Portfolio Selection in Discrete Time with Transaction Costs and Power Utility Function: A Perturbation Analysis

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Abstract: In this paper, we study a multi-period portfolio selection model in which a generic class of probability distributions is assumed for the returns of the risky asset. An investor with a power utility function rebalances a portfolio comprising a risk-free and risky asset at the beginning of each time period in order to maximize expected utility of terminal wealth. Trading the risky asset incurs a cost that is proportional to the value of the transaction. At each time period, the optimal investment strategy involves buying or selling the risky asset to reach the boundaries of a certain no-transaction region. In the limit of small transaction costs, dynamic programming and perturbation analysis are applied to obtain explicit approximations to the optimal boundaries and optimal value function of the portfolio at each stage of a multi-period investment process of any length.

Key Words: Portfolio optimization, discrete time, transaction costs, power utility function, perturbation analysis

1. Introduction

The portfolio selection and consumption problem in a continuous time setting was first studied by Merton (1969). The investor’s objective was to maximize expected utility from consumption, where the price of the risky asset was assumed to be driven by a geometric Brownian motion in a frictionless market. It was shown that, for the case of the power or logarithmic utility function, the optimal investment strategy involved continuously rebalancing the portfolio to maintain a constant proportion of the risky asset. This constant proportion is commonly referred to as the Merton proportion.

However, continuous trading in financial markets could be ruinously expensive due to the impact of transaction costs. Transaction costs incorporated into subsequent research were generally modeled as a constant amount for each transaction (constant costs), an amount proportional to the value of the transaction (proportional costs) or a fixed proportion of the entire portfolio value. Magill and Constantinides (1976) were the first to extend Merton’s model to incorporate trading costs that were proportional to the value of the transaction (proportional transaction costs). Although their argument was heuristic, they provided the insight that “the investor trades in securities when the variation in the underlying security prices forces his portfolio proportions outside a certain region about the optimal proportions in the absence of transactions costs”. Davis and Norman (1990) provided a rigorous formulation and analysis of the portfolio selection and consumption problem with proportional transaction costs.
costs by applying the theory of stochastic singular control. Their work was further generalized by Shreve and Soner (1994) with less restrictive assumptions using the theory of viscosity solutions. Taksar et al. (1988) analyzed the portfolio selection problem with proportional transaction costs, which involved applying stochastic singular control to maximize the long run growth rate of the portfolio value. In the aforementioned models with transaction costs, the typical optimal strategy was not to transact when the proportion of risky asset drifted within a particular no-transaction region. When the proportion of risky asset exceeded the boundary of this region, the investor would transact instantaneously to return to the boundary. Akian et al. (1996) extended the Davis and Norman (1990) model to study the case where there were more than one risky asset. Morton and Pliska (1995) introduced a multi-asset model where the investor paid a transaction cost equal to a fixed proportion of the entire portfolio value. The investor’s objective was to maximize the long run growth rate of the portfolio value and the optimal strategy was shown to be reduced to one that solved a single stopping time problem.

In general, a lack of analytical solutions meant that models with transaction costs had to be solved by numerical methods. These were usually computationally intensive, especially in the case of multiple risky assets. Nonetheless, it was observed that transaction costs were small in practice relative to the value of the transactions. In the limit of small transaction costs, Atkinson and Wilmott (1995) and Mokkhavesa and Atkinson (2002) applied techniques of perturbation analysis about the no transaction costs solution to derive approximate solutions. Janecek and Shreve (2004) provided a rigorous derivation of the asymptotic expansions of the optimal value function and boundaries of the no-transaction region. Their work was recently extended by Gerhold et al. (2012), who obtained power series expansions of arbitrary order for the optimal value function and boundaries of the no-transaction region by using duality theory. However, these perturbation analyses were applied to continuous time models where the prices of risky assets were assumed to be geometric Brownian motions.

It should be noted that Atkinson and Al-Ali (1997) solved the one dimensional portfolio problem exactly for the first correction, a result which seemed to have been repeated by Soner and Touzi (2013) recently by a different method. The multi-asset problem with uncorrelated distribution was solved by Atkinson and Mokkhavesa (2004) who deduced the \(N\)-dimensional cuboid for the no transaction region and found explicit solutions for all the boundaries \((N-1)\) transaction regions, obtaining the regions in the expansion of the solution. Furthermore, Atkinson and Mokkhavesa (2003) applied these methods to situations including stochastic volatility and procedures for any utility function. Atkinson and Papakokkinou (2005) also considered portfolios with a VaR or CVaR constraint as well as transaction costs. The problem of determining a speculator’s utility function from his observed transactions behaviour was investigated by Atkinson and Mokkhavesa (2001). Most of the above analyses were in continuous time and with underlying Brownian behaviour of the risky assets together with (in the multi-asset case) uncorrelated behaviour of the risky assets. The influence of correlation was considered by Atkinson and Ingpochai (2006) and Atkinson and Ingpochai (2010) both for portfolio theory and for option pricing together with stochastic volatility. The option pricing theory is based on an indifference argument given by Davis and Norman (1990) and considers two portfolios, one with the option and one without, both portfolios using utility maximisation with the exponential utility function. For a basket of uncorrelated assets, Atkinson and Alexandropoulos (2006) have solved this problem by a perturbation method and also shown how the Legendre transform can be used to give a single solution to the no transaction region in these alternate variables. This method is also used by Atkinson and Ingpochai (2012) where correlation between assets is taken into account.

In a discrete time framework, Samuelson (1969) studied the portfolio selection and consumption problem in a multi-period model using a dynamic programming approach, which was analogous to the work by Merton (1969). He analyzed the maximization of expected utility
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from consumption, where the return of the risky asset was assumed to follow a general probability distribution. In the case of the power or logarithmic utility function, he showed that the investor’s optimal strategy was to maintain a constant proportion of the risky asset at each time step. Mossin (1968) analyzed the dynamic portfolio selection problem in discrete time and considered the objective of maximizing expected utility of terminal wealth. It was shown that, for the power or logarithmic utility function, the optimal strategy involved making a sequence of single-period decisions which disregarded future reinvestment opportunities. This was described as a myopic decision. However, this was only a special case and it was generally not optimal to make a decision at each time step without considering the steps ahead.

Bobryk and Stettner (1999) incorporated proportional transaction costs in the maximization of expected utility from consumption. In the case of the power or logarithmic utility function, the optimal investment strategy was shown to be characterized by a cone shaped no-transaction region. They derived various bounds on the no-transaction region by specifying upper and lower bounds on the support of the probability measure for the returns of the risky asset. Sass (2005) embedded the Cox et al. (1979) binomial model with a general transaction costs structure. He formulated the objective of maximizing expected utility of terminal wealth as a Markov control problem and gave a multi-period existence result based on the solution of the dynamic programming equation. Explicit results were provided for the one-period problem in the case of a single risky asset with a binomial price process. Atkinson and Storey (2010) studied the discrete time portfolio selection problem by incorporating proportional transaction costs in the Mossin (1968) model. They assumed a general class of underlying probability distributions for the returns of the risky asset and studied the problem of maximizing expected utility of terminal wealth for the power utility function. Perturbation methods were applied to obtain approximations of the optimal boundaries of the no-transaction region in the limits of small and large transaction costs. However, the approximations of the optimal boundaries were only obtained for two time steps and it was not obvious that their approach would allow one to generalize to an arbitrary number of time steps. Atkinson and Quek (2012) considered the case of maximizing expected utility of terminal wealth for the exponential utility function. In the limit of small transaction costs, they applied perturbation analysis to derive approximations of the optimal value function and boundaries of the no-transaction region at all time steps of the problem. An investor with the exponential utility function is characterized by a constant level of absolute risk aversion, which resulted in optimal no-transaction boundaries that were independent of wealth. In practice, one would expect the optimal boundaries to vary with the wealth of the investor. A more realistic description of the investor’s optimal strategy would be provided by using the power utility function.

In this paper, we assume the more realistic case of the power utility function and carry out a perturbation analysis to an arbitrary number of time steps, in the limit of small transaction costs. We present a method for explicitly constructing the approximations of both the optimal value function and optimal boundaries of the no-transaction region. It is a non-trivial extension of the perturbation analysis developed in Atkinson and Quek (2012) as the proportion of risky asset at each time step depends on variations in both the return of the risky asset and the investor’s wealth. It should be stressed that the method of this paper allows for a wide variety of distributions for the returns of the risky asset. In our simplified example discussed in Section 6, we choose a specific distribution which is the same at each timestep, e.g. \( p(s) = 0.7 \times \delta(s - 1.5) + 0.3 \times \delta(s - 0.5) \), with \( \delta(.) \) being the Dirac delta function, but of course many other distributions could be used in our method. Moreover, explicit error bounds are determined in the Appendix. Recent work such as that of Bayer and Veliyev (2014) consider only a Binomial model for the risky asset and a log investor. Similarly, those of Balduzzi and Lynch (1999) and Lynch and Balduzzi (2000) who address numerically quite complex models again are somewhat restricted in terms of the underlying model of the risky asset. We thus conclude that our general approach here should have many applications to situations where a
variety of distributions might be expected for the return probabilities of the underlying risky asset.

The plan of this paper is as follows. In Section 2, we describe the market model and the portfolio selection problem. Section 3 contains a review of the construction of the optimal portfolio via dynamic programming. In Section 4, we state the results for the special case when there are no transaction costs. The main part of the paper is found in Section 5, where we analyze the case of small transaction costs. We devise a perturbation method, which is carried out in two stages, to systematically approximate the optimal value function and optimal boundaries at any time step of the problem. In Section 6, we conclude with a discussion of the main results and a numerical example. The error for the procedure and details of the perturbation analysis are found in the Appendix.

2. Market Model

In this section, we present an overview of the market model that was developed in Atkinson and Storey (2010). Consider a multi-period portfolio selection model with $T$ periods. Assume a financial market with one risk-free asset (bond) and one risky asset (stock), where the price of each asset evolves in discrete time. An investor is assumed to have a risk preference of the constant relative risk aversion class (power utility function). The investor holds a portfolio that is divided between the risk-free and risky assets. A cost proportional to the value of the transaction is incurred each time the investor buys or sells the risky asset. The investor’s objective is to maximize the expected utility of terminal wealth by rebalancing the portfolio optimally at each step of the investment process.

At time $n$, let $W_n$ denote the wealth of the portfolio and let $a_n$ be the dollar value of the risky asset inherited from the previous time step. Therefore, the corresponding value of the risk-free asset is $W_n - a_n$. The investor rebalances the portfolio at time $n$ by buying $l_n$ or selling $m_n$ dollars of the risky asset. Suppose that $s_n$ denotes one plus the random return of the risky asset from time $n$ to $n+1$. Thus, the value of the risky asset at time $n+1$ inherited from time $n$ is

$$a_{n+1} = s_n (a_n + l_n - m_n).$$ (1)

Furthermore, let $\lambda_n$ and $\mu_n$ be the proportion costs of buying and selling the risky asset respectively at time $n$. These costs reduce the wealth invested in the risk-free asset, resulting in a value of $W_n - (a_n + l_n - m_n) - \lambda_n l_n - \mu_n m_n$. Suppose that $r_n$ denotes one plus the sure return of the risk-free asset from time $n$ to $n+1$. The investor’s wealth at time $n+1$ is then given by

$$W_{n+1} = r_n W_n + (s_n - r_n) (a_n + l_n - m_n) - r_n \lambda_n l_n - r_n \mu_n m_n.$$ (2)

Assume that simultaneous buying and selling of the risky asset is not allowed, since it will not be optimal due to the higher costs as compared to only buying or selling the asset. The investor is thus left with three possible choices, which is to buy, to sell or not to transact the risky asset. The investor’s decision at each time step will affect the wealth and risky asset inherited at the next time step.

Suppose that the investor has a risk preference of the power utility type. Let the investor’s utility of wealth be

$$U(W) = \frac{1}{\gamma} W^\gamma,$$ (3)

where $\gamma < 1$ and $\gamma \neq 0$. In this case, it is convenient to parametrize the problem by expressing the original variables in terms of fractions of wealth. Introduce the variables $A_n = a_n/W_n$ (fraction of wealth held in the risky asset), $L_n = l_n/W_n$ (fraction of wealth in buying the risky
asset) and \( M_n = m_n / W_n \) (fraction of wealth in selling the risky asset). Using these variables, Eqs. (1) and (2) can be rewritten as

\[
A_{n+1} = \frac{s_n (A_n + L_n - M_n)}{F_n}
\]

and

\[
W_{n+1} = W_n F_n
\]

respectively, where

\[
F_n = r_n + (s_n - r_n) (A_n + L_n - M_n) - r_n \lambda_n L_n - r_n \mu_n M_n.
\]

Assume that \( 0 < A_n + L_n - M_n < 1 \) and \( 0 < 1 - A_n - (1 + \lambda_n) L_n + (1 - \mu_n) M_n < 1 \), which ensure that the investor’s wealth \( W_n > 0 \).

The investor’s objective is to maximize the expected utility of terminal wealth \( W_T \) given an initial wealth \( W_0 \) and initial proportion of risky asset \( A_0 \), by choosing the optimal strategy at each step of the investment process. The optimal value function at time \( n \) \((n = 0, \ldots, T-1)\) is defined to be

\[
J_n(W_n, A_n) = \max E \left[ U(W_T) \right],
\]

where \( E \) is the conditional expectation operator taken with respect to the random variables \( s_n, \ldots, s_{T-1} \) given \( W_n \) and \( A_n \). The maximization is over the sequence of investments \( (L_n, M_n), \ldots, (L_{T-1}, M_{T-1}) \) in the risky asset. In order to obtain the investor’s optimal value function \( J_0(W_0, A_0) \) and corresponding optimal strategy given an initial wealth \( W_0 \) and initial proportion of risky asset \( A_0 \), we first simplify the problem by applying the principle of dynamic programming.

3. Dynamic Programming

The dynamic programming algorithm for the problem, which starts at terminal time \( T \) and proceeds recursively backwards in time, is given by

\[
J_T(W_T, A_T) = U(W_T)
\]

and

\[
J_{T-k}(W_{T-k}, A_{T-k}) = \max E_{T-k} \left[ J_{T-k+1}(W_{T-k+1}, A_{T-k+1}) \right]
\]

for \( k = 1, \ldots, T \). At time \( T - k \), \( E_{T-k} \) is the conditional expectation operator with respect to the random variable \( s_{T-k} \) given \( W_{T-k} \) and \( A_{T-k} \), while the maximization is over the investment \( (L_{T-k}, M_{T-k}) \) in the risky asset. A detailed description of the construction of the optimal portfolio via dynamic programming can be found in Atkinson and Storey (2010). We shall only state the results of this construction at time \( T - 1 \) and at the general case of time \( T - k \), which are subsequently required for our perturbation analysis in the limit of small transaction costs.

3.1 Time \( T - 1 \)

This is a special case as it is one step before termination of the investment process, which means that there is no further rebalancing opportunity for the investor. Using Eq. (5), the value function at time \( T - 1 \) is written as

\[
J_{T-1}(W_{T-1}, A_{T-1}) = \max E_{T-1} \left[ \frac{1}{\gamma} W_T^\gamma \right] = W_{T-1}^\gamma \max V_{T-1}(A_{T-1}),
\]
where
\[ V_{T-1}(A_{T-1}) = \mathbb{E}_{T-1} \left[ \gamma \frac{1}{F_{T-1}^\gamma} \right]. \] (11)

In Eq. (10), \( W_{T-1} \) is taken out of \( \mathbb{E}_{T-1} \), which is the expectation operator conditional on \( W_{T-1} \) and \( A_{T-1} \). In addition, \( W_{T-1} \) does not depend on the investor’s decision at time \( T - 1 \). Therefore, the problem is reduced to one of maximizing \( V_{T-1}(A_{T-1}) \) with respect to the investment strategy \( (L_{T-1}, M_{T-1}) \). The investor’s choice to buy \( (M_{T-1} = 0) \), to sell \( (L_{T-1} = 0) \) or not to transact \( (M_{T-1} = 0 \text{ and } L_{T-1} = 0) \) in the risky asset affects the definition of \( F_{T-1} \) as given in Eq. (6).

The problem is one of finding the optimal no-transaction region delineated by \( A_{T-1}^- \leq A_{T-1} \leq A_{T-1}^+ \), where \( A_{T-1}^- \) and \( A_{T-1}^+ \) are the optimal buy and sell boundaries respectively. The region to the left of \( A_{T-1}^- \) is the optimal buy region while the region to the right of \( A_{T-1}^+ \) is the optimal sell region. \( A_{T-1}^- \) and \( A_{T-1}^+ \) are given by the first order optimality conditions
\[ \frac{\partial V_{T-1}}{\partial L_{T-1}} = \mathbb{E}_{T-1} \left[ \{s_{T-1} - (1 + \lambda_{T-1}) r_{T-1}\} F_{T-1}^{\gamma-1} \right] = 0 \] (12)
and
\[ \frac{\partial V_{T-1}}{\partial M_{T-1}} = \mathbb{E}_{T-1} \left[ \{(1 - \mu_{T-1}) r_{T-1} - s_{T-1}\} F_{T-1}^{\gamma-1} \right] = 0 \] (13)
respectively. In Eqs. (12) and (13), note that \( F_{T-1} = r_{T-1} + (s_{T-1} - r_{T-1}) A_{T-1}^- \) because we have \( A_{T-1} = A_{T-1}^- \), \( L_{T-1} = 0 \) on the buy boundary and \( A_{T-1} = A_{T-1}^+ \), \( M_{T-1} = 0 \) on the sell boundary. Furthermore, it can be shown that the second order conditions \( \frac{\partial^2 V_{T-1}}{\partial L_{T-1}^2} < 0 \) and \( \frac{\partial^2 V_{T-1}}{\partial M_{T-1}^2} < 0 \) are satisfied, which ensure that these boundaries are optimal. In general, one will solve for \( A_{T-1}^- \) and \( A_{T-1}^+ \) numerically.

Having determined the optimal boundaries \( A_{T-1}^- \) and \( A_{T-1}^+ \), the investor’s optimal strategy and value function are as follows:

In the buy region \( A_{T-1} < A_{T-1}^- \), the investor’s optimal strategy is to buy \( L_{T-1} = A_{T-1}^- - A_{T-1} \) of the risky asset to reach the optimal buy boundary \( A_{T-1}^- \). The corresponding optimal value function \( V_{T-1} \) and its first derivative \( \frac{\partial V_{T-1}}{\partial A_{T-1}} \) are thus given by
\[ V_{T-1}^{(B)} = \mathbb{E}_{T-1} \left[ \frac{1}{\gamma} F_{T-1}^{(B)\gamma} \right] \] (14)
and
\[ \frac{\partial V_{T-1}^{(B)}}{\partial A_{T-1}} = \mathbb{E}_{T-1} \left[ r_{T-1} \lambda_{T-1} F_{T-1}^{(B)\gamma-1} \right], \] (15)
where
\[ F_{T-1}^{(B)} = r_{T-1} + (s_{T-1} - r_{T-1}) A_{T-1}^- - r_{T-1} \lambda_{T-1} \left( A_{T-1}^- - A_{T-1} \right). \] (16)

In the sell region \( A_{T-1} > A_{T-1}^+ \), the investor sells \( M_{T-1} = A_{T-1} - A_{T-1}^+ \) of the risky asset to reach the optimal sell boundary, so that \( V_{T-1} \) and \( \frac{\partial V_{T-1}}{\partial A_{T-1}} \) are given by
\[ V_{T-1}^{(S)} = \mathbb{E}_{T-1} \left[ \frac{1}{\gamma} F_{T-1}^{(S)\gamma} \right] \] (17)
and

\[
\frac{\partial V_T^{(s)}}{\partial A_T} = -\mathbb{E}_{T-1} \left[ r_{T-1} \mu_{T-1} F_T^{(s)\gamma-1} \right], \tag{18}
\]

where

\[
F_T^{(s)} = r_{T-1} + (s_{T-1} - r_{T-1}) A_T^{+} - r_{T-1} \mu_{T-1} \left( A_T - A_T^{-} \right). \tag{19}
\]

In the no-transaction region \( A_T^{-} \leq A_{T-1} \leq A_T^{+} \), where \( L_{T-1} = 0 \) and \( M_{T-1} = 0 \) as the investor does not trade in the risky asset, \( V_{T-1} \) and \( \frac{\partial V_T}{\partial A_T} \) are given by

\[
V_{T-1}^{(N)} = \mathbb{E}_{T-1} \left[ \frac{1}{\gamma} F_{T-1}^{(N)} \right] \tag{20}
\]

and

\[
\frac{\partial V_{T-1}^{(N)}}{\partial A_{T-1}} = \mathbb{E}_{T-1} \left[ \{ s_{T-1} - r_{T-1} \} F_{T-1}^{(N)\gamma-1} \right], \tag{21}
\]

where

\[
F_{T-1}^{(N)} = r_{T-1} + (s_{T-1} - r_{T-1}) A_{T-1}. \tag{22}
\]

It is noted that \( V_{T-1} \) and \( \frac{\partial V_T}{\partial A_T} \) are continuous across the optimal buy and sell boundaries.

At the optimal boundaries \( A_{T-1} = A_{T-1}^{-} \) and \( A_{T-1} = A_{T-1}^{+} \), continuity of the former is simply observed from Eqs. (14), (17) and (20), while the latter is a direct consequence of the first order optimality conditions (12) and (13).

### 3.2 Time \( T - k \)

Applying the dynamic programming algorithm recursively backwards in time allows one to construct the optimal strategy and value function at time \( T - k \) \((k = 2, \ldots, T)\). Using Eq. (5), the optimal value function given by Eq. (9) is expressed as

\[
J_{T-k}(W_{T-k}, A_{T-k}) = \max_{W_{T-k+1}} \mathbb{E}_{T-k} \left[ W_{T-k+1}^{\gamma} V_{T-k+1}(A_{T-k+1}) \right]
\]

\[
= W_{T-k}^{\gamma} \max V_{T-k}(A_{T-k}), \tag{23}
\]

where

\[
V_{T-k}(A_{T-k}) = \mathbb{E}_{T-k} \left[ W_{T-k}^{\gamma} \gamma V_{T-k+1}(A_{T-k+1}) \right]. \tag{24}
\]

The problem is effectively reduced to one of maximizing \( V_{T-k}(A_{T-k}) \) with respect to \( L_{T-k} \) and \( M_{T-k} \), assuming that \( V_{T-k+1}(A_{T-k+1}) \) is optimal by the principle of dynamic programming. The definitions of \( F_{T-k} \) from Eq. (6) and \( A_{T-k+1} \) from Eq. (4) depend on the investor’s decision to buy (\( M_{T-k} = 0 \)), to sell (\( L_{T-k} = 0 \)) or not to transact (\( L_{T-k} = 0 \) and \( M_{T-k} = 0 \)) the risky asset.

The optimal buy boundary \( A_{T-k} = A_{T-k}^{-} \) and sell boundary \( A_{T-k} = A_{T-k}^{+} \) satisfy the corresponding first order optimality conditions

\[
\frac{\partial V_{T-k}}{\partial L_{T-k}} = \mathbb{E}_{T-k} \left[ \gamma \{ s_{T-k} - (1 + \lambda_{T-k}) r_{T-k} \} F_{T-k}^{\gamma-1} V_{T-k+1} + s_{T-k} r_{T-k} (1 + \lambda_{T-k} A_{T-k}) F_{T-k}^{\gamma-2} \frac{\partial V_{T-k+1}}{\partial A_{T-k+1}} \right] = 0 \tag{25}
\]
and
\[
\frac{\partial V_{T-k}}{\partial M_{T-k}} = \mathbb{E}_{T-k}\left[\gamma \left(1 - \mu_{T-k}\right) r_{T-k} - s_{T-k}\right] F_{T-k}^{\gamma-1} V_{T-k+1} - s_{T-k} r_{T-k} \left(1 - \mu_{T-k} A_{T-k}\right) F_{T-k}^{\gamma-2} \frac{\partial V_{T-k+1}}{\partial A_{T-k+1}} = 0, \tag{26}
\]
respectively. In Eqs. (25) and (26), note that \( F_{T-k} = r_{T-k} + \left(s_{T-k} - r_{T-k}\right) A_{T-k} \). Compared to the first order conditions at time \( T-1 \), there is an additional \( \frac{\partial V_{T-k+1}}{\partial A_{T-k+1}} \) term due to the opportunities for the investor to rebalance the portfolio at the time steps ahead. Moreover, the investor is not myopic (unlike the case where there are no transaction costs) and will take into account future rebalancing opportunities when he determines his current investment strategy. In general, one will solve for \( A^{-}_{T-k} \) and \( A^{+}_{T-k} \) by implementing the dynamic programming algorithm numerically, which becomes computationally more intensive as the number of time steps increases.

Having determined the optimal buy and sell boundaries, the investor’s optimal strategy and value function are as follows:

In the buy region \( A^{-}_{T-k} < A^{-}_{T-k} \), the investor’s optimal strategy is to buy \( L_{T-k} = A^{-}_{T-k} - A_{T-k} \) of the risky asset to reach the optimal buy boundary \( A^{-}_{T-k} \). In this case, the optimal value function \( V_{T-k} \) and its first derivative \( \frac{\partial V_{T-k}}{\partial A_{T-k}} \) are given by
\[
V^{(B)}_{T-k} = \mathbb{E}_{T-k}\left[F^{(B)}_{T-k} V_{T-k+1}\right] \tag{27}
\]
and
\[
\frac{\partial V^{(B)}_{T-k}}{\partial A_{T-k}} = \mathbb{E}_{T-k}\left[\gamma r_{T-k} \lambda_{T-k} F^{(B)}_{T-k}^{\gamma-1} V_{T-k+1} - s_{T-k} r_{T-k} \lambda_{T-k} A^{-}_{T-k} F^{(B)}_{T-k}^{\gamma-2} \frac{\partial V_{T-k+1}}{\partial A_{T-k+1}}\right], \tag{28}
\]
where
\[
F^{(B)}_{T-k} = r_{T-k} + \left(s_{T-k} - r_{T-k}\right) A^{-}_{T-k} - r_{T-k} \lambda_{T-k} \left(A^{-}_{T-k} - A_{T-k}\right). \tag{29}
\]
Note that the optimal value function at the time step ahead \( V_{T-k+1} \) is a function of \( A_{T-k+1} \). From Eq. (4), we know that \( A_{T-k+1} \) depends on \( s_{T-k}, A_{T-k} \) and \( A^{+}_{T-k} \) via \( A_{T-k+1} = \frac{s_{T-k} A^{-}_{T-k}}{F^{(B)}_{T-k}} \). Define \( s^{-}_{T-k} \) and \( s^{+}_{T-k} \) as the values of \( s_{T-k} \) which correspond to the optimal buy boundary \( A^{-}_{T-k+1} \) and optimal sell boundary \( A^{+}_{T-k+1} \) at the time step ahead respectively. This implies, after rearranging the expressions, that they are explicitly given by
\[
s^{-}_{T-k} = \frac{r_{T-k} A^{-}_{T-k+1} \left\{ \left(1 - A^{-}_{T-k}\right) \lambda_{T-k} \left(A^{-}_{T-k} - A_{T-k}\right) \right\}}{A^{-}_{T-k} \left(1 - A_{T-k+1}\right)} \text{ and }
s^{+}_{T-k} = \frac{r_{T-k} A^{+}_{T-k+1} \left\{ \left(1 - A^{+}_{T-k}\right) \lambda_{T-k} \left(A^{+}_{T-k} - A_{T-k}\right) \right\}}{A^{+}_{T-k} \left(1 - A_{T-k+1}\right)}. \]
Since the expectation operator \( \mathbb{E}_{T-k} \) is taken with respect to the random variable \( s_{T-k} \), Eq. (27) can thus be written in its
integral form, delineated by $s_{-T-k}$ and $s_{+T-k}$, to give

$$V_{T-k}^{(B)} = \int_{0}^{s_{-T-k}} F_{T-k}^{(B)}(s) \gamma V_{T-k+1}^{(B)} \, ds_{T-k} + \int_{s_{-T-k}}^{s_{+T-k}} F_{T-k}^{(B)}(s) \gamma V_{T-k+1}^{(N)} \, ds_{T-k} + \int_{s_{+T-k}}^{\infty} F_{T-k}^{(B)}(s) V_{T-k+1}^{(S)} \, ds_{T-k}, \tag{30}$$

where $p(s_{T-k})$ is the probability density function of the random variable $s_{T-k}$. We will be using Eq. (30) in the perturbation analysis that follows subsequently.

In the sell region $A_{T-k} > A_{+T-k}$, the investor sells $M_{T-k} = A_{T-k} - A_{+T-k}$ of the risky asset to reach the optimal sell boundary. Thus, $V_{T-k}$ and $\frac{\partial V_{T-k}}{\partial A_{T-k}}$ are given by

$$V_{T-k}^{(S)} = \mathbb{E}_{T-k} \left[ F_{T-k}^{(S)} V_{T-k+1} \right] \tag{31}$$

and

$$\frac{\partial V_{T-k}^{(S)}}{\partial A_{T-k}} = \mathbb{E}_{T-k} \left[ -\gamma r_{T-k} \mu_{T-k} F_{T-k}^{(S)} \gamma^{-1} V_{T-k+1} + s_{T-k} r_{T-k} \mu_{T-k} A_{+T-k} F_{T-k}^{(S)} \frac{\partial V_{T-k+1}}{\partial A_{T-k+1}} \right], \tag{32}$$

where

$$F_{T-k}^{(S)} = r_{T-k} + (s_{T-k} - r_{T-k}) A_{+T-k} - r_{T-k} \mu_{T-k} (A_{T-k} - A_{+T-k}). \tag{33}$$

Similar to the buy region, $A_{T-k+1}$ depends on $s_{T-k}$, $A_{T-k}$ and $A_{+T-k}$ via $A_{T-k+1} = \frac{s_{T-k} A_{+T-k}}{F_{T-k}^{(S)}}$, which implies that

$$s_{-T-k} = \frac{r_{T-k} A_{+T-k+1}}{A_{T-k} \left( 1 - A_{+T-k+1} \right)} \left( 1 - A_{+T-k} \right) \left( A_{T-k} - A_{+T-k} \right) \left( 1 - A_{+T-k} \right), \tag{34}$$

and

$$s_{+T-k} = \frac{r_{T-k} A_{+T-k+1}}{A_{+T-k} \left( 1 - A_{+T-k+1} \right)} \left( 1 - A_{+T-k} \right) \left( A_{T-k} - A_{+T-k} \right) \left( 1 - A_{+T-k} \right). \tag{35}$$

Eq. (31) in its integral form, delineated by the $s_{-T-k}$ and $s_{+T-k}$ given above.

In the no-transaction region $A_{T-k} < A_{T-k} < A_{+T-k}$ where $L_{T-k} = 0$ and $M_{T-k} = 0$, the optimal value function $V_{T-k}$ and its derivative $\frac{\partial V_{T-k}}{\partial A_{T-k}}$ are given by

$$V_{T-k}^{(N)} = \mathbb{E}_{T-k} \left[ F_{T-k}^{(N)} V_{T-k+1} \right] \tag{34}$$

and

$$\frac{\partial V_{T-k}^{(N)}}{\partial A_{T-k}} = \mathbb{E}_{T-k} \left[ \gamma (s_{T-k} - r_{T-k}) F_{T-k}^{(N)} \gamma^{-1} V_{T-k+1} + s_{T-k} r_{T-k} F_{T-k}^{(N)} \frac{\partial V_{T-k+1}}{\partial A_{T-k+1}} \right], \tag{35}$$

where

$$F_{T-k}^{(N)} = r_{T-k} + (s_{T-k} - r_{T-k}) A_{T-k}. \tag{36}$$
Here, \( A_{T-k+1} \) depends on \( s_{T-k} \) and \( A_{T-k} \) via \( A_{T-k+1} = \frac{s_{T-k} A_{T-k}}{E_{T-k}^{(N)}} \), which means that \( s_{T-k} = \frac{r_{T-k} A_{T-k+1}(1 - A_{T-k})}{A_{T-k} (1 - A_{T-k+1})} \) and \( s_{T-k}^+ = \frac{r_{T-k} A_{T-k+1}(1 - A_{T-k})}{A_{T-k} (1 - A_{T-k+1})} \). Similarly, we can write Eq. (34) in its integral form delineated by the above values of \( s_{T-k}^+ \) and \( s_{T-k}^+ \).

It is noted that \( V_{T-k} \) and \( \frac{\partial V_{T-k}}{\partial A_{T-k}} \) are continuous across the optimal boundaries. The former is observable from Eqs. (30), (31) and (34) while the latter is a direct consequence of the first order conditions (25) and (26). Generally, one will need to implement the dynamic programming algorithm recursively to obtain numerical solutions of the optimal boundaries \( A_{T-k}^+ \) and \( A_{T-k}^- \) and the optimal value function \( V_{T-k}(A_{T-k}) \). However, this implementation is computationally intensive particularly when the number of time steps is large. Therefore, this has motivated us to investigate an alternative method of approximating the solutions of \( A_{T-k}^+ \), \( A_{T-k}^- \) and \( V_{T-k}(A_{T-k}) \). Furthermore, in the case where there are no transaction costs, the solution to the portfolio selection problem is easily obtained as the optimal strategy is essentially myopic in nature. Coupled with the knowledge that transaction costs are small in practice, we therefore carry out a perturbation analysis of the small transaction costs model about the no transaction costs case. In addition to deriving more tractable approximations to the solutions, a perturbation analysis may provide some qualitative insights to the nature of the solutions.

4. No Transaction Costs Case

Prior to the perturbation analysis in the limit of small transaction costs, we first consider the special case where there are no transaction costs incurred in buying or selling the risky asset. Setting \( \lambda_{T-k} = 0 = \mu_{T-k} \) and repeating the construction of the optimal portfolio as seen in the previous section, we obtain the following results.

In general \((k = 1, \ldots, T)\), the optimal buy boundary \( A_{T-k}^- \) and sell boundary \( A_{T-k}^+ \) coincide to the same point, which is denoted by \( \tilde{A}_{T-k} \) and given by the first order condition

\[
E_{T-k} \left[ (s_{T-k} - r_{T-k}) \tilde{F}_{T-k}^{T-1} \right] = 0, \quad (37)
\]

where

\[
\tilde{F}_{T-k} = r_{T-k} + (s_{T-k} - r_{T-k}) \tilde{A}_{T-k}. \quad (38)
\]

This optimal point is commonly known as the Merton proportion. Generally, one has to solve Eq. (37) numerically for \( \tilde{A}_{T-k} \), which can be easily done by using standard root finding techniques. Moreover, for the case where the risky asset has a binomial price process, one will be able to obtain an explicit solution for \( \tilde{A}_{T-k} \). The investor’s optimal strategy is thus to transact to the Merton proportion at each time step of the investment process. In addition, the optimal value function \( \tilde{V}_{T-k} \) is given by

\[
\tilde{V}_{T-k} = \frac{1}{\gamma} E_{T-k} \left[ \tilde{F}_{T-k}^{T-1} \right] \cdots \left[ \tilde{F}_{T-k+1}^{T-1} \right] E_{T-k} \left[ \tilde{F}_{T-k}^{T-1} \right]. \quad (39)
\]

It is observed that the optimal value function at time \( T-k \) does not vary with the proportion \( A_{T-k} \) of risky asset inherited from the previous time step, since there is no cost incurred in buying or selling the risky asset to reach the Merton proportion \( \tilde{A}_{T-k} \). The optimal strategy is a myopic one as the investor does not need to consider future rebalancing opportunities at the time steps ahead. If one further assumes that \( s_{T-k} \) are independent and identically distributed random variables and that \( r_{T-k} \) is a constant independent of \( k \), then the Merton
proportion $\tilde{A}_{T-k}$ simplifies to a constant independent of $k$ and the optimal value function becomes $	ilde{V}_{T-k} = \frac{1}{\gamma} \left\{ \mathbb{E}_{T-k} \left[ \tilde{F}_{T-k}^\gamma \right] \right\}^k$. The relatively simple solution of the Merton proportion and the optimal value function motivates one to carry out a perturbation analysis about the no transaction costs solution, in the limit of small transaction costs. Before proceeding further, it is useful to state the following results that

$$
\mathbb{E}_{T-k} \left[ \tilde{F}_{T-k}^\gamma \right] = \mathbb{E}_{T-k} \left[ r_{T-k} \tilde{F}_{T-k}^{\gamma-1} \right] \tag{40}
$$

and

$$
\mathbb{E}_{T-k} \left[ r_{T-k} (s_{T-k} - r_{T-k}) \tilde{F}_{T-k}^{\gamma-2} \right] = -\mathbb{E}_{T-k} \left[ \tilde{A}_{T-k} (s_{T-k} - r_{T-k})^2 \tilde{F}_{T-k}^{\gamma-2} \right]. \tag{41}
$$

The above results are direct consequences of Eq. (37) and will be used extensively to simplify the asymptotic approximations of the value functions in the next section.

5. Small Transaction Costs Case

In practice, transaction costs are usually small compared to the value of the transactions. As observed in the previous section, the no transaction costs problem admits a relatively simple myopic solution. Therefore, one is motivated to analyze the small transaction costs solution as a perturbation about the no transaction costs solution. In Atkinson and Storey (2010), they obtained the leading order approximations to the optimal buy and sell boundaries for two time steps via the expansion of Eqs. (12), (13), (25) and (26) in the limit of small transaction costs. However, it was not obvious that a direct expansion of the first order conditions will enable one to easily obtain leading order approximations to an arbitrary number of time steps. In this section, we present an approach to apply perturbation analysis about the no transaction costs solution in the limit of small transaction costs. The advantages of this approach over the direct expansion approach is that it allows one to systematically obtain approximations of both the optimal value function and optimal boundaries for an arbitrary number of time steps. A similar approach had been adopted in Atkinson and Quek (2012) for an investor with the exponential utility function. A feature of the exponential utility function was that it resulted in optimal boundaries that were independent of the investor’s wealth, which is not usually the case in practice. A more realistic description of the investor’s optimal strategy is provided by using the power utility function. Moreover, it is also more challenging to carry out the perturbation analysis in this context as the proportion of risky asset inherited at each time step depends on variations in both the return of the risky asset and the investor’s wealth.

We start with approximating the optimal value function and follow by determining the optimal boundaries. In order to approximate the optimal value function for an arbitrary number of time steps, we adopt an approach that consists of two stages. The first stage involves making the assumption that the investor buys or sells to reach the Merton proportion at each time step when transaction costs are small. This is clearly a suboptimal strategy as the investor has ignored the presence of the no-transaction region. Consequently, an approximation of the suboptimal value function is derived at each time step. The second stage assumes that the investor behaves optimally by taking into account the no-transaction region. A sequence of corrections are then applied to the suboptimal value function to give us the desired approximation to the optimal value function. After approximating the optimal value function at each time step, the optimal boundaries are then approximated by imposing the condition that the first derivative of the value function is continuous across the boundaries. Finally, the error in this procedure is calculated and we give an example in the Appendix.

Suppose now that transaction costs are small such that $\lambda_{T-k} = \varepsilon \lambda_{T-k}$ and $\mu_{T-k} = \varepsilon \mu_{T-k}$, where $\varepsilon \ll 1$, $\lambda_{T-k} = O(1)$ and $\mu_{T-k} = O(1)$, for $k = 1, \ldots, T$. Here, $O(.)$ is the usual
asymptotic order symbol so that $\bar{\lambda}_T - k$ and $\bar{\mu}_T - k$ are said to be “of the order” 1. Equivalently, $\lambda_T - k$ and $\mu_T - k$ are said to be of the order $\varepsilon$. We now apply a perturbation analysis in the following two stages.

5.1 Stage One: Transacting to the Merton Proportion

In the first stage, assume that the investor follows the suboptimal strategy of transacting to the Merton proportion. This is equivalent to assuming that both the optimal buy and sell boundaries are equal to the Merton proportion, which is suboptimal as we have effectively removed one of the investor’s possible choices of not transacting in the risky asset.

In general, at time $T - k$ for $k = 1, \ldots, T$, the investor is assumed to adopt the suboptimal strategy of buying $L_T - k$ or selling $M_T - k$ of the risky asset to reach the Merton proportion $\tilde{A}_T - k = A_T^+$. Therefore, Eq. (5) becomes

$$ W_{T-k+1} = W_{T-k} \hat{F}_{T-k}, \quad (42) $$

where

$$ \hat{F}_{T-k} = \tilde{F}_{T-k} - \varepsilon \bar{\lambda}_T - k r_{T-k} L_{T-k} - \varepsilon \bar{\mu}_T - k r_{T-k} M_{T-k}. \quad (43) $$

Recall, from the analysis of the no transaction costs solution, that $\tilde{F}_{T-k} = r_{T-k} + (s_{T-k} - r_{T-k}) \tilde{A}_{T-k}$. The proportion of risky asset inherited in the next time step is now given by

$$ A_{T-k+1} = \frac{s_{T-k} \tilde{A}_{T-k}}{\hat{F}_{T-k}}. \quad (44) $$

In particular, the investor’s suboptimal strategy of transacting to the Merton proportion and corresponding value function in the buy and sell regions (compare with Section 3) are as follows:

In the buy region $A_{T-k} < \tilde{A}_{T-k}$, the investor buys $L_{T-k} = \tilde{A}_{T-k} - A_{T-k}$ of the risky asset so that

$$ \hat{F}^{(B)}_{T-k} = \hat{F}_{T-k} - \varepsilon \bar{\lambda}_T - k r_{T-k} L_{T-k} \tilde{A}_{T-k} - A_{T-k} \tilde{A}_{T-k} \hat{V}^{(B)}_{T-k}. \quad (45) $$

The value function now becomes

$$ \hat{V}^{(B)}_{T-k} = \frac{1}{\gamma} \mathbb{E}_{T-k} \left[ \hat{F}^{(B)}_{T-k} \hat{V}^{(B)}_{T-k+1} \right] \quad (46) $$

at time $T - 1$ and

$$ \hat{V}^{(B)}_{T-k} = \mathbb{E}_{T-k} \left[ \hat{F}^{(B)}_{T-k} \hat{V}^{(B)}_{T-k+1} \right] \quad (47) $$

at time $T - k$ ($k = 2, \ldots, T$), where $s_{T-k} = \tilde{s}_{T-k}$ denotes one plus the risky return that results in an inherited proportion of $A_{T-k+1} = \tilde{A}_{T-k+1}$ (the Merton proportion). From Eq.
(44), \( \hat{s}_{T-k} = \frac{r_{T-k} \hat{A}_{T-k+1}}{\hat{A}_{T-k}} \left\{ \left( 1 - \hat{A}_{T-k} \right) - \varepsilon \hat{\lambda}_{T-k} \left( \hat{A}_{T-k} - \hat{A}_{T-k} \right) \right\} \). It is convenient to define

\[
\hat{s}_{T-k} = \frac{r_{T-k} \hat{A}_{T-k+1}}{\hat{A}_{T-k}} \left( 1 - \hat{A}_{T-k} \right)
\]

and then express

\[
\hat{s}_{T-k} = \hat{s}_{T-k} \left\{ 1 - \frac{\varepsilon \hat{\lambda}_{T-k} \left( \hat{A}_{T-k} - \hat{A}_{T-k} \right)}{\left( 1 - \hat{A}_{T-k} \right)} \right\}.
\]

Therefore, note that \( \hat{s}_{T-k} \) is the leading order term of \( \hat{s}_{T-k} \). It is remarked that, if one assumes in the special case that \( s_{T-k} \) are independent and identically distributed random variables and that \( r_{T-k} \) are constant in time, then \( \hat{A}_{T-k} \) are constant in time and \( \hat{s}_{T-k} \) reduces to \( r_{T-k} \).

In the sell region \( A_{T-k} > \hat{A}_{T-k} \), the investor sells \( M_{T-k} = A_{T-k} - \hat{A}_{T-k} \) of the risky asset so that

\[
\hat{F}_{T-k}^{(S)} = \hat{F}_{T-k} - \varepsilon \hat{\mu}_{T-k} r_{T-k} \left( A_{T-k} - \hat{A}_{T-k} \right).
\]

The value function now becomes

\[
\hat{V}_{T-1}^{(S)} = \frac{1}{\gamma} \mathbb{E}_{T-1} \left[ \hat{F}_{T-1}^{(S) \gamma} \right]
\]
at time \( T - 1 \) and

\[
\hat{V}_{T-k}^{(S)} = \mathbb{E}_{T-k} \left[ \hat{F}_{T-k}^{(S) \gamma} \hat{V}_{T-k+1} \right]
\]

\[
= \int_{0}^{\hat{s}_{T-k}} \hat{F}_{T-k}^{(S) \gamma} \hat{V}_{T-k+1} p(s_{T-k}) \, ds_{T-k}
\]

\[
+ \int_{\hat{s}_{T-k}}^{\infty} \hat{F}_{T-k}^{(S) \gamma} \hat{V}_{T-k+1} p(s_{T-k}) \, ds_{T-k}
\]
at time \( T - k \) (\( k = 2, \ldots, T \)), where

\[
\hat{s}_{T-k} = \hat{s}_{T-k} \left\{ 1 - \frac{\varepsilon \hat{\mu}_{T-k} \left( A_{T-k} - \hat{A}_{T-k} \right)}{\left( 1 - \hat{A}_{T-k} \right)} \right\}.
\]

It is of interest to note that the value functions in the buy and sell regions differ by only a change of the transaction cost variables from \( \hat{\lambda}_{T-k} \) to \( -\hat{\mu}_{T-k} \). Essentially, we exploit this observation to deduce the approximation of the value function in the sell region from that in the buy region.

### 5.2 Stage One: Perturbation about the No Transaction Costs Case

Since the parameter \( \varepsilon \ll 1 \) in the limit of small transaction costs, we will derive approximations of the suboptimal value functions as power series in terms of \( \varepsilon \), starting from time \( T - 1 \) and then proceeding to the general time \( T - k \) case. We achieve this by perturbing the suboptimal value function about the no transaction costs solution. Recall that the no transaction costs solution is of a relatively simple form since it is characterized by a myopic investment strategy. In the spirit of dynamic programming, the perturbation will be done recursively, starting from time \( T - 1 \) and proceeding backwards in time. In general, the perturbation at time \( T - k \).
depends on the perturbations from the time steps ahead. The exception is time \( T - 1 \), since it is one step before termination of the investment process.

### 5.2.1 Time \( T - 1 \)

The value function in the buy region, from Eqs. (45) and (46), is given by

\[
\hat{V}_{T-1}^{(B)} = \frac{1}{\gamma} \mathbb{E}_{T-1} \left[ \left\{ \frac{\gamma}{T} \hat{F}_{T-1} - \varepsilon \hat{\lambda}_{T-1} \right\} \left( \hat{A}_{T-1} - A_{T-1} \right) \right].
\]  

(54)

Expanding it as a power series in \( \varepsilon \) and using Eq. (40), we approximate

\[
\hat{V}_{T-1}^{(B)} = \hat{V}_{T-1} \left\{ 1 - \varepsilon \hat{\lambda}_{T-1} \right\} \left( \hat{A}_{T-1} - A_{T-1} \right)
\]

\[+ \frac{1}{2} \varepsilon^2 \hat{\lambda}_{T-1}^2 \alpha_{T-1} \left( \hat{A}_{T-1} - A_{T-1} \right)^2 \] + \( O(\varepsilon^3) \),

\[\alpha_{T-1} = \frac{\gamma (\gamma - 1) \lambda_{T-1}^2 \mathbb{E}_{T-1} \left[ \hat{F}_{T-1}^{\gamma - 2} \right]}{\mathbb{E}_{T-1} \left[ \hat{F}_{T-1}^\gamma \right]}.
\]

(55)

(56)

Note that the leading order term of the expansion is \( \hat{V}_{T-1} = \frac{1}{\gamma} \mathbb{E}_{T-1} \left[ \hat{F}_{T-1}^\gamma \right] \), which we recall is the no transaction costs solution given by Eq. (39). The interested reader is referred to \( A \) for an analysis of the remainder term in the above expansion, which is shown to be bounded. The remainder terms for subsequent expansions will not be provided but are otherwise similar.

Similarly, the value function in the sell region is given by Eqs. (50) and (51), which differs from the value function in the buy region by a change of variable from \( \hat{\lambda}_{T-1} \) to \( -\hat{\mu}_{T-1} \).

Therefore, it can be immediately deduced from Eq. (55) that

\[
\hat{V}_{T-1}^{(S)} = \hat{V}_{T-1} \left\{ 1 - \varepsilon \hat{\mu}_{T-1} \right\} \left( \hat{A}_{T-1} - \hat{A}_{T-1} \right)
\]

\[+ \frac{1}{2} \varepsilon^2 \hat{\mu}_{T-1}^2 \alpha_{T-1} \left( \hat{A}_{T-1} - \hat{A}_{T-1} \right)^2 \] + \( O(\varepsilon^3) \).

(57)

### 5.2.2 Time \( T - k \)

Taking one step back to time \( T - 2 \), the value function in the buy region is given by Eq. (47). Our aim is to delineate the integrals of \( \hat{V}_{T-2}^{(B)} \) by \( \tilde{s}_{T-2} \) rather than \( \hat{s}_{T-2} \), since \( \tilde{s}_{T-2} \) is the leading order term of \( \hat{s}_{T-2} \). Therefore, it is rewritten as

\[
\hat{V}_{T-2}^{(B)} = \int_{0}^{\tilde{s}_{T-2}} \hat{F}_{T-2}^{(B)\gamma} \hat{V}_{T-1}^{(B)} \hat{p}(s_{T-2}) \, ds_{T-2} + \int_{\tilde{s}_{T-2}}^{\infty} \hat{F}_{T-2}^{(B)\gamma} \hat{V}_{T-1}^{(S)} \hat{p}(s_{T-2}) \, ds_{T-2}
\]

\[+ \int_{\tilde{s}_{T-2}}^{\infty} \hat{F}_{T-2}^{(B)\gamma} \left\{ \hat{V}_{T-2}^{(B)} - \hat{V}_{T-2}^{(S)} \right\} \hat{p}(s_{T-2}) \, ds_{T-2},
\]

(58)

where \( \tilde{s}_{T-2} = \tilde{s}_{T-2} + O(\varepsilon) \) from Eq. (49). Recall that \( \hat{V}_{T-1}^{(B)} \) and \( \hat{V}_{T-1}^{(S)} \) are functions of \( A_{T-1} \), where \( A_{T-1} = \frac{s_{T-2} \hat{A}_{T-2}}{\hat{F}_{T-2}^{(B)}} \) from Eq. (44). Also, when \( s_{T-2} = \tilde{s}_{T-2} \), we have \( A_{T-1} = \hat{A}_{T-1} \) by definition. We now derive an estimate of the third integral. Applying the Mean Value Theorem
for an integral,
\[ \int_{\hat{s}_T-2}^{\hat{s}_T-2} \hat{F}_{T-2}^{(B)\gamma} \left\{ \hat{V}_{T-1}^{(S)} - \hat{V}_{T-1}^{(B)} \right\} p(s_{T-2}) d s_{T-2} \]
\[ = (\hat{s}_T-2 - \hat{s}_T-2) \hat{F}_{T-2}^{(B)\gamma} \left\{ \hat{V}_{T-1}^{(S)} - \hat{V}_{T-1}^{(B)} \right\} p(s_{T-2}), \] (59)

which is evaluated at a point \( s_{T-2} \in (\hat{s}_T-2, \hat{s}_T-2) \), i.e. \( s_{T-2} = \hat{s}_T-2 + O(\varepsilon) \). At this point, \( A_{T-1} = \hat{A}_{T-1} + O(\varepsilon) \) as a return that is close to \( \hat{s}_T-2 \) results in a proportion of risky asset that is close to \( \hat{A}_{T-1} \). This implies that
\[ \hat{V}_{T-1}^{(S)} - \hat{V}_{T-1}^{(B)} = -\varepsilon (\overline{\mu}_{T-1} + \overline{\lambda}_{T-1}) \gamma \hat{V}_{T-1} \left( A_{T-1} - \hat{A}_{T-1} \right) + O(\varepsilon^2) \] (60)
is of \( O(\varepsilon^3) \). Since \( \hat{s}_T-2 - \hat{s}_T-2 \) is of \( O(\varepsilon) \), Eq. (59) is of \( O(\varepsilon^3) \).

Therefore, the value function in the buy region is approximated by
\[ \hat{V}_{T-2}^{(B)} = \int_0^{\hat{s}_T-2} \hat{F}_{T-2}^{(B)\gamma} \hat{V}_{T-1}^{(B)} p(s_{T-2}) d s_{T-2} + \int_{\hat{s}_T-2}^{\infty} \hat{F}_{T-2}^{(B)\gamma} \hat{V}_{T-1}^{(S)} p(s_{T-2}) d s_{T-2} + O(\varepsilon^3). \] (61)

We now substitute the expressions for \( \hat{F}_{T-2}^{(B)} \), \( \hat{V}_{T-1}^{(B)} \) and \( \hat{V}_{T-1}^{(S)} \) from Eqs. (45), (55) and (57) into Eq. (61). Expanding in powers of \( \varepsilon \), simplifying with Eq. (40) and collecting the common terms together, it can be shown after some algebra that \( \hat{V}_{T-2}^{(B)} \) is given by Eq. (62) below. In order to approximate the value function in the sell region, recall that it differs from the value function in the buy region by a change of variable from \( \lambda_{T-2} \) to \( -\overline{\mu}_{T-2} \). Therefore, with the aforementioned change of variable, we can immediately deduce the approximation of \( \hat{V}_{T-2}^{(S)} \), which is given below by Eq. (63).

Using Eqs. (47) and (52), the above analysis for the time \( T - 2 \) case is repeated recursively at time \( T - 3 \), time \( T - 4 \) and so on. In general, the suboptimal value function in the buy and sell regions at time \( T - k \) (\( k = 2, \ldots, T \)) can be inductively shown to be approximated by

\[ \hat{V}_{T-k}^{(B)} = \hat{V}_{T-k} \left\{ 1 - \varepsilon \overline{\lambda}_{T-k} \gamma \left( \hat{A}_{T-k} - A_{T-k} \right) + \frac{1}{2} \varepsilon^2 \overline{\lambda}_{T-k} \alpha_{T-k} \left( \hat{A}_{T-k} - A_{T-k} \right)^2 \right. \]
\[ + \varepsilon^2 \overline{\lambda}_{T-k} \left( \beta_{T-k} - \gamma \sum_{i=3}^{k} \zeta_{T-i+1} \right) \left( \hat{A}_{T-k} - A_{T-k} \right) + \varepsilon \sum_{i=2}^{k} \zeta_{T-i} \]
\[ + \varepsilon^2 \left( \frac{1}{2} \sum_{i=2}^{k} \eta_{T-i} \alpha_{T-i+1} - \frac{1}{\gamma} \sum_{i=3}^{k} \zeta_{T-i} \beta_{T-i+1} + \sum_{i=4}^{k} \sum_{j=4}^{i} \zeta_{T-i} \zeta_{T-j+2} \right) \}
\[ + O(\varepsilon^3) \] (62)

and, with a change of variable from \( \overline{\lambda}_{T-k} \) to \( -\overline{\mu}_{T-k} \), by

\[ \hat{V}_{T-k}^{(S)} = \hat{V}_{T-k} \left\{ 1 - \varepsilon \overline{\mu}_{T-k} \gamma \left( A_{T-k} - \hat{A}_{T-k} \right) + \frac{1}{2} \varepsilon^2 \overline{\mu}_{T-k} \alpha_{T-k} \left( A_{T-k} - \hat{A}_{T-k} \right)^2 \right. \]
\[ + \varepsilon^2 \overline{\mu}_{T-k} \left( \beta_{T-k} - \gamma \sum_{i=3}^{k} \zeta_{T-i+1} \right) \left( A_{T-k} - \hat{A}_{T-k} \right) + \varepsilon \sum_{i=2}^{k} \zeta_{T-i} \]
\[ + \varepsilon^2 \left( \frac{1}{2} \sum_{i=2}^{k} \eta_{T-i} \alpha_{T-i+1} - \frac{1}{\gamma} \sum_{i=3}^{k} \zeta_{T-i} \beta_{T-i+1} + \sum_{i=4}^{k} \sum_{j=4}^{i} \zeta_{T-i} \zeta_{T-j+2} \right) \}
\[ + O(\varepsilon^3). \] (63)
Note that in Eqs. (62) and (63), the summation terms are only valid in instances where the upper limit is at least as large as the lower limit. For example, when $k = 2$, $\sum_{i=3}^{k} \zeta_{T-i+1}$ is not valid and assumed to be equal to zero while $\sum_{i=2}^{k} \eta_{T-i} = \eta_{T-2}$. The corresponding definitions of $\alpha_{T-k}$, $\beta_{T-k}$, $\zeta_{T-k}$ and $\eta_{T-k}$ are as follows:

$$\alpha_{T-k} = \frac{\gamma (\gamma - 1) \eta_{T-k}^2 E_{T-k} \left[ \tilde{F}_{T-k}^{-2} \right]}{E_{T-k} \left[ \tilde{F}_{T-k}^{-1} \right]},$$

(64)

$$\beta_{T-k} = \frac{\gamma T_{T-k}}{E_{T-k} \left[ \tilde{F}_{T-k}^{-1} \right]} \times \left\{ T_{T-k+1} \int_{0}^{T_{T-k}} \tilde{F}_{T-k}^{-2} \left( s_{T-k} \tilde{A}_{T-k} - \gamma \left( s_{T-k} \tilde{A}_{T-k} - \tilde{F}_{T-k} \tilde{A}_{T-k+1} \right) \right) p(s_{T-k}) ds_{T-k} \right\}$$

$$- \mu_{T-k+1} \int_{0}^{T_{T-k}} \tilde{F}_{T-k}^{-1} \left( s_{T-k} \tilde{A}_{T-k} - \gamma \left( s_{T-k} \tilde{A}_{T-k} - \tilde{F}_{T-k} \tilde{A}_{T-k+1} \right) \right) p(s_{T-k}) ds_{T-k} \right\},$$

(65)

$$\zeta_{T-k} = \frac{\gamma}{E_{T-k} \left[ \tilde{F}_{T-k}^{-1} \right]} \times \left\{ \tilde{\lambda}_{T-k+1} \int_{0}^{T_{T-k}} \tilde{F}_{T-k}^{-1} \left( s_{T-k} \tilde{A}_{T-k} - \tilde{F}_{T-k} \tilde{A}_{T-k+1} \right) p(s_{T-k}) ds_{T-k} \right\}$$

$$- \mu_{T-k+1} \int_{0}^{T_{T-k}} \tilde{F}_{T-k}^{-1} \left( s_{T-k} \tilde{A}_{T-k} - \tilde{F}_{T-k} \tilde{A}_{T-k+1} \right) p(s_{T-k}) ds_{T-k} \right\},$$

(66)

and

$$\eta_{T-k} = \frac{1}{E_{T-k} \left[ \tilde{F}_{T-k}^{-1} \right]} \times \left\{ \tilde{\lambda}_{T-k+1} \int_{0}^{T_{T-k}} \tilde{F}_{T-k}^{-2} \left( s_{T-k} \tilde{A}_{T-k} - \tilde{F}_{T-k} \tilde{A}_{T-k+1} \right) \right\}^2 p(s_{T-k}) ds_{T-k} \right\}$$

$$+ \mu_{T-k+1} \int_{0}^{T_{T-k}} \tilde{F}_{T-k}^{-2} \left( s_{T-k} \tilde{A}_{T-k} - \tilde{F}_{T-k} \tilde{A}_{T-k+1} \right) p(s_{T-k}) ds_{T-k} \right\}. \quad (67)$$

In the first stage of the perturbation analysis, we have obtained approximations for the suboptimal value functions $\tilde{V}_{T-k}^{(B)}$ and $\tilde{V}_{T-k}^{(S)}$ in the buy and sell regions by ignoring the no-transaction region and perturbing about the no transaction costs solution. The main advantage is that we have derived the approximations, albeit suboptimal, at any time step of the investment process. However, we do not know how the no-transaction region will affect the terms in these approximations and the corrections that may be required. In the second stage of our analysis, we improve on these preliminary approximations by correcting them as we incorporate the no-transaction region.
5.3 Stage Two: Perturbation about the Suboptimal Value Function

In the second stage of the perturbation analysis, we reintroduce the no-transaction region in our approximation of the optimal value function. So instead of transacting to the Merton proportion at all times, the investor will buy to reach the optimal buy boundary when he falls in the buy region. Correspondingly, the investor will sell to reach the optimal sell boundary in the sell region and will choose not to trade in the no-transaction region. In the limit of small transaction costs, one would expect the optimal buy and sell boundaries to be close to the Merton proportion. Assume that the optimal value function in the case of small transaction costs, one would expect the optimal buy and sell boundaries to be close to the Merton proportion. Assume that the optimal value function in the corresponding buy and sell regions, followed by a further perturbation about the no-transaction region, which will be a perturbation about the no transaction costs solution. Applying the dynamic programming principle, this sequence of correction and approximation is achieved recursively backwards in time by using the estimates of the optimal value functions from the time steps ahead. As before, the analysis starts at time \( T - 1 \) before proceeding to the general time \( T - k \) case.

We subsequently demonstrate in our perturbation analysis that these assumptions are indeed self-consistent in this model.

At each time step, we first perturb the optimal value function about the suboptimal buy and sell regions, followed by a further perturbation about the no-transaction costs solution. The aim is to correct the suboptimal value function for the terms that were left out when we assumed a strategy of transacting to the Merton proportion. Recall that for the optimal value function, \( F_{T-k}^{(B)} \), \( F_{T-k}^{(S)} \) and \( F_{T-k}^{(N)} \) are given by Eqs. (29), (33) and (36) respectively. Since we are first perturbing the optimal value function about the suboptimal value function, using Eqs. (68), (69) and (70), they are rewritten as

\[
F_{T-k}^{(B)} = \hat{F}_{T-k}^{(B)} + \varepsilon \omega_{T-k}^- (s_{T-k} - r_{T-k}) - \varepsilon^2 \lambda_{T-k} \omega_{T-k}^- r_{T-k}, \tag{71}
\]

\[
F_{T-k}^{(S)} = \hat{F}_{T-k}^{(S)} + \varepsilon \omega_{T-k}^+ (s_{T-k} - r_{T-k}) + \varepsilon^2 \mu_{T-k} \omega_{T-k}^+ r_{T-k}, \tag{72}
\]

and

\[
F_{T-k}^{(N)} = \hat{F}_{T-k} + \varepsilon \omega_{T-k}^- (s_{T-k} - r_{T-k}). \tag{73}
\]

Define the correction term in the buy and sell regions by

\[
\delta_{T-k}^{(B)} = V_{T-k}^{(B)} - \hat{V}_{T-k}^{(B)}, \tag{74}
\]

and

\[
\delta_{T-k}^{(S)} = V_{T-k}^{(S)} - \hat{V}_{T-k}^{(S)}, \tag{75}
\]

respectively. After correcting for the optimal value function in the buy and sell regions, we proceed to obtain an approximation of the optimal value function \( V_{T-k}^{(N)} \) in the no-transaction region, which will be a perturbation about the no transaction costs solution. Applying the dynamic programming principle, this sequence of correction and approximation is achieved recursively backwards in time by using the estimates of the optimal value functions from the time steps ahead. As before, the analysis starts at time \( T - 1 \) before proceeding to the general time \( T - k \) case.
5.3.1 Time $T - 1$

In the buy region, recall that $\hat{V}^{(B)}_{T-1}$ is given by Eq. (46) and that $V^{(B)}_{T-1}$ is given by Eqs. (14) and (71) as

$$\hat{V}^{(B)}_{T-1} = \frac{1}{\gamma} E_{T-1} \left[ \hat{F}^{(B)}_{T-1} \right]$$

and

$$V^{(B)}_{T-1} = \frac{1}{\gamma} E_{T-1} \left[ \left\{ \hat{F}^{(B)}_{T-1} + \varepsilon\bar{\omega}_{T-1} (s_{T-1} - r_{T-1}) - \varepsilon^2 \bar{\lambda}_{T-1} \bar{\omega}_{T-1} r_{T-1} \right\} \right]$$

respectively. First, we perturb about the suboptimal solution as the correction term is given by $\delta^{(B)}_{T-1} = V^{(B)}_{T-1} - \hat{V}^{(B)}_{T-1}$. Expanding Eq. (77) in powers of $\varepsilon$ up to $O(\varepsilon^2)$ and subtracting Eq. (76), we obtain

$$\delta^{(B)}_{T-1} = \frac{1}{\gamma} E_{T-1} \left[ \gamma \hat{F}^{(B)}_{T-1}^{-1} \left\{ \varepsilon\bar{\omega}_{T-1} (s_{T-1} - r_{T-1}) - \varepsilon^2 \bar{\lambda}_{T-1} \bar{\omega}_{T-1} r_{T-1} \right\} \right]$$

$$+ \frac{1}{2} \gamma (\gamma - 1) \hat{F}^{(B)}_{T-1}^{-2} \varepsilon^2 \bar{\omega}_{T-1}^2 (s_{T-1} - r_{T-1})^2 + O(\varepsilon^3).$$

Next, we perturb about the no transaction costs solution. Substituting $\hat{F}^{(B)}_{T-1} = \hat{F}^{(B)}_{T-1} = \bar{\lambda}_{T-1} \bar{\omega}_{T-1} r_{T-1} \left( \hat{A}_{T-1} - A_{T-1} \right)$ into Eq. (78), expanding further in powers of $\varepsilon$ and simplifying with Eqs. (37), (40) and (41), the correction term is found to be

$$\delta^{(B)}_{T-1} = \bar{V}_{T-1} \epsilon^2 \left\{ \bar{\lambda}_{T-1} \bar{\omega}_{T-1} r_{T-1} \left( \hat{A}_{T-1} - A_{T-1} \right) + \frac{1}{2} \bar{\omega}_{T-1}^2 \right\} \phi_{T-1}$$

$$- \bar{\lambda}_{T-1} \bar{\omega}_{T-1} \gamma \} + O(\varepsilon^3),$$

where

$$\phi_{T-1} = \frac{\gamma (\gamma - 1) E_{T-1} \left[ (s_{T-1} - r_{T-1})^2 \hat{F}^{(B)}_{T-1}^{-2} \right]}{E_{T-1} \left[ \hat{F}^{(B)}_{T-1} \right]}.$$ (80)

In the sell region, note that $V^{(S)}_{T-1}$ is given by Eqs. (17) and (72) as

$$V^{(S)}_{T-1} = \frac{1}{\gamma} E_{T-1} \left[ \left\{ \hat{F}^{(S)}_{T-1} + \varepsilon\bar{\omega}_{T-1} (s_{T-1} - r_{T-1}) + \varepsilon^2 \bar{\mu}_{T-1} \bar{\omega}_{T-1} r_{T-1} \right\} \right]$$

which is equivalent to $V^{(B)}_{T-1}$ with a change of variable from $\bar{\mu}_{T-1}$ to $\bar{\lambda}_{T-1}$ and from $\bar{\omega}_{T-1}$ to $\bar{\omega}_{T-1}$. Therefore, the correction term in the sell region is immediately deduced from Eq. (79) to be

$$\delta^{(S)}_{T-1} = \bar{V}_{T-1} \epsilon^2 \left\{ \bar{\mu}_{T-1} \bar{\omega}_{T-1} \left( \hat{A}_{T-1} - \hat{A}_{T-1} \right) + \frac{1}{2} \bar{\omega}_{T-1}^2 \right\} \phi_{T-1}$$

$$+ \bar{\mu}_{T-1} \bar{\omega}_{T-1} \gamma \} + O(\varepsilon^3).$$ (82)

In the no-transaction region, $V^{(N)}_{T-1}$ is given by Eq. (20) and (73) as

$$V^{(N)}_{T-1} = \frac{1}{\gamma} E_{T-1} \left[ \left\{ \hat{F}^{(N)}_{T-1} + \varepsilon\bar{\omega}_{T-1} (s_{T-1} - r_{T-1}) \right\} \right]$$

$$.$
Expanding in powers of $\varepsilon$ up to $O(\varepsilon^2)$ and simplifying with Eq. (37),

$$V_{T-1}^{(N)} = \tilde{V}_{T-1} - 1 + \frac{1}{2} \varepsilon^2 \omega_{T-1}^2 \phi_{T-1} + O(\varepsilon^3).$$

Note that time $T - 1$ is a special case as it is one step before termination of the investment process. We now take one step back to time $T - 2$ and describe the analysis that is required, followed by the result for the general case.

5.3.2 Time $T - k$

In this section, we describe the key ideas in deriving the correction and approximation of the optimal value function at time $T - 2$. Details of this perturbation analysis can be found in B. One of the key observations is that the analysis in the sell region differs from that in the buy region only by a suitable change of variables. This reduces the analysis that is required for the problem, since one can immediately deduce the result for the sell region from the buy region by a simple change of variables.

Therefore, we focus our attention to the perturbation analysis in the buy region. From Eq. (30), $V_{T-2}^{(B)}$ is expressed in its integral form as a sum of integrals delineated by $s_{T-2}^-$ and $s_{T-2}^+$. We note that $\tilde{s}_{T-2}$ is the leading order term of $s_{T-2}^-$ and $s_{T-2}^+$. Thus, our first objective is to rewrite $V_{T-2}^{(B)}$ as a sum of integrals delineated by $\tilde{s}_{T-2}$, which is achieved by applying the mean value theorem for integrals. This makes it directly comparable with $\tilde{V}_{T-2}^{(B)}$ from Eq. (61), which we have also expressed as a sum of integrals delineated by $\tilde{s}_{T-2}$. In order to estimate the correction $\delta_{T-2}^{(B)} = V_{T-2}^{(B)} - \tilde{V}_{T-2}^{(B)}$, we first perturb the optimal value function $V_{T-2}^{(B)}$ about the suboptimal value function $\tilde{V}_{T-2}^{(B)}$ by taking their difference and using the approximation of the optimal value function from the time step ahead (i.e. time $T - 1$ in this case). This is followed by a perturbation about the no transaction costs solution. After some long algebra and simplification, we will be able to derive the approximation of the correction $\delta_{T-2}^{(B)}$ in the buy region. The corresponding approximation of the correction term in the sell region $\delta_{T-2}^{(S)} = V_{T-2}^{(S)} - \tilde{V}_{T-2}^{(S)}$ can then be immediately deduced from the buy region by a change of variables from $\tilde{\lambda}_{T-2}$ to $-\mu_{T-2}$ and $\tilde{\omega}_{T-2}$ to $\phi_{T-2}^{+}$.

After deriving the correction in the buy and sell regions, we proceed to estimate the optimal value function $V_{T-2}^{(N)}$ for the no-transaction region, which is given by Eq. (34). Once again, by applying the mean value theorem for integrals, we can rewrite $V_{T-2}^{(N)}$ as a sum of integrals delineated by $\tilde{s}_{T-2}$. Substituting in estimates of the optimal value function from the time step ahead, expanding in powers of $\varepsilon$ and simplifying, we will be able to derive the approximation for the optimal value function in the no-transaction region.

Having approximated the optimal value function in the buy, sell and no-transaction regions at time $T - 2$, we then proceed to the next time step. Applying the dynamic programming principle, the above analysis for the time $T - 2$ case is repeated recursively backwards in time at $T - 3$ using the results from $T - 2$, at $T - 4$ using the results from $T - 3$ and so on. By induction, we are then able to derive the results for the general time $T - k$ ($k = 2, \ldots, T$) case, which are stated below. The correction in the buy and sell regions are found to be given by

$$\delta_{T-k}^{(B)} = \tilde{V}_{T-k} \varepsilon^2 \left\{ \tilde{\lambda}_{T-k} \tilde{\omega}_{T-k} A_{T-k} \left( \tilde{A}_{T-k} - A_{T-k} \right) + \frac{1}{2} \omega_{T-k}^2 \phi_{T-k} \right\} \phi_{T-k}$$

$$- \tilde{\lambda}_{T-k} \tilde{\omega}_{T-k} \gamma + \tilde{\omega}_{T-k} \psi_{T-k} + \sum_{i=2}^{k} \theta_{T-i} + O(\varepsilon^3)$$

(85)
and
\[
\delta^{(S)}_{T-k} = \tilde{V}_{T-k} \varepsilon^2 \left\{ \tilde{\mu}_{T-k} \omega^+_{T-k} \tilde{A}_{T-k} \left( \tilde{A}_{T-k} - \tilde{A}_{T-k} \right) + \frac{1}{2} \omega^+_{T-k} \right\} \phi_{T-k} \\
+ \tilde{\mu}_{T-k} \omega^+_{T-k} \gamma + \omega^+_{T-k} \psi_{T-k} + \sum_{i=2}^{k} \theta_{T-i} \right\} + O(\varepsilon^3). \quad (86)
\]

The optimal value function in the no-transaction region is given by
\[
V^{(N)}_{T-k} = \tilde{V}_{T-k}^{(N)} + \delta^{(N)}_{T-k}, 
\]
where
\[
\tilde{V}^{(N)}_{T-k} = \tilde{V}_{T-k} \left\{ 1 + \varepsilon^2 \omega_{T-k} \psi_{T-k} + \frac{1}{2} \varepsilon^2 \omega^2_{T-k} \phi_{T-k} + \varepsilon \sum_{i=2}^{k} \zeta_{T-i} \\
+ \varepsilon^2 \left( \frac{1}{2} \sum_{i=2}^{k} \eta_{T-i} \alpha_{T-i} + \frac{1}{8} \sum_{i=3}^{k} \zeta_{T-i} \beta_{T-i} + \sum_{i=4}^{k} \sum_{j=4}^{k} \zeta_{T-i} \zeta_{T-j} \right) \right\} \\
+ O(\varepsilon^3) \quad (87)
\]
and
\[
\delta^{(N)}_{T-k} = \tilde{V}_{T-k} \varepsilon^2 \sum_{i=2}^{k} \theta_{T-i} + O(\varepsilon^3). \quad (88)
\]

Recall that \(\alpha_{T-k}, \beta_{T-k}, \zeta_{T-k} \) and \(\eta_{T-k}\) are previously defined in Eqs. (64) to (67). The definitions of \(\phi_{T-k}, \psi_{T-k} \) and \(\theta_{T-k}\) in the expressions above are as follows:
\[
\phi_{T-k} = \frac{\gamma (\gamma - 1) \mathbb{E}_{T-k} \left[ (s_{T-k} - r_{T-k})^2 \tilde{F}_{T-k}^{\alpha_{T-k}} \right]}{\mathbb{E}_{T-k} \left[ \tilde{F}_{T-k}^{\gamma_{T-k}} \right]}, \quad (90)
\]
\[
\psi_{T-k} = \frac{\gamma}{\mathbb{E}_{T-k} \left[ \tilde{F}_{T-k}^{\gamma_{T-k}} \right]} \\
\times \left\{ \tilde{\lambda}_{T-k+1} \int_{0}^{s_{T-k}} \tilde{F}_{T-k}^{\gamma_{T-k} - 2} \left[ s_{T-k} r_{T-k} + \gamma (s_{T-k} - r_{T-k}) \left( s_{T-k} \tilde{A}_{T-k} - \tilde{F}_{T-k} \tilde{A}_{T-k+1} \right) \right] \\
\times p(s_{T-k}) \, ds_{T-k} \\
- \tilde{\mu}_{T-k+1} \int_{s_{T-k}}^{\infty} \tilde{F}_{T-k}^{\gamma_{T-k} - 2} \left[ s_{T-k} r_{T-k} + \gamma (s_{T-k} - r_{T-k}) \left( s_{T-k} \tilde{A}_{T-k} - \tilde{F}_{T-k} \tilde{A}_{T-k+1} \right) \right] \\
\times p(s_{T-k}) \, ds_{T-k} \right\}. \quad (91)
\]
and

\[
\theta_{T-k} = \frac{1}{\mathbb{E}_{T-k} \left[ \hat{F}_{T-k}^{\gamma} \right] } \times \left\{ \int_{s_{T-k}}^{\bar{s}_{T-k}} \left[ -\hat{F}_{T-k}^{\gamma-1} \bar{\lambda}_{T-k+1} \omega_{T-k+1}^\gamma \bar{A}_{T-k+1} \left( s_{T-k} \bar{A}_{T-k} - \bar{F}_{T-k} \bar{A}_{T-k+1} \right) \phi_{T-k+1} \\
+ \hat{F}_{T-k}^{\gamma} \left( \frac{1}{2} \bar{\omega}_{T-k+1}^2 \phi_{T-k+1} - \bar{\lambda}_{T-k+1} \omega_{T-k+1}^\gamma + \omega_{T-k+1}^\gamma \psi_{T-k+1} \mathbb{I}_{\{k\neq 2\}} \right) \right] \\
\times p(s_{T-k}) \, ds_{T-k} \\
+ \int_{s_{T-k}}^{\infty} \left[ \hat{F}_{T-k}^{\gamma-1} \bar{\mu}_{T-k+1} \omega_{T-k+1}^\gamma \bar{A}_{T-k+1} \left( s_{T-k} \bar{A}_{T-k} - \bar{F}_{T-k} \bar{A}_{T-k+1} \right) \phi_{T-k+1} \\
+ \hat{F}_{T-k}^{\gamma} \left( \frac{1}{2} \bar{\omega}_{T-k+1}^2 \phi_{T-k+1} + \bar{\mu}_{T-k+1} \omega_{T-k+1}^\gamma + \omega_{T-k+1}^\gamma \psi_{T-k+1} \mathbb{I}_{\{k\neq 2\}} \right) \right] \\
\times p(s_{T-k}) \, ds_{T-k} \right\}, \tag{92}
\]

where \( \mathbb{I}_{\{k\neq 2\}} \) denotes an indicator function with respect to the index \( k \).

In conclusion, we have devised a systematic method to approximate the optimal value function in the buy, sell and no-transaction regions at any time step. In this method, we initially assumed that the investor adopted a suboptimal strategy of transacting to the Merton proportion in the limit of small transaction costs. The second order approximation of the suboptimal value function in the buy and sell regions was then derived by perturbing about the no transaction costs solution. However, by ignoring the no-transaction region, we have missed out on some second order terms in our preliminary approximation. In order to correct our initial approximation in the buy and sell regions, we perturbed the optimal value function about the suboptimal value function, followed by a perturbation about the no transaction costs solution. Thereafter, we derived the approximation for the optimal value function in the no-transaction region by perturbing about the no transaction costs solution. This perturbation scheme was achieved by backwards recursion starting from time \( T - 1 \). Nonetheless, the optimal buy and sell boundaries are as yet unknown, which we determine in the next section.

### 5.4 Approximation of the Optimal Buy and Sell Boundaries

In Sections 5.1 and 5.3, we have derived the approximation of the optimal value function in the buy, sell and no-transaction regions at each time step. In this section, we verify that the optimal value function is continuous across the buy and sell boundaries. We further recall that the first derivative of the optimal value function should also be continuous across the optimal boundaries. An application of this condition allows us to derive estimates for the optimal buy and sell boundaries. In addition, the results that we obtain serve to demonstrate that the perturbation analysis is indeed self-consistent.

We start by considering the case at time \( T - 1 \). Collecting the previous results from Eqs. (55), (57), (79), (82) and (84), the optimal value function in the buy, sell and no-transaction regions at time \( T - 1 \) is given by \( V_{T-1}^{(B)} = \hat{V}_{T-1}^{(B)} + \delta_{T-1}^{(B)} \), \( V_{T-1}^{(S)} = \hat{V}_{T-1}^{(S)} + \delta_{T-1}^{(S)} \) and \( V_{T-1}^{(N)} \) respectively. Recall that \( A_{T-1} = \bar{A}_{T-1} + \varepsilon \omega_{T-1} \) in the no-transaction region. At the optimal
buy boundary, \( A_{T-1} = \bar{A}_{T-1} \) and \( \omega_{T-1} = \omega_{T-1}^- \). Thus, we can verify that

\[
V^{(B)}_{T-1} = \tilde{V}_{T-1} \left\{ 1 + \frac{1}{2} \varepsilon^2 \omega_{T-1}^- \phi_{T-1} \right\} + O(\varepsilon^3) = V^{(N)}_{T-1}.
\]

(93)

Similarly, at the optimal sell boundary, where \( A_{T-1} = A_{T-1}^+ \) and \( \omega_{T-1} = \omega_{T-1}^+ \), we can also verify that \( V^{(S)}_{T-1} = V^{(N)}_{T-1} \). Therefore, the optimal value function is continuous across the buy and sell boundaries up to \( O(\varepsilon^2) \).

Note that we have obtained the approximation of the optimal value function only up to \( O(\varepsilon^2) \). This is so that we can match the first derivative of the optimal value function at the optimal buy and sell boundaries up to \( O(\varepsilon) \). If we wish to obtain higher order estimates of the optimal boundaries, we will need to estimate the optimal value function beyond \( O(\varepsilon^2) \).

By matching the first derivative of the optimal value function at the buy and sell boundaries, we are able to derive the first order approximation of these boundaries. Therefore, in order to determine the optimal boundaries, differentiate \( V^{(B)}_{T-1} \), \( V^{(S)}_{T-1} \) and \( V^{(N)}_{T-1} \) with respect to \( A_{T-1} \) to give

\[
\frac{\partial V^{(B)}_{T-1}}{\partial A_{T-1}} = \tilde{V}_{T-1} \left\{ \varepsilon \lambda_{T-1}^+ \gamma - \varepsilon^2 \tilde{\lambda}_{T-1} \alpha_{T-1} \left( \bar{A}_{T-1} - A_{T-1} \right) \right\} + O(\varepsilon^3),
\]

(94)

\[
\frac{\partial V^{(S)}_{T-1}}{\partial A_{T-1}} = \tilde{V}_{T-1} \left\{ -\varepsilon \mu_{T-1} \gamma + \varepsilon^2 \tilde{\mu}_{T-1} \alpha_{T-1} \left( A_{T-1} - \bar{A}_{T-1} \right) \right\} + O(\varepsilon^3)
\]

(95)

and

\[
\frac{\partial V^{(N)}_{T-1}}{\partial A_{T-1}} = \tilde{V}_{T-1} \varepsilon \omega_{T-1} \phi_{T-1} + O(\varepsilon^2).
\]

(96)

Note that the first derivative of the optimal value function in the no-transaction region is obtained by recalling that \( A_{T-1} \) and \( \omega_{T-1} \) are related via \( A_{T-1} = A_{T-1} + \varepsilon \omega_{T-1} \).

At the buy boundary (i.e. \( A_{T-1} = \bar{A}_{T-1} \) and \( \omega_{T-1} = \omega_{T-1}^- \)), by equating the coefficients of \( \varepsilon \) for \( \frac{\partial V^{(B)}_{T-1}}{\partial A_{T-1}} \) and \( \frac{\partial V^{(N)}_{T-1}}{\partial A_{T-1}} \), we obtain

\[
\omega_{T-1}^- = \frac{\lambda_{T-1}^+ \gamma}{\phi_{T-1}}.
\]

(97)

Similarly, at the sell boundary (i.e. \( A_{T-1} = A_{T-1}^+ \) and \( \omega_{T-1} = \omega_{T-1}^+ \)), by equating the coefficients of \( \varepsilon \) for \( \frac{\partial V^{(S)}_{T-1}}{\partial A_{T-1}} \) and \( \frac{\partial V^{(N)}_{T-1}}{\partial A_{T-1}} \), we obtain

\[
\omega_{T-1}^+ = \frac{-\mu_{T-1} \gamma}{\phi_{T-1}}.
\]

(98)

Substituting these estimates back into the optimal value function, we have completed the derivation of its approximation up to \( O(\varepsilon^2) \). In addition, the corresponding buy and sell boundaries are estimated by \( A_{T-1}^- = \bar{A}_{T-1} + \varepsilon \omega_{T-1}^- \) and \( A_{T-1}^+ = \bar{A}_{T-1} + \varepsilon \omega_{T-1}^+ \) up to \( O(\varepsilon) \).

For the general case of time \( T - k \) \((k = 2, \ldots, T)\), collecting the results from Eqs. (62), (63), (85),(86),(88) and (89),the optimal value function in the three regions is given by \( V^{(B)}_{T-k} = \)

...
\[ \hat{V}^{(B)}_{T-k} + \delta^{(B)}_{T-k}, \ \hat{V}^{(S)}_{T-k} = \tilde{V}^{(S)}_{T-k} + \delta^{(S)}_{T-k} \] and \[ \hat{V}^{(N)}_{T-k} = \tilde{V}^{(N)}_{T-k} + \delta^{(N)}_{T-k}. \]

At the optimal buy boundary, \( A_{T-k} = A^+_{T-k} \) and \( \omega_{T-k} = \omega^-_{T-k} \). Thus, it can be verified that

\[
\begin{align*}
\hat{V}^{(B)}_{T-k} &= \hat{V}_{T-k} \left\{ 1 + \varepsilon \sum_{i=2}^{k} \zeta_{T-i} + \varepsilon^2 \sum_{i=2}^{k} \theta_{T-i} \\
+ &\varepsilon^2 \left( \frac{1}{2} \sum_{i=2}^{k} \eta_{T-i} \alpha_{T-i+1} - \frac{1}{\gamma} \sum_{i=3}^{k} \zeta_{T-i} \beta_{T-i+1} + \sum_{i=1}^{k} \sum_{j=4}^{k} \zeta_{T-i} \zeta_{T-j+2} \right) \\
+ &\frac{1}{2} \sum_{i=2}^{k} \omega^-_{T-k} \phi_{T-k} + \varepsilon^2 \omega^-_{T-k} \psi_{T-k} \right\} + O(\varepsilon^3) = V^{(N)}_{T-k}. \tag{99}
\end{align*}
\]

At the optimal sell boundary, \( A_{T-k} = A^+_{T-k} \) and \( \omega_{T-k} = \omega^+_{T-k} \). Similarly, it can also be verified that \( \hat{V}^{(S)}_{T-k} = V^{(N)}_{T-k} \) up to \( O(\varepsilon^2) \).

In order to estimate the buy and sell boundaries, we differentiate \( V^{(B)}_{T-k}, V^{(S)}_{T-k} \) and \( V^{(N)}_{T-k} \) with respect to \( A_{T-k} \), which give us

\[
\begin{align*}
\frac{\partial V^{(B)}_{T-k}}{\partial A_{T-k}} &= \hat{V}_{T-k} \left\{ \varepsilon \tilde{\lambda}_{T-k} \gamma - \varepsilon^2 \tilde{\lambda}^2_{T-k} \alpha_{T-k} \left( \tilde{A}_{T-k} - A_{T-k} \right) \\
- &\varepsilon^2 \tilde{\lambda}_{T-k} \left( \beta_{T-k} - \gamma \sum_{i=3}^{k} \zeta_{T-i+1} \right) + \varepsilon^2 \tilde{\lambda}_{T-k} \omega^-_{T-k} \tilde{A}_{T-k} \phi_{T-k} \right\} + O(\varepsilon^3), \tag{100}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial V^{(S)}_{T-k}}{\partial A_{T-k}} &= \hat{V}_{T-k} \left\{ -\varepsilon \tilde{\mu}_{T-k} \gamma + \varepsilon^2 \tilde{\mu}^2_{T-k} \alpha_{T-k} \left( A_{T-k} - \tilde{A}_{T-k} \right) \\
+ &\varepsilon^2 \tilde{\mu}_{T-k} \left( \beta_{T-k} - \gamma \sum_{i=3}^{k} \zeta_{T-i+1} \right) + \varepsilon^2 \tilde{\mu}_{T-k} \omega^+_{T-k} \tilde{A}_{T-k} \phi_{T-k} \right\} + O(\varepsilon^3), \tag{101}
\end{align*}
\]

and

\[
\frac{\partial V^{(N)}_{T-k}}{\partial A_{T-k}} = \hat{V}_{T-k} \left\{ \varepsilon \psi_{T-k} + \varepsilon \omega_{T-k} \phi_{T-k} \right\} + O(\varepsilon^2). \tag{102}
\]

At the buy boundary (i.e. \( A_{T-k} = A^+_{T-k} \) and \( \omega_{T-k} = \omega^-_{T-k} \)), by equating the coefficients of \( \varepsilon \) for \( \frac{\partial V^{(B)}_{T-k}}{\partial A_{T-k}} \) and \( \frac{\partial V^{(N)}_{T-k}}{\partial A_{T-k}} \), we obtain

\[
\omega^-_{T-k} = \frac{\tilde{\lambda}_{T-k} \gamma - \psi_{T-k}}{\phi_{T-k}}. \tag{103}
\]

Similarly, at the sell boundary (i.e. \( A_{T-k} = A^-_{T-k} \) and \( \omega_{T-k} = \omega^+_{T-k} \)), by equating the coefficients of \( \varepsilon \) for \( \frac{\partial V^{(S)}_{T-k}}{\partial A_{T-k}} \) and \( \frac{\partial V^{(N)}_{T-k}}{\partial A_{T-k}} \), we obtain

\[
\omega^+_{T-k} = \frac{-\tilde{\mu}_{T-k} \gamma - \psi_{T-k}}{\phi_{T-k}}. \tag{104}
\]

The corresponding optimal buy and sell boundaries are thus given by \( A^-_{T-k} = \tilde{A}_{T-k} + \varepsilon \omega^-_{T-k} \) and \( A^+_{T-k} = \tilde{A}_{T-k} + \varepsilon \omega^+_{T-k} \) up to \( O(\varepsilon) \). Therefore, in the limit of small transaction costs, we have obtained the first-order approximation of the optimal boundaries at any time \( T-k \).
Observe from Eq. (90) that \( \phi_{T-k} \) is determined by the variables \( r_{T-k}, s_{T-k} \) and \( \dot{A}_{T-k} \) at time \( T-k \). The variable \( r_{T-k} \) is deterministic while the random variable \( s_{T-k} \) is characterized by its probability density function \( p(s_{T-k}) \). Observe from Eq. (91) that, in addition to the variables at time \( T-k \), \( \psi_{T-k} \) also depends on the variables \( \dot{\lambda}_{T-k+1}, \dot{\mu}_{T-k+1} \) and \( \dot{A}_{T-k+1} \) at the time step ahead. Moreover, the Merton proportion \( \dot{A}_{T-k} \) at any time \( T-k \) is determined by the specification of \( r_{T-k} \) and \( p(s_{T-k}) \) via Eq. (37). Therefore, one concludes that the first order approximation of the optimal buy and sell boundaries at each time step essentially depend on the transaction costs, returns of the risk-free asset and returns of the risky asset at the current time step and one time step ahead.

In Section 6, we make a few assumptions that allow us to reduce these expressions for the optimal boundaries to simpler forms.

### 6. Results

In order to illustrate our main results with a numerical example, we make the following assumptions:

- Assume that \( r_{T-k} \) is constant in time and say that \( r_{T-k} = r \) for all \( k \).
- Assume that \( s_{T-k} \) are independent and identically distributed to the random variable \( s \) for all \( k \).
- Assume that the costs of buying and selling the risky assets are equal and constant in time and say that \( \dot{\lambda}_{T-k} = \dot{\mu}_{T-k} = \dot{\lambda} \) for all \( k \).

We focus on the results in the general case of time \( T-k \) (\( k = 2, \ldots, T \)) as the time \( T-1 \) case is trivial. With these assumptions, we deduce from Eq. (37) that the Merton proportion \( \dot{A}_{T-k} = \dot{A} \) is a constant, where \( \dot{A} \) satisfies the equation

\[
E \left[ (s-r) \left\{ r + (s-r) \dot{A} \right\}^{\gamma-1} \right] = 0. \tag{105}
\]

Here, the expectation \( E \) is taken with respect to the random variable \( s \). From Eq. (48), we simplify the term \( \dot{s}_{T-k} = r \dot{A} \left( 1 - \dot{A} \right) / \dot{A} \left( 1 - \dot{A} \right) = r \). Furthermore, from Eq. (91), we simplify the expression \( s_{T-k} \dot{A}_{T-k} - \dot{F}_{T-k} \dot{A}_{T-k+1} = s \dot{A} - \left\{ r + (s-r) \dot{A} \right\} \dot{A} = (s-r) \dot{A} \left( 1 - \dot{A} \right) \).

We observe that \( \phi_{T-k} = \phi \) and \( \psi_{T-k} = \psi \) are constants, where

\[
\phi = \frac{\gamma (\gamma - 1) E \left[ (s-r)^2 \left\{ r + (s-r) \dot{A} \right\}^{\gamma-2} \right]}{E \left[ \left\{ r + (s-r) \dot{A} \right\}^{\gamma} \right]} \tag{106}
\]

and

\[
\psi = \frac{\gamma \dot{\lambda}}{E \left[ \left\{ r + (s-r) \dot{A} \right\}^{\gamma} \right]} \times \left\{ \int_0^{\infty} \left\{ r + (s-r) \dot{A} \right\}^{\gamma-2} \left\{ s r + \gamma (s-r)^2 \dot{A} \left( 1 - \dot{A} \right) \right\} p(s) \, ds \right. \\
- \int_r^{\infty} \left\{ r + (s-r) \dot{A} \right\}^{\gamma-2} \left\{ s r + \gamma (s-r)^2 \dot{A} \left( 1 - \dot{A} \right) \right\} p(s) \, ds \right\}. \tag{107}
\]
Consequently, the first order approximations of the optimal boundaries

\[ A_{T-k}^+ = \bar{A} - \varepsilon \frac{\lambda \gamma + \psi}{\phi} \tag{109} \]

and

\[ A_{T-k}^- = \bar{A} + \varepsilon \frac{\lambda \gamma - \psi}{\phi} \tag{108} \]

are also constants over time. This is not a surprising observation as the Merton proportion, which corresponds to the no transaction costs case, is also a constant. Thus, in the limit of small transaction costs, one might expect the optimal buy and sell boundaries (being near the Merton point) to be constants as well. However, note that these results only hold under the given assumptions. One advantage of this portfolio selection model is that the probability distribution for the return of the risky asset at each time step is generic. In general, if the returns of the risky asset are not assumed to be identically distributed, then one will not expect the optimal boundaries to be constants over time.

In order to study the behavior of the optimal boundaries numerically, we consider a simple example and assume that the parameters in our model take the values

\[ T = 6, \gamma = 0.1, r = 1.05, \text{ and the random variable } s \text{ is given by its probability density function } p(s) = 0.7 \times \delta(s - 1.5) + 0.3 \times \delta(s - 0.5), \]

where \( \delta(.) \) is the Dirac delta function.

In Fig. 1, the transaction costs are allowed to vary from 0 to 0.03. This figure shows the relationship between the optimal boundaries at the initial time and transaction costs. The exact boundaries are obtained by implementing the dynamic programming algorithm numerically while the approximate boundaries are given by Eqs. (108) and (109). When transaction costs are 0, the boundaries converge to the Merton proportion and the no-transaction region disappears. Therefore, in the absence of transaction costs, the investor will always trade in the risky asset to reach the Merton proportion. When transaction costs increase, the width (difference between the buy and sell boundaries) of the no-transaction region increases and the investor is less likely to trade in the risky asset. In practice, the level of transaction costs incurred in the financial markets are typically between 0.003 and 0.004. From this example, it can be observed that the approximate boundaries are good estimates of the exact boundaries in the limiting case of small transaction costs.

In Fig. 2, we assume that the transaction costs are fixed at 0.004 and we vary \( \gamma \) from 0.1 to 0.3. The level of risk aversion of the investor, which is measured by \( (1 - \gamma) \), therefore varies from 0.7 to 0.9. In this figure, we investigate how the investor’s risk aversion affects the Merton proportion and the optimal boundaries at the initial time as approximated by Eqs. (108) and (109). It is observed that the Merton proportion and the optimal boundaries are inversely related to the level of risk aversion of the investor. As the investor becomes increasingly risk averse, he would prefer to hold less of the risky asset and more of the risk-free asset, which explains the decrease in the Merton proportion. Although the width of the no-transaction region appears to remain the same, the effects of increasing the investor’s risk aversion are seen by the sell region that is widening and the buy region that is narrowing. This means that there is a greater tendency for the investor to sell rather than to buy the risky asset when his level of risk aversion is high.

In Fig. 3, we assume that transaction costs are 0.004 and \( \gamma = 0.1 \). This figure shows the linear relationship between the optimal holdings in the risky asset and the wealth of the investor. Recall the parametrization \( a_{T-k} = A_{T-k} W_{T-k} \), where \( a_{T-k} \) is the dollar value invested in the risky asset, \( A_{T-k} \) is the proportion of wealth invested in the risky asset and \( W_{T-k} \) is the wealth of the investor. The Merton line \( a_0 = A_0 W_0 \) corresponds to the optimal value to be invested in the risky asset when there are no transaction costs. The buy boundary \( a_0 = A_0 W_0 \) and the sell boundary \( a_0 = A_0 W_0 \) delineate the no-transaction region, which in this case is
narrow since the transaction costs are small. As the investor’s wealth increases, observe that the optimal holdings in the risky asset (in dollar value terms) also increase as depicted by the increasing buy and sell boundaries, which is what one would expect in practice. Therefore, this observation represents a more realistic description of an investor’s behavior as compared to the exponential utility function, where the Merton line, buy and sell boundaries are found to be independent of the investor’s wealth.

In conclusion, we have devised a perturbation method that allows one to obtain approximations of the optimal value function and optimal boundaries to an arbitrary number of time steps in the portfolio selection model. Therefore, when transaction costs are small, one does not need to compute the optimal value function and optimal boundaries via the dynamic programming algorithm, which becomes computationally intensive as the number of time steps
increases. Furthermore, one has the flexibility to specify the probability distribution of the returns of the risky asset as we have kept it generic in the model.

References


Appendix A. Remainder Term of $\hat{V}^{(B)}_{T-1}$

In this appendix, we derive a bound on the remainder term of $\hat{V}^{(B)}_{T-1}$. Applying Taylor’s Theorem to Eq. (54),

$$
\hat{V}^{(B)}_{T-1} = \frac{1}{\gamma} E_{T-1} \left[ \hat{F}_{T-1} - \gamma \hat{F}_{T-1}^{\gamma-1} \epsilon \hat{\lambda}_{T-1} r_{T-1} \left( \hat{A}_{T-1} - A_{T-1} \right) 
\right. 
+ \frac{1}{2} \gamma (\gamma - 1) \hat{F}_{T-1}^{\gamma-2} \epsilon^2 \hat{\lambda}_{T-1}^2 r_{T-1}^2 \left( \hat{A}_{T-1} - A_{T-1} \right)^2 
\left. - \frac{1}{6} \gamma (\gamma - 1) (\gamma - 2) \epsilon^3 \hat{\lambda}_{T-1}^3 r_{T-1}^3 \left( \hat{A}_{T-1} - A_{T-1} \right)^3 
\times \left\{ \hat{F}_{T-1} - \xi \hat{\lambda}_{T-1} r_{T-1} \left( \hat{A}_{T-1} - A_{T-1} \right) \right\}^{\gamma-3} \right], 
$$

(A1)

where $0 < \xi < \epsilon$. Therefore, the absolute value of the remainder is

$$
\left| \hat{F}_{T-1} \right| = \frac{1}{6} (\gamma - 1) (\gamma - 2) \epsilon^3 \hat{\lambda}_{T-1}^3 r_{T-1}^3 \left( \hat{A}_{T-1} - A_{T-1} \right)^3 
\times E_{T-1} \left[ \left\{ \hat{F}_{T-1} - \xi \hat{\lambda}_{T-1} r_{T-1} \left( \hat{A}_{T-1} - A_{T-1} \right) \right\}^{\gamma-3} \right]. 
$$

(A2)

Now, since $\hat{F}_{T-1} = r_{T-1} + (s_{T-1} - r_{T-1}) \hat{A}_{T-1}$, we have

$$
E_{T-1} \left[ \left\{ \hat{F}_{T-1} - \xi \hat{\lambda}_{T-1} r_{T-1} \left( \hat{A}_{T-1} - A_{T-1} \right) \right\}^{\gamma-3} \right] = \int_0^\infty \{ r_{T-1} + (s_{T-1} - r_{T-1}) \hat{A}_{T-1} - \xi \hat{\lambda}_{T-1} r_{T-1} \left( \hat{A}_{T-1} - A_{T-1} \right) \}^{\gamma-3} 
\times p(s_{T-1}) \, ds_{T-1}. 
$$

(A3)
For $\gamma < 1$, note that \( \left\{ r_{T-1} + (s_{T-1} - r_{T-1}) \tilde{A}_{T-1} - \xi \tilde{\lambda}_{T-1} r_{T-1} \left( \tilde{A}_{T-1} - A_{T-1} \right) \right\} \gamma^{-3} \) is a decreasing function of $s_{T-1}$ and has a maximum at $s_{T-1} = 0$. Hence,

\[
\mathbb{E}_{T-1} \left[ \left\{ \tilde{F}_{T-1} - \xi \tilde{\lambda}_{T-1} r_{T-1} \left( \tilde{A}_{T-1} - A_{T-1} \right) \right\} \gamma^{-3} \right] < \int_0^\infty \left\{ r_{T-1} \left( 1 - \tilde{A}_{T-1} \right) - \xi \tilde{\lambda}_{T-1} r_{T-1} \left( \tilde{A}_{T-1} - A_{T-1} \right) \right\} \gamma^{-3} p(s_{T-1}) \, ds_{T-1} = r_{T-1} \gamma^{-3} \left\{ \left( 1 - \tilde{A}_{T-1} \right) - \xi \tilde{\lambda}_{T-1} \left( \tilde{A}_{T-1} - A_{T-1} \right) \right\} \gamma^{-3}. \tag{A4}
\]

Since $0 < \xi < \varepsilon$, we obtain a bound for

\[
\left| \tilde{F}_{T-1}^{(B)} \right| < \frac{1}{6} (\gamma - 1) (\gamma - 2) \varepsilon^3 \tilde{\lambda}_{T-1}^2 r_{T-1} \left( \tilde{A}_{T-1} - A_{T-1} \right)^3 \times \left\{ \left( 1 - \tilde{A}_{T-1} \right) - \varepsilon \tilde{\lambda}_{T-1} \left( \tilde{A}_{T-1} - A_{T-1} \right) \right\} \gamma^{-3}. \tag{A5}
\]

Note that even though $\gamma - 3 < -2$, this bound is finite since we have assumed that $0 < (1 - \tilde{A}_{T-1}) - \varepsilon \tilde{\lambda}_{T-1} \left( \tilde{A}_{T-1} - A_{T-1} \right) < 1$.

### Appendix B. Derivation of $\delta_{T-2}^{(B)}$, $\delta_{T-2}^{(S)}$ and $V_{T-2}^{(N)}$

In this appendix, we apply perturbation analysis to derive the correction in the buy and sell regions and the optimal value function in the no-transaction region.

In the buy region at time $T - 2$, recall that $V_{T-2}^{(B)}$ is given by Eq. (30). We also make the observation that $s_{T-2} = \frac{r_{T-2} A_{T-1} \left\{ \left( 1 - A_{T-2} \right) - \lambda_{T-2} \left( A_{T-2} - A_{T-2} \right) \right\} }{A_{T-2} \left( 1 - A_{T-1} \right)} = \tilde{s}_{T-2} + O(\varepsilon)$ and that $s_{T-2}^+ = \frac{r_{T-2}^+ A_{T-1} \left\{ \left( 1 - A_{T-2} \right) - \lambda_{T-2} \left( A_{T-2} - A_{T-2} \right) \right\} }{A_{T-2} \left( 1 - A_{T-1} \right)} = \tilde{s}_{T-2}^+ + O(\varepsilon)$. Therefore, $\tilde{s}_{T-2}$ is the leading order term of $s_{T-2}$ and $s_{T-2}$, which motivates us to rewrite $V_{T-2}^{(B)}$ in terms of integrals delineated by $\tilde{s}_{T-2}$. Thus, we have

\[
V_{T-2}^{(B)} = \int_{\tilde{s}_{T-2}}^{s_{T-2}} F_{T-2}^{(B)} \gamma V_{T-1}^{(B)} p(s_{T-2}) \, ds_{T-2} + \int_{\tilde{s}_{T-2}}^{s_{T-2}^+} F_{T-2}^{(B)} \gamma V_{T-1}^{(S)} p(s_{T-2}) \, ds_{T-2} + \int_{\tilde{s}_{T-2}}^{s_{T-2}} F_{T-2}^{(B)} \gamma \left( V_{T-1}^{(N)} - V_{T-1}^{(B)} \right) p(s_{T-2}) \, ds_{T-2} + \int_{\tilde{s}_{T-2}}^{s_{T-2}^+} F_{T-2}^{(B)} \gamma \left( V_{T-1}^{(N)} - V_{T-1}^{(S)} \right) p(s_{T-2}) \, ds_{T-2}. \tag{B1}
\]

Applying the Mean Value Theorem to the third integral,

\[
\int_{\tilde{s}_{T-2}}^{s_{T-2}^+} F_{T-2}^{(B)} \gamma \left( V_{T-1}^{(N)} - V_{T-1}^{(B)} \right) p(s_{T-2}) \, ds_{T-2} = (\tilde{s}_{T-2} - s_{T-2}^+) F_{T-2}^{(B)} \gamma \left( V_{T-1}^{(N)} - V_{T-1}^{(B)} \right) p(s_{T-2}) \tag{B2}
\]
evaluated at a point \( s_{T-2} \in (\tilde{s}_{T-2}, \tilde{s}_{T-2}), \) i.e. \( s_{T-2} = \tilde{s}_{T-2} + O(\varepsilon) \). Recall that when \( s_{T-2} = \tilde{s}_{T-2} \), we have \( A_{T-1} = \tilde{A}_{T-1} \) by definition. Therefore, when \( s_{T-2} = \tilde{s}_{T-2} + O(\varepsilon) \), we have \( A_{T-1} = \tilde{A}_{T-1} + O(\varepsilon) = \tilde{A}_{T-1} + O(\varepsilon) \). This implies that the term \( V_{T-1}^{(N)} - V_{T-1}^{(B)} = \varepsilon \lambda_{T-1} (\tilde{A}_{T-1} - A_{T-1}) [\tilde{F}_{T-1}^{(B)}] + O(\varepsilon^2) = O(\varepsilon^2) \). Since \( \tilde{s}_{T-2} - \tilde{s}_{T-2} = O(\varepsilon) \), we can conclude that Eq. (B2) is of \( O(\varepsilon^3) \). Similarly, by applying the Mean Value Theorem to the fourth integral of Eq. (B1), we can also show that it is of \( O(\varepsilon^3) \). In conclusion, Eq. (B1) can be simplified to

\[
V_{T-2}^{(B)} = \int_0^{s_{T-2}} F_{T-2}^{(B)} V_{T-1}^{(B)} p(s_{T-2}) \, ds_{T-2}
+ \int_{s_{T-2}}^{\infty} F_{T-2}^{(B)} \gamma V_{T-1}^{(B)} p(s_{T-2}) \, ds_{T-2} + O(\varepsilon^3). \tag{B3}
\]

Expressing \( V_{T-2}^{(B)} \) in this form allows us to perturb it about the suboptimal value function \( \hat{V}_{T-2}^{(B)} \), which is also in terms of integrals delineated by \( \tilde{s}_{T-2} \). Using Eq. (71) and the approximation of the optimal value function from time \( T - 1 \), express \( V_{T-2}^{(B)} \) as

\[
V_{T-2}^{(B)} = \int_0^{s_{T-2}} \left\{ \hat{V}_{T-2}^{(B)} + \varepsilon \omega_{T-2} (s_{T-2} - r_{T-2}) - \varepsilon^2 \lambda_{T-2} \omega_{T-2} r_{T-2} \right\} \gamma p(s_{T-2}) \, ds_{T-2}
+ \int_{s_{T-2}}^{\infty} \left\{ \hat{V}_{T-2}^{(B)} + \varepsilon \omega_{T-2} (s_{T-2} - r_{T-2}) - \varepsilon^2 \lambda_{T-2} \omega_{T-2} r_{T-2} \right\} \gamma p(s_{T-2}) \, ds_{T-2} + O(\varepsilon^3). \tag{B4}
\]

Here, we note that \( \hat{V}_{T-1}^{(B)}, \hat{V}_{T-1}^{(S)}, \delta_{T-1}^{(B)} \) and \( \delta_{T-1}^{(S)} \) are functions of \( A_{T-1} = \frac{ST-2\tilde{A}_{T-2}}{F_{T-2}^{(B)}} \) since the investor buys to reach the optimal buy boundary \( \tilde{A}_{T-2} \).

In addition, recall that the suboptimal value function given by Eq. (61), is

\[
\hat{V}_{T-2}^{(B)} = \int_0^{s_{T-2}} F_{T-2}^{(B)} \gamma \hat{V}_{T-1}^{(B)} p(s_{T-2}) \, ds_{T-2} + \int_{s_{T-2}}^{\infty} F_{T-2}^{(B)} \gamma \hat{V}_{T-1}^{(S)} p(s_{T-2}) \, ds_{T-2} + O(\varepsilon^3). \tag{B5}
\]

In this case, \( \hat{V}_{T-2}^{(B)} \) and \( \hat{V}_{T-2}^{(S)} \) are functions of \( A_{T-1} = \frac{ST-2\tilde{A}_{T-2}}{F_{T-2}^{(B)}} \) since the investor buys to reach the Merton proportion \( \tilde{A}_{T-2} \). Adopting a similar approach as the derivation of the correction \( \delta_{T-1}^{(B)} \) at time \( T - 1 \), we first perturb the optimal value function about the suboptimal solution, followed by a perturbation about the no transaction costs solution. Subtracting Eq. (B5) from Eq. (B4), expanding in powers of \( \varepsilon \) and simplifying with Eqs. (37), (40) and (41), it can be shown after much algebra that

\[
\delta_{T-2}^{(B)} = \hat{V}_{T-2}^{(B)} \varepsilon^2 \left\{ \lambda_{T-2} \omega_{T-2} \tilde{A}_{T-2} \left( \tilde{A}_{T-2} - A_{T-2} \right) + \frac{1}{2} \omega_{T-2}^2 + \left[ \lambda_{T-2} \omega_{T-2} \gamma + \omega_{T-2} \psi_{T-2} + \theta_{T-2} \right] \right\} + O(\varepsilon^3), \tag{B6}
\]

where \( \phi_{T-2}, \psi_{T-2} \) and \( \theta_{T-2} \) are defined in Eqs. (90), (91) and (92) respectively.

In the sell region, the correction term can be immediately deduced from that in the buy
region by a change of variables from $\tilde{\lambda}_{T-2}$ to $-\tilde{\mu}_{T-2}$ and from $\omega^+_{T-2}$ to $\omega^-_{T-2}$ to give us

$$
\delta^{(S)}_{T-2} = \tilde{V}_{T-2}^2 e^2 \left\{ \left[ \tilde{\mu}_{T-2} \omega^+_{T-2} \tilde{A}_{T-2} \left( A_{T-2} - \tilde{A}_{T-2} \right) + \frac{1}{2} \omega^+_{T-2} \right] \phi_{T-2} + \tilde{\mu}_{T-2} \omega^+_{T-2} \gamma + \omega^+_{T-2} \psi_{T-2} + \theta_{T-2} \right\} + O(\varepsilon^3), \tag{B7}
$$

In the no-transaction region, recall that $V_{T-2}^{(N)}$ is given by Eq. (34). Following the same line of argument as in the buy region, we can show that similar to Eq. (B3),

$$
V_{T-2}^{(N)} = \int_0^{s_{T-2}} F_{T-2}^{(N)\gamma} V_{T-1}^{(B)}(s_{T-2}) \, ds_{T-2}
+ \int_{s_{T-2}}^{\infty} F_{T-2}^{(N)\gamma} V_{T-1}^{(S)}(s_{T-2}) \, ds_{T-2} + O(\varepsilon^3). \tag{B8}
$$

Using Eq. (73) and the approximation of the optimal value function from time $T-1$,

$$
V_{T-2}^{(N)} = \int_0^{s_{T-2}} \left\{ \tilde{F}_{T-2} + \varepsilon \omega_{T-2} (s_{T-2} - r_{T-2}) \right\}^\gamma \left\{ \tilde{V}_{T-1}^{(B)} + \delta^{(B)}_{T-1} \right\} p(s_{T-2}) \, ds_{T-2}
+ \int_{s_{T-2}}^{\infty} \left\{ \tilde{F}_{T-2} + \varepsilon \omega_{T-2} (s_{T-2} - r_{T-2}) \right\}^\gamma \left\{ \tilde{V}_{T-1}^{(S)} + \delta^{(S)}_{T-1} \right\} p(s_{T-2}) \, ds_{T-2} + O(\varepsilon^3), \tag{B9}
$$

where $\tilde{V}_{T-1}^{(B)}$, $\tilde{V}_{T-1}^{(S)}$, $\delta^{(B)}_{T-1}$ and $\delta^{(S)}_{T-1}$ are functions of $A_{T-1} = \frac{s_{T-2} A_{T-2}}{F_{T-2}^{(N)}}$, since the investor does not transact in this region where $A_{T-2}^+ \leq A_{T-2} \leq A_{T-2}^-$. Let us denote

$$
V_{T-2}^{(N)} = \tilde{V}_{T-2} + \delta_{T-2}^{(N)}, \tag{B10}
$$

where

$$
\tilde{V}_{T-2}^{(N)} = \int_0^{s_{T-2}} \left\{ \tilde{F}_{T-2} + \varepsilon \omega_{T-2} (s_{T-2} - r_{T-2}) \right\}^\gamma \tilde{V}_{T-1}^{(B)}(s_{T-2}) \, ds_{T-2}
+ \int_{s_{T-2}}^{\infty} \left\{ \tilde{F}_{T-2} + \varepsilon \omega_{T-2} (s_{T-2} - r_{T-2}) \right\}^\gamma \tilde{V}_{T-1}^{(S)}(s_{T-2}) \, ds_{T-2} \tag{B11}
$$

and

$$
\delta^{(N)}_{T-2} = \int_0^{s_{T-2}} \left\{ \tilde{F}_{T-2} + \varepsilon \omega_{T-2} (s_{T-2} - r_{T-2}) \right\}^\gamma \delta^{(B)}_{T-1}(s_{T-2}) \, ds_{T-2}
+ \int_{s_{T-2}}^{\infty} \left\{ \tilde{F}_{T-2} + \varepsilon \omega_{T-2} (s_{T-2} - r_{T-2}) \right\}^\gamma \delta^{(S)}_{T-1}(s_{T-2}) \, ds_{T-2}. \tag{B12}
$$

Expanding in powers of $\varepsilon$ and simplifying with Eq. (37), it can be shown after some algebra that

$$
\tilde{V}_{T-2}^{(N)} = \tilde{V}_{T-2} \left\{ 1 + \varepsilon^2 \omega_{T-2} \psi_{T-2} + \frac{1}{2} \varepsilon^2 \omega^+_{T-2} \phi_{T-2} + \varepsilon \zeta_{T-2} + \frac{1}{2} \varepsilon^2 \theta_{T-2} \right\} + O(\varepsilon^3) \tag{B13}
$$
and

$$\delta_{T-2}^{(N)} = \bar{\delta}_{T-2} \epsilon^2 \theta_{T-2} + O(\epsilon^3).$$

(B14)

This concludes our derivation of \(\delta_{T-2}^{(B)}, \delta_{T-2}^{(S)}\) and \(V_{T-2}^{(N)}\).