td>
<td>Characteristic buckling load + mode</td>
</tr>
<tr>
<td>GNA</td>
<td>Geometrically Nonlinear</td>
<td>Nonlinear</td>
<td>Linear</td>
<td>Perfect</td>
<td>Characteristic buckling load + mode</td>
</tr>
<tr>
<td>GMNA</td>
<td>Geometrically &amp; Materially Nonlinear</td>
<td>Nonlinear</td>
<td>Nonlinear</td>
<td>Perfect</td>
<td>Characteristic buckling load + mode</td>
</tr>
<tr>
<td>GNIA</td>
<td>Geometrically Nonlinear + Imperfections</td>
<td>Nonlinear</td>
<td>Linear</td>
<td>Imperfect</td>
<td>Characteristic buckling load + mode</td>
</tr>
<tr>
<td>GMNIA</td>
<td>Geometrically &amp; Materially Nonlinear + Imperfections</td>
<td>Nonlinear</td>
<td>Nonlinear</td>
<td>Imperfect</td>
<td>Characteristic buckling load + mode</td>
</tr>
</tbody>
</table>

2.3.9 Conclusion

The important considerations for the buckling analysis (analytical and computational) of cylinder systems have been narrated, together with the diverse types of computational analysis available to understand the particular nonlinearity governing the behaviour of the shell system in isolation. It may be observed that the decomposition of computational analyses into diverse types encourages a more systematic and focussed shell buckling investigation. The subsequent review of buckling investigation is now focussed on cylinder systems that are dominated by axial compression. However, material nonlinearity in the shell and the associated instabilities are not reviewed since the primary focus here is on the fundamental elastic response of the system, where loss of stiffness is primarily due to geometric changes.
2.4 Buckling of elastic cylinders under uniform axial compression

2.4.1 Introduction

A thin-walled cylinder may be regarded as an efficient form for a compression member owing to the relatively far distance between the materials and the neutral axis of its cross-section, which can reduce its susceptibility to Euler buckling significantly based on a higher lever arm and second moment of area (Brazier, 1927) than for a solid section. Consequently, they are found in diverse structural applications in many fields, where the shell experiences some form of axial compression, although cylinders under uniform axial compression are considered the reference case.

The development of axial compression in a practical civil engineering shell structure can take different forms and cylinders dominated by axial compression may be described as the most common. Key examples include the relatively uniform axial compression in a tower due to self-weight, the circumferentially varying, but smooth, axial compression in a horizontal tank due to self-weight bending and the frictional drag from stored solids in a silo (Rotter, 2004). Other structural applications include chimneys & wind turbine support towers under seismic and wind loading, piles & pipelines under gravity and earth pressure loading. As a result of its practical importance and the simplicity of solving its governing differential equations algebraically, the reference shell system of cylinders under uniform axial compression system boasts a rich engineering research history (Hoff, 1966; Yamaki, 1984).

2.4.2 Buckling behaviour of perfect cylinders assuming small displacement theory

It was earlier mentioned that buckling must be a bending phenomenon since only the bending rigidity of the structural system is active in sustaining the critical loads (Eq. (2.2) & Eq. (2.5)). Meanwhile, the displacements necessary to support the same strain energy in bending are higher relative to those in membrane due to the different stiffnesses (Bushnell, 1985). Consequently, buckling usually signifies an engineering failure of the life of a shell structure. Buckling in a perfect cylinder under uniform axial compression is deemed to occur once the critical value of stress $\sigma_{cl}$ or stress resultant $N_{cl}$ (Eq. (2.12)) is attained during loading. These critical stress and stress resultant were independently and simultaneously derived by Lorenz (1908), Timoshenko (1910) and Southwell (1914), based on the assumptions of geometrically linear conditions, i.e. small displacement theory, with the ends of the cylinder under simple
supports, in addition to a pre-buckling stress state that is under pure membrane action, which allows the governing differential equation to be solvable algebraically.

\[
N_{cl} = \frac{Et^2}{r\sqrt{3(1-\nu^2)}} , \quad \sigma_{cl} = \frac{E}{\sqrt{3(1-\nu^2)}} \cdot \frac{t}{r} \approx 0.605E\frac{t}{r} \quad \text{for steel with } \nu = 0.3 \quad (2.12)
\]

The significance of Eq. (2.12) in the elastic stability of axially compressed cylinders is that the critical stress relates directly with the modulus of elasticity and the Poisson coupling, but indirectly with the radius-to-thickness ratio \( r/t \) of the cylinder, therefore this elastic critical stress becomes progressively smaller as the shell grows thinner (as \( r/t \to \infty \)). The linear critical stress \( \sigma_{cl} \) is also known as the ‘classical elastic critical buckling stress’ and, despite the numerous simplifying assumptions made to derive it, remains an important reference critical stress in the buckling analysis of cylinders subject to any form of axial compression. The derivation of \( N_{cl} \) may also be performed, following the procedure of analysis in Fig. 2.8, by minimising the solution \( N \) given in Eq. (2.13) to the uncoupled Donnell linear stability equation (Eq. (2.14)) with respect to \( \eta^2 \) and subject to the shape of the buckling mode (also referred to as eigenmode) condition in Eq. (2.15) (Brush & Almroth, 1975).

\[
N = D\eta^2 + \frac{Et}{r^2\eta^2} , \quad \text{where } \eta^2 = \left( \frac{m^2 r^2 + n^2}{mr^2} \right)^2 \quad \text{and } m = m\frac{\pi}{L} \quad (2.13)
\]

\[
D\nabla^4 w + N\nabla^4 \left( \frac{d^2 w}{dz^2} \right) + \frac{Et}{r^2} \cdot \frac{d^4 w}{dz^4} = 0 \quad (2.14)
\]

\[
\eta^2 = \sqrt{\frac{Et}{Dr^2}} \quad (2.15)
\]

Here, \( D \) (Eq. (2.4)) is the bending rigidity of the cylinder, \( E \) is the Young’s modulus, \( N \) is the membrane stress resultant action on the cylinder, \( L, r \) and \( t \) are the length, radius and thickness of the cylinder wall respectively, \( w \) and \( z \) are the radial deformation and axial coordinate respectively, while \( m \) and \( n \) are the number of meridional half-waves and circumferential full waves admissible on the cylinder at buckling respectively.
The shape of the eigenmode condition in Eq. (2.15) may be simplified by assuming the condition of axisymmetry, which translates into zero variation around the circumference of the cylinder \( \delta \delta \theta \rightarrow 0 \) such that a circular cross-section can be maintained throughout the length i.e. \( n = 0 \). The resulting buckling mode is illustrated in Fig. 2.9a and the corresponding half-wavelength \( \lambda_{\text{axi}} \) of each axisymmetric buckle along the length may be calculated using Eq. (2.16):

\[
m = \frac{L}{\pi} \left( \frac{Et}{Dr^2} \right)^{\frac{1}{4}}
\]

\[
\lambda_{\text{axi}} = \frac{L}{m} = \pi \left( \frac{Dr^2}{Et} \right)^{\frac{1}{4}} \Rightarrow \lambda_{\text{axi}} = \frac{\pi \sqrt{rt}}{12 (1 - \nu^2)^{\frac{1}{4}}} \approx 1.728 \sqrt{rt}
\]

Furthermore by rearranging the expression describing the shape of the eigenmode condition to represent the equation of a circle, a ‘Koiter circle’ (Fig. 2.10) that is named after the inventor, may be drawn to relate the corresponding half-wavelengths in the circumferential and axial directions (Koiter, 1945; Calladine, 1983). Without the condition of axisymmetry, i.e. \( m \) and \( n \) being integers but \( \neq 0 \), the linear buckling eigenmode may assume the chequer-board pattern shown in Fig. 2.9b.

![Fig. 2.9 – Representative buckling modes for perfect cylinders under axial compression: a) axisymmetric mode, (b) non-symmetric mode (after Rotter, 2004)](image-url)
For a cylinder under uniform axial compression, Eq. (2.15) suggests that there are diverse shapes that the cylinder can buckle into at the same load, i.e. several combinations of $m$ and $n$ that satisfy the condition. Consequently, it is almost always possible that the cylinder will exhibit a dimple imperfection in the shape of the critical eigenmode. This competition for criticalness between several modes is indicative of a high imperfection sensitivity in the elastic stability of the shell.

However, it may be easily inferred that the assumption of a geometrically linear condition, i.e. small displacement theory, in the formulation of the critical buckling stress $\sigma_{cl}$ cannot capture the actual point of buckling in this cylinder system since buckling is a subject under geometrically nonlinear mechanics (Brush & Almroth, 1975). The behaviour of the system under geometrically nonlinear conditions is therefore presented below.

### 2.4.3 Buckling behaviour of perfect cylinders under geometrically nonlinear conditions

When a cylinder is uniformly compressed in the axial direction, lateral expansions must develop in the cylinder to compensate for the induced meridional displacements (Poisson effect) and these lateral expansions must also be restrained near the edges by the end boundary condition, giving rise to a region of boundary layer near the edges (Brush & Almroth, 1975; Rotter, 2004). The depth of this boundary layer necessary to satisfy kinematic compatibility
with the end boundary condition is controlled by the linear meridional bending half-wavelength \( \lambda \) (Eq. (2.17)), which is obtained as a solution to the axisymmetric linear bending theory problem (Novozhilov, 1959; Timoshenko & Woinowsky-Krieger, 1959; Calladine, 1983). Altogether, this introduces a pre-buckling rotation of structural elements, especially near the edges, which can become amplified with increasing load.

\[
\lambda = \frac{\pi \sqrt{rt}}{\left[3\left(1-\nu^2\right)\right]^{0.25}} \approx 2.444\sqrt{rt} \quad \text{for} \quad \nu = 0.3
\]  

(2.17)

Under geometrically nonlinear conditions, the secondary effect associated with this influence of pre-buckling rotations is captured and a typical nonlinear equilibrium path for an axially compressed cylinder is illustrated in Fig. 2.11. It is shown that the cylinder may start up with a linear fundamental path but as the load approaches the value of the linear critical stress resultant \( N_{cl} \), the nonlinearity associated with the progressive amplification of pre-buckling rotations near the ends now becomes non-negligible in the load path before a bifurcation occurs onto a sharp descending path, characteristic of an unstable symmetric system. This secondary effect, arising from influence of pre-buckling rotations, accounts for approximately \(~15\%\) difference between the linear and the nonlinear buckling loads of the cylinder system, depending on the type of end boundary condition employed (Brush & Almroth, 1975; Yamaki, 1984; Rotter, 2004).

![Fig. 2.11 – Typical equilibrium path of a finite length axially compressed cylinder](image-url)
2.4.4 Effect of length and end boundary condition on buckling response of perfect cylinders

A quick revisit to the shape of the eigenmode condition presented in Eq. (2.15) and upon which the reference elastic critical buckling stress $\sigma_{cl}$ is based reveals that the stipulated condition may become impossible to fulfil when the cylinder length becomes very short, since $m$ and $n$ must remain positive integers. This difficulty was pointed out by Brush & Almroth (1975) to occur once the cylinder length, defined in terms of the Batdorf (1947) parameter $Z$ (Eq. (2.18)), falls below a certain value ($Z < 2.85$).

\[ Z = \frac{L^2}{rt} \sqrt{1-v^2} = \omega^2 \sqrt{1-v^2} \quad \text{and} \quad \omega = \frac{L}{\sqrt{rt}} \quad (2.18) \]

Furthermore, a very long cylinder may opt to buckle as a column (Euler buckling, see 2.1.1) with undeformed cross-sections i.e. undergo global buckling with $m = n = 1$ (Timoshenko & Gere, 1961; Yamaki, 1984). A similar behaviour is demonstrated by these systems under geometrically nonlinear condition. Consequently, the buckling behaviour of cylinder under uniform axial compression is categorised under three unique length domains, namely: ‘short’, ‘medium’ and ‘long’ (Yamaki, 1984; EN 1993-1-6, 2007; ECCS EDR5, 2013). In the ‘short’ domain, extensive bending deformations develop in the cylinder to satisfy kinematic compatibility with end boundary conditions. As a result, buckling occurs at a stress significantly higher than the theoretical buckling stress predicted on the assumption of pure and uniform membrane actions in the cylinder ($\sigma << \sigma_{cl}$). The behaviour of cylinders in the ‘medium’ domain demonstrates close agreement with $\sigma_{cl}$ since the effect of bending deformations is now localised near the edges, keeping the cylinder now largely under membrane actions, i.e. $\sigma \approx \sigma_{cl}$. ‘Long’ cylinders behave in a manner similar to Euler buckling in columns and fail at a stress much lower than $\sigma_{cl}$ ($\sigma << \sigma_{cl}$).

Using this Batdorf (1947) parameter $Z$, Yamaki (1984) offered a comprehensive view of the influence of length and boundary condition on the nonlinear buckling strength of axially compressed cylinders (Fig. 2.12). However, because $Z$ relates with the square of the cylinder length $L$, it grows in geometric progression to increasing length and consequently the choice of $Z$ as a suitable dimensionless length parameter has now been replaced by a related, but simpler, $\omega$ (Eq. (2.18)) parameter (Rotter, 2004). This parameter $\omega$ is linear in the cylinder length $L$ and permits the buckling behaviour of the shell system to be predicted independent of
the radius-to-thickness ratio $r/t$ within each length domain, offering an efficient means of characterising the buckling strength of the cylinder regardless of varying $r/t$ ratio.

Fig. 2.12 – Influence of length and end boundary conditions on the nonlinear buckling strength of perfect cylinder under uniform axial compression (from Yamaki, 1984)

Several types of end boundary conditions, according to the definitions given in Table 2.2 and Fig. 2.13, were also employed by Yamaki (1984) to understand the sensitivity of the buckling behaviour of axially compressed cylinders to different restraint conditions.

Table 2.2 – Definition of the boundary condition for cylinders (Rotter, 2004)

| Name | $\delta u$ | $\delta v$ | $\delta w$ | $\delta \beta$ | Name | $\delta u$ | $\delta v$ | $\delta w$ | $\delta \beta$
<table>
<thead>
<tr>
<th></th>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>r</td>
<td>r</td>
<td>r</td>
<td>f</td>
<td>C1</td>
<td>r</td>
<td>r</td>
<td>r</td>
<td>r</td>
</tr>
<tr>
<td>S2</td>
<td>r</td>
<td>f</td>
<td>r</td>
<td>f</td>
<td>C2</td>
<td>r</td>
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</tr>
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<td>S3</td>
<td>f</td>
<td>r</td>
<td>r</td>
<td>f</td>
<td>C3</td>
<td>f</td>
<td>r</td>
<td>r</td>
<td>r</td>
</tr>
<tr>
<td>S4</td>
<td>f</td>
<td>f</td>
<td>r</td>
<td>f</td>
<td>C4</td>
<td>f</td>
<td>f</td>
<td>r</td>
<td>r</td>
</tr>
</tbody>
</table>

Notes:
- f – unrestrained displacement during buckling
- r – restrained displacement during buckling
The influence of varying end boundary condition reveals a mild effect on the nonlinear buckling stress of the cylinders, except for a simply-supported boundary condition that does not restrain circumferential displacement during buckling, denoted as S2 & S4 in Fig. 2.12 and Table 2.2. Where this is the case, the nonlinear buckling stress can be as low as 50\% of $\sigma_{cl}$ (Ohira, 1961; 1963; Hoff & Rehfield, 1965; Hoff, 1965; Almroth, 1966).

### 2.4.5 Influence of geometric imperfections on buckling response

Initial geometric imperfections may be described as a deviation from the intended shell geometry and are often a by-product of the construction or fabrication process either in a systemic or random manner. The presence of this unavoidable feature and how it affects the buckling strength of shell structures continues to be an active area of research study. In fact, the wide and long-standing discrepancies observed between $\sigma_{cl}$ (Eq. (2.12)) and the buckling loads recorded during experimental studies on axially compressed cylinders were eventually attributed to the inevitable presence of geometric imperfections as is discussed below.

- **Wide discrepancies between theoretical and real buckling loads**

The discovery that thin cylinders potentially fail during experimental investigations at a load level that is as low as 20\% of $\sigma_{cl}$ (Fig. 2.14) led many researchers to undertake extensive research enquiry into the factors responsible for the unacceptable discrepancy. The beginning of this endeavour saw many researchers attempt to incorporate factors that were originally ignored during the formulation of $\sigma_{cl}$. The first two endeavours tried to rectify the simplifying membrane state of stress assumption. This idea originated from the knowledge that when a real cylinder is axially compressed, lateral expansions must be restrained near the edges, thereby...
inducing local bending deformations, to satisfy kinematic compatibility with the end boundary condition (Rotter, 2004). Therefore, the true state of stress in the cylinder must be a combination of membrane and bending actions, with varying proportion of each action dependent on the cylinder length.

The effort to include the effect of prebuckling rotations and the associated changes in stress was undertaken by numerous researchers (e.g. Fischer, 1965; Almroth, 1966; Gorman & Evan-Iwanowski, 1970; Yamaki & Kodama, 1972). It was revealed that this effect only accounted for a modest (~15%) reduction in the buckling stress and was eventually ruled out as the primary reason behind the huge discrepancy shown in Fig. 2.14. A similar conclusion was drawn for the influence of end boundary conditions (Ohira, 1961; Stein, 1962; Ohira, 1963; Hoff, 1965; Almroth, 1966), except where the end condition does not restrain circumferential displacement during buckling, resulting in a significant reduction that can go as high as 50% of \( \sigma_{cl} \).

![Fig. 2.14 – Experimental buckling stress of elastic cylinder under uniform axial compression (from Harris et al., 1957)](image)

Upon establishing that the simplifying assumptions made in the formulation of \( \sigma_{cl} \) could not be held accountable for this unacceptable discrepancy, other potential factors that do not relate to the formulation of \( \sigma_{cl} \) were thereafter investigated. Finally, the factor responsible for the significant discrepancy between experimental and theoretical buckling loads of axially compressed cylinders is now accepted to be the presence of initial geometric imperfections in
the shell geometry, an important conclusion drawn from the significant efforts of numerous contributors (e.g. von Kármán & Tsien, 1941; Koiter, 1945; Donnell & Wan, 1950; Koiter, 1963; Budiansky & Hutchinson, 1966; Yamaki, 1984; Bushnell, 1985; Rotter & Teng, 1989; Rotter, 1997). This conclusion was also verified experimentally by employing more carefully manufactured specimens, with the reported buckling stress now in the range of 40 – 90% of $\sigma_{cl}$ depending on the quality of fabrication (Babcock & Sechler, 1963; Tennyson, 1963; Almroth et al., 1964; Horton & Durham, 1965).

• Imperfection sensitivity in axially compressed cylinders

Once it was established that the huge discrepancy in the predicted buckling strengths of axially compressed cylinder systems may be largely attributed to the detrimental effect of geometric imperfections, several subsequent studies were conducted to understand how geometric imperfections could play such a crucial role in the elastic stability of this system and characterise the buckling strength predictions. One of the key discoveries was that the degree of imperfection sensitivity in the system is related to the shape of the secondary equilibrium path of the initially perfect structure, therefore leading many researchers to investigate the post-buckling behaviour of the initially perfect system.

Early attempts at explaining this post-buckling behaviour of the initially perfect system for axially compressed cylinders were undertaken by von Kármán & Tsien (1941) and Donnell & Wan (1950) and their analyses revealed a sharp drop in the equilibrium path of the perfect system upon bifurcation (Fig. 2.15a & b respectively). Furthermore, an important evidence for this hypothesis was given in the perturbation analyses of Koiter (1945; 1963), who explored the effect of minor geometric deviations in the shape of the axisymmetric buckling mode (Fig. 2.9), on the buckling stress of the structure. With a strong focus on the initial post-buckling behaviour, the seminal works of Koiter (1945; 1963) revealed that when the initial portion of the secondary path has a positive slope, i.e. the state of equilibrium at the bifurcation point is stable, mild sensitivity to geometric imperfections ensues. However, when the initial portion of the secondary path has a negative slope (i.e. unstable bifurcation), extreme sensitivity to geometric imperfections may represent the elastic stability of that system.

Thompson and Hunt (1973; 1984) also reported similar imperfection sensitivity behaviour for unstable symmetric structural systems, such as a cylinder under uniform axial compression and
spherical shells under uniform external pressure. Brush and Almroth (1975) suggested that the problem of imperfection sensitivity in shell structures is best answered by the nature of the stability or instability of the equilibrium state at the bifurcation point itself.

By considering the state of stress in the system before buckling, Rotter (2016b) concluded that the significant sensitivity to geometric imperfection demonstrated by cylinder systems under the classical load cases of uniform axial compression and, to a lesser extent, external pressure and torsion (Fig. 2.16), is due to the fact that their prebuckling behaviour is strictly dominated by uniform membrane actions.
Rotter (2004) however advised that this sensitivity to initial geometric imperfections depends largely on the form and amplitude of geometric imperfections, as well as the loading and the length of the structure.

- **Forms and amplitudes of geometric imperfections**
  Following the success of the perturbation analyses of Koiter (1945; 1963) in explaining the reason behind the wide discrepancy, many imperfection sensitivity studies that followed similarly employed geometric imperfections in the form of the linear eigenmode as it was defined by a mathematically-convenient trigonometric function. It was long thought that geometric imperfections in the form of the critical linear axisymmetric eigenmode of the perfect shell with wavelength $\lambda_{cl}$ (Eq. (2.16)) caused the greatest reduction in the buckling strength of systems with axisymmetric distribution of axial compression (Amazigo, 1969; Muggeridge & Tennyson, 1969; Hutchinson & Koiter, 1970; Hutchinson *et al.*, 1971; Amazigo & Budiansky, 1972). The historical precedent has led to the eigenmode-affine pattern being prescribed as the ‘default’ imperfection form for the computational analysis of shells of any geometry and loading by the European design standard on metal shell (EN 1993-1-6, 2007) where no other unfavourable form can be justified. Despite the wide usage of the critical buckling eigenmode as a severe form of imperfection, Schneider *et al.* (2001) advised that this form of imperfection should only be used a guide since it does not always turn out to be the most ‘severe’ imperfection form and the ‘worst’ form of imperfection should depend on the shell geometry and loading (Yamaki, 1984; Greiner & Derler, 1995; Rotter, 2011a).

The search for the most ‘practical’ imperfection form to use in a computational analysis poses a significant challenge. This is because in addition to being highly detrimental to the buckling strength of the shell, another important feature of the imperfection is how relatable it is to realistic defects arising from the manufacturing process and tolerances which can be measured on the real structure. Since the axisymmetric weld depression of Rotter and Teng (1989) relates with the shrinkage that occurs at the welded joints during cooling (Fig. 2.17), it is generally considered as representative of this common manufacturing process found in real cylinder structures (Bornscheuer *et al.*, 1983; Berry *et al.*, 2000; Pircher *et al.*, 2001). Being also axisymmetric in design, the effect of this imperfection form agrees with the findings of Hutchinson *et al.*, (1971) from axisymmetric eigenmodes causing the most detrimental effect on cylinders under uniform axial compression.
The definition of the weld depression imperfection is given in Eq. (2.19), where $\lambda$ is the linear meridional bending half-wavelength (Eq. (2.17)), $\delta_0$ is the nominal imperfection amplitude, $z$ represents the meridional axis and $k$ is a parameter that describes the degree of rotational continuity at the weld with $k = 1$ or 0, depicting either the Type A or Type B weld depression respectively (Song et al., 2004).

\[
\delta = \delta_0 \cdot \exp \left(-\frac{\pi}{\lambda} \left| z - \frac{L}{2} \right| \right) \cdot \left\{ \cos \left(\frac{\pi}{\lambda} \left| z - \frac{L}{2} \right| \right) + k \sin \left(\frac{\pi}{\lambda} \left| z - \frac{L}{2} \right| \right) \right\}
\]

The effect of these initial geometric imperfections on the characteristic buckling strength of the cylinder is usually expressed in terms of a knock-down factor, also known as the imperfection reduction factor $\alpha$, which is a ratio of the nonlinear elastic buckling strengths of the imperfect shell structures to those of the perfect shells. The individual variation of $\alpha$ with imperfection amplitude represents an imperfection sensitivity curve, which is illustrated in Fig. 2.16. Under different imperfection forms, the corresponding imperfection sensitivity curves of cylinders under uniform axial compression are presented in Fig. 2.18. A monotonous decrease in buckling strength with increasing imperfection amplitude may be inferred from Fig. 2.18a in addition to the fact that the linear eigenmode imperfection of Koiter (1945; 1963) may, at first sight, be considered to be the most severe form. However, the illustration also demonstrates the complexity of comparing imperfection sensitivity relationships from different forms of imperfection on the basis of adopting the mathematical imperfection amplitude $\delta_0$ as the measure of the extent of geometric deviation.
Contrary to the suggestion that the eigenmode imperfection may be considered as the ‘most’ severe in Fig. 2.18a, it is revealed in Fig. 2.18b that similar imperfection sensitivity relationships can be established across different assumed forms of imperfection through the adoption of another imperfection amplitude definition, this time around, an ‘equivalent geometric deviation’ $\delta_e$ (Rotter, 2004). This measure of the total extent of geometric deviation is defined here as the distance between the most inward and most outward midsurface deviation and for the case of eigenmode imperfections, it is taken as the ‘crest-to-trough’ deviation, owing to its sinusoidal nature.

![Graph showing imperfection sensitivity curves for cylinders under uniform axial compression.](image)

**Fig. 2.18 – Imperfection sensitivity curves for cylinders under uniform axial compression, using a) nominal imperfection amplitude and b) total geometric deviations as a measure of the imperfection amplitude (from Rotter, 2004; 2017a).**

The relationship between the ‘equivalent geometric deviation’ $\delta_e$ and the nominal imperfection amplitude $\delta_o$ for both the eigenmode and weld depression imperfections is given in Eq. (2.20) below, where for the weld depression imperfection, it may be obtained from the local maximum of the algebraic expression (Eq. (2.19)) defining this imperfection form:

$$\delta_e = \begin{cases} 
2\delta_o & \text{(eigenmode imperfection)} \\
1.04\delta_o & \text{(weld depression imperfection)}
\end{cases} \quad (2.20)$$

Furthermore, some studies have demonstrated that cylinder systems do not necessarily follow this monotonous decay in buckling strength with increasing imperfection amplitude, as in effect, a deeper imperfection may cause a rise in buckling strength (e.g. Yamaki, 1984; Rotter, 2004; Sadowski et al., 2017a) as shown in the equilibrium paths of Fig. 2.6 & Fig. 2.19. In other studies (e.g. Sadowski & Rotter, 2011a; 2012; 2013b), this phenomenon was described as being manifested by a smooth transition from pre- to post-buckling along the equilibrium...
path with no numerical indication of instability. Nevertheless, as will be shown later in the current thesis, the reason behind this counter-intuitive behaviour is because the growing buckling mode of the more imperfect cylinder is beginning to become restrained by the end boundary condition (Rotter, 2011b). Although this counter-intuitive phenomenon poses a major challenge in establishing codified imperfection sensitivity relationships for use in design, where it occurs, Yamaki (1984) suggested that a safe buckling load may be conservatively selected as the load value corresponding to the point of inflection along the equilibrium path. This demonstrates that adequate care must be put in place when attempting to predict the failure loads of these structural shell systems at deeper imperfections and when comparing imperfection sensitivity relationships across different forms of geometric imperfection, since the degree of sensitivity also depends on the measure of imperfection amplitude that is adopted.

Fig. 2.19 – Different equilibrium paths followed by cylinders under eccentric discharge at increasing imperfection amplitudes (from Sadowski & Rotter, 2011a)

The nonlinear buckling strength of a perfect cylinder under axial compression was earlier shown to depend significantly on the cylinder length, it may thus be expected that the cylinder length could also influence to sensitivity of this shell system to forms and amplitudes of geometric imperfections as advised by Rotter (2004). However, prior to the start of the current PhD research, such a conclusion could not have been drawn because no form of studies existed that had systematically investigated length effects on the buckling of any imperfect shell systems, regardless of the loading condition or geometry. Although there appears to be a
genuine lack of appreciation for this detailed investigation, the reasons for this research gap may perhaps be attributed to the large amount of computational analyses needed to understand length-effects on imperfection sensitivity and the lack of a robust framework, such as the newly introduced Reference Resistance Design (RRD) that will be presented shortly, to interpret the result with. In fact, it was not until the very recent work of Rotter & Al-Lawati (2016) that the first numerical evidence to support this hypothesis for axially compressed cylinders became available, in spite of the numerous investigations that had been performed to understand the imperfection sensitivity of this particular shell system (e.g. von Kármán & Tsien, 1941; Koiter, 1945; Donnell & Wan, 1950; Koiter, 1963; Amazigo & Budiansky, 1972; Yamaki, 1984). Furthermore, it may be argued that the expense and difficulty in controlling the exact geometry under so many varying conditions are a contributory factor to why such an investigation could not be feasibly performed experimentally. The study of Rotter & Al-Lawati (2016) may also have been inspired by the preliminary investigation of length effects on elastic imperfect cylinders under uniform bending, considering the critical eigenmode imperfection only, a product of the current PhD research that was published as part of the proceedings of the 8th International Conference on Advances in Steel Structures Conference proceeding in 2015.

- **Measure of geometric imperfections**

The previous section has elaborated on the significance of the amplitude of geometric imperfections to the buckling strength predictions in cylinders. It was also revealed that the form of imperfection equally plays a critical role and a continuously increasing imperfection amplitude does not necessarily lead to lesser buckling strength. The major challenge now rests on how to utilise this knowledge and idea for the structural design of cylinders. Since it is impossible to know a priori what form of imperfection the fabricated shell structure will assume, equivalent imperfections forms, such as eigenmode-affine pattern, weld depression, asymmetric modes, etc. are typically assumed. However, there had always been a heavy reliance on empirical rules obtained from laboratory tests in the structural design community and this made verification of buckling strengths using measured tolerances very difficult (Rotter, 2017a). This difficulty arises from the fact that laboratory test models cannot truly represent in-situ full scale civil engineering shells, since the fabrication processes used on full scale shells are very different from those used for the laboratory test models (Michel et al., 2000; Mathon & Limam, 2006). Consequently, the different fabrication processes lead to
different forms and amplitudes of geometric imperfection such that test results may not be applicable to full scale construction (Rotter, 2016a; 2017b).

Perhaps the first attempt to address this difficulty in verification through credible tolerances can be found in ECCS EDR2 (1983) where the imperfection amplitude in a full scale shell is now measured by means of a ‘measuring stick’ whose length is carefully chosen to correspond with the unit of expected wavelength of the buckles. The imperfection sensitivity relationship obtained from this method of stick measurements reveal moderately close agreement with those from Koiter’s (1945) original analysis for the most studied shell system of cylinders under uniform axial compression (Rotter, 1985). Consequently, this concept paved way for the development of tolerance limits to be used in defining the imperfection amplitude of the equivalent geometric imperfection, relative to the quality of fabrication (Table 2.3). Although relating quality of fabrication with imperfection amplitude may have originated from Rotter’s (1985) validation analysis, a widespread use of this idea is now evident in standards, such as the current European design standard for metal shells (EN 1993-1-6+A1, 2017), independent standards on tolerances (EN 10210-2, 2006; EN 1090-2; 2008) and the ECCS EDR5 (2008; 2013). The EN 1993-1-6 (2007) provisions for estimating the amplitude of geometric imperfections for the buckling limit state (LS3), by means of dimple tolerances are presented below.

<table>
<thead>
<tr>
<th>Fabrication tolerance quality class</th>
<th>Description</th>
<th>Value of $U_{o,\text{max}}$</th>
<th>$Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Class A</td>
<td>Excellent</td>
<td>0.006</td>
<td>40</td>
</tr>
<tr>
<td>Class B</td>
<td>High</td>
<td>0.010</td>
<td>25</td>
</tr>
<tr>
<td>Class C</td>
<td>Normal</td>
<td>0.016</td>
<td>16</td>
</tr>
</tbody>
</table>

Dimple tolerances in real cylinder structures must be measured in both the axial and circumferential directions by means of a ‘stick’ having length, which depends on the orientation of the compressive stresses in the cylinder. Where axial compressive stresses dominate (Fig. 2.20a & b), the ‘stick’ length is calculated as:

$$l_{gx} = 4\sqrt{r\ell}$$

$x$ being the meridional axis here
This ‘stick’ length approximately represents the half-wavelength of a classical axisymmetric eigenmode (Fig. 2.9b) in each direction (Rotter, 2004) being the default imperfection form recommended by EN 1993-1-6 (2007) where no other forms can be justified. The ‘stick’ length to be used in measuring the circumferential dimple tolerances (Fig. 2.20d) where there are circumferential compressive or shear stresses and that to be used across welds (Fig. 2.20e & f) in both directions are given in Eq. (2.22)a & b respectively.

\[ l_{g0} = 2.3 \left( \frac{t^2r}{l} \right)^{0.25} \quad \text{subject to} \quad l_{g0} \leq r \quad \text{(a)} \]

\[ l_{gw} = 25t \quad \text{with} \quad l_{gw} \leq 500mm \quad \text{(b)} \]

(Fig. 2.20 – Dimple imperfection measure according to the different ‘stick’ lengths (EN 1993-1-6, 2007))
For cylinders dominated by meridional compression, the imperfection amplitude should be measured on a meridian and across a weld, as illustrated in Fig. 2.20a & c and where there could be circumferential stresses (e.g. ovalising cylinders), measurement on circumferential circle may be ideal.

The dimple parameter $U_o$ is defined as the ratio of the depth of the initial dimples $\Delta w_o$ to the ‘stick’ length $l_g$ for each case (Eq. (2.23)) and the limiting values $U_{o,max}$ representing the different fabrication tolerance quality classes, together with the quality parameter $Q$ are also given in Table 2.3.

$$U_{ox} = \Delta w_{ox} / l_{gx} \quad , \quad U_{o\theta} = \Delta w_{o\theta} / l_{g\theta} \quad \text{and} \quad U_{ow} = \Delta w_{ow} / l_{gw}$$

(2.23)

2.5 Buckling of elastic cylinders under uniform bending

2.5.1 Introduction

The introduction of bending loads on a structural member naturally induces tension and compression stresses on either side of the member’s cross-section as illustrated in Fig. 2.21. In cylinders, the buckling behaviour under bending is a field that has a rich history, which is almost as old as that of cylinder under uniform axial compression. One fundamental peculiarity in the behaviour of cylinders under bending is the cross-sectional ovalisation, a phenomenon that was discovered by Brazier (1927), in his analysis of an infinitely long tube under flexure.

According to this analysis, the bending moment $M$ transmitted by the tube at any section may be related to curvature $\varphi$ of the tube by means of the expression in Eq. (2.24):
\[ M = \frac{E}{2\pi r^3 t} \left( 2\varphi - \frac{3r^4\varphi^3(1-\nu^2)}{t^2} \right) \]  

(2.24)

where \( r \) and \( t \) are the radius and thickness of the tube wall respectively while \( E \) and \( \nu \) are the Young’s modulus and Poisson’s ratio respectively. This relationship demonstrates that a point of maximum moment \( M_{Braz} \) should occur at a corresponding curvature \( \varphi_{Braz} \) (Eq. (2.25)), above which collapse of the tube becomes inevitable. Altogether, the effect constitutes a limit-point instability (e.g. Fig. 2.1a) with significant pre-buckling nonlinearity, owing to the progressive flattening of the tube cross-section in response to increasing meridional curvature. This progressive flattening is a direct consequence of the transverse components of the meridional tensile and compressive stresses (Fig. 2.21), directing the tube fibre towards the neutral axis and with the resultant effect of a progressive reduction in the bending stiffness of the tube, following a decreasing second moment of area of the cross-section (Wood, 1958; Reissner, 1961; Fabian, 1977; Gellin, 1980; Boyle, 1981; Calladine, 1983; Tang et al., 1985; Libai & Bert, 1994; Tatting et al., 1995; 1997; Karamanos, 2002; Li & Kettle, 2002; Elchalakani et al., 2002; Houliara & Karamanos, 2006; Wadee et al., 2006; 2010; Chadli et al., 2012; Rotter et al., 2014).

\[ M_{Braz} \approx \frac{2\sqrt{2}}{9} \left( \frac{E\pi rt^2}{\sqrt{1-\nu^2}} \right) \text{ and } \varphi_{Braz} = \frac{\sqrt{2}}{3} \cdot \frac{t}{r^2\sqrt{1-\nu^2}} \]  

(2.25)

The progressive flattening of the cross-section at the midspan also increases the local circumferential radius of curvature \( \rho \) of the cylinder (\( \rho \to \infty \) as the midspan cross section is becoming more akin to a flat plate) and indirectly reduces the local critical buckling stress \( \sigma_{cl}(\rho) \) of the tube, given in Eq. (2.26). Consequently, a local bifurcation would occur before the elastic stability of tubes under bending can attain the limiting moment described by \( M_{Braz} \).

\[ \sigma_{cl}(\rho) = \frac{E}{\sqrt{3(1-\nu^2)}} \cdot \frac{t}{\rho} \]  

(2.26)

An illustration of the linearized solution by St Venant’s and the coupling between local bifurcation and limit-point instability occurring in tubes under flexure is illustrated in Fig. 2.22, where the pre-buckling nonlinearity due to ovalisation may be seen as the obvious deviation from the linearized solution. This pre-mature local bifurcation was also hinted at by Brazier.
(1927) after spotting some ‘lobed deformations’ developing only on the compressed side of the bent tube during experimental studies. The same conclusion of local buckling preceding limit-point instability was also drawn by these afore-mentioned authors.

Consequently, the limiting $M_{Brax}$ has been found to remain a theoretical upper bound that can never be attained in thin cylinders due to a pre-mature local bifurcation occurring at a moment that is approximately 6% below $M_{Brax}$ (Gellin, 1980; Karamanos, 2002). Although it may sound theoretically possible to achieve this limiting moment in thicker cylinders ($r/t \rightarrow 0$), thick cylinders however fail by plastic collapse before attaining $M_{Brax}$ (Sadowski & Rotter, 2013a). The coupling between pre-buckling ovalisation and local bifurcation in the elastic stability of perfect cylinders under global bending makes the behaviour of these structures inherently complex. One of the numerous complications experienced by researchers while interpreting the result of the buckling analysis of elastic cylinders under global bending is presented below.

![Equilibrium Paths](image)

Fig. 2.22 – Illustration of the equilibrium paths of thin circular tubes under uniform bending according to different solutions.

Shortly after the first report of Brazier ovalisation phenomenon, there was a widespread misconception about the relationship between the critical bending stress in a relatively shorter cylinder under bending and classical elastic critical buckling stress $\sigma_{el}$ (Eq. (2.12)) corresponding to the undeformed radius of curvature in an axially compressed cylinder (Seide & Weingarten, 1961). This misconception was that the theoretical maximum bending stress that will initiate buckling is exactly 1.3 times compressive buckling stress ($\sigma_{el}$). Although this relationship was derived by Flügge (1932) following the result of his buckling analysis on cylinders under combined bending and axial compression with the assumption of a specific $r/t$ and length. Timoshenko (1932) however, in his seminal work ‘Theory of Elastic Stability’ cited
this relationship without the qualifying geometric parameter assumptions, which then became a general rule until the decisive work of Seide & Weingarten (1961) resolved the inconsistencies.

Specifically, Seide & Weingarten (1961) analysed the buckling of relatively short cylinders having length-to-radius ratio $L/r < 0.3$ under bending by means of the Batdorf's modified Donnell shell equations (Donnell, 1935; Batdorf, 1947) together with the Galerkin energy method and with the assumption of linear pre-buckling membrane stress state through small-deflection theory (pre-buckling cross-section ovalisation ignored). The results, illustrated in Fig. 2.23, provided evidence to support a conclusion that the critical buckling stress under uniform bending without ovalisation was similar to the corresponding value under axial compression. This conclusion was also confirmed by the more rigorous analysis of Reddy and Calladine (1978) that followed afterward.

![Fig. 2.23 – The variation of normalised critical bending stress to classical buckling stress with length under two different dimensionless length definitions (Seide & Weingarten, 1961; Rotter et al., 2014)](image)

2.5.2 **The local buckling hypothesis**

Following the finding by Seide & Weingarten (1961) that the linear critical buckling stress may, as a first approximation, be taken as the same for elastic cylinders under bending as for those under uniform axial compression, Axelrad (1965; 1985) employed a concept of local stability in describing the behaviour of tubes under bending and proposed that local bifurcation
may be predicted approximately by considering the stress state ‘at a single point’. This proposition is now known as the ‘local buckling hypothesis’ and it implies that buckling may be deemed to have occurred once the membrane stress in the most compressed fibre of the cylinder attains the same critical value of $\sigma_{cl}$ (Eq. (2.12)) for axially compressed cylinders. However, buckling requires the membrane stress distribution to be close to uniform and within a finite zone that can support an axial compression buckle (Calladine, 1983; Rotter et al., 2014). Nevertheless, this hypothesis is sufficiently representative of the elastic instability in very thin cylinders under a pre-buckling membrane stress state with little or no local bending deformations or cross-section ovalisation. In addition, this concept was employed in formulating the theoretical buckling moment for cylinders under bending, i.e. the classical elastic critical buckling moment $M_{cl}$ and the corresponding mean cross-sectional curvature $\varphi_{cl}$ with the aid of beam bending theory:

$$M_{cl} = W_{el} \cdot \sigma_{cl} \quad \text{where} \quad \begin{cases} W_{el} = \pi r^2 t \quad \text{(elastic section modulus)} \\ \sigma_{cl} = 0.605 Et/r \quad \text{(classical elastic critical stress)} \end{cases}$$

$$\varphi_{cl} = \frac{t}{r^2 \sqrt{3(1-\nu^2)}} \approx 0.605 \frac{t}{r^2}$$

(2.27)

Although it may not representative of the true failure load of every elastic cylinder under bending, the classical elastic critical buckling moment $M_{cl}$ remains an important reference load value used in the buckling analysis of elastic cylinders under uniform bending.

### 2.5.3 Influence of cylinder length on cross-sectional ovalisation

By improving on the analysis of Seide & Weingarten (1961) with the aid of the semi-membrane shell theory, which is now designed to account for the possibilities of circumferential bending (cross-section ovalisation) in the governing equations by including circumferential bending moments and curvatures ($M_\theta$ and $\kappa_\theta$ respectively), Axelrad (1965) was able to show that there is a relationship between pre-buckling cross-section ovalisation and cylinder length. The result revealed that there is a unique point in the relationship between the nonlinear buckling strength and cylinder length where cross-section ovalisation begins to influence the elastic stability of the cylinder (Rotter et al., 2014). On investigating the effect of finite length, Calladine (1983) came up with a different dimensionless length parameter $\Omega$ (Eq. (2.28)) to specify this boundary as being at $\Omega \approx 0.5$. For cylinder lengths greater than this value, prebuckling
ovalisation begins to influence the elastic stability of the structural system and below it, the system experiences no prebuckling ovalisation because the end effect preserves the initial circular cross-section of the cylinder throughout the length.

\[ \Omega = \frac{L}{r} \sqrt{\frac{t}{r}} = \frac{\omega}{r/t} \]  
(2.28)

However, the complete scope of the influence of cylinder length and thickness on pre-buckling ovalisation was only recently offered by the parametric study of Rotter et al. (2014), using the finite element method. The study of Rotter et al. (2014) complemented the work of Seide & Weingarten (1961) and showed that at very short lengths, the linear buckling moment \( M_{cr} \) predictions of clamped tubes under bending do not agree with the ‘local buckling hypothesis’ of Axelrad (1965; 1985), i.e. \( M_{cr} >> M_{cl} \). Instead, the normalised linear bifurcation moment prediction \( M_{cr}/M_{cl} \) tends towards an infinite value as the length \( \omega \to 0 \) because at these very short lengths, the end boundary condition restrains the formation of local buckles severely, similar to the behaviour of short cylinders under uniform axial compression (Fig. 2.24).

As a result, more strain energy is required to overcome the physical constraint, leading to higher buckling moment predictions in the tubes. The additional buckling moment versus length
relationship for simply-supported (S) cylinders shown in Fig. 2.24 was undertaken as a first preliminary validation study and a follow-up to the work of Rotter et al. (2014).

Furthermore, as the length increases the influence of end boundary condition diminishes significantly, and the local buckling hypothesis eventually becomes a realistic representation of the buckling behaviour of the tubes under geometrically linear conditions, i.e. $M_{cr} \approx M_{cl}$ or $M_{cr}/M_{cl} \rightarrow 1$. Since this simple linear eigenvalue analysis ignores the nonlinear stress amplification of pre-buckling rotations near the ends, no further instability phenomena in the form of pre-buckling ovalisation is anticipated in the tubes as the length grows indefinitely. Altogether, this allowed a conservative approximation of the relationship between the normalised linear buckling moment $M_{cl}$ and length for clamped cylinders under bending into the expression of Eq. (2.29) offered by Rotter et al. (2014):

$$\frac{M_{cl}}{M_{cr}} = \frac{1}{1 + 4\omega^{-2}}$$  (2.29)

In addition, Rotter et al. (2014) established that the nonlinear buckling behaviour of isotropic elastic cylinders under global bending may be categorised under four distinct length domains: ‘short’, ‘medium’, ‘transitional’ and ‘long’. Although these domains are qualitatively similar to the ones for cylinders under uniform axial compression (Fig. 2.12), the additional ‘transitional’ domain is due to the Brazier ovalisation phenomenon in cylinders under uniform bending, absent in axially compressed cylinders. The study also provided strong evidence that the absence of pre-buckling ovalisation at some moderate lengths ($\Omega < 0.5$) was a consequence of the power of the end boundary condition that restrains all displacement and rotational degrees of freedom at the ends and any flattening of the cross-section around midspan. Therefore, cylinders that are not prone to ovalisation phenomenon were simply categorised under the ‘short’ and ‘medium’ length domains with the aid of the dimensionless length parameter $\omega$ (Eq. (2.18)). This dimensionless parameter $\omega$ relates directly with $\sqrt{rt}$, the unit of an axisymmetric buckle half-wavelength $\lambda_{axi}$ (Eq. (2.16)) or linear bending half-wavelength $\lambda$ (Eq. (2.17)) and it is the dimensionless length group governing the buckling behaviour of perfect ‘medium’ cylinders in bending.

Cylinders in the ‘transitional’ domain experience the onset of cross-section ovalisation, which grows significantly as the length increases while cylinders in the ‘long’ domain experience a
fully developed cross-section ovalisation, which is as detrimental as possible. The advantage of employing carefully chosen dimensionless groups to describe complex relationships completely may be observed in the relationships between the normalised nonlinear buckling moment and length, shown in Fig. 2.25 with the aid of two different dimensionless lengths ($L/r$ and $\Omega$). The dimensionless parameters $\omega$ & $\Omega$ both permit self-similarity of the nonlinear elastic response to be preserved (e.g. from $\Omega \geq 0.5$ in Fig. 2.25b), akin to the function of the Reynold’s numbers in fluid mechanics, in the moment-length relationship, regardless of the varying $r/t$ ratios.

Fig. 2.25 – Relationship between the nonlinear buckling strength and radius-to-thickness ratio of cylinders in bending using different length dimensionless groups: a) using the dimensionless length $L/r$ and b) using the dimensionless length $\Omega$ (from Rotter et al., 2014)
As a summary, the current characterisation of the nonlinear buckling behaviour of elastic perfect cylinders under uniform bending over the recently identified four unique length domains is presented in Fig. 2.26 below for both ‘clamped’ (C) and ‘simply-supported’ (S) cylinders, with the aid of the dimensionless lengths $\omega$ & $\Omega$ for a single instance of $r/t = 100$. The results of the clamped cylinders were extracted from Rotter et al. (2014) while those of simply-supported cylinders were obtained from the first preliminary validation study. Both Fig. 2.25 and Fig. 2.26 demonstrate that the nonlinear buckling behaviour of perfect cylinders in bending is strongly length-dependent and employing a single buckling moment prediction, whether in terms of the Brazier moment $M_{Braz}$ (Eq. (2.25)) or the classical elastic critical buckling moment $M_{cl}$ (Eq. (2.27)) for all length domains as the controlling buckling moment, may be over- or under-conservative respectively.

![Characteristic buckling resistance of perfect elastic cylinder (r/t = 100)](image)

Fig. 2.26 – Characteristic buckling resistance of perfect elastic cylinder ($r/t = 100$) in bending across the recently identified length domains (a product of the computational work of Rotter et al., (2014) and the first preliminary validation study conducted on perfect elastic cylinders with simply-supported (S) ends during the current PhD research).

The nonlinear buckling moment $M_k$ predictions of perfect clamped cylinders in bending were also characterised into algebraic expressions shown in Table 2.4 by Rotter et al. (2014) over...
all the identified length domains in terms of a geometrical reduction factor $\alpha_G = (M_k / M_c)$ that connotes the single influence of geometric nonlinearity on the linear elastic buckling resistance of the cylinder.

These length-dependent algebraic characterisations are envisaged to ease the work of a structural analyst by becoming a proxy to undertaking onerous computational analyses in estimating the buckling strength of perfect cylinders under bending. The preliminary proposals for clamped perfect cylinders under uniform bending are currently under revision and with further extensions on the work by the current PhD research, the aim of incorporating the findings into the updated version of the European design standard for metal shells (EN 1993-1-6, 2007) may be achieved successfully in 2020.

<table>
<thead>
<tr>
<th>Class</th>
<th>$\omega$ range</th>
<th>$\Omega$ range</th>
<th>Algebraic equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Short</td>
<td>$3 \leq \omega &lt; 4.8$</td>
<td>n/a</td>
<td>$\alpha_G = 1.93 - 0.5(\omega - 3.8)^2 - 0.44(\omega - 3.8)^4$</td>
</tr>
<tr>
<td>Medium</td>
<td>$4.8 \leq \omega &lt; 0.5(r/t)$</td>
<td>$\Omega &lt; 0.5$</td>
<td>$\alpha_G = \begin{cases} 0.85 + 0.029(\omega - 7.1)^2 &amp; \text{when } 4.8 \leq \omega &lt; 8.6 \ 0.92 &amp; \text{for } 8.6 \leq \omega &lt; 0.5(r/t) \end{cases}$</td>
</tr>
<tr>
<td>Transitional</td>
<td>$\omega &gt; 0.5(r/t)$</td>
<td>$0.5 \leq \Omega &lt; 7.0$</td>
<td>$\alpha_G = 1.07 \left(\frac{1 - 0.22\Omega + 0.061\Omega^{2/3}}{1 + 0.12\Omega^{2/3}}\right)$</td>
</tr>
<tr>
<td>Long</td>
<td>n/a</td>
<td>$\Omega \geq 7.0$</td>
<td>$\alpha_G = 0.516$</td>
</tr>
</tbody>
</table>

### 2.5.4 An overview of imperfection sensitivity

The imperfection sensitivity of axially compressed cylinder systems has been extensively studied and various analytical, experimental and numerical results exist, which support their extreme sensitivity to geometric imperfections that is drawn from the predominantly membrane state of stress in the cylinder. However, a similar endeavour is currently lacking in existing research database for cylinder systems under a closely related load case of uniform bending.

In the concept of stress design, it is often assumed that the imperfection sensitivity relationship for uniform bending may be adopted as for uniform compression (EN 1993-1-6, 2007), a conservative choice although no rigorous proof to support such a conservative assumption had yet been established. The first direct evidence of imperfection sensitivity in cylinders under uniform bending is the computational work of Chen et al. (2008) who conducted a limited
parametric study on clamped cylinders with elastic-plastic material property and with the aid of realistic weld depression imperfections. The study proposed that the imperfection sensitivity of these cylinders is as illustrated in Fig. 2.27 and that the imperfection reduction factor $\alpha_I$, which accounts for the influence of geometric imperfections in the elastic response of the cylinder, may be predicted conservatively by means of Eq. (2.30), an approximation that is in the current Amendment to the European design standard for metal shells (EN 1993-1-6+A1, 2017).

$$\alpha_I = \frac{1}{1 + 2\left(\delta_o/t\right)^{0.8}} \tag{2.30}$$

Here, $t$ is the thickness of the cylinder wall, while $\delta_o$ is the nominal amplitude of the Rotter & Teng (1989) Type A axisymmetric weld depression, measured as the distance between the original midsurface and the most inward deviation.

It should be noted however that the imperfection sensitivity relationship in Fig. 2.27 and the subsequently algebraic expressions in Eq. (2.30) were established from limited cylinder models, all maintaining a constant length-to-radius ratio $L/r = 7$ at varying $r/t$ of 500, 700, 1000 and 2000. Therefore, these cylinders may now be described as having dimensionless length
parameter $\omega \approx 157, 185, 221$ and 313 (or $\Omega = 0.31, 0.26, 0.22$ and 0.16) for $r/t$ of 500, 700, 1000 and 2000 respectively and following the later categorisation of Rotter et al. (2014), may all be classified as ‘medium’ length cylinders, since $\Omega < 0.5$ in all cases. This implies that the cylinders do not undergo any pre-buckling nonlinearity due to the cross-sectional ovalisation phenomenon. Consequently, these predictions may not be representative of the imperfection sensitivity exhibited by ovalising cylinders.

However, there is no subsequent study that has validated this imperfection sensitivity relationship to be indeed representative or conservative across all the other length domains or whether the weld depression imperfection does indeed offer the most conservative imperfection form. In addition, a systematic investigation and documentation of the sensitivity of cylinders under bending to multiple imperfection forms across a wide range of parameters has never been performed. These are some of the main aims of the current PhD.

2.6 Current European design rule on Buckling Limit State

2.6.1 Introduction

One of the essences of research in structural engineering is to develop new concepts, understanding and improved methods for the design and assessment of practical structures (Rotter, 2017c). However, a full understanding of the mechanics governing the behaviour of shell structures has always posed a major challenge to analysts dating back several decades. This may be attributed to the numerous complexities associated with the wide structural forms conceivable and the potential coupling between many factors such as length, type of loading, influence of pre-buckling rotations, nature of end boundary conditions, material nonlinearity, forms and amplitudes of imperfections, that affect shell stability (von Kármán & Tsien, 1941; Stein, 1962; Hoff & Rehfield, 1965; Brush & Almroth, 1975; Yamaki, 1984; Bushnell, 1985; Rotter & Teng, 1989; Rotter, 2002; 2004; Chen et al., 2008; Rotter et al., 2014). A possible disconnect between the shell buckling research community and design practice may be due to this persistent difficulty faced by early researchers in devising a full understanding of the behaviour of shell structures (Rotter, 2011a). Nonetheless, significant advancements have been made by the shell buckling research community in devising adequate design concepts to bridge this gap between research and design practice and the following sections summarise the different assessment methods currently available for the buckling limit state of metal shells.
2.6.2 The concept of Stress Design

A shell structure may simply be modelled as a structure in which a total of six stress resultants acts at every point and out of which only three (meridional $N_z$, circumferential $N_\theta$ and shear $N_{z\theta}$ membrane stress resultants) control the stability of the shell (Rotter, 2017b), since only the membrane stress resultants can cause buckling in thin shells. The bending components however, may induce local plasticity and reduce the incremental stiffness, potentially causing a knock-on reduction on the local buckling stress of the shell. Consequently, early experiments and algebraic analyses were mostly performed on shells that develop uniform membrane stress state during loading (thin isotropic cylinders under uniform axial compression, uniform external pressure and uniform torsion), due to the relative ease of performing buckling analyses on such shell systems (Rotter, 2016a; 2016b). Individual nonlinearities that govern the behaviour of each of these fundamental shell systems is then characterised as algebraic expressions, e.g. the influence of geometric nonlinearity and imperfection sensitivity are simply accounted for using empirical reduction factors.

Subsequently, the traditional method of design for the buckling limit state of shells was achieved based on the concept of uniform membrane stress in the shell and is termed ‘stress design’. This method works by expressing the buckling resistance (meridional $\sigma_{z,Rk}$, circumferential $\sigma_{\theta,Rk}$ or shear $\tau_{z\theta,Rk}$) in shell structures in terms of working stresses in the system and deducing the necessary adjustment to the material yield stress $f_y$ through a capacity curve (Fig. 2.28), which describes the relationship between the relative buckling strength $\chi$ and relative slenderness $\lambda$ of the shell structure (ECCS EDR5, 2008; 2013). This capacity curve separates out various nonlinearities that affect the behaviour of general shell structures into different parameters.

The assessment here considers two conditions in the shell as limiting, namely: elastic buckling & full yield and so the characteristic stress at buckling is taken to be a proportion of the material yield stress $f_y$ (Rotter, 2016a), although with necessary adjustment in place where shear stresses are involved. The design value of membrane stress ($\sigma_{z,Ed}$, $\sigma_{\theta,Ed}$ or $\tau_{z\theta,Ed}$) in the shell is taken as the maximum local value and may be obtained through a linear elastic shell analysis (LA) or in cases of simple loading and support conditions, through membrane theory. Each characteristic buckling stress $\sigma_{Rk}$ (meridional $\sigma_{z,Rk}$, circumferential $\sigma_{\theta,Rk}$ or shear $\tau_{z\theta,Rk}$) is then calculated using the capacity curve (Fig. 2.28) & Eq. (2.31).
where $\alpha$ and $\beta$ are the elastic ‘knock-down’ and plastic range factors respectively while $\eta$ is the interaction exponent and $\lambda_0$ is the squash limit relative slenderness. The plastic limit relative slenderness $\lambda_p$ may simply be calculated from Eq. (2.32) while the relative shell slenderness parameter $\lambda$ for the meridional, circumferential and shear stress components should be obtained from Eq. (2.33)a, b & c respectively:

$$\lambda_p = \frac{\alpha}{\sqrt{1 - \beta}}$$  

$$\frac{f_y}{\sigma_{x_{Rcr}}} \quad (a) \quad \frac{f_y}{\sigma_{\theta_{Rcr}}} \quad (b) \quad \frac{f_y}{\tau_{x\theta_{Rcr}}} \quad (c)$$

Here, $\sigma_{x_{Rcr}}, \sigma_{\theta_{Rcr}}$ and $\tau_{x\theta_{Rcr}}$ are the meridional, circumferential and shear elastic critical buckling stresses respectively, obtained by simple expressions describing the shell system under the
buckling relevant combinations of actions (EN 1993-1-6, 2007). The definition of \( \alpha \) and \( \beta \) to be employed in calculating each characteristic buckling stress (meridional \( \sigma_{z,Rk} \), circumferential \( \sigma_{\theta,Rk} \) or shear \( \tau_{z\theta,Rk} \)) must correspond to the algebraic expressions derived for \( \alpha \) and \( \beta \) from the three fundamental shell systems of cylinder under uniform axial compression, uniform external pressure and uniform torsion respectively, without interaction.

Although the stress design method offers a relatively simple approach (hand calculation) to sizing metal shells, there are several drawbacks that limit its applicability to the design of any generic shell for service, all related to the assumption of uniform membrane stress state, which allows the use of the local maximum compressive membrane stress (Rotter, 2016b) as the governing parameter. These limitations are presented below.

The effect of geometric nonlinearity on shell stability is known to be mild under a uniform stress state with a resultant 15% reduction (Fig. 2.12) in buckling strength for axial compression and milder for the other cases (Yamaki, 1984). However, under unsymmetrical loading conditions, geometric nonlinearity plays an important role (Houliara & Karamanos, 2006; Sadowski & Rotter, 2010). Furthermore, under the uniform stress state, both the effect of geometric nonlinearity and imperfections are reduced into a single ‘knock-down factor’ parameter \( \alpha \). This becomes misleading for unsymmetrical load case of uniform bending, where it has been shown that a perfect cylinder of sufficient length, which should lead to \( \alpha = 1 \), undergoes cross-section ovalisation (geometric nonlinearity effect) and fails at a load level that is almost half of the uniform stress state counterpart (compare Eq. (2.25) with Eq. (2.27)).

Meanwhile the range of buckling behaviour that may be obtained from shell structures has burgeoned following the possibility of different complex geometries, loading or end boundary conditions and the different forms of geometric imperfections, thereby, making ‘hand calculation’ through stress design less truly representative of the complex nature of the shell structure, although possibly conservative. Bearing this range of shell buckling behaviour in mind, the EN 1993-1-6 (2007) prescribes two formal methods that exploit computational capabilities through global numerical analyses, either using LBA-MNA approach or GMNIA methodology. These methods are able to generate a conservative and consistent outcome to the calculations of shell buckling for all kinds of geometric and material properties. They may also be adopted for shells that do not currently have any classical expressions for the buckling resistances, or in the event that the designer opts to employ higher-level design concepts to
reduce potential uncertainties through more accurate numerical evaluations (EN 1993-1-6, 2007; ECCS EDR5, 2008; 2013). However, the possible repetitive nature under varying conditions may become highly time-consuming.

2.6.3 Design by global numerical analysis using LBA-MNA methodology

The first of the recommended approaches in this assessment method is in the form of LBA/MNA methodology (Rotter, 2011b). Unlike in stress design where a single point on the shell is considered, this method considers the overall shell structure in establishing resistances. The LBA/MNA method works by relating the overall slenderness $\lambda_{ov}$ (Eq. (2.34)) of the shell structure to two reference resistances, namely: linear elastic critical buckling load factor $R_{cr}$ and an estimate of the plastic collapse load factor $R_{pl}$ of the shell ignoring strain hardening, which are obtained through more accurate LBA and MNA analyses respectively. The definition of the critical elastic buckling resistance and the plastic reference resistance of the shell from global LBA-MNA analyses is illustrated in Fig. 2.29.

$$\lambda_{ov} = \frac{R_{pl}}{R_{cr}}$$

(2.34)

The current LBA-MNA method also employs a capacity curve, similar to Fig. 2.28, whereby the overall buckling reduction factor $\chi_{ov}$ may be calculated by means of similar expressions as those in Eq. (2.31) for buckling stress design concept, except with the replacement of relative
local parameter values with their corresponding overall values. The characteristic buckling resistance $R_k$ and the design buckling resistance $R_d$ may thereafter be calculated as follows and compared against the design loads for strength verification, where $\gamma_{M1}$ is the partial factor for resistance to buckling:

$$R_k = \chi_{ov} R_{pd} \quad \text{and} \quad R_d = \frac{R_k}{\gamma_{M1}}$$

(2.35)

Appropriate values of the elasticity and plasticity parameters ($\alpha, \beta, \eta, \lambda_0$ & $\chi_h$) may be estimated by the designer. However, these estimates must be made considering the imperfection sensitivity, geometric nonlinearity and other aspects of the particular shell buckling case, with judgement enhanced by comparison with known results for comparable shell buckling conditions (Rotter, 2011b). The need to compare the estimated values of these important parameters with ‘known’ cases is the main challenge of LBA-MNA methodology since only very few cases may currently be regarded as ‘known’ (Rotter, 2016a).

### 2.6.4 Design by global numerical analysis using GMNIA methodology

The most sophisticated numerical design procedure is perhaps the GMNIA methodology, which exploits the full power of modern numerical analysis to compute the characteristic elastic-plastic buckling resistance of the shell structure directly. This method of design is especially valuable where prestigious, expensive or high-risk structures are involved and generally for failure investigation. However, because the influence of geometric imperfections in thin shells under compressive stresses is known to be more often the controlling factor (Donnell & Wan, 1950; Koiter, 1963; Budiansky & Hutchinson, 1966; Bushnell, 1985; Rotter & Teng, 1989), this design method thus requires the shell model to be explicitly defined with geometric imperfections. The main challenge with design by GMNIA lies in the existing array of geometric imperfection forms in which one option must be carefully selected, which predicts the most detrimental effect on the buckling strength of the shell and which must be relatable to tolerance measures that are set for fabrication (ECCS EDR5, 2008; Rotter, 2016b).

For the global GMNIA analysis, the equilibrium path of the shell structure may be traced by means of the static Riks (1979) arc-length algorithm, which can accurately trace ascending and descending load paths that are smooth, unlike in static load- or displacement-controlled methods that cannot trace any ‘snap-back’ in the equilibrium path. However, it is also possible
that the arc-length algorithm jumps over from pre- to post-buckling path and becomes unable
to capture the exact point of buckling event, owing to the abrupt changes that occur with sharp
descending post-buckling path. In such a case, loss of positive definiteness of the tangent
stiffness matrix, evident as the presence of negative eigenvalues in the global tangent stiffness
matrix, is recommended as the numerical indicator of buckling event (Schmidt & Rotter, 2008).

Furthermore, this method of design requires other preliminary analyses to be performed on the
shell model, prior to the full GMNIA. The LBA and MNA analyses must first be performed to
establish the overall relative slenderness $\lambda_{ov}$ of the shell, according to Eq. (2.34). The sole
purpose of calculating the overall relative slenderness is to offer the analyst an insight into the
fundamental characteristics of the shell buckling case under consideration (pure elastic
buckling, elastic-plastic behaviour, etc.), as presented in the capacity curve of Fig. 2.28.
Subsequently, a GMNIA should be performed to ascertain that the chosen form of imperfection
has a sufficiently deleterious effect and to give confidence that only the lowest resistance has
been obtained. The imperfect elastic-plastic critical buckling resistance $R_{GMNIA}$ should be
obtained as shown in Fig. 2.30 on the basis of four distinct criteria, defined below.

\[
\text{Fig. 2.30 – Definition of the elastic-plastic buckling resistance for the global GMNIA analysis (from Schmidt & Rotter, 2008).}
\]

**Criterion C1:** The maximum load factor on the load-deformation curve (limit load). This
criterion works well in shell systems demonstrating a limit-point instability.
**Criterion C2**: The bifurcation load factor, where a bifurcation occurs on the load path before attaining the limit point of the load-deformation curve.

**Criterion C3**: The largest tolerable deformation, where this occurs on the load path before reaching a bifurcation or limit load. This criterion is suited for structures that exhibit material strain hardening or geometric hardening (Rotter, 2005b) so that the equilibrium path demonstrates indefinite hardening. This scenario was encountered and reported in the buckling investigation of very short perfect elastic cylinders under uniform bending, contained in Chapter 4 of this thesis.

**Criterion C4**: The load factor at which the equivalent stress at the most highly stressed point on the shell surface reaches the design value of the yield stress. This criterion offers a conservative treatment of the GNIA analysis as a proxy to the full GMNIA analysis on the shell.

The crucial subject of imperfection and how it is accounted for in design by GMNIA is presented below. The imperfections should first be introduced into the model as equivalent geometric imperfections, such that the pattern is carefully pre-chosen to lead the most unfavourable effect on $R_{GMNIA}$ (Schmidt & Rotter, 2008) unless where this cannot be established a priori and iterative analyses must be performed, on the basis of different imperfection forms, to identify the worst case. The current European design standard for metal shells (EN 1993-1-6+A1, 2017) however, prescribes the eigenmode-affine pattern (Fig. 2.9) as the equivalent geometric imperfections, where no other forms can be justified. The pre-computed geometry of eigenmode-affine imperfection, obtained from LBA analysis, can simply be scaled to an appropriate amplitude and imported as a mesh imperfection into the GMNIA analysis. The amplitude of the adopted imperfection should also relate with tolerance measures, dependent on the quality of fabrication and selected as the maximum of Eq. (2.36)a or b:

$$
\Delta w_{0,eq,1} = l_g \ U_{n1} \quad (a) \quad ; \quad \Delta w_{0,eq,2} = n_i \ t \ U_{n2} \quad (b)
$$

(2.36)

where $l_g$ is the relevant gauge length defined in the dimple tolerance measurement of Fig. 2.17, $t$ is the thickness of the shell wall, $n_i$ may simply be taken as 25 and the dimple imperfection parameters $U_{n1}$ and $U_{n2}$ are given in Table 2.5. The reliability of the buckling resistance prediction should then be verified by performing GMNIA, to obtain the ‘control buckling
resistance’ (\(R_{GMNIA,\text{check}}\)) on shell buckling cases with similar fundamental characteristics and having already-established buckling resistance (\(R_{k,\text{known,check}}\)) or test results (\(R_{test,\text{known,check}}\)).

Table 2.5 – Recommended values for dimple imperfection parameters \(U_{n1}\) and \(U_{n2}\) (from Schmidt & Rotter, 2008)

<table>
<thead>
<tr>
<th>Fabrication tolerance</th>
<th>Description</th>
<th>Recommended values of (U_{n1})</th>
<th>Recommended values of (U_{n2})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Class A</td>
<td>Excellent</td>
<td>0.010</td>
<td>0.010</td>
</tr>
<tr>
<td>Class B</td>
<td>High</td>
<td>0.016</td>
<td>0.016</td>
</tr>
<tr>
<td>Class C</td>
<td>Normal</td>
<td>0.025</td>
<td>0.025</td>
</tr>
</tbody>
</table>

A calibration factor \(k_{GMNIA}\), which is calculated as the ratio of the established resistances to the ‘control resistance’ and must be between 0.8 and 1.2 (or 1.0, where test results are used) for the GMNIA approach to be valid, is then used to calculate the characteristic (\(R_{GMNIA}\)) and finally the design (\(R_d\)) buckling resistance, as shown in Eq. (2.37).

\[ R_k = k_{GMNIA} R_{GMNIA} \quad \text{and} \quad R_d = \frac{R_k}{\gamma_{M1}} \]  

(2.37)

where \(\gamma_{M1}\) is the partial factor for resistance to buckling

2.6.5 Reference Resistance Design (RRD)

The Reference Resistance Design (RRD) is a design framework, recently introduced by Rotter (2016a; 2016b) to enhance ‘hand calculation’ while furnishing databases of \(\alpha, \beta\), etc. values that can be used by all design concepts for buckling limit state. It is a form of ‘hand calculation’ method that works directly with resistances rather than working stresses and incorporates the advantages of LBA-MNA design method but does not include its shortcomings in which key parameters may only be estimated nor the afore-mentioned limitations associated with stress design. RRD has now been accepted as a method of design in the latest amendment to the current European design standard for metal shell (EN 1993-1-6+A1, 2017).

RRD also employs the capacity curve in Fig. 2.28 to describe the relationship between the normalised nonlinear characteristic resistance of the shell, expressed now as the ratio of the nonlinear buckling resistance to the reference plastic collapse resistance (\(R_k/R_{pl}\)) and its dimensionless slenderness \(\lambda = \sqrt{(R_{pl}/R_{cr})}\), where \(R_{cr}\) is the reference elastic critical buckling resistance of the shell. Furthermore, the relationship described in Eq. (2.31) for stress design is
employed in RRD method, except that the $\alpha$ parameter is now decomposed into a combination of the geometrical ($\alpha_G$) and imperfection ($\alpha_I$) reduction factors (Rotter, 2011b) to permit a definitive understanding of each nonlinearity:

$$\alpha = \alpha_G \times \alpha_I$$  \hfill (2.38)

The capacity curve of Fig. 2.28 may also be transformed into a modified capacity curve of Fig. 2.31, which herein describes the relationship between the dimensionless resistance ($R_k/R_{pl}$) and relative strength ($R_k/R_{cr}$). This modified capacity curve is highly beneficial within the concept of the RRD framework on the condition that where a suitable dimensionless length group is established such that a self-similarity of the geometric nonlinearity ($\alpha_G$) in the shell system is preserved under changing radius-to thickness $r/t$ ratio, the elastic portion of the capacity curve remains vertically straight. Consequently, the relative elastic nonlinear buckling strength may simply be read off at the intercept of the horizontal axis, regardless of varying $r/t$ ratio.

![Modified capacity curve for shell structures](from Rotter, 2007)

The main benefit of RRD is that it frees the analyst from having to perform complex LBA/MNA/GMNIA modelling but to use the pre-determined algebraic expressions that contain the information that these analyses would have offered, thereby limiting the possible errors from individual analyst’s inadequacies. In addition, RRD has permitted a shift in focus on characterising real behaviour, while separating out various nonlinearities into parameters with physical meaning, like in stress design, and offering researchers a lens through which to treat shell, and metal structural systems, generally. However, the major challenge with RRD is
the level of computing effort necessary to characterise all the reference resistances and key parameters for any specific shell buckling case, a challenge that was addressed in this PhD research by the efficient management strategy developed specifically for RRD characterisations and presented in Chapter 3 of this thesis.

Prior to the current PhD research, the $\alpha G$ parameter for the specific shell system of isotropic and elastic cylinder under uniform bending had been characterised into algebraic expressions, under rigidly clamped end support conditions. The current PhD therefore intends to contribute into the RRD design method for cylinders under uniform bending by investigating the imperfection sensitivity of this shell system completely and characterising its $\alpha$ parameter within the framework of RRD. The plasticity-related parameters $\beta, \lambda_0, \eta$ and $\chi_h$ are the subject of another computational investigation in the research group of Dr Sadowski and thus falls outside the scope of the current PhD research.
Chapter 3 - Research method and preliminary investigations

3.1 Introduction

The analysis of a shell structure is highly complex because even simple shell systems lead to higher order differential equations, which are often in the form of coupled partial differential equation (PDE) systems. Consequently, exact close-form solutions with the aid of algebraic analysis may only be obtained for a few cases of simple geometry and loading condition, which undoubtedly also require some simplifying assumptions. Nevertheless, this challenge has now become ameliorated following the introduction of numerical methods in obtaining the solutions to the governing equations and, ultimately, by finite element analysis (FEA). This FE method stands out in that it has no restriction on the geometry, boundary conditions, material properties or potential variation in properties along any axis, thereby removing the need for many simplifying assumptions, in addition to the fact that it is applicable to any field problem (Felippa & Clough, 1970). With the availability of fast, powerful computers and significant advancement in modern scientific computing, FEA techniques are now widely incorporated into computer software packages, either as proprietary codes e.g. ABAQUS, (2014); ANSYS (Fluent, 2009); ADAPTIC (Izzuddin, 1991) or open-source e.g. OpenSees (Mazzoni et al., 2005). A widespread use of these software packages is evident in the modern engineering research literature.

Although most modern FEA software packages do not have limitations regarding the size or amount of degrees of freedom that may be present in a model to be analysed, it is computationally cheaper when the size of the model is reduced considerably. This modelling procedure is particularly important where there is the need for running several repeated runs such that even minor time differences may accumulate into a heavy time penalty. One of the ways by which this procedure may be achieved is through the exploitation of structural symmetries, whereby the model is cut in halves along a plane of ‘mirror’ symmetry and thereafter introducing symmetry boundary conditions along this plane of cut. Where this is achievable, the computational time can become significantly reduced than when the complete model is employed. This common finite element modelling procedure may be found in several numerical investigations, particularly in the recent computational studies on the buckling behaviour of cylinders (e.g. Song et al., 2004; Chen et al., 2008; Sadowski & Rotter, 2011a; 2013a; Rotter et al., 2014; Xu et al., 2017) using finite elements. These studies all exploited
structural symmetries along either one or two planes for computational efficiency, thereby reducing the cylinder into half or quarter cylinder model respectively. However, such treatments will only be valid under special conditions (Teng & Song, 2001), and Song et al. (2004) warned that the artificial restraint against translational or rotational degrees of freedom arising from the symmetry boundary condition could potentially influence the buckling pattern assumed by the cylinder model and consequently introduce an artificial stiffness that may over-estimate the failure load of the structure being modelled.

The potential sensitivity of the buckling behaviour of perfect elastic cylinder under uniform bending to different structural model treatments is therefore first investigated in detail in the current chapter to verify which numerical treatment may be the cheapest computationally, without jeopardising the accuracy of the predicted results.

3.2 Scope of the current work

The primary focus of the current PhD research is to present a comprehensive assessment of the buckling behaviour of perfect and imperfect elastic cylinders under uniform bending over a wide parametric variation of cylinder length $L$, radius-to-thickness ratio $r/t$, end boundary condition, forms and amplitudes of geometric imperfections. A preliminary comparison of predictions from different computational models is first performed in the current chapter, based on a structural symmetry analysis, to verify that the finite element model, which exploits two symmetry planes, is an appropriate representation of the cylinder structure. Building on the recommendation from this preliminary study, the generic detail of the numerical model adopted in this thesis is illustrated. Furthermore, the range of values for the parameters employed is presented, especially the cylinder length, which is made to span across all the four length domains recently identified by Rotter et al. (2014) for elastic cylinders under uniform bending, namely: ‘short’, ‘medium’, ‘transitional’ and ‘long. It was also established by the same author that the buckling behaviour of perfect elastic cylinders under uniform bending may be expressed in a manner that is independent of the slenderness of the cylinder (defined here as its radius-to-thickness $r/t$ ratio) with the aid of two dimensionless length groups $\omega$ and $\Omega$ (Eq. (3.1)). Consequently, the potential sensitivity to varying $r/t$ ratios of the buckling behaviour of elastic imperfect cylinders under uniform bending is investigated by employing representative $r/t$ ratios spanning between 100 – 1000.
\[
\omega = \frac{L}{\sqrt{rt}} = \frac{L}{r} \left(\frac{r}{t}\right)^{1/2} ; \quad \Omega = \frac{L}{r} \left(\frac{r}{t}\right)^{1/2} = \omega \cdot \frac{t}{r}
\] (3.1)

In addition, a wide variation of the types of analysis and geometric properties of the cylinders is considered in the current thesis and as a result, the need for full and effective automation becomes increasingly important and the details of the automation strategy employed are presented herein. Finally, a prerequisite mesh convergence analysis of the finite element model employed, across all the length domains, diverse types of analysis and amplitudes of geometric imperfections is presented. Based on the results of this mesh convergence analysis, important conclusions relating to the level of mesh refinement adequate for this cylinder systems are drawn.

3.3 Structural symmetry sensitivity analysis

Cylinders under uniform bending may be described as a candidate model for the exploitation of structural symmetry owing to the symmetry of loading along its length and the meridional membrane stress distribution along its circumference (Fig. 3.1).

The complete membrane theory solutions for the stress resultants \(N_z, N_\theta \text{ & } N_{z\theta}\) and deformations \(u, v \text{ & } w\) in cylinder under uniform bending, reproduced in Eq. (3.2), also offer
another insight into the possibility of a mirror symmetry along the circumference. By considering each of these membrane solutions, it may be revealed that the action of uniform bending induces circumferentially varying meridional membrane stress resultant $N_z$ and meridional deformation $u$ at the edges of the cylinder, which are both in the form of harmonic one (i.e. $\cos 1 \cdot \theta$). Consequently, on the basis of this, a circumferential mirror symmetry is exploited. The four distinct structural symmetry configurations generated for this cylinder system are illustrated in Fig. 3.2 and details about these configurations are listed.

$$
N_\theta(z, \theta) = 0 \\
N_z(z, \theta) = 0 \\
N_z(z, \theta) = -\frac{M}{\pi r^2} \cdot \cos \theta \\
w = \frac{M}{2\pi Er^2t} \left(z^2 - zL + 2vr^2\right) \cdot \cos \theta
$$

(3.2)

Fig. 3.2 – Illustration of the different structural symmetry configurations explored.

**a. Full model**, without any structural symmetry exploited. Hence, the full length of the cylinder together with the full circumference of the cross section are modelled;

**b. Half-length model** in which only the meridional symmetry is exploited. Here, only half of the cylinder length is modelled but the entire circumference is employed;
c. **Half circumference model**, where the circumferential symmetry is used to model only half of the circumference, but the full length of the cylinder is employed; and

d. **Quarter model** in which both symmetries about the circumferential and meridional axes are exploited. This model contains only half of the cylinder length and half of its circumference.

The schematics of the cylinder system under both full and half-length models are illustrated in Fig. 3.3. Where meridional symmetry is not exploited, i.e. full-length models, the translational degree of freedom along the meridional axis ($z$) is released at one of the edges (shown as a roller support in Fig. 3.3a) to avoid the possibility of over-constraining the model during the finite element analysis, essentially typifying a simply-supported beam. By contrast, where meridional symmetry is exploited (half-length models), a corresponding symmetry boundary condition, which prevents any in-plane deformation but allows out-of-plane deformations (shown as the roller support for the cylinder length in Fig. 3.3b) along the plane of cut (meridional axis) is provided. A similar symmetry boundary condition is provided where circumferential symmetry is exploited.

![Fig. 3.3 –Free-body diagrams of the cylinder under bending exploiting a) full and b) half-length design models.](image)

The buckling resistance predictions, together with some buckling modes of the shell system under geometrically linear and nonlinear conditions, based on these four unique structural symmetry configurations are therefore first analysed with the aid of the commercial finite element (FE) suite ABAQUS (2014). Although Rotter *et al.*, (2014) employed the same FE suite to offer key insights into the length-dependent buckling behaviour of perfect elastic cylinders under bending, both the study and its precursor (Chapter 3 of the PhD thesis of Chen,
2011) however, skipped this important preliminary analysis perhaps owing to the strong conviction that any behavioural deviation due to the imposed symmetry should be limited to very short cylinders. Consequently, this preliminary verification study follows a similar computational methodology as the one described in Rotter et al., (2014), except for the varying structural symmetry configurations and the additional unrestrained rotational edge condition (Simply-supported, S) employed. A single instance of radius-to-thickness ratio $r/t = 100$ is considered and as a result, the cylinder length $L$ is expressed in terms of the dimensionless length parameter $\omega$ (Eq. (3.1)) alone, which is made to span between a lower limit of $\omega = 1$ and an upper limit of $\omega = 1000$. A bending moment of magnitude corresponding to the classical elastic critical buckling moment $M_{cl}$ (Eq. (3.3)) was applied (Rotter et al., 2014; Xu et al., 2017).

$$M_{cl} = \pi r^2 t \sigma_{cl} \approx 1.813 \frac{E r t^2}{\sqrt{1 - \nu^2}}$$ (3.3)

Two distinct types of end boundary conditions are employed. In one case, all degrees of freedom are kept restrained (Clamped, C) and only the meridional rotation about the circumferential axis is unrestrained (Simply-supported, S) in the other case. Furthermore, two types of computational analyses were performed on each of the design model employed at constant cylinder length, namely: linear elastic bifurcation analysis (LBA) and geometrically nonlinear analysis (GNA). Both analyses employ complete 3D shell theory, which captures all the bending deformations that may arise from the kinematic compatibility requirements with the end boundary condition without or with geometric nonlinearity incorporated in the model respectively. In the geometrically nonlinear analyses, the equilibrium path of the system was traced by means of the modified arc-length algorithm of Riks (1979).

### 3.3.1 Response of the perfect cylinder assuming small displacement theory

In addition to generating the critical buckling mode (eigenmode), the linear elastic bifurcation analysis (LBA) computes the critical buckling moment $M_{cr}$ of the cylinder, distinct from the classical elastic critical buckling moment $M_{cl}$ by virtue of employing a complete 3D shell theory. Under the geometrically linear conditions, the buckling strength predictions against length $\omega$ (Fig. 3.4) for the four unique finite element models employed show marked variations when the cylinder length is very short and, in the range, $\omega < 20$, the magnitude of which also
depends on the nature of the end support condition. The unrestrained (simply-supported, S) end support condition demonstrates a more pronounced variability within this short length range, owing to the weaker restraint offered by the edges. Beyond a cylinder length of $\omega = 20$, the predicted buckling behaviour is now essentially invariant with the choice of symmetry in the finite element model.

Fig. 3.4 – Normalised linear buckling moment $M_{cr}/M_{cl}$ vs. $\omega$ relationship of perfect elastic cylinder under uniform bending employing four unique finite element models.

The critical buckling modes at representative lengths, indicated in Fig. 3.4 by the dotted ellipse, are thereafter extracted and compared for all the cylinder models employed (Fig. 3.5 & Fig. 3.6). The respective numbers of circumferential full waves $n$ and meridional halfwaves $m$ for each eigenmode are shown in parenthesis. For the purpose of illustrating the critical buckling modes in the cylinder models, the ability of FE suite ABAQUS to recreate the full cylinder model was used where structural symmetries have been exploited, although the line of structural symmetry may still be seen on the cylinder as a solid continuous line. It is revealed that indeed the artificial restraint introduced by structural symmetry may play a significant role in the buckling pattern exhibited by the cylinder, but this influence is completely dependent on the length of the shell and the rotational restraint condition at the edges. The introduced meridional symmetry (half-length and quarter design models) constrains the number of admissible meridional half waves $m$ that can develop along the length into odd numbers and a higher moment is therefore required to maintain this constraint at certain lengths. However, where no meridional symmetry is exploited such as in the full model or half circumference, any number of $m$ can develop.
Fig. 3.5 – Selected buckling modes for varying structural symmetries exploited for the linear elastic bifurcation analyses (LBA) under a rotationally restrained (Clamped, C) end support condition and with slenderness ratio $\frac{r}{t} = 100$.
Fig. 3.6 – Selected buckling modes for varying structural symmetries exploited for the linear elastic bifurcation analyses (LBA) under a rotationally unrestrained (S) end support condition and with slenderness ratio $r/t = 100$.\[\omega = 1.8\quad (n, m)\quad \omega = 3.8\quad (n, m)\quad \omega = 7.5\quad (n, m)\quad \omega = 50\quad (n, m)\]
Although this effect may be found in both end boundary conditions, it is more visible in the rotationally unrestrained (simply-supported, S) cylinders as a result of the significantly weaker restraint offered by the cylinder wall in this end condition.

3.3.2 Buckling behaviour of perfect cylinder under geometrically nonlinear conditions

The true nonlinear buckling behaviour of the perfect elastic cylinder may be investigated by means of a geometrically nonlinear analysis (GNA). The GNA analysis, performed in ABAQUS, employs the modified Riks (1979) algorithm, which detects instability in the cylinder system automatically by checking for negative eigenvalues in the global tangent stiffness matrix after every load increment (Song et al., 2004). Where a bifurcation instability is anticipated, a tiny mesh perturbation was introduced in the cylinder model for the sole purpose of facilitating the correct identification of buckling moment. The nonlinear buckling moment $M_k$ of the cylinder is then computed as the moment corresponding to the first bifurcation point along the predicted equilibrium path.

By contrast with the LBA computations, under geometrically nonlinear conditions, the variability in buckling moment versus length relationship (Fig. 3.7) due to varying finite element models is now restricted to quite short lengths ($\omega < 5$). Specifically, the cylinder demonstrates no further dependency on the choice of finite element model beyond a length value of $\omega = 5$, regardless of the type of end boundary condition employed. In a similar manner to the LBA results, the variability in buckling behaviour at these very short lengths is shown to depend on whether a meridional symmetry is induced or not. The particularly complex behaviour of very short elastic cylinders under bending is described in detail in Chapter 4 of the current thesis, where it is revealed that these short elastic cylinders do not fail by local buckling nor experience geometric softening arising from Brazier (1927) cross-section ovalisation phenomenon. Instead, they fail through a limit-point instability resulting from a detrimental meridional fold developing on the compressed side of the cylinder.

In the rotationally fixed (clamped, C) cylinder, the apparent wide gap between the buckling moment predictions within the ‘short’ domain, specifically between $\omega = 4.2 – 4.8$, depends on whether a meridional symmetry is exploited or not. This is because the meridional symmetry similarly restricts the number of meridional half waves $m$ that can develop in a GNA to odd numbers, since the plane of cut must act to produce a mirror symmetry. The lack of this
symmetry condition in the full and half circumference models permits an earlier shift in the
buckling behaviour of the cylinder from the meridional fold-forming ‘short’ cylinder behaviour
to local buckling dominated ‘medium’ cylinder behaviour.

However, a different change in behaviour is revealed in the rotationally unrestrained (simply-supported, S) cylinders between $\omega = 3 - 3.4$, where although the cylinder does not lose its limit-point instability occurring at high buckling moments, the geometric change observed along the compressed side of the cylinder either remains as one half-wave (half-length models) or becomes a complete sinusoid of harmonic 1 (full length models). To enhance a better understanding of the behaviour at these very short lengths, the buckling moment versus length relationship is reproduced in a higher resolution in Fig. 3.8, considering cylinder lengths $\omega \leq 20$.

From the preliminary verification study on the choice of finite element design model that is most efficient for the current research investigation, the following may be established and are thereafter adopted in this thesis.

- Under geometrically linear conditions, the buckling moment predictions for the elastic cylinder vary over a wider range of length than originally suggested by Rotter et al. (2014),

Fig. 3.7 – The predicted nonlinear buckling resistance versus length relationship of perfect elastic cylinder under uniform bending employing four unique finite element design models
depending on the choice of finite element design model and the nature of the end boundary condition, although this variation is limited only to very short lengths.

- In addition, a potential shift in the ‘short’ domain boundary from the original value of \( \omega = 4.8 \) recommended by Rotter et al. (2014) to \( \omega = 4 \) may be necessary in categorising the length-dependent nonlinear buckling behaviour of perfect elastic cylinders under uniform bending.

- Most importantly, it is revealed that any deviation from the behaviour of the full cylinder model arises from the exploitation of meridional symmetry or not, further highlighting the crucial role of length in the buckling response of the cylinder.

Fig. 3.8 – Higher resolution version of the predicted buckling moment versus cylinder length relationship with four distinct finite element design models.

Nevertheless, for practical applications and for cylinders where ovalisation plays a role (\( \omega > 0.5(r/t) \)), the ‘computationally accurate and economical’ quarter cylinder model may be described as the most efficient for the current research endeavour.

3.4 The final finite element model template

The generic details of the adopted quarter shell design model for this research investigation are illustrated in Fig. 3.9 below. The displacement and rotational degrees of freedom at the end of the cylinder model were controlled by a reference node, placed at the centre of the cross-section and rigidly connected to the ends of the cylinder by a rigid body kinematic coupling. In
addition, this reference node serves as the point of application of a bending moment, having the value of the classical elastic critical buckling moment $M_{cl}$. Throughout all of the computational analyses, the ends of the cylinder were kept rigidly circular but were unrestrained against meridional displacements or rotation about the $y$-axis.

The degree of rotational restraint at the ends of the cylinder model was made to correspond to the two afore-mentioned clamped (C) and simply-supported (S) boundary conditions, which are more common in practical shell structures. The algebraic definitions of these two end boundary conditions are expressed in Eq. (3.4):

$$w_{z=0,L} = \frac{dw}{dz} \bigg|_{z=0,L} = 0 \quad (C) \quad ; \quad w_{z=0,L} = \frac{d^2w}{dz^2} \bigg|_{z=0,L} = 0 \quad (S)$$

where $w$ is the radial displacement of the cylinder and $z$ represents the meridional axis. The end rotation of the cylinder ($UR_y$ in Fig. 3.9) is first transformed into the mean curvature $\phi$ (Eq. (3.5)a), which is calculated over the full length $L$ of the cylinder and normalised by the buckling curvature $\phi_{cl}$ (Eq. (3.5)b) predicted through beam theory:
\[
\varphi = \frac{2UR_s}{L} \quad (a) \quad \text{;} \quad \varphi_{eq} = \frac{t}{r^2 \sqrt{3(1-\nu^2)}} \approx 0.605 \frac{t}{r^2} \quad \text{for} \quad \nu = 0.3 \quad (b)
\]

### 3.5 Mesh convergence analysis

The accuracy of the predicted model behaviour through a finite element analysis and the actual behaviour of a structural system is known to be largely controlled by the level of mesh refinement and the choice of element employed (Bjorkman & Molitoris, 2011; Liu & Glass, 2013). Although different elements employ different shape functions and may accommodate different structural theories, the mesh density controls the amount of error associated with discretisation (Cook et al., 2002). In addition, the resolution of the displacement field constrains the shapes that the structure can assume, and thus the buckling mode that it can exhibit. Hence, prior to any computational studies using finite element analysis software, a preliminary investigation into the sensitivity of the model to mesh refinement or choice of element employed is usually first conducted. However, an important pre-requisite to analysing and interpreting the response of any structural system efficiently using finite element analysis software is a strong understanding of the physics of the structure.

In the adopted FE suite ABAQUS (2014), S4R is a robust, general-purpose shell element, which does not suffer from shear or membrane locking, owing to the uniformly reduced integration incorporated in its formulation and can accommodate both the shear flexible ‘thick shell’ theory and the classical ‘thin shell’ theory. Furthermore, S4R has been found capable of reproducing similar buckling and post-buckling behaviour as those from experimental test data, other shell elements or the more complete 3D solid continuum elements for the structural system of cylinders under uniform bending (e.g. Sadowski & Rotter, 2013a). Consequently, the current mesh convergence analysis and all of the subsequent computational analyses performed in this PhD research employs the S4R shell element.

Computational analyses, in the form of linear elastic bifurcation analysis (LBA), geometrically nonlinear analysis of the perfect (GNA) and imperfect structure (GNIA) using a single instance of the Rotter & Teng (1989) Type A weld depression placed at the cylinder midspan, were performed. From these analyses, the minimum mesh density necessary for the computed buckling strength to become invariant with further mesh refinement was established. Specifically, the focus here is on estimating the maximum size of individual finite element that
can accurately and economically model the bending deformations associated with local
buckling and kinematic compatibility requirements with the end boundary conditions, although
it should also be able to model the buckling and post-buckling response accurately. The
numerical model employed (Fig. 3.9) has been efficiently partitioned into different zones,
depending on whether the response is expected to be membrane or local buckling dominated.
A higher mesh resolution must be provided to accurately represent the high local curvatures
associated with local buckling and compatibility bending, thus the variation of mesh density
described herein is limited to the bending dominated regions alone. Outside these bending-
dominated regions, coarser elements are used judiciously.

For the mesh convergence analyses, a single radius-to-thickness ratio \( r/t \) of 100 is employed at
cylinder lengths of \( \omega = 3, 50 \) and \( \Omega = 3, 8 \) which are individually representative of the four
established length domains for cylinders in uniform bending (Fig. 3.7). All displacements and
rotational degrees of freedom at the edges are assumed to be restrained (Clamped, C) and
nominal imperfection amplitudes of 0 (Perfect), 0.25, 0.75 & 1.5, considering only the Rotter
& Teng (1989) Type A weld depression, are introduced for the GNA and GNIA. The
discretisation into finite elements was characterised in terms of the size of the element along
the meridional axis, although approximately square meshes were employed throughout the
bending dominated regions to ensure numerical accuracy and stability. As a rule of thumb, a
controlling element size of \( 0.25\sqrt{rt} \) is adopted to represent fine mesh, which translates into
approximately 10 elements per linear bending meridional half wavelength \( \lambda \) (Eq. (3.6)).

\[
\lambda = \frac{\pi\sqrt{rt}}{3\left(1-v^2\right)^{0.25}} \approx 2.444\sqrt{rt} \quad \text{for isotropic cylinder with } v = 0.3
\]  \hspace{1cm} (3.6)

In what follows, the controlling element size is adopted as a base case and the different mesh
density variations employed were defined as a factor of this base case, such that the level of
mesh refinement increases with increase in the ‘mesh refinement factor’ (Eq. (3.7)), i.e. as the
element size decreases.

\[
\text{mesh refinement factor (MRF)} = \frac{0.25\sqrt{rt}}{\text{element size}}
\]  \hspace{1cm} (3.7)
A mesh refinement factor $MRF = 1$ translates into an element size of exactly $0.25\sqrt{rt}$ and as the mesh refinement factor $MRF \to \infty$, the element size $\to 0$, implying a finer mesh. In the membrane-dominated regions, coarse elements, each having a size of $1\sqrt{rt}$ and which is equivalent to a mesh refinement factor $MRF$ of 0.25, are employed.

### 3.5.1 Predicted buckling moments across each length domain

In the short cylinder, the length ($\omega = 3$ or $L = 3\sqrt{rt}$) is comparable with the linear meridional bending half wavelength $\lambda$ (Eq. (3.6)). Therefore, it is anticipated that the pre-buckling stress state will ultimately be dominated by bending actions arising from kinematic compatibility requirements (Novozhilov, 1959; Timoshenko & Woinowsky-Krieger, 1959; Calladine, 1983). As a result, the convergence of buckling moments was seen to follow an increase in the $MRF$. This is because as a bending-dominated system, at lower $MRF$ (i.e. coarse meshes) the structural system becomes over-constrained and excessively stiff against bending, therefore requiring higher strain energy to develop an admissible buckling mode (Sadowski & Rotter, 2014), evident as artificially higher buckling moments. This phenomenon may also be understood in terms of the Rayleigh-Ritz principle (Cook et al., 2002) and the excessive stiffness against bending only becomes mitigated when the mesh refinement is increased, with the buckling resistance now converging towards a single value. From the relationship between the buckling moment and the level of mesh refinement presented in Fig. 3.10, the following observations may be made:

- The predicted buckling moments of short cylinders demonstrate close agreement with the Rayleigh-Ritz principle as the results begin to converge mostly from above and at an $MRF$ value exceeding 2.

- In the medium cylinder ($\omega = 50$) however, the bending deformations associated with buckling and kinematic compatibility are now localised near the edges and midspan i.e. away from the location of buckling, therefore the convergence of buckling moments with MRF is revealed to occur sooner than in short cylinder since there are now lesser regions in the cylinder length that are prone to the potential overstiffness against bending. In this length domain, the results demonstrate good convergence once the $MRF$ exceeds 1.5.

- Although the detrimental ovalisation phenomenon exists in the transitional ($\Omega = 3$) and long ($\Omega = 8$) cylinders (or $\omega = 300$ & 800 for $r/t = 100$ respectively), the results however both
show a good agreement with the afore-mentioned standard rule of thumb. This may be understood within the context that local bending and buckling under meridional membrane stress resultants \( N_z \)'s lead to behaviour governed by quite short wavelengths (\( \lambda \)) and therefore controls the mesh resolution significantly. However, ovalisation is a long-wave circumferential bending phenomenon that is controlled by significantly larger wavelengths than the ones arising from \( N_z \)'s, therefore, ovalisation does not necessarily control meshing. Consequently, in both cases the results converge as \( MRF \approx 1 \).

![Graph showing relationship between mesh refinement and buckling moment](image)

**Fig. 3.10** – Relationship between the predicted buckling moment and the level of mesh refinement for elastic cylinders under uniform bending with \( r/t = 100 \) and employing the axisymmetric weld depression imperfection.

### 3.5.2 Individual predicted moment-curvature relationship

The mesh-dependent elastic buckling behaviour of these sample cylinder systems is illustrated herein, with the aid of the predicted moment-curvature relationship as a function of the mesh refinement factor (\( MRF \)). An upper limit of \( MRF = 3 \) is utilised here as it becomes too computationally expensive to trace the pre- and post-buckling load path of the transitional and long cylinders having a level of mesh refinement beyond this limit. In the short cylinder, the dependency of the response on the scale of the mesh size employed is revealed to be more pronounced with a significant difference observed between the predicted moment-curvature relationship of the cylinder (Fig. 3.11), regardless of the condition of the cylinder geometry.
being perfect or not. This significant difference is a direct consequence of the excessive stiffness against bending demonstrated by the numerical model at coarser mesh resolution.

**Fig. 3.11** – Moment-curvature relationships of the bending-dominated short cylinder systems over a wide variation of mesh configurations.

However, the variation in elastic buckling behaviour is almost non-perceptible in the medium length cylinder (Fig. 3.12), although higher buckling strength at deeper imperfection amplitude ($\delta_o/t = 1.5$) may be observed. This anomalous response arises as a result of the larger buckling mode of the most imperfect cylinder ($\delta_o/t = 1.5$) now beginning to encroach on the end boundary condition that is still effective within the ‘medium’ domain.

**Fig. 3.12** – Moment-curvature relationships of the local buckling ‘medium’ cylinder systems, under different mesh refinement factors.
It is also interesting to note that this phenomenon bears no relationship with the level of mesh refinement employed since all the cylinders demonstrated the counter-intuitive higher buckling moment at deeper imperfections. A more in-depth discussion about this phenomenon is presented in Chapters 4 & 5 of this thesis.

Almost no visible difference in the predicted moment-curvature relationships may be observed under the different mesh refinements employed in the transitional and long cylinders (Fig. 3.13 & Fig. 3.14).

Fig. 3.13 – Predicted moment-curvature relationship of the transitional cylinder system, showing lesser dependency on the level of mesh refinement.

Fig. 3.14 – Moment-curvature relationship of the ovalisation-dominated long cylinders, showing insensitivity to level of mesh refinement.
A potential reason for this is because at these lengths, the response of the cylinders is mostly governed by the ovalisation phenomenon and although high mesh refinement may be required around the midspan to support local buckles, this is now only a very limited portion of the full cylinder length. Furthermore, as mentioned earlier, the larger wavelengths arising from circumferential bending due to this ovalisation phenomenon do not show any sensitivity to mesh resolution.

Once the results of the buckling moment predictions and the moment-curvature relationships are obtained, the next stage is to deduce the ‘optimum’ mesh refinement factors (MRF), which are to be used for all the subsequent computational analyses in this research work. As a guide to selecting this optimum MRF, the computational expense (time) and the accuracy of the predicted results are carefully reviewed. The accuracy of the predicted result was thereafter judged by constraining the percentage difference between any two adjacent buckling moment predictions, otherwise referred to as the ‘relative error’, to a tolerance of 1%, while the computational cost relates to the time of computation and size of the output database (.odb extension). The relative errors for all the sample cylinder systems that were studied in the current mesh convergence analyses are shown in Fig. 3.15, where, except for the short cylinders that experience extensive bending deformations, these relative errors are almost all within 1% tolerance.
Following the careful and detailed mesh convergence study, it is concluded that the mesh configuration necessary to trace the buckling behaviour of elastic cylinder under uniform bending is length-dependent since different physical mechanisms control the response in different length domains and consequently have different demands with respect to the interpolation field.

This is particularly evident in the shorter cylinders, where the region of length required to capture bending deformations associated with buckling or kinematic compatibility, occupies the most significant portion of the cylinder length. Therefore, a higher value of mesh refinement factor $MRF$ becomes required to predict their buckling behaviours accurately. The converse is true when the length of the cylinder is sufficiently long and the circumferential bending ‘ovalisation’ phenomenon, which is insensitive to meshing, begins to control the response of the cylinder. Finally, for the purpose of the huge amount of computational analyses that are reported in the following chapters, the optimum $MRF$’s for the regions requiring high mesh resolution are selected according to the individual length domain as: 3 for the ‘short’ domain, 2 for the ‘medium’ domain and 1 for the ‘transitional’ and ‘long’ domains. For the membrane-dominated regions in the cylinder, the $MRF$ is maintained at a constant value of 0.25 or the element size is kept at approximately $1\sqrt{\left(t\right)}$. The distinction between when the cylinder has a membrane-dominated region or not is made once the cylinder length exceeds $6\lambda$, where $\lambda$ is the linear bending half-wavelength (Eq. (3.6)). A cylinder length threshold of $6\lambda$ has been specifically selected to ensure that all the regions (2$\lambda$ each) of bending due to kinematic compatibility at both ends and the midspan buckling have been adequately supported within the cylinder length.

### 3.6 Range of parameters

The overall aim of the current research project is to characterise the imperfection sensitivity of cylinders under uniform bending in terms of the elastic imperfection reduction factor parameter $\alpha_i$, within the framework of the Reference Resistance Design (RRD) methodology recently adopted into EN-1993-1-6 (2007). As the name implies, this method of design employs well-established reference resistances, namely: the plastic collapse resistance $R_{pl}$ and the critical buckling resistance $R_{cr}$ and builds a capacity curve (Fig. 2.31) by means of six independent parameters ($\alpha_0$, $\alpha_i$, $\beta$, $\lambda_0$, $\eta$, $\chi$) that relates the dimensionless resistance $\chi$ of a structural system to its relative slenderness $\lambda = \sqrt{(R_{pl}/R_{cr})}$. However, these RRD parameters need to be defined $a$
priori, as algebraic expressions, following a comprehensive study of the specific shell buckling case, which would ideally require a huge amount of computational investigations to be conducted over a wide range of other parameters. Although it is possible to establish these parameters from laboratory testing theoretically, the significant cost penalty associated with using such method for every individual system and across all parameter ranges may be difficult to justify in the modern era of limited test budgets, powerful software and cheap computing power.

Each of the RRD parameters relates to a specific physical characteristic of the shell system. The effect of geometric nonlinearity is accounted for by the ‘geometrical reduction factor’ $\alpha_G$, the imperfection sensitivity, is accounted for by the ‘imperfection reduction factor’ $\alpha_I$ under a linear elastic material regime. Although the RRD method operates best within a framework that assumes typical mild steel and a bilinear material law, it can be adapted to those exhibiting intrinsically nonlinear elastic regimes (e.g. aluminium, alloys, stainless steels, etc.). However, very little research appears to have been done on this at present. The slenderness at which plasticity first affects the overall resistance of the shell is accounted for by the ‘plastic range factor’ $\beta$, while the slenderness value at which the reference full plastic condition is achieved is termed the ‘squash limit’ $\lambda_0$. The severity of plasticity effects at every slenderness value may be accounted for by the ‘interaction exponent’ $\eta$ and the potential strength gain due to the effect of strain or geometric hardening is taken care of by the $\chi_h$ parameter (Rotter, 2016a; 2016b; Sadowski et al., 2017a).

Consequently, several types of computational analyses must be performed in order to establish these RRD parameters, although this procedure can be greatly enhanced first by considering the cylindrical metal shell as a data tree with ordered hierarchy of analysis Levels. In any analysis Level, a physical variable may vary, but while ascending into another Level along the same branch, this physical variable must be held constant. A flowchart illustrating this analysis hierarchy and specifically designed for cylinder under uniform bending, being the shell system of interest in this research, is presented in Fig. 3.16. The two reference resistances ($R_{cr}$ & $R_{pl}$) upon which the RRD capacity curve is built are linear in terms of the Young’s modulus $E$ and yield stress $\sigma_y$ respectively, thus these physical variables may be held constant for any RRD characterisation in the base state (or Level 0) of the shell system. This method of analysis
hierarchy may also be applied to other shell systems, but a different structural form, boundary condition or loading distribution will require a fresh start from the base state (Level 0).

The definition of the constitutive material law to include the beneficial effect of strain hardening on mild carbon steel design is now evident in recent research as simple piecewise-linear characterisations of the stress-strain relationship (Buchanan et al., 2016; Gardner et al., 2017; Sadowski et al., 2017b). But, the simplest and most conservative material law in use is
a bilinear relationship with an initial elastic region of modulus $E$ followed by linear strain hardening region of modulus $E_h$ (expressed as a percentage of $E$). Therefore, since the variation in $E$ or $\sigma_y$ is already accounted for in the base state, the Level 1 variable constitutes the post-yield material behaviour in terms of $E$ and $E_h$. Furthermore, it is well known that the presence of geometric imperfections affects the true buckling resistance of a slender shell system, especially one in which the pre-buckling state of stress is dominated by membrane actions. Consequently, the Level 2 variable relates with the amplitude of the imperfection form, a physical variable that must be carefully chosen to be practical and deleterious to the shell system (EN 1993-1-6, 2007; ECCS EDR5, 2013; Sadowski & Rotter, 2013b).

Other physical variables such as the slenderness ($r/t$ ratio) and cylinder length ($L$) that relate with the geometry of the shell system are contained in Level 3 and above. Since the main parameters that are considered in this research study include the cylinder length $L$, $r/t$ ratios, forms and amplitudes of geometric imperfections, the primary focus of the current research in the designed flowchart of Fig. 3.16 is limited to Level 2 – 4. However, since this research considers the behaviour of the shell system under linear elastic material property regime alone, where the complexity originates purely from the geometry, the lower branch of Level 1 path is not followed, a consideration that is outside the scope of this research.

The reference critical buckling moment of the cylinder $M_{cr}$ was computed through a linear elastic bifurcation analysis (LBA). In addition to offering a holistic view of the buckling strength of the cylinder under a geometrically linear and perfect condition, it is an important prerequisite to a full understanding of the single influence of geometric nonlinearity on the buckling resistance of the shell structure. Consequently, the geometrical reduction factor $\alpha_G (= M_{k,GNA} / M_{cr})$ is calculated by comparing the buckling moment obtained through a geometrically nonlinear analysis (GNA) of the perfect shell system $M_{k,GNA}$ with $M_{cr}$. The GNA analysis in turn becomes crucial to a comprehensive understanding of the single influence of geometric imperfection in the elastic stability of the cylinder and in calculating the imperfection reduction factor $\alpha_I (= M_{k,GNIA} / M_{k,GNA})$, when the buckling moment from a geometrically nonlinear analysis of the imperfect structure (GNIA) $M_{k,GNIA}$ is compared with $M_{k,GNA}$. The act of decomposing the geometric nonlinearities in the shell system ($\alpha = M_{k,GNIA} / M_{cr}$) into two distinct parameters ($\alpha_G \& \alpha_I$) offers an important platform through which the nonlinearity that affects the behaviour of the shell system more and under what conditions may be understood,
since it is possible for a long cylinder under uniform bending to behave nearly as perfect as possible (i.e. \( \alpha \rightarrow 1 \)) but still ovalise (\( \alpha \rightarrow \frac{1}{2} \)). In special exploratory cases, other computational analyses involving nonlinear constitutive material relations such as the materially nonlinear analysis (MNA) of the perfect structure, geometrical and materially nonlinear analysis with or without imperfections (GMNIA or GMNA respectively) were performed on the cylinder model.

It should be noted that for every parameter value of length, \( r/t \) ratio and end boundary condition employed in this research study, at least 5 types of computational analyses are performed on each cylinder model, a single instance of LBA and GNA as well as 3 instances of GNIA (at varying level of \( \delta/t \)), from introducing three unique forms of geometric imperfections. Additionally, the effect of varying boundary condition on \( M_{cr} \), one of the two reference resistances used in RRD is investigated while the effect on the plastic collapse resistance \( M_{pl} \) is presented in Appendix A. Consequently, an initial estimate of the total number of computational analyses to be performed was in excess of \(~35,000\) runs. The actual amount of individual computational investigations performed is presented in each subsequent chapter. An overview of a novel model management strategy that was recently devised, during the course of the current PhD research, for managing such a large amount of computational analyses and specially adapted for numerical investigations on cylinders under bending – a focal point of this PhD research study – is presented shortly.

### 3.7 Management strategy for the research work

Each computational analysis performed on a numerical model involves the following set of processes: model generation, submission, termination and processing. In addition, the results from each computational analysis must be pre-processed, analysed and vetted to ensure they conform with the expected physics of the structure or investigated further, if necessary. Altogether, the current research work could become difficult to manage within the timeframe of the PhD program if all the afore-mentioned processes are to be performed manually. Therefore, recourse was made to automation, an advancement made possible with the aid of modern scientific computing and the ability of many modern research FE suites to interface with programming languages such as C++, Python and FORTRAN. The management strategy developed here is designed for cases where the sizes/runtimes of individual models are modest, but there are large number of them.
The first two processes involved in each computational investigation, i.e. model generation and submission were automated through Python interfacing with ABAQUS and details of the programming scripts for these processes may be found in Appendix C. For output processing, three different data assessment tools were employed namely: MATLAB, Python and Excel spreadsheet, and details of the python scripts involved may be found in Appendix D. The particularly challenging process of terminating an ongoing computational analysis however, deserves a special mention. Firstly, it is important to highlight the necessary conditions that must be satisfied before an ongoing analysis is considered to be due for termination. Furthermore, it should be understood that only the computational investigations, where the modified Riks (1979) arc-length algorithm have been used to trace the equilibrium path of the structure (i.e. GNA, GMNA, GNIA, etc.), require this user-induced termination. A load- or displacement-controlled static non-Riks solver is not used since it does not offer access to the history record of the load proportionality factor, in addition to the fact that path-tracing through these non-Riks solvers only allow forward increment of loads or displacements respectively, with the consequential inability to model a ‘snap-back’ effect correctly in a bifurcation buckling. These limitations are non-existent in static Riks (1979) solver.

The first two distinct conditions, sufficient to trigger the termination of an ongoing Riks analysis are described below in terms of ‘Kill Conditions’ (KC) and derived on the basis of the first buckling event along the equilibrium path of the structure. These two ‘Kill Conditions’ are as follows:

- **KC1**: Terminate an ongoing analysis when the current increment’s load proportionality factor is less than that of the previous increment. This kill condition detects bifurcation or limit point instability with unstable post-buckling equilibrium paths (Fig. 3.17a & b).

- **KC2**: Terminate an ongoing analysis when the change in load proportionality factor from one increment to another is less than a specified tolerance. This kill condition ensures that the analysis does not become stuck in an unending loop of cutbacks and increases in the arc-length increment size due to repeated attempts at convergence in the vicinity of a numerically ‘problematic’ bifurcation. This ‘Kill Condition’ is relevant for situations where the solver is unable to proceed into the post-buckling domain for numerical reasons,
regardless of the type of buckling (limit-point or bifurcation) although it has correctly detected the location of the first buckling event.

Numerous other studies (e.g. Yamaki, 1984; Rotter, 2004; 2007; Sadowski & Rotter, 2011a; 2012; 2013b) have reported different scenarios where a deeper imperfection amplitude turns a bifurcation on the equilibrium path of an imperfect structure into a ‘kink’ with a smooth transition from pre- to post-buckling, accompanied by a visible growth in buckling deformations but with the solver reporting no loss of positive-definiteness in the global tangent stiffness matrix (e.g. the GNIA $\delta_o/t = 1.5$ plots in Fig. 3.12). This situation is anticipated to occur very many times during this PhD work and is also illustrated in Fig. 3.18 for very slender imperfect metal silos under eccentric discharge.

A smooth transition from pre-to post-buckling path was noticed in the equilibrium path of these silos at deeper imperfections, with the solver reporting no loss in positive definiteness in the global tangent stiffness matrix, even at the point where there is a significant change in the slope of the path. The possible reason for this counter-intuitive behaviour was hinted at in Rotter (2011b) as a consequence of deeper imperfections now inducing buckling modes that occupy a larger length of the shell (Rotter, 2004) so that the end boundaries begin to influence the predicted buckling strength. In such a case, buckling event has in the past been conservatively selected as the point of inflection along the equilibrium path.
Since the current research study primarily focusses on imperfect cylinders with varying amplitudes of geometric imperfections, the third and final ‘Kill Condition’ KC3 is designed to detect any ‘kink’ along the equilibrium path of the shell system. However, it is believed that in the most general case, a ‘kink’ and an inflection point (change in sign of curvature of the equilibrium path), can be mutually exclusive where, for example, a softening path may abruptly transition to another softening path while the tangent stiffness maintains positiveness, coupled with no drop in the load proportionality factor \((M/M_{cl})\). This is illustrated in Fig. 3.19 and in such a case, it is assumed that the same criterion of ‘change in sign’ in the curvature\(^1\) for detecting inflection point will not necessarily suffice as the ‘kink detector’.

Therefore, the implementation of the KC3 or ‘kink detector’ works directly with the equilibrium path of the structure and not just the history record of the load proportionality factor \((M/M_{cl})\), hence, the reference node of the cylinder model was additionality employed as a tracker node, such that the history record of the cylinder end rotations \((UR_y)\) in Fig. 3.9) may easily be accessed after every load increment. By fitting a circle to the three most-recent points on the equilibrium path, the local radius of curvature at any increment can be computed and an ongoing analysis is terminated if the following conditions for KC3 are satisfied:

---

\(^1\) Here, **curvature** refers to the **second derivative** of the equilibrium path, using a finite different scheme.
• **KC3**: Terminate an ongoing analysis when the absolute relative difference between the current and previous incremental radii of curvature of the equilibrium curve exceeds a tolerance (GNIA).

![Graphs showing 'point of inflection' and 'general kink']

Fig. 3.19 – Comparison of the ‘kink detector’ concept with the general inflection point.

In automating the three effective ‘Kill Conditions’ described above, a preliminary technical requirement is the installation of another proprietary Intel (2016) FORTRAN compiler, operated through Visual Studio 2013 platform, on the personal computer before a full incorporation into the FE suite ABAQUS (2014). On successful installation and incorporation, scripts were then written in FORTRAN programming language as user subroutines, to terminate an ongoing analysis subject to the narrated ‘Kill Conditions’ and employed during any submission process. Details of the adopted user-subroutines are presented in Appendix E.
Chapter 4 - Buckling of very short elastic cylinders under uniform bending

4.1 Introduction and general outline

The first attempt to document length effects in elastic imperfect cylinders under uniform bending is offered here by carefully studying the response of very short elastic cylinders under uniform bending, following the outline narrated below. As an introduction, a brief overview of the length-dependent buckling behaviour of perfect cylinders under uniform axial compression is presented to set the tone for a concise review of existing literatures on the response of a closely-related structural system of perfect cylinders under uniform bending. By limiting the cylinder lengths to be in the very short range such that the end boundary condition is able to influence the response of the cylinder and with the aid of two different types of end boundary condition, the behaviour of this structural system under geometrically linear and nonlinear conditions is thereafter presented extensively, following a detailed description of the research method. As a follow-up to the recommendations from the structural symmetry analysis of Chapter 3, the responses of the perfect cylinders are compared under the full and quarter cylinder design models respectively. Furthermore, the sensitivity of these very short elastic cylinders under uniform bending to geometric imperfection is investigated by means of two distinct forms of imperfection, namely: eigenmode and axisymmetric weld depression. Finally, the influence of geometric nonlinearity alone on the buckling behaviour of perfect short cylinders under uniform bending, denoted by the $\alpha_G$ parameter, as a function of the cylinder length is characterised into simple algebraic expressions to aid structural design.

The nature of the elastic stability of a cylinder under the fundamental loads of uniform axial compression, external pressure and torsion is known to be strongly controlled by its length. To facilitate structural design, cylinders are therefore first categorised into a length domain and dimensioned according to procedures that capture the buckling behaviour governing that particular domain. Each one of the above classical load cases qualitatively exhibits three length domains, namely ‘short’, ‘medium’ and ‘long’ (Yamaki, 1984; Rotter, 2004; Greiner, 2004; Schmidt & Winterstetter, 2004; EN 1993-1-6, 2007; ECCS EDR5, 2013), although the length boundaries defining each domain occur at different numerical values for each load case.

For ‘short’ perfect cylinders under uniform axial compression (Fig. 4.1), the end boundary condition induces extensive compatibility bending deformations and completely restrains the
formation of any local buckling mode. Failure occurs by global limit point buckling at a considerably higher buckling stress than that predicted by the classical elastic critical buckling stress $\sigma_{cl}$ (Eq. (4.1)), derived assuming a pure pre-buckling membrane stress state achievable only in a theoretical cylinder with special boundary conditions:

$$\sigma_{cl} = \frac{E}{\sqrt{3(1-\nu^2)}} \frac{t}{r} \approx 0.605E \frac{t}{r} \text{ for } \nu = 0.3. \quad (4.1)$$

where $E$ is the Young’s modulus and $\nu$ is the Poisson’s ratio. In the ‘medium’ length domain, by contrast, the effect of the boundary restraint is localised near the ends and no longer sufficient to restrain the formation of a local short-wave buckling mode. Since the cylinder is largely under membrane action, buckling occurs at a stress only slightly below $\sigma_{cl}$. The approximately 15% reduction is caused by geometric nonlinearity due to the pre-buckling amplification of compatibility bending rotations near the end boundaries (Brush & Almroth, 1975; Rotter, 2004). ‘Long’ cylinders fail in a different manner usually by Euler column buckling at a stress significantly below $\sigma_{cl}$. These three domains are illustrated in Fig. 4.1
(Yamaki, 1984) where the dimensionless length is expressed in terms of the Batdorf (1947) parameter $Z$, where:

$$Z = \frac{L^2}{rt\sqrt{1-v^2}} = \omega^2\sqrt{1-v^2}. \quad (4.2)$$

This choice of dimensionless length group has since been superseded by a related but simpler parameter $\omega$ (Rotter, 2004), defined in Eq. (4.3)a, which is directly related to both the linear bending half-wavelength $\lambda$ (Eq. (4.3)b) and the classical axisymmetric buckle half-wavelength $\lambda_{cl}$ (Eq. (4.3)c).

$$\omega = \frac{L}{\sqrt{rt}} = \frac{\sqrt{Z}}{\sqrt{1-v^2}} \quad (a)$$

$$\lambda = \frac{\pi\sqrt{rt}}{3(1-v^2)^{0.25}} \approx 2.444\sqrt{rt} \text{ for } \nu = 0.3 \quad (b)$$

$$\lambda_{cl} = \frac{\pi\sqrt{rt}}{12(1-v^2)^{0.25}} \approx 1.728\sqrt{rt} \text{ for } \nu = 0.3 \quad (c)$$

The parameter $\omega$ relates directly with the cylinder length $L$ and permits the influence of the shell slenderness, associated with the radius-to-thickness $r/t$ ratio, to be examined independently of the specific cylinder length within each length domain. Using this notation, the European Standard for metal shells EN-1993-1-6 (2007) defines the length ranges for cylinders under uniform axial compression in a remarkably compact manner as simply $\omega \leq 1.7$ for ‘short’ cylinders, $1.7 < \omega \leq 0.5(r/t)$ for ‘medium’ and $\omega > 0.5(r/t)$ for ‘long’ cylinders. External pressure and torsion are characterised similarly. The value of $\omega = 1.7$ for the ‘short-medium’ boundary corresponds to $Z = 2.85$ in Fig. 4.1 and is a conservative choice to cover all possible edge boundary conditions. In reality, however, it may be seen that under clamped conditions (with restrained meridional edge rotations, denoted in Fig. 4.1 by ‘C’) the ‘short’ domain persists over a wider range of lengths than under simply-supported conditions (with unrestrained edge rotations, denoted by ‘S’).
4.2 Buckling at different lengths under global bending

Building on this theory, a recent computational study of perfect clamped cylinders under uniform global bending by Rotter et al. (2014) characterised the effect of length on the nonlinear elastic stability across a wide range of $r/t$ ratios. The study established that the behaviour under global bending displays length domains that are qualitatively similar to those of the three fundamental load cases, except for the presence of an additional ‘transitional’ region between the ‘medium’ and ‘long’ domains (Fig. 2.26). This new domain is a direct consequence of the Brazier (1927) cross-section ovalisation phenomenon, absent in the other load cases, that detrimentally affects the pre-buckling deformations of ‘long’ cylinders under bending.

Specifically, the ‘transitional’ domain identifies the region of lengths below which ovalisation is completely restrained by the end boundaries, and above which ovalisation is fully developed and cannot become more detrimental. The Brazier moment prediction $M_{Braz}$ (Eq. (4.4)a), describes the limit point cross-sectional failure of long cylinders through nonlinear ovalisation and is only approximately half of the classical elastic critical buckling moment $M_{cl}$ (Eq. (4.4)b) that describes local buckling in shorter cylinders, a significant penalty. A more in-depth discussion of the nonlinear mechanics of ovalisation may be found in Chapter 2 of the current PhD thesis.

\[
M_{Braz} = \frac{2\sqrt{2}}{9} \cdot \frac{E \pi r^2}{\sqrt{1-\nu^2}} \approx 1.035 Er^2 \approx 0.54M_{cl} \text{ for } \nu = 0.3 \quad (a)
\]

\[
M_{cl} = \pi r^2 t \sigma_{cl}(r) \approx 1.901 Er^2 \approx 1.84M_{Braz} \text{ for } \nu = 0.3 \quad (b)
\]

Prior to Rotter et al. (2014), the only evidence available on the existence of ‘short’ cylinders under bending was a logical hint on the potential deformed geometry (Calladine, 1983) of shorter cylinders under the actions of opposing tensile and compressive membrane stress resultants at the ends, which is built on the assumption that one generator remains straight (cylinder side under tension) and the other assumes a ‘bow’ shape due to compression. Consequently, the recent study by Rotter et al. (2014) appears to have been the first to formally confirm and briefly explore the existence of a ‘short’ domain for cylinders under bending. For clamped end conditions, the domain boundary between ‘short’ and ‘medium’ cylinders was deemed to occur at $\omega = 4.8$, somewhat longer than the value of $\omega = 1.7$ for the closely related
load case of uniform axial compression. A preliminary extension of this pivotal study to simply-supported end conditions revealed that this boundary fell at a shorter value of $\omega = 3.2$ due to the weaker rotational restraint offered by the simple supports. These ‘short’ domain predictions for both end support conditions appear to be consistent with the effect seen for cylinders under uniform axial compression in Fig. 4.1. Furthermore, in both cases, cylinders in the ‘short’ domain were found to exhibit a limit point instability at moments and mean curvatures significantly in excess of the classical elastic critical buckling values $M_{cl}$ and $\varphi_{cl}$ (Eqs. (4.4)b and (4.5)) that have been derived on the basis of the simple ‘local buckling hypothesis’ of Axelrad (1965; 1985). This hypothesis assumes that a local short-wave buckle develops as soon as the most compressed fibre in the cylinder reaches the classical elastic critical stress for uniform axial compression $\sigma_{cl}$, corresponding to the undeformed radius $r$ in Eq. (4.1).

$$\varphi_{cl} = \frac{t}{r^2 \sqrt{3(1-\nu^2)}} \approx 0.605 \frac{t}{r^2} \quad \text{for steel with} \; \nu = 0.3$$

(4.5)

![Diagram showing nonlinear equilibrium paths of perfect ‘short’ and ‘long’ elastic cylinders under global bending with clamped end support conditions and $r/t = 100$.](image)

This concept is clearly no longer representative for the ‘short’ domain where the boundary condition restraint prevents the development of an admissible local buckling mode. Further, as is presented in the current chapter, the limit point instability in this length domain is not
attributable to Brazier ovalisation but rather to the progressive growth of a destabilising meridional fold on the compressed side of the cylinder (Fig. 4.2).

Furthermore, a first preliminary study conducted to investigate the sensitivity of elastic cylinders under global bending to the critical eigenmode form of geometric imperfections revealed that the length domain boundaries, originally established based on the behaviour of ‘perfect’ cylinders by Rotter et al. (2014), potentially change. This variation however, depends on imperfection amplitude and the sensitivity is markedly different across the different length domains. Similar findings were recently published for cylinders under uniform compression by Rotter & Al-Lawati (2016). The ‘short’ domain appeared to exhibit particularly counter-intuitive behaviour, with imperfections seemingly acting in a beneficial manner depending on the amplitude. The peculiarities of this behaviour are explored in more detail herein.

4.3 Scope of the study

The current chapter offers a detailed numerical exploration of the behaviour of short elastic cylinders under uniform bending that are strongly influenced by the end boundary conditions. Lengths of up to $\omega = 15$ are investigated, sufficiently long to cover the initial portion of the ‘medium’ domain and its transition to the ‘short’ domain, but too short to be affected by the cross-sectional ovalisation phenomenon, which lies outside the scope of this chapter but will be investigated in Chapter 5 of this PhD thesis. Cylinders shorter than $\omega = 1$ were not considered since they make for ill-conditioned numerical models, the meridional dimension here being disproportionately smaller than the circumferential one. They also fall outside the scope of any known practical application. A small length increment of $\omega = 0.2$ was used between $\omega = 1$ and 5 to allow for a high resolution of data points within the ‘short’ domain proper, while beyond $\omega = 5$ a larger length increment of $\omega = 0.5$ was used. A representative range of practical $r/t$ ratios was investigated, from 100 to 1000.

In addition to linear and nonlinear elastic critical buckling strengths, obtained from linear bifurcation analyses (LBA) and geometrically nonlinear analyses (GNA) respectively, the sensitivity of the nonlinear elastic buckling behaviour to two types of geometric imperfection is investigated (GNIA). These consist of the classical buckling eigenmode and a realistic manufacturing-related axisymmetric ‘weld depression’ placed at the cylinder midspan (Rotter & Teng, 1989). Two different end boundary conditions, which relate to the condition of
meridional rotations of the shell middle surface about the circumferential edge (Fig. 4.3), are employed. The first one is the clamped (C) boundary condition, where these rotations are kept restrained and the second type is the simply-supported (S) boundary condition, where these meridional rotations are free. The current research study is particularly relevant to long cylindrical multi-segment shells with closely spaced stiffening rings (e.g. Singer, 1967), where each individual segment is a very short individual cylinder, often found in the aerospace and marine industries (ECCS EDR5, 2013) and specialised civil engineering applications such as LIPP® silos.

![Diagram of end boundary conditions and imperfection profiles](image)

**Fig. 4.3 – Details of the end boundary conditions, imperfection profiles and amplitudes.**

4.4 Finite element model

The computational analyses were conducted using the commercial finite element program ABAQUS v.6.14-2 (2014), although one additional validation was performed using the specialised ADAPTIC software (Izzuddin, 1991). The numerical model links the end displacements and rotations of the cylinder through a rigid body kinematic coupling to a reference node at the centroid of the cross-section, which keeps the ends rigidly circular and serves as the point of application of the bending moment. The edges are free to displace meridionally but not transversely to the cylinder. Planes of symmetry through the midspan and in the plane of bending were exploited to create a computationally efficient quarter-shell model.
Further details on this computational treatment, which has been successfully used in the previous studies for cylinders under uniform bending (Rotter et al., 2014; Xu et al., 2017), may be found in Chapter 3. In another validation study, full cylinder models, without any recourse to plane(s) of symmetry, were employed. The general-purpose bi-linear shell element with reduced integration S4R was used, while the cylinder was assumed to be made of homogenous and isotropic steel with a Young’s modulus of \( E = 2 \times 10^5 \) N/mm\(^2\) and a Poisson’s ratio of \( \nu = 0.3 \).

The lengths of ‘short’ domain cylinders (\( \omega < 5 \) or \( L < 5\sqrt{rt} \)) are of the same order of magnitude as the linear bending half wavelength \( \lambda \) (Eq. (4.3)b). Consequently, the pre-buckling stress state is dominated by bending action arising from kinematic compatibility with the end boundary conditions (Novozhilov, 1959; Timoshenko & Woinowsky-Krieger, 1959; Calladine, 1983). To capture the high local curvatures associated with this local bending, the models were discretised with fine and approximately square meshes at an element density of 30 linear elements per bending half wavelength \( \lambda \) following the recommendation of a detailed mesh convergence study presented earlier in Chapter 3 of the current thesis. Where relevant, the nonlinear equilibrium path of moment against curvature was followed using the modified arc-length algorithm (Riks, 1979) and the curvature was taken as the mean value \( \varphi \) (Eq. (4.6)) over the full length of the cylinder:

\[
\varphi = \frac{2 \cdot UR_y}{L} \quad (4.6)
\]

Here, \( UR_y \) (in Fig. 3.9) is the rotation of the cross-sectional plane at either end.

The imperfection sensitivity analyses consider two types of geometric imperfections (Fig. 4.3). One is the classical linear buckling eigenmode, which may be easily imported from an LBA analysis, while the second one is a circumferential weld depression, whose geometry must be generated explicitly following the ‘Type A’ (Eq. (4.7)) specification of Rotter & Teng (1989):

\[
\delta = \delta_o \cdot \exp \left( -\frac{\pi}{\lambda} \left| y - \frac{L}{2} \right| \right) \cdot \left\{ \cos \left( \frac{\pi}{\lambda} \left| y - \frac{L}{2} \right| \right) + \sin \left( \frac{\pi}{\lambda} \left| y - \frac{L}{2} \right| \right) \right\} \quad (4.7)
\]

In Eq. (4.7), \( \delta_0 \) is the nominal amplitude of the imperfection, \( y \) represents the meridional coordinate, \( L/2 \) represents the meridional coordinate with origin at the centre of the depression,
and \( \lambda \) is the linear bending half wavelength (Eq. (4.3)b). This definition of a weld depression imperfection is considered representative of a common manufacturing process found in real cylinder structures (Bornscheuer et al., 1983), and has been argued to result in the ‘most detrimental’ effect under uniform axial compression (Rotter, 2004; Berry et al., 2000; Pircher et al., 2001; Song et al., 2004).

To achieve a direct comparison of the imperfection sensitivities under the two very different imperfection forms, the nominal amplitudes \( \delta_0 \) were reformulated in terms of ‘equivalent geometric deviation’ \( \delta_e \). This is defined here as the distance from the most outward radial position of the initial imperfection to the most inward radial position. For the eigenmode imperfection, the amplitude is traditionally described in mathematical terms as the distance between the original position of the shell and peak radial deviation. However, due to the sinusoidal nature of these eigenmodes (illustrated in Fig. 4.3), this is generally half of the equivalent geometric deviation (Rotter, 2013a), and for this reason \( \delta_e \) is double this amplitude for the eigenmode imperfection. Similarly, for the weld depression imperfection, \( \delta_e \) is 1.04 multiplied by the nominal amplitude \( \delta_0 \) in Eq. (4.7). For both imperfections, \( \delta_e/t \) ratios of 0 (GNA), 0.10, 0.25, 0.35, 0.50, 0.75, 1.0, 1.5 and 2.0 were investigated. The total number of individual computational analyses conducted in the current chapter of the PhD thesis is 5,904 as presented in Table 4.1 below.

<table>
<thead>
<tr>
<th>Boundary condition</th>
<th>Lengths</th>
<th>Ratios</th>
<th>Imperfection forms</th>
<th>Amplitudes</th>
<th>LBAs</th>
<th>GNAs</th>
<th>GNIAs</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C)</td>
<td>41</td>
<td>4</td>
<td>2</td>
<td>8</td>
<td>41×4</td>
<td>41×4</td>
<td>41×4×2×8</td>
</tr>
<tr>
<td>(S)</td>
<td>41</td>
<td>4</td>
<td>2</td>
<td>8</td>
<td>41×4</td>
<td>41×4</td>
<td>41×4×2×8</td>
</tr>
<tr>
<td><strong>Total:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>328</td>
<td>328</td>
<td>5,248</td>
</tr>
</tbody>
</table>

To manage this substantial number of computational analyses, especially the ones requiring the use of modified Riks (1979) arc-length algorithm in equilibrium path tracing (GNA & GNIAM), the management strategy described in Chapter 3 is employed. However, owing to the bending-dominated nature of the pre-buckling stress state of these very short cylinders, the job termination process was performed with extreme care since it is possible that, in the geometrically imperfect state, the cylinder experiences a premature local reversal in the load path without any net reduction in stiffness. Where such occurs, the job termination process is
performed manually in the advanced stage of the equilibrium path, where an easily perceptible instability may be justified.

### 4.5 Linear buckling behaviour of perfect ‘short’ cylinders under uniform bending

A linear elastic bifurcation analysis (LBA) is an eigenvalue calculation based on a full linear shell bending theory and so includes the effects of local bending caused by kinematic compatibility with the end boundary condition that affects the pre-buckling membrane stress state. This analysis was performed first to compute the lowest eigenvalue corresponding to the linear critical moment $M_{cr}$, which is distinct from the classical elastic critical moment $M_{cl}$ (Eq. (4.4)b) that assumes a meridionally uniform pre-buckling membrane stress state. The calculation also yielded the shape of the eigenmode imperfection for later use.

![Linear buckling behaviour of perfect short cylinders under bending](image)

The relationship between the normalised moment $M_{cr} / M_{cl}$ and the dimensionless length $\omega$ for clamped (C) cylinders (Fig. 4.4) shows $M_{cr}$ to exceed $M_{cl}$ for very short cylinders ($\omega \leq 5.0$) significantly and become unbounded as $\omega \rightarrow 0$. The reason for this is that a buckle requires a finite portion of the shell to attain the critical membrane stress necessary for it to be supported, but the membrane stress state is so severely affected by the local compatibility bending that
much higher moments are required for its effect to be overcome. The formation of the buckle itself is also physically restrained by the edge boundaries (Rotter et al., 2014). In the other direction, as $\omega$ grows, the ratio $M_{cr} / M_{cl}$ converges to unity suggesting that the ‘local buckling hypothesis’ where buckling occurs when a single point reaches the critical stress becomes a close representation of the realistic behaviour. The LBA and classical moments $M_{cr}$ and $M_{cl}$ are never affected by ovalisation regardless of length because both assume no changes in geometry which prohibits cross-sectional flattening (i.e. ‘long’ cylinder effects), such that under linear conditions there are effectively only two length domains under global bending: ‘short’ and ‘medium’.

The simply-supported boundary condition (Fig. 4.5) restrains only radial and circumferential displacements and is thus significantly more flexible than the clamped (C) condition, which permits the local buckling mode to develop at shorter lengths. The effect of this more relaxed boundary restraint has effectively vanished by $\omega = 2$, where the ‘local buckling hypothesis’ of $M_{cr} / M_{cl} \approx 1$ becomes a close representation of the behaviour. The decaying peaks thereafter at $\omega \approx 3.8, 7.5, 11$ and $14.5$ etc. relate to the changing relationship between the buckling mode and the cylinder length, important at first but increasingly weaker thereafter.

Fig. 4.5 – Linear buckling behaviour of perfect short cylinders under bending, with various $r/t$ ratios, simply-supported (S) condition and selected LBA modes.
Specifically, the dimensionless length $\omega$ may be expressed, in terms of axisymmetric buckle half-wavelength $\lambda_{cl}$ (Eq. (4.3)c), as:

$$\omega \approx 1.728 \left( \frac{L}{\lambda_{cl}} \right).$$

(4.8)

When $L$ is an integer multiple of $\lambda_{cl}$ it is exactly at the value required to support a kinematically admissible buckling mode. For this reason, when $L/\lambda_{cl} = 1, 3, 5$ and $7$ the resulting $\omega = 1.7, 5.2, 8.6$ and $12.1$ correspond very closely to the minima of the curves in Fig. 4.5 where $M_{cr} \approx M_{cl}$.

As established in the structural symmetry analysis presented in the preceding chapter (Chapter 3) of this thesis, the apparent restriction to odd integers is due to the condition of meridional symmetry at midspan, as may be compared with the eigenmodes from the full cylinder model in Fig. 4.5. Between these minima, the complementary effect is lost, and the optimal buckling half-wavelength is constrained by the cylinder geometry, requiring slightly higher moments $M_{cr} > M_{cl}$ to be overcome. This effect is also present for clamped cylinders but is much reduced due to the full rotational restraint and decays even faster.

Lastly, it is interesting to note from Fig. 4.5 that the computed moment $M_{cr}$ exhibits a minor dependency on the $r/t$ ratio despite the normalisation of cylinder length $L$ by $\sqrt{rt}$, particularly near the first and strongest peak at $\omega \approx 3.8$. This relates to a varying circumferential wave number $n$ in the critical eigenmode with the $r/t$ ratio (Fig. 4.6), even if the number of meridional
half-waves at $\omega \approx 3.8$ is always unity. From the equation of the Koiter (1945; 1963) circle expressed in terms of buckle wavelengths:

\[
\left(k - 0.909 \sqrt{\frac{r}{L}}\right)^2 + n^2 \approx 0.909 \frac{r}{L}^2 \quad \text{where} \quad k = m\pi \left(\frac{r}{L}\right),
\]

it may be shown that $n \approx 9, 13, 20$ and $28$ for $r/t = 100, 200, 500$ and $1000$ respectively when $m = 1$, corresponding closely to the equivalent circumferential wave number observed on the buckles forming in the compressive regions in Fig. 4.6 for both boundary conditions. The slight rise in $M_{cr}$ with decreasing $r/t$ is due to the increasing importance of transverse shear deformations in a thicker shell, captured by the finite element calculation but naturally not by the simple analytical expression for $M_{cl}$.

The relationship between the ratio of the computed critical moment $M_{cr}$ to the calculated buckling moment $M_{cl}$ and the dimensionless length $\omega$ of the clamped cylinder was previously characterised in an accurate but conservative manner by Rotter et al. (2014) using a simple power-law relation (Eq. (4.10)a) and a corresponding characterisation for the simply-supported end boundary condition is proposed (Eq. (4.10)b):

\[
\frac{M_{cr}}{M_{cl}} = 1 + \frac{4}{\omega^2} \quad \text{for Clamped (C)} \quad \text{(a)} \quad ; \quad \frac{M_{cr}}{M_{cl}} = 1 + \frac{1}{\omega^{2.5}} \quad \text{for Simply-supported (S)} \quad \text{(b)}
\]

Both expressions are lower bound fits to the computed data points, ignoring the waviness in the relationship due to the changing buckling mode with length. They replace the need for a computational analysis to obtain the linear buckling resistance of a cylinder under bending, one of the two reference resistances used in estimating the overall slenderness of a shell structure (EN 1993-1-6, 2007), permitting a simple hand calculation instead.

4.6 Geometrically nonlinear analyses of perfect short cylinders

4.6.1 Introduction

Geometrically nonlinear analyses (GNA) of perfect ‘quarter shell’ short cylinders were first performed in ABAQUS using the modified Riks algorithm (Riks, 1979), which automatically checks the global tangent stiffness matrix for negative eigenvalues after every load increment,
a numerical indicator of structural instability (Song et al., 2004). At the end of each computational analysis, the nonlinear buckling moment $M_k$ of the cylinder is taken as the maximum moment attained on the equilibrium path, corresponding to either a limit point or the first bifurcation and should also be the point at which negative eigenvalues are first detected in the global tangent stiffness matrix. Further information on the detection of the buckling moment is presented in the management strategy for the research work in Chapter 3 of this thesis. Although no explicit imperfection was defined at this stage, a tiny mesh perturbation in the shape of the LBA eigenmode with amplitude $10^{-3}$ of the wall thickness was included in some models to facilitate the correct identification of the buckling moment. While this is a common modelling procedure (e.g. Stephens et al., 1975; Tafreshi, 2002; Rotter et al., 2014), it is only meaningful when the elastic post-buckling mode is likely to be of a similar shape and triggered from the same location as the LBA eigenmode. This perturbation procedure is simply performed to avoid the danger that the GNA analysis will jump over the lowest bifurcation point and continue erroneously along the pre-buckling path. However, where the elastic post-buckling mode has a snap-through characteristic, the perturbation procedure is no longer necessary since the response does not need to switch to an alternative and distinct equilibrium path.

### 4.6.2 Predicted moment-curvature relationship for very short elastic cylinders under uniform bending

By contrast with the LBA calculation, where the development of a local buckle is merely delayed by the boundary restraint, it can be eliminated entirely under geometrically nonlinear conditions (Rotter, 2007) and short elastic cylinders are able to sustain applied loading at moments greatly in excess of the $M_{cl}$ or $M_{cr}$ predictions with no occurrence of local buckling (Fig. 4.7). After an initial linear load path, once $M_{cl}$ is exceeded the cylinder begins to develop an increasingly deep circumferential fold on the compressed meridian owing to the stress amplification of the compatibility boundary layer of meridional bending. This growing fold progressively reduces the stiffness of the cylinder against global bending until a limit point is reached, and snap-through buckling occurs. In shorter cylinders, the limit point is also eliminated and the elastic cylinder experiences indefinite geometric hardening. A deep circumferential fold deformation mode was hinted at in Fig. 16.9c of Calladine (1983) for short cylinders, but it does not yet seem to have been properly documented. Although both boundary conditions were found to exhibit the snap-through instability at short lengths, the clamped
condition offers a noticeably stiffer path and buckles at higher moments than the simply-supported one (e.g. of $M_k / M_{cl} = 2.75$ at $\omega = 3$ for clamped (C) versus 2.41 for simply-supported (S)), owing to the stiffer rotational restraint offered. It should be recognised that this geometric softening is completely unrelated to Brazier-type flattening of the cross-section with circumferential bending, which only occurs in long cylinders.

Buckling at such high moments requires the attainment of very high mean meridional curvatures, up to approximately 10 or 20 times $\phi_{cl}$ at buckling, and such deformations can only remain fully elastic for very thin shells. This is illustrated in Fig. 4.7 by the equilibrium paths from a geometrically and materially nonlinear analysis (GMNA) of clamped cylinders with $r/t = 100, 1000 \ & 3000$ and a yield stress $\sigma_y = 355$ MPa, where the full extent of geometric softening only becomes visible for $r/t$ in excess of 1000. The choice of the rather low $r/t$ ratio of 100 for this and other more detailed illustrations of short cylinder behaviour in this study should therefore be understood in the context of computational efficiency. This is because the element size was defined in all analyses as a fixed portion of the meridional bending half-wavelength $\lambda$ and thus, equivalently, of the cylinder length $\omega$ since the two are related by a...
constant. To keep the finite element mesh approximately square for numerical accuracy and stability it was necessary to maintain the same element size circumferentially, but this becomes an increasingly smaller portion of the circumference for large \(r/t\) leading to a significant increase in the necessary number of elements, proportional to \(r / \sqrt{(rt)} = \sqrt{(r/t)}\), which also grows with \(r/t\). The choice of \(r/t = 100\) should therefore be treated as one of convenience, though the illustrations are representative of all elastic thin shell behaviour in the 'short' domain.

4.6.3 Moment-length relationship of perfect short elastic cylinders under uniform bending

The relationship between the nonlinear elastic buckling moment \(M_k\), normalised by \(M_{cr}\), and the cylinder dimensionless length \(\omega\) in the range \(2 \leq \omega \leq 15\) for the quarter shell model is shown in Fig. 4.8 for both clamped and simply-supported cylinders and different \(r/t\) ratios. There is only a brief transition between the 'short' and 'medium' domains, characterised by limit point and conventional bifurcation behaviour respectively, falling at approximately \(\omega \approx 4.8\) and \(3.8\) for clamped and simply-supported conditions respectively (confirming Rotter et al. (2014) and improving on the afore-mentioned preliminary study on simply-supported cylinders).

Fig. 4.8 – Computed geometrically nonlinear buckling moment-length relationships for clamped (C) and simply supported (Simply-supported (S)) perfect cylinders having varying \(r/t\) ratios.
The absence of a rotational restraint in the latter permits the local buckling mode to persist for a greater range of lengths in short cylinders, which is consistent with previous findings. However, the adoption of a full shell model for both boundary conditions in Fig. 4.8 shows a marked variation in the predicted buckling moment for the ‘short’ cylinders and a potential shift in the proposed boundary between the ‘short’ and ‘medium’ domains originally established by Rotter et al. (2014). However, the dependency on the choice of finite element model in the predicted buckling strengths does not appear to persist beyond a cylinder length of $\omega = 5$ for both end support conditions. Further details on the relationship between the predicted buckling moments and the choice of finite element model (full or quarter-shell) will be presented shortly. A minor dependency on the $r/t$ ratio is also seen for the buckling moments of short cylinders but the domain boundaries are still very well defined in terms of $\omega$. Lastly, the absolute value of $M_k$ grows without bound with decreasing $\omega$, and the apparent tendency of $M_k/M_{cr}$ to zero as $\omega \to 0$ below approximately 3 is caused by the normalising $M_{cr}$ moment becoming similarly unbounded (see Fig. 4.4 and Fig. 4.5).

![Fig. 4.9 – Computed GNA load paths of selected quarter-shell clamped (C) and simply-supported (S) cylinders with $r/t = 100$, illustrating the transition between ‘short’ and ‘medium’ domains, where for Simply-supported (S) cylinders, as $\omega$ increases, the interaction of pre-buckling geometric softening decreases gradually.](image)
The transition from ‘short’ to ‘medium’ buckling behaviour is illustrated in detail in Fig. 4.9, for the quarter shell model, using two sets of equilibrium paths taken at \( \omega = 4.8 \) and 5.0 (clamped) and from \( \omega = 3.0 \) to 3.8 (simply-supported). A similar transition behaviour is demonstrated by the full shell model, although at different cylinder lengths. This transition appears to be very abrupt for clamped cylinders, where a change in length \( \omega \) of just 0.2 (e.g. 20 mm for \( r = 1000 \) mm and \( t = 10 \) mm) separates limit point buckling behaviour at moments and curvatures well in excess of \( M_{cl} \) and \( \varphi_{cl} \) after an extensive and slowly softening pre-buckling path from local buckling behaviour at moments and curvatures very close to \( M_{cl} \) and \( \varphi_{cl} \) with very little pre-buckling nonlinearity. For simply-supported cylinders, this transition is slightly more gradual, with a more obvious interaction starting for lengths \( \omega \) approximately above 3 between the geometric softening behaviour and the occurrence of the bifurcation triggering the local buckling mode, which occurs earlier as the length increases. For \( \omega \) approximately above 3.8, there is almost no pre-buckling geometric softening attributable to the ‘short’ cylinder folding mechanism.

### 4.6.4 Geometric hardening in shorter perfect cylinders under uniform bending

The behaviour in progressively shorter cylinders deserves closer examination. The transition from local buckles in medium-length cylinders to snap-through buckling in slightly shorter cylinders has been described above. By contrast, as the cylinder is made even shorter, the limit point behaviour progressively changes from the distinct peak seen in Fig. 4.9 for clamped (C) at \( \omega = 4.8 \) towards a smooth transition from geometric softening into geometric hardening at larger displacements (Rotter, 2007), regardless of the choice of finite element model employed. A safeguard against this indefinite stiffening behaviour, presented in Fig. 4.10, was advised in EN 1993-1-6 (2007) in the ‘buckling’ criterion C3 (Fig. 2.29), where only a limitation on the magnitude of displacements can be classed as a failure state. This phenomenon was found here in both clamped and simply-supported cylinders shorter than \( \omega = 2.6 \) and 1.4 respectively. Significant efforts were made to ensure that no bifurcation had been missed by changing both the element and the software. In ABAQUS, the 8-node quadratic S8R reduced-integration thick-shell element, the 8-node linear C3D8I solid continuum brick element with incompatible modes and the 20-node quadratic C3D20R reduced-integration brick element were all explored to ensure no bifurcation had been missed. An independent validation was also performed using the CVS9 shell element from the specialised ADAPTIC software (Izzuddin, 1991; Izzuddin & Liang, 2017) and the same nonlinear response is demonstrated when the locking-free shell
finite element CVS9 of ADAPTIC element library is employed, both for the distinct peak limit point behaviour at $\omega = 3$ and the indefinite geometric hardening when the length is made even shorter.

Further investigations were performed on the buckling response of these extremely short cylinders using ABAQUS solid continuum and shell elements. For the 3D stress elements (C3D8I and C3D20R), five elements were modelled through the thickness of the cylinder wall to enable an accurate assessment of the through-thickness variation of bending stresses, which were initially thought to be potentially necessary to guard against the observed ‘indefinite’ stiffening behaviour. The predictions of the full 3D stress distribution were compared across all elements, with results extracted at the midspan of the most compressed meridian and load levels of 2, 2.5 and 3$\times M_{cl}$ (Fig. 4.11).

Fig. 4.10 – Equilibrium paths of very short clamped (C) elastic cylinders under bending showing a distinct limit point (at $\omega = 3$) and the geometric hardening for shorter cylinders ($\omega = 2.4$) with $r/t = 100$.

The comparison primarily suggests that in an advanced stage on the equilibrium path, the solid continuum elements begin to develop a significant transverse stress component owing to the severe deformation encountered through the now explicitly modelled thickness. The shell element formulation naturally assumes a negligible transverse normal stress component at all times (Donnell, 1933; Sadowski & Rotter, 2013a; Rotter & Al-Lawati, 2016) and therefore maintained an absolute value of zero.
Fig. 4.11 – Nonlinear equilibrium paths of very short clamped elastic cylinders (ω = 2.4) under uniform bending and a through-thickness stress distribution comparison of three stress components extracted at midspan and along the most compressed meridian (i.e. y = L/2, θ = 0°).
Despite this growing discrepancy, the shell elements (S4R and S8R) appear capable of modelling much the same highly nonlinear response in nearly as accurate a manner as their solid continuum counterparts, and at a significantly reduced computational cost. Overall, this demonstrates that very short elastic cylinders may initially show some geometric softening but suffer no type of instability and go on to stiffen indefinitely.

4.6.5 Comparison of buckling response with full cylinder shell model

The nonlinear buckling behaviour of the perfect cylinder presented from 4.6.1 to 4.6.4 of the current chapter had been the buckling response from a quarter design shell model. To establish the extent to which this buckling response is representative of the full cylinder model, a separate validation study is conducted on the clamped (C) cylinders. The validation study computes the nonlinear buckling moments of the full cylinder models, following the methodology used for the quarter shell model and across the same cylinder length range. The resulting incremental buckling modes at carefully selected lengths are compared, as illustrated in Fig. 4.12.

![Diagram of buckling profile](image)

Fig. 4.12 – Schematics (not-to-scale) of the buckling profile of clamped (C) short elastic cylinder under bending (full and quarter design models) along the most compressed meridian. Here, F represents a full cylinder model while Q indicates a quarter model.
It is again revealed that by enforcing a meridional symmetry condition in the quarter shell models, the ‘short’ length domain persists over a much wider range of length since the symmetry boundary condition naturally introduces artificial restraint, which disallows the development of alternating half-waves, in the form of a complete sinusoid, as an admissible incremental buckling mode (Fig. 4.12). This difference in behaviour from the adopted design model is exhibited in the incremental buckling modes of short cylinder in the length range \( \omega = 4.2 – 4.8 \), where the variation in the buckling moment predictions by both models is by a factor of \( \sim 2 \). Within this length range, the quarter shell models predict a limit-point buckling response that is similar to the detrimental meridional fold mechanism, typical of ‘short’ cylinders under bending while the full cylinder model predicts a local buckling instability. Beyond \( \omega = 4.8 \) however, there is no visible difference in the response predicted by both models since failure through local buckling becomes a true representation of the elastic stability of the cylinder (Fig. 4.13).

![Fig. 4.13](image)

The same is observed in the simply-supported (Simply-supported (S)) cylinders, where the cylinder demonstrates significant variability between \( \omega = 3 – 5 \) for the same reason of the restriction in the admissible buckling modes, imposed by the symmetry boundary condition. The dependency on the symmetry condition becomes insignificant beyond a cylinder length of \( \omega = 5 \).
4.6.6 Algebraic characterisation of the nonlinear buckling behaviour of perfect short cylinders under uniform bending

The relationship between the normalised buckling moment $M_b/M_{cr}$ and cylinder length $\omega$ & $\Omega$ for fully restrained (clamped) perfect cylinder under uniform bending was characterised in Rotter et al., (2014) using a conservative lower bound approach. The fundamental essence of this algebraic characterisation is to act as a proxy for undertaking any onerous computational analysis when estimating the characteristic buckling strengths of such cylinder systems. However, the numerical model employed in Rotter et al., (2014) is only of the quarter shell design type and thus did not capture the afore-mentioned variation in buckling moments at lengths $\omega = 4.2 - 4.8$, arising from different buckling modes when compared with the full cylinder models. Nevertheless, two proposals are presented in the current chapter to predict the moment-length relationship of clamped (C) cylinders under bending but only within the ‘short’ and ‘medium’ length domains ($\omega \leq 15$). The first proposed characterisation (Eq. (4.11)) employs the same functional relationship presented in Rotter et al. (2014) but shifts the ‘short’ domain boundary to $\omega = 4$. Hence, the provision for the short cylinder remains functional when the cylinder length is in the range $\omega \leq 4$ but beyond this length value, the provision of the medium domain may be conservatively adopted.

Another algebraic characterisation is also presented in Eq. (4.12) for the clamped cylinders. This alternative algebraic characterisation predicts the moment-length relationship as demonstrated by the ‘full’ cylinder model and is not as conservative as the first proposal, although it comes with the complexity associated with more length sub-divisions. A corresponding characterisation of the buckling moment versus length relationship for rotationally unrestrained (Simply-supported (S)) perfect cylinders under uniform bending is also presented below, where in the first part of the equation (Eq. (4.13)a), a simple linear function fit with an intercept value of 0 is employed to describe the rising relationship between the buckling moment and length, in the cylinder length range of $2 \leq \omega \leq 3$:

$$\alpha_{C} = \begin{cases} 
1.93 - 0.5(\omega - 3.8)^2 - 0.44(\omega - 3.8)^3 & \text{for } 3 \leq \omega \leq 4 \\
0.85 + 0.029(\omega - 7.1)^2 & \text{for } 4 < \omega < 8.6 \quad \text{short & medium only} \\
0.92 & \text{for } 8.6 \leq \omega < 0.5(r/t)
\end{cases} \quad (4.11)$$
Beyond this limit, a conservative lower bound function (Eq. (4.13)b) is offered to predict the moment-length relationship in the remaining part of the short domain and the entirety of the medium domain accurately. This provision ignores the wavy nature of the predicted moment-length relationship when the cylinder length $\omega$ exceeds 7 (Fig. 4.13) since it plateaus the predicted buckling strength of the cylinder to a stable value of $M_k/M_{cr} = 0.865$ upon attaining $\omega = 7$.

### 4.7 Elastic imperfection sensitivity of short and medium cylinders

#### 4.7.1 Introduction

The response of medium-length cylinders under uniform axial compression is widely known to exhibit perhaps the most detrimental sensitivity to geometric imperfections owing to an axisymmetric pre-buckling stress state dominated by uniform membrane action (Koiter, 1945; Koiter, 1963; Lord et al., 1997; Hunt et al., 2003; Rotter, 2004). Since global bending exhibits only a ‘gentle’ circumferential variation in membrane stress that is close to uniform on the compressed side, it may be expected to develop buckles of similar wavelengths and thus exhibit a similar imperfection sensitivity in thin cylinders of sufficient length. The nature of the imperfection sensitivity of these short cylinders under uniform bending is investigated presently in detail using an extended set of GNIA analyses with the critical eigenmode and an additional practically realistic weld depression imperfection.
4.7.2 Individual imperfection sensitivity relationships

The behaviour is first illustrated on a plot of the elastic imperfection reduction factor $M_{k,GNIA} / M_{k,GNA}$ against the total deviation from the original perfect geometry $\delta_e/t$, shown in Fig. 4.14, for representative lengths of $\omega = 3$ (short) and 15 (medium) cylinders under both boundary conditions. The short cylinder displays a neutral sensitivity to eigenmode imperfections at small amplitudes ($\delta_e/t < \sim 0.5$), and for deeper amplitudes this sensitivity could become mildly detrimental, mildly beneficial, or in other cases remain neutral. This is because perfect short cylinders exhibit limit point buckling through meridional folding, and the imposition of a local buckling mode as an imperfection has little physical basis to be detrimental except perhaps by causing stress concentrations or decreasing the initial tangent stiffness.

![Fig. 4.14 – Imperfection sensitivity relationships for ‘short’ and ‘medium’ cylinders under global bending with $r/t = 100$ under eigenmode and weld depression imperfections.](image)

The effect may go either way, and is dependent on the degree of rotational restraint, with the clamped (C) condition eventually causing a strength rise and the simply-supported (S) condition permitting a strength drop. The converse is true for longer cylinders (e.g. $\omega = 15$), since the first linear eigenmode is now analogous to the post-buckling profile of the perfect...
shell and the imperfection sensitivity becomes correspondingly more damaging, though the decrease in buckling strength is only approximately 25% at high amplitudes.

The axisymmetric weld depression imperfection, by contrast, exhibits a more consistently detrimental effect for both cylinder lengths and boundary conditions, though the relationship does not necessarily decrease monotonically with amplitude. For $\omega = 3$, the rather wavy curve suggests only an approximately 13% drop in buckling moment at amplitudes as high as $\delta_{e}/t = 2$ for clamped (C), though for simply-supported (S) this rises to approximately 37%. For $\omega = 15$, after an initial steady decrease, the relationship begins to rise again after attaining a peak approximately 55% reduction at $\delta_{e}/t \approx 1$. At these lengths, neither imperfection form exhibits a sensitivity as severe as under uniform axial compression, for which both forms effectively give the same predictions when expressed in terms of $\delta_{e}$ (Rotter, 2013a). Prior to the current research study, similar effects have not yet been found in very short cylinders under axial compression (Rotter & Al-Lawati, 2016).

A detailed look into the equilibrium paths of short cylinders (e.g. $\omega = 3.6$) with the more severe weld depression imperfection reveals a gradual decrease in the initial tangent stiffness with increasing imperfection amplitude (Fig. 4.15), but no change in the nature of the overall limit point behaviour. Oddly, at small amplitudes of $\delta_{e}/t = 0.25$ and 0.35 the shell was found to undergo a very minor local bifurcation at a moment $M/M_{cl}$ of ~1.2 onto an adjacent equilibrium path whereby the load briefly dropped before resuming its ascent. In this way, deformations corresponding to a small local buckling mode were ‘locked in’ to the shape and became amplified higher up the path. This almost imperceptible local reversal in the load path causes no net reduction in stiffness and thereafter the load path rises without further incident to pass through a limit point. Further, as this anomalous local mode vanishes at deeper imperfection amplitudes, it was decided not to use this first bifurcation as a criterion of failure. Consequently, all critical buckling moments for short shells reported in what follows relate to the moment at the limit point.
Fig. 4.15 – Nonlinear GNIA load paths of perfect and imperfect ‘short’ cylinders with $\omega = 3.6$ and $r/t = 100$ with the weld depression imperfection.

At a ‘medium’ length of $\omega = 15$, it was found that the same moment-curvature response is followed regardless of the end support condition (Fig. 4.16). The eventual rise in strength shown in Fig. 4.14 can be related to a gradual disappearance of the primary bifurcation point with growing imperfection amplitude, and the equilibrium path eventually exhibits a smooth transition from pre- to post-buckling without any negative eigenvalues detected in the global stiffness matrix near the former bifurcation point. This type of stable post-buckling behaviour under deep amplitudes has been observed before in cylinders under various load cases (Yamaki, 1984; Rotter, 2007; Sadowski & Rotter, 2011a) and is a consequence of the growing buckling mode of the more imperfect cylinder now encroaching on the end boundary condition. More is presented on this anomalous phenomenon in Chapter 5 of the current PhD thesis. It thus raises the important question of what criterion of failure to choose when this is the case. The conservative choice is made here in such cases to report the buckling moment at the first point where the equilibrium path abruptly changes slope, even if the transition is stable, to avoid sharp discontinuities in the imperfection sensitivity relationship. At high amplitudes ($\delta_e/t = 2$), the cylinder experiences significant pre-buckling nonlinearity evidently caused by the deep imperfection altering the initial tangent stiffness. The challenges involved in the implementation of a ‘change of slope’ criterion (or a more general ‘kink detector’) are
discussed in detail in Sadowski et al. (2017a), where it is illustrated that systems with stable post-buckling behaviour may exhibit very smooth transitions that are difficult to detect.

**Fig. 4.16** – Nonlinear GNIA load paths of perfect and imperfect ‘medium’ cylinders with $\omega = 15$ and $r/t = 100$ with the weld depression imperfection.

### 4.7.3 Imperfection sensitivity over a wide length range

Detailed relationships between the normalised buckling moment $M_k / M_{cr}$ and the length $\omega$ under both boundary conditions are shown in Fig. 4.17 for the critical eigenmode and Fig. 4.18 for the weld depression imperfection forms respectively. These are presented here for completeness and to illustrate the complexity of attempting to establish a single representative imperfection sensitivity relationship even for such a seemingly simple geometry and load case. The behaviour of imperfect cylinders longer than $\omega = 15$ begins to be influenced by ovalisation, and the results for these are presented in Chapter 5 of the current thesis. The significant variability apparent in the relationships between $M_k / M_{cr}$ and $\omega$ for the eigenmode imperfection is caused by the changing number of buckling waves relative to the cylinder length in the critical mode, which defines the shape of the imperfection. Of the two boundary conditions considered currently, this is significantly more pronounced for the simply-supported (S)
condition (Fig. 4.5) than the clamped (C) condition (Fig. 4.4), and for this reason Fig. 4.17b exhibits a greater variability than Fig. 4.17a.

The eigenmode is therefore not a geometrically well-defined imperfection form, and importantly it does not exhibit the most detrimental imperfection sensitivity for any considered length except near the ‘short’ to ‘medium’ boundary. By contrast, the analytical formulation of the weld depression permits the profile of the imperfect geometry to remain constant at all lengths (Fig. 4.18), clearly resulting in a less ambiguous, more consistent and above all more conservative moment-length relationship upon which a lower-bound design characterisation should be established. Finally, the definition of the position of the ‘short’ to ‘medium’ boundary, proposed on the basis of perfect shell behaviour and the quarter shell model to occur at $\omega = 4.8$ and 3.8 for the clamped (C) by Rotter et al. (2014) and simply-supported (S) conditions respectively, is consistent with that used for axial compression and external pressure in EN 1993-1-6 (2007) and ECCS EDR5 (2013). Nevertheless, with the adoption of a full shell finite element model, the proposed ‘short’ domain boundary is seen to shift from $\omega = 4.8$ to 4.0 for the clamped (C) cylinders but stable at $\omega = 3.8$ for the simply-supported (S) cylinders. Furthermore, with the introduction of either imperfection, the initially proposed boundary becomes increasingly ill-defined. Although for the eigenmode imperfection under the clamped (C) condition this boundary is surprisingly stable at $\omega = 4.8$, for the simply-supported (S) condition it shifts from $\omega = 3.8$ to approximately 3.0. For the weld depression imperfection under the clamped (C) condition, there is an initial drop from $\omega = 4.8$ to approximately 4.0 at small amplitudes, but at deeper amplitudes the boundary is eliminated entirely due to a changing buckling mode. For the simply-supported (S) condition, by contrast, the boundary again shifts from $\omega = 3.8$ to approximately 3.0.

Care should therefore be taken when assuming a sudden rise in the buckling moment once the ‘short’ domain is attained, because the local buckling behaviour that characterises the ‘medium’ domain may persist for shorter lengths than Fig. 4.8 would suggest. Since all real shells are always at least slightly imperfect, it is strongly recommended to characterise the ‘short’ to ‘medium’ boundary as $\omega = 4$ and 3 instead for the Clamped (C) and Simply-supported (S) conditions respectively.
Fig. 4.17 – Computed relationships between $M_k/M_{cr}$ and $\omega$ for perfect and imperfect a) clamped (C) and b) simply-supported (S) cylinders, with $r/t = 100$ and eigenmode imperfections.
Fig. 4.18 – Computed relationships between $M_k/M_{cr}$ and $ω$ for perfect and a) clamped (C) and b) simply-supported (S) cylinders, with $r/t = 100$ and weld depression imperfections.

4.8 Conclusions

The current chapter has presented a thorough investigation of the behaviour of very short elastic cylinders subject to uniform bending accounting for geometric nonlinearity using two
alternative boundary conditions, two forms of geometric imperfections, and covering parametric variations of length and imperfection amplitude.

The response of very short cylinders under uniform bending to an enforced meridional symmetry, under geometrically linear conditions, shows variability from full cylinder model, which depends on the condition of the rotational restraint at the ends, although this variability does not persist beyond a cylinder length of $\omega = 10$. Under geometrically nonlinear conditions however, this variability is limited to cylinder length shorter than $\omega = 5$. Hence, a more computationally cheaper ‘quarter-shell’ cylinder model may be employed with great accuracy in investigating the buckling behaviour of ‘medium’ and longer length cylinders under global bending.

The linear buckling behaviour of very short cylinders under bending is strongly influenced by the restraint of the end boundary condition, which restricts the formation of a local buckling mode. For this reason, a finite element eigenvalue calculation assuming a full pre-buckling stress state attains higher moments than their respective analytical predictions, which assume a simple membrane, stress state and neglect the influence of boundary effects. Rotationally-free or ‘simply-supported’ boundary conditions naturally offer less restraint than rotationally-restrained or ‘clamped’ boundary conditions, and for these the boundary restraint decays faster with increasing length.

Under geometrically nonlinear conditions, medium-length cylinders under global bending fail by local buckling on the compressed side. However, for short lengths this mode is restrained and the shell exhibits snap-through buckling behaviour at moments and curvatures well in excess of the classical local buckling predictions through the development of a deep destabilising fold on the compressed meridian. In particularly short cylinders, this mechanism disappears, and the equilibrium path exhibits indefinite stiffening. The characteristic limit point behaviour is a geometrically nonlinear effect but is completely different from ‘Brazier’ ovalisation, which is entirely prevented at such short lengths.

An investigation into the nonlinear buckling behaviour of elastic cylinders with critical eigenmode and realistic manufacturing-related weld depression imperfections revealed a particularly complex variability in imperfection sensitivity. This is owing to a complex
interaction between the local bifurcation and global limit point buckling modes, depending on the length, the boundary conditions and the imperfection form itself. Deeper imperfections did not necessarily cause a monotonic reduction in buckling strength, though the weld depression consistently exhibited a more damaging influence than the eigenmode. At the lengths considered presently, the imperfection sensitivity under uniform bending is not as severe as under uniform axial compression, and for very short cylinders, the sensitivity is very mild indeed, and sometimes even beneficial. The imperfection sensitivity found in longer cylinders, which is strongly affected by ovalisation, is presented in the following chapter.

A key lesson that can be derived from this study is that, in the verification of a design, it can be dangerous to use nonlinear calculations of an imperfect structure (GMNIA) if the sensitivity to minor changes in the geometry or the form and amplitude of the imperfections has not been fully explored.
Chapter 5 - Imperfection sensitivity in elastic cylinders under uniform bending

5.1 General outline

The preceding chapter has shown that the response of perfect elastic cylinders under bending, which are short enough for any cross-sectional ovalisation phenomenon to be completely restrained by the end boundary condition, may not be well explainable by simple beam bending theory owing to the increasingly significant role played by pre-buckling bending deformations. In addition, a complex sensitivity to geometric imperfections is demonstrated by these short cylinders, although these cylinders are ultimately insensitive to geometric imperfections, for the same reason of dominant pre-buckling bending deformations. The current chapter extends on the above, according to the outline below, to longer cylinders such that the cross-sectional ovalisation phenomenon begins to play a marked role in the behaviour of the cylinder.

The applications of the structural system of cylinders under uniform bending are first highlighted and a brief overview on the mechanics governing the behaviour of these systems are introduced. The main objective of the current chapter is then pointed out together with the range of parameters investigated. Details of the computational methods employed, the three distinct forms of geometric imperfections, and the adjustment introduced to allow for a commensurate comparison of buckling behaviour based on these different imperfection forms are thereafter narrated. A comprehensive summary of all the computational analyses performed and the strategy employed to manage these analyses are subsequently presented. The elastic stability of the perfect and imperfect systems is first presented for a single instance of radius-to-thickness ratio $r/t = 100$, with the aid of predicted moment-curvature equilibrium curves for each of the imperfection forms employed, followed by an illustration of the imperfection sensitivity relationship at representative cylinder lengths. The influence of cylinder length and ovalisation on the predicted buckling moments of the imperfect systems is then explored across the ‘medium’, ‘transitional’ and ‘long’ domains to establish a definitive evidence for the length-dependent imperfection sensitivity demonstrated by these systems. The imperfection sensitivity relationship for the ‘transitional’ length cylinders over a range of $r/t$ ratios is also explored and presented in terms of modified but purely elastic capacity curves, to investigate the dependency of the cylinder buckling strength on its $r/t$ ratio. Finally, an attempt to establish
a suitable and governing dimensionless geometric group for imperfect and strongly ovalising cylinders under bending is narrated.

5.2 Introduction

Although analytical and computational studies investigating the particularly detrimental imperfection sensitivity of cylinders under uniform compression boast a history that extends back by almost a century (Koiter, 1945; 1963; Calladine, 1983; Rotter, 2004; Rotter & Al-Lawati, 2016), a similar effort has not yet been undertaken for imperfect cylinders under uniform bending. In the concept of stress design, it is often assumed that the imperfection sensitivity relationship for uniform bending may be taken to be the same as for uniform compression (EN 1993-1-6, 2007), a conservative choice but one for which no rigorous proof had long been forthcoming. An evidence of this may be found in the computational work of Chen et al. (2008) who conducted a limited parametric study on elastic-plastic clamped cylinders with realistic weld depression imperfections. There is also the work of Vasilikis et al. (2016), who validated a set of thirteen four-point bending tests of thick tubes with a selection of shell finite element models that included imperfections relevant to the spiral welding manufacturing process, including eigenmode dimples, girth weld misalignment and residual stresses. However, a systematic investigation and documentation of the sensitivity of general cylinders under bending to multiple imperfection forms across a wide range of parameters has never been performed.

5.3 Scope of the current study

The main objective of the current chapter is to present a comprehensive assessment of the sensitivity of elastic cylinders under uniform bending to three distinct forms of geometric imperfection. It was recently demonstrated by Rotter et al. (2014) that the cylinder length $L$ is responsible for controlling the extent of pre-buckling ovalisation of perfect elastic cylinders with rigidly-circular clamped ends. When transformed into dimensionless parameters $\omega$ or $\Omega$ (Eq. (5.1)), the length allows a categorisation of geometric nonlinearity for the perfect system into four distinct domains termed ‘short’, ‘medium’, ‘transitional’ and ‘long’ (Fig. 2.26) in a manner that is independent of the radius to thickness ($r/t$) ratio. This is in contrast with only three length domains that were defined for cylinders under uniform compression (Rotter, 2004), which do not include a ‘transitional’ domain as they do not undergo ovalisation. The afore-mentioned four length domain categorisations are retained herein for consistency.
The response of cylinders under the ‘short’ domain is the subject of the preceding chapter (Chapter 4) and was found to exhibit either negligible or indeed beneficial imperfection sensitivity, dependent on the form of imperfection, due to a pre-buckling stress state that is dominated by local compatibility bending with stable post-buckling behaviour. Consequently, cylinders shorter than $\omega = 5$ (approximately two linear bending half-wavelengths, $2\lambda$) are not considered further here. Furthermore, the elastic behaviour of ‘long’ cylinders with fully-developed ovalisation is effectively invariant with further increases in length (Rotter et al., 2014), and the upper limit of interest is set at $\Omega = 10$. These bounds span several orders of magnitude of cylinder lengths and encompass all known practical applications.

Three distinct forms of geometric imperfection with varying amplitude were investigated in this study. The first is the classical critical linear buckling eigenmode, a staple of imperfection sensitivity studies in shells since the first asymptotic analyses of Koiter (1945; 1963) and presented as a ‘default’ imperfection form in computational analyses by the Eurocode on metal shells EN 1993-1-6 (2007). The second is an imposed initial ovalisation (in the form of circumferential harmonic two, i.e. $\cos 2\theta$) to investigate the possible effect of uniform bending on an already slightly flattened cylinder (Sadowski & Rotter, 2013a). The third is the axisymmetric circumferential ‘weld depression’ of Rotter and Teng (1989), a realistic representation of a manufacturing-related defect that has been widely used in computational studies of imperfection sensitivity in cylinders (e.g. Song et al., 2004; Sadowski & Rotter, 2011a). In the current study, the influence of each form of geometric imperfections on the buckling behaviour of the cylinder has been studied individually to allow for a definitive understanding of the severity of each imperfection form, although a combination of these imperfection forms may be employed in varying proportion. More details of each imperfection form are presented shortly.

Since one of the main aims of the current research is to offer a comprehensive insight into the imperfection sensitivity of elastic cylinders under uniform bending, the findings have been contextualised within the framework of Reference Resistance Design (RRD), recently developed by Rotter (2016a; 2016b). This method of manual dimensioning of shells, which has
now been accepted as a method of design through an Amendment to EN 1993-1-6 (Rotter, 2013b), is based on resistances rather than working stresses. RRD employs a ‘capacity curve’ functional form (Fig. 5.1) which relates a shell’s dimensionless characteristic resistance $M_k / M_{pl}$, where $M_{pl}$ is the reference full plastic moment, to its dimensionless slenderness $\sqrt{M_{pl} / M_{cr}}$, where $M_{cr}$ is the reference elastic critical buckling moment. The continuous relationship is characterised by a set of dimensionless algebraic parameters, each accounting for a distinct physical phenomenon: geometric nonlinearity ($\alpha_G$), imperfection sensitivity ($\alpha_I$) and material nonlinearity ($\beta$, $\eta$, $\lambda_0$ and $\chi_h$). In addition to being a method of design, RRD offers a powerful research lens through which the nonlinearities governing a structural system may be established and understood in isolation.

A full characterisation of cylinders under bending within the RRD framework is part of a major ongoing research effort by the PhD candidate’s supervisor and research group that will ultimately allow an analyst to predict the nonlinear bending resistance of a cylinder accurately and conservatively without recourse to an onerous finite element analysis suggested in the concept of design by global LBA/MNA or GMNIA methodology (EN 1993-1-6, 2007). However, establishing each of these RRD parameters comes with the challenge that a vast number of parametric finite element analyses must be performed, interpreted and processed (Sadowski et al., 2017a). Consequently, the current chapter only attempts to offer numerical evidence for the elastic imperfection sensitivity demonstrated by these systems across a wide range of geometric parameters. An algebraic characterisation of the imperfection sensitivity

Fig. 5.1 – a) Generalised and b) Modified capacity curve functional form for cylinders under uniform bending (after Rotter, 2007). Only those nonlinearities relating to the elastic region (right-hand part of the curve, shown as a thick line) are considered in this study.
parameter ($\alpha$), following the successful characterisation of geometric nonlinearity parameter ($\alpha_G$) by Rotter et al. (2014), is the focus of the next chapter (i.e. Chapter 6) of the current PhD thesis. The plasticity-related parameters $\beta$, $\lambda_0$, $\eta$ and $\chi_h$ are outside the scope of this PhD research.

5.4 Computational methodology

5.4.1 Finite element model

Using the commercial ABAQUS v. 6.14-2 (2014) finite element software, each individual finite element model followed the quarter-shell design illustrated in Fig. 3.9. This approach follows the recommendations from the preliminary structural symmetry analysis presented in section 3.3, where it was shown that for GNAs of elastic cylinders within the confines of geometric properties employed herein, the quarter cylinder model is the most efficient computationally, which can correctly capture the behaviour in medium length cylinders and longer ones. A moment of magnitude equal to the predicted buckling moment under the local buckling hypothesis of Axelrad (1965; 1985), and otherwise termed the reference elastic critical buckling resistance $M_{cl}$ (Eq. (5.2)), was applied. The reference node, which is the point of application of the bending moment, was linked kinematically to the displacement and rotational degrees of freedom at the ends of the cylinder. The loaded edge was maintained rigidly circular throughout the analysis but was allowed to rotate and displace in the meridional direction. Material properties for isotropic steel were assumed with a Young’s modulus $E$ of 200 GPa and a Poisson ratio $\nu$ of 0.3, although the analyses are linear in $E$ and may be extended to any isotropic elastic material.

\[
M_{cl} = \frac{\pi}{\sqrt{\frac{3(1-\nu^2)}}} Er^2 \approx 1.901Er^2 \quad \text{for } \nu = 0.3
\]  

(5.2)

It was shown in chapter 4 that the degree of restraint of the rotations in the meridional direction about the circumferential loaded edge had a significant effect on the elastic buckling behaviour within the ‘short’ and the initial portion of the ‘medium’ length domains (Fig. 4.9), with an unrestrained condition leading to lower buckling moments across a wider range of lengths. Consequently, the present study of imperfection sensitivity also investigated the influence of allowing the rotational degrees of freedom at the loaded edge to be either restrained or
unrestrained, respectively representing Clamped (C) and Simply-Supported (S) boundary conditions (EN 1993-1-6, 2007).

The robust, general-purpose 4-node doubly curved shell element with reduced integration S4R was used in every model. The mesh resolution was refined significantly in the vicinity of the loaded edge to model the high local compatibility bending deformations correctly. Similarly, the mesh was also refined at the midspan where both eigenmode and weld depression imperfections were situated, and local buckling was anticipated. In both of these regions, the mesh was assigned a meridional density of 20 or 10 elements per linear bending half-wavelength $\lambda$ (Eq. (5.3)) for the ‘medium’ or ‘transitional and long’ domain respectively, such that the element length was approximately $0.125\sqrt{rt}$ or $0.25\sqrt{rt}$ respectively. Outside these zones of refinement, the cylinder is expected to be predominantly under membrane action and a coarser mesh resolution each with an element size of $1\sqrt{rt}$ or approximately 2.5 elements per $\lambda$, was applied for efficiency.

$$\lambda = \frac{\pi\sqrt{rt}}{\sqrt{3(1-\nu^2)}} \approx 2.444\sqrt{rt} \quad \text{for} \quad \nu = 0.3$$

(5.3)

### 5.4.2 Geometric imperfection forms

- **Critical linear bifurcation eigenmode imperfection**

The widespread adoption of linear eigenmode imperfections defined by mathematically-convenient trigonometric functions originates from Koiter’s (1945; 1963) perturbation analyses, the first studies to provide a reasonably accurate theory explaining the long-standing discrepancies between theoretical buckling stress predictions and experimental buckling loads of cylinders under uniform axial compression (Bushnell, 1985; Rotter, 2004). It was long adopted thereafter that imperfections in the form of the critical linear eigenmode of the perfect shell, and axisymmetric eigenmodes in particular, caused the greatest reduction in buckling strength in cylinders under uniform axial compression (Rotter, 2004), coincidentally also arguably the most common and imperfection-sensitive shell system. Consequently, the eigenmode-affine pattern is prescribed as the ‘default’ imperfection form for the computational analysis of shells of any geometry and loading by EN 1993-1-6 (2007) where no other unfavourable form can be justified. For cylinders under uniform bending in the ‘medium’ domain or longer, the critical linear eigenmode computed by a linear bifurcation analysis
(LBA) exhibits a series of closely-spaced local axial compression buckles on the compressed meridian (Fig. 5.2a; Rotter et al., (2014)) forming at a moment very close to the \( M_{cl} \) prediction (Eq. (5.2)). The finite element (FE) suite ABAQUS conveniently permits this pre-computed geometry to be scaled to an appropriate amplitude \( \delta_0 \) and imported as a mesh imperfection into a subsequent geometrically nonlinear analysis.

- **Imposed ovalisation imperfection**

Cross-section ovalisation in long cylindrical elastic shells under bending is known to reduce the theoretical nonlinear buckling resistance of a cylinder by almost a half (Brazier, 1927; Tatting et al., 1997; Karamanos, 2002; Li & Kettle, 2002). However, ovalisation was shown to be preventable if the dimensionless length of the cylinder falls below \( \Omega = 0.5 \) (Eq. (5.1); Fig. 2.26; Calladine, (1983); Rotter et al., (2014)) because the rigid-circular boundary condition at the ends of the cylinder effectively restrains flattening of the cross-section at midspan. Given the potentially severe penalty to the bending resistance caused by this phenomenon, it is explored here if the bending of an already slightly ovalised cylinder causes large buckling strength reductions to initiate at lengths shorter than \( \Omega = 0.5 \). The functional form defined in Eq. (5.4) (Sadowski & Rotter, 2013a) is used to generate the imperfect geometry explicitly. The single meridional half-wave ensures the cylinder remains circular at the edges \( (z = 0, L) \) but exhibits a circumferential ovalising harmonic two at midspan \( (z = L/2) \) of maximum amplitude \( \delta_0 \) (Fig. 5.2b).

\[
\delta = \delta_0 \cdot \sin \left( \frac{\pi z}{L} \right) \cdot \cos(2\theta)
\]  

(5.4)

- **Axisymmetric circumferential weld depression imperfection**

Uniform bending induces a sinusoidally-varying meridional membrane stress state in the form of circumferential harmonic one (i.e. \( \cos 1.\theta \)), but the region of buckling-inducing membrane compression is wide and smooth enough to produce conditions approaching that of uniform compression. Consequently, it may be expected that an axisymmetric imperfection, so deleterious for uniform compression (Hutchinson & Koiter, 1970; Rotter, 2004), may be similarly deleterious for uniform bending. For this reason, the ‘Type A’ axisymmetric circumferential weld depression imperfection of Rotter and Teng (1989) was adopted (Eq. (5.5)), a realistic representation of common manufacturing defects found in civil engineering shells (Berry et al., 2000; Pircher et al., 2001), with a single instance placed at midspan (Fig. 5.2c).
\[ \delta = \delta_0 \cdot \exp \left( -\frac{\pi}{\lambda} \left| z - \frac{L}{2} \right| \right) \cdot \left\{ \cos \left( \frac{\pi}{\lambda} \left| z - \frac{L}{2} \right| \right) + \sin \left( \frac{\pi}{\lambda} \left| z - \frac{L}{2} \right| \right) \right\} \]  

(5.5)

- **Definition of an ‘equivalent geometric deviation’ \( \delta_e \)**

To allow for a commensurate comparison of buckling sensitivity to varying imperfections, the imperfection amplitudes \( \delta_0 \) of all three imperfection forms were reformulated in terms of an ‘equivalent geometric deviation’ parameter \( \delta_e \) (Eq. (5.6)) more akin to a tolerance measure. This is defined here in terms of the maximum variation from peak to trough of the imperfection form (Fig. 5.2), or the distance from the most inward to most outward radial position along the most compressed meridian. This adjustment accounts for the fact that, for example, though the amplitude of a sinusoidal wave is defined mathematically as \( \delta_0 \) (Eq. (5.4)), it is in fact only half of amplitude of the total geometrical deviation that the shell is subject to (Rotter, 2004).

\[ \delta_e = \begin{cases} 
2\delta_0 & \text{(eigenmode)} \\
\delta_0 & \text{(imposed ovalisation)} \\
1.04\delta_0 & \text{(weld depression)}
\end{cases} \]  

(5.6)

Fig. 5.2 – Illustration of the geometric imperfection forms employed and the ‘equivalent geometric deviation’ \( \delta_e \) considered along the most compressed meridian.

Since meridional compressive stresses are expected to develop in the cylinder system of interest, the ‘equivalent geometric deviation’ parameter \( \delta_e \) is made to relate directly with the
depth of initial dimple $\Delta w_0$, which is described in Fig. 8.4 of EN 1993-1-6 (2007) and can simply be measured on the meridian or first measured on the meridian across the weld (in the case of the weld depression imperfection form) as illustrated in Fig. 2.19c.

5.4.3 Summary of parameter studies and automation of computational analyses

A linear elastic bifurcation analysis (LBA) is performed in ABAQUS as a matrix eigenvalue calculation employing a full 3D shell theory with both bending and membrane action. It is used to compute the critical linear bifurcation moment $M_{LBA} = M_{cr}$ of the perfect shell and to generate a file containing the scalable geometry of the eigenmode imperfection forms. A geometrically nonlinear analysis of the perfect (GNA) or imperfect (GNIA) shell may be performed as an equilibrium path-tracing analysis in ABAQUS with the modified Riks (1979) arc-length algorithm. It identifies the nonlinear characteristic buckling load $M_{GNA}$ or $M_{GNIA} = M_k$ and the corresponding incremental buckling mode. The elastic stability of the system is observed by means of the equilibrium path followed, which is a plot of the applied end moment $M$ against the end rotation $UR_y$ (Fig. 3.9).

The current study is divided into two parts. The first part presents the buckling moment predictions for perfect and imperfect elastic cylinders of varying length ($\omega$ and $\Omega$; Eq. (5.1)), end rotational restraint conditions (Clamped (C) and Simply-Supported (S)), imperfection form and imperfection amplitude. As the dimensionless lengths $\omega$ and $\Omega$ were found by Rotter et al. (2014) to permit the moment-length relationship (Fig. 2.26) for perfect cylinders to be represented independently of the $r/t$ ratio, a single representative value of $r/t = 100$ was employed at the initial stages of this study. For clamped cylinders, 32 lengths were investigated between $\omega = 5$ and 50 (i.e. $\omega = 50$ is $\Omega = 0.5$ if $r/t = 100$) within the ‘medium’ domain, with a higher resolution of lengths between $\omega = 5$ and 15 to explore the waviness of the relationship in this region (Fig. 2.26). For simply-supported cylinders, 37 lengths were investigated between $\omega = 4$ and 50, since the ‘medium’ length domain has been shown in Chapter 4 to initiate at shorter lengths for this end boundary condition. In the ‘transitional’ and ‘long’ domains governed by $\Omega$, the lengths were varied from $\Omega = 1$ to 10 in steps of 0.5 to a total of 16. For every combination of length, boundary condition and imperfection form, 8 normalised equivalent geometric deviations $\delta e/t$ of 0.1, 0.25, 0.35, 0.5, 0.75, 1.0, 1.5 and 2.0 were investigated with GNIAAs, although LBAs and GNAs were also performed on the perfect shell at each combination of length and boundary condition. Normalised equivalent geometric
deviations $\delta_e/t$ deeper than 2.0 are purposely omitted in the investigation for the imposed ovalisation imperfection form as such geometries are closer to elliptical cylinders than circular ones and were the focus of another recent dedicated study by Xu et al. (2017). A full summary of the individual computational analyses is presented in Table 5.1.

Table 5.1 – Balance of analyses for the first part of the study ($r/t = 100$)

<table>
<thead>
<tr>
<th>Boundary condition</th>
<th>Lengths $\omega$</th>
<th>Lengths $\Omega$</th>
<th>Imperfection forms</th>
<th>Amplitudes $\delta_e$</th>
<th>LBAs</th>
<th>GNAs</th>
<th>GNIAs</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C)</td>
<td>32</td>
<td>16</td>
<td>3</td>
<td>8</td>
<td>32+16</td>
<td>32+16</td>
<td>$(32+16)\times 8 \times 3$</td>
</tr>
<tr>
<td>(S)</td>
<td>37</td>
<td>16</td>
<td>3</td>
<td>8</td>
<td>37+16</td>
<td>37+16</td>
<td>$(37+16)\times 8 \times 3$</td>
</tr>
<tr>
<td><strong>Total:</strong></td>
<td><strong>101</strong></td>
<td><strong>101</strong></td>
<td><strong>2,424</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In the second part of this investigation, the extent to which the findings from the first part may be considered to be independent of the $r/t$ ratio are explored through an additional set of GNIAs performed at varying $\Omega$ (i.e. only in the ‘transitional’ length domain), $\delta_e/t$, $r/t$ and by considering the critical eigenmode and weld depression imperfection forms, as summarised in Table 5.2. Since the focus in this second part of the study is on ‘transitional’ domain cylinders that were shown in Fig. 2.26 to be unaffected by the nature of the end boundary condition, only the Clamped (C) boundary condition was used.

Table 5.2 – Balance of analyses for the second part of the study (Clamped cylinders)

<table>
<thead>
<tr>
<th>Constant</th>
<th>Lengths $\Omega$</th>
<th>Ratios $r/t$</th>
<th>Imperfection forms</th>
<th>Amplitudes $\delta_e$</th>
<th>GNIAs $14 \times 34 \times 2 \times 8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta_e/t$</td>
<td>14</td>
<td>34</td>
<td>2</td>
<td>8</td>
<td><strong>7,616</strong></td>
</tr>
<tr>
<td><strong>Total:</strong></td>
<td><strong>7,616</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The total number of individual analyses summarised in Table 5.1 and Table 5.2 is 10,040, an overwhelming number for an analyst to process individually if model generation, submission, termination and processing were to be performed manually. Consequently, the analysis management strategy described in Chapter 3 is employed, which exploits the Python and FORTRAN programming languages interfacing with ABAQUS to automate each of the above operations. Key to the strategy is enforcing a set of ‘kill conditions’, any one of which automatically terminates an ongoing GNA or GNIA when triggered, the most important ones being:
• Terminate analysis when the current increment’s load proportionality factor is less than that of the previous increment. This kill condition detects bifurcation or limit point buckling with unstable post-buckling equilibrium paths.

• Terminate analysis when the absolute relative difference between the current and previous incremental radii of curvature of the equilibrium curve exceeds a tolerance.

The latter kill condition was found to be particularly valuable since it terminates an ongoing analysis immediately as soon as a ‘kink’ is detected on the equilibrium path. It was shown in Chapter 4 that a deeper imperfection amplitude indeed turns a bifurcation on the equilibrium path into a ‘kink’ with a smooth transition from pre- to post-buckling, but without any loss of positive-definiteness in the global tangent stiffness matrix. The latter kill condition, otherwise referred to as the ‘kink detector’ is specifically designed to detect this ‘kink’ and should not be understood as a detector of the inflection point, which is signalled by abrupt change in the curvature (or second derivative) of the equilibrium path.

5.5 Individual predicted moment-curvature relationships

The elastic buckling behaviour of perfect cylinders in the ‘medium’, ‘transitional’ and ‘long’ length domains is introduced here with the aid of predicted moment-curvature equilibrium curves (Fig. 5.3). The mean curvature \( \varphi \) over the full length of the cylinder was obtained from the computed rotation \( UR_y \) (Fig. 3.9) of the rigid circular cross-section at the ends (Eq. (5.7)a).
It was subsequently normalised by the buckling curvature \( \varphi_{cl} \) (Eq. (5.7)b) from beam theory:

\[
\varphi = \frac{2 \cdot UR_y}{L} \quad \text{(a)}; \quad \varphi_{cl} = \frac{t}{r^2 \sqrt{3(1-\nu^2)}} \approx 0.605 \frac{t}{r^2} \text{ for steel with } \nu = 0.3 \quad \text{(b)} \quad (5.7)
\]

In the ‘medium’ length domain, the effect of employing a weaker rotational restraint at the edges is manifest as a minor reduction in the buckling moment from \( \sim 0.92M_{cl} \) to \( \sim 0.90M_{cl} \) for the clamped and simply-supported conditions respectively (Fig. 2.26 & Fig. 5.3). The slightly lower predictions for the simply-supported condition are due to the buckle forming near the loaded edge rather than at midspan as for the clamped condition (Fig. 5.4), a simultaneous consequence of a locally-perturbed membrane stress state and a lack of rotational restraint within the bending boundary layer at the cylinder edges. It should be recognised that realistic
practical boundary conditions lie somewhere between the limiting cases of Clamped (C) and Simply-Supported (S), both of which anyway predict very similar behaviour for perfect cylinders in the ‘medium’ length domain (Fig. 5.3a), and that a finite rotational stiffness restraint would not lead to different behaviour.

Fig. 5.3 – Predicted moment-curvature relationships for perfect cylinders under bending in three length domains and under two sets of end restraint conditions ($r/t = 100$).

Fig. 5.4 – Incremental buckling modes for cylinders of ‘medium’ length ($\omega = 50$) showing the different location of buckles under a) clamped (C) and b) simply-supported (S) end support conditions.

Similarly, for the predictions of imperfect cylinders that follow, these two boundary conditions should equally be seen as limiting cases with the ‘real’ condition falling somewhere in-between. Within the ‘transitional’ domain, the influence of varying rotational restraint
condition at the edges has no further influence at all on either the equilibrium path, the critical buckling moment or the buckling location (Fig. 5.3b). Instead, as pre-buckling ovalisation becomes progressively more severe with length, the moment-curvature relationship becomes increasingly nonlinear and buckling occurs at midspan at lower moments (Fig. 5.3c), with ovalisation eventually reaching a fully-developed state within the ‘long’ domain.

The predicted elastic buckling resistance of cylinders of ‘medium’ length appears to be very dependent on the form and amplitude of the applied initial geometric imperfections (Fig. 5.5). For example, at very small amplitudes ($\delta_e/t < 0.50$), clamped cylinders appear to be most susceptible to eigenmode imperfections, while the imposed ovalisation imperfection has a neutral effect on the predicted buckling strength. However, at deeper imperfections ($\delta_e/t \geq 0.5$), the weld depression becomes the most deleterious and reduces the buckling strength by up to 65% at $\delta_e/t = 1.5$, while the imposed ovalisation imperfection becomes either only mildly detrimental or remains neutral.

![Diagram](image)

Fig. 5.5 – Normalised moment-curvature relationships for increasingly imperfect cylinders of ‘medium’ length ($\omega = 50$ or $\Omega = 0.5$) under the Clamped (C) end support condition with the bifurcation points marker-coded as: diamond – perfect cylinder, square – eigenmode, triangle – imposed ovalisation and circle – weld depression.

In addition, from $\delta_e/t \geq 1.0$, the equilibrium path of the cylinder under the weld depression gradually changes from one exhibiting obvious bifurcation buckling with a steeply-descending post-buckling path (red curve in Fig. 5.5) to one exhibiting only a ‘kink’ with a smooth transition from pre- to post-buckling and a corresponding growth in buckling deformations,
with no negative eigenvalues detected in the global tangent stiffness matrix by the solver at any point. In such cases the buckling moment was conservatively taken as that corresponding to the ‘kink’ as advised by EN 1993-1-6 (2007), automatable using the second of the aforementioned ‘kill conditions’.

In the ‘transitional’ length domain, there is a complex interaction between local buckling, cross-sectional ovalisation and imperfections potentially resulting in a combined loss of up to 75% of the theoretical buckling strength $M_{cl}$ of the cylinder (Fig. 5.6). While this imperfect behaviour is again greatly dependent on both the form and amplitude of the imperfections, the weld depression imperfection consistently controls as the most deleterious imperfection. The imposed ovalisation imperfection has an almost entirely neutral effect in this length domain, suggesting that initially slightly ovalised cylinders are not especially vulnerable to reductions in buckling load arising from further ovalisation under bending than initially perfect cylinders.

![Diagram showing normalised moment-curvature relationships for imperfect cylinders of ‘transitional’ length ($\omega = 150$ or $\Omega = 1.5$) for both end boundary conditions with the bifurcation points marker-coded as: diamond – perfect cylinder, square – eigenmode, triangle – imposed ovalisation and circle – weld depression.](image)

Fig. 5.6 – Normalised moment-curvature relationships for imperfect cylinders of ‘transitional’ length ($\omega = 150$ or $\Omega = 1.5$) for both end boundary conditions with the bifurcation points marker-coded as: diamond – perfect cylinder, square – eigenmode, triangle – imposed ovalisation and circle – weld depression.

5.6 Influence of cylinder length on imperfection sensitivity

Classical imperfection sensitivity relationships for selected lengths across three length domains and for the three different imperfection forms are illustrated in Fig. 5.7 as plots of the ratio of the buckling strength of the imperfect cylinder to that of the perfect cylinder $\alpha_I = \frac{M_{GNIA}}{M_{GNA}}$.
against the equivalent geometric deviation $\delta e/t$. These results suggest that the longest cylinders still in the ‘medium’ length domain (i.e. as $\omega \to 50$ or $\Omega \to 0.5$) demonstrate the most severe sensitivity to any of the three considered imperfection forms. At the ‘medium’ to ‘transitional’ domain boundary, the pre-buckling behaviour is dominated by smooth and near-uniform membrane action with little geometric nonlinearity, which is characteristic of other well-known systems with severe imperfection sensitivity such as cylinders under uniform compression or spherical shells under uniform external pressure (Thompson & Hunt, 1973; 1984).

Cylinders shorter than $\omega = 50$ but still in the ‘medium’ domain become increasingly constrained by the edge boundary condition where compatibility bending has a greater contribution to the pre-buckling stress state, manifest as a gradual mollifying of imperfection sensitivity as $\omega \to 0$. None of these computed relationships suggest as severe an imperfection sensitivity as for cylinders under uniform compression, as illustrated through the comparison in Fig. 5.7 with well-known relationships from Koiter (1945) and Rotter & Teng (1989). Also shown is a comparison with the computational study by Chen et al. (2008), who proposed the first known relationship of $\alpha_I$ for cylinders under uniform bending with the same weld depression imperfections. Quite fortuitously, the imperfection sensitivity relationship proposed
by these authors appears to correspond to the worst possible imperfection sensitivity for this system, despite being established on the basis of a single dimensionless length of \( L/r = 7 \).

The imposed ovalisation imperfection was found to have a consistently neutral effect, with a predicted reduction in buckling moment of only \( \sim 10\% \) for \( \delta_0/t = 2 \) at the ‘most severe’ length of \( \omega = 50 \) and with the sensitivity to this imperfection form almost vanishing for longer ovalising cylinders. An apparent exception may be seen for cylinders on the short side of the ‘medium’ length domain (\( \omega \approx 8 \)), where this imperfection is seen to cause up to approximately \( \sim 40\% \) reduction in the buckling moment for \( \delta_0/t = 2 \). However, while the predictions for such short cylinders are presented here for completeness, it should be stressed that such cylinders are unlikely to be subject to such artificially-imposed ovalisation imperfections in practice as the end boundary conditions, which maintain circularity of the cross-section, would be very effective in restraining such deformations. Consequently, these predictions are not considered further.

The complete computed relationships between the normalised buckling moment \( M_k / M_{cr} \) and the dimensionless cylinder length \( \omega \) (log scale) are presented in Fig. 5.8 – Fig. 5.10 for the eigenmode, imposed ovalisation and weld depression imperfections respectively for both Clamped (C) and Simply-supported (S) sets of boundary conditions. Data from these figures at constant cylinder length \( \omega \) used to form the individual imperfection sensitivity relationships shown in Fig. 5.7 are identified by a dotted oval. A closer inspection of the moment-length relationships for the eigenmode and weld depression imperfections (Fig. 5.8 & Fig. 5.10) confirms that the imperfection sensitivity becomes increasingly severe with increasing length within ‘medium’ length domain, reaching a maximally detrimental effect at the boundary of the ‘medium’ to ‘transitional’ domains (\( \omega \approx 50 \) or \( \Omega \approx 0.5 \)), and then becoming milder with increasing length in the ‘transitional’ length domain as pre-buckling ovalisation becomes more important. All three imperfection forms tend to an ‘asymptotic’ and relatively mild imperfection sensitivity relationship within the ‘long’ length domain which is invariant with further changes in length. Here, ovalisation is fully-developed and is the main mechanism responsible for the significant reduction in the stiffness of the cylinder’s fundamental response.
Fig. 5.8 – Computed relationships between $M_k / M_{cr}$ and $\omega$ for elastic cylinders under uniform bending and the critical linear buckling eigenmode imperfection form.
Fig. 5.9 – Computed relationships between $M_k / M_{cr}$ and $\omega$ for elastic cylinders under uniform bending and the imposed ovalisation imperfection form.
Fig. 5.10 – Computed relationships between $M_k / M_{cr}$ and $\omega$ for elastic cylinders under uniform bending and the axisymmetric circumferential weld depression imperfection form.

Although the response does not strictly pass a limit point due to local buckling always occurring on the flattened side at a critical moment approximately 5% below the predicted limit point
moment (Karamanos, 2002; Xu et al., 2017), the system essentially behaves like a limit point one with imposed imperfections having only a modestly deleterious influence. Similar behaviour is documented in other systems with a highly nonlinear fundamental path that similarly exhibit a milder imperfection sensitivity than those with a linear fundamental path (Thompson & Hunt, 1973; 1984). The relationship between the moment and length is significantly smoother and better defined for the weld depression than for the eigenmode imperfection. This is because the geometry of the weld depression follows a strict mathematical definition that is invariant with length (Eq. (5.5)), whereas the shape of the eigenmode imperfection is computed anew for each length from an LBA and the imperfection is thus slightly different every time.

More importantly, the weld depression appears to almost always be the most severe imperfection form across all lengths, amplitudes and boundary conditions. This may be attributed to the fact that the region of pre-buckling membrane compression is both smooth and wide enough circumferentially to approach conditions that are approximately uniform, and under uniform compression it is well documented that axisymmetric imperfection forms are the most damaging (Hutchinson & Koiter, 1970; Rotter, 2004). The shape of the critical eigenmode is always localised in nature with a high circumferential wave number (Fig. 5.2), thus it cannot be as detrimental in a region of smooth and near-uniform membrane compression as the weld depression, which is axisymmetric by design.

For increasingly shorter cylinders, the eigenmode and, in particular, the weld depression imperfections may exhibit a non-decreasing relationship with increasing imperfection amplitude $\delta_e/t$ where, in effect, a deeper imperfection causes a rise in buckling strength. This stiffening effect may be substantial, causing rises in buckling strength to well above that of the perfect cylinder, such that $\alpha > 1$. This is because a deeper imperfection causes the buckling mode to involve more of the length of the shell (shown in Fig. 8 of Rotter & Teng (1989)), and for these rather short cylinders, the buckling mode eventually begins to encroach upon and become constrained by the edge boundary conditions, as illustrated in Fig. 5.11.

The phenomenon of deeper imperfections causing a rise in buckling strength has been documented before in computational studies of imperfection sensitivity in cylinders under unsymmetrical loading conditions (Sadowski & Rotter, 2011c; 2013b) and poses a problem in
establishing codified imperfection sensitivity relationships for use in design. The Eurocode on Metal Shells EN 1993-1-6 (2007) provides a safeguard against this eventuality for designers using the ‘GMNIA’ procedure to design a shell by requiring the analysis to investigate an imperfection amplitude 10% smaller than the codified value. Where this second analysis is found to predict a lower buckling strength, the analyst is obliged to adopt an iterative procedure and effectively reproduce full imperfection sensitivity relationship to establish the minimum. A strategy that does not burden the analyst with this onerous procedure whilst leading to a conservative codified relationship for $\alpha_I$ for use in RRD is presented in the following chapter (Chapter 6) of this PhD thesis.

![Selected incremental buckling modes of quite short cylinders still in the ‘medium’ length domain ($\omega = 15$) with the weld depression imperfection form showing the growing size of the buckle with imperfection amplitude $\delta_e/t$ to the extent that it interferes with and is constrained by the end boundary conditions.](image)

### 5.7 Influence of cylinder $r/t$ ratio on imperfection sensitivity

The preceding section presented detailed relationships between the predicted buckling moment and the cylinder length at a single $r/t$ ratio of 100, on the hypothesis that formulating these results in terms of the dimensionless length variables $\omega$ (and $\Omega$) allows the behaviour to be expressed independently of the $r/t$ ratio. By way of verification, an additional set of GNIAs for varying $r/t$ from 100 to 500 were performed within the ‘transitional’ length domain for $\Omega$ from 0.5 to 7, as summarised in Table 5.2. It was sufficient to restrict the current analysis to this length domain only because the behaviour here was independent of the boundary condition and because its shortest portion (i.e. at $\Omega = 0.5$) contained the length region with the ‘most severe’ imperfection sensitivity as identified previously. The critical buckling moment predictions are presented in Fig. 5.12 in the form of ‘modified’ capacity curves of $M_k / M_{pl}$ vs $M_k / M_{cr}$, a
construct designed to allow a convenient extraction of the RRD algebraic parameters, though in the absence of plasticity it is possible to establish only $\alpha = a_G \times a_I$.

Assuming that the correct dimensionless group governing the geometrically nonlinear buckling behaviour has been identified (i.e. $\Omega$), computed capacity curves should appear as vertical lines in this space (Fig. 5.1b) indicating an invariant relationship between $M_k / M_{cr}$ and the cylinder slenderness (obtained by varying the $r/t$ ratio) and allowing the corresponding $\alpha$ value to be simply read off the horizontal axis. Thicker cylinders with $r/t$ near 100 have the highest $M_k / M_{pl}$ resistances and are thus in the upper regions of each curve, while thinner cylinder with $r/t$ approaching 500 have the lower $M_k / M_{pl}$ resistances and thus may be found lower down. It should be added that the yield strength of the material is not relevant to this discussion as changing it would only alter the scaling of the vertical axis. For the purposes of constructing Fig. 5.12, a generic steel grade with a 460MPa yield stress was assumed. Only the eigenmode and weld depression imperfections were considered in this analysis.

The elastic ‘modified’ capacity curves shown in Fig. 5.12 illustrate that the dimensionless group $\Omega$, found by Calladine (1983) to arise naturally in the analysis of perfect cylinders (represented by a near-perfect cylinder having a mesh perturbation of amplitude $\delta_e/t = 0.01$) under unsymmetrical loading with small circumferential wave numbers lead to curves that are
vertical. The same dimensionless group is shown to lead to curves that are also quite vertical with the eigenmode imperfections, depicting that an invariant geometric nonlinearity for ovalising cylinders under bending is mostly maintained in this case. For the more severe weld depression imperfection, by contrast, grouping the data in terms of $\Omega$ no longer maintains verticality, especially for increasingly longer and increasingly imperfect ovalising cylinders (i.e. simultaneously higher $\delta/t$ and $\Omega$). In particular, capacity curves in the lower right-hand part of Fig. 5.12 exhibit an increase in $M_k / M_{cr}$ with increasing $r/t$ beyond $r/t \approx 300$, suggesting that the weld depression acts as a stiffening corrugation against the circumferential bending induced by ovalisation (a similar effect was previously documented in Sadowski & Rotter, (2011b)). It may thus be inferred that an RRD characterisation based on predictions for the weld depression imperfection form performed at $r/t = 100$ will, with limited exceptions, constitute a conservative prediction to the possible behaviour. This inference will form the basis of the data set used in the algebraic characterisation of imperfection sensitivity in elastic cylinders under uniform bending that is presented in the Chapter 6 of this PhD thesis.

5.8 Conclusions

Exhaustive computational investigations into the imperfection sensitivity of cylinders under uniform bending has just been presented, covering a wide parametric variation of cylinder length, end support conditions, forms and amplitudes of geometric imperfections. The following key conclusions may be drawn:

- The sensitivity to geometric imperfections demonstrated by these systems is strongly length-dependent, with the most severe sensitivity predicted at a length where ovalisation is just about to begin to influence the fundamental response. Very long cylinders dominated by fully-developed pre-buckling ovalisation are significantly less sensitive to geometric imperfections.

- This system exhibits an imperfection sensitivity that does not necessarily suggest a monotonously decreasing buckling moment with growing imperfection amplitude. This is especially the case for shorter cylinders where the end boundary condition may be very effective in constraining larger buckles characteristic of imperfect cylinders.

- Of the three imperfection forms considered (eigenmode, imposed ovalisation and weld depression), the axisymmetric circumferential weld depression of Rotter & Teng (1989)
consistently appears to be the most deleterious to the strength of the cylinder at all lengths, in addition to being a realistic model of typical defects found in many cylinders in service. It is recommended as an optimal form for similar explorations in other shell systems whose buckling behaviour is governed by significant pre-buckling meridional compression.

- This study appears to be the first to systematically document the length dependency of imperfection sensitivity in any shell system. The shell buckling research community is therefore encouraged to explore the imperfection sensitivity of other systems, even very classical and otherwise well-studied ones, for a similarly strong dependency on a global parameter such as, in the case of cylinders, the length. This study employed a novel automation strategy specially designed to facilitate this task.
Chapter 6 - Algebraic characterisation of imperfection sensitivity in elastic cylinders under uniform bending

6.1 Introduction and general outline

The important role of length on the elastic stability of perfect and imperfect cylinders under uniform bending has been described in Chapters 4 & 5 of this thesis. It is revealed that when the cylinder length is very short, the ensuing compatibility bending deformations that arose in the system begin to deviate the response of the system from the well-established membrane-dominated shell systems. In longer cylinders, cross-sectional ovalisation dictates the elastic stability significantly. With the introduction of geometric imperfections, the most detrimental effect is shown to occur around the medium-to-transitional domain boundary, a consequence of the membrane-dominated pre-buckling stress state at this domain boundary and a gentle circumferential variation of meridional membrane stress resultants, comparable to those from uniform axial compression loading. For shorter imperfect cylinders still in the ‘medium’ domain, increasing the imperfection amplitude does not necessarily translate into a monotonous decrease in buckling strength, especially at deeper amplitudes, since the imperfect cylinder profile now encroaches upon and becomes constrained by the end boundary condition. Altogether, the combined effect of pre-buckling nonlinearity, cross-sectional ovalisation and geometric imperfections is seen to account for a loss in true buckling strength that can go as high as 75% of the theoretical value $Mcl$ at $\delta_e/t = 1.5$ and the weld depression imperfection. The current chapter extends on the above by characterising only the influence of geometric imperfections on the buckling behaviour of elastic cylinders under uniform bending, denoted by the $\alpha_I$ parameter, into algebraic expressions to facilitate structural design, following the outline narrated below.

The output data for each end support condition and for the three distinct forms of geometric imperfections from Chapter 5 are first reduced into a single lower-bound relationship corresponding to the minimum of the predicted buckling strengths by individual imperfection form at constant amplitude, herein described by the equivalent geometric deviation $\delta_e/t$, and cylinder length. Synthetic imperfection sensitivity relationships are thereafter developed for each boundary condition by constraining the buckling resistance at higher amplitude never to exceed that at the preceding amplitude value. The power law functional form, which is widely taken as representative of the imperfection sensitivity relationship demonstrated by other
cylinder systems (Rotter, 2004; EN 1993-1-6, 2007), is then employed to describe the relationship between the imperfection reduction factor $\alpha$ and the imperfection amplitude $\delta e/t$ at each cylinder length using constrained least squares minimisation approach. The relationship between the fitted scaling parameters and the cylinder length are then further characterised also using unconstrained least squares minimisation fittings with the aid of appropriate functional forms that ensure limiting asymptotic values of scaling parameters are achieved as the cylinder length approaches an infinite value.

By employing a validated computational tool, in-depth numerical evidence for the sensitivity of cylinders under uniform bending to three different forms of geometric imperfections was provided in chapter 5 of this thesis. Specifically, it was revealed that the buckling strength of the imperfect shell is strongly dependent on both the form and amplitude of the imperfection, as well as on the cylinder length, the end boundary conditions and, to a lesser extent, the radius-to-thickness $r/t$ ratio. The current chapter attempts to reduce the output data from chapter 5 into a single lower-bound relationship more amenable to characterisation into conservative algebraic expressions for use in manual dimensioning within the Reference Resistance Design (RRD) framework, intended for incorporation into the next version of EN 1993-1-6 (2007) that is currently under revision. The processing and characterisation of the generated result data was conducted using Excel Spreadsheet, with a combination of the MATLAB (2014) and Python programming environments.

6.2 Construction of a lower-bound, length-dependent imperfection sensitivity relationship

The normalised buckling moment $M_k / M_{cr}$ vs cylinder length $\omega$ or $\Omega$ relationships at $r/t = 100$ (Figs. 5.9 – 5.11) for each of the three imperfection forms were first harmonised into a single set of relationships per boundary condition by identifying the minimum buckling strength out of the three imperfection forms at every combination of cylinder length $\omega$ and equivalent geometric deviation $\delta e/t$. This procedure loosely reflects the conservative lower-bound approach used in establishing the imperfection sensitivity relationship for axially-compressed cylinders (Rotter, 2004), where the minimum buckling strengths were identified from a large database of test results. In addition to ensuring that the most detrimental prediction at each length is used as the basis for a conservative characterisation of the buckling behaviour of cylinders under uniform bending, this procedure allows a definitive identification of which
imperfection form is likely to be most critical and where. The resulting lower-bound $M_k / M_{cr}$ vs $\omega$ relationship is shown in Fig. 6.1 where it may be seen that, for modest to deep imperfections and across all length domains, the axisymmetric weld depression consistently controls as the most deleterious form of geometric imperfection.

Fig. 6.1 – Lower-bound $M_k / M_{cr}$ vs $\omega$ relationships for a) Clamped (C) and b) Simply-supported (S) imperfect elastic cylinders under uniform bending
This figure is colour- and marker-coded to enable quick identification of the origin imperfection form for each data point.

6.3 Construction of a synthetic, length-dependent imperfection sensitivity relationship

It was also shown in Chapter 5 that by employing the weld depression imperfection, the shorter parts of the ‘medium’ domain may exhibit an increase in buckling strength for deeper imperfections (Fig. 5.7 & Fig. 5.10), a consequence of a buckling mode that grows with imperfection amplitude eventually encroaching upon and becoming constrained by the boundary condition (Fig. 5.11). This poses a challenge for any algebraic characterisation of imperfection sensitivity, as a conservative design rule must not lead an analyst to attempt to design an ‘imperfect’ shell for a higher buckling resistance than the ‘perfect’ one. Consequently, at any length $\omega$, a synthetic imperfection sensitivity relationship of $M_k / M_{cr}$ vs $\delta_e/t$ was established where a buckling resistance at a larger value of $\delta_e/t$ was constrained to never exceed that at the preceding value of $\delta_e/t$, as shown in Fig. 6.2. This adjustment was found necessary only within the ‘medium’ length domain controlled by $\omega$, but not in the ‘transitional’ or ‘long’ domains controlled by $\Omega$ as these latter domains are free of end boundary effects (see Fig. 2.26).

![Image of synthetic imperfection sensitivity relationships](image-url)

Fig. 6.2 – Synthetic imperfection sensitivity relationships of $M_k / M_{cr}$ vs $\delta_e/t$ for clamped (C) and simply-supported (S) ‘medium’ length cylinders under uniform bending with $r/t = 100$.

The resulting ‘synthetic’ lower-bound non-increasing moment-length relationships are illustrated in Fig. 6.3 for both end boundary conditions. As a fortuitous side-effect, the resulting
curves are more amenable for algebraic characterisation, especially in the ‘medium’ length domain where boundary effects previously led to rather messy moment-length relationships (compare Fig. 6.3 with Fig. 6.1).

Fig. 6.3 – Synthetic $M_k / M_{cr}$ vs. $\omega$ relationships for a) Clamped and b) Simply-supported imperfect elastic cylinders under uniform bending.
6.4 Proposal for the elastic imperfection reduction factor $\alpha_I$

In addition to presenting informative details on the behaviour of imperfect elastic cylinders under uniform bending to a wide parametric variation of length, end boundary condition, forms and amplitude of geometric imperfections, the current research effort aims to characterise the buckling behaviour of this system into simple algebraic expressions that can be easily used for conservative yet reasonably accurate prediction of the buckling resistance.

It is envisaged that this proposal for the imperfection reduction factor $\alpha_I$ will be used within the RRD framework together with the geometrical reduction factor $\alpha_G$ already established by Rotter et al., (2014) and plasticity-related parameters $\lambda_0$, $\eta_0$, $\eta_p$ and $\chi_h$ currently in preparation by Wang et al. (2018) to spare the designer from having to undertake a rigorous computational analysis to determine the fully nonlinear resistance of a cylinder under uniform bending. Furthermore, it is hoped that more structural systems will gradually be processed in a similar manner. The synthetic data set (Fig. 6.3) obtained as described above was used as the basis for the proposals of $\alpha_I$ through a power law functional relationship, to capture the relationship between $\alpha_I$ and $\delta_e/t$ accurately, as illustrated in Fig. 6.4. As a rough guide on the quality of the fitted relationships, the coefficient of determination ($R^2$) values are calculated and compared across the length domains. Although it is revealed that these $R^2$ values tend towards a perfect value of 100% as the cylinder length grows beyond the medium, $R^2$ values of at least 80% between the fitted relationship and the synthetic data set (not the lower bound data set) is also achievable in the medium domain.

The general power law functional relationship that was adopted to capture the dependency of the imperfection reduction factor $\alpha_I$ on the imperfection amplitude $\delta_e/t$ is given in Eq. (6.1). This functional form has been used before to define the elastic imperfection reduction factor for cylinders under uniform axial compression in EN 1993-1-6 (2007) as well as by Chen et al. (2008) and Rotter (2013c) to obtain the first trial expression for $\alpha_I$ for cylinders under uniform bending (see Eq. 2.29 & Fig. 2.27). It has the advantage that it is monotonically decreasing and conservatively predicts $\alpha_I \to 0$ as $\delta_e/t \to \infty$. Further, $\alpha_I = 1$ when $\delta_e/t = 0$, indicating perfect shell behaviour where $\alpha = \alpha_G$ only.

$$\alpha_I(L, \delta_e) = \frac{1}{1 + x_1 (\delta_e/t)^{x_2}} \text{ where } x_1, x_2 \text{ are } f(L) \quad (6.1)$$
The length-dependency of the physical relationship is achieved through the scaling parameters $x_1$ and $x_2$ that are allowed to vary with $\omega$ or $\Omega$ depending on the length domain. A least-squares minimisation was performed at every length to fit the expression for $\alpha_I$ (Eq. (6.1)) to the synthetic imperfection sensitivity relationships (Fig. 6.2 & Fig. 6.3) with the constraint that the fitted $\alpha_I$ expression cannot predict a higher buckling moment than that given by any of the data points. This resulted in sets of $(x_1, x_2)$ pairs for different lengths and boundary conditions. A
second, unconstrained, least-squares minimisation was then performed to fit convenient expressions to each of these $x_1$ or $x_2$ vs. length data sets as illustrated in Fig. 6.5 for both end boundary conditions.

![Fig. 6.5](image)

Fig. 6.5 – Construction of the relationship between the scaling parameters ($x_1$ and $x_2$) and length ($\omega$ and $\Omega$) for the clamped (C) and simply-supported (S) cylinders.

This novel ‘fit to a fit’ procedure ensures that the design expression for $\alpha$ captures the dominant physics of the underlying behaviour and predicts a realistic length-dependent imperfection sensitivity. The second, unconstrained least squares minimisation was performed using a
The variation of $x_1$ with length is revealed to be more sensitive, with values ranging between 0.7 and 1.7 depending on the length domain. However, a mild sensitivity is shown in the distribution of $x_2$ with length with values between 0.6 and 1. Consequently, a single value is adopted for $x_2$ while another unconstrained, least squares minimisation was performed to predict the relationship between $x_1$ and length. From the power law functional form adopted, it may be understood that as the scaling parameter $x_1 \to 0$, the imperfection reduction factor $\alpha_I \to 1$, which implies that the structure exhibits a higher nonlinear buckling strength ($M_{k}/M_{cr}$).

A scenario like this must be avoided in any conservative characterisation and so the algebraic characterisation of scaling parameter $x_1$ is performed to ensure that the predicted $x_1$ value at any length ($\omega$ or $\Omega$) is greater than or equal to the actual value, as shown in Fig. 6.5. For the $x_2$ scaling parameter in the adopted power law form, the effect of any value of $x_2$ depends largely on the $\delta/e/t$ of interest. In a case where $\delta/e/t < 1$, if $x_2$ increases then the denominator of Eq. (6.1) approaches a value of 1 and $\alpha \to 1$. By contrast, where $\delta/e/t > 1$, as $x_2$ increases, then $\alpha \to 0$. However, despite the strong dependency on the particular $\delta/e/t$, the influence of scaling parameter $x_2$ in characterising the relationship between nonlinear buckling moment and length of imperfect cylinders under uniform bending is found to minimal and a single value of $x_2 = 0.7$ is proposed for all the length domains.

The proposed algebraic expressions for the $x_1$ and $x_2$ parameters, to be used together with the power law functional form for $\alpha_I$ in Eq. (6.1), are presented in Table 6.1 and the moment-length relationships computed on their basis are presented in Fig. 6.6 & Fig. 6.7.

<table>
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<th>Table 6.1 – Proposed algebraic expressions for a length-dependent $\alpha_I$ for imperfect elastic cylinders under uniform bending</th>
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One set of $x_1$ parameters is presented in terms of $\omega$ for Clamped (C) and Simply-supported (S) conditions within the ‘medium’ length domain (Fig. 6.5a – d), and another one set is presented in terms of $\Omega$ within the ‘transitional’ and ‘long’ domains (Fig. 6.5e & f). The expression for $x_1$ in the ‘transitional’ domain tends to an asymptotic value of 0.7 as $\Omega \to \infty$, representative of invariant imperfection sensitivity in (infinitely) ‘long’ cylinders. A continuity of $x_1$ value from ‘medium’ to ‘transitional’ domains is also ensured. This enables the $x_1$ and $x_2$ values to remain constant very close to the medium-to-transitional domain boundary regardless of the $r/t$ ratio, since this domain boundary is defined as mainly $\omega = 0.5(r/t)$ or $\Omega = 0.5$. Altogether, this generates the conservative $M_k / M_{cr}$ vs $\omega$ relationships at $r/t = 100$ presented in Fig. 6.6 for both end boundary conditions.

Alternatively, it is suggested, by way of further simplification, that for design purposes the imperfection sensitivity in the ‘medium’ length domain may be conservatively represented by $\alpha_I$ established at the ‘most severe’ length, which is at the beginning of the ‘transitional’ domain, i.e. at $\Omega = 0.5$ (or $\omega = 0.5(r/t)$, as shown in Fig. 6.7. This permits a simpler overall $\alpha_I$ expression represented by just one set of $x_1$ and $x_2$ parameters that are dependent on $\Omega$ only and independent of the nature of the rotational restraint at the end boundary condition. This is the form of the current proposal submitted for consideration to the Technical Committee (TC250/SC3/PT5) working on the latest revision to the EN 1993-1-6 (2007). Lastly, it is suggested that cylinders with boundary conditions that do not maintain rigidly circular ends may be treated as susceptible to fully-developed ovalisation at all lengths, which would translate to an imperfection sensitivity corresponding to the asymptotic or ‘long’ conditions where $x_1 = x_2 = 0.7$ for the power law function.

### 6.5 Conclusions

This chapter has proposed a characterisation for the imperfection sensitivity relationship of elastic cylinders under uniform bending in a compact manner by means of the power law functional form following a careful post-processing of output data from three distinct forms of geometric imperfections, two different end boundary conditions and a single instance of $r/t = 100$. Only the output data corresponding to $r/t = 100$ have been employed here since it was shown in chapter 5 that they constitute a conservative lower bound to the possible behaviour of thin cylinders under uniform bending.
Fig. 6.6 – Resulting characterised relationship between $M_k / M_{cr}$ vs. $\omega$ for a) Clamped (C) and b) Simply-supported (S) elastic imperfect cylinders under uniform bending.
Fig. 6.7 – Alternative characterised relationship between \( \frac{M_k}{M_{cr}} \) vs. \( \omega \) for a) Clamped (C) and b) Simply-supported (S) elastic imperfect cylinders under uniform bending.
A conservative relationship between the buckling moment and the cylinder length can be established by utilising a lower-bound approach, which identifies the minimum buckling strength for every imperfect cylinder at constant length and imperfection amplitude but under a variation of imperfection forms. In addition, an imperfection sensitivity behaviour that is more amenable to algebraic characterisation and appropriate for design purposes may be developed by constraining the buckling behaviour to a non-increasing buckling strength as the imperfection amplitude grows.

Of the three imperfection forms considered (eigenmode, imposed ovalisation and weld depression), the axisymmetric circumferential weld depression of Rotter & Teng (1989) nearly consistently controls as the most deleterious to the strength of the cylinder.

Realistic but conservative closed-form algebraic design expressions have been formulated to describe the reduction in buckling moment due to the effect of geometric imperfections. The proposed relationships are dependent on the cylinder length, \( r/t \) ratio and imperfection amplitude \( \delta_e/t \) and are ready for implementation within the Reference Resistance Design framework recently adopted by EN 1993-1-6 (2007).
Chapter 7 - Conclusion and further research

7.1 Introduction

Thin-walled metal cylinders are widely considered as an optimised structural form in transmitting loads from one point to another because of the higher strength-to-weight ratio they tend to offer in structural design, in addition to the unique aesthetical appeal they possess. Like any structure, the failure of these cylinders comes with a huge economic cost during service if not properly designed, and the economic cost of such a potential failure is of great concern. The engineering failure of these metal structures under compressive loading is usually induced by buckling, a phenomenon that has been the focal point of the current PhD research investigations. However, there are several other factors that can individually influence the resistance to buckling in cylinders, key geometric parameters being the cylinder length and the unavoidable presence of geometric imperfections. Recently, the role of length on the elastic stability of perfect thin-walled metal cylinders under global bending was characterised into four distinct length domains, namely: short, medium, transitional and long; depending on the dominant buckling behaviour (see Rotter et al., 2014). However, a systematic study of the influence of geometric imperfections, in isolation or coupled with other geometric or material factors, on the elastic stability of this structural system is lacking in research literature. The need to address some of these research gaps has been one of the main aims of the current PhD research study.

This research dissertation has presented extensive theoretical background and systematic sets of computational investigations into the buckling behaviour of thin elastic metal cylinders under a non-symmetrical load case of uniform bending covering a wide parametric variation of length, boundary condition, radius-to-thickness $r/t$ ratios, forms and amplitudes of geometric imperfections. The current thesis has focused on exploring the fundamental elastic response of the system where the chief source of complexity is attributable to the consequences of nonlinear changes of geometry, without the influence of material nonlinearities. This is a necessary prerequisite to understanding the total response of a structural system of any slenderness whose initial response is always elastic. All the computational analyses were performed using the commercial finite element analysis software ABAQUS (2014). Several types of computational analyses, within the framework of EN 1993-1-6 (2007), have been performed on each cylinder model to understand the geometric nonlinearities governing the buckling behaviour of that
cylinder in isolation. Altogether, this amounted to a vast number of computational analyses necessary to be performed, so significant that an efficient management strategy had to be developed in order to complete these investigations within the allotted duration of the PhD programme.

7.2 Structural symmetry sensitivity analysis of elastic cylinders under uniform bending

While it is possible to model the complete geometry of any structure to be analysed using ABAQUS, it is computationally more efficient to reduce the size of the model by cutting into halves along plane(s) of ‘mirror’ symmetry and introducing appropriate symmetry boundary conditions. Consequently, a preliminary investigation of the sensitivity of the buckling response, under linear and nonlinear geometric conditions, of elastic cylinders under uniform bending to exploitation of structural symmetries along different planes of cut is first conducted. The results from this preliminary investigation showed that adequate care must be put in place when attempting to interpret the results from finite element models in which a mirror symmetry has been exploited along the meridional axis, especially at very short lengths ($\omega < 10$), because the artificial restraint introduced by the symmetry boundary condition could begin to interfere with the natural buckling pattern of the cylinder. Nevertheless, for longer and ovalising cylinders, a more computationally efficient quarter-shell design model, generated by exploiting structural symmetries along the meridional and circumferential axes of the cylinder, may be used since it yields the same level of accuracy as the full cylinder model.

7.3 Mesh convergence analysis of elastic cylinders under uniform bending

It is well known that the accuracy of the buckling behaviour predicted in a finite element model may be acutely influenced by the level of mesh refinement employed in the model. Consequently, it is a standard pre-requisite to undertake some preliminary investigations on the structure, employing different mesh refinement until the predicted response becomes essentially invariant with further increase in refinement. This same standard procedure was performed on elastic cylinders under uniform bending and the predicted response was monitored by means of a linear bifurcation analysis (LBA), geometrically nonlinear analysis without or with imperfections (GNA or GNIA respectively).
The discretisation into finite elements was performed on the global model but the variation of mesh refinement in this convergence study was performed only on the regions where local bending deformations, due to compatibility bending or local buckling, are expected to develop in the cylinder model during loading. This discretisation was characterised in terms of the number of elements within a linear bending meridional wavelength \( \lambda \) (Eq. (2.17)) – a parameter that measures the depth of boundary layer at each end of the cylinder necessary to satisfy kinematic compatibility with the end boundary condition. The results of the analysis showed that the requirement for mesh refinement is not uniform across all the length domains, a consequence of the different physical mechanisms governing the buckling behaviour of the cylinder in different length domains and which require different demands with respect to the interpolation field.

Where the cylinder length is very short such that the pre-buckling stress state is dominated by bending actions arising from kinematic compatibility requirements with the end boundary condition, a minimum of 30 elements per \( \lambda \) is necessary. Cylinders longer than these require 20 elements per \( \lambda \) while very long and ovalising ones require only 10 elements per \( \lambda \). The minimal number of elements per \( \lambda \) required by very long and ovalising cylinders is due to the fact that ovalisation is governed by a much longer wavelength arising from circumferential bending than meridional compatibility bending or buckling. The regions of cylinder length that are purely under membrane actions and where local short-wave buckles are not expected to develop, coarser meshes with mesh refinement that can go as low as 2.5 elements per \( \lambda \) may be used without jeopardising the predicted load path or buckling strength.

### 7.4 Management strategy for computational analyses in ABAQUS

Over the course of this PhD research, the amount of computational analyses necessary to achieve the objectives of the study steadily burgeoned, owing to the repetitive investigations that were necessary to be conducted over a wide range of parameters. Nevertheless, a computational strategy was developed to aid the design of programme of parametric finite element (FE) analyses with the sole purpose of establishing complete characterisation of the individual nonlinearities governing the response of the cylinder system, within the framework of Reference Resistance Design (RRD). The generation, submission, termination and processing of FE analyses was automated simply by exploiting the capability of FE suite interfacing with Python and FORTRAN programming languages. The important capability of
terminating an ongoing FE analysis automatically once a criterion of failure is met has been explored and procedures for the automatic detection of failure conditions have been proposed, depending on the type of computational analysis.

7.5 Behaviour of very short elastic cylinders under uniform bending

Ring stiffeners may sometimes be introduced along the length of a cylinder at calculated intervals to restrain the global flattening of the cylinder cross-section. Consequently, cylinders may occur as multi-segment shells with closely spaced stiffening rings (Singer, 1967) such that each individual segment represents a very short cylinder. The behaviour of these very short individual cylinders under uniform bending has been explored in detail, employing both small and large deformation theories and accounting for the influence of varying rotational restraint condition at the edges (Clamped (C) and Simply-supported (S)) together with sensitivity to different forms and amplitude of geometric imperfections.

The key finding from this buckling investigation is that since the cylinder length is comparable with the linear bending meridional half wavelength ($\lambda$), the pre-buckling state of stress is largely under bending actions. Consequently, there are significant deviations between the response predicted for these short cylinders through a more complete finite element analysis and the analytical predictions, which were mostly derived on the assumption of dominant membrane actions in the state of stress before buckling.

The buckling behaviour of short cylinder systems assuming small displacement theory reveals that the formation of a local buckling mode can be significantly constrained by the influence of boundary restraint at the edges. Consequently, a linear elastic bifurcation analysis, which employs a complete shell theory that accounts for all membrane and bending deformations in the shell but without incorporating the influence of prebuckling rotation of structural members, predicts buckling moments that are significantly higher than the theoretical prediction, which assumes a pure membrane state of stress that is typical of infinitely long cylinders. Furthermore, as the cylinder length increases, a point is reached where the local buckling hypothesis (Axelrad, 1965; 1985) now becomes a close representation of the behaviour of the cylinder such that the FE computed buckling moment is now approximately equal to the theoretical value. This pattern of behaviour was exhibited under both end boundary conditions although at different length boundaries.
Under geometrically nonlinear analyses, the afore-mentioned local buckling mode is found to be eliminated in short cylinders under bending. Consequently, the cylinder exhibits snap-through buckling following the detrimental geometric changes in the form of a meridional fold, which form on the compressed side of the cylinder, and fails only at moments and curvatures that are significantly greater than the theoretically predicted values. The limit-point behaviour exhibited by these short cylinders is revealed to be a meridional bending phenomenon and should not be confused with the well-known Brazier (1927) cross-section ovalisation, a circumferential bending phenomenon that occurs only in sufficiently long cylinders under bending. While buckling at such very high mean curvatures may be deemed unfeasible in metal shells as a result of material plasticity, it is revealed, through a geometrically and materially nonlinear analysis (GMNA) under ideal elastic-plastic material law, that this is indeed achievable in much thinner cylinders \( r/t > 1000 \). However, in the advanced stage of the equilibrium path, extremely short and geometrically perfect cylinders \( \omega < 2.6 \) or 1.4 for clamped (C) or simply-supported (S) cylinders respectively) exhibit geometric hardening, that is evident as indefinite stiffening on the equilibrium path, such that only a tolerable deformation may be used to establish the failed state of the shell system (see Criterion C3 under 2.6.4).

A complex sensitivity to the critical eigenmode and weld depression imperfections is exhibited by these short cylinders, which may appear mildly detrimental, neutral or even beneficial, depending on the length, boundary conditions and imperfection form. A potential reason for this sensitivity is the complex interaction between local bifurcation and global limit-point buckling modes in the imperfect short cylinders. Furthermore, the buckling response does not suggest a monotonous decrease in buckling strength with increasing imperfection amplitude and generally the effect of imperfections on buckling strength appears milder than the corresponding effect on cylinders under uniform axial compression, although the weld depression imperfection seems more damaging.

### 7.6 Imperfection sensitivity in thin elastic cylinders under uniform bending

The imperfection sensitivity of elastic cylinders under a reference load case of uniform axial compression is a subject that has received a lot of attention in the past following the severe sensitivity demonstrated by this cylinder system to minor geometric imperfections. However, a similar effort has never been put in place for cylinders under uniform bending. The elastic stability of short individual cylinders in multi-segmented shells under uniform bending has
been narrated, together with the sensitivity to critical eigenmode and weld depression geometric imperfections. By extension to common cylinder systems in length domains that of direct practical interest in the industry, the sensitivity to three distinct forms of geometric imperfections – critical eigenmode, imposed ovalisation and weld depression – was investigated in detail here in addition to the influence of varying rotational restraint condition at the edges and global parameters – radius, thickness and above all length – on imperfection sensitivity. Prior to this PhD research, an investigation into the influence of length on the imperfection sensitivity of a cylinder system does not appear to have been performed before.

It was established that similar to other shell systems, these cylinder systems demonstrate sensitivity to forms and amplitudes of geometric imperfection. The imposed ovalisation imperfection, which was specifically introduced to explore the effect of global bending on an already slightly ovalised cylinder following the severe penalty, caused by cross-sectional ovalisation, on the buckling strength of perfect cylinder systems, maintains either mild sensitivity or neutral effect on the predicted buckling strengths of the imperfect systems. The ‘imposed ovalisation’ was eventually ruled out as a potentially ‘severe’ form of geometric imperfections for cylinders under uniform bending. By contrast, the critical eigenmode and axisymmetric weld depression caused significant detrimental effects that can go as high as a 75% reduction in the theoretical buckling strengths predicted for the systems, dependent on the amplitude of imperfections and cylinder length. However, these significant penalties do not appear to be as severe as the corresponding ones predicted for cylinder system under uniform axial compression, which is perhaps the most imperfection-sensitive shell system owing to an axisymmetric pre-buckling stress state that is dominated by uniform membrane actions, (Koiter, 1945; 1963; Rotter & Teng, 1989).

Furthermore, in medium-length cylinders, where the edge condition is still very effective in restraining any flattening of the cross-section, the boundary restraint offered by this edge condition constrains the larger buckling mode feature of the more imperfect system, making the system require more strain energy to attain a ‘failed state’ since more of the cylinder length is now involved. Consequently, the equilibrium path of the system changes from one exhibiting obvious bifurcation with a steeply-descending post-buckling path to one exhibiting only a ‘kink’ with a smooth transition from pre- to post-buckling and a corresponding growth in buckling deformations, but without any numerical indication of instability reported by the
solver. This effect was found responsible for the non-monotonous decrease in buckling strength exhibited by ‘short’ imperfect cylinder systems with increasing imperfection amplitude. Additionally, the effect of employing different end boundary conditions that ensure the circularity of the cross-section is preserved at the edges, on the elastic stability of the imperfect system was found mostly negligible even in medium-length cylinders because the main mechanism now responsible for the significant reduction in the stiffness of the cylinder’s fundamental response is the presence of geometric imperfections and not the rotational restraint condition at the edges.

The sensitivity to geometric imperfections was found to be markedly different across each length domain. In the medium domain, the sensitivity grows with increasing length and reaches the most severe at the boundary of the ‘medium’ to ‘transitional domains. However, this sensitivity becomes milder with increasing length in the ‘transitional’ length domain since pre-buckling ovalisation now becomes more crucial to the elastic stability of the system. The behaviour approaches an ‘asymptotic’ and relatively mild imperfection sensitivity in the ‘long’ length domain, which is invariant with further increases in length, since the system now behaves like a limit point type with nonlinear fundamental path and thus exhibit a milder imperfection sensitivity than systems with a linear fundamental path (Thompson & Hunt, 1973; 1984). Across all the length domains explored, the Rotter and Teng (1989) Type A axisymmetric weld depression appears to control as the most damaging form of geometric imperfection and is therefore recommended as an optimal form for explorations of imperfection sensitivity in cylinder or other systems. Overall, this PhD study is the first research endeavour to formally confirm and establish that the sensitivity to geometric imperfections in elastic cylinders under uniform bending is strictly length-dependent, an unprecedented conclusion for any shell system.

The importance of establishing a suitable dimensionless length group for any shell system such that the predicted buckling strength is independent of the radius-to-thickness $r/t$ ratio was pointed out by Rotter (2005c; 2016a), especially for the characterisation of Reference Resistance Design (RRD) parameters and a suitable dimensionless length group ($\Omega$ in Eq. (5.1) ) for ovalising, near-perfect cylinders under uniform bending was established by Calladine (1983). The extent to which this dimensionless length group $\Omega$ may also be true for imperfect cylinders under uniform bending was then carefully investigated by studying the imperfection
sensitivity of ‘transitional’ length cylinders considering the critical eigenmode and weld depression imperfections, across varying $r/t$ ratios. It is revealed that the same dimensionless length group that allows geometric nonlinearity to be preserved in near-perfect ovalising cylinder may be considered also suitable for imperfect cylinders with the critical eigenmode imperfection. However, for the more damaging weld depression imperfection, the same dimensionless group does not appear suitable as strongly ovalising cylinders with higher $r/t$ appear stiffer than those with lower $r/t$. This is because at higher $r/t$ ratios (approximately $r/t \geq 300$), the weld depression imperfection now acts as a stiffening corrugation against the circumferential bending induced by ovalisation, resulting in a higher buckling strength. Nevertheless, this exploration offers a platform to infer that the buckling strength predictions at $r/t = 100$ represent a conservative dataset for characterising the imperfection sensitivity of elastic thin-walled cylinders under uniform bending.

7.7 Algebraic characterisation of imperfection sensitivity in elastic cylinders under uniform bending

The preceding sections have offered insights into the influence of geometric imperfections on the response of elastic cylinders under uniform bending, considering multiple forms of geometric imperfections. Additionally, the vital role of global parameters such as cylinder length, radius and thickness, on imperfection sensitivity has been narrated. Altogether, this generated a vast amount of output dataset even under a constant $r/t = 100$, which now requires a novel way of managing and processing into comprehensible formats, describing the subject of imperfection sensitivity on a single system of elastic cylinders under uniform bending and its dependency on many other factors.

A conservative lower-bound approach, more akin to the type used on experimental dataset for axially compressed cylinders, was found capable of reducing the corresponding output data from multiple forms of geometric imperfection into a single dataset, although for a specific end boundary condition. In addition to presenting the minimum buckling strength across the different imperfection forms considered, this approach offered a definitive deduction that the Rotter and Teng (1989) Type A axisymmetric weld depression largely controls as the most deleterious to the strength of the system, regardless of the nature of the restraint condition at the edges. Furthermore, the afore-mentioned challenges associated with interpreting the counter-intuitive behaviour of increasing buckling strengths at deeper imperfections, where the
end boundary condition begins to influence the stability of the more imperfect cylinder, were eliminated by introducing a ‘logical constraint’ such that the buckling strength predicted at a larger imperfection amplitude is never allowed to exceed the corresponding value predicted at a preceding amplitude.

By employing a power-law functional form, which had been used in the past to describe the imperfection sensitivity in axially compressed cylinder systems, the relationship between the imperfection reduction factor (or knock-down factor) and amplitude for elastic cylinders under uniform bending was characterised into realistic but conservative closed-form algebraic design expressions. The scaling parameters in this functional form were subsequently made individual functions of the cylinder length, radius and thickness using appropriate functional forms. This novel ‘fit-to-a-fit’ approach was designed to capture the underlying physics of the system and the dependency on global parameters but at the same time not to be overly conservative for efficient structural designs. Although only two practical end boundary conditions have been employed here, the algebraic expressions proposed however, have been made to be independent of the nature of the end boundary condition. It is therefore believed that these proposals will suffice for any real cylinders under uniform bending that maintain rigidly circular ends, otherwise such cylinders may be treated as susceptible to fully-developed ovalisation at any length and designed according to the proposed expressions for ‘long’ cylinders.

Overall, it is shown that indeed closed-form algebraic expressions can be formulated to describe imperfection sensitivity in elastic cylinders under uniform bending accurately, thereby removing the need for an analyst to undertake any further onerous or iterative computational analyses to establish the sensitivity of such cylinder systems to different forms of geometric imperfections.

7.8 Further research

This PhD research study has presented robust methods of undertaking systemic buckling investigations in elastic imperfect cylinders under uniform bending using modern numerical analyses and considering all the diverse factors that potentially influence the buckling resistance in such systems. This study appears to be the first to systematically document length effects in imperfection sensitivity of a typical shell system. However, the research approach
used may also be adopted for buckling investigations of any structural systems and there are indications that the context or methodology described in the current research study is now being used in other studies (see Rotter & Al-Lawati, 2016; Wang & Sadowski, 2018).

While it may be argued that laboratory testing could in theory, be undertaken to establish the ‘real’ buckling parameters for a given shell system under investigation, the huge financial and time cost associated with this method to cover the wide parametric ranges conceivable would make such an endeavour very difficult to justify. Furthermore, the fabrication processes used in producing full-scale civil engineering thin shells are different from those used for laboratory test models (Rotter, 2017b), consequently giving rise to different forms and amplitudes of geometric imperfections, which the shell system may be highly sensitive to, dependent on the state of stress before buckling.

It is therefore believed that the option of vast and automated computational programmes using validated and robust numerical models is the only defensible, feasible and economic option to make comprehensive progress that will expand the scope and improve the cost-effectiveness of shell designs (and other structural forms) across a wide range of applications in different industries, ranging from aerospace to vehicle manufacturing, food processing, mining, steel fabrication, chemical plants and hydro-electricity. This option offers immediate and ready access to reliable, accurate and economic, yet simple, predictions for the design of structures and components in these industries.

Although a reliability study of the predicted buckling strengths using finite element computation and under consistent conditions has not been conducted in this research endeavour, it is highly recommended for future research to quantify the potential random errors in these strength predictions under minor changes in conditions.
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Appendix A - Plastic collapse behaviour of very short cylinders under uniform bending

A.1 Introduction

One of the two key resistances upon which the recently devised Reference Resistance Design (RRD) framework is built is the reference plastic collapse resistance $R_{pl}$ of the shell structure (Rotter, 2016a; 2016b; 2017b). In generic shell systems, this resistance may be computed directly through a materially nonlinear analysis (MNA) of the perfect shell, which employs small displacement theory and an ideal elastic-plastic material law. However, this resistance corresponds to the load value once a full plastic mechanism is formed, an eventuality that occurs only at very large displacements and which requires a potentially very slow process of plateauing of the load displacement curve. Hence, an accurate assessment of the plastic collapse load $R_{pl}$ is not easily achievable through finite element computation. Nevertheless, recent studies have proposed different methods of estimating the true $R_{pl}$ of a structural system efficiently through numerical investigations using only a modest portion of the equilibrium path (e.g. Holst et al., 2005; Doerich & Rotter, 2011; dos Santos et al., 2018).

The full plastic collapse mechanism of a cylinder under bending is classically considered as the attainment of the yield stress throughout the full cross-section $\sigma_y$ in meridional membrane yield. The corresponding full plastic moment $M_p$ may easily be obtained as:

$$M_p = \frac{4}{3} \sigma_y \left[ \left( r + \frac{t}{2} \right)^3 - \left( r - \frac{t}{2} \right)^3 \right]$$

(A.1)

For sufficiently long cylinders, the above expression for $M_p$ is likely to be a true representation of the full plastic moment. However, the presence of compatibility bending stresses for very short cylinders has been shown in Chapter 4 of this thesis to alter the local membrane stress state of such cylinders significantly. As a result, very short elastic cylinders under uniform bending were found to fail at an elastic buckling moment that is significantly higher than the analytical predictions under small or large deformation theories. Hence, this mini study additionally aims to explore the potential influence of compatibility bending deformations on the plastic collapse behaviour of very short ideal elastic-plastic cylinders under uniform bending. Specifically, it was investigated whether there may be an equivalent ‘short shell’
effect on the full plastic mechanism similar to the one on the linear elastic buckling mode (presented in Chapter 4 of this thesis), which would potentially make the corresponding plastic moment diverge from the classical analytical prediction, currently stated in Eq. (A.1).

A.2 Scope of this mini-study

Detailed numerical investigations were performed on elastic-plastic cylinders having lengths that span between \( \omega \) (Eq. (A.2)) of 1 and 15 and radius-to-thickness \( r/t \) between 100 & 1000, although at a constant thickness \( t \) of unity.

\[
\omega = \frac{L}{\sqrt{rt}}
\]  

(A.2)

The full three-dimensional reference plastic collapse mechanism of a cylinder may be obtained computationally through a materially nonlinear finite element analysis (MNA), based on a small displacement but full shell bending theory and assuming an ideal elastic-plastic material law. Consequently, only MNA analyses were performed on these cylinders at this stage. Steel grades of S235, S355 and S460, having yield stresses \( \sigma_y \) of 235, 355 and 460 N/mm\(^2\) respectively and representing the common steel grades in civil engineering practice, were employed. The effect of rotational restraint condition at the cylinder edges on the full plastic collapse mechanism was investigated by employing the clamped (C) and simply-supported (S) end boundary conditions.

A.3 Details of the finite element model

The MNA computational analyses were conducted using ABAQUS (2014) finite element analysis software. The finite element model employed for this mini study is similar to the generic details described in Fig. 3.9, although the model was mostly discretised with approximately square meshes for stability and with a density of at least 30 elements per linear bending meridional half-wavelength \( \lambda \) (Eq. (A.3)), following the recommendations of the mesh convergence analysis presented in Chapter 3 of this thesis.

\[
\lambda = \frac{\pi\sqrt{rt}}{\left[3\left(1-\nu^2\right)\right]^{0.25}} \approx 2.444\sqrt{rt} \text{ for } \nu = 0.3
\]  

(A.3)
The general-purpose S4R shell element was used throughout, having been widely used in other studies of shell structures involving extensive plasticity (e.g. Sadowski & Rotter, 2013; dos Santos et al., 2018). In addition to the afore-mentioned yield stresses, each cylinder model is assumed to possess a Young’s modulus of $E = 2 \times 10^5$ N/mm$^2$ and a Poisson’s ratio of $\nu = 0.3$. Furthermore, the geometric state of the cylinder models was assumed to be perfect and the equilibrium path was followed using the modified Riks algorithm (Riks, 1979), without any geometric nonlinearity, until the development of a clear yield plateau at the plastic collapse moment, denoted currently as $M_{\text{MNA}}$. The cross-sectional rotation at the edges of the model was transformed into a mean curvature that is defined over the full length of the cylinder as $\phi$ (from Eq. (3.5)).

### A.4 Plastic collapse behaviour of very short cylinders of varying length under uniform bending

The plastic collapse behaviour of these cylinders was first investigated by observing the predicted moment-curvature relationship in selected systems under varying end boundary condition, but at a constant $r/t$ of 500, as illustrated in Fig. A.1.

![Fig. A.1 – Normalised moment-curvature relationship of the short cylinders showing yield plateaus at moments, $M \geq M_p$ for $r/t = 500$ and geometric lengths $\omega = 1.4$ and 3.](image-url)

Fig. A.1 – Normalised moment-curvature relationship of the short cylinders showing yield plateaus at moments, $M \geq M_p$ for $r/t = 500$ and geometric lengths $\omega = 1.4$ and 3.
A careful selection of MNA equilibrium paths for three short cylinders with $\omega = 1.4$, 2 and 3 reveals that following an initial linear-elastic path, yielding begins approximately at the first yield moment $M_y \approx (\pi/4)M_p$, as defined by classical beam theory and after which the curve gradually begins to level off and tend towards a plastic plateau. However, while for $\omega = 3$ the simply-supported cylinder plateaus at approximately the analytical $M_p$ value, the clamped cylinder plateaus at a moment that is 2.5% higher. For the shortest $\omega = 1.4$ computation, this effect is even more pronounced, with the simply-supported and clamped cylinders attaining a moment 4% and 8% above $M_p$ respectively, suggesting a clear and beneficial boundary effect that requires higher moment for the formation of a plastic collapse mechanism.

Fig. A.2 – Computed moment-length relationships for ideal elastic-plastic cylinders of varying lengths and $r/t$ ratios under two types of end support conditions.

The behaviour of full plastic collapse occurring at a moment higher than the analytically predicted value $M_p$ is further illustrated in Fig. A.2 above, which shows the $M_{MNA} / M_p$ versus $\omega$ relationship tending to infinity as the cylinder length $\omega \rightarrow 0$ for $r/t$ varying between 100 and 1000 under both support conditions and three different yield stresses, a complementary effect to that seen in Fig. 4.4 for the linear buckling behaviour. For $\omega$ greater than approximately 4.8
and 2.4 for clamped and simply supported respectively, the computed and analytical predictions for the plastic collapse resistance are within 1% of one another.

It may perhaps be surprising that the same dimensionless length $\omega$ is a suitable normalisation for the plastic collapse behaviour as well as the linear buckling behaviour of very short cylinders under bending. It may be recalled that in $\omega$ the length $L$ is divided by $\sqrt{rt}$, effectively the ‘unit’ of the half-wavelength of an axial compression buckle as defined on the Koiter circle (Koiter, 1945; Calladine, 1983; Rotter, 2004). Normalising the $M_{cr}$ versus $L$ relationship by $\sqrt{rt}$ therefore serves to isolate out the effect of the varying buckle size in proportion to the length $L$, thus making the length-dependent behaviour independent of the $r/t$ ratio. The half-wavelength of the bending boundary layer that affects the plastic collapse behaviour at such short lengths is also defined in terms of this ‘unit’ (Eq. (A.3)). Hence, normalising the $M_{MNA}$ versus $L$ relationship by $\sqrt{rt}$ similarly serves to isolate out the effect of the varying boundary layer width in proportion to the length, again achieving an independence of $r/t$. However, this is only likely to be reliable for the case of the linear plastic collapse behaviour, since the boundary layer width is known to grow disproportionately with the applied load under geometrically nonlinear conditions (Rotter, 1983).

The relationship between the ratio of the computed $M_{MNA}$ and calculated $M_p$ and the dimensionless length $\omega$ of the cylinder may be characterised algebraically for either boundary condition as follows:

$$\frac{M_{MNA}}{M_p} = \frac{1}{1 - 0.2e^{-0.74\omega}} \quad \text{for Clamped (C)} \quad (A.4)$$

$$\frac{M_{MNA}}{M_p} = \frac{1}{1 - 0.4e^{-1.83\omega}} \quad \text{for Simply-supported (S)} \quad (A.5)$$

These algebraic expressions capture the rise in computed collapse moment $M_{MNA}$ at very short lengths accurately and predict a value of unity as the length increases.
A.5 Detailed exploration of the plastic collapse behaviour of individual very short cylinders under uniform bending

The reasons for the discrepancy between the MNA predictions for very short cylinders and the classical analytical $M_p$ expression relate to the simple assumption of pure meridional membrane yield around the circumference. Long cylinders under bending are indeed under membrane action at midspan, and the entire thickness of a point of a thin shell is fully yielded. For short cylinders, however, the compatibility bending stresses induced by the boundary restraint cause early local yielding only at the surface of the shell. Moments in excess of $M_p$ must therefore be attained in order to propagate the yielding through the thickness so that it reaches the shell midsurface, but only then is the load-carrying capacity of the material truly exhausted and the plastic collapse mechanism can be said to have developed. In problems with such a complex interaction of membrane and bending action, the equilibrium curve also plateaus only very gradually, as seen in Fig. A.2, and requires significant computing time to capture fully. In such cases, the Convergence Indicator Plot (CIP) detailed in Doerich & Rotter (2011) was used to obtain an accurate estimate of the plateau $M_{MNA}$ moment with only a modest portion of the equilibrium curve. This method essentially uses information about the rate of gradient loss present in the available portion of a decaying curve to make a projection about where the gradient will be zero.

Further investigations were performed on a very short cylinder with $\omega = 1.4$ in the advanced stage of the plateau ($\phi / \phi_y = 10$). The deformed shapes of such cylinders reveal that the maximum deflection occurs at the midspan and with yielding (shown in red in Fig. A.3) already occupying most of the cylinder geometry under both boundary conditions.

Fig. A.3 – Illustration of the von Mises stress contour plot of the deformed cylinder ($\omega = 1.4$) under both end boundary conditions and in the advanced stage of the equilibrium path at $\phi / \phi_y = 10$. 

$\omega = 1.4$, $r/t = 100$; Clamped  $\omega = 1.4$, $r/t = 100$; Simply-supported
The membrane and bending stress resultant distributions of longer cylinders ($\omega = 10$) that fail at the classical collapse moment $M_{pl}$ predicted on the basis of simple beam theory were extracted from the loaded edge, midspan circumference, most compressed and tensed meridian of the cylinder at a large deformation corresponding to $\varphi / \varphi_y = 10$ on the yield plateau. The results are presented in Fig. A.4 and Fig. A.5 for the clamped (C) and simply-supported (S) end boundary conditions respectively. It is revealed that at these longer lengths, the axial membrane stress resultant $n_z$ attained but never exceeded the stress resultant $n_{pl}$ corresponding to membrane yield (Eq. (A.6)) throughout the regions of extraction, regardless of the end boundary condition. This epitomises the plastic collapse behaviour of general beams in which a full plastic collapse mechanism corresponds to the membrane stress in the beam attaining the material yield stress $\sigma_y$ value. Additionally, the shell bending stress resultants $m_z$ & $m_\theta$ and circumferential membrane stress resultants $n_\theta$ are mostly negligible all through the regions of extractions except for the tiny values observed close to the loaded edge of the clamped (C) cylinder (Fig. A.4), which is a consequence of compatibility bending deformations around these edges present for cylinders of any length.

$$n_{pl} = \sigma_y t \quad \text{and} \quad m_{pl} = \frac{\sigma_y t^2}{4} \quad (A.6)$$

By contrast, in the very short cylinders ($\omega = 1.4$), significant values of circumferential membrane stress resultant $n_\theta$ and bending stress resultants $m_z$ & $m_\theta$ were recorded from these short cylinders at midspan which require additional strain energy to form beyond that necessary to cause meridional yielding in $n_z$, even if these values do not attain $n_{pl}$ or $m_{pl}$ (Eq. (A.6)), a consequence of the boundary restraint severely influencing the propagation of yielding through the thickness of the cylinder as illustrated in Fig. A.6 and Fig. A.7 for the clamped (C) and simply-supported (S) conditions respectively. Despite the ability of these short cylinders to attain higher plastic collapse moment than the theoretical value $M_{pl}$, the relationships between the maximum and minimum surface principal stresses ($\sigma_1$ & $\sigma_2$) in these short cylinders and at very large deformations do not violate the provisions of the von Mises yield condition as the stresses all fall within the von Mises yield ellipse (Fig. A.8). It is also shown that the meridional membrane stress resultant $n_z$ exceeds the analytically predicted plastic collapse stress resultant $n_{pl}$ throughout the regions where the stress resultants have been extracted: this excess of $n_z$ relative to $n_{pl}$ is due to the presence of significant $n_\theta$ acting in the same sense and the shape of the von Mises yield criterion.
Fig. A.4 – Distribution of normalised meridional a) stress resultant $n_z/n_{pl}$, b) bending moment resultant $m_z/m_{pl}$ and circumferential c) stress resultant $n_θ/n_{pl}$, d) bending moment resultant $m_θ/m_{pl}$, over the edges of a longer ($ω = 10$) clamped (C) quarter cylinder model in the advanced stage of the yield plateau, corresponding to $ϕ/ϕ_y = 10$. 
Fig. A.5 – Distribution of normalised meridional a) stress resultant $n_z/n_{pl}$, b) bending moment resultant $m_z/m_{pl}$ and circumferential c) stress resultant $n_\theta/n_{pl}$, d) bending moment resultant $m_\theta/m_{pl}$, over the edges of a longer ($\omega = 1.4$) simply-supported (S) quarter cylinder model in the advanced stage of the yield plateau corresponding to $\phi / \phi_y = 10$. 

$\omega = 10$ Simply-supported (S)
Fig. A.6 – Distribution of normalised meridional a) stress resultant $n_z/n_{pl}$, b) bending moment resultant $m_z/m_{pl}$ and circumferential c) stress resultant $n_\theta/n_{pl}$, d) bending moment resultant $m_\theta/m_{pl}$, over the edges of a very short ($\omega = 1.4$) clamped (C) quarter cylinder model in the advanced stage of the yield plateau corresponding to $\phi / \phi_y = 10$. 

$\omega = 1.4$ Clamped (C)
Fig. A.7 – Distribution of normalised meridional a) stress resultant $n_z/n_{pl}$, b) bending moment resultant $m_z/m_{pl}$ and circumferential c) stress resultant $n_\theta/n_{pl}$, d) bending moment resultant $m_\theta/m_{pl}$, over the edges of a very short ($\omega = 1.4$) simply-supported (S) quarter cylinder model in the advanced stage of the yield plateau corresponding to $\phi / \phi_y = 10$. 

$\omega = 1.4$ Simply-supported (S)
Fig. A.8 – Relationship between the maximum and minimum principal stress of these very short cylinders ($\omega = 1.4$) under uniform bending together with the von Mises yield ellipse with a) clamped (C) and b) simply-supported (S) end boundary conditions.
A.6 Conclusion

The plastic collapse behaviour of very short cylinders under uniform bending has been studied in detail, with the aid of materially nonlinear analyses (MNA) of the perfect cylinder and considering both the rigidly clamped (C) and rotationally unrestrained simply-supported (S) boundary conditions. The following conclusions are drawn:

- It is revealed that the boundary restraint at the edges can influence the collapse mechanism of very short cylinders whereby materially nonlinear moments in excess of $M_{pl}$ becomes necessary to achieve a full plastic collapse mechanism in the cylinder system. This behaviour is similar to the linear buckling behaviour of very short elastic cylinders under uniform bending in which the end boundaries severely restrain the formation of local short wave buckling mode, resulting in a higher bifurcation moment.

- The higher plastic collapse moment predicted for the very short cylinders decay rapidly with increasing length and the analytical prediction according to the classical theory $M_{pl}$ becomes a true representation as soon as the cylinder length $\omega$ exceeds 6 or 3 for the clamped (C) or simply-supported (S) end boundary conditions respectively.

- Compact and conservative algebraic expressions can be formulated to predict the reference plastic collapse resistance of cylinders of all lengths under uniform bending, one of the two reference resistances used both in dimensioning the global slenderness of general shells and in predicting the overall resistance of general shells through the Reference Resistance Design (RRD) framework.
Appendix B - Exploration of suitable dimensionless length for imperfect cylinders under uniform bending

B.1 Introduction

In the characterisation of the buckling behaviour of shells with the conceptual device of the capacity curve, it is beneficial to employ a suitable dimensionless group that can maintain a constant geometric nonlinearity (self-similarity) when the geometry of the shell is varied, here achieved by changing radius-to-thickness $r/t$ ratios. Where such a dimensionless group is maintained, the buckling strength predicted at varying $r/t$ but constant imperfection amplitude maintains the same portion of the reference elastic critical buckling resistance. Consequently, when the typical capacity curve of a shell system (Fig. B.1a) is transformed into an elastic modified capacity curve (Fig. B.1b), a straight vertical line is maintained, such that the elastic buckling strength may simply be read off the intercept on the horizontal axis of $M_k / M_{cr}$.

Achieving a constant geometric nonlinearity in a perfect cylinder system, evident as a straight vertical line in the elastic modified capacity curve, is particularly important for the characterisation of the buckling strength of shells within the framework of the recently developed Reference Resistance Design (RRD) method (Rotter, 2005; 2016) and numerous suitable dimensionless length parameters exist for different shell systems all in a geometrically perfect state. Key examples include the dimensionless length parameter $\omega$ (Eq. (B.8)) and Batdorf (1947) parameter $Z$ (Eq. (B.7)) for perfect cylinders under uniform axial compression.
(Yamaki, 1984; Rotter, 2004); $\omega$ and $\Omega$ (Eq. (B.9)) for the local ‘short-wave’ buckling found in ‘medium’ length perfect cylinders and the global ‘long-wave’ ovalisation response phenomenon typical of ‘transitional’ and ‘long’ length perfect cylinders under bending respectively (Calladine, 1983; Rotter et al., 2014; Sadowski et al., 2017). Furthermore, the computational study of Chen & Rotter (2012) established $\xi$ (Eq. (B.10)) as a suitable dimensionless length parameter governing the buckling behaviour of perfect cylindrical shells under wind loading.

\[
Z = \frac{L^2}{rt} \sqrt{1-v^2} = \omega^2 \sqrt{1-v^2} \quad \text{(B.7)}
\]

\[
\omega = \frac{L}{\sqrt{rt}} = \frac{L}{r} \cdot \left(\frac{r}{t}\right)^{1/2} \quad \text{(B.8)}
\]

\[
\Omega = \frac{L}{r} \cdot \left(\frac{r}{t}\right)^{-1/2} \quad \text{(B.9)}
\]

\[
\xi = \frac{L}{r} \cdot \left(\frac{t}{r}\right)^{3/2} \quad \text{(B.10)}
\]

However, there is no known prior study that has either explored or established the suitable dimensionless length for imperfect shell systems under any pattern of loading. Furthermore, it was shown in Chapter 5 of this thesis that the straightness of the elastic portion of the modified capacity curve cannot be achieved for imperfect elastic cylinders under bending with the adoption of an imperfection amplitude defined by $\delta_{el}/t$ and a dimensionless grouping according to the $\Omega$ group. Consequently, this mini study explores whether the main objective of a suitable dimensionless length group can be achieved by employing other definitions of a normalised imperfection amplitude which either relates directly with the radius of the cylinder or can be inspired by dimensionless groups involved in the dimple tolerance parameter $U_0$ as described in Chapter 2 of this thesis.

### B.2 Imperfection sensitivity with different definitions of the normalised imperfection amplitude performed at constant dimensionless group $\Omega$

Attempts to verify whether straightness of the elastic portion of the modified capacity curve can be preserved for long and imperfect cylinders under uniform bending grouped according
to the Ω dimensionless group were made by exploring the imperfection sensitivity of ‘transitional’ length cylinders of varying \( r/t \) ratios, using three new definitions of imperfection amplitude. The focus of this exploration is on ‘transitional’ length cylinders because the geometric bounds \( 0.5 \leq \Omega \leq 7 \) feature cylinders in which ovalisation is either completely restrained or has become fully developed. Only the rotationally restrained clamped (C) boundary condition is employed since transitional length cylinders are unaffected by the nature of the end boundary condition. However, the imperfection sensitivity study considers both the critical eigenmode and weld depression imperfection forms. Elastic cylinders with varying \( r/t \) from 100 to 500 were modelled using the finite element analysis software ABAQUS (2014). Although the yield strength of the material is irrelevant in an elastic analysis, for the purpose of constructing the modified capacity curves of these cylinders a generic steel grade with a 460MPa yield stress was assumed and the corresponding plastic collapse moment \( M_p \) was calculated using Eq. (B.11).

\[
M_p = \frac{4}{3} \sigma_y \left[ \left( r + \frac{t}{2} \right)^3 - \left( r - \frac{t}{2} \right)^3 \right] \quad \text{(B.11)}
\]

Here, \( \sigma_y \) is the yield stress of the material while \( r \) and \( t \) are the radius to midsurface and thickness of the cylinder wall respectively. The critical buckling moment computed through a linear bifurcation analysis (LBA) of the cylinder is denoted as \( M_c \), while the buckling resistance from a geometrically nonlinear analysis of the near-perfect (GNA) or imperfect (GNIA) structure is simply denoted as \( M_k \). It should be understood that in the actual finite element (FE) computations, no material nonlinearity in the cylinder was considered and the material yield stress was strictly used to calculate the theoretical plastic collapse moment of the cylinder and build the elastic modified capacity curves.

In the first instance, the imperfection amplitude was defined simply as a constant relationship relative to the radius of the middle surface of the cylinder \( (\delta_e/r) \), to investigate if a constant depth of dimple (or indentation) in the radial direction, despite varying \( r/t \), may produce straight vertical lines in the elastic portions of the computed modified capacity curves. The remaining two imperfection amplitudes relate directly with the measure of dimple parameter \( U_o \), presented in Chapter 2 of this thesis, where a suitable ‘stick length’ \( l_g \) must first be defined and applied as a denominator against the depth of initial dimple \( \Delta w_o \) (or \( \delta_e \) in this case). Consequently, in
the second case, the new imperfection amplitude employs $\sqrt{rt}$, which is the unit of axisymmetric buckle half-wavelength $\lambda_{\text{axi}}$ (Eq. (2.16)) and that of the linear meridional bending half-wavelength $\lambda$ (Eq. (2.17)) as a suitable denominator, leading to an equivalent geometric deviation of $\delta_e/\sqrt{rt}$. Finally, the unit of circumferential buckling half-wavelength $(l^2rt)^{0.25}$ is employed to investigate whether a constant geometric nonlinearity can be preserved in imperfect cylinders with fully developed ovalisation, which is a circumferential bending phenomenon, ultimately leading to an equivalent geometric deviation of $\delta_e/(l^2rt)^{0.25}$. Here, the imperfection amplitude varies with the cylinder length. The summary of the GNIA computational analyses, amounting to a total of 14,280 individual analyses, is presented in Table B.1. These new imperfection amplitudes span between $10^{-5}$ and $10^{-1}$ in geometric progression with a common factor of $10^1$ and it covers all geometric cases of near-perfect, moderately imperfect to deeply imperfect cylinders.

Table B.1 – Balance of analyses for the investigations involving diverse definitions of imperfection amplitude.

<table>
<thead>
<tr>
<th>Constant Amplitudes</th>
<th>Lengths $\Omega$</th>
<th>Ratios $r/t$</th>
<th>Imperfection forms</th>
<th>GNIAs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta_e/r$</td>
<td>14</td>
<td>34</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>$\delta_e/\sqrt{(rt)}$</td>
<td>14</td>
<td>34</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>$\delta_e/(l^2rt)^{0.25}$</td>
<td>14</td>
<td>34</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td><strong>Total:</strong></td>
<td></td>
<td></td>
<td></td>
<td><strong>14,280</strong></td>
</tr>
</tbody>
</table>

The results of the current exploration and the original dataset are presented in Fig. B.2 – Fig. B.5 for the $\delta_e/t$, $\delta_e/r$, $\delta_e/\sqrt{(rt)}$ and $\delta_e/(l^2rt)^{0.25}$ definitions respectively, where it can be seen that a straight vertical line remains achievable for the near-perfect geometric cases (thus confirming $\Omega$ as the appropriate dimensionless group even with different definitions of the imperfection amplitude) but as the cylinder becomes more imperfect and with increasing $r/t$, the more slender cylinders (i.e. higher $r/t$) fail at considerably lower buckling strengths, resulting in a leftward deviation from verticality. Overall, no clear benefit of employing any of these new definitions of imperfection amplitude to ensure that the geometric nonlinearity of the imperfect system is preserved can be identified.
Fig. B.2 – Elastic modified capacity curves of ‘transitional’ length imperfect cylinders under bending with the imperfection amplitude corresponding to $\delta / t$.

Fig. B.3 – Elastic modified capacity curves of ‘transitional’ length imperfect cylinders under bending with the imperfection amplitude corresponding to $\delta / r$. The corresponding value of $\delta / t$ at the least slender (lowest $r/t$) and most slender (highest $r/t$) cylinder is annotated.
Fig. B.4 – Elastic modified capacity curves of ‘transitional’ length imperfect cylinders under bending with the imperfection amplitude corresponding to $\delta_e / \sqrt{(rt)}$. The corresponding value of $\delta_e / t$ at the least slender (lowest $r/t$) and most slender (highest $r/t$) cylinder is annotated.

Fig. B.5 – Elastic modified capacity curves of ‘transitional’ length imperfect cylinders under bending with the imperfection amplitude corresponding to $\delta_e / (l^2 rt)^{0.25}$. The corresponding value of $\delta_e / t$ at the least slender (lowest $r/t$) and most slender (highest $r/t$) cylinder is annotated.
B.3 An attempt at fitting a suitable dimensionless parameter to the existing dataset for imperfect cylinder systems

Given the governing dimensionless length groups governing the behaviour of perfect cylinders systems under different loading conditions above, it may be noted that with the exception of the Batdorf parameter $Z$, all the other dimensionless length parameters may be defined by means of a generic geometric group $\zeta_{ab}$ (Eq. (B.12)), such that $a$, $\Omega$ and $\xi$ all correspond to a scaling parameter $a$ of 1 and $b$ of 1/2, -1/2 and 4/7 respectively:

$$\zeta_{ab} = \left( \frac{L}{r} \right)^a \cdot \left( \frac{r}{t} \right)^b$$  \hspace{1cm} (B.12)

Using this generic geometric group as a functional relationship together with the existing GNIA dataset (Fig. B.2) for the imperfect cylinder systems under the traditional imperfection amplitude definition of $\delta_e/t$, attempts were made to fit suitable scaling parameters $a$ and $b$ for the imperfect system. In the first instance, the scaling parameter $a$ was constrained to 1, similar to the prediction for the perfect cylinder systems, while $b$ was allowed to vary without bound until verticality of the elastic modified capacity curve is achieved. The condition for verticality was set by minimising the range between the maximum and minimum buckling strengths in a typical modified capacity curve of the imperfect system at constant length and $\delta_e/t$.

This tentative effort involving optimisation and root finding produced a trial algebraic expression for the scaling parameter $b$, which employs both the $\Omega$ (Eq. (B.9)) geometric group and the imperfection amplitude $\delta_e/t$ as defined in Eq. (B.13), where a value of $b = -\frac{1}{2}$ is automatically generated for a perfect system ($\delta_e/t \to 0$) or at a cylinder length of $\Omega = 0.5$.

$$b = -\frac{1}{2} + \frac{1}{5} \ln \left( \Omega + \frac{1}{2} \right) \cdot \left( 1 - e^{100(\delta_e/t)^b} \right)$$ \hspace{1cm} (B.13)

It would thus appear that a geometric group, simply denoted as $\Omega_b$ and defined in terms of the $\zeta_{ab}$ parameter in which $a = 1$ and $b$ as is defined above, preserved the geometric nonlinearity of the imperfect system, as illustrated in Fig. B.6 for the axisymmetric weld depression imperfection. However, a closer inspection of this outcome and wider GNIA data range revealed that the apparent verticality of the modified capacity curve observed is an illusion, traceable from the fact that this $\Omega_b$ geometric group merely scaled the GNIA dataset (shown in
Fig. B.7 as a plot of the relative strength $M_k/M_{cr}$ against the new group $\Omega_b$, using only a tiny portion of the results, to the extent that it appears to be a controlling dimensionless group. Overall, it appears that there may not be a suitable dimensionless group preserving self-similarity for deeply imperfect, long and ovalising cylinders.

Fig. B.6 – Characterised elastic modified capacity curves for cylinders having the weld depression form of imperfection and under the new dimensionless length group $\Omega_b$ but with the classical definition of imperfection amplitude ($\delta_e/t$).

Fig. B.7 – Original relationship between the relative elastic strength $M_k/M_{cr}$ and cylinder length under varying $r/t$ ratios, but with the length now defined in terms of $\Omega_b$ dimensionless group and constrained to a maximum of $\Omega_b = 7$. 

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Appendix C - Programming script for input file generation and job submission in ABAQUS

C.1 Generation of FE-model input files

Each input file corresponding to a cylinder model and having a particular set of parameters, such as a constant length, radius, thickness, material properties, imperfection form, amplitude of imperfection or type of computational analysis is generated by means of the python script in Fig. C.1 below.

```python
from abaqus import *
from abaqusConstants import *
from math import *
from caeModules import *
from driverUtils import executeOnCaeStartup
import sketch
import part
from math import *
import job
import numpy
import os
import shutil
executeOnCaeStartup()

Mdb()

# Geometry

t = 1.0 # mm - thickness
RS = [100., 200, 500., 1000.] # array of r/t
delta = 0.10*t # mm - imperfection amplitude

# Material (Elastic steel only)
E = 2e5 # MPa - Young's modulus
nu = 0.3 # Poisson's ratio

# Analysis

step = 4 # 1 = LBA; 2 = GNA; 3 = MNA; 4 = GNIA
pert = 0 # include perturbation? 0 - No; 1 - Yes
# ovalise = 0 # allow ends to ovalise? 0 - No; 1 - Yes
simple_support = 0 # allow meridional rotations at the ends? 0 - No; 1 - Yes
threads, cpus = 4, 2

Model, RS = 0, list(RS)
for R in RS:
    if step == 1: py_root = 'LBA_rot'+str(int(R/t))
    if step == 2: py_root = 'GNA_rot'+str(int(R/t))
    if step == 3: py_root = 'MNA_rot'+str(int(R/t))+'.S'+str(int(sy))
    if step == 4: py_root = 'GNIA_rot'+str(int(R/t))+'.dot'+str(round(delta/t,2)).replace('.',',','p')

    # Constants
    step_size = 0.04
    sigma_cl = E*t/(R*sqrt(3*(1-nu*nu))) # dcl
    N_cl = pi*R**2*sigma_cl # Ncl
    lamda = math.pi*math.sqrt(R*t)/((1-nu*nu))**0.25

    Files, PY = [], 0
    for TT in range(threads): # Automatic .py files for execution in the cmd prompt
        Files.append(open(py_root+'_runme'+str(TT)+'.py','w'))
        Files[TT].write('from os import system

# Invoked in command line as:
# python '+py_root+'_runme'+str(TT)+'.py

# Length design, to be filled with intended length ranges, e.g.
list_om = []
small_omega = numpy.linspace(4., 4.8, 5)
for ak in range(len(small_omegaal)):
    list_om.append(small_omegaal[ak])
```

...
LENTHS = []
for LL in list_cm:
    LENGTHS.append(LL*np.sqrt(R*t))
for LL in range(len(LENGTHS)):
    Model = 1
    L = LENGTHS[LL]
    omega = L / math.sqrt(R*t) # little omega
    Omega = omega*R/R # big Omega

    mdb.Model(name='Model-1' + str(Model), modelType=STANDARD_EXPLICIT)
    session.viewports['Viewport: 1'].setValues(displayedObject=None)

    if step == 1:
        job_name = 'LBA_rot' + str(int(R/t)) + '_om' str(round(omega,2)).replace('.',',','p')
    if step == 2:
        job_name = 'GNIA_rot' + str(int(R/t)) + '_om' str(round(omega,2)).replace('.',',','p')
    if step == 4:
        job_name = 'LBA_rot' + str(int(R/t)) + '_om' str(round(omega,2)).replace('.',',','p')

    # Defining tube geometry
    coords, nz = [], 100.0
    z_bot, z_seg = 0.0, 0.5*L
    dz = z_seg/nz # Meridional & radial increment sizes per segment
    for z in range(0, int(nZ)+1):
        z_loc = z_bot + float(z)*dz
        z_weld = 0.0
        amp = delta/1.0432 # Maximum deviation of the geometry

    # Computation of imperfect local outer surface coordinate
    r0_loc = R*amp*exp(-pi*abs(z_loc - z_weld)/lamba)*cos(pi*abs(z_loc - z_weld)/lamba) + sin(pi*abs(z_loc - z_weld)/lamba)
    coords.append((r0_loc, z_loc))
    s = mdb.models['Model-1' + str(Model)].ConstrainedSketch(name='__profile__',
        sheetSize=1000.0)
    g, v, d, c = s.geometry, s.vertices, s.dimensions, s.constraints
    s.setPrimaryObject(option=STANDALONE)
    s.ConstructionLine(point1=(0.0, -500.0), point2=(0.0, 500.0))
    s.FixedConstraint(entity=g[2])
    session.viewports['Viewport: 1'].view.setValues(nearPlane=892.591,
        farPlane=1003.03, width=506.048, height=421.39, cameraPosition=(-20.8563,
        30.5729, 942.809), cameraTarget=(-20.8563, -20.8563, 0))
    s.Spline(points=coords)
    s.VerticalConstraint(entity=g[2], addUndoState=False)
    p = mdb.models['Model-1' + str(Model)].Part(name='Part-1', dimensionality=THREE_D,
        type=DEFORMABLE_BODY)
    p = mdb.models['Model-1' + str(Model)].parts['Part-1']
    p.BaseShellRevolve(sketches, angle=180.0, flipRevolveDirection=OFF)
    s.unsetPrimaryObject()
    session.viewports['Viewport: 1'].setValues(displayedObject=p)
    del mdb.models['Model-1' + str(Model)].sketches['__profile__']

# Defining material
mdb.models['Model-1' + str(Model)].Material(name='Steel')
mdb.models['Model-1' + str(Model)].materials['Steel'].Elastic(table=((E, nu),))
# Partitioning
if 0.5*L > 9.0*lamda:
a = mdb.models['Model-1'+str(Model)].rootAssembly
a.DatumPointByCoordinate(coords=(R, 3.0*lamda, 0,0))
a.DatumPointByCoordinate(coords=(-R, 3.0*lamda, 0,0))
a.DatumPointByCoordinate(coords=(0.0, 3.0*lamda, R))
a.DatumPointByCoordinate(coords=(0.0, 3.0*lamda, R)))
d11 = a.datums
a.DatumPlaneByThreePoints(point1=dim1[8], point2=dim1[9], point3=dim1[7])
d21 = a.datums
a.DatumPlaneByThreePoints(point1=dim1[5], point2=dim1[6], point3=dim1[4])
f1 = a.instances['Part-1-1'].faces
pickedFaces = f1.getSequenceFromMask(mask=('[#1]'), )
d11 = a.datums
a.PartitionFaceByDatumPlane(datumPlane=dim1[10], faces=pickedFaces)
f1 = a.instances['Part-1-1'].faces
pickedFaces = f1.getSequenceFromMask(mask=('[#1]'), )
d21 = a.datums
a.PartitionFaceByDatumPlane(datumPlane=dim1[11], faces=pickedFaces)
# Defining interaction and kinematic coupling
a = mdb.models['Model-1'+str(Model)].rootAssembly
rl = a.referencePoints
refPoints1=(rl[15],)
region1=regionToolset.Region(referencePoints=refPoints1)
a = mdb.models['Model-1'+str(Model)].rootAssembly
el = a.instances['Part-1-1'].edges
datum = mdb.models['Model-1'+str(Model)].rootAssembly.datums[14]
ans = mdb.models['Model-1'+str(Model)].rootAssembly.Set(name='m_set-1',
referencePoints=(
mdb.models['Model-1'+str(Model)].rootAssembly.referencePoints[15],))
if simple_support:
    mdb.models['Model-1'+str(Model)].Coupling(name='Constraint-1',
controlPoint=region1,
surface=region2, influenceRadius=WHOLE_SURFACE, couplingType=KINEMATIC,
localCsys=datum, u1=ON, u2=ON, u3=OFF, ur1=ON, ur2=OFF, ur3=OFF)
else:
    mdb.models['Model-1'+str(Model)].Coupling(name='Constraint-1',
controlPoint=region1,
surface=region2, influenceRadius=WHOLE_SURFACE, couplingType=KINEMATIC,
localCsys=datum, u1=ON, u2=ON, u3=OFF, ur1=ON, ur2=OFF, ur3=ON)
else:
a.DatumCsysByThreePoints(point2=dim1[2], name='Datum csys-2',
coordsType=CYLINDRICAL, origin=(0.0, 0.0, 0.0))
point1=el.instances['Part-1-1'].InterestingPoint(edge=el[2]),)
region2=regionToolset.Region(edges=edges1)
datum = mdb.models['Model-1'+str(Model)].rootAssembly.datums[14]
ans = mdb.models['Model-1'+str(Model)].rootAssembly.Set(name='m_set-1',
referencePoints=(
mdb.models['Model-1'+str(Model)].rootAssembly.referencePoints[15],))
if simple_support:
    mdb.models['Model-1'+str(Model)].Coupling(name='Constraint-1',
controlPoint=region1,
surface=region2, influenceRadius=WHOLE_SURFACE, couplingType=KINEMATIC,
localCsys=datum, u1=ON, u2=ON, u3=ON, ur1=ON, ur2=OFF, ur3=OFF)
else:
```
# Defining BCs
if 0.5*L > 9.0*lambda:
    # Midspan meridional symmetry
    a = mdb.models['Model-1'].rootAssembly
    region = a.instances['Part-1-1'].edges
    el = el.getSequenceFromMask(mask=('[#4 ]', ), )
    mdb.models['Model-1'].YsymmBC(name='BC-1', createStepName='Initial',
        region=region)
    # Circumferential symmetry
    a = mdb.models['Model-1'].rootAssembly
    el = a.instances['Part-1-1'].edges
    edgels = el.getSequenceFromMask(mask=('[#2 ]', ), )
    region = regionToolset.Region(edges=edgels)
    mdb.models['Model-1'].ZsymmBC(name='BC-1', createStepName='Initial',
        region=region)
    # Reference Point BC
    createStepName='Initial',
    region=region, ul=SET, u2=UNSET, u3=SET, ur1=SET, ur2=SET, ur3=UNSET,
    displacement=DISPLACEMENT, distributionType=UNIFORM, fieldName='', localCsys=NONE)
    else:
        # Midspan meridional symmetry
        a = mdb.models['Model-1'].rootAssembly
        el = a.instances['Part-1-1'].edges
        edgels = el.getSequenceFromMask(mask=('[#2da ]', ), )
        region = regionToolset.Region(edges=edgels)
        mdb.models['Model-1'].YsymmBC(name='BC-3', createStepName='Initial',
            region=region)
        # Circumferential symmetry
        a = mdb.models['Model-1'].rootAssembly
        el = a.instances['Part-1-1'].edges
        edgels = el.getSequenceFromMask(mask=('[#5 ]', ), )
        region = regionToolset.Region(edges=edgels)
        mdb.models['Model-1'].ZsymmBC(name='BC-3', createStepName='Initial',
            region=region)
    # Step
    if step == 1:
        # End bending moment load
        a = mdb.models['Model-1'].rootAssembly
```

if 0.5*L > 9.0*lambda:
    refPoints1=(r1[15],)
else:
    RefPoints1=(r1[5],)
region = regionToolset.Region(referencePoints=refPoints1)
mdb.models['Model-1-3'].Moment(name='Mcl1', createStepName='Step',
region=region, cm=0.5*M_c1, distributionType=UNIFORM, field='', localCsys=None)

# Mesh controls
circ = 50 # 50
merid = 100 # 100
if 0.5*L > 9.0*lambda:
    a = mdb.models['Model-1-3'].rootAssembly
    fl = a.instances['Part-1-1'].faces
    pickedRegions = fl.getSequenceFromMask(mask=('[#7 ]',), )
    a.setMeshControls(regions=pickedRegions, elemShape=QUAD, technique=STRUCTURED)
    elemType1 = mesh.ElemType(elemCode=S4R, elemLibrary=STANDARD,
    secondOrderAccuracy=OFF, hourglassControl=DEFAULT)
    elemType2 = mesh.ElemType(elemCode=S3, elemLibrary=STANDARD)
a = mdb.models['Model-1-3'].rootAssembly
    fl = a.instances['Part-1-1'].faces
    faced = fl.getSequenceFromMask(mask=('[#7 ]',), )
    pickedRegions = (faced, )
    a.setElementType(regions=pickedRegions, elemType1=(elemType1, elemType2))
else:
a = mdb.models['Model-1-3'].rootAssembly
    fl = a.instances['Part-1-1'].faces
    faced = fl.getSequenceFromMask(mask=('[#7 ]',), )
    pickedRegions = (faced, )
    a.setElementType(regions=pickedRegions, elemType1=(elemType1, elemType2))

# Adding keywords
if step == 1 and pert == 1:

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Fig. C.1 – Python programming script for the generation of cylinder finite element models.
C.2 Submission of jobs for FE analysis

The process of job submission is further divided into two stages. In the first stage, all the generated input files from Fig. C.1 are arranged into 4 different scripts, to utilise 2 cores for each of the 4 threads available in the computer system’s CPU. This arrangement of generated input files is performed to loop over the varying global parameters – length, radius or both – and store the prompt for running ABAQUS/Standard analysis through the command prompt. Details of this python script are presented in Fig. C.2 below. In the final stage, the 4 scripts generated from Fig. C.2 are then run through any suitable python programming environment.

```python
from numpy import *

ROTS = [100., 200., 300., 400.]

# array of r/t

DOT = 0.1

# imperfection amplitude

THREADS = 4

which = 'GNIA'

Files = []

for TH in range(THREADS):

    Files.append(open(TH.write('ALL_runme_'+str(TH)+'.py', 'w')))

    Files[TH].write('from os import system\n')

    Files[TH].write('import os\n')

    Files[TH].write('** Invoked in command line as: \n')

    Files[TH].write('python ' + 'ALL_runme_+str(TH)+.py\n')

    Files[TH].write('path = os.getcwd()\n')

    for ROT in ROTS:

        cdrot, count = 'rot'*str(int(ROT)), 0

        Files[TH].write('os.chdir("'+cdrot+'")\n')

        if which == 'LBA':

            if not os.path.isfile('.'+plt+''+'.com', True, False)

                root = 'path'+line[23:34]

                Files[TH].write('if not os.path.isfile('+root+''+'.fil')

        else:

            root = 'path'+line[23:34]

            Files[TH].write('if not os.path.isfile('+root+''+'.sta')

        Files[TH].write(''+line)

        for line in fid:

            count += 1

            if count > 3:

                if which == 'LBA':

                    if len(plt) > 0:

                        Files[TH].write(os.remove('+root+''+'.com')

                    else:

                        if which == 'GNIA':

                            Files[TH].write(os.remove('+root+''+'.com')

                        else:

                            Files[TH].write(os.remove('+root+''+'.com')

                    Files[TH].write(os.remove('+root+''+'.odb')

                    Files[TH].write(os.remove('+root+''+'.sim')

                    Files[TH].write(os.remove('+root+''+'.sta')

                    Files[TH].write(os.remove('+root+''+'.prt')

                    Files[TH].write(os.remove('+root+''+'.dat')

                    Files[TH].write(os.remove('+root+''+'.com')

                    Files[TH].write('else: print "+root[1:]+" skipped\n')

                first, second = open(file_name,'r',0), True, False

                # Removing all GNIA analysis files except the .sta files

                Files[TH].write('os.chdir(path)\n')

                Files[TH].close()
```

Fig. C.2 – Python programming script for the ordered arrangement of jobs, ready for submission.
Appendix D - Programming script for pre-processing and algebraic characterisation of output data from ABAQUS

The scripts written in Python programming language to perform the pre-processing of output data from ABAQUS computational analyses in order to enforce a non-increasing imperfection sensitivity at deeper imperfections, otherwise referred to as the synthetic imperfection sensitivity in Chapter 6 of this thesis is presented below. Furthermore, the algorithm for the algebraic characterisation of imperfection sensitivity mainly as a function of the equivalent geometric deviation using a power-law functional form is presented.

import os
import os.path
from math import *
from numpy import *
from scipy.optimize import curve_fit
import matplotlib.pyplot as plt
plt.rcParams.update({'figure.max_open_warning': 0})
plt.figure()
for BC in range(0,1):
    subplot_dummy += 1
    if BC == 0:
        filename, end_cond = 'Clamped BC IRF.txt', 'BC1r'
    elif BC == 1:
        filename, end_cond = 'SS BC IRF.txt', 'BC2f'
    constrain = True # SETS THE SYNTHETIC DATA EXTRACTION AS TRUE OR FALSE
    print('USING POWER LAW FUNCTION FORM')
    if BC == 0:
        res_file = 'Clamped BC coefficients (power law).txt'
    else:
        res_file = 'SS BC coefficients (power law).txt'
    with open(filename, r') as fid:
        content = fid.readlines()
        content = [x.strip() for x in content]
    fid2 = open(res_file, 'w+')

    xdata = [0.00, 0.10, 0.25, 0.35, 0.50, 0.75, 1.00, 1.50, 2.00] # δe/t
    ydata = content().split()
    if constrain:
        if ydata[m] > ydata[m-1]:
            ydata[m] = ydata[m-1]

    def func2(x, a, b):
        return 1.0/(1.0+(a*(x**b)))
    ini_guess = [0.1, 0.1]
    popt, pcov = curve_fit(func2, xdata, ydata, ini_guess)
    coeffs = tuple(popt)
    y_fit_ini = array(list(map(func2,xdata,coeffs[0]*ones(len(xdata)),coeffs[1]*ones(len(xdata))))))
    resid = tuple(array(ydata) - y_fit_ini)
    if min(resid) < 0.0:
        sigma = ones(len(xdata))
        indx = [resid.index(sorted(resid)[1]), resid.index(min(resid))]
        sigma[indx] = 0.01
        popt = curve_fit(func2,xdata,ydata,[coeffs[0],coeffs[1]],sigma=sigma)
        coeffs_new = tuple(popt)
y_fit = 
array(list(map(func2,xdata,coeffs_new[0]*ones(len(xdata)),coeffs_new[1]*ones(len(xdata)))))

else:
    pass

plt.figure()

plt.scatter(xdata, array(raw_ydata), marker='x', m=50, label='FE - Data')
plt.plot(xdata, y_fit, 'r--', label='FIT Data')
plt.legend(); plt.xlim(0.00, 2.00); plt.ylim(0.00, 1.20
plt.ylabel('Imprecision reduction factor, \( \alpha_I \)', fontsize=16)
plt.xlabel('Equivalent geometric deviation, \( \delta_e/t \)', fontsize=16)
plt.grid(b=True, which='both', color='0.55', linestyle='--')

y_bar = mean(ydata)
res = array(ydata) - array(y_fit[0:len(xdata)])
tot = array(ydata) - y_bar
SSres, SSTot = dot(res, res), dot(tot, tot)
R_squared = round((1 - (SSres/SSTot))*100, 2)
res_array = list()

x1, x2 = "(0.3f)".format(popt[0]), "(0.3f)".format(popt[1])

x3 = "(0.1f)".format(small_omega)

x4 = "(0.2f)".format(R_squared)

fid2.write(str(x3))
fid2.write( ' ')

fid2.write(str(x2))
fid2.write( ' ')

fid2.write(str(x2))
fid2.write( ' ')

fid2.write(str(x4))
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Appendix E - Algorithms for the automatic termination of ongoing geometrically nonlinear analysis in ABAQUS

In a geometrically nonlinear analysis with finite element (FE) suite ABAQUS (2014), the equilibrium path of the structure may be more accurately traced by means of the static Riks (1979) arc-length algorithm, which allows the tracing of both ascending and descending load factors or displacements, unlike the static load-controlled and displacement-controlled algorithms. However, the default stopping criteria available in ABAQUS, while employing the Static Riks step, only consider the maximum load proportionality factor (LPF) on the whole structure or a maximum displacement on a specific tracker node, or a maximum number of tolerable increments in the load factor. Consequently, it becomes difficult to terminate an ongoing analysis automatically, except where the stopping conditions, defined in terms of these maximum load or displacements, are known ab initio.

Nevertheless, ABAQUS allows the introduction of user subroutine scripts, written in FORTRAN fixed form (with .f or .for extension) programming language, to enhance some of the processes involved in any given computational analysis, although with a pre-requisite installation of a suitable FORTRAN compiler. This additional capability of the FE suite allows a continuous run of parametric studies through repetitive computational analyses, provided that a carefully defined set of stopping criteria, hereafter referred to as Kill Conditions (KC’s), is available. This capability was exploited in all of the Static Riks analyses performed during this PhD research and the subroutine scripts employed are broken down into two. The first script (Fig. E.1 – GN_Riks_killer2) terminates an ongoing analysis under two sets of conditions, namely: a noticeable reduction in the load factor, or when the rate of increase is load factor becomes negligible, a judgement made using a tolerance value of $10^{-12}$. In addition to the two afore-mention Kill conditions, the second script (Fig. E.2 – GN_Riks_Killer3) incorporates a third kill condition, which detects any general ‘kink’ on the equilibrium path of the structure through an investigation of the percentage relative change in local radii of curvature of the equilibrium path.
SUBROUTINE URDFIL(LSTOP,LOVRWRT,KSTEP,KINC,DTIME,TIME)
  INCLUDE 'ABA_PARAM.INC'
  DIMENSION ARRAY(513),JRRAY(NPRECD,513),TIME(2)
  EQUIVALENCE (ARRAY(1),JRRAY(:,1))
  CHARACTER XINDIR*,XFNAME*,RESNAME*
  DATA OLDMAX/0.D0/
  CURRMAX = 0.D0
  TOL = 1.E-12
C RESNAME is the character content of the directory and name of the job
LEXFNAME = 0
LXINDIR = 0
XFNAME = '
XINDIR = '
RESNAME = '
CALL GETJOBNAME(XFNAME,LEXFNAME)
CALL GETOUTDIR(XINDIR,LXINDIR)
RESNAME = TRIM(XINDIR) /"\" TRIM(XFNAME)".txt"
C FIND CURRENT INCREMENT.
CALL POSFIL(KSTEP,KINC,ARRAY,JRCD)
C GET VALUE OF LOAD FACTOR AND STORE THIS IN CURRMAX
CALL DBFILE(0,ARRAY,JRCD)
IF (JRCD.NE.0) GO TO 110
KEY=JRRAY(1,2)
IF (KEY.SG.0.000) CONTINUE
CURRMAX = ARRAY(11)
110 CONTINUE
C COMPLETED READING OF CURRENT INCREMENT. NOW CHECK TO SEE IF VALUE OF
C LOAD FACTOR HAS INCREASED SINCE LAST INCREMENT
PRINT *, "INC = ",KINC
PRINT *, "THE CURRENT LPF is ",CURRMAX
PRINT *, "dLPF = ",CURRMAX-OLDMAX
C KILL CONDITION 1 - NEGATIVE dLPF
IF (CURRMAX.LE.OLDMAX) THEN
  CONTINUE
ELSE
  PRINT *, 'KILL CONDITION 1 REACHED'
  LSTOP=1
  OPEN(UNIT = 2, FILE = RESNAME, FORM = "FORMATTED", STATUS = UNKNOWN", ACTION =
''READWRITE")
  WRITE(2,*)(OLDMAX)
  CLOSE(2)
END IF
C KILL CONDITION 2 - dLPF less than TOL
IF (ABS(CURRMAX-OLDMAX).GE.TOL) THEN
  CONTINUE
ELSE
  PRINT *, 'KILL CONDITION 2 REACHED'
  LSTOP=1
  OPEN(UNIT = 2, FILE = RESNAME, FORM = "FORMATTED", STATUS = "NEW", ACTION =
''READWRITE")
  WRITE(2,*)(OLDMAX)
  CLOSE(2)
END IF
OLDMAX=ARRAY(11)
LOVRWRT=1
RETURN
END

Fig. E.1 – ABAQUS user subroutine for the automatic termination of an ongoing Riks analysis under two sets of Kill conditions (GN_Riks_Killer2)
SUBROUTINE URFIL(LSTOP,LOVRWRT,KSTEP,KINC,DTIME,TIME)
    INCLUDE 'ABA_PARAM.INC'
    DIMENSION ARRAY(513),JARRAY(NPRED3,513),TIME(2)
    EQUIVALENCE (ARRAY(),JARRAY(1,1))
    REAL, DIMENSION(100) :: ALL_LP,ALL_DOF,ALL_RCURV
    REAL CHANGE_RCURV,S1,S2,S3,A1,A2,PRODUCT_TRIANGLE_SIDES,AREA_TRIANGLE
    DATA ALL_DOF(KINC)=ALL_DOF(JSTEP),ALL_RCURV,JSTEP=0.00/
    PARAMETER (TOL_INC_LPFW0.01D-7,TOL_CHANGE_RCURV=1.0DD)

C RESNAME is the character content of the directory name of the job
CHARACTER XINDIR*255, XFNAME*40, RESNAME*400
LXNAME = 0
LXINDIR = 0
XFNAME = ' '
XINDIR = ' '
RESNAME = ' '

CALL GETJOBNAME(XFNAME,LXNAME)
CALL GETOUTDIR(XINDIR,LXINDIR)
RESNAME = TRIM(XINDIR) //""/ TRIM(XFNAME)//".txt"

C Loop over a large number and call the .odb file
DO K = 1, 999999
    CALL DBFILE(0,ARRAY,JRCR)
    IF (JRCR.NE.1) GO TO 111
    KEY = JARRAY(1,1)

C LPF is stored in Record Key 2000 attribute 9 + offset 2 = array position 11
    IF (KEY.EQ.2000) THEN
        ALL_LP(KINC) = ARRAY(11)
        PRINT *, 'INCREMENT', KINC, 'CURRENT LPF', ALL_LP(KINC)
    END IF

C DOFs are stored in Record Key 101 attributes 2-7 + offset 2 = array positions 4-9
    IF (KEY.EQ.101) THEN
        ALL_DOF(KINC) = ABS(ARRAY(KINC))
    END IF

C The radius of curvature requires the last 3 points on the equilibrium path
    IF (KINC.GE.3) THEN
        KINC = KINC - 3
        S1 = SQRT((ALL_LP(KINC) - ALL_LP(KINC-1))**2 + (ALL_DOF(KINC) - ALL_DOF(KINC-1))**2)
        S2 = SQRT((ALL_LP(KINC) - ALL_LP(KINC-2))**2 + (ALL_DOF(KINC) - ALL_DOF(KINC-2))**2)
        S3 = SQRT((ALL_LP(KINC) - ALL_LP(KINC-3))**2 + (ALL_DOF(KINC) - ALL_DOF(KINC-3))**2)
        A1 = (ALL_DOF(KINC) - ALL_DOF(KINC-1))**2 + (ALL_DOF(KINC) - ALL_DOF(KINC-2))**2 + (ALL_DOF(KINC) - ALL_DOF(KINC-3))**2
        A2 = ALL_LP(KINC)**2 + ALL_DOF(KINC)**2
        AREA_TRIANGLE = S1*S2*S3
        PRODUCT_TRIANGLE_SIDES = S1 + S2 + S3
        AREA_TRIANGLE = 0.25D0*ABS(A1-A2)
        ALL_RCURV(KINC-2) = 0.25D0*PRODUCT_TRIANGLE_SIDES/AREA_TRIANGLE
    END IF
END IF
END DO

C Kill Condition 1 (KC1) is when Current LPF < Previous LPF
    IF (KINC.GE.2) THEN
        IF (ALL_LP(KINC).GE.ALL_LP(KINC-1)) THEN
            CONTINUE
        ELSE
            LSTOP = 1; PRINT *, "Kill Condition 1 met."
            OPEN(UNIT = 2, FILE = RESNAME, FORM = "FORMATTED", STATUS = "UNKNOWN", ACTION = "READWRITE")
            WRITE(2,*), ALL_LP(KINC)
            CLOSE(2)
        END IF
    END IF

C Kill Condition 2 (KC2) is when Current LPF Previous LPF < TOL_INC_LPFW
    IF (KINC.GE.2) THEN
        IF ((ALL_LP(KINC) - ALL_LP(KINC-1)).GE.TOL_INC_LPFW) THEN
            CONTINUE
        ELSE
            LSTOP = 1; PRINT *, "Kill Condition 2 met."
            OPEN(UNIT = 2, FILE = RESNAME, FORM = "FORMATTED", STATUS = "NEW", ACTION = "READWRITE")
            WRITE(2,*), ALL_LP(KINC)
            CLOSE(2)
        END IF
    END IF

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END IF
END IF
C     Kill Condition 3 (KC3) is when $\text{CHANGE}_R\text{CURV} > \text{TOL}_\text{CHANGE}_R\text{CURV}$, delayed to
C     at least increment 10
IF (KINC.GE.10) THEN
   CHANGE_RCURV = $\text{ABS}((\text{ALL}_R\text{CURV}(\text{KINC}-3)-\text{ALL}_R\text{CURV}(\text{KINC}-2))/\text{ALL}_R\text{CURV}(\text{KINC}-3))$
   PRINT *, "CURRENT CHANGE IN RCURV", CHANGE_RCURV
IF (CHANGE_RCURV.LE.TOL_CHANGE_RCURV) THEN
   CONTINUE
ELSE
   LSTOP = 1; PRINT *, "Kill Condition 3 met."
   OPEN(UNIT = 2, FILE = RESNAME, FORM = "FORMATTED", STATUS = "NEW", ACTION =
   "READWRITE")
   WRITE(2,*) ALL_LPF(KINC)
   CLOSE(2)
END IF
END IF
LOVRWRT = 1
RETURN
END

Fig. E.2 –ABAQUS user subroutine for the automatic termination of an ongoing Riks analysis under three sets of Kill conditions (GN_Riks_Killer3).