A partial differential equation formulation of Vickrey's bottleneck model, part I: Methodology and theoretical analysis ☆

Ke Han[a], Terry L. Friesz[b,*], Tao Yao[b]

[a]Department of Mathematics, Pennsylvania State University, PA 16802, USA.
[b]Department of Industrial and Manufacturing Engineering, Pennsylvania State University, PA 16802, USA.

Abstract

This paper conducts rigorous mathematical analysis of the continuous-time Vickrey's model originally proposed in Vickrey (1969). The essential part of this model is an *ordinary differential equation* (ODE) with a right-hand side which is discontinuous in the state variable. Such formulation induces difficulties in both theoretical analysis and numerical computation. Moreover it is widely suspected that an explicit solution to this ODE does not exist. In this paper we advance the knowledge and understanding of the continuous-time Vickrey's model by taking a novel approach of formulating it as a partial differential equation (PDE) and applying the variational method to obtain an explicit solution representation. Such an explicit solution is then shown to be the strong solution to the ODE in full mathematical rigor. Our methodology also leads to the *generalized Vickrey's model* (GVM) which allows the flow to be a distribution. This is a desirable feature of traffic models in the context of analytical *dynamic traffic assignment* (DTA). The PDE formulation provides new insights and interpretation of the Vickrey's model, which leads to a number of extensions as well as connection with the first-order traffic flow models such as *Lighthill-Whitham-Richards* (LWR) model. The explicit solution representation also leads to a new computational method, which will be discussed in an accompanying paper.

Keywords:
continuous-time Vickrey's model, ordinary differential equation, partial differential equation, explicit solution, Lax-Hopf formula

1. Introduction

1.1. General background

Vickrey's Model (VM) is one of the most commonly used link models in the current dynamic traffic assignment literature. It was original proposed in Vickrey (1969) and has been extensively studied in the DTA context, for example in Drissi-Kaïouni and Hameda-Benchekroun (1992); Heydecker and Addison (1996); Kuwahara and Akamatsu (1997); Li et al. (2000). Vickrey's

☆This work was partially supported by NSF through grant EFRI-1024707, "A theory of complex transportation network design."

*Corresponding author
Email addresses: kxh323@psu.edu (Ke Han), tfriesz@psu.edu (Terry L. Friesz), tyy1@engr.psu.edu (Tao Yao)

model is based on the assumption that the queue has negligible size and the travel time on the link consists a free-flow travel time plus a congestion related queuing time. The key component of the model is an ordinary differential equation (ODE) with discontinuous dependence on the state variable. This leads to several theoretical difficulties since a classical solution no longer exists in this case, and it is widely doubted that such ODE does not admit an explicit solution representation. For a general review on ODEs with discontinuous state-dependence, the readers are referred to Filippov (1988); Stewart (1990).

Research based on the Vickrey's model mainly focuses on two aspects, (1) modifications of the original model in continuous-time and (2) discrete-time version of the model. The first direction is primarily represented by the work of Armbruster et al. (2006a); Ban et al. (2011); Pang et al. (2011). In Pang et al. (2011) a *linear complementarity system* (LCS) was used as an approximation to Vickrey's model in the study of continuous-time DUE problems on a single link. Such LCS does not have an explicit ODE representation. Yet it was shown in Ban et al. (2011) that the LCS yields an absolutely continuous solution that satisfies the complementarity condition almost everywhere. In the attempt to approximate the LCS with an explicit ODE, Ban et al. (2011) proposed the α-model which includes a smoothing parameter $\alpha \gg 1$. The α-model admits an explicit ODE representation. Moreover, it exhibits an asymptotic behavior as α approaches infinity and the limit is precisely the LCS studied in Pang et al. (2011). The point-queue dynamic has also been used in the modeling of continuous supply chain networks. For example, Armbruster et al. (2006a) considered an ODE very similar to the α-model: such ODE also contains a smoothing parametert $\varepsilon \ll 1$. Such a system, which we will call the ε-model, has been verified extensively against *deterministic discrete event simulations* (DDES) in Armbruster et al. (2006b). Notice that all of the above continuous-time models deviate from Vickrey's original model in exchange for the theoretical and computational convenience. In addition, the smoothing parameters introduced by the α-model of Ban et al. (2011) and the ε-model of Armbruster et al. (2006a) still need to be further understood in terms of the physical meanings and underlying modeling implications. A more detailed review on these models can be found in Section 3.

So far the Vickrey's or Vickrey-like models are mainly studied in discrete time and in a numerical framework. However, fundamental issues such as convergence and physical realism of the discrete solutions remain relatively less understood. It was shown in Ban et al. (2011) that either a forward or a backward finite-difference discretization of the Vickrey's ODE may yield negative queue lengths or negative flows. A few modifications to this numerical algorithm were proposed that would avoid such deficiency, e.g. the work of Armbruster et al. (2006a); Ban et al. (2011); Pang et al. (2011). However, these improved algorithms still bear some numerical limitations. For example, the α-model of Ban et al. (2011) could still lead to negative queues if a forward scheme is used. Furthermore, the forward scheme of the α-model exhibits only conditional stability. The ε-model proposed in Armbruster et al. (2006a) could potentially lead to non-physical solutions in some rare circumstances, see Han et al. (submitted for publication) for an example. In addition, the performances of both the α-model and the ε-model will be greatly influenced by the parameters chosen, which could potentially add complexity and uncertainty to the system.

In summary of the above literature review, a mathematically rigorous investigation of the original Vickrey's model in continuous time will not only provide new insights of the physics of the model, but also plow new ground for the analysis and computation of Vickrey's model in the context of analytical DTAs. This is precisely the motivation of our two-part work.

1.2. Methodology

Due to the mathematical irregularity of the ODE formulation, the continuous-time Vickrey's model is rarely studied in the current literature. It has not been noted before that one may analyze this model from the approach of partial differential equations. The PDEs are playing an essential role in the fluid-based traffic flow models, e.g. the Lighthill-Whitham-Richards model of Lighthill and Whitham (1955); Richards (1956). A list of selected references includes Bressan and Han (2011a,b); Claudel and Bayen (2010a,b); Daganzo (1994, 1995, 2005); Garavello and Piccoli (2006); Han et al. (submitted for publication); Newell (1993). The PDE formulation of Vickrey's model arises from the observation that this model, like many other fluid models, is based on conservation of cars. Such a principle manifests itself in the generic form:

$$\partial_t \rho(t,\, x) + \partial_x f(t,\, x) \;=\; 0 \tag{1.1}$$

where $\rho(t,\, x)$ and $f(t,\, x)$ denotes respectively the density and flow at time t and location x. The extra spatial dimension can be naturally embedded in the Vickrey's model due to the presence of a free-flow phase (although our analysis works even if the free-flow time is zero). The drop of capacity (bottleneck) at the exit of the link can be modeled by an x-dependent flow capacity constraint imposed on the entire link. This implies that the flow is not only a function of density (which is the case for LWR model), but also a function of the x variable. A more detailed and formal discussion based on the above observations can be found in Section 4.1.

Once we obtain a conservation law describing the dynamics of the Vickrey's model, we will immediately derive a Hamilton-Jacobi equation by integrating the scalar conservation law. The key component of our methodology is the variational method for Hamilton-Jacobi equations known as the Lax-Hopf formula, see Bressan and Han (2011a); Daganzo (2005); Evans (2010); Le Floch (1988); Lax (1957). The formula was originally proposed as a semi-analytic solution representation of the scalar conservation law and Hamilton-Jacobi equation. Its application to fluid based traffic flow models are recently investigated in Aubin et al. (2008); Bressan and Han (2011a,b); Claudel and Bayen (2010a,b); Daganzo (2005). The variational approach formulates the viscosity solution to the Hamilton-Jacobi equation as an optimization problem. However, we note that the classical Lax-Hopf formula does not immediately apply to our problem because the Hamiltonian of our H-J equation has a discontinuous dependence on the spatial variable x. Thus the H-J equation does not admit solutions in the viscosity sense, see Evans (2010) for more details. In other words, one can only expect the solution to be defined in the distributional sense, i.e. the densities $\rho(t,\, x)$ can contain dirac-delta which is precisely the mathematical abstraction of the "point queue". Fortunately, in this paper we are able to apply the Lax-Hopf formula in a novel way even the resulting solution is a distribution. Technical details of the analysis are presented in Section 4.2.

1.3. Findings and contributions

The PDE approach proposed in this manuscript provides a new level of understanding of the relation between the Vickrey's model and other PDE-based macroscopic traffic models. Moreover, the PDE theory and solution method enables us to derive the explicit solution to the Hamilton-Jacobi equation and eventually tackle the original Vickrey's ODE:

$$\frac{dq(t)}{dt} \;=\; u(t-t_0) - \begin{cases} \min\{u(t-t_0),\, M\} & \text{if } q(t) \;=\; 0 \\ M & \text{if } q(t) \;\neq\; 0 \end{cases} \tag{1.2}$$

where $q(\cdot)$ denotes queue length, $u(\cdot)$ denotes flow into the link which is given. M denotes the flow capacity and t_0 denotes the free flow time. As part of the results in this paper, an explicit solution to the above ODE does exist

$$q(t) = \int_{-\infty}^{t-t_0} u(s)\,ds - M(t-t_0) - \min_{0\leq\tau\leq t-t_0}\left\{\int_{-\infty}^{\tau} u(s)\,ds - M\tau\right\} \tag{1.3}$$

This will be shown to satisfy the ODE (1.2) almost everywhere later in Theorem 5.2.

As the first part of a two-part comprehensive study of the continuous-time Vickrey's model, this paper provides, besides the explicit solution, analysis on the mathematical properties of the solution (1.3). We will show analytical results regarding physical realism of the solution, solution regularity and *first-in-first-out* (FIFO).

The Vickrey's model is originally presented as an ODE which explicitly requires that the inflow $u(\cdot)$ must be L^1 integrable. Equivalently, the cumulative entering vehicle count must be absolutely continuous. In this paper, we will relax this assumption and extend the model to incorporate flows that are distributions. This is achieved by working with the cumulative vehicle counts, also known as the Newell-curve due to Newell (1993), instead of flow or density. The resulting model will be called *generalized Vickrey's model* (GVM). The GVM can handle cumulative entering vehicle count with up to countably many upward jumps. It will reduce to the original Vickrey's model provided that the cumulative entering vehicle count is absolutely continuous. Such extension is necessary in the study of analytical DTA problems. For example, in Bressan and Han (2011a) a *dynamic user equilibrium* (DUE) problem was investigated that only admits solutions containing dirac-delta in the departure flow.

In the end, the proposed PDE formulation leads to a number of extensions of Vickrey's model that will be discussed in Section 7. By modifying the flux function (Hamiltonian) of the conservation law (H-J equation), we are able to consider road inhomogeneity and time-varying flow capacity. Insights on the existence of solutions and mathematical properties are provided. In addition, we will discuss the connection of Vickrey's model to LWR model via a comparison of their respective fundamental diagrams.

The contribution of this article can be summarized as follows

1. We formulate the continuous-time Vickrey's model first as a scalar conservation law and then a Hamilton-Jacobi equation. Such formulation sheds light on the physics of the Vickrey's model and provides insights of its connection to the LWR model.
2. We apply the variational method to the Hamilton-Jacobi equation and derive an explicit solution. This solution leads to the explicit solution to the ODE with discontinuous right hand side.
3. Mathematical properties of the closed-form solution in terms of regularity and physical implications are established and analyzed.
4. We present the generalized Vickrey's model using cumulative vehicle counts. The generalized model is independent of any ODE formulation and allows the flows to be distributions.

The generalized Vickrey's model proposed in this paper is quite convenient for the analysis of network problems arising in the DTA context. Han et al. (submitted for publication) used the GVM to show the continuity of the effective delay operator in the context of dynamic user equilibrium of Friesz et al. (1993). Such an analytical result would be difficult to establish if based on the ODE formulation.

1.4. Organization

The rest of this article is organized as follows, Section 2 presents the original Vickrey's model. Section 3 briefly reviews a few continuous-time modifications of Vickrey's model proposed in Armbruster et al. (2006a); Ban et al. (2011); Pang et al. (2011). Section 4 takes the novel approach of reformulating the Vickrey's model as a partial differential equation and applying the variational method to solve it. An explicit solution is obtained as a result. Section 5 returns to the original ODE formulation and shows that the solution obtained from Section 4 indeed solves the ODE with discontinuous right hand side. Section 6 presents the generalized Vickrey's model which allows for a more general flow profile. In Section 7 several extensions of the Vickrey's model are discussed based on the PDE representation. Further insights on the relation between Vickrey's model and other first-order PDE models are also provided. Finally Section 8 presents two examples that illustrate the unique features of the closed-form solution as well as the Hamilton-Jacobi equation put forward in this paper.

2. The Vickrey's model

The Vickrey's model originally introduced in Vickrey (1969) is based on two key assumptions, (i) the vehicles have negligible sizes, therefore any queue does not occupy any physical space; (ii) the link travel time consists of a fixed travel time plus a congestion-related queuing time. This model takes various mathematical forms in the literature, see Drissi-Kaïouni and Hameda-Benchekroun (1992); Heydecker and Addison (1996); Kuwahara and Akamatsu (1997); Li et al. (2000). One of the most recognized mathematical formulations is given in Kuwahara and Akamatsu (1997); Nie and Zhang (2005) which we will present here and use as the starting point of our analysis. Let us introduce the notations

$$
\begin{aligned}
u(t):&\quad \text{the link entering flow}\\
w(t):&\quad \text{the link exiting flow}\\
q(t):&\quad \text{the queue size}\\
M:&\quad \text{the flow capacity of the bottleneck}\\
t_0:&\quad \text{the fixed free flow travel time}\\
\lambda(t):&\quad \text{the link traversal time of drivers entering the link at } t
\end{aligned}
$$

Consider a time horizon of $[0, T]$ for any $T > 0$. The dynamics governing the model can be described as: (1) vehicles entering the link move with free flow speed before arriving at the exit which is a bottleneck; (2) a queue with zero physical size (point queue) forms at the exit if the arriving flow exceeds the bottleneck capacity. The rate at which vehicles are released from the queue (exit rate) is described as follows.

$$
w(t) = \begin{cases} \min\{u(t-t_0),\, M\} & \text{if } q(t) = 0 \\ M & \text{if } q(t) > 0 \end{cases} \qquad t_0 \leq t \leq T \tag{2.4}
$$

The rule for releasing vehicles from the queue is straightforward: vehicles are released at maximum rate as allowed by the link capacity and vehicles supplied to the queue. Once the flow arriving at the queue $u(t-t_0)$ and the flow leaving the queue $w(t)$ are determined, the rate of change of the queue can be described simply by conservation

$$
\frac{dq(t)}{dt} = \begin{cases} 0 & \text{if } 0 \leq t < t_0 \\ u(t-t_0) - w(t) & \text{if } t_0 \leq t \leq T \end{cases} \tag{2.5}
$$

Here we are assuming that the road is initially empty, therefore the queue remains empty up to time t_0. Notice that if the queue $q(t)$ is nonzero, then the time it takes to traverse this queue is proportional to $q(t)$, thus the total travel time can be expressed as

$$\lambda(t) = t_0 + \frac{q(t+t_0)}{M} \qquad 0 \leq t \leq T \tag{2.6}$$

Vickrey's model is expressed mathematically by (2.4)-(2.6). Notice that (2.4) and (2.5) can be combined to form an ordinary differential equation:

$$\frac{dq(t)}{dt} = u(t-t_0) - \begin{cases} \min\{u(t-t_0),\, M\} & \text{if } q(t) = 0 \\ M & \text{if } q(t) > 0 \end{cases}, \quad t_0 \leq t \leq T \tag{2.7}$$

The case when $0 \leq t \leq t_0$ is trivial since $q(t) \equiv 0$. One important observation is that the right hand side of ODE (2.7) is discontinuous in its unknown $q(t)$. To see this, consider the case where $q(t) > 0$ and $u(t-t_0) < M$. When $q(t)$ approaches zero, the exit flow $w(t)$ will jump from M down to $u(t-t_0)$. Such discontinuous dependence on the state variable implies that the ODE (2.7) does not admit any classical solution, i.e. solutions that are locally continuously differentiable. Instead one may consider the solution in the integral sense, i.e. solution that satisfies the differential equation at almost every $t \in [0, T]$ or more generally, solution in the distributional (weak) sense.

3. Modifications of the Vickrey's model

Due to the irregularity within the ODE, a direct analysis of the aforementioned Vickrey's model in continuous-time is difficult: one will need to rely on the mathematical tools for discontinuous ODEs, e.g. Filippov (1988); Stewart (1990). For this reason Vickrey's model has been mainly studied and used in discrete time in the current literature. Yet a few modifications of the Vickrey's model have been proposed in Armbruster et al. (2006a); Ban et al. (2011); Pang et al. (2011) that will allow further analysis of the dynamics in continuous-time. In this section, we will briefly review these models with some minor discussions.

3.1. Linear Complementarity System formulation of Pang et al. (2011)

In the study of a DUE problem at a bottleneck, Pang et al. (2011) used a time-dependent *linear complementarity system* (LCS) as an alternative to the ODE (2.7). The LSC can be explicitly written as

$$\dot{q}(t) = s(t) + u(t-t_0) - M \tag{3.8}$$

$$0 \leq s(t) \perp q(t) \geq 0 \tag{3.9}$$

Here $s(t)$ is a time-dependent slack variable. The system (3.8)-(3.9) does not admit an explicit ODE formulation but has the implicit ODE form

$$\min\{q(t),\; \dot{q}(t) + M - u(t-t_0)\} = 0 \tag{3.10}$$

The LCS model explicitly imposes the non-negativity of the queue length $q(t)$. As mentioned in Ban et al. (2011), this model deviates from Vickrey's original model when $q(t) = 0$. Indeed, the Vickrey's model implies in this case that $\dot{q}(t) = \max\{u(t-t_0) - M,\, 0\}$, but (3.10) merely implies $\dot{q}(t) \geq u(t-t_0) - M$.

Notice that in terms of time-discretization of (3.8) and (3.9), a forward (explicit) scheme is not well-defined. That is, given the state variables at current time step, the state variables at the next time step are underdetermined, see Ban et al. (2011) for more detail. However a backward (implicit) scheme is well-defined and coincides with the numerical scheme proposed in Nie and Zhang (2005). In addition, the queue length is guaranteed to be non-negative under such scheme because of (3.9).

3.2. *The α-model of Ban et al. (2011)*

In Ban et al. (2011) a novel approach was used which transformed the above LCS into an explicit ODE. This was done by introducing a parameter $\alpha \gg 1$ and writing

$$\dot{q}(t) \;=\; \max\{u(t-t_0) - M,\; -\alpha\, q(t)\} \tag{3.11}$$

Hereafter, we will refer to this model as the α-model, a name suggested by Ban et al. (2011). It was shown that the solution to (3.11) in continuous-time will guarantee that both the queue size and the exit flow are non-negative. Such conformity to the physical realism is not obvious from (3.11). Moreover, as $\alpha \to +\infty$, the ODE (3.11) approaches the LCS asymptotically. However, we note that this model also deviates from Vickrey's model: if $q(t) \neq 0$ and $u(t-t_0) - M + \alpha q(t) < 0$ then

$$\dot{q}(t) \;=\; -\alpha\, q(t) \;>\; u(t-t_0) - M$$

while the Vickrey's model and the LCS both stipulate that $\dot{q}(t) = u(t-t_0) - M$. On the numerical side of the α-model, it was demonstrated in Ban et al. (2011) that an explicit discretization scheme for (3.11) may result in negative queues. In addition the explicit scheme is only conditional stable. The implicit (backward) scheme, on the other hand, ensures the non-negativity of the queue and the flow while accurately approximating the LCS model if the α is appropriately selected.

The remaining question for the α-model is how to appropriately select the numerical scale for α to ensure the proper functioning of this model. In a dynamic traffic assignment setting where the flows on each link are highly variable, the selection of the α value is crucial. In addition, it is interesting to further explore the physical meaning or underlying modeling implications of the parameter α.

3.3. *The ε-model of Armbruster et al. (2006a)*

Models built upon the point-queue concept also appear in the study of continuous supply chains networks. In particular, Armbruster et al. (2006a); Fügenschuh et al. (2008) proposed the continuous supply chain models by considering a flow dynamic very similar to that of Vickrey's model.

$$\frac{d}{dt} q(t) \;=\; u(t) - \begin{cases} \min\{u(t),\, M\} & \text{if } q(t) \;=\; 0 \\ M & \text{if } q(t) \;\neq\; 0 \end{cases} \tag{3.12}$$

Notice that by (3.12) it is assumed that the queue is located at the entrance of each link (processor) rather than the exit. In order to smooth out the discontinuity in the ODE (3.12), a smoothing parameter $\varepsilon \ll 1$ is used and (3.12) is rewritten as

$$\frac{d}{dt} q(t) \;=\; u(t) - \min\left\{M,\; \frac{q(t)}{\varepsilon}\right\} \tag{3.13}$$

From now on, we will call this model the ε-model. We observe that the ODE (3.13) is stiff whenever $0 < q(t) < \varepsilon M$. A stiffness condition needs to be imposed for the explicit scheme:

$$\Delta t \;\leq\; \varepsilon \tag{3.14}$$

where Δt denotes time step size in the discretization. The ε-model has a similar structure as the α-model. Such observation is aided by the following slightly different presentation of (3.13):

$$\frac{d}{dt}q(t) \;=\; \max\left\{u(t) - M,\;\; \frac{1}{\varepsilon}\Big(\varepsilon u(t) - q(t)\Big)\right\} \tag{3.15}$$

where ε can be treated as $1/\alpha$. In regimes of large α or small ε, the systems (3.15) and (3.11) tend to exhibit the same qualitative behavior.

The ε-model, like the α-model, has a an explicit ODE representation. It is a very practical alternative to (3.12), as verified extensively against Discrete Event Simulation in Armbruster et al. (2006a). However, the solution quality deteriorates for relatively large values of ε, see Han et al. (submitted for publication) for an example. Therefore in practice ε is usually chosen to be very small. Such fact indicates that there is a constant trade-off between solution quality and computational efficiency, due to (3.14).

We note here that both the α-model and the ε-model rely on the appropriate choice of an endogenous parameter, which is influenced by the numerical scales of variables such as t, $q(t)$, $u(t)$. Such a fact could potentially raise the issue of scalability of these modele. In other words, a case-dependent strategy for choosing the parameters is necessary to ensure the overall performance of the models.

3.4. summary

All three models have their own merits in analysis and application. Each one approximates the Vickrey's model under certain conditions and the numerical methods manage to fix the problems of negative queue and negative flow arising from the time-discretization of the original Vickrey's ODE. In addition, the approximate formulations provide valuable analytical and computational insights of the Vickrey's model.

Unlike the aforementioned work, the study in this paper aims to directly tackle the exact Vickrey's ODE in continuous-time and provide further insights of the model from the view point of partial differential equations.

4. A PDE formulation of Vickrey's model

In this section, we will reformulate the Vickrey's model using partial differential equation. This novel approach provides unique insights and analytical tools that are unavailable through theoretical or numerical study of the ODE.

4.1. The PDE model

Recall the Vickrey's model in the ODE form:

$$\frac{dq(t)}{dt} \;=\; u(t - t_0) - w(t), \quad q(t_0) \;=\; 0, \qquad t_0 \;\leq\; t \;\leq\; T + t_0 \tag{4.16}$$

$$
w(t) = \begin{cases} 0 & 0 \le t \le t_0 \\ \begin{cases} \min\{u(t-t_0),\, M\} & \text{if } q(t) = 0 \\ M & \text{if } q(t) > 0 \end{cases} & t_0 < t \le T + t_0 \end{cases} \tag{4.17}
$$

$$
\lambda(t) = \frac{q(t+t_0)}{M} + t_0 \qquad 0 \le t \le T \tag{4.18}
$$

The key ingredient of a PDE re-formulation of system (4.16)-(4.18) is a virtual spatial dimension $x \in [0, L]$, where L is the arc length. We introduce the free flow speed v_0 such that $L = v_0\, t_0$. Similar to the Lighthill-Whitham-Richards model of Lighthill and Whitham (1955); Richards (1956), the arc dynamics of Vickrey's model can be described based on mass conservation:

$$
\partial_t \rho(t, x) + \partial_x f(t, x) = 0, \qquad (t, x) \in [0, T] \times [0, L] \tag{4.19}
$$

where $\rho(t, x)$, $f(t, x)$ denotes respectively the vehicle density and flow at a point in the temporal-spatial domain. Notice that a PDE of the form (4.19) is always true for any dynamics prescribed by conservation of mass. What distinguishes the Vickrey's model from LWR model is the dependence of flow on the density. In the classical LWR model flow is a pure function of the density, which is given by the fundamental diagram. However, in Vickrey's model the flow depends not only on density but also on the spatial variable x, due to the presence of a bottleneck at the exit $x = L$. Indeed, the flow $f(t, x)$ can be expressed via the flux function ϕ as

$$
f(t, x) = \phi\big(x, \rho(t, x)\big) \doteq \begin{cases} v_0\, \rho(t, x) & \text{if } x \in [0, L) \\ \min\{M,\, v_0\, \rho(t, x)\} & \text{if } x = L \end{cases} \tag{4.20}
$$

Another way of interpreting the flux function ϕ is the following. Introduce the x-dependent capacity function $\mathcal{M}(x)$ defined as

$$
\mathcal{M}(x) \doteq \begin{cases} +\infty & \text{if } x \in [0, L) \\ M & \text{if } x = L \end{cases} \tag{4.21}
$$

This definition is straightforward: in the free flow phase $x \in [0, L)$ vehicles always travel at a constant speed no matter how large the flow may get, this is like saying the flow capacity is infinite; whereas at the exit $x = L$ a finite capacity constraint applies. The flux function can be defined as

$$
\phi\big(x, \rho(t, x)\big) \doteq \min\big\{\mathcal{M}(x),\;\; v_0\, \rho(t, x)\big\}, \tag{4.22}
$$

explicitly imposing the flow capacity constraint. On can easily check that the two definitions (4.20) and (4.22) of ϕ are equivalent. Summarizing what is discussed, we present the conservation-based PDE formulation of Vickrey's model.

$$
\begin{cases} \partial_t \rho(t, x) + \partial_x \min\big\{\mathcal{M}(x),\, v_0\, \rho(t, x)\big\} = 0, & (t, x) \in [0, T] \times [0, L] \\ v_0\, \rho(t, 0) = u(t) \end{cases} \tag{4.23}
$$

where $\mathcal{M}(x)$ is given in (4.21). $u(t)$ is the flow into the link.

Remark 4.1. *(4.23) is a scalar conservation law with space-dependent flux function. A review of mathematical results on such type of PDE can be found in Evans (2010). Notice that due to* *the discontinuous dependence of the flux function on x (see (4.21)), the solution to this PDE can only be considered in the distributional sense. In other words $\rho(t,\ x)$ may contain δ-distributions – a mathematical abstraction of the notion of "point queue"*[1].

Next, we present the Hamilton-Jacobi equation which is obtained by integrating the conservation law (4.23). Introduce the cumulative vehicle counts

$$N(t,\ x) \ \doteq\ \int_0^t f(s,\ x)\,ds, \qquad U(t) \ \doteq\ \int_0^t u(s)\,ds, \qquad t \in [0,\ T],\ \ x \in [0,\ L] \tag{4.24}$$

The $N(t,\ x)$, usually called the Moskowitz function or Newell-curve, measures the number of vehicles that have passed location x by time t. $U(\cdot) \equiv N(\cdot,\ 0)$ is the cumulative entering vehicle count. The $N(\cdot,\ \cdot)$ satisfies the following Hamilton-Jacobi equation

$$\begin{cases} \partial_t N(t,\ x) - \min\Big\{\mathcal{M}(x),\ -v_0\,\partial_x N(t,\ x)\Big\} \ =\ 0 \\ N(t,\ 0) \ =\ U(t) \end{cases} \tag{4.25}$$

Classical results on Hamilton-Jacobi equation with viscosity solution can be found in Bardi and Capuzzo-Dolcetta (1997); Evans (2010). For a review of its recent extensions and applications to traffic flow theory, the readers are referred to Aubin et al. (2008); Bressan and Han (2011a,b); Claudel and Bayen (2010a,b); Daganzo (2005); Friesz et al. (in press). In the next subsection, we will apply the Lax-Hopf formula to obtain an explicit solution representation of (4.25).

4.2. The Lax-Hopf formula

Initially introduced by Lax (1957, 1973), then extended by Aubin et al. (2008); Bardi and Capuzzo-Dolcetta (1997); Le Floch (1988) and applied to traffic theory by Claudel and Bayen (2010a); Daganzo (2005), the Lax-Hopf formula provides a new characterization of the solution to the scalar conservation law and Hamilton-Jacobi equation. The Lax-Hopf formula is derived from the characteristics equations associated with the Hamilton-Jacobi PDE, which arises from the calculus of variations and classical mechanics. See Evans (2010) for a complete discussion.

For the completeness of our presentation, we will review the Lax-Hopf formula for a Hamilton-Jacobi equation with x-independent Hamiltonian, in the context of the following Cauchy problem (initial-value problem).

$$\begin{cases} \partial_t \mathcal{N}(t,\ x) + \mathcal{H}\Big(\partial_x \mathcal{N}(t,\ x)\Big) \ =\ 0 & (t,\ x) \in \mathbb{R} \times (0,\ \infty) \\ \mathcal{N}(0,\ x) \ =\ g(x) & x \in \mathbb{R} \end{cases} \tag{4.26}$$

Theorem 4.2. (Lax-Hopf formula) *Suppose $\mathcal{H}$ is continuous and convex, $g(\cdot) : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, then*

$$\mathcal{N}(t,\ x) \ =\ \inf_{y \in \mathbb{R}} \left\{ t \cdot \mathcal{L}\left(\frac{x-y}{t}\right) + g(y) \right\} \tag{4.27}$$

is the unique viscosity solution to the initial-value problem (4.26). Where $\mathcal{L}$ is the Legendre transformation of $\mathcal{H}$:

$$\mathcal{L}(q) \ =\ \inf_p \{\mathcal{H}(p) - qp\} \tag{4.28}$$

[1]For this reason, many scholars refer to Vickrey's model as *point queue model* (PQM). However, the authors of this paper are reluctant to use this name since the name PQM was first used by Daganzo (1994) to describe the *link delay model* (LDM) proposed by Friesz et al. (1993).

Proof. See Evans (2010). □

Remark 4.3. *We note that formula (4.27) does not immediately apply to our problem, i.e. equation (4.25). This is because 1) the Hamiltonian $\mathcal{H}(\cdot)$ is x-independent; 2) (4.26) contains initial condition instead of boundary condition; and 3) $U(\cdot)$ is in general not Lipschitz continuous since $u(\cdot)$, its derivative, can be arbitrarily large.*

The major effort made in this paper is to overcome the difficulties described in Remark 4.3. Fortunately, all these problems can be solved by introducing a flow-based conservation law whose primary variable is the flow $f(t, x)$:

$$\begin{cases} \partial_x f(t, x) + \partial_t \left(\dfrac{1}{v_0} f(t, x)\right) = 0, & (t, x) \in [0, T] \times [0, L) \\ f(t, 0) = u(t), & t \in [0, T] \end{cases} \tag{4.29}$$

We remind the readers to pay attention to the domain for the spatial variable x. The PDE in (4.29) is obviously equivalent to (4.19) as long as the boundary point $x = L$ is excluded. Our next step is to apply the formula from Theorem 4.2 to the following Hamilton-Jacobi equation which is equivalent to (4.29):

$$\begin{cases} \partial_x N(t, x) + \dfrac{1}{v_0} \partial_t N(t, x) = 0, & (t, x) \in [0, T] \times [0, L) \\ N(t, 0) = U(t), & t \in [0, T] \end{cases} \tag{4.30}$$

Remark 4.4. *Notice that the system (4.30) can be interpreted as an initial value problem by switching the roles of t and x. This technique enables us to apply Theorem 4.2.*

Lemma 4.5. *Assume that $U(\cdot)$ is Lipschitz continuous with Lipschitz constant M. Then the viscosity solution to (4.30) is given by the following formula*

$$N(t, x) = \min_{0 \leq \tau \leq t - \frac{x}{v_0}} \left\{U(\tau) - M\tau\right\} + M \cdot \left(t - \frac{x}{v_0}\right), \quad (t, x) \in [0, T] \times [0, L) \tag{4.31}$$

Proof. By assumption, the flow $f(t, x) = \partial_t N(t, x)$ is uniformly bounded by M. Using the notation of Theorem 4.2, we have

$$\mathcal{H}(p) = \frac{1}{v_0} p, \qquad p \in [0, M]$$

Then the Legendre transformation becomes

$$\mathcal{L}(q) = \begin{cases} 0, & q \leq \frac{1}{v_0} \\ \left(q - \frac{1}{v_0}\right) M, & q > \frac{1}{v_0} \end{cases} \tag{4.32}$$

Then applying (4.27) with switched t and x, we readily deduce

$$N(t, x) = \inf_{\tau \in \mathbb{R}} \left\{x \cdot \mathcal{L}\left(\frac{t - \tau}{x}\right) + U(\tau)\right\} \tag{4.33}$$

In view of (4.32), two cases may arise for (4.33)

1. $\frac{t-\tau}{x} \leq \frac{1}{v_0}$, then (4.33) becomes

$$N(t,\, x) \;=\; \inf_{\tau \geq t - \frac{x}{v_0}} U(\tau) \;=\; U\Big(t - \frac{x}{v_0}\Big) \tag{4.34}$$

The last inequality is due to the monotonicity of $U(\cdot)$.

2. $\frac{t-\tau}{x} > \frac{1}{v_0}$, then (4.33) becomes

$$N(t,\, x) \;=\; \inf_{0 \leq \tau < t - \frac{x}{v_0}} \Big\{\Big(t - \tau - \frac{x}{v_0}\Big)M + U(\tau)\Big\} \tag{4.35}$$

(4.34), (4.35) together yield

$$N(t,\, x) \;=\; \min\Big\{U\Big(t - \frac{x}{v_0}\Big),\;\; \inf_{0 \leq \tau < t - \frac{x}{v_0}} \Big\{\Big(t - \tau - \frac{x}{v_0}\Big)M + U(\tau)\Big\}\Big\} \tag{4.36}$$

It is useful to observe that, by left-continuity of $U(\cdot)$,

$$\inf_{0 \leq \tau < t - \frac{x}{v_0}} \Big\{\Big(t - \tau - \frac{x}{v_0}\Big)M + U(\tau)\Big\} \;\leq\; \lim_{\tau \to (t - \frac{x}{v_0})-} \Big\{\Big(t - \tau - \frac{x}{v_0}\Big)M + U(\tau)\Big\} \;=\; U\Big(t - \frac{x}{v_0}\Big)$$

We immediately derive from (4.36) that

$$N(t,\, x) \;=\; \inf_{0 \leq \tau < t - \frac{x}{v_0}} \Big\{\Big(t - \tau - \frac{x}{v_0}\Big)M + U(\tau)\Big\} \;=\; \min_{0 \leq \tau \leq t - \frac{x}{v_0}} \Big\{\Big(t - \tau - \frac{x}{v_0}\Big)M + U(\tau)\Big\}$$

The last equality is due to the left-continuity of $U(\cdot)$. This shows (4.31). □

Remark 4.6. *Formula (4.31) in fact has a much simpler representation when $U(\cdot)$ is Lipschitz continuous with constant M because in this case it is always true that the minimum in (4.31) is attained at the right boundary, thus*

$$N(t,\, x) \;=\; U\Big(t - \frac{x}{v_0}\Big) - M\Big(t - \frac{x}{v_0}\Big) + M\Big(t - \frac{x}{v_0}\Big) \;=\; U\Big(t - \frac{x}{v_0}\Big), \quad \forall\, x \in [0,\, L)$$

That is, the flow profile at location $x \in [0,\, L)$ is only a translation of the profile at $x = 0$, which is consistent with the dynamic of the free-flow phase. In addition, by setting $x = L-$ we have that the cumulative count of vehicles arriving at the queue (if any) by time t is

$$N(t,\, L-) = U\Big(t - \frac{L}{v_0}\Big) \;=\; U(t - t_0)$$

The purpose of us invoking the variational formula (4.31) is to extend it to include the boundary point $x = L$ and show that (4.31) holds true even if $U(\cdot)$ is not Lipschitz continuous or not continuous at all. To this end, we relax the assumption on $U(\cdot)$ as follows.

Definition 4.7. (Entering vehicle count)
The cumulative count of vehicles entering the link is defined to be a non-decreasing, left-continuous function $U(\cdot) : [0,\, T] \to \mathbb{R}_+$ with possibly countably many upward jumps.

Remark 4.8. *An immediate consequence of the left-continuity is that* $U(0) = 0$*, even if* $U(0+) > 0$*. This is easier to see if we extend the function* $U(\cdot)$ *to the negative axis* $(-\infty, 0)$ *and assign* $U(t) \equiv 0 \ \forall\, t < 0$*.*

If we can represent the link entry flow as an integrable function $u(t)$, then the entering vehicle count $\int_0^t u(s)\,ds$ is absolutely continuous. Yet Definition 4.7 defines a much more general function $U(\cdot)$, whose assumption implies that the entry flow can be a distribution.

Let $w(\cdot)$ be the flow leaving the queue as defined in (4.17), we define the cumulative exiting vehicle count

$$W(t) \doteq \int_0^t w(s)\,ds \tag{4.37}$$

which is Lipschitz continuous since $w(t) \leq M$.

Remark 4.9. *The subtlety of the PDE representation of the Vickrey's model lies in the cumulative vehicle counts* $N(t, L-)$ *and* $N(t, L+)$ *which are not the same quantity. The former is identical to* $U(t - t_0)$ *which is not necessarily continuous; while the latter is identical to* $W(t)$ *which is Lipschitz continuous. Thus the solution* $N(t, \cdot)$ *for every* t *has a possible downward jump at* $x = L$*, which implies the presence of a concentration of mass (dirac-delta) at the exit of the link. This is precisely the "point queue".*

By (4.16), (4.17) and (4.37) we have that $W(\cdot)$ is Lipschitz continuous with constant M. Moreover

$$U(t - t_0) \geq W(t), \qquad q(t) = U(t - t_0) - W(t) \tag{4.38}$$

$$U(t - t_0) > W(t) \implies \frac{d}{dt}W(t) = M, \quad \text{for almost every } t \tag{4.39}$$

The next theorem is one of the main results of this paper. It states that the formula for $N(t, x)$ from Lemma 4.5 can be extended to include the boundary point $x = L$. As a result, a closed-form solution for $W(\cdot)$ is available.

Theorem 4.10. (Analytical representation of the solution) *Given cumulative entering vehicle count* $U(\cdot) : [0, T] \to \mathbb{R}_+$ *as in Definition 4.7, the cumulative exiting vehicle count is given by*

$$W(t) = \min_{0 \leq \tau \leq t - t_0} \Big\{U(\tau) - M\tau\Big\} + M(t - t_0), \qquad t \in [t_0, T + t_0] \tag{4.40}$$

Proof. We start by showing the following based on (4.38), (4.39).

$$\min_{t_0 \leq \tau \leq t} \{W(\tau) - M\,\tau\} = \min_{t_0 \leq \tau \leq t} \{U(\tau - t_0) - M\,\tau\}, \qquad t \in [t_0, T + t_0] \tag{4.41}$$

By (4.38), $W(\tau) \leq U(\tau - t_0)$. It thus remains to show

$$\min_{t_0 \leq \tau \leq t} \{W(\tau) - M\,\tau\} \geq \min_{t_0 \leq \tau \leq t} \{U(\tau - t_0) - M\,\tau\}$$

Fix any $t \in [t_0, T + t_0]$, let $\Delta_t \doteq \min_{t_0 \leq \tau \leq t} \Big\{W(\tau) - M\tau\Big\}$ and denote

$$\tau_t^* \doteq \inf\Big\{t_0 \leq \tau \leq t : \ W(\tau) - M\tau = \Delta_t\Big\} \tag{4.42}$$

Clearly $\tau_t^* \in [t_0, t]$ exists and is unique. By continuity of $W(\cdot)$, we have

$$W(\tau_t^*) - M\tau_t^* = \Delta_t$$

We claim that $U(\tau_t^* - t_0) = W(\tau_t^*)$. Clearly we only need to prove this claim for $\tau_t^* > t_0$ since it is always true that $U(0) = W(t_0) = 0$. By contradiction assume $U(\tau_t^* - t_0) > W(\tau_t^*)$, then the left-continuity of $U(\cdot - t_0)$ at τ_t^* implies there exists $\delta > 0$ such that $U(\tau - t_0) > W(\tau)$, $\forall \tau \in (\tau_t^* - \delta, \tau_t^*]$. By (4.39), we deduce

$$W(\tau_t^*) - W(\tau) \;=\; \int_\tau^{\tau_t^*} M\,ds \;=\; M(\tau_t^* - \tau) \;\Longrightarrow\; W(\tau) - M\tau \;=\; \Delta_t, \qquad \forall \tau \in (\tau_t^* - \delta,\, \tau_t^*]$$

This yields contradiction to (4.42), thereby the claim is substantiated. Finally $U(\tau_t^* - t_0) = W(\tau_t^*)$ leads to

$$\min_{t_0 \le \tau \le t} \{U(\tau - t_0) - M\tau\} \;\le\; U(\tau_t^* - t_0) - M\tau_t^* \;=\; \Delta_t \;=\; \min_{t_0 \le \tau \le t} \{W(\tau) - M\tau\}$$

This proves (4.41).

Next we observe that by Lipschitz continuity of $W(\cdot)$, it follows that

$$\min_{t_0 \le \tau \le t} \{W(\tau) - M\tau\} \;=\; W(t) - M\,t \tag{4.43}$$

Therefore we deduce from (4.43) and (4.41) that

$$W(t) \;=\; \min_{t_0 \le \tau \le t} \{W(\tau) - M\tau\} + M\,t \;=\; \min_{t_0 \le \tau \le t} \{U(\tau - t_0) - M\,\tau\} + M\,t \;=\; \min_{0 \le \tau \le t - t_0} \{U(\tau) - M\,\tau\} + M(t - t_0)$$

□

Remark 4.11. *Although the above theorem is a consequence of the PDE approach, (4.40) is independent of any spatial variable or any assumption related to the PDE. Such a formula remains true even if* $t_0 = 0$.

Theorem 4.10 combined with Lemma 4.5 provides a closed-form solution to the H-J equations (4.25). The fact that the formula for $x \in [0, L)$ can be extended to include $x = L$ is non-trivial. Later in Section 5, we will show that the quantity $q(t) \doteq U(t - t_0) - W(t)$ where $W(t)$ is given in (4.40) indeed solves the Vickrey's ODE in the integral sense. The impact of equation (4.40) is three-fold. (1) The solution $W(t)$ is given in closed-form, providing opportunity for further mathematical analysis that is relatively difficult to conduct based on the ODE. (2) The exit vehicle count $W(\cdot)$ can be computed directly from the entering vehicle count without any intermediate computation of the queue. (3) The solution given by (4.40) works for very general flow profiles such as the one in Definition 4.7.

For the rest of this section, we will establish mathematical properties of the resulting solution in continuous time.

Proposition 4.12. (Mathematical properties of the solution) *Given* $U(\cdot) : [0, T] \to \mathbb{R}_+$ *defined in Definition 4.7, the following holds:*

(1) The exiting vehicle count

$$W(t) \;=\; \min_{0 \le \tau \le t - t_0} \Big\{U(\tau) - M\tau\Big\} + M(t - t_0), \qquad t \in [t_0,\, T + t_0] \tag{4.44}$$

is non-negative, non-decreasing and Lipschitz continuous with Lipschitz constant M;

(2) The queue size

$$q(t) \;=\; U(t-t_0)-W(t) \;=\; U(t-t_0)-M(t-t_0)-\min_{0\le\tau\le t-t_0}\Big\{U(\tau)-M\tau\Big\}, \qquad t\in[t_0,\,T+t_0] \quad (4.45)$$

is non-negative. In addition, if $U(\cdot)$ is absolutely continuous then so is $q(\cdot)$.

(3) (First-in-first-out) Let the link traversal time of vehicle entering at time t be

$$\lambda(t) \;\doteq\; t_0+\frac{q(t+t_0)}{M} \qquad t\in[0,\,T] \quad (4.46)$$

where $q(t)$ satisfies (4.45). Then for any $0\le t_1<t_2\le T$,

$$t_1+\lambda(t_1) \;\le\; t_2+\lambda(t_2) \quad (4.47)$$

The equality of (4.47) holds if and only if $U(t_1)=U(t_2)$ and $w(t)\equiv M,\;\; t\in[t_0+t_1,\,t_0+t_2]$.

Proof. (1) We first show $W(\cdot)$ is non-decreasing. Fix any $t_0\le t_1\le t_2\le T+t_0$

$$\begin{aligned} W(t_2)-W(t_1) \;&=\; \min_{0\le\tau\le t_2-t_0}\{U(\tau)-M\tau\}-\min_{0\le\tau\le t_1-t_0}\{U(\tau)-M\tau\}+M(t_2-t_1) \\ &\ge\; -M(t_2-t_1)+M(t_2-t_1) \;=\; 0 \end{aligned}$$

Next the non-negativity follows from monotonicity and the fact that $W(t_0)=0$. To see that $W(\cdot)$ given by (4.44) is Lipschitz continuous, we observe for any $t_0\le t_1\le t_2\le T+t_0$,

$$\begin{aligned} 0 \;\le\; W(t_2)-W(t_1) \;&=\; \min_{0\le\tau\le t_2-t_0}\{U(\tau)-M\tau\}-\min_{0\le\tau\le t_1-t_0}\{U(\tau)-M\tau\}+M(t_2-t_1) \\ &\le\; M(t_2-t_1) \end{aligned}$$

(2) The non-negativity of $q(\cdot)$ is obvious from (4.45). Since $q(\cdot)$ is the difference between $U(\cdot-t_0)$ and a Lipschitz (hence absolutely) continuous function $W(\cdot)$, thus it will be absolutely continuous provided that $U(\cdot)$ is absolutely continuous.
(3) Using (4.45) and (4.46), we equivalently write (4.47) as

$$U(t_1)-\min_{0\le\tau\le t_1}\{U(\tau)-M\tau\} \;\le\; U(t_2)-\min_{0\le\tau\le t_2}\{U(\tau)-M\tau\}$$

which is always true since $U(t_1)\le U(t_2)$ and $\min_{0\le\tau\le t_1}\{U(\tau)-M\tau\}\ge\min_{0\le\tau\le t_2}\{U(\tau)-M\tau\}$. In order for the equality in (4.47) to hold, one must have

$$U(t_1) \;=\; U(t_2), \quad \text{and} \quad \min_{0\le\tau\le t_1}\{U(\tau)-M\,\tau\} \;=\; \min_{0\le\tau\le t_2}\{U(\tau)-M\,\tau\}$$

The second identity implies the following

$$W(t_0+t_2)-W(t_0+t_1) \;=\; M(t_2-t_1)$$

This means $w(t)\equiv M$ for $t\in[t_0+t_1,\,t_0+t_2]$. □

Remark 4.13. *The implication of the sufficient and necessary condition in (3) is that if two drivers entering at different times were to exit the link at the same time, then 1) there must be no other cars between them and 2) the driver that enters first must remain in the queue until the second driver catches up with him/her.*

5. Explicit solution to the ordinary differential equation

In this section, we return to the original system (4.16)-(4.17). Recall the ODE with discontinuous dependence on the state variable $q(\cdot)$:

$$\frac{dq(t)}{dt} = u(t-t_0) - \begin{cases} \min\{u(t-t_0),\, M\} & \text{if } q(t) = 0 \\ M & \text{if } q(t) > 0 \end{cases}, \qquad t \in [t_0,\, T+t_0] \tag{5.48}$$

We assert that the solution we derived from the PDE approach indeed solves this ODE. To make our assertion precise, let us define the strong solution to the ODE.

Definition 5.1. (Strong Solution of the ODE) *Given an integrable function $u(\cdot) : [0,\, T] \to \mathbb{R}_+$ as in (5.48), the strong solution (also called the integral solution) to the ODE (5.48) is defined to be an absolutely continuous function $q(\cdot) : [t_0,\, T+t_0] \to \mathbb{R}_+$ with $q(0) = 0$ that satisfies (5.48) at almost every $t \in [t_0,\, T+t_0]$.*

Theorem 5.2. *The explicit formula for the queue length $q(t)$ given by Proposition 4.12*

$$q(t) = U(t-t_0) - M\,(t-t_0) - \min_{0\leq\tau\leq t-t_0} \Big\{U(\tau) - M\tau\Big\}, \qquad t \in [t_0,\, T+t_0] \tag{5.49}$$

is the strong solution to the ODE (5.48) in the sense of Definition 5.1, where the absolutely continuous function

$$U(t) \doteq \int_0^t u(s)\,ds, \qquad t \in [0,\, T]$$

Proof. Define the Lipschitz continuous function with Lipschitz constant M:

$$W(t) \doteq \min_{0\leq\tau\leq t-t_0} \Big\{U(\tau) - M\tau\Big\} + M(t-t_0), \qquad t \in [t_0,\, T+t_0] \tag{5.50}$$

Notice that we have by definition $U(\cdot - t_0) \geq W(\cdot)$. Then (5.49) becomes $q(t) = U(t-t_0) - W(t)$, it follows immediately that $q(\cdot)$ is absolutely continuous and

$$\frac{d}{dt}q(t) = u(t-t_0) - \frac{d}{dt}W(t) \qquad \text{almost every } t \in [t_0,\, T+t_0]$$

Notice that all the above derivatives exist almost everywhere due to absolute continuity. It remains to show that for almost every $t \in [t_0,\, T+t_0]$,

$$\frac{d}{dt}W(t) = \begin{cases} \min\{u(t-t_0),\, M\} & \text{if } q(t) = 0 \\ M & \text{if } q(t) > 0 \end{cases} \tag{5.51}$$

Fix any $t \in (t_0,\, T+t_0] \setminus \Omega$ where Ω is the set of points where $U(\cdot - t_0)$ and $W(\cdot)$ are not differentiable. Ω has zero measure. Two cases may arise:
1. $q(t) = 0$. Then $U(t-t_0) = W(t)$. By definition (5.50) we have

$$U(t-t_0) - M(t-t_0) = \min_{0\leq\tau\leq t-t_0} \Big\{U(\tau) - M\tau\Big\} \tag{5.52}$$

We claim that $\frac{d}{dt}\Big(U(t-t_0) - M(t-t_0)\Big) \leq 0$. Otherwise if $\frac{d}{dt}U(t-t_0) > M$, consider $\varepsilon > 0$ small enough such that $\frac{d}{dt}U(t-t_0) > M + \varepsilon$. Then there exists $\delta > 0$ such that

$$\frac{U(t-t_0) - U(t_1 - t_0)}{t - t_1} > M + \varepsilon, \qquad \forall\, t_1 \in [t-\delta,\, t)$$

contradicting (5.52), thereby the claim is substantiated.

Next we will show that $\frac{d}{dt}W(t) = \frac{d}{dt}U(t-t_0)$. Indeed, consider any sequence $t_n > t$, $n \geq 1$ such that $t_n \to t$. Recalling $W(\cdot) \leq U(\cdot - t_0)$ we have

$$\frac{W(t_n) - W(t)}{t_n - t} \leq \frac{U(t_n - t_0) - U(t - t_0)}{t_n - t} \qquad \forall\ n \geq 1.$$

Thus $\frac{d}{dt}W(t) \leq \frac{d}{dt}U(t-t_0)$. Similar argument applies to any other sequence $t_\nu < t$, $\nu \geq 1$ such that $t_\nu \to t$, and we have $\frac{d}{dt}W(t) \geq \frac{d}{dt}U(t-t_0)$. We have shown what is promised.

It then follows that $\frac{d}{dt}W(t) = u(t-t_0)$ where $u(t-t_0)$ is bounded by M according to Lipschitz continuity of $W(\cdot)$. This finishes the proof of the first case of (5.51).

2. $q(t) > 0$. In this case

$$U(t-t_0) - M(t-t_0) > \min_{0 \leq \tau \leq t-t_0} \Big\{ U(\tau) - M\tau \Big\}$$

We define the right hand side of the above inequality to be ζ. By continuity, there exists a right neighborhood $\mathcal{N}_t^+$ of t such that $U(t_1 - t_0) - M(t_1 - t_0) > \zeta$ for all $t_1 \in \mathcal{N}_t^+$. This implies

$$W(t_1) = \min_{0 \leq \tau \leq t_1 - t_0} \{U(\tau) - M\,\tau\} + M(t_1 - t_0) = \zeta + M(t_1 - t_0), \qquad \forall\ t_1 \in \mathcal{N}_t^+$$

Therefore take any sequence $t_n \in \mathcal{N}_t^+$, $n \geq 1$ with $t_n \to t$

$$\frac{d}{dt}W(t) = \lim_{t_n \to t} \frac{W(t_n) - W(t)}{t_n - t} = M$$

This shows the second case of (5.51). □

6. Generalized Vickrey's model

The original Vickrey's model is based on an ODE formulation (2.7) or equivalent form, which implicitly requires that the inflow $u(\cdot)$ is an integrable function, or equivalently, the entering vehicle count $U(\cdot)$ is absolutely continuous. This assumption however, might be too restrictive in the context of analytical dynamic traffic assignment. For example, recent studies Bressan and Han (2011a,b) show that in certain cases, a user equilibrium exists only when $U(\cdot)$ is allowed to have discontinuities. Such a need for extension motivates the following Generalized Vickrey's Model (GVM).

Notice that the formulae in Proposition 4.12 do not require continuity of $U(\cdot)$. In fact by Definition 4.7, $U(\cdot)$ only needs to be non-decreasing and left-continuous. This suggests a way of extending Vickrey's model to a more general setting without invoking the ODE. We first consider the scenario where the arc is initially empty.

Definition 6.1. (GVM with zero initial condition)

The generalized Vickrey's model is defined in terms of cumulative vehicle counts. Assume the arc is empty at $t = 0$. Given the entering vehicle count $U(\cdot) : [0,\, T] \to \mathbb{R}_+$ which is non-decreasing and left-continuous. Then The exiting vehicle count $W(\cdot) : [0,\, T] \to \mathbb{R}_+$ is given by

$$W(t) = \begin{cases} 0 & t \in [0,\, t_0) \\ \min\limits_{0 \leq \tau \leq t - t_0} \Big\{ U(\tau) - M\tau \Big\} + M(t - t_0), & t \in [t_0,\, T + t_0] \end{cases} \tag{6.53}$$

The arc traversal time function $\lambda(\cdot) : [0, T] \to [0, +\infty)$ *is given by*

$$\lambda(t) \;=\; t_0 + \frac{1}{M} \cdot \Big(U(t) - Mt - \min_{0 \le \tau \le t} \{ U(\tau) - M\tau \} \Big) \qquad t \in [0, T] \tag{6.54}$$

In the case where $U(\cdot)$ is absolutely continuous and $\frac{d}{dt} U(t) \doteq u(t)$, the generalized Vickrey's model reduces to the original ODE-based system (4.16)-(4.18). In the GVM, the link inflow is no longer assumed to be a classical function, instead it is considered in the distributional sense.

6.1. Initial-boundary value problem

We consider the initial-boundary value problem for GVM. In particular, we assume an initial distribution of cars on the link and a non-zero initial queue size.

$$\rho(0, x) \;=\; \rho_0(x), \qquad x \in [0, L), \qquad q(0) \;=\; q_0 \tag{6.55}$$

This is mathematically expressed as the following H-J equation with initial-boundary conditions

$$\begin{cases} \partial_t N(t, x) - \min\big\{ \mathcal{M}(x), \, -v_0 \, \partial_x N(t, x) \big\} \;=\; 0 \\ N(t, 0) \;=\; U(t), & t \in [0, T] \\ N(0, x) \;=\; \int_x^L \rho_0(y) \, dy + q_0, & x \in [0, L] \end{cases} \tag{6.56}$$

Proposition 6.2. (GVM with initial-boundary conditions) *Assume the initial conditions (6.55) and boundary condition* $U(\cdot)$. *Then the cumulative exiting vehicle count* $W(\cdot)$ *and the arc traversal time function* $\lambda(\cdot)$ *are given by*

$$W(t) \;=\; \min_{0 \le \tau \le t} \{ \overline{U}(\tau) - M\tau \} + Mt, \qquad t \in [0, T + t_0] \tag{6.57}$$

$$\lambda(t) \;=\; t_0 + \frac{1}{M} \left(\overline{U}(t + t_0) - M(t + t_0) - \min_{0 \le \tau \le t + t_0} \{ \overline{U}(\tau) - M\tau \} \right), \qquad t \in [0, T] \tag{6.58}$$

Where $\overline{U}(\cdot)$ *is defined as*

$$\overline{U}(t) \;\doteq\; \begin{cases} 0 & t \;=\; 0 \\ q_0 + \displaystyle\int_{L - t v_0}^{L} \rho_0(y) \, dy & t \in (0, t_0] \\ q_0 + \displaystyle\int_0^L \rho_0(y) \, dy + U(t - t_0) & t \in (t_0, T + t_0] \end{cases} \tag{6.59}$$

Proof. Let $\tilde{U}(\cdot) : [0, T + t_0] \to \mathbb{R}_+$ count the cumulative number of vehicles arriving at the queue. Then clearly $\tilde{U}(0) = q_0$. In addition, during $(0, t_0]$ the cars initially on the link will arrive at the queue at a rate

$$f(t, L-) \;=\; v_0 \, \rho_0(L - v_0 \, t), \qquad t \in (0, t_0].$$

The above identity is a simple application of method of characteristics since the dynamic on $x \in [0, L)$ is governed by a linear advection equation. Thus we have for $t \in (0, t_0]$,

$$\tilde{U}(t) \;=\; q_0 + \int_0^t v_0 \rho_0(L - v_0 s) \, ds \;=\; q_0 + \int_{L - v_0 t}^{L} \rho_0(y) \, dy$$

Finally during $(t_0,\, T + t_0]$ the flow continues to arrive is determined by $U(\cdot - t_0)$. We have for $t \in (t_0,\, T + t_0]$,

$$\tilde{U}(t) \;=\; q_0 + \int_0^L \rho_0(y)\, dy + U(t - t_0)$$

Notice that we need to pick the left-continuous version of $\tilde{U}(\cdot)$, which is given by (6.59). Identities (6.57), (6.58) follow immediately. □

Remark 6.3. *Notice that the left-continuity dictates that $\overline{U}(0) = 0$. The formulae would not be correct if we use $\tilde{U}(\cdot)$ instead.*

7. Extensions and discussion

In this section we will discuss a few extensions of Vickrey's model based on the PDE formulation (4.23) and relate it to the Lighthill-Whitham-Richards model. All of these extensions as well as the transformation to the LWR model can be done by modifying the capacity function $\mathcal{M}(\cdot)$ appearing in the PDE representation of Vickrey's model:

$$\partial_t \rho(t,\, x) + \partial_x \min\Big\{\mathcal{M}(x),\; v_0\, \rho(t,\, x)\Big\} \;=\; 0 \tag{7.60}$$

7.1. Piecewise-constant $\mathcal{M}(x)$

In the formulation (4.23), $\mathcal{M}(\cdot)$ is defined in (4.21). One natural extension is to consider piecewise-constant $\mathcal{M}(\cdot)$ as a function of x. Such a PDE could represent inhomogeneous link with, for example, lane add/drop. It could also account for a network consisting of a series of homogenous links.

Similar to the generalized Vickrey's model, the equation (7.60) with piecewise-constant $\mathcal{M}(\cdot)$ does not admit solution in the L^1 space. The solution however does exist in the distributional sense. It is useful to notice that the solution $\rho(t,\, x)$ for such system can generate δ-distribution only at location x such that $\mathcal{M}(x-) > \mathcal{M}(x+)$, i.e. whenever there is a drop of capacity. To show solution existence, one only needs to solve the problem at a single bottleneck and proceed with induction. For a more detailed discussion of conservation laws with (possibly discontinuous) x-dependence of the flux, the readers are referred to Garavello et al. (2007); Herty and Klar (2007).

7.2. Time-varying flow capacity $M(t)$

We consider the Vickrey's model where the flow capacity is a (piecewise-constant) function $M(\cdot)$ of time. So we may write

$$\mathcal{M}(t,\, x) \;=\; \begin{cases} +\infty & \text{if } \; x \in [0,\, L) \\ M(t) & \text{if } \; x = L \end{cases}$$

The resulting conservation law then has a $t,\, x$-dependent flux function. However this is quite easy to analyze using our explicit formula. Without loss of generality we consider piecewise-constant $M(t)$ with discontinuities $0 < s_1 < s_2 < \ldots < s_n < \ldots$ and denote $M_i \equiv M(t)$, $t \in (s_{i-1},\, s_i)$. By convention we let $s_0 = 0$. For simplicity we assume $t_0 = 0$.

Assume the inflow profile $U(\cdot)$ is given by Definition 4.7. Then for the first time period $t \in [s_0,\ s_1]$ the cumulative exiting vehicle count

$$W(t) \;=\; \min_{0\leq\tau\leq t}\{U(\tau)-M_1\,\tau\}+M_1 t, \qquad t\in[s_0,\ s_1]$$

We apply the GVM to the next time interval $[s_1,\ s_2]$ with initial queue size $q(s_1)=U(s_1)-W(s_1)$ and inflow profile

$$\overline{U}(t) \;=\; \begin{cases} 0 & t=s_1 \\ U(t)-W(s_1) & t\in(s_1,\ s_2] \end{cases}$$

Then the cumulative exiting vehicle count during $(s_1,\ s_2]$ is

$$W(t) \;=\; U(t)-W(s_1)-M_2(t-s_1)-\min\Big\{0,\ \ \min_{s_1\leq\tau\leq t}\{U(\tau)-W(s_1)-M_2(\tau-s_1)\}\Big\}, \quad t\in(s_1,\ s_2]$$

We distinguish between two cases

- $\min\limits_{s_1\leq\tau\leq t}\{U(\tau)-W(s_1)-M_2(\tau-s_1)\}\geq 0$, in this case

  $$\begin{aligned} W(t) \;&=\; U(t)-W(s_1)-M_2(t-s_1) \;=\; U(t)-(M_1 s_1+M_2(t-s_1))-\min_{0\leq\tau\leq s_1}\{U(\tau)-M_1\tau\} \\ &=\; U(t)-V(t)-\min_{0\leq\tau\leq t}\{U(\tau)-V(\tau)\} \end{aligned}$$

  where the function
  $$V(t) \;\doteq\; \int_0^t M(s)\,ds$$

- $\min\limits_{s_1\leq\tau\leq t}\{U(\tau)-W(s_1)-M_2(\tau-s_1)\}< 0$, in this case

  $$\begin{aligned} W(t) \;&=\; U(t)-W(s_1)-M_2(t-s_1)-\min_{s_1\leq\tau\leq t}\{U(\tau)-W(s_1)-M_2(\tau-s_1)\} \\ &=\; U(t)-(M_1 s_1+M_2(t-s_1))-\min_{s_1\leq\tau\leq t}\{U(\tau)-(M_1 s_1+M_2(\tau-s_1))\} \\ &=\; U(t)-V(t)-\min_{0\leq\tau\leq t}\{U(\tau)-V(\tau)\} \end{aligned}$$

In either case we have

$$W(t)=U(t)-V(t)-\min_{0\leq\tau\leq t}\{U(\tau)-V(\tau)\} \tag{7.61}$$

A simple mathematical induction will show that (7.61) is true for all $t\geq 0$.

The potential cumulative out flow function $V(t)$ is a generalization of the function $M\cdot t$. The case when the capacity is time-dependent also admits an explicit solution whose structure is similar to that of the GVM.

7.3. $\mathcal{M}(\cdot)$ *that depends only on density*

As our final discussion, we will recover the LWR model by letting $\mathcal{M}$ to depend on density. This allows us to establish the connection between Vickrey's model and the LWR model.

Define

$$\mathcal{M}(\rho) \;=\; -\,\omega\cdot(\rho-\rho_{jam}), \qquad \rho\leq\rho_{jam} \tag{7.62}$$

where ω and ρ_{jam} are any positive constants. It is then easy to verify that $\phi(\rho) \doteq \min\{\mathcal{M}(\rho),\ v_0\rho\}$ coincides with the triangular fundamental diagram and we have the classical LWR conservation law

$$\partial_t\rho(t,\ x) + \partial_x \min\{\mathcal{M}(\rho(t,\ x)),\ v_0\rho(t,\ x)\} \ =\ 0$$

The relation between Vickrey's model and LWR model can be illustrated as follows. They are both represented by evolution equation based on mass conservation. The different is that Vickrey's model explicitly imposes bounds on the flows, whereas the LWR model explicitly imposes bounds on the densities. This can be see from their respective fundamental diagrams (Figure 1). The Vickrey's model bounds the flows, allowing density to be arbitrarily large. Once the density exceeds the critical value M/v_0 a wave with zero speed will be created corresponding to a δ-distribution in the density. The LWR model on the other hand, stipulates that the density cannot exceed ρ_{jam}. Thus when the density reaches the critical value where the flow is maximized, a wave with negative speed will be created preventing cars from piling up at a single location. Such a mechanism will eleminate concentration of mass at one point.

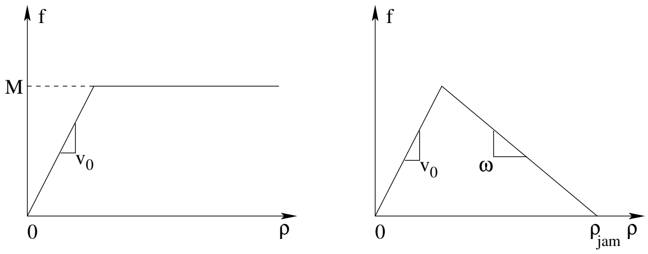

Figure 1: Fundamental diagrams of the Vickrey's model (left) and the LWR model (right).

8. Example

8.1. Continuous-time solutions

We illustrate the geometric meaning of formula (5.49) using a continuous-time example. Without loss of generality we assume the free flow time $t_0 = 0$. Consider the inflow

$$u(t) \ =\ \cos(t-\pi)+1, \qquad U(t)\ =\ \sin(t-\pi)+t, \qquad t\in[0,\ 10] \tag{8.63}$$

The key step of computing the GVM is to find $\min\limits_{0\leq\tau\leq t}\{U(\tau) - M\tau\}$. Four different cases where $M = 0.5,\ 1,\ 1.5,\ 2$ are depicted in Figure 2, 3, 4, 5, respectively.

1. $M = 0.5$. Then $U(\tau) - M\tau$ has a global minimizer $\tau^* = \pi/3$ and $U(\tau^*) - M\tau^* = -0.3424$ therefore

$$\min_{0\leq\tau\leq t}\{U(\tau) - M\tau\} \ =\ \begin{cases} U(t) - Mt & \text{if } 0 \ \leq\ t\ \leq\ \pi/3 \\ -0.3424 & \text{if } \pi/3\ <\ t\ \leq\ 10 \end{cases}$$

   and the solution to ODE (5.48) is

$$q(t) \ =\ \begin{cases} 0 & \text{if } 0\ \leq\ t\ \leq\ \pi/3 \\ \sin(t-\pi)+0.5t+0.3424 & \text{if } \pi/3\ <\ t\ \leq\ 10 \end{cases}$$

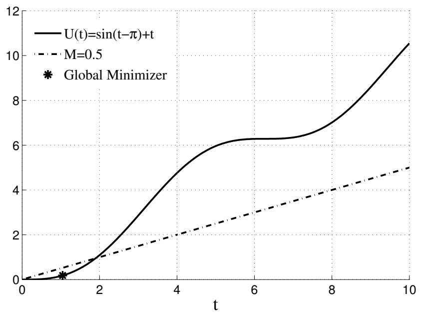

Figure 2: Case 1: $M = 0.5$

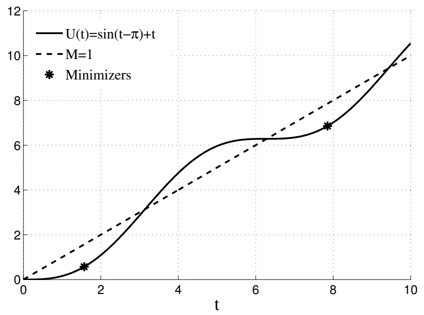

Figure 3: Case 2: $M = 1$

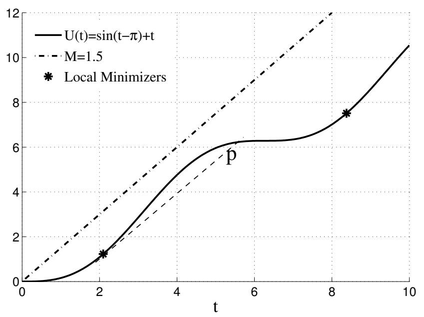

Figure 4: Case 3: $M = 1.5$

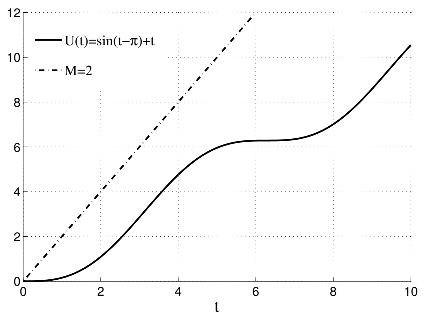

Figure 5: Case 4: $M = 2$

2. $M = 1$. Similar to previous case we have two global minimizers $\tau_1^* = \pi/2$, $\tau_2^* = 5\pi/2$ with minimum value -1, therefore

$$\min_{0\le\tau\le t}\{U(\tau) - M\tau\} = \begin{cases} U(t) - Mt & \text{if } 0 \le t \le \pi/2 \\ -1 & \text{if } \pi/2 < t \le 10 \end{cases}$$

$$q(t) = \begin{cases} 0 & \text{if } 0 \le t \le \pi/2 \\ \sin(t-\pi) + 1 & \text{if } \pi/2 < t \le 10 \end{cases}$$

3. $M = 1.5$. There exists a local minimum $\tau_1^* = 2\pi/3$ with value -1.9132 and a global minimum $\tau_2^* = 8\pi/3$ with value -5.0548. There exists a point $p = (5.3876, 6.1682)$ between τ_1^* and τ_2^* such that p yields the same objective value as τ_1^*, therefore

$$\min_{0\le\tau\le t}\{U(\tau) - M\tau\} = \begin{cases} U(t) - Mt & \text{if } 0 \le t \le 2\pi/3 \\ -1.9132 & \text{if } 2\pi/3 < t \le 5.3876 \\ U(t) - Mt & \text{if } 5.3876 < t \le 8\pi/3 \\ -5.0548 & \text{if } 8\pi/3 < t \le 10 \end{cases}$$

$$
q(t) = \begin{cases} 0 & \text{if } 0 \le t \le 2\pi/3 \\ \sin(t-\pi) - 0.5t + 1.9132 & \text{if } 2\pi/3 < t \le 5.3876 \\ 0 & \text{if } 5.3876 < t \le 8\pi/3 \\ \sin(t-\pi) - 0.5t + 5.0548 & \text{if } 8\pi/3 < t \le 10 \end{cases}
$$

4. $M = 2$, then since $U'(t) \le 2$, the global minimizer is at $\tau^* = 0$, with value 0, so

$$
q(t) = U(t) - Mt = \sin(t-\pi) - t \qquad 0 \le t \le 10
$$

Remark 8.1. *The above four solutions for queue sizes satisfy the ODE everywhere except case 3. From Figure 4 we observe that the quantity* $\min_{0 \le \tau \le t} \{U(\tau) - M\tau\}$ *has a kink at point p. This makes* $q(\cdot)$ *not differentiable there. This is confirmed by the plot of* $q(\cdot)$ *shown in Figure 6.*

Figure 6: Queue length, case 3. $q(t)$ is not differentiable at $t = 5.3876$.

8.2. Three consecutive arcs

In this example, we interpret the Vickrey's model from a PDE point of view. As shown in Section 4, the continuous-time Vickrey's model can be formulated as a Hamilton-Jacobi equation by introducing an artificial spatial dimension $x \in [0, L]$ and free flow speed v_0 such that $t_0 = L/v_0$. Consider a sequence of three links, e_1, e_2, e_3 with a combined length of 2000 meters, see Figure 7. The parameters for each link are shown in Table 1.

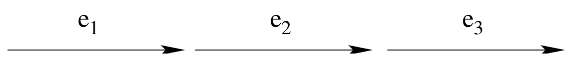

Figure 7: A simple network of three consecutive arcs

Link	e_1	e_2	e_3
L_i (meter)	667	667	667
$v_{0,i}$ (meter/min)	1500	1500	1500
M_i (veh/min)	40	30	24

Table 1: Link parameters

We introduce the cumulative vehicle count $N(\cdot, \cdot) : [0, T] \times [0, L] \to \mathbb{R}_+$, $N(t, x)$ measures the number of vehicles that have passed location x up to time t. Then the dynamics on this three-link network can be described by the single H-J equation (with proper initial and boundary value conditions)

$$\partial_t N(t, x) - \min\{M(x), -v_0 \partial_x N(t, x)\} = 0 \tag{8.64}$$

Notice that (8.64) has an x-dependent Hamiltonian:

$$M(x) = \begin{cases} 40, & x \in [0, 667] \\ 30, & x \in [667, 1333] \\ 20 & x \in [1333, 2000] \end{cases}$$

The network inflow are dipicted in Figure 8. Denote by $U(t)$ the cumulative entering vehicle count at $x = 0$. The PDE (8.64) is readily solved using the Lax-Hopf formula with boundary condition $N(t, 0) = U(t)$.

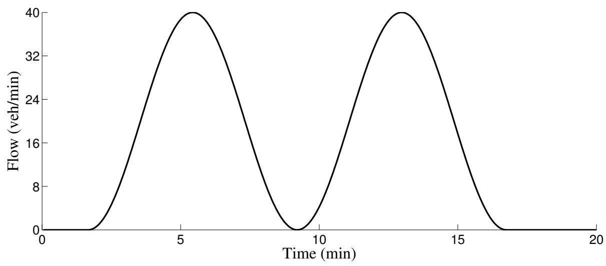

Figure 8: Entering flow of link e_1.

Figure 9 shows the quantity $N(t, 0) - N(t, x)$, $(t, x) \in [0, 20] \times [0, 2000]$, which measures the number of vehicles located in the spatial domain $[0, x]$ at time t. We clearly see several discontinuities in Figure 9, which corresponds to the point-mass located in front of link e_2, e_3, i.e. at $x = 667$, $x = 1333$. The time-varying magnitude of the jump measures the respective queue size. Figure 10 shows the flow $u(t, x)$ obtained by numerically differentiating $N(t, x)$ w.r.t. time [2]. The plateau indicate that the flow reaches maximum for downstream links e_2, e_3.

9. Conclusion

The continuous-time ODE formulation of Vickrey's model is analytically and computationally problematic due to the discontinuous right hand side of the ODE. So far most research on the Vickrey's model is either discrete-time in nature or relies on a modification of the original one, e.g. Armbruster et al. (2006a); Ban et al. (2011); Pang et al. (2011). In this article, we directly tackle such an ODE problem by reformulating it as a partial differential equation (Hamilton-Jacobi equation). This new formulation allows the Vickrey's model to be analyzed in continuous time with well-developed PDE theory, as a result, the ODE with discontinuous right hand side can be solved in closed form.

[2] Although $N(t, x)$ has discontinuities w.r.t. location, it is Lipschitz continuous w.r.t. time since its time derivative, the flow, is uniformly bounded. This is the unique feature of models that explicitly bound the flow but not the density.

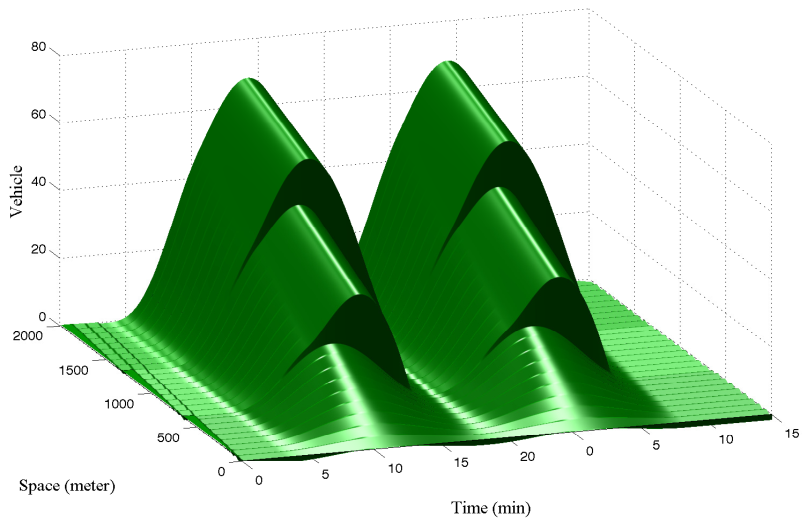

Figure 9: $N(t,\ 0) - N(t,\ x)$, the number of vehicles within $[0,\ x]$ at time t.

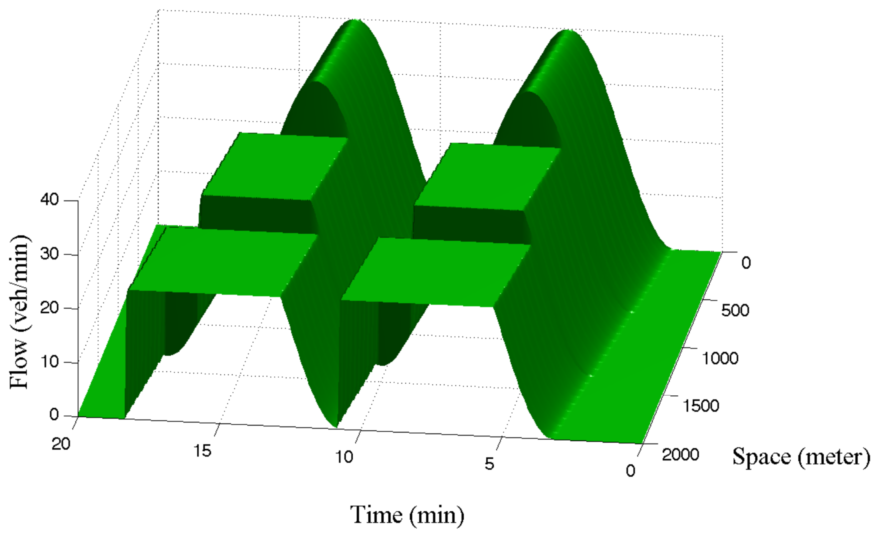

Figure 10: Vehicle flow $\frac{d}{dt}N(t,\ x)$

We start by adding a virtual spatial variable to the model and transform the ODE to a scalar conservation law whose flux function depends on the spatial variable. Then we apply the variational method to the corresponding Hamilton-Jacobi equation for an explicit solution representation. The closed-form solution is later verified to satisfy the original ODE almost everywhere. We complete our analysis by providing analytical properties of the solution and extending the model to incorporate more general flow profiles, i.e. distributions. The resulting generalized Vickrey's model is given by closed-form representation of all its variables and does not invoke any ODE. A few extensions based on the PDE formulation are discussed that also relate the Vickrey's model to the LWR model.

The work in this paper provides a new level of understanding of the Vickreys' model by relating it to first-order PDE models such as LWR model. This novel point of view enables us to understand the model using more powerful mathematical tools, with fruitful results such as closed-form solution, solution regularity, stability and convergence. Such methodology also benefits the study of DTA models by presenting the model in closed form for direct analysis. The work in this paper has already led to a few important findings regarding the network performance model that are reported in Han et al. (submitted for publication).

Further study of this approach in the discrete-time framework and its applications to dynamic traffic assignment will be discussed in an accompanying paper Han et al. (in press).

References

Armbruster, D., De Beer, C., Freitag, M., Jagalski, T., Ringhofer, C., 2006a. Autonomous control of production networks using a pheromone approach. Physica A 363 (1), 104-114.

Armbruster, D., Degond, P., Ringhofer, C., 2006b. A model for the dynamics of large queuing networks and supply chains, SIAM Journal on Applied Mathematics 66 (3), 896-920.

Aubin, J.P., Bayen, A.M., Saint-Pierre, P., 2008. Dirichlet problems for some Hamilton-Jacobi equations with inequality constraints. SIAM Journal on Control and Optimization 47 (5), 2348-2380.

Ban, X., Pang, J.S., Liu, H.X., Ma, R., 2011. Continuous-time point-queue models in dynamic network loading. Transportation Research Part B 46 (3), 360-380.

Bardi, M., Capuzzo Dolcetta, I., 1997. Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations, Birkhäuser.

Bressan, A., Han, K., 2011a. Optima and equilibria for a model of traffic flow. SIAM Journal on Mathematical Analysis 43 (5), 2384-2417.

Bressan, A., Han, K., 2011b. Nash equilibria for a model of traffic flow with several groups of drivers. ESAIM: Control, Optimization and Calculus of Variations, DOI: 10.1051/cocv/2011198

Claudel, C.G., Bayen, A.M., 2010a. Lax-Hopf based incorporation of internal boundary conditions into Hamilton-Jacobi equation. Part I: theory. IEEE Transactions on Automatic Control 55 (5), 1142-1157.

Claudel, C.G., Bayen, A.M., 2010b. Lax-Hopf based incorporation of internal boundary conditions into Hamilton-Jacobi equation. Part II: computational methods. IEEE Transactions on Automatic Control 55 (5), 1158-1174.

Daganzo, C.F., 1994. The cell transmission model. Part I: a simple dynamic representation of highway traffic. Transportation Research Part B 28 (4), 269-287.

Daganzo, C.F., 1995. The cell transmission model. Part II: network traffic. Transportation Research Part B 29 (2), 79-93.

Daganzo, C.F., 2005. A variational formulation of kinematic waves: basic theory and complex boundary conditions. Transportation Research Part B 39 (2), 187-196.

Drissi-Kaïouni O., Hameda-Benchekroun A., 1992. A dynamic traffic assignment model and a solution algorithm. Transportation Science 26 (2), 119-128.

Evans, L.C., 2010. Partial Differential Equations. Second edition. American Mathematical Society, Providence, RI.

Filippov, A.F., 1988, Differential Equations with Discontinuous Right-Hand Sides. Kluwer Academic Publishers.

Le Floch, P., 1988. Explicit formula for scalar non-linear conservation laws with boundary condition. Mathematical Models and Methods in Applied Sciences 10 (3), 265-287.

Friesz, T.L., Bernstein, D., Smith, T., Tobin, R., Wie, B., 1993. A variational inequality formulation of the dynamic network user equilibrium problem. Operations Research 41 (1), 80-91.

Friesz, T.L., Han, K., Meimand, A., Neto, P., Yao, T., in press. Dynamic user equilibrium based on a hydrodynamic model. Transportation Research Part B.

Fügenschuh, A., Göttlich, S., Herty, M., Klar, A., Martin, A., 2008. A discrete optimization approach to large scale supply networks based on partial differential equations. SIAM Journal on Scientific Computing 30 (3), 1490-1507.

Garavello, M., Natalini, R., Piccoli, B., Terracina, A., 2007. Conservation laws with discontinuous flux Networks and Heterogenous Media 2 (1), 159-179.

Garavello, M., Piccoli, B., 2006. Traffic Flow on Networks. Conservation Laws Models. AIMS Series on Applied Mathematics, Springfield, MO..

Han, K., Yao, T., Friesz, T.L., submitted for publication. Lagrangian-based hydrodynamic model: freeway traffic estimation. Transportation Research Part B.

Han, K., Friesz, T.L., Yao, T., submitted for publication. A variational approach for continuous supply chain networks. Submitted to SIAM Journal on Control and Optimization.

Han, K., Friesz, T.L., Yao, T., in press. A partial differential equation formulation of the continuous-time Vickrey's model: computational method and application. Transportation Research Part B.

Han, K., Friesz, T.L., Yao, T., submitted for publication. Existence of simultaneous route and departure choice dynamic user equilibrium. Transportation Research Part B.

Herty, M., Klar, A., Piccoli, B., 2007. Existence of solutions for supply chain models based on partial differential equations. SIAM Journal on Mathematical Analysis 39 (1), 160-173.

Heydecker, B.G., Addison J.D., 1996. An exact expression of dynamic traffic equilibrium. In J.B. Lesort, editor. Transportation and Traffic Theory. Pergamon Press (Oxford, UK 1996) 359-383.

Kuwahara, M., Akamatsu, T., 1997. Decomposition of the reactive dynamic assignments with queues for a many-to-many origin-destination pattern. Transportation Research Part B 31 (1), 1-10.

Lax, P.D., 1957. Hyperbolic systems of conservation laws II. Communications on Pure and Applied Mathematics 10 (4), 537-566.

Lax, P.D., 1973. Hyperbolic systems of conservation laws and the mathematical theory of shock waves, SIAM.

Li, J., Fujiwara, O., Kawakami, S., 2000. A reactive dynamic user equilibrium model in network with queues. Transportation Research Part B 34 (8), 605-624.

Lighthill, M.J., Whitham, J.B., 1955. On kinematic waves II: A theory of traffic flow in long crowded roads. Proceedings of the Royal Society A229, 317-345.

Newell, G.F., 1993. A simplified theory of kinematic waves in highway traffic, part I: General theory. Transportation Research Part B 27 (4), 281-287.

Nie, X., Zhang, H.M., 2005. A comparative study of some macroscopic link models used in dynamic traffic assignment. Networks and Spatial Economics 5 (1), 89-115.

Pang, J.S., Han, L.S., Ramadurai, G., Ukkusuri, S., 2011. A continuous-time linear complementarity system for dynamic user equilibria in single bottleneck traffic flows. Mathematical Programming, Series A 133 (1-2), 437-460.

Richards, P.I., 1956. Shockwaves on the highway. Operations Research 4 (1), 42-51.

Stewart, D.E., 1990. A high accuracy method for solving ODEs with discontinuous right-hand side. Numerische Mathematik 58, 299-328.

Vickrey, W.S. 1969. Congestion Theory and Transport Investment. The American Economic Review 59 (2), 251-261.