A Linear Program to Compare Path-Complete Lyapunov Functions

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***Abstract*— We provide an algorithmic procedure allowing to compare stability certificates for discrete-time switching systems and in specific Path-Complete Lyapunov functions (PCLFs). These mathematical objects consist of a set of positive definite functions and a set of Lyapunov inequalities, encoded in a directed, labeled graph. Given two such graphs, we formulate necessary and sufficient conditions to decide if the existence of a PCLF for the first graph implies existence of a PCLF for the second graph, where the corresponding set of functions is constructed by conic combinations of the set of functions related to the first PCLF. The conditions depend only on the topologies of the two graphs and can be verified by solving a linear program. It is the first systematic approach to compare the conservativeness of PCLFs.**

I. Introduction and Preliminaries

Discrete-time switching systems [1]–[4] present major theoretical challenges [5], provide an accurate modeling framework for many processes [6]–[9] and are good approximations of complex hybrid dynamical systems [10]. We consider switching systems of the form

$$x(t+1) = f_{\sigma(t)}(x(t)), \tag{1}$$

where the state $x(t)$ evolves in $\mathbb{R}^n$. The switching signal $\sigma(\cdot) : \mathbb{N} \to \{1, ..., M\}$ assigns at each time instant one of $M \geq 1$ modes, each associated with a continuous map $f_i : \mathbb{R}^n \to \mathbb{R}^n$, $1 \leq i \leq M$, such that $f_i(x) = 0 \Leftrightarrow x = 0$. We assume for simplicity that the switching signal is arbitrary, however, all results extend to a wider class of switching signals, such as the ones considered in [11].

We consider the stability analysis problem, focusing on global uniform stability, see e.g. [12, Definition 1] for a standard definition. Although verifying stability is undecidable even for linear dynamics [5], the problem has been studied extensively due to its importance in control [4]. A standard approach to address the problem is to search for a Lyapunov function [13]. A well known stability certificate concerns the existence of a *common quadratic Lyapunov function* [4, Section II-A].

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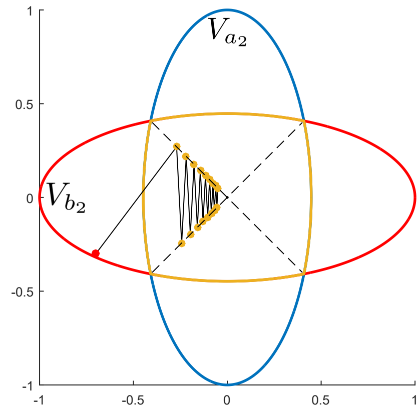


Fig. 1: Geometric representation of the path-complete stability criterion using the graph G_2 in Example 1. A trajectory for the system of Example 1, with the red point as initial condition and with the switching sequence $212121\cdots$, is presented. One can show that the intersection (in yellow) of the level sets of the functions V_{a_2} and V_{b_2} (resp. in blue and red) is the level set of a common Lyapunov function for that system [12].

More complex however less conservative criteria exist involving, e.g., sum-of-squares polynomials [14], max-of-quadratics [15] or polytopic Lyapunov functions [16]. *Multiple Lyapunov functions* [2], [17]–[19], that are composed of several pieces that together form a stability certificate are also an attractive alternative. Additionally, there are converse results, see e.g., [20], that induce *semi–algorithms*, using hierarchies of less and less conservative classes of Lyapunov functions [21], or families of multiple Lyapunov functions [11], [19], [22]–[24], that are guaranteed to eventually provide a stability certificate when a system is stable.

The main reason for the existence of so many different tools is that they provide stability certificates that are only sufficient, and the converse results induce algorithmic procedures that are non-conservative only asymptotically. In view of this, it is crucial to understand which performances can *a priori* be expected from a given criterion.

In an effort to unify and generalize many of the existing techniques for discrete-time switching systems, the framework of Path-Complete Lyapunov functions was recently introduced in [25]. A Path-Complete Lyapunov function (PCLF) is a type of multiple Lyapunov function that boils down to two objects: one is a finite set of functions, called the *pieces* of the PCLF, and the other one is a directed graph that encodes Lyapunov inequalities between these pieces. We define such a graph as $\mathbf{G} = (S, E)$, where S is the set of nodes of the graph and $E \subseteq S \times S \times \{1, \ldots, M\}$ is a set of directed edges, each one being labeled by one of the modes of the system (1). In order to form a valid stability certificate under arbitrary switching, it has been shown [25], [26] that the graph $\mathbf{G}$ needs to be *path-complete*:

*Definition 1 (**Path-completeness**):* A graph $\mathbf{G} = (S, E)$ is *path-complete* if for any $k \geq 1$ and any sequence $\sigma = \sigma_1 \ldots, \sigma_k$, $\sigma_i \in \{1, \ldots, M\}$, there is a *path* in the graph, $(s_i, s_{i+1}, \sigma'_i)_{i=1,2,\ldots,}$, with $(s_i, s_{i+1}, \sigma'_i) \in E$, such that the sequence σ is contained in the sequence $\sigma' = (\sigma'_i)_{i=1,2,\ldots}$.

Unless stated otherwise, the graphs in this paper are considered path-complete. For a PCLF on a graph $\mathbf{G} = (S, E)$, the pieces are members of a set of functions $\{V_s\}_{s \in S}$. Each element V_s of the set is associated to a node of the graph $\mathbf{G}$ and is a Lyapunov Function Candidate, see e.g., [17], a class of functions defined below.

*Definition 2 (**LFC**):* A *Lyapunov Function Candidate* (LFC) $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a continuous function for which there exist two functions α, β, of class $\mathcal{K}_\infty$ [1] satisfying

$$\forall x \in \mathbb{R}^n : \alpha(\|x\|) \leq V(x) \leq \beta(\|x\|), \tag{2}$$

where $\|x\|$ denotes the Euclidean norm of $x \in \mathbb{R}^n$.

The graph of a PCLF encodes Lyapunov inequalities between its pieces:

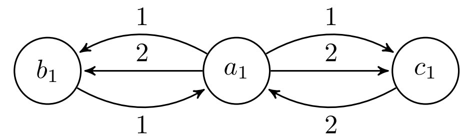
*Definition 3 (**PCLF**):* Consider the system (1) with dynamics $\{f_\sigma\}_{\sigma \in \{1,\ldots,M\}}$. The path-complete graph $\mathbf{G} = (S, E)$, and the set of LFCs $\{V_s\}_{s \in S}$ induce a *Path-Complete Lyapunov Function* if

$$\forall (s, d, \sigma) \in E, \forall x \in \mathbb{R}^n : V_d(f_\sigma(x)) \leq V_s(x). \tag{3}$$

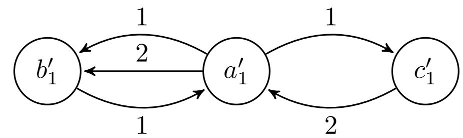
In that case, we write that the property $\mathrm{pclf}(\mathbf{G}, \{V_s\}_{s \in S}, \{f_\sigma\}_{\sigma \in \{1,\ldots,M\}})$ holds.

Theorem 1.1 ([25], [26]): Consider the system (1), a graph $\mathbf{G} = (S, E)$ with M labels, and a set of LFCs $\{V_s\}_{s \in S}$. Then, the satisfaction of the inequalities (3) is a sufficient condition for the stability of the system if and only if $\mathbf{G}$ is path-complete.

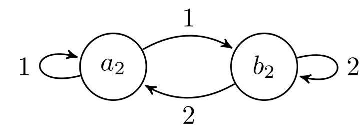
Example 1: Consider the graphs $\mathbf{G}_1$, $\mathbf{G}'_1$ and $\mathbf{G}_2$ in Figures 2a, 2b and 2c respectively. These graphs have two labels, namely 1 and 2, on their edges, corresponding to switching systems on two modes. The graph $\mathbf{G}'_1$ is not path-complete, but the others are. The graph $\mathbf{G}_1$ encodes six Lyapunov inequalities (one per edge) between three Lyapunov Function candidates (one per node), which we denote by $\{V_{a_1}, V_{b_1}, V_{c_1}\}$, as shown in Figure 2a. For example, since $(a_1, b_1, 1) \in E$, then $\forall x \in \mathbb{R}^n$, $V_{b_1}(f_1(x)) \leq V_{a_1}(x)$. We consider the following



(a) $\mathbf{G}_1$: it is path-complete.



(b) $\mathbf{G}'_1$: it is not path-complete, since the sequence 222 cannot be formed with a path in the graph.



(c) $\mathbf{G}_2$: it is path-complete.

Fig. 2: Graphs for Example 1.

linear switching system consisting of $M = 2$ modes: $x_{t+1} = f_{\sigma(t)}(x_t) = A_{\sigma(t)} x_t$, $\sigma(t) \in \{1, 2\}$, with

$$A_1 = \alpha \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } A_2 = \alpha \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \alpha = 0.9. \tag{4}$$

This choice is inspired from [25, Example 5.2]. For our choice of α, no common quadratic Lyapunov functions exists [25, Example 5.2]. Furthermore, we verify numerically that we cannot find a set of quadratic pieces satisfying the inequalities of $\mathbf{G}_1$[2]. However we can verify that all the 4 inequalities of the graph $\mathbf{G}_2$ are satisfied for the pieces

$$\left\{ \begin{array}{l} V_{a_2}\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = 5x_1^2 + x_2^2, \\ V_{b_2}\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = x_1^2 + 5x_2^2 \end{array} \right\}. \tag{5}$$

They are illustrated in Figure 1 along with a trajectory of the system with initial condition $x(0) = (-0.7, -0.3)^\top$ and with a periodic switching sequence $2121\cdots$. Since $\mathbf{G}_2$ is path complete, this provides us with a proof of stability for our linear switching system from Theorem 1.1.

As shown in Example 1, and reported in previous works, e.g. [25, Section 4], [12, Section 4], the conservativeness of a Path-Complete Lyapunov function depends on the choice of the graph. Our goal is to provide a better understanding of when, for two given graphs $\mathbf{G}_1 = (S^1, E^1)$, $\mathbf{G}_2 = (S^2, E^2)$, and for arbitrary dynamics $f := \{f_\sigma\}_{\sigma \in [M]}$, the existence of pieces $\{V_s\}_{s \in S^1}$ satisfying $\mathrm{pclf}(\mathbf{G}_1, \{V_s\}_{s \in S^1}, f)$ implies that of pieces $\{U_r\}_{r \in S^2}$ such that $\mathrm{pclf}(\mathbf{G}_2, \{U_r\}_{r \in S^2}, f)$ holds true as well. In

[1] A function $\alpha(z) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}_\infty$ if it is continuous, radially unbounded, strictly increasing, with $\alpha(0) = 0$.

[2] The codes for reproducing Examples 1, 2, 3 and 4 are available at `sites.uclouvain.be/scsse/cdc2017-codesExamples.zip`

short, we aim at understanding when we can certify that $\mathbf{G}_2$ provides a less conservative criterion than $\mathbf{G}_1$.

Most of the works on path complete Lyapunov functions, namely [11], [19], [22], [25, Section 4], [12, Section 4], focus on linear dynamics and quadratic pieces. Therein, all comparisons between PCLFs rely on showing that we can construct the pieces of one PCLF as conic combinations of the pieces that form another PCLF, and in some cases their compositions with the system dynamics [25, Proposition 4.2]. Motivated by this observation, we explore the comparison between graphs in the setting described above. In specific, our main contribution is a necessary and sufficient condition, verifiable by linear programming, that considers two graphs $\mathbf{G}_1$ and $\mathbf{G}_2$, and allows us to decide when one can form a PCLF for a graph $\mathbf{G}_2$ with pieces that are constructed as conic combinations of the pieces that form a PCLF for $\mathbf{G}_1$ (whenever these pieces exist). Our condition does not require any assumption on the dynamics or the parametrization of the pieces of the PCLF.

Structure: In Section II, we introduce and illustrate the property we wish to capture. In Section III, we present the developments towards our main result, Theorem 3.1, while Section IV concludes our work.

Notations: Given a matrix $A \in \mathbb{R}^{m\times n}$, we let $(A)_{k,\ell}$ be the element on the kth row and ℓth column of A. The transpose of A is written $A^\top$. For two matrices $A, B \in \mathbb{R}^{m\times n}$, $A \leq B$ holds componentwise. We denote the matrices with all elements equal to zero and one with $\mathbf{0}$ and $\mathbf{1}$ respectively. For any integer K, we let $[K] = \{1,\dots,K\}$. For a finite set Z, we let $|Z|$ denote the cardinality of the set. Finally, we implicitly associate to each finite discrete set Z an ordering of its element through a bijection $k_Z : Z \to \{1,\dots,|Z|\}$. Using these orderings, given two sets Z_1 and Z_2 and a matrix $A \in \mathbb{R}^{|Z_1|\times|Z_2|}$, for any $z_1 \in Z_1$ and $z_2 \in Z_2$, we use the shortcut notation $(A)_{z_1,z_2}$ to refer to the element $(A)_{k_{Z_1}(z_1),k_{Z_2}(z_2)}$. Given a system (1), we refer to the dynamics as a set of maps $f = \{f_\sigma\}_{\sigma\in[M]}$.

II. Comparing Graphs via Conic Combinations

Let us start with an example.

Example 2: In Example 1, for the choice of linear switching system (4) (with $\alpha = 0.9$) we conclude that there is no PCLF with quadratic pieces for the graph $\mathbf{G}_1$, but there is one for the graph $\mathbf{G}_2$. It turns out that it cannot be the opposite. In fact, in [25] it is shown that if $\{V_{a_1}, V_{b_1}, V_{c_1}\}$ together with $\mathbf{G}_1$ induce a PCLF, then the functions

$$V_{a_2} = V_{a_1} + V_{b_1} \text{ and } V_{b_2} = V_{a_1} + V_{c_1} \tag{6}$$

satisfy the inequalities of the graph $\mathbf{G}_2$. If the functions in the first set are quadratics, since those of the second set are expressed as *conic combinations* of those of the first, they are quadratic as well. To illustrate this graphically, we consider the parametrized system of Example 1

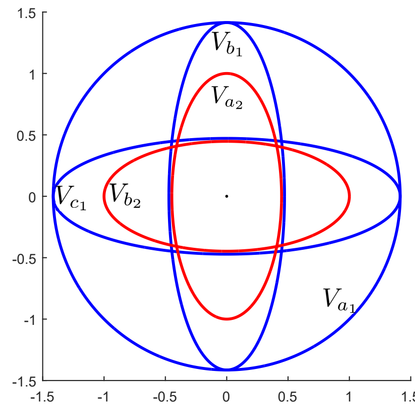


Fig. 3: Example 3, the level sets of the functions defined in both (5) (in red) and (7) (in blue). These functions are valid pieces for the PCLF for the graph $\mathbf{G}_2$ and $\mathbf{G}_1$ respectively allowing to prove stability of the linear system defined through (4) with $\alpha = 0.3$.

setting $\alpha = 0.3$. We can verify that there is a PCLF with quadratic pieces for $\mathbf{G}_1$, with

$$\left\{\begin{aligned} V_{a_1}\left(\begin{pmatrix}x_1\\x_2\end{pmatrix}\right) &= \frac{1}{2}(x_1^2 + x_2^2),\\ V_{b_1}\left(\begin{pmatrix}x_1\\x_2\end{pmatrix}\right) &= \frac{1}{2}(9x_1^2 + x_2^2),\\ V_{c_1}\left(\begin{pmatrix}x_1\\x_2\end{pmatrix}\right) &= \frac{1}{2}(x_1^2 + 9x_2^2)\end{aligned}\right\}, \tag{7}$$

and that the pieces (5) continue to form a valid PCLF for the graph $\mathbf{G}_2$. Remark that if we combine the functions in (7) according to (6), we obtain the set of functions (5). This is represented graphically in Figure 3.

We introduce notations allowing to represent the set of Lyapunov function candidates of a graph in a vector. In this way, the subsequent algebraic manipulations become easier by allowing to express (3) with vector inequalities.

*Definition 4 (**VLFC**):* A *Vector Lyapunov Function Candidate (VLFC)* is a vector function $\mathbf{V} : \mathbb{R}^n \to \mathbb{R}^N_{\geq 0}$, where each element $(\mathbf{V})_i : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $i \in [N]$, is a Lyapunov Function candidate.

Given a graph $\mathbf{G} = (S, E)$ with a set of labels $[M]$, and $\sigma \in [M]$, we define the two matrices $\mathbf{S}_\sigma(\mathbf{G}) \in \{0,1\}^{|E_\sigma|\times|S|}$ and $\mathbf{D}_\sigma(\mathbf{G}) \in \{0,1\}^{|E_\sigma|\times|S|}$ as follows:

$$\begin{aligned}(\mathbf{S}_\sigma)_{e,s} = 1 &\Leftrightarrow \exists d \in S : e = (s,d,\sigma) \in E,\\ (\mathbf{D}_\sigma)_{e,d} = 1 &\Leftrightarrow \exists s \in S : e = (s,d,\sigma) \in E,\end{aligned} \tag{8}$$

where $E_\sigma \subset E$ corresponds to the edges with label σ[3].

Example 3: We construct the matrices $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{D}_1$, $\mathbf{D}_2$ for the graphs $\mathbf{G}_1$ and $\mathbf{G}_2$ of Figures 2a and 2c. We start from $\mathbf{G}_1$ and let a_1, b_1 and c_1 be the first, second and third node of the graph respectively. The edges

[3]For a graph $\mathbf{G}$ and label σ, the matrix $\mathbf{D}_\sigma(\mathbf{G}) - \mathbf{S}_\sigma(\mathbf{G})$ recovers the incidence matrix [27] of the subgraph of $\mathbf{G}$ where we keep only the edges with label σ.

are ordered counter-clockwise starting from the edge $(a_1, b_1, 1)$. With these conventions, we have

$$\mathbf{S}_1(\mathbf{G}_1) = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}, \mathbf{D}_1(\mathbf{G}_1) = \begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}. \quad (9)$$

The edges for mode 2 are also ordered counter-clockwise starting from the edge $(a_1, b_1, 2)$, leading to

$$\mathbf{S}_2(\mathbf{G}_1) = \begin{pmatrix}1 & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}, \mathbf{D}_2(\mathbf{G}_1) = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}. \quad (10)$$

Using similar ordering conventions for $\mathbf{G}_2$, we have take a_2 as the first node and b_2 as the second. For mode 1, the first edge (corresponding to the first row) is $(a_2, b_2, 1)$ and the second $(a_2, a_2, 1)$ leading to

$$\mathbf{S}_1(\mathbf{G}_2) = \begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}, \mathbf{D}_1(\mathbf{G}_2) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \quad (11)$$

For mode 2, the first edge is $(b_2, a_2, 2)$ and the second edge is $(b_2, b_2, 2)$, leading to

$$\mathbf{S}_2(\mathbf{G}_2) = \begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}, \mathbf{D}_2(\mathbf{G}_2) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \quad (12)$$

Proposition 2.1 restates the Lyapunov decrease conditions (3) in a vector form which will be convenient in the sequel and it is presented without a proof.

Proposition 2.1: Given a graph $\mathbf{G} = (S, E)$, dynamics $f = \{f_\sigma\}_{\sigma\in[M]}$ and a set of pieces $\{V_s\}_{s\in S}$, $\mathrm{pclf}(\mathbf{G}, \{V_s\}_{s\in S}, f)$ holds if and only if

$$\forall x \in \mathbb{R}^n,\ \forall \sigma \in [M],\ \mathbf{D}_\sigma(\mathbf{G})\mathbf{V}(f_\sigma(x)) \le \mathbf{S}_\sigma(\mathbf{G})\mathbf{V}(x),$$

where $\mathbf{V} : \mathbb{R}^n \to \mathbb{R}^{|S|}_{\ge 0}$ is the VLFC with $(\mathbf{V})_s = V_s$.

*Definition 5 (**Conic comparison**):* Consider two graphs $\mathbf{G}_1 = (S^1, E^1)$ and $\mathbf{G}_2 = (S^2, E^2)$, with a set of labels $[M]$ and a *conic combination matrix*

$$C \in \mathbb{R}^{|S^2|\times|S^1|}_{\ge 0},\ \forall s_2 \in S^2 : \sum_{s_1} (C)_{s_2,s_1} \ge 1. \quad (13)$$

We write $\mathbf{G}_1 \le_C \mathbf{G}_2$ if for any dimension $n \in \mathbb{N}$, for any choice of dynamics $\{f_\sigma\}_{\sigma\in[M]}$, $f_\sigma : \mathbb{R}^n\ \mathbb{R}^n$, for any choice of VLFCs $\mathbf{V} : \mathbb{R}^n \to \mathbb{R}^{|S^1|}$, any point $x \in \mathbb{R}^n$, and any $\sigma \in [M]$, the following implication holds

$$\begin{aligned}&\mathbf{S}_\sigma(\mathbf{G}_1)\mathbf{V}(x) - \mathbf{D}_\sigma(\mathbf{G}_1)\mathbf{V}(f_\sigma(x)) \ge \mathbf{0}\\ &\Rightarrow \mathbf{S}_\sigma(\mathbf{G}_2)C\mathbf{V}(x) - \mathbf{D}_\sigma(\mathbf{G}_2)C\mathbf{V}(f_\sigma(x)) \ge \mathbf{0}.\end{aligned} \quad (14)$$

The following result shows that conic comparisons indeed allow to compare conservativeness of Path-Complete Lyapunov functions.

Theorem 2.2: Consider two graphs $\mathbf{G}_1 = (S^1, E^1)$, $\mathbf{G}_2 = (S^2, E^2)$ and a matrix $C \in \mathbb{R}^{|S^2|\times|S^1|}_{\ge 0}$ satisfying (13). The following statements are equivalent.

(i): $\mathbf{G}_1 \le_C \mathbf{G}_2$.

(ii): For any integer n, any set of dynamics $f = \{f_\sigma\}_{\sigma\in[M]}$, $f_\sigma : \mathbb{R}^n \to \mathbb{R}^n$ in any dimension n, and any choice of LFCs $\{V_s\}_{s\in S^1}$,

$$\mathrm{pclf}(\mathbf{G}_1, \{V_r\}_{r\in S^1}, f) \Rightarrow \mathrm{pclf}(\mathbf{G}_2, \{U_s\}_{s\in S^2}, f) \quad (15)$$

where for any $s \in S^2$, $U_s := \sum_{r\in S^1} (C)_{s,r} V_r$.

Remark 1: We point out that the difference between the two statements of Theorem 2.2 is significant, yet subtle. When we write $\mathbf{G}_1 \le_C \mathbf{G}_2$, the implication (14) holds *pointwise*, i.e., if a point $x \in \mathbb{R}^n$ satisfies the left hand side of the implication, it also satisfies the right hand side. On the other hand, (ii) is truly the property that we need to capture in order to compare PCLFs, namely that for all $n \in \mathbb{N}$, for all choices of dynamics $f = \{f_\sigma\}_{\sigma\in[M]}$, $f_\sigma : \mathbb{R}^n \to \mathbb{R}^n$, for all VLFCs $\mathbf{V} : \mathbb{R}^n \to \mathbb{R}^{|S^1|}_{\ge 0}$,

$$\begin{aligned}&\forall x \in \mathbb{R}^n, \mathbf{S}_\sigma(\mathbf{G}_1)\mathbf{V}(x) - \mathbf{D}_\sigma(\mathbf{G}_1)\mathbf{V}(f_\sigma(x)) \ge \mathbf{0}\\ &\Rightarrow \forall x \in \mathbb{R}^n, \mathbf{S}_\sigma(\mathbf{G}_2)C\mathbf{V}(x) - \mathbf{D}_\sigma(\mathbf{G}_2)C\mathbf{V}(f_\sigma(x)) \ge \mathbf{0}.\end{aligned}$$

Summarizing, Theorem 2.2 shows that the concept of conic comparison, which is much easier to handle algebraically, is equivalent to the notion of comparison by conic combination that we wish to capture.

III. An LP-formulation for Conic Comparisons

In this section we prove our main result, which shows that given two graphs $\mathbf{G}_1$ and $\mathbf{G}_2$ we can decide efficiently whether a conic comparison in the sense of Definition 5 is possible. In specific, we establish that if a conic combination matrix C exists, it can be obtained as a solution to a linear program.

Theorem 3.1: Consider two graphs $\mathbf{G}_1 = (S^1, E^1)$ and $\mathbf{G}_2 = (S^2, E^2)$ with the same set of labels $[M]$. There is a matrix $C \in \mathbb{R}^{|S^2|\times|S^1|}$ satisfying (13) such that $\mathbf{G}_1 \le_C \mathbf{G}_2$ if and only if there are M nonnegative matrices $K_\sigma \in \mathbb{R}^{|E^2|\times|E^1|}_{\ge 0}$, $\sigma \in [M]$, such that

$$\begin{aligned}\forall \sigma \in [M], \mathbf{S}_\sigma(\mathbf{G}_2)C &\ge K_\sigma \mathbf{S}_\sigma(\mathbf{G}_1),\\ \mathbf{D}_\sigma(\mathbf{G}_2)C &\le K_\sigma \mathbf{D}_\sigma(\mathbf{G}_1),\end{aligned} \quad (16)$$

with $\mathbf{S}_\sigma$ and $\mathbf{D}_\sigma$ defined in (8).
The proof of the theorem is presented after two intermediate results, Lemmas 3.2 and 3.3.

Example 4: Consider again the graphs $\mathbf{G}_1$ and $\mathbf{G}_2$ from Example 1. In Example 2, we showed that if $\mathbf{V} = \begin{pmatrix}V_{a_1} & V_{b_1} & V_{c_1}\end{pmatrix}^\top$ satisfied to the inequalities of $\mathbf{G}_1$, then

$$\mathbf{U} = \begin{pmatrix}U_{a_2}\\ U_{b_2}\end{pmatrix} = C\mathbf{V}, \quad C := \begin{pmatrix}1 & 1 & 0\\ 1 & 0 & 1\end{pmatrix}, \quad (17)$$

satisfy the inequalities of $\mathbf{G}_1$. Hence, $\mathbf{G}_1 \le_C \mathbf{G}_2$, for the matrix C defined in (17). Considering the matrices $\mathbf{S}$ and $\mathbf{D}$ defined in Example 3 for these graphs, for that matrix C, the inequalities (16) are satisfied (with equality) with

$$K_1 = \begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 1\end{pmatrix}, \quad K_2 = \begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 1\end{pmatrix}.$$

In order to further ease the algebraic manipulations of our inequalities, we express the Lyapunov decrease conditions in matrix form

$$\begin{pmatrix}\mathbf{S}_1(\mathbf{G}) & -\mathbf{D}_1(\mathbf{G}) & \mathbf{0}\\ \mathbf{S}_2(\mathbf{G}) & \mathbf{0} & -\mathbf{D}_2(\mathbf{G})\end{pmatrix}\begin{pmatrix}\mathbf{V}(x)\\ \mathbf{V}(f_1(x))\\ \mathbf{V}(f_2(x))\end{pmatrix} \ge \mathbf{0},$$

where $\mathbf{S}_i, \mathbf{D}_i$ are provided in (8). The above inequalities are non-linear in x, V and the dynamics $f = \{f_\sigma\}_{\sigma\in[M]}$. Nevertheless, they are linear with respect to the vector $\begin{pmatrix}\mathbf{V}(x)^\top & \mathbf{V}(f_1(x))^\top & \mathbf{V}(f_2(x))^\top\end{pmatrix}^\top$. This motivates us to study the set of all such vectors.

To this purpose, consider an integer $n \geq 1$, a VLFC $\mathbf{V} : \mathbb{R}^n \to \mathbb{R}^N_{\geq 0}$, a set of M maps $f = \{f_\sigma\}_{\sigma\in[M]}$, $f_\sigma : \mathbb{R}^n \to \mathbb{R}^n$, and a vector $x \in \mathbb{R}^n$. We define the vector $y(x, f, \mathbf{V}) \in \mathbb{R}^{(M+1)N}$,

$$y(x, f, \mathbf{V}) := \begin{pmatrix} y^0 \\ y^1 \\ \vdots \\ y^M \end{pmatrix} = \begin{pmatrix} \mathbf{V}(x) \\ \mathbf{V}(f_1(x)) \\ \vdots \\ \mathbf{V}(f_M(x)) \end{pmatrix}. \tag{18}$$

Additionally, we let

$$\mathbf{Y}_{n,M,N} = \begin{Bmatrix} y(x, \{f_\sigma\}_{\sigma\in[M]}, \mathbf{V}) : x \in \mathbb{R}^n, \\ f_\sigma : \mathbb{R}^n \to \mathbb{R}^n, \mathbf{V} \text{ is a VLFC.} \end{Bmatrix}, \tag{19}$$

be the set of all such vectors for a fixed dimension n, and finally, we let

$$\mathcal{Y}_{M,N} = \bigcup_{n=1}^{\infty} \mathbf{Y}_{n,M,N}. \tag{20}$$

Notice that in the definition of $\mathcal{Y}_{M,N}$, we no longer explicitly take into account the dynamics of the system (1), its dimensions, and the nature of VCLF. The only remaining elements M and N actually correspond to the number of modes, or labels, to a number of nodes in a graph.

Lemma 3.2: For any $M \geq 1$, $N \geq 1$, it holds that

$$\mathbb{R}^{(M+1)N}_{\geq 0} \supset \mathcal{Y}_{M,N} \supset \mathbb{R}^{(M+1)N}_{>0}.$$

We are now in position to characterize the relation between graphs of Definition 5 without explicitly involving dynamics or Lyapunov functions.

Lemma 3.3: Consider two graphs $\mathbf{G}_1 = (S^1, E^1)$ and $\mathbf{G}_2 = (S^2, E^2)$ with labels $\sigma \in [M]$. There is a matrix $C \in \mathbb{R}^{|S^2|\times|S^1|}_{\geq 0}$ satisfying (13) such that $\mathbf{G}_1 \leq_C \mathbf{G}_2$ if and only if *for all nonnegative* vector $y = \begin{pmatrix}(y^0)^\top & (y^1)^\top & \dots & (y^M)^\top\end{pmatrix}^\top \in \mathbb{R}^{(M+1)|S^1|}_{\geq 0}$ where $y^i \in \mathbb{R}^{|S^1|}_{\geq 0}$, $0 \leq i \leq M$, and for all $\sigma \in [M]$, it holds that

$$\begin{aligned} &\mathbf{S}_\sigma(\mathbf{G}_1)y^0 - \mathbf{D}_\sigma(\mathbf{G}_1)y^\sigma \geq \mathbf{0} \\ &\Rightarrow \mathbf{S}_\sigma(\mathbf{G}_2)Cy^0 - \mathbf{D}_\sigma(\mathbf{G}_2)Cy^\sigma \geq \mathbf{0}. \end{aligned} \tag{21}$$

Lemma 3.3 shows that the conic comparison formulated in Definition 5 is equivalent to verifying a set inclusion between two polyhedral sets. The remainder of the proof of Theorem 3.1 is based on an extended version of Farkas' Lemma, see e.g. [28, Lemma II.2], that transforms this geometric characterization into an algebraic relation.

Lemma 3.4 ([28]): Consider two matrices $A \in \mathbb{R}^{p\times n}$ and $B \in \mathbb{R}^{q\times n}$. The following are equivalent:

$$\begin{aligned} &(\{y \in \mathbb{R}^n : Ay \geq \mathbf{0},\, y \geq \mathbf{0}\} \subseteq \{y \in \mathbb{R}^n : By \geq \mathbf{0}\}) \\ &\Leftrightarrow \exists K \in \mathbb{R}^{m\times p} : KA \leq B,\, K \geq \mathbf{0}. \end{aligned}$$

We are now in position to prove Theorem 3.1.

Proof: [Theorem 3.1] *Sufficiency:* Given a system (1) in dimension n, assume that, for a vector $x \in \mathbb{R}^n$ and a VLFC $\mathbf{V}$, it holds that for all $\sigma \in [M]$

$$\mathbf{D}_\sigma(\mathbf{G}_1)\mathbf{V}(f_\sigma(x)) \leq \mathbf{S}_\sigma(\mathbf{G}_1)\mathbf{V}(x).$$

Since K_σ is nonnegative, it holds

$$\forall\sigma \in [M], K_\sigma\mathbf{D}_\sigma(\mathbf{G}_1)\mathbf{V}(f_\sigma(x)) \leq K_\sigma\mathbf{S}_\sigma(\mathbf{G}_1)\mathbf{V}(x).$$

Applying (16) we have for all $\sigma \in [M]$,

$$\begin{aligned} \mathbf{D}_\sigma(\mathbf{G}_2)C\mathbf{V}(f_\sigma(x)) &\leq K_\sigma\mathbf{D}_\sigma(\mathbf{G}_1)\mathbf{V}(f_\sigma(x)) \\ &\leq K_\sigma\mathbf{S}_\sigma(\mathbf{G}_1)\mathbf{V}(x) \leq \mathbf{S}_\sigma(\mathbf{G}_2)C\mathbf{V}(x), \end{aligned}$$

where C satisfies (13). Therefore, (14) holds for the graphs $\mathbf{G}_1$ and $\mathbf{G}_2$, hence $\mathbf{G}_1 \leq_C \mathbf{G}_2$ by Definition 5.
Necessity: The result follows from Lemma 3.3, Lemma 3.4, and algebraic manipulations of the sets of linear constraints on the vectors y in (21). Let us assume that there is a matrix C satisfying (13) such that $\mathbf{G}_1 \leq_C \mathbf{G}_2$. From Lemma 3.3, this implies that for all $\sigma \in [M]$, the set

$$\{y \in \mathbb{R}^{(M+1)|S^1|}_{\geq 0} : \mathbf{D}_\sigma(\mathbf{G}_1)y^\sigma \leq \mathbf{S}_\sigma(\mathbf{G}_1)y^0\} \tag{22}$$

is a subset of

$$\{y \in \mathbb{R}^{(M+1)|S^1|}_{\geq 0} : \mathbf{D}_\sigma(\mathbf{G}_2)Cy^\sigma \leq \mathbf{S}_\sigma(\mathbf{G}_2)Cy^0\}, \tag{23}$$

for $\sigma \in [M]$, where

$$y = \begin{pmatrix}(y^0)^\top & (y^1)^\top & \dots & (y^M)^\top\end{pmatrix}^\top,$$

with $y^i \in \mathbb{R}^{|S^1|}, 0 \leq i \leq M$.
The result then follows directly from Lemma 3.4. ■

IV. Conclusion

Path-complete Lyapunov functions have proved useful for designing stability criteria for complex systems. It has been noticed in the literature that some of these criteria are less conservative than others, and entire hierarchies have been proposed, with better performance of the criterion when going upper in the hierarchy at the cost of a higher computational effort. However, the relationship between complexity and efficiency of the criteria, and the understanding of what makes a criterion better than another, have remained elusive until now.

This work is the first systematic attempt towards comparing two given such criteria, in a setting independent of the dynamics, the choice of the vector Lyapunov function candidates and the dimension of the system. We propose a general necessary and sufficient condition that allows to conclude that one criterion is better than another, which is solely based on the topologies of the automata describing the criteria. The condition is algebraic and can be verified by the solution of a Linear Program.

In the future, we wish to extend the established conic combination setting to include compositions of the pieces of the PCLF with the dynamics. Moreover, we wish to generalize the established theory towards a universal characterization of ordering PCLFs.

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