Power Spectrum Estimation from Noisy Autocorrelations Values

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Abstract

The problem of estimating the power spectrum from noisy autocorrelation values is considered in this paper, and it is proposed that in order to reduce errors, oversampling of the available time domain data should be employed. The oversampling problem is discussed from the Frame Theory point of view, and it is shown that the frame reconstruction represents an improvement upon the standard correlogram, windowing, and autoregressive modelling approaches.

1 Introduction

Estimation of the power spectrum is a fundamental problem in signal processing and there are many techniques available for its solution. A summary of these with many references can be found in [7]. There are also books completely dedicated to the subject [8, 9].

In this paper we wish to address the power spectrum estimation problem with the additional complication that the autocorrelation values are themselves noisy. We maintain that, to estimate the power spectrum from a single realization of each measurement, a price to be paid is oversampling the data at a density larger than the Nyquist one.

The oversampling problem is discussed from the Frame Theory point of view [4, 15, 3]. Within this framework, power spectrum reconstruction appears as a *tight* frame superposition that looks very much like the classical correlogram estimation from oversampling data. In fact, Fast

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Fourier Transform can also be used to obtain this reconstruction, but we wish to stress the considerable theoretical difference between these two cases: the oversampling problem involves a set of functions which constitute a frame but not a basis. Indeed, this implicit redundancy of the frame appears to be the main reason for the noise reduction, but is also the reason that the representation is not unique.

The paper is organized as follows: the problem to be addressed is introduced in section 2 and some comments about classical power spectrum estimation are given in section 3. In section 4 the oversampling problem is discussed as a particular case of the general Frame Theory. Simulations are included in section 5, where the frame reconstruction is compared with the correlogram and windowing approaches and also with autoregressive (AR) modelling [8, 9]. Finally conclusions are drawn in section 6.

2 The problem to be addressed

For a zero mean wide-sense stationary random process, x(t), the autocorrelation function $R(\tau)$ is defined in terms of ensemble averages as [6]:

$$R(\tau) = \frac{1}{\sigma^2}\boldsymbol{E}(x(t)x(t+\tau)^*), \tag{1}$$

where

$$\sigma^2 = \boldsymbol{E}(x(t)x(t)^*). \tag{2}$$

The normalized power spectral density function, $P(\omega)$, is the Fourier Transform of the autocorrelation function, i.e.,

$$P(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} R(\tau)\exp(-i\omega\tau)d\tau. \tag{3}$$

The normalized power spectral density (henceforth referred to as power spectrum) is a positive function which is normalized to unity [13], i.e.,

$$P(\omega) \geq 0 \tag{4}$$

$$\int_{-\infty}^{+\infty} P(\omega)d\omega = 1. \tag{5}$$

If the process is ergodic, the autocorrelation function, which is defined as in (1), can be estimated as the time average:

$$R(\tau) = \lim_{T\to\infty} R^T(\tau), \tag{6}$$

where

$$R^T(\tau) = \frac{1}{\sigma^2(2T-\tau)} \int_{-T}^{T-\tau} x(t)x(t+\tau)^* dt \tag{7}$$

is an unbiased estimator of $R(\tau)$ as

$$R(\tau) = \boldsymbol{E}(R^T(\tau)). \tag{8}$$

However for T fixed, the approximation $R^T(\tau)$ holds not for every τ but only for $\tau \ll T$. Although $\lim_{T\to\infty} R^T(\tau) \to R(\tau)$ the convergence is not uniform in τ [11]. For τ near T, $R^T(\tau)$ is not a reliable estimator of $R(\tau)$ and the Fourier Transform of $R^T(\tau)$ does not tend to $P(\omega)$ as a consequence of the large variance of $R^T(\tau)$ [6, 11].

The autocorrelation function often in practice can be directly measured through some experiment, for example as in a Michelson interferometer [5] and therefore, the measurements are influenced by noise or imprecisions. The problem we address then may be formulated as follows: *Given a finite set of observed values of the autocorrelation function* $R^o(\tau_i)$; $i = 1, \ldots, N$, *where each value* $R^o(\tau_i)$ *is known to within an error* $\epsilon^o(\tau_i)$, *our purpose is to estimate robustly the power spectrum* $P(\omega)$. We deal only with bandlimited autocorrelation functions for which

$P(\omega) = 0$ for $|\omega| > \omega_C$. The model for the data at lag τ_i will be taken to be

$$R(\tau_i) = \int_{-\omega_C}^{\omega_C} P(\omega) \exp(i\omega\tau_i) d\omega \tag{9}$$

3 Some comments on classical Power Spectrum Estimation

A bandlimited autocorrelation function of bandwidth ω_C, can be uniquely determined in terms of its sample values $R(\frac{n\pi}{\omega_C})$ as [14]

$$R(\tau) = \sum_{n=-\infty}^{+\infty} R(\frac{n\pi}{\omega_C}) \frac{\sin(\omega_C(\tau - \frac{n\pi}{\omega_C}))}{\omega_C(\tau - \frac{n\pi}{\omega_C})}, \tag{10}$$

where $(\frac{\omega_C}{\pi})$ is the Nyquist frequency. Equation (10) enables us to calculate exactly the power spectrum from the Fourier Transform of the discrete data. Indeed, replacing $R(\tau)$ by its equivalent expression (10) in the power spectrum of (3), we have:

$$P(\omega) = \sum_{n=-\infty}^{+\infty} R(\frac{n\pi}{\omega_C}) \int_{-\infty}^{+\infty} \frac{1}{2\pi} \frac{\sin(\omega_C(\tau - \frac{n\pi}{\omega_C}))}{\omega_C(\tau - \frac{n\pi}{\omega_C})} \exp(-i\omega\tau) d\tau. \tag{11}$$

The integral in (11) is the Fourier Transform of the function $\frac{\sin(\omega_C(\tau - \frac{n\pi}{\omega_C}))}{\omega_C(\tau - \frac{n\pi}{\omega_C})}$, which is equal to $\frac{U_{\omega_C}(\omega)}{2\omega_C} \exp(-i\frac{n\pi}{\omega_C}\omega)$, where $U_{\omega_C}(\omega)$ is a function that takes the value one if $|\omega| \leq \omega_C$ or zero otherwise. Thus, the power spectrum can be recast as:

$$P(\omega) = \frac{U_{\omega_C}(\omega)}{2\omega_C} \sum_{n=-\infty}^{+\infty} R(\frac{n\pi}{\omega_C}) \exp(-i\frac{n\pi}{\omega_C}\omega). \tag{12}$$

It is clear that by knowing the exact values of the autocorrelation function sampled at the Nyquist frequency for $n = -\infty, \ldots, \infty$, the power spectrum can be determined exactly. However, in practice we face two basic limitations. First, we can only know and process a finite number of data, and second, these data are noisy.

Let us consider first the finite data problem assuming that we have at our disposal the exact values of $R(\frac{n\pi}{\omega_C})$ for $|\frac{n\pi}{\omega_C}| < T$ or $|n| < N = \lfloor \frac{T\omega_C}{\pi} \rfloor$ (the notation $\lfloor \eta \rfloor$ is chosen to denote the largest integer not exceeding η). In these circumstances, the equation below provides the approximate, correlogram representation $P^N(\omega)$ of $P(\omega)$ i.e.,

$$P^N(\omega) = \frac{U_{\omega_C}(\omega)}{2\omega_C} \sum_{n=-N}^{N} R(\frac{n\pi}{\omega_C}) \exp(-i\frac{n\pi}{\omega_C}\omega). \tag{13}$$

In the limit $N \to \infty$, $P^N(\omega) \to P(\omega)$ and obviously this would also be true in the very special case for which $R(\frac{n\pi}{\omega_C})$ were zero for $|n| > N$. This exceptional case can never be a consequence of a situation such as $R(\tau) = 0$ for $|\tau| > T$ since the autocorrelation function is bandlimited, and is therefore an analytical function in the entire complex plane. Such functions are either identically zero or they have isolated zeros but they cannot vanish in a continuous interval [11]. Thus, in general, for finite data the quantity $P^N(\omega)$ can only be an approximation of $P(\omega)$. In this paper we restrict our consideration to those cases for which the autocorrelation function has fast decay and therefore the truncation error, committed by assuming that $R(\tau) = 0$ for $T > 0$, can be disregarded. However, the purpose of our contribution is to deal with noisy data for which $P^N(\omega)$ above is not a good estimator of $P(\omega)$. In the early work in this area, to reduce the variance of the estimate it has been proposed to deemphasize the contribution of $R(\tau)$ for τ near T through the use of a "window" function $W(\tau)$ [1]. This proposal accepts a smoothed version $P_W(\omega)$ of $P(\omega)$, calculated as:

$$P_W(\omega) = \frac{U_{\omega_C}(\omega)}{2\omega_C} \sum_{n=-N}^{N} R(\frac{n\pi}{\omega_C}) W(\frac{n\pi}{\omega_C}) \exp(-i\frac{n\pi}{\omega_C}\omega), \tag{14}$$

where $W(\tau)$ decreases gradually to zero as $|\tau| \to T$. The window can be selected from several functions defined for such a purpose [1, 13, 12]. However, within the windowing approach, resolution is inevitably sacrificed. In order to reduce the variance of the estimate with significantly improved resolution, we propose to oversample the autocorrelation values at a density larger than the Nyquist. In the next section we discuss the oversampling problem within the context of Frame Theory.

4 The oversampling problem. Power Spectrum Estimation as a tight frame superposition

Frames were introduced by Duffin and Shaffer (1952) [4] and are also reviewed in Young (1980) [15]. More recent results on Frame Theory are given in [3]. Here we only give the frame definition and some of the properties related with our problem.

Definition: A family of functions $\phi_j; j = -\infty, \ldots, \infty$ *in a Hilbert space* H *is called a frame for* H *if there exist* $A > 0$*,* $B < \infty$ *so that, for all* $f \in \mathsf{H}$

$$A\|f\|^2 \leq \sum_{j=-\infty}^{\infty} |\langle f, \phi_j \rangle|^2 \leq B\|f\|^2 \tag{15}$$

where the angle brackets denote inner product and $\|f\|^2$ the square norm of f. The numbers A and B are called frame bounds.

From the definition it is clear that a frame is a complete set of functions since the relations $\langle f, \phi_j \rangle = 0; j = -\infty, \ldots, \infty$ imply that $f \equiv 0$. The removal of an element from a frame leaves either a frame or an incomplete set [15]. A frame that ceases to be complete if an arbitrary function ϕ_j is removed, is called "exact" (note that the last property implies that only exact frames are bases; in the general case the functions ϕ_j are typically not linearly independent).

If the two frame bounds are equal ($A = B$), then the frame is called a "tight frame". In a tight frame, for all $f \in \mathsf{H}$

$$\sum_{j=-\infty}^{\infty} |\langle f, \phi_j \rangle|^2 = A\|f\|^2 \tag{16}$$

which implies that knowing the inner products $\langle f, \phi_j \rangle$, f can easily be reconstructed as [2, 3]

$$f = \frac{1}{A} \sum_{j=-\infty}^{\infty} \langle f, \phi_j \rangle \phi_j. \tag{17}$$

Formula (17) is reminiscent of the expansion of f into an orthogonal basis, but it is important to realize that tight frames are not necessarily orthogonal bases. Only in the cases for which the frame bound $A = 1$ and $\|\phi_j\| = 1\ ; j = -\infty, \ldots, \infty will the set \phi_j$ constitute an orthonormal basis. In most other cases the ϕ_j may not be linearly independent. We now show that the oversampling problem involves a set of functions which form a tight frame, but not a basis. In order to simplify notation we denote:

$$\varphi_n^a(\omega) \;=\; U_{\omega_C}(\omega)\frac{\exp(-ia\frac{n\pi}{\omega_C}\omega)}{\sqrt{2\omega_C}} \tag{18}$$

where $a < 1$. (Note that $\|\varphi_n^a(\omega)\| = 1$).

Proposition: For any function $P(\omega)$ belonging to a Hilbert space of ω_C-band limited functions, i.e., $P(\omega) = 0$ for $|\omega| > \omega_C$, the set $\varphi_n^a(\omega); n = -\infty, \ldots, \infty$ constitutes a tight frame with frame bound $A = a^{-1}$.

This follows from

$$\sum_{n=-\infty}^{\infty} |\langle P, \varphi_n^a\rangle|^2 \;=\; \sum_{n=-\infty}^{\infty} |\int_{-\infty}^{+\infty} P(\omega)U_{\omega_C}(\omega)\frac{\exp(ia\frac{n\pi}{\omega_C}\omega)}{\sqrt{2\omega_C}}d\omega|^2. \tag{19}$$

In fact, since $\{U_{\frac{\omega_C}{a}}(\omega)\frac{\exp(-ia\frac{n\pi}{\omega_C}\omega)}{\sqrt{\frac{2\omega_C}{a}}}\}$ is an othonormal basis in $(\frac{\omega_C}{a}, -\frac{\omega_C}{a})$, we can express $P(\omega)U_{\omega_C}(\omega)$ as:

$$P(\omega)U_{\omega_C}(\omega) = U_{\frac{\omega_C}{a}}(\omega)\sum_{n=-\infty}^{\infty} g_n\frac{\exp(-ia\frac{n\pi}{\omega_C}\omega)}{\sqrt{\frac{2\omega_C}{a}}} \tag{20}$$

with

$$g_n = \int_{-\frac{\omega_C}{a}}^{\frac{\omega_C}{a}} P(\omega)U_{\omega_C}(\omega)\frac{\exp(ia\frac{n\pi}{\omega_C}\omega)}{\sqrt{\frac{2\omega_C}{a}}}d\omega = \int_{-\infty}^{+\infty} P(\omega)U_{\omega_C}(\omega)\frac{\exp(ia\frac{n\pi}{\omega_C}\omega)}{\sqrt{\frac{2\omega_C}{a}}}d\omega. \tag{21}$$

Hence, from (20) and the orthonormality condition we have:

$$\begin{aligned}\int_{-\frac{\omega_C}{a}}^{\frac{\omega_C}{a}} |P(\omega)U_{\omega_C}(\omega)|^2 d\omega \;&=\; \int_{-\omega_C}^{\omega_C} |P(\omega)|^2 d\omega = \sum_{n=-\infty}^{\infty} |g_n|^2 \\ &=\; \sum_{n=-\infty}^{\infty} |\int_{-\infty}^{+\infty} P(\omega)U_{\omega_C}(\omega)\frac{\exp(ia\frac{n\pi}{\omega_C}\omega)}{\sqrt{\frac{2\omega_C}{a}}}d\omega|^2 \end{aligned} \tag{22}$$

and replacing (22) in (19) we obtain:

$$\sum_{n=-\infty}^{\infty} |\langle P, \varphi_n^a\rangle|^2 d\omega = \frac{1}{a}\int_{-\omega_C}^{\omega_C} |P(\omega)|^2 d\omega = a^{-1}\|P\|^2 \tag{23}$$

Using equation (17) and the assumed model (9) we are in a position to reconstruct the power spectrum as

$$P(\omega) = a \sum_{n=-\infty}^{+\infty} \langle P, \varphi_n^a \rangle \varphi_n^a(\omega) = \frac{a}{\sqrt{2\omega_C}} \sum_{n=-\infty}^{+\infty} R(a\frac{n\pi}{\omega_C})\varphi_n^a(\omega). \tag{24}$$

If a were to increase to $a' = ka < 1$, where k is an integer number, then the resulting situation may be considered as equivalent to removing some functions φ_n^a from the original set. Equation (23) therefore still holds with a new frame bound a'^{-1}. This means that the set $\varphi_n^a\ ; a < 1$ no longer constitutes an exact frame, and hence the functions are linearly dependent and thus they do not form a basis. As a consequence, equation (24) gives only one way of reconstructing the power spectrum. In other words, there may exist other sets of coefficients $c_n; n = -\infty, \ldots, \infty$ for which

$$P(\omega) = \sum_{n=-\infty}^{+\infty} c_n \varphi_n^a(\omega). \tag{25}$$

However, it is a property of frames that the superposition formula (24), derived from (17), is the most "economical" in the sense that for any other coefficients for which $c_n \neq a\langle P, \varphi_n^a \rangle$ we have [4, 15]

$$\sum_{n=-\infty}^{+\infty} |c_n|^2 > a^2 \sum_{n=-\infty}^{+\infty} |\langle P, \varphi_n^a \rangle|^2. \tag{26}$$

Thus, the assumed model (9) gives the set of coefficients with minimum norm.

We have seen that, any power spectrum with $P(\omega) = 0$ for $|\omega| > \omega_C$, can be exactly reconstructed from an infinite but discrete set of samples of the autocorrelation function, either as a superposition of orthonormal functions ($a = 1$) or as a superposition in a tight frame ($a < 1$). In the real situation in which only samples for $\tau < T$ are known we have $2M + 1 = \lfloor \frac{2T\omega_C}{a\pi} \rfloor + 1 = \lfloor \frac{2N}{a} \rfloor + 1$ data and the approximate representation of the power spectrum in terms of $2M + 1$ frame elements is

$$P^M(\omega) = \frac{a}{\sqrt{2\omega_C}} \sum_{n=-M}^{M} R(a\frac{n\pi}{\omega_C})\varphi_n^a(\omega). \tag{27}$$

Since in this paper we focus our attention on the cases for which the truncation error can be disregarded, in comparison to the error due to the noise of the data, the frame decomposition (27) becomes much more appropriate.

The frame property of reducing errors is pointed out in [2, 3] and it is extensively studied in [10] for tight Weyl-Heisenber frames, where it is showed that the noise contribution to the reconstruction of a function decreases with the oversampling parameter. In the next section we exemplify this fact by numerical simulations. Moreover, we indicate the gain in resolution which a tight frame reconstruction represents in relation to the windowing approach. The results are also compared to those obtained from AR models which reproduce the exact power spectrum only for noiseless autocorrelation values.

5 Simulations

The particular error model to be chosen is the standard variance of the estimator (7). Such a variance is exactly derivable for a normal process, but it is known to be a good approximation for other processes as well, and the form is given by [6, 11]

$$\sigma^2_{R^T} = \frac{1}{2T-\tau}\int_{2T+\tau}^{2T-\tau}(R(\tau+\alpha)R(\tau-\alpha)+R^2(\alpha))(1-\frac{|\alpha|}{2T-\tau})d\alpha \tag{28}$$

The estimator $R^T(\tau)$ is unbiased, and hence the errors are assumed to be zero mean and will then be simulated as:

$$R^o(\tau_n) = R(\tau_n) + \epsilon^o(\tau_n) \quad ; n = 0, \ldots, M \tag{29}$$

where $\epsilon(\tau_n)$ is a random variable generated by a Gaussian distribution whose variance $\sigma^2(\tau_n) = \sigma^2_{R^T}(\tau_n)$ is given by (28). With $\tau_M = T$, the T-interval is discretized as: $\tau_n = a\frac{n\pi}{\omega_C}$; $n = 0, \ldots, M = \lfloor \frac{T\omega_C}{a\pi} \rfloor$. For all the examples we shall set $\omega_C = \pi$ and $a = 0.025$ in the oversampling case or $a = 1$ for the Nyquist density.

The continuous curve in Fig. 1a, 1b, 1c, and 1d is the exact power spectrum we wish to obtain and that can be estimated with high accuracy, from the noiseless data, by all the approaches that we compare for the case of noisy data.

Since in the simulations given here the autocorrelation values are real, the power spectrum is an even function and thus we have plotted it in the $(0, \omega_C)$ domain only. The squares, triangles and diamonds of Fig 1a represent the classic correlogram estimations for three different sets of noisy data sampled at the Nyquist density in the time domain $T = 13$. (For the sake of clarity in Fig 1a, 1b, 1c, 1d and 1e the corresponding results are given for only three set of data). The squares, triangles and diamonds of Fig 1b represent the estimations that are obtained from the same data as in Fig 1a but multipling them by the Tukey-Hanning window [1, 13]. The results of Fig 1d are obtained by using Papoulis window [12] . The dot-lines of Fig 1d show the frame reconstructions for six different sets of oversampled data. By comparing Fig 1d with 1a, 1b, and 1c, it can be seen clearly that oversampling leads to a significant reduction of noise effect in comparison not only with the correlogram estimation but also with windowing approaches. The squares, triangles and diamonds of Fig 1e represent the estimations obtained from the same data as in Fig 1a, 1b and 1c but from an AR model of order 13. As it can be seen in Fig 1e, the introduction of noise renders this model, as expected, totally inadequate.

The figures 2a, 2b, 2c, 2d and 2e are as above but for $T = 12$.

6 Conclusions

The problem of estimation of the power spectrum from uncertain autocorrelation values has been addressed. The cases that have been considered are those in which, due to the unreliability of the data, the classic correlogram cannot be applied over a single set of data. In those cases, oversampling the time domain of the data has been proposed as a way of reducing noise effects. The price to be paid in order to be able to determine the power spectrum using a single realization of each measurement is to process more samples than the minimum that are needed if the data are noiseless.

The oversampling problem has been discussed from the frame Theory point of view. Reconstruction of the power spectrum as a superposition in a tight frame is in fact a trivial case within the general Frame Theory. It is very easy and fast to compute through the use of the Fast Fourier Transform. However, from Frame Theory it is possible to obtain a deep understanding of the concepts involved when Fast Fourier Transform is applied to estimate the power spectrum from oversampling of data.

Simulations support the view that oversampling of the data can produce important improvement in the spectral estimation over other standard approaches.

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Figure Captions

Figure 1a: Power spectrum vs. angular frequency ω. The continuous curve represents the true power spectrum $P(\omega)$. The squares, triangles and diamonds are correlogram estimations for three different set of data sampling the $T = 13$ interval at the Nyquist density.

Figure 1b: Power spectrum vs. angular frequency ω. The continuous curve represents the true power spectrum $P(\omega)$. The squares, triangles and diamonds are the estimations obtained from the same data as in Fig 1.a but using Tukey-Hanning window.

Figure 1c: Same description as in Fig 1b, but using Papoulis window

Figure 1d: Power spectrum vs. angular frequency ω. The continuous curve represents the true power spectrum $P(\omega)$. The dots-lines the frame reconstructions for six different set of data oversampling the $T = 13$ interval.

Figure 1e: Power spectrum vs. angular frequency ω. The continuous curve represents the true power spectrum $P(\omega)$. The squares, triangles and diamonds are the the estimations obtained from an AR model of order 13.

Figure 2a: Same details as in Fig.1.a, but for a $T = 12$ data domain.

Figure 2b: Same details as in Fig.1.b, but for a $T = 12$ data domain.

Figure 2c: Same details as in Fig.1.c, but for a $T = 12$ data domain.

Figure 2d: Same details as in Fig.1.d, but for a $T = 12$ data domain.

Figure 2e: Same details as in Fig.1.e, but for a $T = 12$ data domain, and with an AR model of order 12.

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