A PRESENTATION OF THE DEFORMED $W_{1+\infty}$ ALGEBRA

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Abstract. We provide a generators and relation description of the deformed $W_{1+\infty}$-algebra introduced in previous joint work of E. Vasserot and the second author. This gives a presentation of the (spherical) cohomological Hall algebra of the one-loop quiver, or alternatively of the spherical degenerate double affine Hecke algebra of $GL(\infty)$.

INTRODUCTION

In the course of their work on the cohomology of the moduli space of $U(r)$-instantons on $\mathbb{P}^2$ in relation to $W$-algebras and the AGT conjecture (see [SV]) E. Vasserot and the second author introduced a certain one-parameter deformation $\mathcal{SH}^c$ of the enveloping algebra of the Lie algebra $W_{1+\infty}$ of algebraic differential operators on $\mathbb{C}^\ast$. The algebra $\mathcal{SH}^c$—which is defined in terms of Cherednik’s double affine Hecke algebras—acts on the above mentioned cohomology spaces (with a central character depending on the rank $n$ of the instanton space). For the same value of the central character, $\mathcal{SH}^c$ is also strongly related to the affine $W$ algebra of type $\mathfrak{gl}_n$, and has the same representation theory (of admissible modules) as the latter. The same algebra $\mathcal{SH}^c$ arises again as the (spherical) cohomological Hall algebra of the quiver with one vertex and one loop, and as a degeneration of the (spherical) elliptic Hall algebra (see [SV, Sec. 4, 8]). It also independently appears in the work of Maulik and Okounkov on the AGT conjecture, see [MO].

The definition of $\mathcal{SH}^c$ given in [SV] is in terms of a stable limit of spherical degenerate double affine Hecke algebras, and does not yield a presentation by generators and relations. In this note, we provide such a presentation, which bears some resemblance with Drinfeld’s new realization of quantum affine algebras and Yangians. Namely, we show that $\mathcal{SH}^c$ is generated by families of elements in degrees $-1, 0, 1$, modulo some simple quadratic and cubic relations (see Theorems 3.1, 3.2).

The definition of $\mathcal{SH}^c$ is recalled in Section 1. In the short Section 2 we briefly recall the links between $\mathcal{SH}^c$ and Cherednik algebras, resp. $W$-algebras. The presentation of $\mathcal{SH}^c$ is given in Section 3, and proved in Section 4. Although we have tried to make this note as self-contained as possible, there are multiple references to statements in [SV] and the reader is advised to consult that paper (especially Sections 1 and 8) for details.

1. Definition of $\mathcal{SH}^c$

1.1. Symmetric functions and Sekiguchi operators. Let $\kappa$ be a formal parameter, and let us set $F = \mathbb{C}(\kappa)$. Let us denote by $\Lambda_F$ the ring of symmetric polynomials in infinitely many variables with coefficients in $F$, i.e.

$$
\Lambda_F = F[X_1, X_2, \ldots]^{\otimes\infty} = F[p_1, p_2, \ldots].
$$

For $\lambda$ a partition, we denote by $J_\lambda$ the integral form of the Jack polynomial associated to $\lambda$ and to the parameter $\alpha = 1/\kappa$. It is well-known that $\{J_\lambda\}$ forms a basis of $\Lambda_F$ (see e.g. [S], or [SV, Sec. 1.3, 1.6]).

The polynomials $J_\lambda$ arise as the joint spectrum of a family of commuting differential operators $\{D_{0,l}\}, l \geq 1$ called Sekiguchi operators. In the above normalization, these may be characterized through the following relations:

1
where $s$ runs through the set of boxes in the partition $\lambda$, and where $c(s) = x(s) - \kappa y(s)$ is the content of $s$. Here $x(s), y(s)$ denote the $x$ and $y$-coordinates of the box $s$, when $\lambda$ is drawn according to the continental convention (see [SV, Sec. 0.1]).

We denote by $D_{l,0} \in \text{End}(\Lambda_F)$ the operator of multiplication by the power-sum function $p_l$.

1.2. The algebras $\text{SH}^+$ and $\text{SH}^>$. Let $\text{SH}^+$ be the unital subalgebra of $\text{End}(\Lambda_F)$ generated by $\{D_{0,l}, D_{l,0} \mid l \geq 1\}$. For $l \geq 1$ we set $D_{1,l} = [D_{0,l+1}, D_{1,0}]$. This relation is still valid when $l = 0$, and we furthermore have

$$[D_{0,1}, D_{1,k}] = D_{1,k+l-1} \quad l \geq 1, k \geq 0.$$  

We denote by $\text{SH}^>$ the unital subalgebra of $\text{SH}^+$ generated by $\{D_{1,l} \mid l \geq 0\}$, and by $\text{SH}^0$ the unital subalgebra of $\text{SH}^+$ generated by the Sekiguchi operators $\{D_{0,l} \mid l \geq 1\}$. It is known (and easy to check from (1.1)) that the $D_{0,l}$ are algebraically independent, i.e. $\text{SH}^0 = F[D_{0,1}, D_{0,2}, \ldots]$. Observe that by (1.2), the operators $ad(D_{0,l})$ preserve the subalgebra $\text{SH}^>$. This allows us to view $\text{SH}^+$ as a semi-direct product of $\text{SH}^0$ and $\text{SH}^>$. In fact, the multiplication map induces an isomorphism

$$(1.3) \quad \text{SH}^> \otimes \text{SH}^0 \simeq \text{SH}^+$$

(see [SV] Prop. 1.18).

1.3. Grading and filtration. The algebra $\text{SH}^+$ carries an $\mathbb{N}$-grading, defined by setting $D_{0,l}, D_{1,k}$ in degrees zero and one respectively. This grading, which corresponds to the degrees as operators on polynomials will be called the rank grading. It also carries an $\mathbb{N}$-filtration compatible with the rank grading, induced from the filtration by the order of differential operators. It may be characterized as follows, see [SV] Prop. 1.2 : $\text{SH}^+\langle \leq d \rangle$ is the space of elements $u \in \text{SH}^+$ satisfying

$$ad(z_1) \circ \cdots \circ ad(z_{d+1})(u) = 0$$

for all $z_1, \ldots, z_{d+1} \in F[D_{1,0}, D_{2,0}, \ldots]$. We have $\text{SH}^+\langle \leq 0 \rangle = F[D_{1,0}, D_{2,0}, \ldots]$. The following is proved in [SV] Lemma 1.21. Set $D_{r,d} = [D_{0,d+1}, D_{r,0}]$ for $r \geq 1, d \geq 0$.

Proposition 1.1. (i) The associated graded algebra $gr\text{SH}^+$ is equal to the free commutative polynomial algebra in the generators $D_{r,d} \in gr\text{SH}^+\langle r,d \rangle$, for $r \geq 0, d \geq 0, (r,d) \neq (0,0)$.

(ii) The associated graded algebra $gr\text{SH}^>$ is equal to the free commutative polynomial algebra in the generators $D_{r,d} \in gr\text{SH}^+\langle r,d \rangle$, for $r \geq 1, d \geq 0$.

We will need the following slight variant of the above result, which can easily be deduced from [SV] Prop. 1.38. For $r \geq 1$, set $D'_{r,d} = ad(D_{0,2})^d(D_{r,0})$. Then

$$D'_{r,d} \in r^{d-1} D_{r,d} \otimes \text{SH}^+\langle r, \leq d-1 \rangle.$$  

In particular, $gr\text{SH}^>$ is also freely generated by the elements $D'_{r,d} \in gr\text{SH}^+\langle r,d \rangle$.

1.4. The algebra $\text{SH}^\circ$. Let $\text{SH}^<$ be the opposite algebra of $\text{SH}^>$. We denote the generator of $\text{SH}^>$ corresponding to $D_{1,l}$ by $D_{-1,l}$. The algebra $\text{SH}^\circ$ is generated by $\text{SH}^>, \text{SH}^0, \text{SH}^<$ together with a family of central elements $c = (c_0, c_1, \ldots)$ indexed by $\mathbb{N}$, modulo a certain set of relations involving the commutators $[D_{-1,k}, D_{1,l}]$ (see [SV] Sec. 1. 8). In order to write down these relations, we need a few notations. Set $\xi = 1 - \kappa$ and

$$G_0(s) = -\log(s), \quad G_l(s) = (s^{-1} - 1)/l, \quad l \geq 1, \quad \varphi_l(s) = \sum_{q=1,-\xi,-\kappa} s^l(G_l(1 - qs) - G_l(1 + qs)), \quad l \geq 1.$$
\[ \phi_l(s) = s^l G_l(1 + \xi s) \]

We may now define \( \text{SH}^c \) as the algebra generated by \( \text{SH}^>, \text{SH}^<, \text{SH}^0 \) and \( F[c_0, c_1, \ldots] \) modulo the following relations:

\begin{align}
[D_0,1, D_{1,k}] &= D_{1,k+1}, \quad [D_{-1,k}, D_{0,l}] = D_{-1,k+l-1}, \\
[D_{-1,k}, D_{1,l}] &= E_{k+l}, \quad l, k \geq 0,
\end{align}

where the elements \( E_k \) are determined through the formulas

\[ 1 + \xi \sum_{l \geq 0} E_l s^{l+1} = \exp \left( \sum_{l \geq 0} (-1)^{l+1} c_l \phi_l(s) \right) \exp \left( \sum_{l \geq 0} D_{0,l+1} \phi_l(s) \right) \]

Set \( \text{SH}^{0,c} = \text{SH}^0 \otimes F[c_0, c_1, \ldots] \). One can show that the multiplication map provides an isomorphism of \( F \)-vector spaces

\[ \text{SH}^> \otimes \text{SH}^{0,c} \otimes \text{SH}^< \cong \text{SH}^c. \]

Putting the generators \( D_{k,1} \) in degree \( \pm 1 \) and the generators \( D_{0,l}, c_i \) in degree zero induces an \( \mathbb{Z} \)-grading on \( \text{SH}^c \). One can show that the order filtration on \( \text{SH}^>, \text{SH}^< \) can be extended to a filtration on the whole \( \text{SH}^c \), but we won’t need this last fact.

2. Link to \( W \)-algebras, Cherednik algebras and shuffle algebras

2.1. Relation the Cherednik algebras. Let \( \omega \) be a new formal parameter and let \( \text{SH}^\omega \) be the specialization of \( \text{SH} \) at \( c_0 = 0, c_1 = -\kappa \omega' \). Let \( H_n \) be Cherednik’s degenerate (or trigonometric) double affine Hecke algebra with parameter \( \kappa \) (see [C]). Let \( \text{SH}_n \subset H_n \) be its spherical subalgebra. The following result shows that \( \text{SH}^\omega \) may be thought of as the stable limit of \( \text{SH}_n \) as \( n \) goes to infinity (see [SV, Sec. 1.7]) :

**Theorem.** For any \( n \) there exists a surjective algebra homomorphism \( \Phi_n : \text{SH}^\omega \to \text{SH}_n \) such that \( \Phi_n(\omega) = n \). Moreover \( \bigcap_n \ker \Phi_n = \{0\} \).

2.2. Realization as a shuffle algebra. Consider the rational function

\[ g(z) = \frac{h(z)}{z}, \quad h(z) = (z + 1 - \kappa)(z - 1)(z + \kappa). \]

Following [FO], we may associate to \( g(z) \) an \( \mathbb{N} \)-graded associative \( F \)-algebra \( A_{g(z)} \), the symmetric shuffle algebra of \( g(z) \) as follows. As a vector space,

\[ A_{g(z)} = \bigoplus_{n \geq 0} F[z_1, \ldots, z_n]^{S_n} \]

with multiplication given by

\[ P(z_1, \ldots, z_r) \ast Q(z_1, \ldots, z_s) = \sum_{\sigma \in S_{r+s}} \sigma \cdot \left( \prod_{1 \leq i < j \leq r+s} g(z_i - z_j) \cdot P(z_1, \ldots, z_r)Q(z_{r+1}, \ldots, z_{r+s}) \right) \]

where \( S_{r+s} \subset S_{r+s} \) is the set of \( (r, s) \) shuffles inside the symmetric group \( S_{r+s} \). Let \( S_{g(z)} \subset A_{g(z)} \) denote the subalgebra generated by \( A_{g(z)}[1] = F[z] \). The following is proved in [SV, Cor. 6.4] :

**Theorem.** The assignment \( S_{g(z)}[1] \ni z^l \mapsto D_{l,1}^l \), \( l \geq 0 \) induces an isomorphism of \( F \)-algebras

\[ S_{g(z)} \cong \text{SH}^>. \]
2.3. Relation to $W$-algebras. Let $W_{1+\infty}$ be the universal central extension of the Lie algebra of all differential operators on $\mathbb{C}^*$ (see e.g. [PKRW]). This is a $\mathbb{Z}$-graded and $\mathbb{N}$-filtered Lie algebra. The following result shows that $\text{SH}$ may be thought of as a deformation of the universal enveloping algebra $U(W_{1+\infty})$ of $W_{1+\infty}$ (see [SV, App. F]):

**Theorem.** The specialization of $\text{SH}^c$ at $\kappa = 1$ and $c_i = 0$ for $i \geq 1$ is isomorphic to $U(W_{1+\infty})$.

More interesting is the fact that, for certain good choices of the parameters $c_0, c_1, \ldots$, a suitable completion of $\text{SH}^c$ is isomorphic to the current algebra of the (affine) $W$-algebra $W(\mathfrak{gl}_r)$ (see e.g. [A, Sec. 3.11]). Fix an integer $r \geq 1$, $k \in \mathbb{C}$ and let $(\varepsilon_1, \ldots, \varepsilon_r)$ be new formal parameters. Let $\mathfrak{U}(W_k(\mathfrak{gl}_r))'$ be the formal current algebra of $W(\mathfrak{gl}_r)$ at level $k$, defined over the field $F(\varepsilon_1, \ldots, \varepsilon_r)$ (see [SV] Sec. 8.4 for details). Let $\text{SH}^{(r)}$ be the specialization of $\text{SH}^c$ to $\kappa = k + r$, $c_i = \varepsilon_1^i + \cdots + \varepsilon_r^i$ for $i \geq 0$. The following is proved in [SV, Cor. 8.24], to which we refer for details.

**Theorem.** There is an embedding $\text{SH}^{(r)} \to \mathfrak{U}(W_k(\mathfrak{gl}_r))'$ with a dense image, which induces an equivalence between the category of admissible $\text{SH}^{(r)}$-modules and the category of admissible $\mathfrak{U}(W_k(\mathfrak{gl}_r))'$-modules.

3. Presentation of $\text{SH}^+$ and $\text{SH}^c$

3.1. Generators and relations for $\text{SH}^+$. Consider the $F$-algebra $\widetilde{\text{SH}}^+$ generated by elements $\{\tilde{D}_{0,l} \mid l \geq 1\}$ and $\{\tilde{D}_{1,k} \mid k \geq 0\}$ subject to the following set of relations:

\[
\begin{align*}
&[\tilde{D}_{0,l}, \tilde{D}_{0,k}] = 0, \quad \forall \, l, k \geq 1, \\
&[\tilde{D}_{0,l}, \tilde{D}_{1,k}] = \tilde{D}_{1,l+k+1}, \quad \forall \, l \geq 1, k \geq 0, \\
&(3)[\tilde{D}_{1,2}, \tilde{D}_{1,1}] - [\tilde{D}_{1,3}, \tilde{D}_{1,0}] + [\tilde{D}_{1,1}, \tilde{D}_{1,0})] + \kappa(\kappa - 1)(\tilde{D}_{1,0}^2 + [\tilde{D}_{1,1}, \tilde{D}_{1,0}] = 0 \\
&[\tilde{D}_{1,0}, [\tilde{D}_{1,0}, \tilde{D}_{1,1}] = 0.
\end{align*}
\]

Let $\widetilde{\text{SH}}^0 = F[\tilde{D}_{0,1}, \tilde{D}_{0,2}, \ldots]$ denote the subalgebra of $\widetilde{\text{SH}}^+$ generated by $\tilde{D}_{0,l}, l \geq 1$, and let $\widetilde{\text{SH}}^>$ be the subalgebra generated by $\tilde{D}_{1,k}, k \geq 0$. The algebras $\text{SH}^+, \text{SH}^0, \text{SH}^>$ are all $\mathbb{N}$-graded, where $\tilde{D}_{0,l}$ and $\tilde{D}_{1,k}$ are placed in degrees zero and one respectively. According to the terminology used for $\text{SH}^+$, we call this grading the rank grading.

**Theorem 3.1.** The assignment $\tilde{D}_{0,l} \mapsto D_{0,l}, \tilde{D}_{1,k} \mapsto D_{1,k}$ for $l \geq 1, k \geq 0$ induces an isomorphism of graded $F$-algebras

$$\phi : \widetilde{\text{SH}}^+ \xrightarrow{\sim} \text{SH}^+.$$ 

Obviously, the map $\phi$ restricts to isomorphisms $\widetilde{\text{SH}}^0 \simeq \text{SH}^0, \widetilde{\text{SH}}^> \simeq \text{SH}^>$. Note however that $\widetilde{\text{SH}}^>$ is not generated by the elements $D_{1,k}$ with the sole relations (3.3). Theorem 3.1 is proved in Section 4.
3.2. Generators and relations for $\text{SH}^c$. For the reader’s convenience, we write down the presentation of $\text{SH}^c$, an immediate corollary of Theorem 3.1 above. Let $\text{SH}^c$ be the algebra generated by elements $\{\tilde{D}_{0,l} \mid l \geq 1\}, \{\tilde{D}_{\pm 1,k} \mid k \geq 0\}$ and $\{\tilde{c}_i \mid i \geq 0\}$ subject to the following set of relations:

\[(3.5)\] \[\tilde{D}_{0,l}, \tilde{D}_{0,k} = 0, \quad \forall \, l, k \geq 1,\]

\[(3.6)\] \[\tilde{D}_{0,l} \tilde{D}_{1,k} = \tilde{D}_{1,l+k-1}, \quad \tilde{D}_{-1,k} \tilde{D}_{0,l} = \tilde{D}_{-1,l+k-1} \quad \forall \, l, k \geq 1, k \geq 0,\]

\[(3.7)\] \[(3 \tilde{D}_{1,2}, \tilde{D}_{1,1}) - [\tilde{D}_{1,3}, \tilde{D}_{1,0}] + [\tilde{D}_{1,1}, \tilde{D}_{1,0}] + \kappa (\kappa - 1) (\tilde{D}_{1,0}^2 + [\tilde{D}_{1,1}, \tilde{D}_{1,0}]) = 0\]

\[(3.8)\] \[(3 \tilde{D}_{-1,2}, \tilde{D}_{-1,1}) - [\tilde{D}_{-1,3}, \tilde{D}_{-1,0}] + [\tilde{D}_{-1,1}, \tilde{D}_{-1,0}] + \kappa (\kappa - 1) (-\tilde{D}_{1,0}^2 + [\tilde{D}_{-1,1}, \tilde{D}_{-1,0}]) = 0\]

\[(3.9)\] \[\tilde{D}_{1,0} \tilde{D}_{1,0} \tilde{D}_{1,1} = 0, \quad [\tilde{D}_{-1,0}, [\tilde{D}_{-1,0}, \tilde{D}_{-1,1}]] = 0,\]

\[(3.10)\] \[\tilde{D}_{-1,k} \tilde{D}_{1,l} = \tilde{E}_{k+l}, \quad l, k \geq 0,\]

where the $\tilde{E}_l$ are defined by the formula (1.7).

**Theorem 3.2.** The assignment $\tilde{D}_{0,l} \mapsto D_{0,l}, \tilde{D}_{\pm 1,k} \mapsto D_{\pm 1,k}$ for $l \geq 1, k \geq 0$ and $\tilde{c}_i \mapsto c_i$ for $i \geq 0$ induces an isomorphism of $F$-algebras

$$\phi : \text{SH}^c \xrightarrow{\sim} \text{SH}^e.$$

Coupled with the Theorems in Section 2.3., this provides a potential 'generators and relations' approach to the study of the category of admissible modules over the W-algebras $W_k(\mathfrak{gl}_r)$.

4. Proof of Theorem 3.1

4.1. Let us first observe that $\phi$ is a well-defined algebra map, i.e. that relations (3.1)–(3.4) hold in $\text{SH}^c$. For (3.1)–(3.2) this follows from the definition of $\text{SH}^c$ and [SV (3.18)]. Equation (3.3) may be checked directly, e.g. from the Pieri rules (see [SV (2.26)]), or from the shuffle realization of $\text{SH}^c$ (see 4.2. below). As for equation (3.4), we have by [SV (3.15)], $[[D_{1,1}, D_{1,0}], D_{1,0}] = [D_{2,0}, D_{1,0}] = 0$. The map $\phi$ is surjective by construction; in the rest of the proof, we show that it is injective as well.

4.2. Using relation (3.2) it is easy to see that any monomial in the generators $\tilde{D}_{0,l}, \tilde{D}_{1,k}$ may be expressed as a linear combination of similar monomials, in which all $\tilde{D}_{0,l}$ appear on the right of all $\tilde{D}_{1,k}$. Hence the multiplication map $\text{SH}^c \otimes \text{SH}^0 \to \text{SH}^c$ is surjective. Since $\phi$ clearly restricts to an isomorphism $\text{SH}^0 \simeq \text{SH}^0$ we only have to show, by (1.3), that $\phi$ restricts to an isomorphism $\text{SH}^c \simeq \text{SH}^c$. Our strategy will be to construct a suitable filtration on $\text{SH}^c$ mimicking the order filtration of $\text{SH}^c$ and to pass to the associated graded algebras.

4.3. We begin by proving directly, using the shuffle realization of $\text{SH}^c$, that $\phi$ is an isomorphism in ranks one and two. This is obvious in rank one since $\phi$ is a graded map and the only relation in rank one is (3.2).

Suppose $\sum \alpha_i D_{1,k_i} D_{1,l_i} = 0$ is a relation in rank two. The shuffle realization then implies $\sum \alpha_i z_i^{k_i} \ast z_i^{l_i} = 0$ so that

$$h(z_1 - z_2) \left( \sum \alpha_i z_i^{k_i} z_i^{l_i} \right) = h(z_2 - z_1) \left( \sum \alpha_i z_i^{l_i} z_i^{k_i} \right).$$
Therefore $\sum \alpha_i z_i^k z_j^l = h(z_2 - z_1)P(z_1, z_2)$ where $P(z_1, z_2)$ is some symmetric polynomial in $z_1, z_2$. Hence $\sum \alpha_i z_i^k z_j^l$ is a linear combination of polynomials of the form $h(z_2 - z_1)(z_1^k z_2^l + z_1^j z_2^k)$ so that $\sum \alpha_i D_{1,k} D_{1,l}$ is a linear combination of expressions of the form

\[(4.1)\]

\[3[D_{1,t+2}, D_{1,k+1}] - [D_{1,t+1}, D_{1,k+2}] - [D_{1,t+1}, D_{1,k+3}] + [D_{1,t+1}, D_{1,k}] - [D_{1,t}, D_{1,k+1}] + k(\kappa - 1)(D_{1,k} D_{1,l} + D_{1,l} D_{1,k}) = [D_{1,t}, D_{1,k+1}].\]

If $I$ denotes the image of $[3.3]$ under the action of $F[\ad D_{0,2}, \ad D_{0,3}, \ldots]$ then using [3.2] we see that each such expression lies in $\phi(I)$ so that $\phi$ is indeed an isomorphism in rank two.

We remark that the relations (4.1) may be written in a more standard way using the generating functions $\sum D(z) = \sum D_{1,t} z^{-t}$ as follows:

\[(4.2)\]

\[k(z - w) D(z) D(w) = - k(w - z) D(w) D(z)\]

where $k(u) = (u - 1 + \kappa)(u + 1)(u - \kappa) = - h(-u)$. In particular, the defining relation (3.3) may be replaced by the above (4.2), of which it is a special case.

4.4. We now turn to the definition of the analog, on $\tilde{SH}$, of the order filtration on $SH^>$. We will proceed by induction on the rank $r$. For $r = 1, d \geq 0$, we set

\[SH^> [1, \leq d] = \bigoplus_{k \leq d} F\tilde{D}_{1,k}.\]

Assuming that $SH^> [r', \leq d']$ has been defined for all $r' < r$ we let $\tilde{SH}^> [r, \leq d]$ be the subspace spanned by all products

\[SH^> [r', \leq d'] \cdot \tilde{SH}^> [r'', \leq d''], \quad r' + r'' = r, d' + d'' = d\]

and by the spaces

\[ad(\tilde{D}_{1,t})(\tilde{SH}^> [r - 1, \leq d - t + 1]), \quad l = 0, \ldots, d + 1.\]

From the above definition, it is clear that $\tilde{SH}^>$ is a $\mathbb{Z}$-filtered algebra. Note that it is not obvious at the moment that $\tilde{SH}^> [r, \leq d] = \{0\}$ for $d < 0$. Because the associated graded $gr\tilde{SH}^>$ is commutative, it follows by induction on the rank $r$ that $\phi : \tilde{SH}^> \to SH^>$ is a morphism of filtered algebras. We denote by $gr\tilde{SH}^>$ the associated graded of $\tilde{SH}^>$ and we let $\tilde{\phi} : gr\tilde{SH}^> \to grSH^>$ be the induced map. The map $\tilde{\phi}$ is graded with respect to both rank and order. Moreover $\tilde{\phi}$ is an isomorphism in ranks 1 and 2 (indeed, that the filtration as defined above coincides with the order filtration in rank 2 can be seen directly from [SV, (1.84)]). The rest of the proof of Theorem 4.1 consists in checking that $\tilde{\phi}$ is an isomorphism. Once more, we will argue by induction. So in the remainder of the proof, we fix an integer $r \geq 3$ and assume that $\tilde{\phi}$ is an isomorphism in ranks $r' < r$.

4.5. By our assumption above, the algebra $gr\tilde{SH}^>$ is commutative in ranks less than $r$, that is $ab = ba$ whenever $\text{rank}(a) + \text{rank}(b) < r$. Our first task is to extend this property to the rank $r$.

**Lemma 4.1.** The algebra $gr\tilde{SH}^>$ is commutative in rank $r$.

**Proof.** We have to show that for $a \in \tilde{SH}^> [r_1, \leq d_1], b \in \tilde{SH}^> [r_2, \leq d_2]$ and $r_1 + r_2 = r$ we have

\[(4.3)\]

\[[a, b] \in \tilde{SH}^> [r, \leq d_1 + d_2 - 1].\]

We argue by induction on $r_1$. If $r_1 = 1$ then (4.3) holds by definition of the filtration. Now let $r_1 > 1$ and let us further assume that (4.3) is valid for all $r_1', r_2'$ with $r_1' + r_2' = r$ and $r_1' < r_1$. We will now prove (4.3) for $r_1, r_2$, thereby completing the induction step. According to the definition of the filtration, there are two cases to consider:
Case 1) We have $a = a_1a_2$ with $a_1 \in \mathcal{SH}^>[s', \leq d'], a_2 \in \mathcal{SH}^>[s'', \leq d'']$ such that $s' + s'' = r_1, d' + d'' = d_1$. Then $[a, b] = a_1[a_2, b] + [a_1, b]a_2$. By our induction hypothesis on $r$, $[a_2, b] \in \mathcal{SH}^>[s'+r_2, \leq d''+d_2-1]$ hence $a_1[a_2, b] \in \mathcal{SH}^>[r, \leq d_1 + d_2 - 1]$. The term $[a_1, b]a_2$ is dealt with in a similar fashion.

Case 2) We have $a = [\tilde{D}_{l,t}, a']$ with $a' \in \mathcal{SH}^>[r_1-1, \leq d_1 - l + 1]$. Then $[a, b] = [[\tilde{D}_{l,t}, a'], b] = [\tilde{D}_{l,t}, [a', b]] - [a', [\tilde{D}_{l,t}, b]]$. By our induction hypothesis on $r$, $[a', b] \in \mathcal{SH}^>[r_1+r_2-1, \leq d_1 + d_2 - l]$ hence $[\tilde{D}_{l,t}, [a', b]] \in \mathcal{SH}^>[r, \leq d_1 + d_2 - 1]$. Similarly, $[\tilde{D}_{l,t}, b] \in \mathcal{SH}^>[r_2 + 1, \leq d_2 + l - 1]$. The inclusion $[a', [\tilde{D}_{l,t}, b]] \in \mathcal{SH}^>[r, \leq d_1 + d_2 - 1]$ now follows from the induction hypothesis on $r_1$.

We are done.

4.6. We now focus on the filtered piece of order $\leq 0$ of $\mathcal{SH}^>$. We inductively define elements $\tilde{D}_{l,0}$ for $l \geq 2$ by

$$\tilde{D}_{l,0} = \frac{1}{l - 1}[[\tilde{D}_{l,1}, \tilde{D}_{l-1,0}].$$

From [SV (1.35)] we have $\phi(\tilde{D}_{l,0}) = D_{l,0}$. Since we assume are assuming that $\overline{\phi}$ is an isomorphism in ranks less than $r$, we have $[\tilde{D}_{l,0}, \tilde{D}_{l',0}] = 0$ whenever $l + l' < r$.

**Lemma 4.2.** We have $[\tilde{D}_{l,0}, \tilde{D}_{l',0}] = 0$ for $l + l' = r$.

**Proof.** If $r = 3$ this reduces to the cubic relation (3.3). For $r = 4$ we have to consider

$$[\tilde{D}_{3,0}, \tilde{D}_{1,0}] = \frac{1}{2}[[\tilde{D}_{1,1}, \tilde{D}_{2,0}], \tilde{D}_{1,0}]$$

$$= \frac{1}{2}[\tilde{D}_{1,1}, [\tilde{D}_{2,0}, \tilde{D}_{1,0}]] - \frac{1}{2}[\tilde{D}_{2,0}, [\tilde{D}_{1,1}, \tilde{D}_{1,0}]]$$

$$= -\frac{1}{2}[\tilde{D}_{2,0}, \tilde{D}_{2,0}] = 0.$$

Now let us fix $l, l'$ with $l + l' = r$. We have

$$[\tilde{D}_{l,0}, \tilde{D}_{l',0}] = \frac{1}{l - 1}[[\tilde{D}_{l,1}, \tilde{D}_{l-1,0}], \tilde{D}_{l',0}]

(4.4)

= \frac{1}{l - 1}[\tilde{D}_{l,1}, [\tilde{D}_{l-1,0}, \tilde{D}_{l',0}]] - \frac{1}{l - 1}[\tilde{D}_{l-1,0}, [\tilde{D}_{l,1}, \tilde{D}_{l',0}]]$$

$$= -\frac{l'}{l - 1}[\tilde{D}_{l-1,0}, \tilde{D}_{l'+1,0}].$$

If $r = 2k$ is even then by repeated use of (4.4) we get

$$[\tilde{D}_{l,0}, \tilde{D}_{l',0}] = c[\tilde{D}_k, \tilde{D}_k] = 0$$

for some constant $c$. Next, suppose that $r = 2k + 1$ is odd, with $k \geq 2$. Applying $ad(\tilde{D}_{l,1})$ to $[\tilde{D}_{k+1,0}, \tilde{D}_{k-1,0}] = 0$ yields the relation

$$\frac{k+1}{l}[[\tilde{D}_{k+2,0}, \tilde{D}_{k-1,0}] + (k - 1)[\tilde{D}_{k+1,0}, \tilde{D}_{k,0}] = 0.$$

Similarly, applying $ad(\tilde{D}_{2,1})$ to $[\tilde{D}_{k,0}, \tilde{D}_{k-1,0}] = 0$ and using the relation $[D_{k,1}, D_{l,0}] = klD_{l+k,0}$ in $\mathcal{SH}^>$ (see [SV (1.91), (8.47)]) we obtain the relation

$$\frac{k}{l}[[\tilde{D}_{k+2,0}, \tilde{D}_{k-1,0}] + (k - 1)[\tilde{D}_{k,0}, \tilde{D}_{k+1,0}] = 0.$$

Equations (4.5) and (4.6) imply that $[\tilde{D}_{k+2,0}, \tilde{D}_{k-1,0}] = [\tilde{D}_{k+1,0}, \tilde{D}_{k,0}] = 0$. The general case of $[\tilde{D}_{l,0}, \tilde{D}_{l',0}] = 0$ is now deduced, as in the case $r = 2k$, from repeated use of (4.4). \[\square\]

Note that Lemma 4.2 above implies that $\mathcal{SH}^>[r, \leq -1] = \{0\}$. 
4.7. Recall that \( grSH^> \) is a free polynomial algebra in generators in the generators \( D'_{s,d} \) for \( s \geq 1, d \geq 0 \). In order to prove that \( \bar{\phi} \) is an isomorphism in rank \( r \), it suffices, in virtue of Lemma 1.1, to show that the factor space

\[
U_{r,d} = grSH^> [r,d] / \big\{ \sum_{r'' + r'''' = r} grSH^> [r'', d''] \cdot grSH^> [r'', d''] \big\}
\]

is one dimensional for any \( d \geq 0 \). Let us set, for any \( s \geq 1, d \geq 0 \)

\[
\hat{D}'_{s,d} = ad(\hat{D}_{0,2})^d(\hat{D}_{s,0}) \in SH^> [s, \leq d].
\]

We will denote by the same symbol \( \hat{D}'_{s,d} \) the corresponding element of \( grSH^> [s,d] \). Note that \( \hat{D}'_{s,0} = \hat{D}_{s,0} \) We claim that in fact \( U_{r,d} = F\hat{D}'_{r,d} \). Observe that \( \phi(\hat{D}'_{s,d}) = D'_{s,d} \) for any \( s, d \), hence \( \hat{D}'_{s,d} \in U_{s,d} \) for any \( s \leq r, d \geq 0 \). Moreover, by our general induction hypothesis on \( r \) we have \( U_{s,d} = F\hat{D}'_{s,d} \) for any \( s < r \) and \( d \geq 0 \).

We will prove that \( U_{r,d} = F\hat{D}'_{r,d} \) by induction on \( d \). For \( d = 0 \), this comes from Lemma 1.2. So fix \( d > 0 \) and let us assume that \( U_{r,l} = F\hat{D}'_{r,l} \) for all \( l < d \). By definition of the filtration on \( SH^> \), \( U_{r,d} \) is linearly spanned by the classes of the elements

\[
[D_{1,0}, \hat{D}'_{r-1,d+1}], [D_{1,1}, \hat{D}'_{r-1,d}], \ldots, [D_{1,d+1}, \hat{D}'_{r-1,0}].
\]

By our induction hypothesis on \( d \), the elements

\[
[D_{1,0}, \hat{D}'_{r-1,d}], [D_{1,1}, \hat{D}'_{r-1,d-1}], \ldots, [D_{1,d}, \hat{D}'_{r-1,0}]
\]

all belong to \( F\hat{D}'_{r,d-1} \oplus SH^> [r, \leq d - 1] \). Applying \( ad(\hat{D}_{0,2}) \), we see that

\[
[D_{1,0}, \hat{D}'_{r-1,d+1}] + [D_{1,1}, \hat{D}'_{r-1,d}] + \ldots + [D_{1,d}, \hat{D}'_{r-1,1}] + [D_{1,d+1}, \hat{D}'_{r-1,0}]
\]

all belong to \( F\hat{D}'_{r,d} \oplus SH^> [r, \leq d - 1] \). Next, applying \( ad(\hat{D}_{0,d+2}) \) to the equality \( [D_{1,0}, \hat{D}_{r-1,0}] = 0 \) yields

\[
[D_{1,0}, \hat{D}_{r-1,d+1}] + [D_{1,d+1}, \hat{D}_{r-1,0}] = 0
\]

which implies, by (1.4), that

\[
[D_{1,0}, \hat{D}_{r-1,d+1}] + r^d[D_{1,d+1}, \hat{D}_{r-1,0}] \in [D_{1,0}, SH^> [r-1, \leq d]] \subseteq SH^> [r, \leq d - 1].
\]

The collection of inclusions (4.7), (4.8) may be considered as a system of linear equations in \( U_{r,d} \) modulo \( F\hat{D}'_{r,d} \) in the variables \([D_{1,0}, \hat{D}'_{r-1,d+1}], \ldots, [D_{1,d+1}, \hat{D}'_{r-1,0}] \) whose associated matrix is

\[
M = \begin{pmatrix}
1 & 0 & \cdots & 0 & 1 \\
1 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 1 & 0 \\
0 & \cdots & 0 & 1 & -r^d
\end{pmatrix}
\]

is invertible. We deduce that \([D_{1,0}, \hat{D}'_{r-1,d+1}], \ldots, [D_{1,d+1}, \hat{D}'_{r-1,0}] \) all belong to the space \( F\hat{D}'_{r,d} \oplus SH^> [r, \geq d - 1] \) as wanted. This closes the induction step on \( d \). We have therefore proved that \( U_{r,d} = F\hat{D}'_{r,d} \) for all \( d \geq 0 \), and hence that \( \bar{\phi} \) and \( \phi \) is an isomorphism in rank \( r \). This closes the induction step on \( r \). Theorem 3.1 is proved.

\[\square\]

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A PRESENTATION OF THE DEFORMED $W_{1+\infty}$ ALGEBRA

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