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Supplement to "A central limit theorem for the realised covariation of a bivariate Brownian semistationary process"

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Contents

1. Introduction

We have collected the supplementary material for the article "A central limit theorem for the realised covariation of a bivariate Brownian semistationary process" in this document. The key background material needed for deriving our main results is the so-called Fourth Moment Theorem by [7]. Hence we provide a brief introduction to this important result in Section 2. Moreover, since the type of convergence arising in our central limit theorem is the so-called *stable convergence in law*, we devote Section 3 to reviewing its definition and basic properties which we apply in the proofs of our results. Finally, we have relegated some of the proofs of the main article to this supplementary material, which are provided in Section 4.

2. Pathway to the Fourth Moment Theorem

The purpose of this section is to illustrate the background necessary to illustrate the techniques developed by Nualart and Peccati that led them to proving the celebrated *Fourth Moment Theorem*.

We start with an introduction to Malliavin calculus. A good source for this material is Section 2 of [5]. A good summary of the necessary tools is also presented in [4]. The standard comprehensive reference for Malliavin calculus is [6].

2.1. Wiener Chaos decomposition

We fix a real, separable Hilbert space $\mathcal{H}$, with its scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and norm $\|\cdot\|_{\mathcal{H}} := \langle \cdot, \cdot \rangle_{\mathcal{H}}^{\frac{1}{2}}$. We denote by $X = \{X(h) \colon h \in \mathcal{H}\}$ an *isonormal* Gaussian process over $\mathcal{H}$ defined on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$, by which we mean a stochastic process indexed over $\mathcal{H}$ such that $\mathbb{E}[X(g)X(h)] = \langle g, h \rangle_{\mathcal{H}}$, for every $f, g \in \mathcal{H}$. We will assume that $\mathscr{F}$ is generated by X.

The first important result is the granted existence of an isonormal process:

Proposition 2.1. *Given a real, separable Hilbert space $\mathcal{H}$, there exists an isonormal process over $\mathcal{H}$.*

Proof. See [5], Theorem 2.1.1. □

We now introduce the fundamental notion of Wiener chaos, which plays a crucial role in our derivation of results. First, we recall:

Definition 2.2 (Hermite polynomials). *Let $p \geq 0$ be an integer. We define the p-th Hermite polynomial as $H_0 := 1$, for $p = 0$, and $H_{p+1}(x) := xH_p(x) - pH_{p-1}(x)$, for $p > 0$.*

Remark 2.3. *This is just one of many equivalent definitions for the Hermite polynomials. See [5], Definition 1.4.1 and Proposition 1.4.2 for alternative equivalent formulations and characterisations.*

Definition 2.4. *For each $n \geq 0$, $\mathcal{H}_n$ denotes the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_n(X(h)) : h \in \mathcal{H}, \|h\|_{\mathcal{H}} = 1\}$. The space $\mathcal{H}_n$ is called the n-th* Wiener chaos of X.

Wiener chaoses of different order on a Gaussian space are orthogonal as the next proposition shows.

Proposition 2.5. *Let $Z, Y \sim \mathscr{N}(0, 1)$ be jointly Gaussian. then, for all $n, m \geq 0$:*

$$\mathbb{E}[H_n(Z)H_m(Y)] = \begin{cases} n! \left(\mathbb{E}[ZY]\right)^n, & \text{if } n = m, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. See [5], Proposition 2.2.1. □

The next theorem states the fundamental fact that the L^2-space of random variables can be orthogonally decomposed as a direct sum of Wiener chaoses.

Theorem 2.6 (Wiener-Itô chaos decomposition). *The following decomposition holds:*

$$L^2(\Omega) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n.$$

So, every variable $F \in L^2(\Omega)$ can be written uniquely as:

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} F_n,$$

where $F_n \in \mathcal{H}_n$ and the series converges in $L^2(\Omega)$.

Proof. See Theorem 2.2.4 in [5]. □

2.2. Tensor products

In this section we give a very brief definition of tensor products of Hilbert spaces. The reference that we use here is [9].

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces with inner products $\langle \cdot, \cdot \rangle_{\mathcal{H}_1}$ and $\langle \cdot, \cdot \rangle_{\mathcal{H}_2}$. For $g \in \mathcal{H}_1$ and $h \in \mathcal{H}_2$, denote the bilinear form $g \otimes h \colon \mathcal{H}_1 \times \mathcal{H}_2 \to \mathbb{R}$ by:

$$[g \otimes h](x, y) = \langle x, g \rangle \langle y, h \rangle, \qquad (x, y) \in \mathcal{H}_1 \times \mathcal{H}_2.$$

Let $\mathcal{E}$ be the set of all finite linear combinations of such bilinear forms.

Lemma 2.7. *The bilinear form $\ll \cdot, \cdot \gg$ on $\mathcal{E}$ defined by:*

$$\ll g_1 \otimes h_1, g_2 \otimes h_2 \gg := \langle g_1, g_2 \rangle_{\mathcal{H}_1} \langle h_1, h_2 \rangle_{\mathcal{H}_2} \tag{1}$$

is symmetric, well defined and positive definite, and thus defines a scalar product on $\mathcal{E}$.

Proof. See [9]. □

The space $\mathcal{E}$ with the scalar product $\ll \cdot, \cdot \gg$ is obviously not complete. Hence we give the following definition.

Definition 2.8 (Tensor product)**.** *The* tensor product *of the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ is the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ defined to be the completion of $\mathcal{E}$ under the scalar product in* (1)*.*

Furthermore, we denote by $\mathcal{H}^{\otimes n}$ the n-fold tensor product between $\mathcal{H}$ and itself.

Symmetric tensors will play an important role in our discussion, and are defined next:

Definition 2.9 (Symmetrisation of a tensor product)**.** *If $f \in \mathcal{H}^{\otimes n}$ is of the form:*

$$f = h_1 \otimes \cdots \otimes h_n,$$

then the symmetrisation *of f, denoted by $\tilde{f}$, is defined to be:*

$$\tilde{f} := \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} h_{\sigma(1)} \otimes \cdots \otimes h_{\sigma(n)},$$

where the sum is taken over all permutations of $\{1, \ldots, n\}$. The closed subspace of $\mathcal{H}^{\otimes n}$ generated by the elements of the form $\tilde{f}$, is called the n-fold symmetric tensor product *of $\mathcal{H}$, and is denoted by $\mathcal{H}^{\odot n}$.*

A recurrent construction that we will encounter is that of *contracting* a tensor product, defined as follows:

Definition 2.10 (Contraction of tensors)**.** *Let $g = g_1 \otimes \cdots \otimes g_n \in \mathcal{H}^{\otimes n}$ and $h = h_1 \otimes \cdots \otimes h_m \in \mathcal{H}^{\otimes m}$. For any $0 \leq p \leq \min(n, m)$, we define the $p-th$ contraction of $g \otimes h$ as the following element of $\mathcal{H}^{\otimes m+n-2p}$:*

$$g \otimes_p h := \langle g_1, h_1 \rangle_{\mathcal{H}} \ldots \langle g_p, h_p \rangle_{\mathcal{H}} g_{p+1} \otimes \cdots \otimes g_n \otimes h_{p+1} \otimes \ldots h_m.$$

Note that, even if g and h are symmetric, their p-th contraction is not, in general, a symmetric tensor. We therefore denote by $g \widetilde{\otimes}_p h$ its symmetrisation.

2.3. The derivative operator

In this section we define the Malliavin derivative operator. We will need this to define its adjoint operator, the *multiple integral*, that we will use later for the proof of the central limit theorem.

Let $\mathscr{S}$ denote the set of *smooth* random variables, i.e. of the form:

$$f\left(X(h_1), \ldots, X(h_m)\right), \tag{2}$$

where $m \geq 1$, f is a test function, i.e. $f \in C^\infty$ and f and all of its derivatives have at most polynomial growth and $h_i \in \mathcal{H}$, for $i \in \{1, \ldots, m\}$.

Lemma 2.11. *The space $\mathscr{S}$ is dense in $L^q(\Omega)$ for every $q \geq 1$.*

Proof. See Lemma 2.3.1 in [5]. □

We need one last technical definition before we can introduce the Malliavin derivative.

Definition 2.12. *Given a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and a generic real, separable Hilbert space $\mathcal{H}$, we denote by $L^q(\Omega, \mathcal{H}) := L^q(\Omega, \mathscr{F}, \mathbb{P}; \mathcal{H})$ the class of those $\mathcal{H}$-valued random elements Y that are $\mathscr{F}$-measurable and such that $\mathbb{E}\left[\|Y\|_{\mathcal{H}}^q\right] < \infty$.*

We proceed to define the Malliavin derivative of a smooth variable.

Definition 2.13 (Malliavin Derivative)**.** *Let $F \in \mathscr{S}$ be given by (2), and $p \geq 1$ an integer. The p-th* Malliavin derivative *of F with respect to X is the element of $L^2(\Omega, \mathcal{H}^{\odot p})$ defined by:*

$$D^p F := \sum_{i_1, \ldots, i_p = 1}^{m} \frac{\partial^p}{\partial x_{i_1} \ldots \partial x_{i_p}} f\left(X(h_1), \ldots X(h_m)\right) h_{i_1} \otimes \cdots \otimes h_{i_p}.$$

In order for us to define the adjoint of the Malliavin derivative, we need to make sure that the latter operator is at least closable, or else its adjoint could be defined in too small a subset of $L^2(\Omega, \mathcal{H}^{\odot p})$. Indeed, recall the following result from functional analysis (we denote by A^* the adjoint of a linear operator A):

Proposition 2.14. *A linear operator $A \colon D(A) \to H$ is closable if and only if A^* is densely defined.*

The following theorem establishes the fundamental fact that the Malliavin operator is indeed closable.

Theorem 2.15. *Let $q \in [1, \infty)$, and let $p \geq 1$ be an integer. Then the operator $D^p \colon \mathscr{S} \subset L^q(\Omega) \to L^q(\Omega, \mathcal{H}^{\odot p})$ is closable.*

Proof. See Proposition 2.3.4 in [5]. □

2.4. The multiple integral

We are ready to give the formal definition of the multiple divergence operator δ^p.

Definition 2.16. *Let $p \geq 1$ be an integer. Denote by Dom δ^p the subset of elements $u \in L^2(\Omega, \mathcal{H}^{\otimes p})$ such that there exists a constant c satisfying:*

$$\left|\mathbb{E}\left[\langle D^p F, u\rangle_{\mathcal{H}^{\otimes p}}\right]\right| \leq c\sqrt{\mathbb{E}\left[F^2\right]}, \tag{3}$$

for all $F \in \mathscr{S}$.

Condition (3) ensures that, for a fixed $u \in \mathrm{Dom}\ \delta^p$, the linear operator $F \mapsto \mathbb{E}\left[\langle D^p F, u\rangle_{\mathcal{H}^{\otimes p}}\right]$ is continuous from $\mathscr{S}$ equipped with the $L^2(\Omega)$ norm into $\mathbb{R}$. Therefore it can be extended to a linear operator from $L^2(\Omega)$ into $\mathbb{R}$. By the Riesz representation theorem, then there exists a unique element in $L^2(\Omega)$, denoted $\delta^p(u)$, such that: $\mathbb{E}\left[\langle D^p F, u\rangle_{\mathcal{H}^{\otimes p}}\right] = \mathbb{E}\left[F\delta^p(u)\right]$. Thus, we can give the following definition:

Definition 2.17 (Multiple divergence). *The multiple divergence operator $\delta^p \colon Dom\ \delta^p \subset L^2(\Omega, \mathcal{H}^{\otimes p}) \to L^2(\Omega)$ is defined to be the adjoint operator of D^p. That means that if $u \in Dom\ \delta^p$ then $\delta^p(u)$ is defined to be that only element of $L^2(\Omega)$ such that:*

$$\mathbb{E}\left[F\delta^p(u)\right] = \mathbb{E}\left[\langle u, D^p F\rangle_{\mathcal{H}^{\otimes p}}\right],$$

for all $F \in \mathscr{S}$.

Finally, we define the multiple integral operator, which is the object we will need the most in our discussion:

Definition 2.18 (Multiple integral). *Let $p \geq 1$ and $f \in \mathcal{H}^{\odot p}$. The p-th* multiple integral *of f with respect to X is defined to be $I_p(f) := \delta^p(f)$.*

We further write $I_0 := I$ for the identity in $\mathbb{R}$.

The connection between multiple integrals and the Wiener chaos decomposition is asserted by the following theorem:

Theorem 2.19. *Let $f \in \mathcal{H}$, with $\|f\|_{\mathcal{H}} = 1$. Then, for any integer $p \geq 1$, we have:*

$$H_p((X(f)) = I_p\left(f^{\otimes p}\right).$$

As a consequence, the linear operator I_p is an isometry from $\mathcal{H}^{\odot p}$ onto the $p-$th Wiener chaos $\mathcal{H}_p$ of X.

Proof. See Theorem 2.2.7 in [5]. □

In particular, crucially, the image of a $p-$th multiple integral lies in the $p-$th Wiener chaos of X.

We will also make use of the following product formula:

Theorem 2.20 (Product formula for multiple integrals). *Let $p, q \geq 1$. If $f \in \mathcal{H}^{\odot p}$ and $g \in \mathcal{H}^{\odot q}$, then:*

$$I_p(f)I_q(g) = \sum_{r=0}^{p\wedge q} r!\binom{p}{r}\binom{q}{r} I_{p+q-2r}(f\,\widetilde{\otimes}_r g).$$

Proof. See [5], Theorem 2.7.10. □

2.5. The Fourth Moment Theorem

With our arsenal of technical tools, we can start to prepare the statement of the fourth moment theorem. We begin by stating another very remarkable fact. For a vector of L^2-variables belonging to a fixed Wiener chaos, joint weak convergence to the Gaussian distribution is equivalent to marginal convergence. More precisely, we have the following theorem:

Theorem 2.21. *Let $d \geq 2$ and $q_d, \dots, q_1 \geq 1$ be some fixed integers. Consider vectors:*

$$\mathbf{F}_n := (F_{1,n}, \dots, F_{d,n}) = (I_{q_1}(f_{1,n}), \dots, I_{q_d}(f_{d,n})), \qquad n \geq 1,$$

with $f_{i,n} \in \mathcal{H}^{\odot q_i}$. Let $C \in \mathscr{M}_d(\mathbb{R})$ be a symmetric, nonnegative definite matrix, and let $\mathbf{N} \sim \mathscr{N}_d(0, C)$. Assume that:

$$\lim_{n\to\infty} \mathbb{E}\left[F_{r,n} F_{s,n}\right] = C(r,s), \qquad 1 \leq r, s \leq d. \tag{4}$$

Then, as $n \to \infty$ the following two conditions are equivalent:

a) $\mathbf{F}_n$ converges in law to $\mathbf{N}$.
b) For every $1 \leq r \leq d$, $F_{r,n}$ converges in law to $\mathscr{N}(0, C(r,r))$.

Proof. See Theorem 6.2.3 in [5]. □

We can finally present the statement of the fourth moment theorem, which gives us equivalent conditions for convergence in law when the sequence of variables belongs to a fixed Wiener chaos.

Theorem 2.22 (Fourth moment theorem). *Let $F_n = I_q(f_n), n \geq 1$, be a sequence of random variables belonging to the q-th chaos of X, for some fixed integer $q \geq 2$ (so that $f_n \in \mathcal{H}^{\odot q}$). Assume, moreover, that $\mathbb{E}[F_n^2] \to \sigma^2 > 0$ as $n \to \infty$. Then, as $n \to \infty$, the following assertions are equivalent:*

1. *$F_n \stackrel{\mathscr{L}}{\to} N(0, \sigma^2)$,*
2. *$\lim_{n\to\infty} \mathbb{E}[F_n^4] = 3\sigma^2$,*
3. *$\|f_n \otimes_r f_n\|_{H^{\otimes(2q-2r)}} \to 0$, for all $r = 1, \dots, q-1$.*

Proof. This is a simplified version of Theorem 5.2.7 in [5]. □

3. Some remarks on stable convergence

In the central limit theorem, we use the notion of *stable convergence*. Here we briefly recall its definition and key properties. In this section we take definitions and results from [1] and from the survey on uses and properties of stable convergence in [8].

Definition 3.1 (Stable convergence). *Let a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ be fixed. Suppose the sequence of variables $Y^{(n)}$ converges weakly to Y, denoted by:*

$$Y^{(n)} \Rightarrow Y.$$

We say that $Y^{(n)}$ converges stably *to Y and write $Y^{(n)} \stackrel{st.}{\Rightarrow} Y$ if, for any $\mathscr{F}-$measurable set B, we have:*

$$\lim_{n\to\infty} \mathbb{P}\left(\{Y^{(n)} \leq x\} \cap B\right) = \mathbb{P}\left(\{Y \leq x\} \cap B\right),$$

for a countable, dense set of points x.

It is easy to see that $Y^{(n)} \stackrel{st.}{\Rightarrow} Y$, if and only if for any f bounded Borel function, and for any $\mathscr{F}-$measurable *fixed* variable Z:

$$\lim_{n\to\infty} \mathbb{E}\left[f\left(Y^{(n)}\right) Z\right] = \mathbb{E}\left[f(Y)Z\right].$$

Yet another characterisation is the following:

$$Y^{(n)} \stackrel{st.}{\Rightarrow} Y \iff (Y^{(n)}, Z) \Rightarrow (Y, Z),$$

for any $\mathscr{F}-$measurable *fixed* variable Z.

An obvious consequence of the previous characterisation is the following *continuous mapping theorem* for stable convergence:

Theorem 3.2 (Continuous mapping theorem)**.** *Suppose that $Y_n \overset{st.}{\Rightarrow} Y$, that σ is any fixed $\mathscr{F}$-measurable random variable and that $g(x,y)$ is a continuous function of two variables. Then:*

$$g(Y_n, \sigma) \overset{st.}{\Rightarrow} g(Y, \sigma).$$

When the limiting variable Y can be taken to be independent of $\mathscr{F}$, we say that the stable convergence is *mixing*, and we write:

$$Y^{(n)} \Rightarrow T \qquad \text{(mixing)}.$$

Finally, there is a useful criterion that can be used to establish mixing convergence:

Proposition 3.3. *Suppose that $Y^{(n)} \Rightarrow Y$. Then the following are equivalent:*

1. $Y^{(n)} \Rightarrow Y \qquad$ *(mixing),*
2. *For all fixed $k \in \mathbb{N}$ and $B \in \sigma\left(Y^{(1)} \ldots, Y^{(k)}\right)$ such that $\mathbb{P}(B) > 0$,*

$$\lim_{n\to\infty} \mathbb{P}\left(Y^{(n)} \le x \middle| B\right) = F_Y(x).$$

Proof. See Proposition 2 in [1]. □

4. Proofs

4.1. Proofs for Section 2

4.1.1. Proof of Lemma 2.10

We start off by proving the very useful Lemma 2.10.

Proof of Lemma 2.10. Note that we can express $c(x)$ as follows:

$$\begin{aligned} c(x) = &\int_0^x g^{(1)}(s)g^{(2)}(s)\,ds + \int_0^\infty g^{(1)}(s+x)g^{(2)}(s+x)\,ds \\ &- \int_0^\infty g^{(1)}(s)g^{(2)}(s+x)\,ds - \int_0^\infty g^{(1)}(s+x)g^{(2)}(s)\,ds + \int_0^\infty g^{(1)}(s)g^{(2)}(s)\,ds. \end{aligned}$$

After a change of variable, we can write the second integral as: $\int_x^\infty g^{(1)}(s)g^{(2)}(s)\,ds$, and therefore we can simplify the expression as:

$$\begin{aligned} c(x) &= 2\int_0^\infty g^{(1)}(s)g^{(2)}(s)\,ds - \int_0^\infty g^{(1)}(s)g^{(2)}(s+x)\,ds - \int_0^\infty g^{(1)}(s+x)g^{(2)}(s)\,ds \\ &= \int_0^\infty g^{(1)}(s)(g^{(2)}(s) - g^{(2)}(s+x))\,ds + \int_0^\infty g^{(2)}(s)(g^{(1)}(s) - g^{(1)}(s+x))\,ds. \end{aligned} \tag{5}$$

Assumption 2.2 implies that

$$c(x) = x^{\delta^{(1)}+\delta^{(2)}+1}\frac{1}{2}\left(L_0^{(1,2)}(x) + L_0^{(2,1)}(x)\right).$$

Note that $L_4^{(1,2)}(x) := \frac{1}{2}\left(L_0^{(1,2)}(x) + L_0^{(2,1)}(x)\right)$ is itself a slowly varying function and the constant $H = \frac{1}{2}\left(H^{(1,2)} + H^{(2,1)}\right)$. □

4.1.2. Proof of Example 2.11

Let us next provide the details of the computation of H for the case of two Gamma kernels, as discussed in Example 2.11.

Proof of Example 2.11. We start with the expression for $c(x)$ given in (5):

$$c(x) = 2\int_0^\infty g^{(1)}(s)g^{(2)}(s)\,ds - \int_0^\infty g^{(1)}(s)g^{(2)}(s+x)\,ds - \int_0^\infty g^{(1)}(s+x)g^{(2)}(s)\,ds.$$

If we plug in the explicit expression for the Gamma kernel, we obtain:

$$\begin{aligned} c(x) = 2\int_0^\infty s^{\delta^{(1)}+\delta^{(2)}} e^{-(\lambda^{(1)}+\lambda^{(2)})s}\,ds - \int_0^\infty (s^{\delta^{(1)}} e^{-\lambda^{(1)}s}(s+x)^{\delta^{(2)}} e^{-\lambda^{(2)}(s+x)}\,ds \\ - \int_0^\infty (s+x)^{\delta^{(1)}} e^{-\lambda^{(1)}(s+x)} s^{\delta^{(2)}} e^{-\lambda^{(2)}s}\,ds. \quad (6) \end{aligned}$$

The first integral can be easily evaluated:

$$2\int_0^\infty g^{(1)}(s)g^{(2)}(s)\,ds = 2\frac{\Gamma(\delta^{(1)}+\delta^{(2)}+1)}{(\lambda^{(1)}+\lambda^{(2)})^{\delta^{(1)}+\delta^{(2)}+1}}.$$

The other two integrals can be computed analytically in terms of a power series using formula (12) in [3][p. 234]. We will use the notation: $(a)_n = a(a+1)\dots(a+n-1) := \prod_{k=0}^{n-1}(a+k) = \frac{\Gamma(a+n)}{\Gamma(a)}$, with $(a)_0 := 1$.

For the first one of the two, for example, the final result is:

$$\begin{aligned} K_1^{(1)} e^{-\lambda^{(2)}t} x^{\delta^{(1)}+\delta^{(2)}+1} \sum_{k=0}^\infty \frac{(1+\delta^{(1)})_k}{(\delta^{(1)}+\delta^{(2)}+2)_k} \frac{\left((\lambda^{(1)}+\lambda^{(2)})x\right)^k}{k!} \\ + K_2 e^{-\lambda^{(2)}x} \sum_{k=0}^\infty \frac{(\delta^{(2)})_k}{(\delta^{(1)}+\delta^{(2)})_k} \frac{((\lambda^{(i)}+\lambda^{(j)})x)^k}{k!}, \quad (7) \end{aligned}$$

for constants $K_1^{(1)}, K_2$:

$$K_1^{(1)} = \frac{\Gamma(\delta^{(1)}+1)\Gamma(-1-\delta^{(1)}-\delta^{(2)})}{\Gamma(-\delta^{(1)})}, \qquad K_2 = \frac{\Gamma(\delta^{(1)}+\delta^{(2)}+1)}{(\lambda^{(1)}+\lambda^{(2)})^{\delta^{(1)}+\delta^{(2)}+1}}.$$

Swapping the variables $\delta^{(1)}, \delta^{(2)}$, we obtain the result for the second integral. Summing up, we conclude that (6) equals:

$$\begin{aligned} c(x) = 2K_2 - x^{\delta^{(1)}+\delta^{(2)}+1}\left(K_1^{(1)} e^{-\lambda^{(1)}x} f^{(1)}(x) + K_1^{(2)} e^{-\lambda^{(2)}x} f^{(2)}(x)\right) \\ - K_2\left(e^{-\lambda^{(1)}x} f^{(3)}(x) + e^{-\lambda^{(2)}x} f^{(4)}(x)\right), \end{aligned}$$

where $f^{(1)}, f^{(2)}$ are power series such that $\lim_{x\to 0} f^{(1)}(x) = \lim_{x\to 0} f^{(2)}(x) = 1$, while:

$$f^{(3)}(x) = \sum_{k=0}^\infty \frac{(\delta^{(1)})_k}{(\delta^{(1)}+\delta^{(2)})_k} \frac{\left((\lambda^{(1)}+\lambda^{(2)})x\right)^k}{k!}, \qquad f^{(4)}(x) = \sum_{k=0}^\infty \frac{(\delta^{(2)})_k}{(\delta^{(1)}+\delta^{(2)})_k} \frac{\left((\lambda^{(1)}+\lambda^{(2)})x\right)^k}{k!}.$$

Using the Taylor expansion: $e^{-\lambda^{(i)}x} = 1 - \lambda^{(i)}x + o(x)$, some of the terms simplify to give:

$$\begin{aligned} c(x) &= -x^{\delta^{(1)}+\delta^{(2)}+1}\left(K_1^{(1)} e^{-\lambda^{(1)}x} f^{(1)}(x) + K_1^{(2)} e^{-\lambda^{(2)}x} f^{(2)}(x)\right) + O(x^2) \\ &= x^{\delta^{(1)}+\delta^{(2)}+1}\left(-K_1^{(1)} e^{-\lambda^{(1)}x} f^{(1)}(x) - K_1^{(2)} e^{-\lambda^{(2)}x} f^{(2)}(x) + f^{(5)}(x)\right), \end{aligned}$$

and we know that $f^{(5)}(x) = O\left(x^{1-\delta^{(1)}-\delta^{(2)}}\right)$. If we call $L_4^{(1,2)}(x) = -K_1^{(1)}e^{-\lambda^{(1)}x}f^{(1)}(x) - K_1^{(2)}e^{-\lambda^{(2)}x}f^{(2)}(x) + f^{(5)}(x)$, then $L_4^{(1,2)}(x)$ is continuous and we also have:

$$\lim_{x\to 0+} L_4^{(1,2)}(x) = -K_1^{(1)} - K_1^{(2)},$$

which in particular implies that $L_4^{(1,2)}(x)$ is slowly varying at zero.

We know by [2] that:

$$\lim_{x\to 0+} L_0^{(i,i)}(x) = 2^{-1-2\delta^{(i)}} \frac{\Gamma\left(\frac{1}{2}-\delta^{(i)}\right)}{\Gamma\left(\frac{3}{2}+\delta^{(i)}\right)},$$

and so:

$$\lim_{x\to 0+} \tilde{L}_0^{(1,2)}(x) = K_0 := 2^{-1-\delta^{(1)}-\delta^{(2)}} \sqrt{\frac{\Gamma(\frac{1}{2}-\delta^{(i)})\Gamma(\frac{1}{2}-\delta^{(j)})}{\Gamma(\frac{3}{2}+\delta^{(i)})\Gamma(\frac{3}{2}+\delta^{(j)})}}.$$

Finally, we can then find an expression for H:

$$H = \frac{-K_1^{(1)} - K_1^{(2)}}{K_0} = \left(-\frac{\Gamma(\delta^{(1)}+1)\Gamma(-1-\delta^{(1)}-\delta^{(2)})}{\Gamma(-\delta^{(1)})} - \frac{\Gamma(\delta^{(2)}+1)\Gamma(-1-\delta^{(1)}-\delta^{(2)})}{\Gamma(-\delta^{(2)})}\right) \times 2^{1+\delta^{(1)}+\delta^{(2)}} \sqrt{\frac{\Gamma(\frac{3}{2}+\delta^{(i)})\Gamma(\frac{3}{2}+\delta^{(j)})}{\Gamma(\frac{1}{2}-\delta^{(i)})\Gamma(\frac{1}{2}-\delta^{(j)})}}.$$

□

4.2. Proofs for Section 6

4.2.1. Proof of Lemma 6.1

Proof of Lemma 6.1. Simply write Taylor's formula twice, with Lagrange remainder:

$$\begin{cases} u(x+h) = u(x) + hu'(x) + \frac{1}{2}h^2 u''(\zeta^+), & \zeta^+ \in (x, x+h), \\ u(x-h) = u(x) - hu'(x) + \frac{1}{2}h^2 u''(\zeta^-), & \zeta^- \in (x-h, x). \end{cases}$$

Adding the two equations:

$$u(x+h) - 2u(x) + u(x-h) = h^2 \left(\frac{u''(\zeta^+) + u''(\zeta^-)}{2}\right).$$

By continuity of u'' and the intermediate value theorem:

$$\frac{u''(\zeta^+) + u''(\zeta^-)}{2} \in u''\left((\zeta^-, \zeta^+)\right) \subseteq u''\left((x-h, x+h)\right),$$

which implies the result. □

4.2.2. Proof of Theorem 3.2

Proof of Theorem 3.2. We start from the last statement of the theorem, i.e. the limiting covariance matrix. The limit: $\lim_{n\to\infty} \mathbb{E}\left[F_{i,n}F_{j,n}\right]$ has been computed in the last few sections, where we picked intervals $[a_k, b_k]$ of length 1 and showed that the matrix is diagonal, with diagonal elements all equal to $C(1,1)$. It is straightforward to change the summation indices in (31) from $\sum_{i=1}^{n}$ to $\sum_{i=\lfloor na_k\rfloor+1}^{\lfloor nb_k\rfloor}$. Since $\lim_{n\to\infty} \frac{\lfloor nb_k\rfloor - \lfloor na_k\rfloor}{n} = \lim_{n\to\infty} \frac{nb_k - \{nb_k\} - na_k + \{na_k\}}{n} = b_k - a_k$, we get the limit as in the statement.

The weak convergence is now implied by an application of Theorem 2.21. In order to show that condition (b) there is satisfied, we need to check one of the equivalent conditions provided by Theorem 2.22. Employing condition 3 in our case accounts to verifying that, for $1 \leq k \leq d$:

$$\left\| \left(\frac{1}{\sqrt{n}} \sum_{i=\lfloor na_k \rfloor+1}^{\lfloor nb_k \rfloor} \frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \widetilde{\otimes} \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}} \right) \otimes_1 \left(\frac{1}{\sqrt{n}} \sum_{i=\lfloor na_k \rfloor+1}^{\lfloor nb_k \rfloor} \frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \widetilde{\otimes} \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}} \right) \right\|_{\mathcal{H}^{\otimes 2}} \to 0.$$

Without loss of generality, we look at $d = 1$ and assume $a_1 = 0, b_1 = 1$:

$$\frac{1}{n} \left\| \sum_{i,j=1}^{n} \left(\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \widetilde{\otimes} \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}} \right) \otimes_1 \left(\frac{\Delta_j^n G^{(1)}}{\tau_n^{(1)}} \widetilde{\otimes} \frac{\Delta_j^n G^{(2)}}{\tau_n^{(2)}} \right) \right\|_{\mathcal{H}^{\otimes 2}}.$$

Let us examine the following:

$$\begin{aligned}
&\left(\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \widetilde{\otimes} \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}} \right) \otimes_1 \left(\frac{\Delta_j^n G^{(1)}}{\tau_n^{(1)}} \widetilde{\otimes} \frac{\Delta_j^n G^{(2)}}{\tau_n^{(2)}} \right) \\
&= \left(\frac{1}{2} \frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \otimes \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}} + \frac{1}{2} \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}} \otimes \frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \right) \otimes_1 \left(\frac{1}{2} \frac{\Delta_j^n G^{(1)}}{\tau_n^{(1)}} \otimes \frac{\Delta_j^n G^{(2)}}{\tau_n^{(2)}} + \frac{1}{2} \frac{\Delta_j^n G^{(2)}}{\tau_n^{(2)}} \otimes \frac{\Delta_j^n G^{(1)}}{\tau_n^{(1)}} \right) \\
&= \frac{1}{4} \mathbb{E} \left[\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \frac{\Delta_j^n G^{(1)}}{\tau_n^{(1)}} \right] \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}} \otimes \frac{\Delta_j^n G^{(2)}}{\tau_n^{(2)}} + \frac{1}{4} \mathbb{E} \left[\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \frac{\Delta_j^n G^{(2)}}{\tau_n^{(2)}} \right] \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}} \otimes \frac{\Delta_j^n G^{(1)}}{\tau_n^{(1)}} \\
&+ \frac{1}{4} \mathbb{E} \left[\frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}} \frac{\Delta_j^n G^{(1)}}{\tau_n^{(1)}} \right] \frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \otimes \frac{\Delta_j^n G^{(2)}}{\tau_n^{(2)}} + \frac{1}{4} \mathbb{E} \left[\frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}} \frac{\Delta_j^n G^{(2)}}{\tau_n^{(2)}} \right] \frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \otimes \frac{\Delta_j^n G^{(1)}}{\tau_n^{(1)}} \\
&= \frac{1}{4} \sum_{\substack{\{a,a'\}=\{1,2\} \\ \{b,b'\}=\{1,2\}}} r_{a,b}^{(n)}(j-i) \frac{\Delta_i^n G^{(a')}}{\tau_n^{(a')}} \otimes \frac{\Delta_j^n G^{(b')}}{\tau_n^{(b')}}.
\end{aligned}$$

We hence obtain:

$$\begin{aligned}
&\frac{1}{n^2} \left\| \sum_{i,j=1}^{n} \left(\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \widetilde{\otimes} \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}} \right) \otimes_1 \left(\frac{\Delta_j^n G^{(1)}}{\tau_n^{(1)}} \widetilde{\otimes} \frac{\Delta_j^n G^{(2)}}{\tau_n^{(2)}} \right) \right\|_{\mathcal{H}^{\otimes 2}}^2 \\
&= \frac{1}{16n^2} \left\| \sum_{i,j=1}^{n} \left(\sum_{\substack{\{a,a'\}=\{1,2\} \\ \{b,b'\}=\{1,2\}}} r_{a,b}^{(n)}(j-i) \frac{\Delta_i^n G^{(a')}}{\tau_n^{(a')}} \otimes \frac{\Delta_j^n G^{(b')}}{\tau_n^{(b')}} \right) \right\|_{\mathcal{H}^{\otimes 2}}^2 \\
&= \frac{1}{16n^2} \sum_{i,j,i',j'=1}^{n} \Bigg\langle \sum_{\substack{\{a,a'\}=\{1,2\} \\ \{b,b'\}=\{1,2\}}} r_{a,b}^{(n)}(j-i) \frac{\Delta_i^n G^{(a')}}{\tau_n^{(a')}} \otimes \frac{\Delta_j^n G^{(b')}}{\tau_n^{(b')}}, \qquad (8) \\
&\qquad \sum_{\substack{\{\alpha,\alpha'\}=\{1,2\} \\ \{\beta,\beta'\}=\{1,2\}}} r_{\alpha,\beta}^{(n)}(j'-i') \frac{\Delta_{i'}^n G^{(\alpha')}}{\tau_n^{(\alpha')}} \otimes \frac{\Delta_{j'}^n G^{(\beta')}}{\tau_n^{(\beta')}} \Bigg\rangle_{\mathcal{H}^{\otimes 2}} \\
&= \frac{1}{16n^2} \sum_{\substack{\{a,a'\}=\{1,2\} \\ \{b,b'\}=\{1,2\} \\ \{\alpha,\alpha'\}=\{1,2\} \\ \{\beta,\beta'\}=\{1,2\}}} \sum_{i,j,i',j'=1}^{n} r_{a,b}^{(n)}(j-i) r_{\alpha,\beta}^{(n)}(j'-i') \left\langle \frac{\Delta_i^n G^{(a')}}{\tau_n^{(a')}} \otimes \frac{\Delta_j^n G^{(b')}}{\tau_n^{(b')}}, \frac{\Delta_{i'}^n G^{(\alpha')}}{\tau_n^{(\alpha')}} \otimes \frac{\Delta_{j'}^n G^{(\beta')}}{\tau_n^{(\beta')}} \right\rangle_{\mathcal{H}^{\otimes 2}}
\end{aligned}$$

$$= \frac{1}{16n^2} \sum_{\substack{\{a,a'\}=\{1,2\} \\ \{b,b'\}=\{1,2\} \\ \{\alpha,\alpha'\}=\{1,2\} \\ \{\beta,\beta'\}=\{1,2\}}} \sum_{i,j,i',j'=1}^{n} r^{(n)}_{a,b}(j-i) r^{(n)}_{\alpha,\beta}(j'-i') r^{(n)}_{a',\alpha'}(i'-i) r^{(n)}_{b',\beta'}(j'-j). \tag{9}$$

We need to show that the quantity in (9) converges to zero. It is sufficient to show that the sum of the absolute values converges to zero. If we apply Hölder inequality, we get:

$$\frac{1}{16n^2} \sum_{\substack{\{a,a'\}=\{1,2\} \\ \{b,b'\}=\{1,2\} \\ \{\alpha,\alpha'\}=\{1,2\} \\ \{\beta,\beta'\}=\{1,2\}}} \sum_{i,j,i',j'=1}^{n} \left| r^{(n)}_{a,b}(j-i) r^{(n)}_{\alpha,\beta}(j'-i') r^{(n)}_{a',\alpha'}(i'-i) r^{(n)}_{b',\beta'}(j'-j) \right|$$

$$\leq \frac{1}{16n^2} \sum_{\substack{\{a,a'\}=\{1,2\} \\ \{b,b'\}=\{1,2\} \\ \{\alpha,\alpha'\}=\{1,2\} \\ \{\beta,\beta'\}=\{1,2\}}} \sum_{i,j,i',j'=1}^{n} \left| r^{(n)}_{a,b}(j-i) r^{(n)}_{a',\alpha'}(i'-i) \left[\left(r^{(n)}_{\alpha,\beta}(j'-i') \right)^2 + \left(r^{(n)}_{b',\beta'}(j'-i') \right)^2 \right] \right|.$$

So we can split the sum into two components. Let us perform the substitution

$$(i,j,i',j') \to (i,j,i',l) := (i,j,i',i'-j').$$

We have:

$$\frac{1}{16n^2} \sum_{\substack{\{a,a'\}=\{1,2\} \\ \{b,b'\}=\{1,2\} \\ \{\alpha,\alpha'\}=\{1,2\} \\ \{\beta,\beta'\}=\{1,2\}}} \sum_{i,j,i',j'=1}^{n} \left| r^{(n)}_{a,b}(j-i) r^{(n)}_{a',\alpha'}(i'-i) \right| \left(r^{(n)}_{\alpha,\beta}(j'-i') \right)^2$$

$$= \frac{1}{16n^2} \sum_{\substack{\{a,a'\}=\{1,2\} \\ \{b,b'\}=\{1,2\} \\ \{\alpha,\alpha'\}=\{1,2\} \\ \{\beta,\beta'\}=\{1,2\}}} \sum_{i,j,i'=1}^{n} \sum_{|l|<n} \left| r^{(n)}_{a,b}(j-i) r^{(n)}_{a',\alpha'}(i'-i) \right| \left(r^{(n)}_{\alpha,\beta}(l) \right)^2$$

$$\leq \frac{1}{16n^2} \sum_{\substack{\{a,a'\}=\{1,2\} \\ \{b,b'\}=\{1,2\} \\ \{\alpha,\alpha'\}=\{1,2\} \\ \{\beta,\beta'\}=\{1,2\}}} \sum_{l\in\mathbb{Z}} \left(r^{(n)}_{\alpha,\beta}(l) \right)^2 \sum_{i,j,i'=1}^{n} \left| r^{(n)}_{a,b}(j-i) r^{(n)}_{a',\alpha'}(i'-i) \right|$$

$$= \frac{1}{16n^2} \sum_{\substack{\{a,a'\}=\{1,2\} \\ \{b,b'\}=\{1,2\} \\ \{\alpha,\alpha'\}=\{1,2\} \\ \{\beta,\beta'\}=\{1,2\}}} \sum_{l\in\mathbb{Z}} \left(r^{(n)}_{\alpha,\beta}(l) \right)^2 n \sum_{|i|<n} \left| r^{(n)}_{a,b}(i) \right| \sum_{|j|<n} \left| r^{(n)}_{a',\alpha'}(j) \right|$$

$$= \frac{1}{16n} \sum_{\substack{\{a,a'\}=\{1,2\} \\ \{b,b'\}=\{1,2\} \\ \{\alpha,\alpha'\}=\{1,2\} \\ \{\beta,\beta'\}=\{1,2\}}} \sum_{l\in\mathbb{Z}} \left(r^{(n)}_{\alpha,\beta}(l) \right)^2 \sum_{|i|<n} \left| r^{(n)}_{a,b}(i) \right| \sum_{|j|<n} \left| r^{(n)}_{a',\alpha'}(j) \right|$$

$$= \frac{1}{16} \sum_{\substack{\{a,a'\}=\{1,2\} \\ \{b,b'\}=\{1,2\} \\ \{\alpha,\alpha'\}=\{1,2\} \\ \{\beta,\beta'\}=\{1,2\}}} \sum_{k\in\mathbb{Z}} \left(r^{(n)}_{\alpha,\beta}(k) \right)^2 \left[\frac{1}{\sqrt{n}} \sum_{|i|<n} \left| r^{(n)}_{a,b}(i) \right| \right] \left[\frac{1}{\sqrt{n}} \sum_{|j|<n} \left| r^{(n)}_{a',\alpha'}(j) \right| \right].$$

Now, fix $\delta > 0$:

$$\frac{1}{\sqrt{n}}\sum_{|i|<n}\left|r^{(n)}_{a,b}(i)\right| = \frac{1}{\sqrt{n}}\sum_{|i|\le\lfloor n\delta\rfloor}\left|r^{(n)}_{a,b}(i)\right| + \frac{1}{\sqrt{n}}\sum_{\lfloor n\delta\rfloor<|i|<n}\left|r^{(n)}_{a,b}(i)\right|.$$

Thanks to Hölder's inequality, the first term is bounded by:

$$\frac{1}{\sqrt{n}}\sqrt{2\lfloor n\delta\rfloor+1}\sqrt{\sum_{i\in\mathbb{Z}}\left|r^{(n)}_{a,b}(i)\right|^2},$$

and the second one by:

$$\frac{1}{\sqrt{n}}\sqrt{2(n-\lfloor n\delta\rfloor-1)}\sum_{\lfloor n\delta\rfloor<|i|<n}\left|r^{(n)}_{a,b}(i)\right|^2.$$

For a fixed δ, the second one converges to 0 as n tends to infinity. The first one is bounded by $K\sqrt{\delta}$ (for a positive constant $K<\infty$), thus letting $\delta\to 0$ we have:

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\sum_{|i|<n}\left|r^{(n)}_{a,b}(i)\right| = \lim_{\delta\to 0}\lim_{n\to\infty}\frac{1}{\sqrt{n}}\sum_{|i|<n}\left|r^{(n)}_{a,b}(i)\right| = 0,$$

provided that the following series

$$\sum_{i\in\mathbb{Z}}\left(r^{(n)}_{\alpha,\beta}(i)\right)^2,\qquad \sum_{i\in\mathbb{Z}}\left(r^{(n)}_{a,b}(i)\right)^2,\qquad \sum_{i\in\mathbb{Z}}\left(r^{(n)}_{a',b'}(i)\right)^2,$$

converge. In a completely analogous way, we can show that the second component of the original sum converges to zero, provided that also $\sum_{i\in\mathbb{Z}}\left(r^{(n)}_{b',\beta'}(i)\right)^2$ is finite. But since from Theorem 3.1 we have:

$$\left|r^{(n)}_{i,j}(k)\right|^2 \le C(k-1)^{2\delta^{(i)}+2\delta^{(j)}+2\varepsilon-2},\quad \text{for } k\ge 2,$$

it is sufficient to ask that $\delta^{(i)}+\delta^{(j)}<\frac{1}{2}$ for all possible choices of i,j. Explicitly, it is sufficient to ask that: $\delta^{(1)}<\frac{1}{4}$ and $\delta^{(2)}<\frac{1}{4}$.

The statement of the theorem is proved.

□

4.3. Extending the CLT for the Gaussian core to bivariate stationary processes

As already mentioned in Remark 3.5 in the main article, we note that the central limit theorem for the Gaussian core, see Theorem 3.4, can be formulated under more general conditions which do not require the particular integral representation we are working with throughout the paper[1]: Consider a bivariate Gaussian stationary process $\mathbf{G}=(G^{(1)},G^{(2)})^\top$. Define for $i,j\in\{1,2\}$ and $t>0$: $\bar{R}^{(i,j)}(t) := \mathbb{E}[(G^{(j)}_t - G^{(i)}_0)^2]$. We note that, due the stationarity of the process, we can write

$$\bar{R}^{(i,j)}(t) = \underbrace{\mathbb{E}(G^{(j)2}_t)+\mathbb{E}(G^{(i)2}_0)-2\mathbb{E}(G^{(j)}_0G^{(i)}_0)}_{=:C_{ij}} -2\mathbb{E}[(G^{(j)}_t-G^{(j)}_0)G^{(i)}_0],$$

where only the last term depends on t. That means that Assumption 2.2 is effectively an assumption on the asymptotic behaviour of $\mathbb{E}[(G^{(j)}_t-G^{(j)}_0)G^{(i)}_0]$. More precisely, if we assume that $\mathbf{G}$ satisfies Assumption 2.2, it implies in particular that

$$-2\mathbb{E}[(G^{(j)}_t-G^{(j)}_0)G^{(i)}_0] = \rho_{i,j}t^{\delta^{(i)}+\delta^{(j)}+1}L^{(i,j)}_0(t),$$

[1] We thank the referee for pointing this out.

where $\rho_{i,j} \in [-1,1]$ is linked to the correlation between $G^{(i)}$ and $G^{(j)}$ and is such that $\rho_{i,j} = 1$ for $i = j$ and $\rho_{i,j} = 0$ if $G^{(i)}$ and $G^{(j)}$ are uncorrelated.

Under Assumption 2.2, one can easily show that an adapted version of Lemma 2.10 holds, where the function c (denoted by $\widetilde{c}$ now to avoid confusion) in that lemma is now defined as $\widetilde{c}(t) = -\left(\mathbb{E}[(G_t^{(2)} - G_0^{(2)})G_0^{(1)}] + \mathbb{E}[(G_t^{(1)} - G_0^{(1)})G_0^{(2)}]\right)$. In particular,

$$\begin{aligned}\widetilde{c}(t) &= -\left(\mathbb{E}[(G_t^{(2)} - G_0^{(2)})G_0^{(1)}] + \mathbb{E}[(G_t^{(1)} - G_0^{(1)})G_0^{(2)}]\right) &(10)\\ &= \rho_{1,2} t^{\delta^{(1)}+\delta^{(2)}+1} \underbrace{\frac{1}{2}\left(L_0^{(1,2)}(t) + L_0^{(2,1)}(t)\right)}_{=L_4^{(1,2)}(t)} = \rho_{1,2} t^{\delta^{(1)}+\delta^{(2)}+1} L_4^{(1,2)}(t), &(11)\end{aligned}$$

where all the slowly varying functions are chosen as before. I.e. $\widetilde{c}(t) = \rho_{1,2} c(t)$.

We observe that Theorem 3.1 carries over for the general setting as well: We remark that $\rho_\vartheta^{(i,j)}(0) = \rho_{i,j} H$ for $i \neq j$, $\rho_\vartheta^{(i,j)}(0) = 1$ for $i = j$ in that theorem and all the other quantities are defined as before. To see this, note that when going through the steps of the proof of Theorem 3.1, we notice that the only quantity we need to consider specifically is $r_{i,j}^{(n)}(0)$ for $i \neq j$. In that case, we have

$$r_{i,j}^{(n)}(0) = \mathbb{E}\left[\frac{\Delta_1^n G^{(i)}}{\tau_n^{(i)}} \frac{\Delta_1^n G^{(j)}}{\tau_n^{(j)}}\right] = -\frac{1}{\tau_n^{(i)} \tau_n^{(j)}} \left(\mathbb{E}[(G_t^{(j)} - G_0^{(j)})G_0^{(i)}] + \mathbb{E}[(G_t^{(i)} - G_0^{(i)})G_0^{(j)}]\right).$$

Hence, using the adapted version of Lemma 2.10, see equation (10), we have $r_{1,2}^{(n)}(0) = r_{2,1}^{(n)}(0) = \frac{\widetilde{c}(\Delta_n)}{\xi_n}$. Then we can continue as in the proof of Theorem 3.1 to conclude that

$$r_{1,2}^{(n)}(0) = r_{2,1}^{(n)}(0) = \frac{\widetilde{c}(\Delta_n)}{\xi_n} = \frac{\rho_{12} \Delta_n^{\delta^{(1)}+\delta^{(2)}+1} L_4^{(1,2)}(\Delta_n)}{\Delta_n^{\delta^{(1)}+\delta^{(2)}+1} \tilde{L}_0^{(1,2)}(\Delta_n)} \to \rho_{1,2} H, \quad \text{as } n \to \infty.$$

Also, Theorem 3.2 carries over to the general setting (with ρ being replaced by $\rho_{1,2}$): When we consider the proof of that theorem. We observe that the integral representation was actually not used in that theorem so the core of the proof goes through one-to-one and then in the final step we use our adapted version of Theorem 3.1 to conclude.

Similarly, we can then conclude that also Theorems 3.3 and 3.4 can be extended to a setting of a bivariate Gaussian process using the adapted versions of Lemma 2.10 and Theorem 3.1.

References

[1] D. J. Aldous and G. K. Eagleson. On mixing and stability of limit theorems. *The Annals of Probability*, 6(2):325–331, 1978. ISSN 00911798. URL `http://www.jstor.org/stable/2243222`.

[2] O. E. Barndorff-Nielsen, J. M. Corcuera, and M. Podolskij. Multipower variation for brownian semistationary processes. *Bernoulli*, 17(4):1159–1194, 2011.

[3] H. Bateman. Tables of integral transforms. *California Institute of Technology Bateman Manuscript Project, New York: McGraw-Hill, 1954, edited by Erdelyi, Arthur*, 2, 1954.

[4] J. M. Corcuera. New central limit theorems for functionals of gaussian processes and their applications. *Methodology and Computing in Applied Probability*, 14(3):477–500, 2012. ISSN 1387-5841. URL `http://dx.doi.org/10.1007/s11009-011-9236-9`.

[5] I. Nourdin and G. Peccati. *Normal approximations with Malliavin calculus: from Stein's method to universality*, volume 192. Cambridge University Press, 2012.

[6] D. Nualart. *The Malliavin calculus and related topics*, volume 1995. Springer, 2006.

[7] D. Nualart and G. Peccati. Central limit theorems for sequences of multiple stochastic integrals. *The Annals of Probability*, 33(1):177–193, 2005. ISSN 00911798. URL `http://www.jstor.org/stable/3481767`.

[8] M. Podolskij and M. Vetter. Understanding limit theorems for semimartingales: a short survey. *Statistica Neerlandica*, 64(3):329–351, 2010.

[9] M. Reed and B. Simon. *Methods of modern mathematical physics*, volume 2. Academic press, 1975.