Numerical DEM analysis of small-strain stiffness under anisotropic stress states

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Abstract.

This contribution assesses the influence of stress anisotropy on stiffness using discrete element method (DEM) simulations of true-triaxial tests supplemented with analytical studies. The samples considered comprised normally consolidated random monodisperse samples. The simulations were carried out at four different mean stress levels; at each stress level various combinations of the three principal stresses were considered. Stiffness was measured using planar wave propagation simulations. Using regression analysis it is shown that density effects can be considered using void ratio correction factors derived for isotropically compressed samples. However, a void ratio correction factor that considers coordination number is seen to be more marginally appropriate than the conventional form used in geotechnical experimental work. Material exponents that quantify the influence of each stress component on the stiffness were then determined. Analytical expressions derived from effective medium theory are less effective than the correction functions following the form used in current experimental practice.

Keywords: small-strain stiffness, stress anisotropy, true-triaxial tests.

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Introduction

The power-law relationship between elastic shear stiffness and mean effective stress, $G_0 \propto p'^n$ is well established (Hardin & Richart 1963; Houlsby & Wroth 1991). The relative magnitudes of the principal stress components also impact the stiffness values. Bellotti et al. (1996) experimentally related stiffness to all three principal stresses using the following expression (modified from the original form):

$$G_0 = C_p F(e)\left(\sigma_{prop}\right)^{n_{prop}}(\sigma_{osc})^{n_{osc}}(\sigma_{third})^{n_{third}} \quad (1)$$

where $F(e)$ is a normalizing function to eliminate void ratio effects, and C_p is a (dimensional) experimentally determined material constant. The stresses in the directions of propagation and oscillation of the stress wave are denoted σ_{prop} and σ_{osc}, respectively, while the remaining orthogonal stress is denoted σ_{third}; typically these stresses correspond to the principal stresses. The material exponents n_{prop}, n_{osc} and n_{third} quantify the influence of each principal stress on G_0.

Referring to prior studies of stiffness using discrete element method (DEM) (e.g. Khalili et al. 2017; Magnanimo et al. 2008), where particle-scale data are available a more appropriate form of Equation 1 might be:

$$G_0 = F\left(e, C_N\,\chi, \upsilon_p, E_p\right)\left(\sigma_{prop}\right)^{n_{prop}}(\sigma_{osc})^{n_{osc}}(\sigma_{third})^{n_{third}} \quad (2)$$

where C_N is the average coordination number, χ is a measure of the orientation fabric and υ_p and E_p are the Poisson's ratio and Young's modulus of the solid particle material, and the units of $F\left(e, C_N\,\chi, \upsilon_p, E_p\right)$ must ensure dimensional consistency.

Most prior research has concurred that only σ_{prop} and σ_{osc} measurably influence the elastic stiffness (e.g. Roesler 1979). O'Donovan et al. (2015) used DEM simulation data considering lattice packings and a limited number of stress combinations to argue that σ_{third} also has finite influence on G_0. These limitations compromised the clarity of their data and consequently the generality of their conclusions. This contribution addresses the resultant gap in understanding by using an improved DEM simulation approach to assess the material behaviour over a range of p' values and a large number of permutations/combinations of the three principal stresses in each case. The shear stiffnesses were determined using wave propagation, reflecting the popularity of bender element testing in experimental soil mechanics research. Dynamic bender element testing is attractive as is difficult to measure stiffnesses at very small strains accurately using conventional, load-deformation techniques even where local deformation transducers are mounted to measure locally (Clayton, 2011); furthermore when used in a triaxial cell bender elements can give the shear moduli in the principal stress planes (Lings et al. , 2000) and they can be mounted in a wide range of apparatuses (Clayton, 2011; Hamlin, 2014). The DEM wave propagation simulations were supplemented with and stress probes.

DEM simulation approach

The simulation approach broadly follows that documented in Otsubo et al. (2017) and Nguyen et al. (2017). For completeness, a brief overview is provided here. A modified version of the granular LAMMPS code was used (Plimpton, 1995), with a simplified Hertz-Mindlin contact model; simulation parameters are summarized in Table 1. Rigid wall boundaries were applied perpendicular to the direction of wave propagation while the remaining lateral boundaries were periodic to minimize boundary effects (Figure 1).

Random monodisperse dense and loose samples (RMD and RML respectively) were used. The randomly packed specimens were created from an initial cloud of 34986 non-contacting particles; a two-stage compression process was used. The dense random samples (RMD) were initially isotropically compressed to a mean effective stress, p' , of 1 kPa with $\mu_p = 0$ while $\mu_p = 0.15$ for the first compression stage for the loose samples (RML). The interparticle friction was then increased to $\mu_p = 0.35$. After modifying μ_p , there was a subsequent servo-controlled isotropic compression to a target confining pressure of $p' = 70$ kPa, $p' = 200$ kPa, $p' = 700$ kPa, or $p' = 7000$ kPa (depending on the final p' required). Then starting from each of these isotropic stress states, the principal stresses were either maintained constant or increased to generate various combinations of stress anisotropy with target values of $p' = 100$ kPa, $p' = 300$ kPa, $p' = 1000$ kPa, or $p' = 10000$ kPa. Using this approach each sample was then isotropic and normally consolidated; there was inherent anisotropy induced by previous loading or by simulation of a deposition process. The stress combinations were constrained so that the minimum and maximum principal stresses considered were $\sigma_3 = 0.75p'$ and $\sigma_1 = 1.25p'$ respectively, to avoid gross yield. These stress combinations are similar to those that are considered in physical cubical cell experiments (e.g. Hamlin, 2014)

Figure 2(a) illustrates the variation in C_N, with the geometric mean of the stresses in oscillation and propagation directions($\sqrt{\sigma_{osc}\sigma_{prop}}$) for each simulation considered, while Figure 2(b) illustrates the variation in the average mechanical coordination number, C_N^* , (calculated by excluding particles with 0 or 1 contacts) with $\sqrt{\sigma_{osc}\sigma_{prop}}$. Both sets of data indicate that, at a given value of p', there is not a significant variation in the number of contacts in the sample as a function of the stress anisotropy; nor are there significant variations in void ratio (Figure 2(c)). Note that the relationship between void ratio and coordination number is not unique; Magnanimo et al. (2008) who also considered monodisperse samples, applied a particular DEM simulation approach which achieved random dense samples with C_N value that varied between 4.88 and 6.65 all with a void ratio ≈ 0.64.

Once the desired stress state was achieved, shear waves were generated by applying a single-period sine motion to one rigid wall and the change in stress measured at the opposite wall was used to determine the shear wave velocity (V_s) using the peak-peak method as described in Otsubo et al. (2017) and Nguyen et al. (2017). To avoid inter-particle sliding, following Magnanimo et al. (2008) during stress wave propagation, $\mu_p =$ 0.45 was used for both RMD and RML samples. The corresponding shear modulus was then calculated as $G_0^{DEM} = \rho V_s^2$ where ρ is the sample density ($\rho = \frac{\rho_p}{1+e}$, ρ_p is the particle density). At each p' value 14 to 16 simulations were performed for each sample type.

To confirm that the wave propagation simulations give reasonable stiffness values, small stress increment triaxial probes were simulated. In these probes the vertical stress was increased by $\delta\sigma_z$ =1 kPa while the horizontal stresses were maintained constant ($\delta\sigma_x = \delta\sigma_y = 0$) to obtain the equivalent stiffness, $G_{equ}^{prob} = \delta q/3\delta\varepsilon_s$ ($\delta \mathrm{q} = \delta\sigma_\mathrm{z} - \delta\sigma_\mathrm{x}$) and the shear strain increment is $\delta\varepsilon_s = 2(\delta\varepsilon_\mathrm{z} - \delta\varepsilon_\mathrm{x})/3$; $\delta\varepsilon_\mathrm{z}$ and $\delta\varepsilon_\mathrm{x}$ are the principal strain increments in z and x directions respectively). For a real soil with an anisotropic fabric this stiffness value does not correspond to either G_{zx} *or* G_{zy} (e.g. Lings et al. 2011). However for the isotropic random samples here there should be agreement; for the random dense sample with $p' = 300$ kPa, $G_{equ}^{prob} = 279.3$ MPa, $G_{zx}^{DEM} = 280.9$ MPa and

$G_{zy}^{DEM} = 281.6$ MPa; confirming the ability of the dynamic simulations to give a correct measure of stiffness. Note that one could also use the analytical approach proposed by Agnolin and Roux, (2007) to determine the stiffness tensor from the particle-scale data available from the DEM simulations.

Simulation results

Previous empirical investigations (Hardin & Blandford 1989; Sadek et al. 2007) assumed $n_{prop} = n_{osc}$ and neglected the contribution of σ_{third}; consequently correlations of the form $G_{oscprop} \propto (\sigma_{osc}\sigma_{prop})^n$ were explored as in Figure 3. The data for the random samples give $n = 0.355$ for RMD samples and $n = 0.388$ for RML samples; these are close to the exponent $n = 0.333$ that is obtained by applying effective medium theory (EMT) to a random packing of spherical particles for this Hertzian contact model (Walton, 1987; Chang et al., 1991; Liao et al., 2000) .

For the three sample types considered, the exponent, n, observed for the scenario where p' remains constant differs from the n value that fits the data when p' varies. In all three cases the relationship between G_0 and $\sigma_{oscp}\sigma_{prop}$ depends on the stress anisotropy. The n values (obtained from least squares regression) for the isotropic stress cases are measurably higher than n values obtained by fitting a line to the data obtained for the various stress combinations examined at a given p' value. In comparison with the RMD samples, there is more scatter in the data for the RML sample.

Here υ_p and E_p are constant for all the simulations and so, to facilitate examination of the G_0 dependancy on σ_{prop} and σ_{osc}, referring to Equation 2, an appropriate function $F^*(e, C_N, \chi)$ was sought so that the influence of stress could be appropriately isolated.

To remove the influence of stress and fabric anisotropy equivalent simulations on isotropic samples with the same particle parameters documented in Otsubo (2016) were considered. Referring to Figure 4, 7 DEM samples were created and the friction coefficient was varied so that the void ratio at 1kPa ranged from 0.545 to 0.689. Each of these samples was then isotropically compressed to 10 MPa and G_0 values were determined systematically during the compression. Following earlier soil mechanics research

(Hardin and Richart 1963), assuming such correction function, considering void ratio alone takes the form $\mathrm{F(e)} = \frac{(B-e)^2}{1+e}$ and, applying regression analysis to the entire dataset gives a value of $B = 1.186$ (Otsubo 2016). Referring to Figure 5(a) when the $\frac{G_0}{\mathrm{F(e)}}$ data are considered a clearer correlation with $\sqrt{\sigma_{osc}\sigma_{prop}}$ emerges, and the differences between loose and dense and isotropic and anisotropic stress states are less apparent. Revisiting the data on Figure 4, B appears to be stress-level dependent and a simple curve fitting gives $B^p = exp(-0.1 + 0.046p')$; use of this approach effectively unifies the loose and dense data (Figure 5(b)); however both the correction and the stiffness now have a stress- dependency. One can also isolate particles that contributed to the stress transmission network and consider only particles with at least two contacts, so that $e^* = \frac{V_{Total}-\sum_{i=2}^{\infty} V_S^i}{\sum_{i=2}^{\infty} V_S^i}$ in which V_S^i is the sum of the volume of particles with a coordination number i, and V_{Total} is the total sample volume. Using e^* and again assuming a functional form $\mathrm{F(e^*)} = \frac{(B^*-\mathrm{e}^*)^2}{1+\mathrm{e}^*}$, gives B^*=1.484; application of this correction factor is less effective at unifying the data than the pressure-sensitive correction (Figure 5(c)).

To assess the relative merits of these void ratio corrections linear regression analyses were applied to Equation 1, where the unknown parameters are C_p, n_{prop}, n_{osc} ,and n_{third}. These regressions are compared with the DEM data on Figures 6(a) and (c); while the resulting exponent values are provided on Table 2.

Effective medium theory (EMT) indicates that for an isotropic stress state a correction factor of the form $F^{EMT}(e, C_N) = [C_N/(1+e)]^{2/3}$ can be used. The results support an idea that the normalized stiffness (G/(F(e*)) has a power−law dependency on C_N; i.e.

$$F(e, C_N) = \frac{(B^* - \mathrm{e}^*)^2}{1 + \mathrm{e}^*} {C_N}^{\alpha} \quad (3)$$

Figure 6 (b) gives the resulting regression; and includes comparison of a regression including only $F(e^*)$. Comparing values of the ratio of the prediction of the regression and the simulation data are plotted as a function of $\sqrt{\sigma_{osc}\sigma_{prop}}$ on Figure 6(c) and (d); confirming that including C_N improves the prediction, but that the discrepancy can be as large as 5.6%. Referring to the summary of the relative errors on Table 3, there are minor improvements in the predictive ability of the regression when comparing the case with

$B = 1.186$ and e with the other correction options considered. It is important to recognize however that the dataset does not include samples at the same void ratio with differing coordination numbers.

Excluding the case where B is varied with p', whichever correction function is applied the contributions of n_{prop} and n_{osc} greatly exceed the contribution of n_{third} however it is difficult to clearly say whether σ_{prop} or σ_{osc} dominates.

The correction function F(e) is clearly empirical and based on experimental practice. Informed by EMT, Figure 7 considers the data from an alternative perspective. EMT suggests a correlation between G_0 and $[C_N/(1+e)]^{2/3}$ and this is confirmed in Figure 7(a) (neglecting consideration of the stress anisotropy G_0 is normalized by $E_p^{2/3}p'^{1/3}$) . For the relatively small stress anisotropies considered here a clear trend emerges; these data indicate a dependency of G_0 on both the C_N and e in line with EMT. However the data on Figure 7(b) show that normalization of the G_0 data by the product $E_p^{2/3}p'^{1/3} \times [C_N/(1+e)]^{2/3}$ is less effective than use of the $\mathrm{F(e)} = \frac{(B-e)^2}{1+e}$ or its variants considered above; these observations also hold for the isotropic dataset developed by Otsubo (2016). Yimsiri and Soga (2000) attributed the inability of EMT to capture the stress-dependancy of soil stiffness to its inability to capture variations in fabric with stress. In their DEM simulations, Magnanimo et al (2008) found that the variation of $G_0\,/p'^{1/3}$ with C_N differs from the EMT prediction. More sophisticated theoretical EMT expressions (e.g. Thornton, 1993) also failed to capture the observations noted here; assumptions such as affine deformations and irrotational particles limit use of this theory.

Referring to Figure 8(a) there is a clear link between the stiffness ratio and the stress ratio; as would be expected from Equation 1. Inclusion of various measures of fabric anisotropy in the regression analyses did not support further extension of the correction function to include a χ term. This can be understood by reference to Figure 8(b); for these samples with only stress induced anisotropy the fabric and stress anisotropies are closely linked and so the χ term cannot be considered an independent variable.

Conclusions

DEM simulations of dynamic plane wave propagation have been used to consider the influence of stress anisotropy on sample stiffness in a fundamental manner and the following observations are made.

1. In the absence of a void ratio correction relationships between stiffness and the principal stresses (i.e. between G_0 and $\sqrt{\sigma_{osc}\sigma_{prop}}$) differ for the cases of isotropic compression, where p' is varied, and anisotropic compression, at a constant p'.
2. Application of void ratio functions derived using equivalent samples subject to isotropic stress states unifies the dense and loose sample at each stress level for all the stress anisotropies considered and confirms the negligible influence of σ_{third} .
3. The empirical void ratio correction function with a form $\mathrm{F(e)} = \frac{(B-e)^2}{1+e}$ served better to unify the data than a normalization derived from isotropic EMT which has already been shown to have limited validity.
4. Regression analysis that incorporates the $F(e^*)$ correction function and assumes a power-law correlation with the coordination number give a only slightly better match to the simulation data when compared with a macro-scale, conventional void ratio correction function. However these conclusions may not be universally applicable; the C_N-e relationship is not unique, and this data set does not include samples with similar void ratios and very different coordination numbers as were examined by Magnanimo et al. (2008).
5. For the samples considered here the stress and fabric anisotropies are closely linked and so inclusion of a fabric term, independent of stress, in the overall expression for stiffness cannot be justified.

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