Nonlinear valuation under credit, funding, and margins: existence, uniqueness, invariance, and disentanglement[1] [2]

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Abstract

Since the 2008 global financial crisis, the banking industry has been using valuation adjustments to account for default risk and funding costs. These adjustments are computed separately and added together by practitioners as if the valuation equations were linear. This assumption is too strong and does not allow to model market features such as different borrowing and lending rates and replacement default closeout. Hence we argue that the full valuation equations are nonlinear, and this paper is devoted to studying the nonlinear valuation equations introduced in Pallavicini et al (2011).

We illustrate all the cash flows exchanged by the parties involved in a derivative contract, in presence of default risk, collateralisation with re-hypothecation and funding costs. Then we show how to obtain semi-linear PDEs or Forward Backward Stochastic Differential Equations (FBSDEs) from present-valuing said cash flows in an arbitrage-free setup, and we study the well-posedness of these PDEs and FBSDEs in a viscosity and classical sense.

Moreover, from a financial perspective, we discuss cases where classical valuation adjustments (XVA) can be disentangled. We show how funding costs are offset by treasury valuation adjustments when one takes a whole–bank perspective in the valuation, while the same costs are not offset by such adjustments when taking a shareholder perspective. We show that although we use a risk-neutral valuation framework based on a locally risk-free bank account, our final valuation equations do not depend on the risk-free rate. Finally, we show how to consistently derive a netting set valuation from a portfolio level one.

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1 Introduction

Before the financial crisis of 2007-2008, the credit risk of counterparties with a high rating was not rigorously accounted for in the valuation of over-the-counter (OTC) derivatives. The credit events on Fannie Mae, Freddie Mac, Lehman Brothers, Washington Mutual, Landsbanki, Glitnir, and Kaupthing were definitely a hint that things would have needed to change. Another event that modified the financial landscape was the abrupt rise in the difference between the rate implied by overnight indexed swaps (OISs) and the LIBOR rate. The increment of this basis was a sign of the sudden realisation by market participants of the credit and liquidity risk present in the interbank market. These risks lead the dealers to reconsider the valuation of OTC claims and amend the book value of their deals by various adjustments. After the financial crisis the size of OTC markets has shrank but still remains important. For example the market value of outstanding OTC derivative contracts was reported to be \$24.7 trillion end of 2012, with \$632.6 trillion in notional value (BIS2013).

1.1 A Concise Introduction to Valuation Adjustments

In the classical theory of derivative pricing, one usually evaluates a financial contract (for example an option or an interest rate swap) without taking into account the default risk of the contracting parties. Hence, the value of a contract can be in general determined by taking a conditional expectation, under a risk-neutral measure, of the discounted promised cash flows. As we have illustrated above, after the financial crisis of 2007-2008 the exposition towards counterparty risk for OTC contracts became of first importance. This is not to say that credit or counterparty risk were ignored before (see for example the reviews of the field [10, 32], or the early work [22]), but after the financial crisis it became definitely a major topic in both practitioners and academic literature. More specifically, a particular way to account for default and credit risk gained a lot of traction, leading to the so called *Valuation Adjustments*. In particular banks already had a big infrastructure of pricing libraries to compute the value of contracts without accounting for default risks (the so called *default-free* value). Hence, it made a lot of sense to define *adjustments* to be added to the default-free price in order to recover the actual value of the contract being examined. The literature on valuation adjustments is very vast and has many contributions both from practitioners and academics. For a broad introduction to the topic, we suggest the monographs [23] and [37]. Here we just highlight the aspects of the problem which are closer to this work.

The first valuation adjustment to consider is the so called credit valuation adjustment (CVA), which adjusts the value of a deal to take into account the possible losses due to the default of the counterparty in the considered contract. More precisely, CVA is a positive quantity, subtracted from the default-free price that is increasing with respect to the counterparty default probability. Clearly, in order to suffer a loss due to the default of a counterparty, the deal should generate a positive exposure towards said counterparty. For example if a bank sells a call option to a client, the bank is not exposed to the client default and hence this position does not need a CVA. On the other hand if the bank buys a call option from a client, then if the client defaults the bank might suffer losses due to the inability of the client of delivering the option's payoff. CVA is very intuitive, in fact we are simply saying that the value of an option bought from a risky party is lower than one traded with a default free one. CVA is a well known adjustment, and market players account for it since a while and is well established in the literature (see for example [22]).

As we said CVA is the adjustment due to the default risk of our counterparty. On the other hand there are cases (e.g. we sell an option) where our counterparty is at risk because of our default. From our counterparty's perspective, the price should be adjusted by a CVA. This adjustment, due to *our* default risk is commonly referred to as debit valuation adjustment (DVA). It can be also seen as the CVA seen from the eyes of the other party. Building on the previous call option example, a very natural thing to do is to consider contracts that have a bilateral exposure, for example forward rate agreements (FRAs). In this case the contract exposes *both* parties to default risk.

This kind of contracts present both a CVA and a DVA. While DVA has the merit of keeping valuation bilateral (our CVA is the counterparty DVA, and our DVA is the counterparty CVA), this adjustment has some counterintuitive consequences: for example, a dealer could book a profit as a consequence of a decrease of her credit quality. This is not just a theoretical possibility: for example Citigroup reported in a press release on the first quarter revenues of 2009 that "Revenues also included [...] a net 2.5$ billion positive CVA on derivative positions, excluding monolines, mainly due to the widening of Citi's CDS spreads". Accounting standards by the FASB accept DVA while the Basel Committee does not recognise it in the context of regulatory capital accounting. Another aspect of DVA is the difficulty of hedging it. In fact, one cannot sell protection on itself, and hence banks are left just with proxy hedging in order to cover changes in their own default intensity, leaving jump to default uncovered. See again [23] for a discussion. Valuation adjustments such as CVA and DVA aim at pricing default risk, on the other hand, one could adopt a more conservative approach and compute the economic capital relative to default risk. Clearly, this has to be done taking into account the dependence structure of market and credit risk, with the intrinsic computational burden that this implies, see for example [38]. Since default risk became a prominent risk factor, the market and regulators have tried to implement ways to reduce it. In particular, the use of collateralisation is now widespread in order to mitigate the exposure at default. In practice, the parties involved in a contract are usually required to post some assets every day as a guarantee in case they default (see Section 2.1 for a more detailed explanation). Collateralisation is definitely effective in reducing counterparty risk, and regulators such as the European Market Infrastructure Regulation (EMIR) are pushing for a high level of collateralisation in the market, enforcing stricter collateral rules for over the counter contracts. Another way to reduce the impact of default risk are netting agreements. Such an agreement between two parties allows them to net the exposures resulting from trades in the netting agreement. This is beneficial because it reduces outstanding exposures and avoids double counting. The last tools against default risk that we will mention are Central Counterparties (CCP). A central counterparty is a well capitalised financial institution that provides clearing and settlement services for derivatives. In particular CCPs act as intermediaries between two trading parties taking the risk of the counterparty default and ensuring that the payments are performed even in case of a default. To achieve this CCPs use both netting across clients and high margining standards. For more details on CCPs see [27, 33] while for valuation of CCP cleared contracts we refer the reader to [24]. Clearly collateralisation plays a role in reducing default risk and consequently CVA and DVA charges. On the other hand, contagion and gap risk effects may lead to residual CVA and DVA, as was shown for the case of credit default swap trades in [17]. More general contagion at a more systemic level has been investigated for example in [5]. Another important aspect, investigated in the literature, is the analysis of the so called wrong-way risk, i.e. when the exposure to a counterparty is negatively correlated with the credit quality of the counterparty itself. Analyses of the dependence structure between exposure and default risk are treated for example in [6], while [12] and [17, 18] specialise the problem to particular products: Longevity Swaps and Credit Default Swaps respectively.

In more recent years, the funding valuation adjustment (FVA) was introduced. This adjustment takes into account the cost of funding the deal. Traders hedge the deal with a client on the market, maintaining a portfolio of positions in the underlying asset and cash. To maintain these positions, and to feed the collateral account that is usually attached to an OTC trade, the trader needs funds from the treasury of the bank, which in turn needs to raise this money from external funders. All the interest charges on these borrowing and lending activities contribute to the valuation of the deal, and this is precisely what the FVA accounts for. This adjustment can be quite sizable, for example Michael Rapoport reported in the Wall Street Journal, on Jan 14, 2014, that funding valuation adjustments costed J.P. Morgan Chase $1.5 billion in the fourth quarter results. The rise of this valuation adjustment is probably due to the fact that the spread between secured rates and unsecured ones has widened due to the perceived increase in the riskiness of the banking sector. This has risen the costs associated with the carrying of a derivative position. The literature that analyses the valuation of derivatives, accounting for funding and default risk is vast: [45, 46] have a derivation of the valuation equations, [21] analyses the nonlinearity in said equations, while [13] deals with multicurve modeling (for a credit risk model in a multi-curve setup see for example [35]). From the practitioners side we have [25, 50] that frame the valuation problem into a PDE setting. For a backward stochastic differential equation approach, see instead [28] and [43]. Another interesting aspect treated in the literature is how to reconcile the perspective of different agents related to the bank for what concerns funding and default costs. In particular see [4, 39] for an analysis of the shareholder/bondholder problem in assessing the costs of financing (this aspect is treated in Section 5.1) and [1, 26] for a preliminary analysis of how to deal with netting sets (this subject will be treated in Section 5.2). Finally, for what concerns the numerical aspects of the problem see for example [42].

More recently, a capital valuation adjustment (KVA) is being discussed to account for the regulatory capital absorbed by the considered trade, see for example [36]. We will not treat KVA here since the industry didn't reach a consensus on its definition yet.

Valuation adjustments are a useful instrument to partially assess the risk of a position but they might lead to think that the above mentioned risks and costs of a trading position are linear. The design of a framework for valuing trades in presence of all these market imperfections can easily become quite challenging. For example, even the simple fact that borrowing and lending do not happen at the same rate, or the use of replacement closeout upon first default, leads to nonlinear equations and hence to the sophisticated mathematical tools needed to handle them (the market features leading to these nonlinearities will be analysed in Section 2.1).

1.2 Our Contribution and Structure of the Paper

This paper builds on the valuation equations first introduced in Pallavicini et al (2011) [45] extending their scope and showing for the first time well posedness results for said equations. The valuation equations of [45] are the first to highlight the fixed point nature of the valuation in presence of funding costs and default risk. They represent also the first approach that allows to use both a risk-free or replacement closeout model (compare with [50] and [25]). More specifically, we illustrate all the cash flows exchanged by the parties involved in a derivative contract, in presence of default risk, collateralisation with re-hypothecation and funding costs. Then, we show how to obtain a partial differential equaiton (PDE) and a forward backward differential equation (FBSDE) from present-valuing said cash flows in an arbitrage-free setup, and we study the well-posedness of these equations in a viscosity and classical sense. We highlight that our paper is the first work to treat both the PDE and the FBSDE of the valuation problem at a great detail level. The PDE perspective is useful since it highlights the links between our valuation problem and standard PDE based valuation techniques. On the other hand, the FBSDE formalism provides a connection with valuation via risk neutral expectations, powerful theorems for the regularity of the price (important for hedging), and alternative computational methods that can be used in high dimensions. We also show an interesting invariance result: while our starting point to derive our pricing equations is a classical risk-neutral valuation based on a locally risk-free rate, said rate does not appear in the final pricing equations that we obtain. This result frees us from having to infer the risk-free rate from market rates. In fact, if we choose as our hedging strategy a natural generalisation of the Black-Scholes delta hedging, our final PDEs and FBSDEs depend only on market quantities and contractual quantities. From a financial point of view we analyse the funding component of the value of a contract and deduce that it is offset by the funding benefits the bank faces in servicing trades when taking a whole-bank perspective. When these are negated, as in the case where one takes a shareholder perspective, funding costs remain. The whole bank versus shareholders debate was first brought to the general attention of the public in a basic setup in [39], and here we rigorously show under which assumptions it can be actually proven. Lastly, we also tackle the problem of how our pricing framework can be used in presence of multiple netting sets. This is a very realistic case of practical importance, and it is often neglected in the literature. In this work we show that, if we suppose that the bank is always net borrower on the market (banks usually are very leveraged companies), we can attribute a funding cost to each netting set and we can then carry on the valuation of the whole bank's portfolio. This result is a rigorous improvement of the ones in [1, 26].

The rest of the paper is structured as follows. In Section 2 we introduce the financial problem that we aim to solve: the pricing of a derivative contract subject to counterparty risk, collateralisation and funding. We describe the mathematical setup, and we derive a first valuation equation based on conditional expectations. In Section 3, we start by showing how, with a change of filtration technique, we derive a pricing equation for the pre-default value process of our derivative. Furthermore we show how this pricing equation leads naturally to a Markovian FBSDE. Finally we derive a semi-linear PDE by assuming regularity of the BSDE solution. In Section 4, we study conditions for well-posedness of the nonlinear valuation FBSDE and the associated PDE. We then present our invariance result: under a delta-hedging assumption, the solution does not depend on the risk-free rate. In Section 5, we analyse the Funding Valuation Adjustment, disentangling the different contribution and highlighting the cancellation between the funding term and the DVA on the funding strategy. Furthermore we reconcile the whole bank view with the netting sets one. In Section 6 we show how our theoretical contributions can be applied in a numerical example to concrete derivatives such as options or forwards. Lastly in Section 7 we summarise our contributions and illustrate future research directions. A number of longer proofs, an example of how funding cash flows originate, and a brief introduction to BSDEs are found in appendices.

2 Cash Flows Analysis and First Valuation Equation

In this section we explain in detail what is the financial problem that we will solve in the next sections. We take the perspective of a bank that enters an OTC derivative contract with a counterparty, for example an interest rate swap or an option.

2.1 The Cash Flows

The objective of the bank is to compute the value of the above mentioned contract, taking into account all the cash flows that the derivative position generates from inception to maturity:

- The first cash flows to take into account are the ones promised by the contract: for example in a typical interest rate swap the two parties will exchange over time floating rate payments for fixed rate ones, while in a European call option the payment will only happen at the maturity of the contract. Being interested in derivatives, we suppose that the payments promised by our contract depend on the value S_t of an underlying asset, and we allow both for a dividend process $\pi(S_t, t)$ and a payment at maturity $\Phi(S_T)$ that depend deterministically on time and the value of the underlying asset. This specification clearly includes many derivatives of practical interest such as European vanilla options, interest rate swaps or basis swaps.

- The second type of cash flows generated by a derivative position are the ones related to default risk. These include both on default cash flows and the ones relative to the collateralisation procedure being implemented to mitigate the default risk itself. Let us start from the collateralisation procedure: every day the two parties compute an estimate of the value of the contract and compare it to the estimate they had the day before. The party whose perspective shows that the contract has depreciated will have to post cash or assets for a value equal to the magnitude of the depreciation. On the other hand, the party which is receiving the collateral will have to remunerate it at a certain interest rate agreed by both parties. The collateralisation procedure ensures that the party which is positively exposed to the other one has some assets of the other party as a guarantee in case of default. In many practical cases the assets received as collateral can be *re-hypotecated*, i.e. traded on the market, and do not have to be kept segregated. In scenarios of mark to market inversion re-hypothecation can exacerbate default risk, since collateral may be not fully recovered. However, re-hypothecation has also benefits since in general receiving collateral is a source of funding for the bank. There are many details of the collateralisation procedure that we are omitting here (such as the presence of minimum transfer thresholds, haircuts and so on) because we want to focus only on the features that are essentials to our framework. In practice we will model the collateral procedure as a continuous stream of cash flows C_t that will be positive if the bank is receiving collateral and negative otherwise. We furthermore assume that collateral is paid in cash, and is remunerated at a rate c_t. We do not assume a specific rate such as LIBOR or OIS for c_t since our model does not depend on such a choice at this stage, but we will discuss possible implications in the numerical experiment of Section 6.

- The second type of cash flows due to default risk are the ones due to the default of one of the two parties. The exchanges of cash flows between the two parties continue up to the first default time, i.e. up to τ. At the default of one of the two parties the close-out value ε_τ, representing the residual value of the claim at τ, is computed. Moreover the parties also compute the quantity $\varepsilon_\tau - C_\tau$, i.e. the net exposure at the moment of default. If we exclude the possibility of simultaneous defaults (more details on this can be found in Section 2.2), then either the bank defaulted before the counterparty ($\tau_I < \tau_C$) or the counterparty defaulted before the bank ($\tau_C < \tau_I$). Suppose first that we are in the case $\tau_C < \tau_I$, then if $\varepsilon_\tau - C_\tau \leq 0$ the bank is net debtor and will have to repay the whole ε_τ to the counterparty. If on the other hand $\varepsilon_\tau - C_\tau > 0$, the bank is net creditor and will recover just a fraction REC_C of its credits, namely the corresponding cash flow will be equal to $C_\tau + REC_C(\varepsilon_\tau - C_\tau)$. It is often more practical to write formulae where the loss given default, defined as $LGD_X := 1 - REC_X$, appears instead of the recovery.

  The case in which the bank defaults before the counterparty is symmetrical. We assume for simplicity that the recovery fractions are deterministic exogenous quantities, this is a common practice in the literature and it considerably simplifies the exposition. The present value of these cash flows will give origin to CVA and DVA adjustments as stated in Section 1.1. Notice, however, that if the cash flow on default of the counterparty will always be paid or received by the shareholders of the bank, the same cannot be said for the cash flow on the bank's own default. In fact, in case of default shareholders are wiped out, and all the cash flows are distributed to creditors. A stream of literature on valuation adjustments, see for example [4], emphasizes that the valuation done by shareholders is different from the valuation done by bondholders. However, note that taking the whole bank perspective makes the valuation symmetric at closeout cash-flows level: if we charge something to our counterparty due to her default event, the counterparty will charge us something for our own default event, see for example the discussion in [23]. From a trader perspective, then the point of shareholders versus bondholder may appear to be a moot point if she needs to agree on a bilateral price. If instead, the valuation is carried out to take into account profitability analysis, fund transfer pricing or internal allocation exercises, then distinguishing different perspectives can be very helpful.

- Other cash flows come from the hedging of the derivative. We suppose that the bank enters repurchase agreements (repo) to hedge its exposure. A repo is a contract between two parties where one party lends out (usually on an overnight basis) some assets in exchange for cash. When the contract terminates the asset is returned to the owner which will have to pay an interest rate (repo rate) over the sum he received. This lending mechanism can be used to gain exposure to the underlying of the contract to hedge against the price movements of the derivative. We call H_t the value of the risky asset position that the bank has via the repo, C_t^H the cash received or posted in the repo and h_t the repo rate. We assume that the banks continuously rolls over repo contracts and that at each point in time $H_t + C_t^H = 0$, meaning that the bank receives in the repo the exact value of the assets it is lending. On the other hand the gain of the repo position will be given by the growth of the assets that are being repoed minus $h_t C_t^H = -h_t H_t$, i.e. the repo rate times the amount of cash received. This argument can be made symmetric since the bank can enter a repo contract both by lending out assets or by lending out cash depending on the sign of the exposure needed for hedging.

- The last bit of cash flows is due to the funding activity of the bank. Namely to pay the derivative cash flows, or to hedge it the bank needs cash. Exactly as in the Black-Scholes model the bank had access to an account to finance its expenses or to invest in a *risk free* way. Similarly we suppose that the bank can borrow or lend money at two different rates, respectively f_t^- and f_t^+. For a concrete specification of the two funding rates see Section 5. We indicate F_t the amount that the bank is borrowing ($F_t > 0$) or investing ($F_t < 0$) at each time t.

Finally (refining the reasoning of [46]) we consider also the on default cash flow on the funding strategy. We consider only the bank's default on the funding strategy since we assume that if the bank has a cash surplus, it will invest it into very safe assets. This is a reasonable model and does not introduce the complication of having to model several default times for the bank's funders. The present value of this cash flow will be denoted as DVA_F, "DVA of the funding operation", in analogy with the usual DVA (see Section 5.1 for more details). It is the so called DVA2 of [39].

2.2 The Mathematical Formulation

We consider a probability space $(\Omega, \mathscr{A}, \mathbb{Q})$, endowed with a filtration $(\mathscr{G}_u)_{u\geq 0}$ that represents the time evolution of the available information on the market. We indicate $\mathscr{G}$ the whole filtration $(\mathscr{G}_u)_{u\geq 0}$. As we as we are working in continuous time we will assume that all filtrations satisfy the *usual conditions*, i.e. that they are complete and right-continuous. We want to price a generic derivative contract, with maturity T, between two financial entities, an investor I and a counterparty C. We will refer to I either as the investor or the bank since the kind of investor that we generally have in mind is an investment bank and we will call "the derivative" or "the contract" a generic derivative between I and C. We suppose that both I and C can default. In particular we introduce two $\mathscr{G}$-stopping times τ_I, τ_C to model I and C default times respectively. Furthermore, we suppose that the default-free information on the market is encoded in the Brownian filtration $\mathscr{F} = (\mathscr{F}_u)_{u\geq 0}$. We assume that τ_I, τ_C are the first jump times of two Cox processes with stochastic $\mathscr{F}$-predictable intensities λ^I and λ^C (note that we are using a reduced form model for the credit spreads of both the bank and the counterparty and we are not considering macroeconomic factors, see for example [7]). More precisely, we suppose that there exist ξ_I and ξ_C, two independent standard (mean 1) exponential random variables, and two $\mathscr{F}$-predictable positive processes λ^I_t and λ^C_t. Then we define

$$\tau_i = \inf\left\{t \geq 0 \,|\, \int_0^t \lambda^i_u du > \xi_i\right\} \quad i \in \{I, C\}.$$

The so defined default times have many interesting properties (see for example the classic framework of Duffie and Huang [31] and the *Canonical Construction* in Example 9.1.5 of [10]) and we review the most important ones for our work in the following. First, the conditional survival probability associated to each default time has a bond-like expression given by

$$\mathbb{E}\left[1_{\{\tau_i>t\}} \,|\, \mathscr{F}_s\right] = e^{-\int_0^t \lambda^i_u du} \quad \forall s \geq t,\, i \in \{I, C\}.$$

Second, the two default times τ_I and τ_C are conditionally independent with respect to $\mathscr{F}$, meaning that

$$\mathbb{E}\left[1_{\{\tau_C>t_1\}}1_{\{\tau_I>t_2\}} \,|\, \mathscr{F}_s\right] = \mathbb{E}\left[1_{\{\tau_C>t_1\}} \,|\, \mathscr{F}_s\right]\mathbb{E}\left[1_{\{\tau_I>t_2\}} \,|\, \mathscr{F}_s\right] \quad \forall t_1, t_2 \in [0, s].$$

This seemingly technical property basically says that once we know the intensities of default of the two times, they are independent, or in other words that their intensities encode all their dependence structure. A consequence of the conditional independence property is that for the first to default time $\tau = \tau_I \wedge \tau_C$ we have

$$\mathbb{E}\left[1_{\{\tau>t\}} \,|\, \mathscr{F}_s\right] = e^{-\int_0^t \lambda_u du} \quad \forall s \geq t,$$

where $\lambda = \lambda^I + \lambda^C$. Lastly we have that the probability of simultaneous defaults is 0, i.e.

$$\mathbb{Q}[\tau_I = \tau_C] = 0.$$

For convenience of notation we indicate $\overline{\tau} = \tau \wedge T$. We further assume that for all t we have $\mathscr{G}_t = \mathscr{F}_t \vee \mathscr{H}^I_t \vee \mathscr{H}^C_t$ where

$$\mathscr{H}^I_t = \sigma(1_{\{\tau_I \leq s\}},\; s \leq t),$$
$$\mathscr{H}^C_t = \sigma(1_{\{\tau_C \leq s\}},\; s \leq t).$$

We will write $\mathbb{E}_t^{\mathscr{G}}[\cdot] := \mathbb{E}[\cdot|\mathscr{G}_t]$ and similarly for $\mathscr{F}$. We further assume that there exists an $\mathscr{F}$-adapted process r_t, called *risk-free* rate, that represents the time value of money in the market and we denote B_t the process accruing at rate r_t:

$$dB_t = r_t B_t dt \quad B_0 = 1.$$

Moreover we suppose that $\mathbb{Q}$ is a martingale (or risk-neutral) measure associated with the deflator B_t, i.e. a probability measure under which the price processes of the non-dividend-paying traded assets discounted at the risk-free rate are $\mathscr{G}$martingales. For convenience of notation we indicate $D(s,t,x) = e^{-\int_s^t x_u du}$ the discount factor associated to the rate x_u and in the case of the risk-free rate we define $D(s,t) := D(s,t,r)$. Lastly we introduce the $\mathscr{F}$-adapted price process S_t of the underlying of our derivative contract.

Remark 1. We notice that an $\mathscr{F}$ adapted $\mathscr{G}$-martingale can be shown to be also an $\mathscr{F}$-martingale. On the other hand It is possible to show that square integrable $\mathscr{F}$-martingale are in this setup also $\mathscr{G}$-martingale (see for example Section 7.3 of [40]).

2.3 The Valuation Equation

In Section 2.1 we explained in detail the cash flows generated by entering a generic derivative contract in presence of funding costs, collateral and default risk. In Table 1 we summarise these cash flows and our mathematical assumptions about them. Starting from these cash flows, here we derive a pricing equation for the value of the derivative we are pricing. A possible alternative to our approach is discussed for example in [11, 16].

Symbol	Role	Assumptions
$\Phi()$	Payoff at maturity	Lipschitz function of S_T
π	Contract dividends	$\mathscr{F}$-predictable
C	Collateral process	$\mathscr{F}$-predictable
H	Hedging process	$\mathscr{G}$-predictable
ε	Closeout amount	$\mathscr{F}$-predictable
c	Collateral remuneration rate	$c_t = 1_{\{C_t>0\}}c_t^+ + 1_{\{C_t\leq 0\}}c_t^-$, $\mathscr{F}$-predictable
ℓ^+, ℓ^-	Liquidity basis	Real numbers
f	Funding rate	$f_t = 1_{\{F_t\leq 0\}}f_t^- + 1_{\{F_t>0\}}f_t^+$, $\mathscr{G}$-adapted
h	Hedging rate	$h_t = 1_{\{H_t>0\}}h_t^+ + 1_{\{H_t\leq 0\}}h_t^-$, $\mathscr{G}$-adapted

Table 1: Summary of cash flows and their mathematical modeling

Remark 2. We note that the assumption on the $\mathscr{F}$ predictability of ε is not realistic for some credit derivatives, such as CDS, which could have a significant jump of value at the default of one of the parties. See for instance [17].

We wish to analyse the deal both from the perspective of the bank's shareholders and the whole bank. From the shareholders' point of view we should not include any cash flow that does not benefit them. In particular, since shareholders are the most junior claimants to the bank's cash flows we suppose that if the bank defaults, they will receive no money at all while more senior creditors will receive a portion of their investment. It then makes sense to run our analysis including all cash flows and we will put the indicator $1_{\{\mathfrak{b}\}}$ in front of the cash flows that do not benefit the shareholders. The aforementioned indicator has hence to be interpreted as equal to 1 if we are considering the whole bank's perspective, and 0 otherwise. Note that this is just a notation that we use, instead of writing the same formula twice. Hence the sign $1_{\{\mathfrak{b}\}}$ doesn't play any role when checking regularity assumptions, it is there just to denote the cash flows that are received only by the bondholders.

We start by supposing that we can write the value process V_t of our derivative as the sum of the funding, collateral and hedging accounts:

$$V_t = F_t + C_t + H_t + C_t^H.$$

Since we supposed the hedge to be perfectly collateralised we see that the amount that the bank needs to finance at each time is given by

$$F_t = V_t - C_t.$$

To find the value V_t we compute the conditional expectation under the measure $\mathbb{Q}$ of the sum of all future cash flows discounted at the risk-free rate:

$$\begin{aligned}
V_t =& \mathbb{E}_t^{\mathscr{G}}\Bigg[\underbrace{1_{\{\tau>T\}}D(t,T)\Phi(S_T) + \int_t^{\overline{\tau}} D(t,u)\pi_u du}_{\text{Contractual cash flows}}\Bigg] - \mathbb{E}_t^{\mathscr{G}}\Bigg[\underbrace{\int_t^{\overline{\tau}} D(t,u)(c_u - r_u)C_u du}_{\text{Cost of carry of collateral account}}\Bigg] \\
&- \mathbb{E}_t^{\mathscr{G}}\Bigg[\underbrace{\int_t^{\overline{\tau}} D(t,u)(f_u - r_u)(V_u - C_u) du}_{\text{Costs due to funding account}}\Bigg] - \mathbb{E}_t^{\mathscr{G}}\Bigg[\underbrace{\int_t^{\overline{\tau}} D(t,u)(r_u - h_u)H_u du}_{\text{Costs due to hedging}}\Bigg] \\
&+ \mathbb{E}_t^{\mathscr{G}}\Bigg[D(t,\tau)1_{\{t<\tau<T\}}\Big(\underbrace{\theta_\tau}_{\text{On default cash flows due to contract}} + \underbrace{1_{\{\mathfrak{b}\}}1_{\{\tau=\tau_I\}}LGD_I(V_{\tau_I} - C_{\tau_I})^+}_{\text{On default cash flow due to funding strategy}}\Big)\Bigg],
\end{aligned} \tag{1}$$

where we have postulated the following form for the on default cash flow

$$\theta_t = \varepsilon_t - 1_{\{\tau_C<\tau_I\}}LGD_C(\varepsilon_t - C_t)^+ + 1_{\{\mathfrak{b}\}}1_{\{\tau_I<\tau_C\}}LGD_I(\varepsilon_t - C_t)^-,$$

where we are assuming that the collateral account can be re-hypotecated as in [17]. For more details on the financial meaning of the cash flows appearing in Equation (1) see Section 2.1.

Remark 3. We remark that the process θ_t is not just a mathematical construct, but it represents what the on default cash flow would be if the first to default time would happen at time t. Along this line we can interpret ε_t as the residual value of the contract at time t.

Remark 4. We notice that, even if perfectly collateralised, the hedge still contributes to the gains of the position, as shown in Equation (1). For a financial explanation see Section 2.1, while for a concrete example on a call option see Appendix A.

Remark 5. Notice how (1) is implicitly defined on $\{\tau > t\}$, since all the integrals appearing on the right hand side are defined on the stochastic interval $(t, \overline{\tau}]$.

Equation (1) will be the starting point of our analysis in the following sections and builds on the valuation equations found in [45,46]. The equations in [45,46] are very interesting from a financial point of view because they cast a nonlinear, recursive valuation problem in a form that is very similar to what usual valuation formulae look like, i.e. a conditional expectation of discounted cash flows. Equation (1) is a fixed point problem since the process V appears on both side of the equality. This kind of fixed point problem that arise from taking a conditional expectation of the future values of a process, are usually translated in the formalism of (forward) backward stochastic differential equations and this will be the aim of the following two sections. For a short introduction to (F)BSDEs see Appendix C.

It should be mentioned that the terms due to contractual cash flows, cost of carry of the collateral and on default cash flow due to contract are well understood in the classical literature on default and credit risk (see for example [10] and references therein). On the other hand the components due to funding, hedging and the on default cash flow of the funding strategy, are still discussed and debated in the literature and in the next sections we will try to analyse them both from a mathematical and a financial point of view.

3 Pre-Default Value Process

In this section we want to obtain a BSDE for the *pre-default* value of the contract we are considering. Financially, this means that we want to obtain a formula for the price up to default, since we are not really interested in what happens after the default event. Mathematically, this means that we can simplify our problem. In fact saying that we are interested in the pre-default price means that we are interested in the quantity $1_{\{t<\tau\}}V_t$, and while V_t will in general be $\mathscr{G}$-adapted, we know that there exists an $\mathscr{F}$-measurable process $\widetilde{V}_t$ such that $1_{\{t<\tau\}}V_t = 1_{\{t<\tau\}}\widetilde{V}_t$. This is a considerable simplification since the $\mathscr{G} = (\mathscr{G}_t)_{t\geq 0}$ filtration is a complicated object that entails the information relative to default times. Hence, in order to obtain a more tractable FBSDE, we are interested in switching from the complete filtration to the much simpler, Brownian, default free one. This will also allow us to substitute default times and default indicators in our formula with default intensities and discount factors.

3.1 Change of Filtration

We now review the needed tools to achieve the above mentioned switch of filtration.

The following lemma, also known as *key lemma* in credit risk literature, (see for example [10] Section 5.1) plays a big role in understanding how to switch from filtration $\mathscr{G}$to filtration $\mathscr{F}$.

Lemma 3.1 (Key Lemma)**.** *For any $\mathscr{A}$-measurable random variable X and any $t \in \mathbb{R}_+$, we have:*

$$\mathbb{E}_t^{\mathscr{G}}[1_{\{t<\tau\leq s\}}X] = 1_{\{\tau>t\}}\frac{\mathbb{E}_t^{\mathscr{F}}[1_{\{t<\tau\leq s\}}X]}{\mathbb{E}_t^{\mathscr{F}}[1_{\{\tau>t\}}]}. \tag{2}$$

In particular we have that for any $\mathscr{G}_t$-measurable random variable Y there exists an $\mathscr{F}_t$-measurable random variable Z such that

$$1_{\{\tau>t\}}Y = 1_{\{\tau>t\}}Z.$$

The intuition behind the Lemma is the following: the only part of $1_{\{t<\tau\leq s\}}X$ that is measurable under $\mathscr{G}_t$ but not $\mathscr{F}_t$ is the indicator $1_{\{t<\tau\}}$. Hence if we want an expression involving only $\mathscr{F}$ expectation to coincide with $\mathbb{E}_t^{\mathscr{G}}[1_{\{t<\tau\leq s\}}X]$ we need to factor $1_{\{t<\tau\}}$ out. The lemma tells us that the price that we have to pay to do this is dividing $\mathbb{E}_t^{\mathscr{F}}[1_{\{t<\tau\leq s\}}X]$ by the "probability given $\mathscr{F}_t$" of the event $\{\tau > t\}$.

Now we will illustrate how we can apply the previous Lemma when there is a richer structure (e.g. a stochastic process) to exploit. In particular we want to perform the change of filtration in two main cases: the case of a stopped integral (i.e. of the form $\mathbb{E}_t^{\mathscr{G}}\left[\int_t^{\overline{\tau}}\varphi_u du\right]$) treated in Lemma 3.2, or the case of a process valued at a default time (i.e. of the form $\mathbb{E}_t^{\mathscr{G}}\left[1_{\{t<\tau<T\}}1_{\{\tau_I<\tau_C\}}\varphi_\tau\right]$), treated in Lemma 3.3.

Lemma 3.2. *Suppose that φ_u is a $\mathscr{G}$-adapted process. We consider a default time τ with intensity λ_u. If we indicate $\overline{\tau} = \tau \wedge T$ we have:*

$$\mathbb{E}_t^{\mathscr{G}}\left[\int_t^{\overline{\tau}}\varphi_u du\right] = 1_{\{\tau>t\}}\mathbb{E}_t^{\mathscr{F}}\left[\int_t^T D(t,u,\lambda)\widetilde{\varphi_u}du.\right]$$

where $\widetilde{\varphi_u}$ is an $\mathscr{F}_u$ measurable variable such that $1_{\{\tau>u\}}\widetilde{\varphi_u} = 1_{\{\tau>u\}}\varphi_u$.

Proof. See Appendix B.1. □

A useful result in the same spirit as the previous one is the following (Lemma 3.8.1 in [9])

Lemma 3.3. *Suppose that φ_u is an $\mathscr{F}$-predictable process. We consider two conditionally independent default times τ_I, τ_C generated by Cox processes with $\mathscr{F}$-intensity rates λ_t^I, λ_t^C. If we indicate $\tau = \tau_C \wedge \tau_I$ we have:*

$$\mathbb{E}_t^{\mathscr{G}}\left[1_{\{t<\tau<T\}}1_{\{\tau_I<\tau_C\}}\varphi_\tau\right] = 1_{\{\tau>t\}}\mathbb{E}_t^{\mathscr{F}}\left[\int_t^T D(t,u,\lambda^I+\lambda^C)\lambda_u^I\varphi_u du\right].$$

Now we apply the previous lemmas to Equation (1). More precisely we obtain the following expression for V_t:

$$\begin{aligned} V_t =&1_{\{\tau>t\}}\mathbb{E}_t^{\mathscr{F}}\left[D(t,T,r+\lambda)\Phi(S_T)+\int_t^T D(t,u,r+\lambda)(\pi_u-(c_u-r_u)C_u-(\widetilde{f}_u-r_u)(\widetilde{V}_u-C_u)-(r_u-\widetilde{h}_u)\widetilde{H}_u)du\right]\\ &+1_{\{\tau>t\}}\mathbb{E}_t^{\mathscr{F}}\left[\int_t^T D(t,u,r+\lambda)(\widetilde{\theta}_u+1_{\{\mathfrak{b}\}}LGD_I\lambda_u^I(\widetilde{V}_u-C_u)^+)du\right], \end{aligned} \tag{3}$$

where, we have (using Lemma 3.3):

$$\widetilde{\theta}_u=\varepsilon_u\lambda_u-LGD_C\lambda_u^C(\varepsilon_u-C_u)^+ + 1_{\{\mathfrak{b}\}}LGD_I\lambda_u^I(\varepsilon_u-C_u)^-. \tag{4}$$

We are now able to see the financial consequences of the previous lemmas. In fact Equation (3) shows that the value V_t up to default can be computed in the default free filtration $\mathscr{F}$, getting rid of all default indicators, but then we have to discount all cash flows by default intensity terms, lowering their value in absolute terms, or in other words we weight them by their probability of happening given $\mathscr{F}_t$.

Remark 6. We note that if a rate is of the form

$$x_t=x^+1_{\{g(V_t,H_t,C_t)>0\}}+x^-1_{\{g(V_t,H_t,C_t)\le 0\}},$$

for a linear function g, then on the set $\{\tau>t\}$ it coincides with the rate

$$\widetilde{x}_t=\widetilde{x}^+1_{\{g(\widetilde{V}_t,\widetilde{H}_t,C_t)>0\}}+\widetilde{x}^-1_{\{g(\widetilde{V}_t,\widetilde{H}_t,C_t)\le 0\}}.$$

Equation (3) says that V_t is zero on $\{\tau\le t\}$ and that on the set $\{\tau>t\}$ it coincides with some $\mathscr{F}_t$-measurable random variable, but we already know that $\widetilde{V}_t$ coincides with V_t on $\{\tau>t\}$. Hence we can write the following:

$$\begin{aligned} \widetilde{V}_t =&\mathbb{E}_t^{\mathscr{F}}\left[D(t,T,r+\lambda)\Phi(S_T)+\int_t^T D(t,u,r+\lambda)(\pi_u-(c_u-r_u)C_u-(\widetilde{f}_u-r_u)(\widetilde{V}_u-C_u)-(r_u-\widetilde{h}_u)\widetilde{H}_u)du\right]\\ &+\mathbb{E}_t^{\mathscr{F}}\left[\int_t^T D(t,u,r+\lambda)(\widetilde{\theta}_u+1_{\{\mathfrak{b}\}}LGD_I\lambda_u^I(\widetilde{V}_u-C_u)^+)du\right]. \end{aligned} \tag{5}$$

Notice that now we have the process $\widetilde{H}_u$ instead of H_u, meaning that in this equation we are considering the predictable version of the hedging process. This can be interpreted as the part of the hedging strategy that hedge default-free risks.

3.2 Markovian FBSDE and PDE

Now that we have an equation for $\widetilde{V}_t$, we are ready to transform it in an equivalent BSDE. On one hand this has the benefit of letting us use powerful existence and uniqueness theorems to guarantee the well posedness of our problem. On the other hand from the BSDE we can derive a PDE that makes easy to compare our problem to the usual Black-Scholes PDE and get a deeper intuition of what is happening. Equation (5) is a non-linear, implicit equation for $\widetilde{V}$ that we aim to translate into the BSDE formalism. For an introduction to BSDE and their use in finance see Section C. In the next Proposition we show the BSDE satisfied by $\widetilde{V}$.

Proposition 3.1. *The value process $\widetilde{V}_t$ satisfies the following BSDE:*

$$\begin{aligned} \widetilde{V}_t =&\Phi(S_T)+\int_t^T\left[\pi_u-(f_u+\lambda_u)\widetilde{V}_u+\widetilde{\theta}_u+(f_u-c_u)C_u-(r_u-\widetilde{h}_u)\widetilde{H}_u+1_{\{\mathfrak{b}\}}LGD_I1_{\{\widetilde{V}_u-C_u>0\}}\lambda_u^I(\widetilde{V}_u-C_u)\right]du\\ &-\int_t^T Z_udW_u. \end{aligned} \tag{6}$$

Proof. See Appendix B.2. □

Remark 7. Another possible approach (without default risk) to BSDE modelling of funding costs is shown for example in [43].

Remark 8. Notice that even if from Equation (6) it looks like $\widetilde{V}_t$ depends on the future, it is just how the BSDE is formulated (see Section C). In the next section we will obtain an $\mathscr{F}_t$-adapted process as a solution to (6).

Equation (6) is too general for our purposes. Hence, in order to guarantee existence and uniqueness of a solution, we will make some simplifying assumptions. Since in this manuscript we are not focusing on path dependent derivatives, it makes sense to set our problem in a Markovian framework. More precisely we assume that:

- π_u is a deterministic function $\pi(u, S_u)$ of u and S_u, Lipschitz continuous in S_u. This means that coupons just depend on time and on the value of the underlying ate the time they are being paid;
- $r, f^{\pm}, c^{\pm}, \lambda^I, \lambda^C, h^{\pm}$ are deterministic bounded functions of time;
- $C_u = \alpha_u \widetilde{V}_u$, where $0 \leq \alpha_u \leq 1$ is a deterministic function of time. This means that we assume that the collateral account is a fraction of the value process;
- the close-out value ε_t is equal to $\widetilde{V}_t$ (i.e. we are using a so called replacement closeout hypothesis that adds non-linearity with respect to a risk-free closeout, see for example [23] and [21]);
- $\widetilde{H}_u = H(u, S_u, \widetilde{V}_u, Z_u)$, where $H(u, s, v, z)$ is a deterministic function, Lipschitz-continuous in v, z uniformly in u. Moreover we assume that $H(u, s, 0, 0)$ is continuous in s. This means that we are postulating a particular form for the pre default hedging process (see Remark 11);
- we suppose that S_t is a solution to the following SDE

$$dS_t = r_t S_t dt + \sigma(t, S_t) dW_t,$$

  where $\sigma(t, S_t)$ is Lipschitz continuous uniformly in time and in S_t.

Remark 9. There are two main conditions that are needed to apply the well posedness theorems of Section 4. The first is some form of Lipschitz regularity in the coefficient of the BSDE and the second is that the BSDE should be Markovian, i.e. all the stochastic dependence should be on S, V or Z. In the light of this restrictions we picked some convenient assumptions that would allow us to apply said well posedness theorems and that would not burden the notation. An example of convenient choices are the deterministic and boundedness assumptions on the rates. We could have instead opted to introduce a multidimensional factor process instead of a one dimensional SDE for S and we could have made all the rated depend on it in a *regular* manner. This woud of course have been more general, but would have partially hidden the parallel between Black-Scholes theory and our framework. A similar observation can be made for the collateral and the close-out value processes. In fact we could have chosen different specifications for them, for example $\varepsilon = \varepsilon_t^{Deafult-free} = \mathbb{E}_t\left[D(t,T)\Phi(S_T) + \int_t^T D(t,u)\pi_u du\right]$ and $C_t = \alpha_t \varepsilon_t$ would represent the case of the so called risk-free closeout, where the residual value of the deal is computed as if it was between default-free counterparties. Also in this case there would be assumptions that would allow us to apply well posedness theorems but for simplicity we picked the replacement close-out hypothesis.

Remark 10. Note that assuming $\varepsilon_t = \widetilde{V}_t$ means that the term $\lambda_u \varepsilon_u$ in $\widetilde{\theta}_u$ will cancel the term $-\lambda_t \widetilde{V}_t$ in Equation (6).

Remark 11. We can assume such a specific form for the pre-default hedging process since the default intensities are deterministic, hence when we switched to filtration $\mathscr{F}$ the only risk left to be hedged is market risk. For a discussion on how to include bonds see [8] and [16].

Under above assumptions, equation (6) can be written as:

$$
\begin{aligned}
dS_t^{q,s} &= r_t S_t^{q,s} dt + \sigma(t, S_t^{q,s}) dW_t \quad q < t \le T \\
S_q &= s_q \quad 0 \le t \le q \\
dV_t^{q,s} &= -\underbrace{\left[\pi_t + \theta_t + ((1-\alpha_t)(1_{\{\mathfrak{b}\}} LGD_I 1_{\{V_t^{q,s}>0\}}\lambda_t^I - f_t) - \lambda_t - c_t\alpha_t)V_t^{q,s} - (r_t - h_t) H(t, S_t^{q,s}, V_t^{q,s}, Z_t^{q,s})\right]}_{B(t,S_t^{q,s},V_t^{q,s},Z_t^{q,s})} dt \\
&\quad + Z_t^{q,s} dW_t \\
V_T^{q,s} &= \Phi(S_T^{q,s}),
\end{aligned}
\tag{7}
$$

where we have emphasised the dependence on the initial data and we have omitted the tildes to ease the notation.

Now we want to give an intuition on how to obtain a Black-Scholes like PDE from our FBSDE. Suppose that the value of our contract can be written as $V_t^{q,s} = u(t, S_t^{q,s})$, where $u()$ is a $C^{1,2}$. Then we can apply Ito's formula to $u(t, S_t^{q,s})$, obtaining:

$$
du(t, S_t^{q,s}) = \left(\partial_t u(t, S_t^{q,s}) + r_t S_t^{q,s} \partial_s u(t, S_t^{q,s}) + \frac{1}{2}\sigma(t, S_t^{q,s})^2 \partial_s^2 u(t, S_t^{q,s})\right) dt + \sigma(t, S_t^{q,s}) \partial_s u(t, S_t^{q,s}) dW_t. \tag{8}
$$

Then by comparing expressions (8) and (7) we have the following

$$
\begin{aligned}
&\partial_t u(t, S_t^{q,s}) + r_t S_t^{q,s} \partial_s u(t, S_t^{q,s}) + \frac{1}{2}\sigma(t, S_t^{q,s})^2 \partial_s^2 u(t, S_t^{q,s}) = -B(t, S_t^{q,s}, \widetilde{V}_t, Z_t^{q,s}) \\
&\sigma(t, S_t^{q,s}) \partial_s u(t, S_t^{q,s}) = Z_t^{q,s}.
\end{aligned}
\tag{9}
$$

So, $V_t^{t,s} = u(t, s)$ satisfies the following semilinear PDE:

$$
\begin{aligned}
&\partial_t u(t, s) + \frac{1}{2}\sigma(t, s)^2 \partial_s^2 u(t, s) + r_t s \partial_s u(t, s) + B(t, s, u(t, s), (\partial_s u \sigma)(t, s)) = 0 \\
&u(T, s) = \Phi(s).
\end{aligned}
\tag{10}
$$

Moreover we see from (9) that the process $Z_t^{q,s}$ is in a certain sense, a multiple of the delta-hedging process.

4 FBSDE Well-Posedness and Invariance Results

We now analyse the well-posedness of both the FBSDE and the PDE of the previous section. In particular we will show existence and uniqueness results in a strong sense, while for weak existence see for example [3]. Moreover we will analyse the dependence of the solution from the risk-free rate r_t, and show that under a delta hedging hypothesis, the value process V_t does not depend on the risk-free rate.

4.1 Existence and Uniqueness results

More specifically as done in Pardoux and Peng [48] we have (for the case of a non Brownian driven FBSDE see [2], while see [30] for a generalization to fully coupled FBSDE):

Theorem 4.1. *Consider the following FBSDE on the interval* $[0, T]$

$$
\begin{aligned}
&dX_t^{q,x} = \mu(t, X_t^{q,x}) dt + \sigma(t, X_t^{q,x}) dW_t \quad q < t \le T \\
&X_t = x \quad 0 \le t \le q \\
&dY_t^{q,x} = -f(t, X_t^{q,x}, Y_t^{q,x}, Z_t^{q,x}) dt + Z_t^{q,x} dW_t \\
&Y_T^{q,x} = g(X_T^{q,x})
\end{aligned}
\tag{11}
$$

Assume that there exists a constant K *such that* $\forall t$

- $|\mu(t,x)-\mu(t,x')|+|\sigma(t,x)-\sigma(t,x')|\leq K|x-x'|$;
- $|\mu(t,x)|+|\sigma(t,x)|\leq K(1+|x|)$;
- $|f(t,x,y,z)-f(t,x,y',z')|\leq K(|y-y'|+|z-z'|)$;

Moreover suppose that there exists a constant $p\geq 1/2$ such that:

$$|g(x)|+|f(t,x,0,0)|\leq K(1+|x|^p),$$

and that the map

$$x\mapsto (f(t,x,0,0),g(x)),$$

is continuous,then there exist two measurable deterministic functions $u(t,x)$, $d(t,x)$ such that the unique solution $(X_t^{q,x},Y_t^{q,x},Z_t^{q,x})$ of (11) *is given by*

$$Y_t^{q,x}=u(t,X_t^{q,x})\qquad Z_t^{q,x}=d(t,X_t^{q,x})\sigma(t,X_t^{q,x}),$$

and moreover $u(t,x)=Y_t^{t,x}$ is the unique viscosity solution to following PDE

$$\begin{aligned}&\partial_t u(t,x)+\frac{1}{2}\sigma(t,x)^2\partial_x^2 u(t,x)+\mu(t,x)\partial_x u(t,x)+f(t,x,u(t,x),\sigma(t,x)\partial_x u(t,x))=0\\&u(T,x)=g(x).\end{aligned}\tag{12}$$

To obtain a classical solution to equation (12) we need to require more smoothness of the coefficients of equation (11). Since due to the positive parts present in the default term we cannot require regularity in the backward part of the FBSDE we have to resort to the following (see J.Zhang [54] Theorem 2.4.1 on page 41):

Theorem 4.2. *Consider Equation* (11)*. If we assume that there exists a positive constant K such that*

- $\sigma(t,x)^2\geq \frac{1}{K}$;
- $|f(t,x,y,z)-f(t,x',y',z')|+|g(x)-g(x')|\leq K(|x-x'|+|y-y'|+|z-z'|)$;
- $|f(t,0,0,0)|+|g(0)|\leq K$;

and moreover the functions $\mu(t,x)$ and $\sigma(t,x)$ are C^2 in x, with bounded derivatives, then equation (11) *has a unique solution $(X_t^{q,x},Y_t^{q,x},Z_t^{q,x})$ and $u(t,x)=Y_t^{t,x}$ is the unique* classical *(i.e. $C^{1,2}$) solution to the semilinear PDE* (12)*.*

We now apply Theorem 4.1 to our FBSDE, to obtain the following:

Theorem 4.3. *Suppose that the rates λ_t, f_t, c_t, h_t, r_t are bounded. Moreover assume that there exists a $p\geq 1/2$ such that $|B(t,s,0,0)|+\Phi(s)\leq K(1+|s|^p)$. Then* (7) *has a unique solution, and moreover $u(t,s)=V_t^{t,s}$ is a viscosity solution to the following semilinear PDE:*

$$\begin{aligned}&\partial_t u(t,s)+\frac{1}{2}\sigma(t,s)^2\partial_s^2 u(t,s)+r_t s\partial_s u(t,s)+B(t,s,u(t,s),\sigma(t,s)\partial_s u(t,s))=0\\&u(T,s)=\Phi(s).\end{aligned}\tag{13}$$

Proof. We want to show that $|B(t,s,v,z)-B(t,s',v',z')|\leq K(|v-v'|+|z-z'|)$ so that the assumptions of Theorem 4.1 are satisfied and we can use it to obtain the thesis. We start by rewriting the term

$$B(t,s,v,z)=\pi_t(s)+\theta_t(v)+((1-\alpha_t)(1_{\{\mathfrak{b}\}}LGD_I 1_{\{v>0\}}\lambda_t^I-f_t)-\lambda_t-c_t\alpha_t)v-(r_t-h_t)H(t,s,v,z).$$

This term is a sum of many terms, hence is sufficient to show that every summand is Lipschitz continuous. The term π_t is Lipschitz continuous in s by assumption. The θ term and the $((1-\alpha_t)(1_{\{\mathfrak{b}\}}LGD_I 1_{\{v>0\}}\lambda_t^I - f_t)-\lambda_t-c_t\alpha_t)v$ term are continuous and piece-wise linear, hence Lipschitz continuous. The last term is piece-wise linear as a function of H which is a Lipschitz function of v,z. □

4.2 Invariance Theorem

We now focus on the case in which we use delta-hedging. More precisely we have shown that, under enough regularity, Ito's formula identifies $Z_t^{q,s} = \sigma(t, S_t^{q,s})\partial_s u(t, S_t^{q,s})$ (see (9)) where $V_t^{q,s} = u(t, S_t^{q,s})$. Hence, let us consider the specific hedging strategy

$$\widetilde{H}_t^{q,s} = H(t, S_t^{q,s}, \widetilde{V}_t^{q,s}, Z_t^{q,s}) = S_t^{q,s} \frac{Z_t^{q,s}}{\sigma(t, S_t^{q,s})}. \tag{14}$$

Since here we want to focus on market risk, this hedging strategy does not include terms relative to bonds and hence it is not a perfect hedge. For a similar approach see for example [26]. Now we prove that with this choice equation (7) has a solution $V_t^{q,s}$ such that $V_t^{q,s} = u(t, S_t^{q,s})$ with $u \in C^{1,2}$. Due to the hedging term, $B(t, s, v, z)$ is not Lipschitz continuous in s. Hence the application of Theorem 4.2 to our FBSDE is not straightforward. Our strategy to solve (7) with the hedging term given by (14) is the following: we build a FBSDE equivalent to (7) to which we can apply Theorem 4.2. The equivalence of these two FBSDEs will be given by the fact that they have the same associated PDE. To build our equivalent FBSDE we can exploit the linearity in Z_t of the hedging term and we can absorb it in the drift of the forward equation. More precisely consider the following:

$$\begin{aligned}
&dS_t^{q,s} = h_t S_t^{q,s} dt + \sigma(t, S_t^{q,s}) dW_t \quad q < t \le T \\
&S_q = x \quad 0 \le t \le q \\
&dV_t^{q,s} = -\underbrace{\left[\pi_t + \theta_t + ((1-\alpha_t)(1_{\{\mathfrak{b}\}} LGD_I 1_{\{V_t^{q,s}>0\}} \lambda_t^I - f_t) - \lambda_t - c_t\alpha_t) V_t^{q,s}\right]}_{B'(t, S_t^{q,s}, V_t^{q,s})} dt + Z_t^{q,s} dW_t \\
&V_T^{q,s} = \Phi(S_T^{q,s}).
\end{aligned} \tag{15}$$

Note that the drift of the forward equation in (15) is the rate h. This could be a problem if one wanted to model different borrowing and lending repo rates. In fact the state dependency of h would add some nonlinearity to the problem (see for example [21]).

Finally, one verifies that the assumptions of Theorem 4.2 are indeed satisfied for this equation:

Theorem 4.4. *Assume that the rates λ_t, f_t, c_t, h_t, r_t are bounded, and that $\sigma(t, s)$ is a positive C^2 function with bounded derivatives and that $h^+ = h^-$, then we can apply Theorem 4.2 and hence equation* (15) *has a unique solution, and moreover $V_t^{t,s} = u(t, s) \in C^{1,2}$ and satisfies the following semilinear PDE:*

$$\begin{aligned}
&\partial_t u(t, s) + \frac{1}{2}\sigma(t, s)^2 \partial_s^2 u(t, s) + h_t s \partial_s u(t, s) + B'(t, s, u(t, s)) = 0 \\
&u(T, s) = \Phi(s).
\end{aligned} \tag{16}$$

We now go backward from a solution to the PDE (16) to a solution of the FBSDE (7). Under our assumptions on drift and diffusion terms the following forward SDE has a unique solution:

$$dS_t = r_t S_t dt + \sigma(t, S_t) dW_t \quad S_0 = s. \tag{17}$$

We define $V_t = u(t, S_t)$ and $Z_t = \sigma(t, S_t)\partial_s u(t, S_t)$. By Theorem 4.4 we know that $u(t, s) \in C^{1,2}$ and by applying Ito's formula and (16) we obtain:

$$\begin{aligned}
dV_t =& du(t, S_t) = \left(\partial_t u(t, S_t) + r_t S_t \partial_s u(t, S_t) + \frac{1}{2}\sigma(t, S_t)^2 \partial_s^2 u(t, S_t)\right) dt + \sigma(t, S_t)\partial_s u(t, S_t) dW_t \\
&= \left((r_t - h_t) S_t \partial_s u(t, S_t) - B'(t, S_t, u(t, S_t))\right) dt + \sigma(t, S_t)\partial_s u(t, S_t) dW_t \\
&= \left((r_t - h_t) S_t \frac{Z_t}{\sigma(t, S_t)} - \pi_t(S_t) - \theta_t(V_t) - ((1-\alpha_t)(1_{\{\mathfrak{b}\}} LGD_I 1_{\{V_t>0\}} \lambda_t^I - f_t) - \lambda_t - c_t\alpha_t) V_t\right) dt + Z_t dW_t.
\end{aligned}$$

Hence we found the following:

Theorem 4.5 (Solution to the Valuation Equation). *Let S_t be the solution to equation* (17) *and $u(t,s)$ the classical solution to equation* (16). *Then the process $(S_t, u(t,S_t), \sigma(t,S_t)\partial_s u(t,S_t))$ is the unique solution to equation* (7).

Proof. From the reasoning above we found that $(S_t, u(t,S_t), \sigma(t,S_t)\partial_s u(t,S_t))$ solves the equation (7). Then from Theorem 4.3 we know that equation (7) has a unique solution and hence we have the thesis. □

Remark 12. Our proof that $V_t = u(t,S_t)$ with $u(t,s) \in C^{1,2}$ has as by-product that the reasoning we used to choose $\widetilde{H}_t^{q,s} = H(t, S_t^{q,s}, \widetilde{V}_t^{q,s}, Z_t^{q,s}) = S_t^{q,s}\frac{Z_t^{q,s}}{\sigma(t,S_t^{q,s})}$ can now be made fully rigorously.

Remark 13. In case of perfect collateralisation ($\alpha = 1$), notice how we obtain

$$B'(t,s,u(t,s)) = \pi(t,s) + (1 - \lambda_t - c_t)u(t,s).$$

This makes the PDE (16) linear, and very similar to the original Black Scholes PDE.

Moreover notice that (15) or equivalently (16) do not depend on the risk-free rate r_t. This happens because the delta-hedging strategy leads to $B(t,s,u(t,s),\partial_s u(t,s)) = (\dots) + (r_t - h_t)s\frac{\partial_s u(t,s)}{\sigma(t,s)}$. This ensures that the $r_t \partial_s u(t,s)$ term of (13) cancels out with the one embedded in $B(t,s,u(t,s),\partial_s u(t,s))$ leading to (16). Hence we have proven the following:

Theorem 4.6 (Invariance Theorem). *If we choose $\widetilde{H}_t^{q,s} = H(t, S_t^{q,s}, \widetilde{V}_t^{q,s}, Z_t^{q,s}) = S_t^{q,s}\frac{Z_t^{q,s}}{\sigma(t,S_t^{q,s})}$, i.e. a delta-hedging strategy to back our deal and we apply either Theorem 4.3 or Theorem 4.4 to Equation* (15), *then we see that the price V_t does* not *depend on the risk-free rate r_t but only on market rates.*

This proposition shows that even if we started from a risk neutral approach, the valuation equation (15) (or equivalently (16)) does not depend directly on the risk-free rate but only on rates available on the market. This is satisfactory from a theoretical perspective because it shows that we can introduce the risk-free rate as a convenience tool to compute our price, but since our cash-flows do not allow for investing or lending at that rate, it does not appear in the evaluation. This in a sense is a sanity check that tells us that what we are doing is not arbitrary and the choices we are making are not influencing the final value. On the other hand in the next section we show that if we want to decompose the price into the default-free price and a sum of valuation adjustments, then the way this decomposition is usually done, depends on the choice of the risk-free rate.

Lastly we want to highlight the financial interpretation of the stochastic calculus ideas that we used. The equivalence between Equation (15) and Equation (7) with the choice $\widetilde{H}_t = S_t\frac{Z_t}{\sigma(t,S_t)}$ is similar to a change of measure that modifies the drift of the underlying. If we were to include bonds in our hedging strategy, we would have needed to change measure (and consequently Brownian motions and default intensities) to obtain a result equivalent of Equation (16). See for example [8] and [16]. The equivalence between Equation (5) and Equation (6) shows that the discount factor has the same function as the coefficients in front of V_t. In particular we notice that in Equation (5) we are discounting at the rate $r_t + \lambda_t$ and in the equivalent Equation (6) we eliminated the discount factors but we had to add the $(r_t + \lambda_t)V_t$ term. We are substituting the discount factor in our equation with the correspondent growth term.

5 Disentanglement of Valuation Adjustments

In this section we analyse the financial implications of what we have shown so far. In particular we deal with the interplay between funding terms and DVA ones, and we tackle the problem of evaluating many netting sets. As we will show in the numerical section, funding terms are the same order of magnitude as CVA terms and hence their overlap with other adjustments should be properly accounted for. Netting set evaluation, is a problem of big practical relevance that, to the best of our knowledge, has not been carefully treated in the literature as yet. In this section we would like to obtain more explicit formulae, and in order to do so we pick a specific, but still quite general, form for the funding rate f. In particular we suppose that the bank invests its surplus of cash in a relatively risk free asset (think of safe government bonds or even better of a strategy where the bank buys a bond and the associated CDS) and hence we suppose that f_t^+ is given by the risk free rate r plus a liquidity basis ℓ^+. For what concerns the borrowing rate, we suppose that the bank borrows money unsecuredly, from an external funder on the market, at a rate given by its zero coupon bond rate plus a liquidity basis ℓ^-. Hence we obtain the following form for the funding rate:

$$f_t = 1_{\{V_t - C_t \geq 0\}}(r_t + LGD_I \lambda_t^I + \ell^+) + 1_{\{V_t - C_t < 0\}}(r_t + \ell^-). \tag{18}$$

5.1 Funding Valuation Adjustments

The overlap of DVA terms and funding ones is a very important point has generated a lot of debate (see [4,39] and references therein) on the role of funding costs in the valuation of derivatives. To see how our model helps us to understand this issue we write Equation (15) in the conditional expectation form, substituting the expression for f from Equation (18), in the case $1_{\{\mathfrak{b}\}} = 1$ (i.e. taking the whole bank perspective).

Even if Equation (15) is an implicit equation, for the discussion of this section it makes sense to refer to particular terms of it. In particular, in the next proposition, we identify the cash flows contributing to Credit Valuation Adjustment (CVA), the Debt Valuation Adjustment (DVA), the Collateral Valuation Adjustment (ColVA), the Funding Valuation Adjustment (FVA = FBA-FCA, funding benefit minus funding cost), and the treasury Debt Valuation Adjustment (DVA$_F$).

Proposition 5.1. *The pre-default value $\widetilde{V}_t$ of a contract subject to default risk, collateralisation and funding costs can be written as:*

$$\begin{aligned}
\widetilde{V}_t = & \underbrace{\mathbb{E}_t^{\mathscr{F},h}\left[D(t,T,r)\Phi(S_T) + \int_t^T D(t,u,r)\pi_u du\right]}_{\textit{Default-free price}} + \underbrace{\mathbb{E}_t^{\mathscr{F},h}\left[\int_t^T D(t,u,r+\lambda)\lambda_u(\varepsilon_u - \varepsilon_u^{Default-free})du\right]}_{Mismatch} \\
& \underbrace{-\mathbb{E}_t^{\mathscr{F},h}\left[\int_t^T D(t,u,r+\lambda)(c_u - r_u)C_u du\right]}_{ColVA} \\
& \underbrace{-\mathbb{E}_t^{\mathscr{F},h}\left[\int_t^T D(t,u,r+\lambda)LGD_C\lambda_u^C(\varepsilon_u - C_u)^+ du\right]}_{-CVA} + \underbrace{\mathbb{E}_t^{\mathscr{F},h}\left[\int_t^T D(t,u,r+\lambda)LGD_I\lambda_u^I(\varepsilon_u - C_u)^- du\right]}_{+DVA} \\
& + \underbrace{\mathbb{E}_t^{\mathscr{F},h}\left[\int_t^T D(t,u,r+\lambda)LGD_I\lambda_u^I(\widetilde{V}_u - C_u)^+ du\right]}_{+DVA_F} \\
& \underbrace{-\mathbb{E}_t^{\mathscr{F},h}\left[\int_t^T D(t,u,r+\lambda)(1_{\{\widetilde{V}_u - C_u \leq 0\}}\ell^- + 1_{\{\widetilde{V}_u - C_u > 0\}}(LGD_I\lambda_u^I + \ell^+))(\widetilde{V}_u - C_u)du\right]}_{+FVA=+FBA-FCA},
\end{aligned} \tag{19}$$

where: we used $\mathbb{E}^{\mathscr{F},h}[\cdot]$ to indicate that the drift of the underlying is h as in Equation (15), $\varepsilon_t^{Deafult-free} = \mathbb{E}_t\left[D(t,T)\Phi(S_T) + \int_t^T D(t,u)\pi_u du\right]$ is the risk free value of the contract from u up to T and we recall that C_t is the collateralisation account while ε_t is the exposure process.

Proof. See Appendix B.3. □

We can see now the price as formed by a risk free component, a mismatch term proportional to the difference between $\varepsilon_t - \varepsilon_t^{Default-free}$, and the valuation adjustments. There are two funding components, one due to lending (FBA) and the other due to borrowing (FCA), analogous to the lending (CVA) and borrowing (DVA and DVA$_F$) components of the on default cash flow. Finally we have a ColVA component that could be similarly split into margin benefit minus margin cost, both being positive. Note that all the basic adjustments CVA, DVA, FCA, FBA, DVA$_F$ are positive.

Let us now focus on the funding terms and the DVA_F. We see that the DVA$_F$ compensates the funding costs of the portfolio, in fact assuming $\ell^+, \ell^- = 0$ we have DVA$_F$−FCA= 0. This equation means that funding costs are real if we take the shareholder perspective, while they vanish (or correspond just to a liquidity cost) if we examine the deal from the whole bank's perspective, i.e. including also the bondholder in the picture. What we find in our framework is coherent with the work of [39], in the sense that if we include the DVA$_F$ term then funding costs vanish. In accordance with [4] we find that funding costs are a matter of perspective. In particular they derive from accounting only for cash flows that benefit the shareholders. It is important to notice that despite arriving to analogous conclusions, our approach is different from the one of [4] and is more similar to the earlier valuation adjustments literature presented in Section 1.1.

How to deal with the funding costs then? In practice a bank treasury will implement a Funding Transfer Protocol (FTP in short) to align traders' decisions and shareholders' point of view. Derivative prices in the trader book will be adjusted to reflect funding costs so that traders will mark a loss in their book unless they charge a correspondent adjustment to the client (exactly as a shareholder would do). Naive FTP policies may result in arbitrages as shown by [39] for example. On the other hand funding costs are relevant only in non collateralised markets (see [24]), and the only remaining instance of such a market is the non liquid, asymmetric OTC market between banks and corporate. In such a context it might be actually difficult to lock in a real arbitrage due to the market imperfections just highlighted.

The need for a practical policy gave rise to all the FBA, FCA implementations that approximate the contribution of the funding term in Equation (15). An initial discussion of different stylized treasury and FTP policies had been given in [46].

Let us conclude this section by showing what happens if we assume that $\varepsilon_u = \varepsilon_u^{Default-free}$ and we also suppose that $\ell^+ = \ell^- = 0$ and $c_u = r_u$. In this simplified case, we have:

$$\begin{aligned}\widetilde{V}_t =& \underbrace{\mathbb{E}_t^{\mathscr{F}}\left[D(t,T)\Phi(S_T) + \int_t^T D(t,u)\pi_u du\right]}_{Default-free\, price} \\ & \underbrace{-\mathbb{E}_t^{\mathscr{F}}\left[\int_t^T D(t,u,r+\lambda) LGD_C \lambda_u^C (\varepsilon_u - C_u)^+ du\right]}_{-CVA} + \underbrace{\mathbb{E}_t^{\mathscr{F}}\left[\int_t^T D(t,u,r+\lambda) LGD_I \lambda_u^I (\varepsilon_u - C_u)^- du\right]}_{DVA}.\end{aligned} \tag{20}$$

Hence under these hypothesis we have seen that the price is obtained as in the case where only default-risk is present.

5.2 Netting Sets Valuation

Until now we always treated the deal we were pricing as if it was the only deal of our investor, or equivalently, we were treating the whole portfolio of the bank as part of a single netting set with a single counterparty. In this section we show how our reasoning can be easily applied to the valuation of multiple netting sets. In particular we look for a version of Equation (1) that can be applied to a single netting set instead that to the whole portfolio. In particular in this section, to make everything clearer we avoid discussing the change of filtration and the BSDE in order to make the presentation easier to follow. As a straightforward generalisation of the setup of Section 2 we consider n counterparties $C^1, \dots, C^n$ and our investor I. Hence we introduce the $n+1$ default times $\tau_{C^1}, \dots, \tau_{C^n}, \tau_I$, generated as first jump times of Cox processes with $\mathscr{F}$-predictable intensities $\lambda^{C^1}, \dots, \lambda^{C^n}, \lambda^I$. The whole information on the space is given by the filtration $\mathscr{G}$ defined at each time by $\mathscr{G}_t = \mathscr{H}_t^{C^1} \vee \dots \vee \mathscr{H}_t^{C^n} \vee \mathscr{H}_t^I \vee \mathscr{F}_t$. It is worth to notice that this construction is what is called *Canonical Construction* in Example 9.1.5 of [10]. It can be shown that in this setting the stopping time

$$\tau_C = \max_i \tau_{C^i}$$

admits an $\mathscr{F}$-predictable intensity λ^C and is conditionally independent from τ_I (see for example [10]). For convenience of notation is useful to introduce

$$\tau = \tau_I \wedge \tau_C, \quad \overline{\tau} = \tau \wedge T, \quad \tau^i = \tau_I \wedge \tau_{C^i}, \quad \overline{\tau}^i = \tau^i \wedge T.$$

In the following we consider the valuation of a portfolio composed by n netting sets, one for each counterparty. If we indicate with V_t the value at time t of such a portfolio we have the following generalisation of the valuation formula (1):

$$\begin{aligned} V_t =& \sum_{i=1}^n \mathbb{E}_t^{\mathscr{G}} \left[1_{\{\tau^i > T\}} D(t,T) \Phi^i(S_T) + \int_t^{\overline{\tau}} D(t,u) 1_{\{\tau_{C^i} > u\}} (\pi_u^i - (c_u^i - r_u) C_u^i) du \right] \\ & - \mathbb{E}_t^{\mathscr{G}} \left[\int_t^{\overline{\tau}} D(t,u) ((f_u - r_u) \left(V_u - \sum_{i=1}^n C_u^i 1_{\{\tau_{C^i} > u\}} \right) + (r_u - h_u) \sum_{i=1}^n H_u^i)) du \right] \\ & + \sum_{i=1}^n \mathbb{E}_t^{\mathscr{G}} \left[D(t,\tau^i) 1_{\{t < \tau^i < T\}} \theta^i \right] + \mathbb{E}_t^{\mathscr{G}} \left[D(t,\tau) 1_{\{\mathfrak{b}\}} 1_{\{t < \tau < T\}} 1_{\{\tau = \tau_I\}} LGD_I \left(V_{\tau_I} - \sum_{i=1}^n C_{\tau_I}^i 1_{\{\tau_{C^i} > \tau_I\}} \right)^+ \right]. \end{aligned} \tag{21}$$

where

$$\theta^i = \varepsilon_{\tau^i}^i \underbrace{-1_{\{\tau_{C^i} < \tau_I\}} LGD_{C^i} (\varepsilon_{\tau_{C^i}}^i - C_{\tau_{C^i}}^i)^+}_{-CVA^i} \underbrace{+1_{\{\tau_I < \tau_{C^i}\}} LGD_I (\varepsilon_{\tau_I}^i - C_{\tau_I}^i)^-}_{+DVA^i}. \tag{22}$$

In this formula we have that $\Phi^i, \pi^i, C^i, c^i, H^i, \theta^i$ indicate respectively the cash flow at maturity, the dividends, the collateral account, the collateral rate, the hedging portfolio, and the on default cash flow relative to the i-th netting set. We can observe that by definition of netting sets the on default cash flows given by the sum of the n on default cash flows while the funding is netted across all the netting sets. Now we want to find $V_t^1 \dots V_t^n$ such that $V_t = \sum_{i=1}^n V_t^i 1_{\{t < \tau_{C^i}\}}$, i.e. a distribution of the value of the portfolio to the single netting sets. This attribution is not straightforward since we are supposing that the investor funds its portfolio as a whole. A possible solution is to make the *reduced borrowing hypothesis*, meaning that we suppose that the bank is always net borrower on the market (this is almost always the case in practice).

If we suppose that the bank is always net borrowing, we can simplify the definition of f (see Equation (18)) by removing the indicators, as in

$$f_t = r_t + LGD_I \lambda_t^I + \ell^+. \tag{23}$$

This choice can be motivated by observing that the bank is funding its portfolio as a whole, regardless of whether the single netting sets are short or long cash. Notice that this choice affects also how we calculate the DVA of the operations needed to fund the hedging strategy. In particular, we have to drop the positive part from the DVA_F term since the bank is net borrowing, regardless of whether each netting set is long or short cash. Under this assumption, and using (23) we obtain:

$$\begin{aligned} V_t =& \sum_{i=1}^n \mathbb{E}_t^{\mathscr{G}} \left[1_{\{\tau^i > T\}} D(t,T) \Phi^i(S_T) + \int_t^{\overline{\tau}} D(t,u) 1_{\{\tau_{C^i} > u\}} (\pi_u^i - (c_u^i - r_u) C_u^i) du \right] \\ & - \mathbb{E}_t^{\mathscr{G}} \left[\int_t^{\overline{\tau}} D(t,u) ((LDG_I \lambda^I + \ell^+) \left(V_u - \sum_{i=1}^n C_u^i 1_{\{\tau_{C^i} > u\}} \right) + \sum_{i=1}^n (r_u - h_u^i) H_u^i)) du \right] \\ & + \sum_{i=1}^n \mathbb{E}_t^{\mathscr{G}} \left[D(t,\tau) 1_{\{t < \tau < T\}} \theta^i \right] + \mathbb{E}_t^{\mathscr{G}} \left[D(t,\tau) 1_{\{\mathfrak{b}\}} 1_{\{t < \tau < T\}} 1_{\{\tau = \tau_I\}} LGD_I \left(V_{\tau_I} - \sum_{i=1}^n C_{\tau_I}^i 1_{\{\tau_{C^i} > \tau_I\}} \right) \right]. \end{aligned} \tag{24}$$

As we said at the beginning of this section, we now want to split the portfolio value V_t as a sum of netting-set values. To do this we define V_t^i as the solution to the following equation:

$$
\begin{aligned}
V_t^i =&\mathbb{E}_t^{\mathscr{G}}\left[1_{\{\tau^i>T\}}D(t,T)\Phi^i(S_T)+\int_t^{\overline{\tau}^i}D(t,u)(\pi_u^i-(c_u^i-r_u)C_u^i)du\right]\\
&-\mathbb{E}_t^{\mathscr{G}}\left[\int_t^{\overline{\tau}^i}D(t,u)((LDG_I\lambda^I+\ell^+)(V_u^i-C_u^i)+(r_u-h_u^i)H_u^i))du\right]\\
&+\mathbb{E}_t^{\mathscr{G}}\left[D(t,\tau^i)1_{\{t<\tau^i<T\}}\theta^i\right]+\mathbb{E}_t^{\mathscr{G}}\left[D(t,\tau^i)1_{\{\mathfrak{b}\}}1_{\{t<\tau^i<T\}}1_{\{\tau^i=\tau_I\}}LGD_I\left(V_{\tau_I}^i-C_{\tau_I}^i\right)\right].
\end{aligned}
\tag{25}
$$

Then we have that $V_t=\sum_{i=1}^n V_t^i 1_{\{t<\tau_{C^i}\}}$.

We now briefly analyse Equation (25), to see if we can recognise in it, some of the valuation adjustments we discussed before, and in particular we will put emphasis on the funding ones. Let us note that Equation (25) can be seen as the equivalent of Equation (1) for the i-th netting set in the case of a symmetric funding policy. Notice however that the on own default cash flow

$$
\begin{aligned}
\overline{DVA_F^i}&=\mathbb{E}_t^{\mathscr{G}}\left[D(t,\tau^i)1_{\{\mathfrak{b}\}}1_{\{t<\tau^i<T\}}1_{\{\tau^i=\tau_I\}}LGD_I\left(V_{\tau_I}-C_{\tau_I}^i\right)\right]\\
&=\underbrace{\mathbb{E}_t^{\mathscr{G}}\left[D(t,\tau^i)1_{\{\mathfrak{b}\}}1_{\{t<\tau^i<T\}}1_{\{\tau^i=\tau_I\}}LGD_I\left(V_{\tau_I}-C_{\tau_I}^i\right)^+\right]}_{DVA_F^i}\\
&-\mathbb{E}_t^{\mathscr{G}}\left[D(t,\tau^i)1_{\{\mathfrak{b}\}}1_{\{t<\tau^i<T\}}1_{\{\tau^i=\tau_I\}}LGD_I\left(V_{\tau_I}-C_{\tau_I}^i\right)^-\right]
\end{aligned}
\tag{26}
$$

does not have a positive part as in equation (1) and it cannot be seen as a DVA_F minus a CVA of the funding strategy since it depends on the investor's default intensity only (for a discussion of the CVA term on the funding strategy see for example [15]). This point is crucial since $\overline{DVA_F^i}$ still cancels out with FVA^i (in the case of no liquidity basis), where:

$$
FVA^i=-\mathbb{E}_t^{\mathscr{G}}\left[\int_t^{\overline{\tau}^i}D(t,u)(LDG_I\lambda^I+\ell^+)(V_u^i-C_u^i)du\right].
$$

Another thing worth noticing is the double counting that at first sight appears in Equation (25). In fact one could split the FVA^i in $-FCA^i+FBA^i$ as follows:

$$
FVA^i=\underbrace{-\mathbb{E}_t^{\mathscr{G}}\left[\int_t^{\overline{\tau}^i}D(t,u)(LDG_I\lambda^I+\ell^+)(V_u^i-C_u^i)^+du\right]}_{FCA^i}+\underbrace{\mathbb{E}_t^{\mathscr{G}}\left[\int_t^{\overline{\tau}^i}D(t,u)(LDG_I\lambda^I+\ell^+)(V_u^i-C_u^i)^-du\right]}_{FBA^i},
$$

and then the FBA^i term overlaps with the DVA^i term given in Equation (22) (this is apparent when changing expectations to filtration $\mathscr{F}$ as done in Section 3.1). This issue is clearly solved by the fact that FVA^i as a whole cancels out with $\overline{DVA_F^i}$, while if one would price the netting set just applying the netting set DVA_F^i that would cancel only the FCA^i part leaving the double counting issue there.

Remark 14. It interesting at this point to note how general is the reduced borrowing hypothesis. In particular we observe that under the funding specification (18), and with $1_{\{\mathfrak{b}\}}=1$, Equations (25) and (1) differ only if $\ell^+\neq\ell^-$. In fact in Equation (25), $\overline{DVA_F^i}$ partially cancels out with the FVA^i and leaves just the integral of $-\ell^+(V_u-C_u)$, while in Equation (1) the term $FVA+DVA_F$ produces the integral of $\ell^-(V_u-C_u)^- -\ell^+(V_u-C_u)^+$. Another important point is that what we have said is true because we are accounting for the DVA_F term. If we disregard the DVA_F term then there is a difference between Equations (25) and (1) even if $\ell^+=\ell^-$.

Lastly in the general case (i.e. without assuming the reduced borrowing hypothesis) or if one wanted to assign a value to each deal, then it will be needed a way to attribute to each netting set or deal a portion of the positive part of a sum of the netting sets or deal's values. The problem is then analogous to the one of assigning a CVA or DVA to a single deal in a netting set, and this can be done following different policies. A possible one for example, based on the so called *Euler allocation* is given by [49].

6 A Numerical Example

In this section we give an illustrative example of the size of the different valuation adjustments in a simple setting. Even if, generally, banks use more sophisticated models, we believe that our stylised model is able to capture the relative size of valuation adjustments, and make us understand impact of netting without the clutter of a full-blown model. For this purpose we consider a six months European call option and a six months equity forward. The underlying is a stock modeled by the following equation under the real world measure:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \quad S_0 = 100\$,$$

where $\sigma = 40\%$. We assume that the bank has a constant default intensity λ^I of 2% and a constant loss given default LGD_I of 60%, while the counterparty has a default intensity λ^C of 4% and a loss given default LGD_C of 60%. Furthermore we assume that the banks hedges its market risk via repo and reverse repo contracts on the underlying stock, with associated rate $h = 0.5\%$. We assume that the collateral fraction is constant in time ($\alpha_t = \alpha$) and that collateral is remunerated at a constant rate c of 0.2%. Finally we assume the funding specification (18). We perform our valuation under the measure $\mathbb{Q}$ using an explicit Euler scheme for the system (15) (for a detailed analysis of the scheme see [14]):

$$\begin{aligned}
&S_{t_i} = S_{t_{i-1}}(1 + (t_i - t_{i-1})h + (W_{t_i} - W_{t_{i-1}})\sigma),\\
&S_0 = 100\$,\\
&V_{t_i} = \mathbb{E}_{t_i}\left[V_{t_{i+1}} + \left[\theta_{t_{i+1}} + ((1-\alpha)(1_{\{\mathfrak{b}\}} LGD_I 1_{\{V_{t_{i+1}}>0\}}\lambda^I - f) - \lambda - c\alpha)V_{t_{i+1}}\right](t_{i+1} - t_i)\right],\\
&V_T = \Phi(S_T),
\end{aligned} \tag{27}$$

Where $\Phi(S_T) = (S_T - K)^+$ in the case of the option and $\Phi(S_T) = (S_T - K)$ in the case of the forward.

Remark 15. The calibration of our model consists of two parts, which are generally independent. In the first part, one has to calibrate the forward model (the Black Scholes SDE in our case) to market data. We observe that our framework does not, in practice, introduce any new complexity in this procedure. In fact, our framework complexity is very reduced when we are dealing with fully collateralised or centrally traded products. For fully collateralised liquid products every term depending on the difference $V - C$ vanishes and we recover the Black and Scholes equation as shown in Remark 13. The second part of the calibration (common to any valuation framework that includes counterparty risk and funding costs) consists in finding a value for the different rates and the LGD. In practice the rates to be used in the pricing equation are available either from contractual documents (e.g. collateral rate), from the treasury (funding rates), or can be bootstrapped from the market . For example default intensities can be obtained from traded CDSs if available or proxied using data from similar companies or even data from rating agencies. Clearly, proxies are not perfect and the constitute a source of model uncertainty. To bootstrap default intensities from CDS market data, one needs to know the relevant *LGD*. This is usually taken from a ratings agency's or a data provider's estimate. For a concrete example of calibration to interest rate instruments see for example [13].

We suppose that the investor wants to buy a derivative (either the call or the forward), and wants to understand what is the value of the option. In the tables below we report the total value of the deal, the risk-free price, and the adjustments defined as in Equation (19). Let us start analysing the case of the call option.

	K=90			K=100			K=110		
$\ell^+ = \ell^- = 0$	$\alpha = 0$	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 0$	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 0$	$\alpha = 0.5$	$\alpha = 1$
Total	16.3592	16.4534	16.5481	11.2210	11.2858	11.3509	7.4567	7.4999	7.5434
Risk-free	16.5559	16.5559	16.5559	11.3563	11.3563	11.3563	7.5469	7.5469	7.5469
CVA	0.1943	0.0974	0.0000	0.1336	0.0670	0.0000	0.0891	0.0447	0.0000
DVA	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
FCA	0.0972	0.0487	0.0000	0.0668	0.0335	0.0000	0.0445	0.0223	0.0000
FBA	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
DVA_F	0.0972	0.0487	0.0000	0.0668	0.0335	0.0000	0.0445	0.0223	0.0000
ColVA	0.0000	-0.0040	-0.0081	0.0000	-0.0027	-0.0055	0.0000	-0.0018	-0.0037
Mismatch	-0.0024	-0.0010	0.0004	-0.0016	-0.0007	0.0003	-0.0011	-0.0004	0.0002
$\ell^+ = \ell^- = 0.001$									
Total	16.3510	16.4493	16.5481	11.2154	11.2830	11.3509	7.4530	7.4980	7.5434
Risk-free	16.5559	16.5559	16.5559	11.3563	11.3563	11.3563	7.5469	7.5469	7.5469
CVA	0.1943	0.0974	0.0000	0.1335	0.0670	0.0000	0.0891	0.0447	0.0000
DVA	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
FCA	0.1052	0.0528	0.0000	0.0723	0.0363	0.0000	0.0482	0.0242	0.0000
FBA	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
DVA_F	0.0971	0.0487	0.0000	0.0668	0.0335	0.0000	0.0445	0.0223	0.0000
ColVA	0.0000	-0.0040	-0.0081	0.0000	-0.0027	-0.0055	0.0000	-0.0018	-0.0037
Mismatch	-0.0025	-0.0010	0.0004	-0.0017	-0.0007	0.0003	-0.0011	-0.0004	0.0002

Table 2: Valuation of a call option with different strikes K, different levels of collateralisation α, and different liquidity basis ℓ^+, ℓ^-. Spot price 100$, maturity 6 months, annual volatility 40%.

In Table 2 we show the magnitudes of different valuation adjustments varying the strike of the option, the level of collateralisation and the liquidity bases ℓ^+, ℓ^-. For a given strike is clear how a higher collateralisation reduces the impact of valuation adjustments, zeroing them out in the case of perfect collateralisation. There are although some caveats to the last observation: first of all here we are considering the very stylised case of $C_t = \alpha V_t$ while in practice C_t can be different because it has to bee agreed between the two parties; secondly we are here disregarding gap-risk issues, i.e. the fact that the collateral present in the collateral account at the moment of default can be very different from the value of the deal, because of delays in the default procedure or of jump-to-default effects (see for example [24]). Nevertheless it remains true that collateralisation reduces the need for many valuation adjustments. Of course there is a trade-off: increased levels of collateral mean increased costs due to collateral remuneration (in fact to be even, the investor adds a bigger in absolute value, negative, ColVA). These costs could be even higher if we accounted for the so called initial margin, i.e. an additional collateral amount proportional to the value at risk of the exposure relative to the deal being collateralised(see [24]).

Another point that we can make from Table 2 is that both CVA and FCA are increasing with the moneyness of the option. This is intuitive, since the more the option is in the money the more valuable the deal is for the bank and hence more costly to fund and more impacting in case of the counterparty default. From Table 2 it can also be seen that the Mismatch term behaves as expected, roughly tracking the integral difference between V_t and the default-free price. In fact we can see that it decreases in absolute value both with the increase of collateral level and with the increase of value of the contract.

The analysis we carried on for the option can be also applied to the case of a forward, illustrated in Table 3.

	K=90			K=Par			K=110		
$\ell^+ = \ell^- = 0$	$\alpha = 0$	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 0$	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 0$	$\alpha = 0.5$	$\alpha = 1$
Total	10.1005	10.1686	10.2372	-0.0452	-0.0226	0.0000	-9.7140	-9.7291	-9.7434
Risk-free	10.2420	10.2420	10.2420	0.0000	0.0000	0.0000	-9.7480	-9.7480	-9.7480
CVA	0.1593	0.0798	0.0000	0.0887	0.0445	0.0000	0.0484	0.0243	0.0000
DVA	0.0197	0.0098	0.0000	0.0442	0.0221	0.0000	0.0816	0.0409	0.0000
FCA	0.0796	0.0399	0.0000	0.0443	0.0222	0.0000	0.0242	0.0121	0.0000
FBA	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
DVA_F	0.0796	0.0399	0.0000	0.0443	0.0222	0.0000	0.0242	0.0121	0.0000
ColVA	0.0000	-0.0025	-0.0050	0.0000	0.0000	0.0000	0.0000	0.0024	0.0048
Mismatch	-0.0018	-0.0008	0.0002	-0.0007	-0.0003	0.0000	0.0000	0.0000	-0.0002
$\ell^+ = \ell^- = 0.001$									
Total	10.0955	10.1661	10.2372	-0.0452	-0.0226	0.0000	-9.7099	-9.7266	-9.7434
Risk-free	10.2420	10.2420	10.2420	0.0000	0.0000	0.0000	-9.7480	-9.7480	-9.7480
CVA	0.1592	0.0798	0.0000	0.0887	0.0444	0.0000	0.0485	0.0243	0.0000
DVA	0.0197	0.0098	0.0000	0.0442	0.0221	0.0000	0.0816	0.0409	0.0000
FCA	0.0862	0.0432	0.0000	0.0480	0.0241	0.0000	0.0262	0.0132	0.0000
FBA	0.0016	0.0008	0.0000	0.0037	0.0018	0.0000	0.0068	0.0034	0.0000
DVA_F	0.0796	0.0399	0.0000	0.0443	0.0222	0.0000	0.0242	0.0121	0.0000
ColVA	0.0000	-0.0025	-0.0050	0.0000	0.0000	0.0000	0.0000	0.0024	0.0048
Mismatch	-0.0019	-0.0008	0.0002	-0.0007	-0.0003	0.0000	0.0002	0.0000	-0.0002

Table 3: Valuation of a forward contract with different forward prices K, different levels of collateralisation α, and different liquidity basis ℓ^+, ℓ^-. Spot price 100$, maturity 6 months, annual volatility 40%.

The main differences are in the presence of DVA and the change in sign of ColVA. For what concerns DVA we can see that it behaves oppositely to CVA with respect to the moneyness of the forward, because the less the forward is in the money for the bank ($K = 110$), the more it is in the money for the counterparty and hence the more the counterparty would like to be compensated for the bank's default. Similarly the ColVA is negative for the party which is in the money, since it is the one more likely to post collateral and a positive for the party which is out of the money.

	$\ell^+ = 0$			$\ell^+ = 0.001$		
	K=90	K=100	K=110	K=90	K=100	K=110
Total	16.3591	11.2209	7.4564	16.3509	11.2152	7.4526
Risk-free	16.5560	11.3563	7.5469	16.5560	11.3563	7.5469
CVA	0.1943	0.1336	0.0891	0.1943	0.1335	0.0891
DVA	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
FCA	0.0972	0.0668	0.0445	0.1052	0.0723	0.0482
FBA	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
DVA_F	0.0970	0.0666	0.0442	0.0970	0.0665	0.0442
ColVA	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Mismatch	-0.0024	-0.0017	-0.0011	-0.0025	-0.0017	-0.0012

Table 4: Valuation of an uncollateralised call option, under *reduced borrowing hypothesis*, with different strikes K, and different liquidity basis ℓ^+, ℓ^-. Spot price 100$, maturity 6 months, annual volatility 40%.

	$\ell^+ = 0$			$\ell^+ = 0.001$		
	K=90	K=Par	K=110	K=90	K=Par	K=110
Total	10.1006	-0.0452	-9.7147	10.0955	-0.0452	-9.7099
Risk-free	10.2420	0.0000	-9.7480	10.2420	0.0000	-9.7480
CVA	0.1593	0.0887	0.0485	0.1592	0.0887	0.0485
DVA	0.0197	0.0442	0.0816	0.0197	0.0442	0.0816
FCA	0.0796	0.0443	0.0242	0.0862	0.0480	0.0262
FBA	0.0197	0.0442	0.0816	0.0213	0.0479	0.0884
DVA_F	0.0599	0.0001	-0.0574	0.0599	0.0001	-0.0573
ColVA	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Mismatch	-0.0019	-0.0007	8.6806	-0.0019	-0.0007	0.0002

Table 5: Valuation of an uncollateralised forward, under *reduced borrowing hypothesis*, with different forward prices K, and different liquidity basis ℓ^+, ℓ^-. Spot price 100$, maturity 6 months, annual volatility 40%.

We now want to numerically analyse the problem of Section 5.2, i.e. how the valuation changes if we take a netting set perspective instead of a portfolio perspective. While Tables 2 and 3 show the pricing of our option and forward as if they constituted the whole bank portfolio, Tables 4 and 5 and show the price of the above mentioned derivatives as if each of them was a netting set. To compute these prices we solve (25). Remember that (25) lies on the assumption of reduced borrowing, i.e. on assuming that the bank as a whole is net borrowing. In particular notice that, as we noted in Section 5.2, if $\ell^+ = \ell^-$, or if the exposure of the netting set is always positive (the case of the option) the values computed with or without reduced borrowing hypothesis are the same. This can be seen comparing the columns of Tables 4 and 5 with the corresponding ones in Tables 2 and 3 (i.e. the ones with $\alpha = 0$). On the other hand if liquidity bases are different ($\ell^+ \neq \ell^-$), we obtain a different result. To see this, compare Table 6(where $\ell^+ = 0.001$ and $\ell^- = 0$) with the columns of Tables 4 and 5, relative to the case $\ell^+ = 0.001$. We can see that for the option nothing changes, since in this case the exposure of the investor is always positive, while for the forward we see different results due to ℓ^+ being different from ℓ^-. As explained in Section 5.2 if we disregard the DVA_F term then there is a mismatch in pricing even if $\ell^+ = \ell^-$. Is thus clear that funds transfer protocol of a bank should properly account for DVA_F, otherwise it creates discrepancies between netting sets and portfolio evaluation, roughly proportional to $LGD_I \lambda_I + (\ell^- - \ell^+)$.

The last point that we want to make is that we have shown how these adjustments are sizable, and therefore the need of a sophisticated framework to handle them is, in our opinion, justified.

	Forward			Call		
$\ell^+ = 0.001, \ell^- = 0$	K=90	K=Par	K=110	K=90	K=100	K=110
Total	10.0939	-0.0490	-9.7168	16.3510	11.2155	7.4530
Risk-free	10.2420	0.0000	-9.7480	16.5560	11.3563	7.5469
CVA	0.1592	0.0886	0.0484	0.1943	0.1335	0.0891
DVA	0.0197	0.0442	0.0816	0.0000	0.0000	0.0000
FCA	0.0862	0.0480	0.0262	0.1052	0.0723	0.0482
FBA	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
DVA2	0.0796	0.0443	0.0242	0.0971	0.0668	0.0445
ColVA	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Mismatch	-0.0020	-0.0009	4.7322	-0.0025	-0.0017	-0.0011

Table 6: Valuation of an uncollateralised forward and an uncollateralised call option, with different strikes K, and different liquidity basis ℓ^+, ℓ^-. Spot price 100$, maturities 6 months, annual volatility 40%.

7 Conclusions

In summary, we proposed a framework for the evaluation of derivatives subject to counterparty risk, collateralisation ad funding costs. Via a thorough analysis of all the cash flows happening during the life span of such a derivative, we have shown how the evaluation problem is intrinsically nonlinear, since many of these cash flows depend on the value of the derivative itself. To properly account for those nonlinearities, the valuation of a derivative in our framework requires the solution of a Backward Stochastic Differential Equation or an equivalent semilinear Partial Differential Equation. We show under which conditions such a solution exists and is unique, both in a classical and in a viscosity sense. From a financial and operative perspective, we illustrate how the funding component of the valuation (FVA) is offset by the on default funding benefit of the bank (DVA_F) when taking a whole bank perspective, while this is not the case from the shareholder's point of view. Moreover we conduct an analysis of how different valuation adjustments can be reconciled from a netting set level to a portfolio level. In this respect, we propose a way to assign valuation adjustments to each netting set, relying on a reduced borrowing hypothesis. Finally we conduct numerical experiments to show the practical impact and relevance of our theoretical findings. We show how the netting set and portfolio level valuation formulae lead to different results unless liquidity basis are equal. Furthermore we highlight the size of funding components in the valuation of a deal and hence the importance of being clear and consistent in taking a shareholder point of view or a whole bank one when performing the evaluation of a contract.

Appendices

A Adjusted Cash Flows Under a Simple Trading Model

In this section we show in a simple example how one can intuitively derive the pricing equation (1) in a simple case of a call option without credit risk. This will highlight the contribution of hedging and funding cash flows, without the complications introduced by default risk.

Example A.1. Consider a European call option on an equity asset S with strike K and maturity T. Supposing we are able to replicate the call, we assume

$$V_t = \frac{H_t}{S_t} S_t - C_t^H + F_t + C_t.$$

We show which cash flows are transferred between a trader, the bank treasury and the repo market to fund the trade. We analyse these cash flows on a small time interval $[t, t+dt]$, seen from the trader's perspective (i.e. the option buyer). This is based in large part on conversations with traders working with their bank treasuries.

Time t:

1. The trader wishes to buy a call option with maturity T whose current price is $V_t = V(t, S_t)$. He needs V_t cash to do that. So he borrows V_t cash from the bank treasury and buys the call.
2. He then receives the collateral amount C_t for the call, that he gives to the treasury.
3. Now he wishes to hedge the call option he just bought. To do this, he plans to repo-borrow $\frac{H_t}{S_t}$ stocks on the repo-market. Hence, he borrows $C_t^H = H_t$ cash at time t from the treasury.
4. He repo-borrows and amount $\frac{H_t}{S_t}$ of stock, posting cash $C_t^H = H_t$ as a guarantee.
5. He sells on the market the stock he just obtained from the repo, getting back the price H_t in cash.
6. He gives H_t back to treasury, so the outstanding debt to the treasury is $V_t - C_t$ since $H_t - C_t^H = 0$

Time $t + dt$:

7. The trader needs to close the repo. To do that he needs to give back $\frac{H_t}{S_t}$ stock that he plans to buy on the market. So he borrows $\frac{H_t}{S_t}S_{t+dt}$ cash from the bank treasury.
8. He buys $\frac{H_t}{S_t}$ stock and he gives it back to close the repo, in exchange he gets back the cash $C_t^H = H_t$ deposited at time t plus interest $h_t C_t^H = h_t H_t$.
9. He gives back to the treasury the cash $C_t^H = H_t$ he just obtained, so that the net value of the repo operation has been
$$H_t(1 + h_t\,dt) - \frac{H_t}{S_t}S_{t+dt} = -\frac{H_t}{S_t}\,dS_t + h_t H_t\,dt.$$
Notice that this $-\frac{H_t}{S_t}$ is the right amount one needed to hedge V in a classic delta hedging setting.
10. He closes the derivative position, the call option, and gets V_{t+dt} cash.
11. He has to pay back the collateral plus interest, so he asks the treasury the amount $C_t(1 + c_t\,dt)$ that he gives back to the counterparty.
12. The outstanding debt plus interest (at rate f) to the treasury is
$V_t - C_t + C_t(1 + c_t\,dt) + (V_t - C_t)f_t\,dt = V_t(1 + f_t\,dt) + C_t(c_t - f_t\,dt)$.
He then gives to the treasury the cash V_{t+dt} he just obtained, the net effect being
$$V_{t+dt} - V_t(1 + f_t\,dt) - C_t(c_t - f_t)\,dt = dV_t - f_t V_t\,dt - C_t(c_t - f_t)\,dt.$$
13. The total amount of flows is :
$$-\frac{H_t}{S_t}\,dS_t + h_t H_t\,dt + dV_t - f_t V_t\,dt - C_t(c_t - f_t)\,dt.$$

If we present–value the above flows in t in a risk neutral setting, we obtain

$$\mathbb{E}_t[-\frac{H_t}{S_t}\,dS_t + h_t H_t\,dt + dV_t - f_t V_t\,dt - C_t(c_t - f_t)\,dt] = -\frac{H_t}{S_t}(r_t - h_t)S_t\,dt + (r_t - f_t)V_t\,dt - C_t(c_t - f_t)\,dt - d\varphi(t)$$
$$= -H_t(r_t - h_t)\,dt + (r_t - f_t)(V_t - C_t)\,dt + C_t(r_t - c_t)\,dt - d\varphi(t)$$

This derivation holds assuming that $\mathbb{E}_t[dS_t] = r_t S_t\,dt$ and $\mathbb{E}_t[dV_t] = r_t V_t\,dt - d\varphi(t)$, where $d\varphi$ is a dividend of V in $[t, t+dt)$ expressing the funding costs. Setting the above expression to zero we obtain

$$d\varphi(t) = -H_t(r_t - h_t)\,dt + (r_t - f_t)(V_t - C_t)\,dt + C_t(r_t - c_t)\,dt$$

which coincides with the definition given earlier in (1).

B Proofs

B.1 Proof of Lemma 3.2

$$\mathbb{E}_t^{\mathscr{G}}\left[\int_t^{\overline{\tau}} \varphi_u du\right] = \mathbb{E}_t^{\mathscr{G}}\left[\int_t^T 1_{\{\tau>t\}}1_{\{\tau>u\}}\varphi_u du\right] = \int_t^T \mathbb{E}_t^{\mathscr{G}}\left[1_{\{\tau>t\}}1_{\{\tau>u\}}\varphi_u\right] du,$$

then by using Lemma 3.1 we have

$$= \int_t^T 1_{\{\tau>t\}}\frac{\mathbb{E}_t^{\mathscr{F}}\left[1_{\{\tau>t\}}1_{\{\tau>u\}}\varphi_u\right]}{\mathbb{Q}[\tau > t\,|\mathscr{F}_t]}du = 1_{\{\tau>t\}}\int_t^T \mathbb{E}_t^{\mathscr{F}}\left[1_{\{\tau>u\}}\varphi_u\right] D(0,t,\lambda)^{-1}du,$$

now we choose an $\mathscr{F}_u$ measurable variable such that $1_{\{\tau>u\}}\widetilde{\varphi_u} = 1_{\{\tau>u\}}\varphi_u$ and obtain

$$= 1_{\{\tau>t\}}\int_t^T \mathbb{E}_t^{\mathscr{F}}\left[\mathbb{E}_u^{\mathscr{F}}\left[1_{\{\tau>u\}}\right]\widetilde{\varphi_u}\right] D(0,t,\lambda)^{-1}du = 1_{\{\tau>t\}}\int_t^T \mathbb{E}_t^{\mathscr{F}}\left[D(0,u,\lambda)\widetilde{\varphi_u}\right] D(0,t,\lambda)^{-1}du$$
$$= 1_{\{\tau>t\}}\mathbb{E}_t^{\mathscr{F}}\left[\int_t^T D(t,u,\lambda)\widetilde{\varphi}_u du\right].$$

□

B.2 Proof of Proposition 3.1

We introduce the process:

$$X_t = \int_0^t D(0,u,r+\lambda)\pi_u du + \int_0^t D(0,u,r+\lambda)(\widetilde{\theta}_u + 1_{\{\mathfrak{b}\}} LGD_I \lambda_u^I (\widetilde{V}_u - C_u)^+) du$$
$$- \int_0^t D(0,u,r+\lambda)\left[(c_u - r_u)C_u + (f_u - r_u)(\widetilde{V}_u - C_u) + (r_u - \widetilde{h}_u)\widetilde{H}_u\right] du.$$

Now we can obtain a martingale summing up X_t and $D(0,t,r+\lambda)\widetilde{V}_t$:

$$D(0,t,r+\lambda)\widetilde{V}_t + X_t = \mathbb{E}_t^{\mathscr{F}}[X_T + D(0,T,r+\lambda)\Phi(S_T)].$$

So, dividing by $D(0,u,r+\lambda)$ and using integration by parts formula we have:

$$\widetilde{V}_t + \int_0^t \frac{1}{D(0,u,r+\lambda)} dX_u + \int_0^t X_u d\frac{1}{D(0,u,r+\lambda)}$$
$$= \widetilde{V}_0 + \int_0^t \mathbb{E}_u^{\mathscr{F}}[X_T + D(0,T,r+\lambda)\Phi(S_T)] d\frac{1}{D(0,u,r+\lambda)} + \int_0^t \frac{1}{D(0,u,r+\lambda)} d\mathbb{E}_u^{\mathscr{F}}[X_T + D(0,T,r+\lambda)\Phi(S_T)].$$

That leads to:

$$\widetilde{V}_t + \int_0^t \frac{1}{D(0,u,r+\lambda)} dX_u \quad = \widetilde{V}_0 + \int_0^t \widetilde{V}_u (r_u + \lambda_u) du + \int_0^t \frac{1}{D(0,u,r+\lambda)} d\mathbb{E}_u^{\mathscr{F}}[X_T + D(0,T,r+\lambda)\Phi(S_T)].$$

The process $\mathbb{E}_t^{\mathscr{F}}[X_T + D(0,T,r+\lambda)\Phi(S_T)]$ is clearly a closed $\mathcal{F}$-martingale, and hence

$$\mathcal{M}_t = \int_0^t D(0,u,r+\lambda)^{-1} d\mathbb{E}_u^{\mathscr{F}}[X_T + D(0,T,r+\lambda)\Phi(S_T)].$$

is a local $\mathscr{F}$-martingale. Then, since $\mathcal{M}_t$ is adapted to the Brownian driven filtration $\mathscr{F}$, by the martingale representation theorem we have $\mathcal{M}_t = \int_0^t Z_u dW_u$ for some $\mathscr{F}$-predictable process Z_u. Hence, substituting for X_t, we can write :

$$\widetilde{V}_t = \widetilde{V}_0 - \int_0^t \left[\pi_u - (f_u + \lambda_u)\widetilde{V}_u + \widetilde{\theta}_u + (f_u - c_u)C_u - (r_u - \widetilde{h}_u)\widetilde{H}_u + 1_{\{\mathfrak{b}\}} LGD_I 1_{\{\widetilde{V}_u - C_u > 0\}} \lambda_u^I (\widetilde{V}_u - C_u)\right] du$$
$$+ \int_0^t Z_u dW_u.$$

To conclude we just have to notice that

$$\widetilde{V}_T = \widetilde{V}_0 - \int_0^T \left[\pi_u - (f_u + \lambda_u)\widetilde{V}_u + \widetilde{\theta}_u + (f_u - c_u)C_u - (r_u - \widetilde{h}_u)\widetilde{H}_u + 1_{\{\mathfrak{b}\}} LGD_I 1_{\{\widetilde{V}_u - C_u > 0\}} \lambda_u^I (\widetilde{V}_u - C_u)\right] du$$
$$+ \int_0^T Z_u dW_u.$$

And hence by computing $\widetilde{V}_T - \widetilde{V}_t$ we obtain:

$$\widetilde{V}_t = \Phi(S_T) + \int_t^T \left[\pi_u - (f_u + \lambda_u)\widetilde{V}_u + \widetilde{\theta}_u + (f_u - c_u)C_u - (r_u - \widetilde{h}_u)\widetilde{H}_u + 1_{\{\mathfrak{b}\}} LGD_I 1_{\{\widetilde{V}_u - C_u > 0\}} \lambda_u^I (\widetilde{V}_u - C_u)\right] du$$
$$- \int_t^T Z_u dW_u.$$

We highlight that this backward expression for $\widetilde{V}_t$ is obtained from two forward expressions and is not in contrast with the $\mathscr{F}$-adaptedness of $\widetilde{V}_t$. □

B.3 Proof of Proposition 5.1

We start by rewriting Equation (15) in conditional expectation form. Moreover we recover the terms C_t and ε_t that we identified with $\alpha_t \widetilde{V}_t$ and $\widetilde{V}_t$ at the beginning of Section 3.2 (see Remark 9 for a comment on why Equation (15) still makes sense also with different choices of C_t and ε_t). The equation we obtain looks as follows:

$$\begin{aligned}
\widetilde{V}_t =& \mathbb{E}_t^{\mathscr{F},h}\left[D(t,T,r+\lambda)\Phi(S_T) + \int_t^T D(t,u,r+\lambda)\pi_u du\right] + \mathbb{E}_t^{\mathscr{F},h}\left[\int_t^T D(t,u,r+\lambda)\lambda_u \varepsilon_u du\right] \\
& \underbrace{-\mathbb{E}_t^{\mathscr{F},h}\left[\int_t^T D(t,u,r+\lambda)(c_u - r_u)C_u du\right]}_{ColVA} \\
& \underbrace{-\mathbb{E}_t^{\mathscr{F},h}\left[\int_t^T D(t,u,r+\lambda)LGD_C \lambda_u^C (\varepsilon_u - C_u)^+ du\right]}_{-CVA} + \underbrace{\mathbb{E}_t^{\mathscr{F},h}\left[\int_t^T D(t,u,r+\lambda)LGD_I \lambda_u^I (\varepsilon_u - C_u)^- du\right]}_{+DVA} \\
& + \underbrace{\mathbb{E}_t^{\mathscr{F},h}\left[\int_t^T D(t,u,r+\lambda)LGD_I \lambda_u^I (\widetilde{V}_u - C_u)^+ du\right]}_{+DVA_F} \\
& \underbrace{-\mathbb{E}_t^{\mathscr{F},h}\left[\int_t^T D(t,u,r+\lambda)(1_{\{\widetilde{V}_u - C_u \le 0\}}\ell^- + 1_{\{\widetilde{V}_u - C_u > 0\}}(LGD_I \lambda_u^I + \ell^+))(\widetilde{V}_u - C_u)du\right]}_{+FVA=+FBA-FCA}.
\end{aligned} \tag{28}$$

To obtain the thesis we sum and subtract $\varepsilon_u^{Default-free}$ inside the integral in the second term of (28) and we group the resulting $\mathbb{E}_t^{\mathscr{F},h}\left[\int_t^T D(t,u,r+\lambda)\lambda_u \varepsilon_u^{Default-free} du\right]$ with the first term of (28).
□

C Backward Stochastic Differential Equations

Solutions of Stochastic differential equations (SDEs) describe the stochastic evolution of a process from a known starting condition (deterministic, or random with a known distribution) to an unknown random terminal one. Hence if we have a system for which we know the starting state (or its distribution) and whose dynamics are affected by some stochastic factor, we can try to model it with an SDE. Sometimes instead of having the starting state of a system, we just have a terminal state and with non-stochastic ordinary differential equations we can address this situation by simply reverting time. A time reversed ODE is still an ODE. In dealing with stochastic processes things are a little more complicated as a time inverted SDE is not a standard SDE anymore, and we need to introduce the concept of Backward Stochastic Differential Equations (BSDEs). As we mentioned BSDEs are not simply time reverted stochastic differential equations (SDEs), since time reversion has bad effects on the measurability of the solution. To understand why let us try to define the simplest backward stochastic differential equation on $[0,T]$:

$$dY_t = 0 \quad Y_T = \xi,$$

where we would like our solution to be adapted to the Brownian filtration $\mathscr{G}_t$ (peeking into the future is not desirable in financial modeling), and we suppose that the random variable ξ is $\mathscr{G}_T$ measurable. Clearly the simplest idea of solution $Y_t = \xi$ presents the problem of not being adapted to the filtration $\mathscr{G}$. The process Y_t closest to the constant one is given by $Y_t := \mathbb{E}[\xi|\mathscr{G}_t]$ and it makes sense to modify our simple BSDE in order for $Y_t := \mathbb{E}[\xi|\mathscr{G}_t]$ to be an acceptable solution. It is easy to see that Y_t is a martingale under $\mathscr{G}$, hence using the Brownian representation theorem we can write

$$dY_t = Z_t dW_t, \quad Y_0 = \mathbb{E}[\xi],$$

for some process Z_u. Hence we see that a definition for our simple BSDE could be

$$Y_t = \xi - \int_t^T Z_u dW_u \quad \text{or in differential form:} \quad dY_t = Z_t dW_t, \quad Y_T = \xi,$$

where Y_t, Z_t are *both* unknown. In this simple case the solution to our equation is given as desired by $Y_t := \mathbb{E}[\xi|\mathscr{G}_t]$ and Z_t is the process such that $\mathbb{E}[\xi|\mathscr{G}_t] = \int_0^t Z_u dW_u$. In general then a BSDE is an equation in two unknown processes (Y_t, Z_t) such that:

$$Y_t = \xi + \int_t^T (f(Y_u, Z_u, u)du - Z_u dW_u),$$

where f is a possibly stochastic function and ξ is our terminal condition at time T. Notice how taking a conditional expectation leads to

$$Y_t = \mathbb{E}_t^{\mathbb{Q}}\left[\xi + \int_t^T f(Y_u, Z_u, u)du\right], \tag{29}$$

a fixed point equation, very similar to Equation (1). We argue now that BSDEs are a quite natural way to formulate pricing problems in finance for two reasons: the first one is that usually when pricing a contract one knows its value at maturity (typically the payoff) and hence is obviously facing a backward problem, and the second is that the process Z_u offers a natural interpretation in terms of hedging. To illustrate our points let us consider a call option on a stock that follows the usual Black-Scholes dynamics:

$$dS_t = rS_t dt + \sigma S_t dW_t \tag{30}$$

Furthermore let us introduce also a bank account that accrues at rate r:

$$dB_t = rB_t dt \quad B_0 = 1.$$

Consider a self financing strategy η_t, ζ_t that tries to replicate the deal, i.e. is such that its value process V_t follows:

$$\begin{aligned} V_t &= \eta_t B_t + \zeta_t S_t \quad V_T = (S_T - K)^+ \ \mathbb{Q}\text{-almost-surely}, \\ dV_t &= \eta_t r B_t dt + \zeta_t dS_t. \end{aligned} \tag{31}$$

If we define the discounted wealth $\overline{V}_t = V_t B_t^{-1}$ then we can write the following BSDE for $\overline{V}_t$ and $Z_t := B_t^{-1}\zeta_t \sigma S_t$:

$$\overline{V}_t = B_T^{-1}(S_T - K)^+ - \int_t^T Z_u dW_u. \tag{32}$$

Notice that we obtain (32) by subtracting the forward expressions (31) for V_T and V_t and substituting the expression for the call payoff. This doesn't alter the measurability of our processes. The system given by an SDE and a BSDE such as Equations (30),(32) is usually called forward backward stochastic differential equation (FBSDE).

From (32) we can obtain the usual risk neutral valuation formula by taking a conditional expectation (under suitable regularity of the process ζ_u):

$$V_t = \mathbb{E}_t\left[B_t B_T^{-1}(S_T - K)^+\right].$$

Moreover note that the second unknown in the BSDE is a multiple of the hedging strategy.

To gain more insight let us assume that $V_t = u(S_t, t)$, where $u(s,t) \in C^{2,1}$. Again under suitable regularity we can apply Ito's formula and obtain:

$$d\overline{V}_t = -rB_t^{-1}u(S_t,t)dt + B_t^{-1}\left[\left(\frac{\partial u}{\partial t}(S_t,t) + rS_t\frac{\partial u}{\partial s}(S_t,t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 u}{\partial s^2}(S_t,t)\right)dt + \sigma S_t \frac{\partial u}{\partial s}(S_t,t)dW_t\right].$$

By matching the drift and the diffusion terms with the ones in the BSDE we obtain:

$$\begin{aligned} \frac{\partial u}{\partial t}(S_t,t) + rS_t\frac{\partial u}{\partial s}(S_t,t) + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 u}{\partial s^2}(S_t,t) - ru(S_t,t) = 0 \\ \frac{\partial u}{\partial s}(S_t,t) = \zeta_t. \end{aligned} \tag{33}$$

Roughly speaking, the first equation is Black-Scholes PDE, while the second one shows that the hedging process is the usual Black-Scholes delta hedging. BSDEs arise quite naturally in financial applications and optimisation: see for example the seminal paper [34] for a general survey, and [29, 53] for applications in the operations research literature.

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