Topics in Volatility Models

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A Thesis submitted for the degree of Doctor of Philosophy

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March 2009
To My Parents: Yingsen and Wentao
ABSTRACT

In this thesis I will present my PhD research work, focusing mainly on financial modelling of asset’s volatility and the pricing of contingent claims (financial derivatives), which consists of four topics:

1. Several changing volatility models are introduced and the pricing of European options is derived under these models;
2. A general local stochastic volatility model with stochastic interest rates (IR) is studied in the modelling of foreign exchange (FX) rates. The pricing of FX options under this model is examined through the use of an asymptotic expansion method, based on Watanabe-Yoshida theory. The perfect/partial hedging issues of FX options in the presence of local stochastic volatility and stochastic IRs are also considered. Finally, the impact of stochastic volatility on the pricing of FX-IR structured products (PRDCs) is examined;
3. A new method of non-biased Monte Carlo simulation for a stochastic volatility model (Heston Model) is proposed;
4. The LIBOR/swap market model with stochastic volatility and jump processes is studied, as well as the pricing of interest rate options under that model.

In conclusion, some future research topics are suggested.

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ACKNOWLEDGEMENTS

I would like to thank my Ph.D supervisor, Dr. Chris Barnett, from the Financial Mathematics programme, Institute for Mathematical Sciences, for his guidance, ideas and constant support for my research; Dr. Gerry Salkin, Mr James Selfe and the rest of the Risk & Product Development team in Mitsubishi UFJ Securities International for sponsoring my Ph.D studies and teaching me the practice of financial markets. Special thanks go to Dr. Yanmin Li for his help and the many fruitful discussions with him.

Mitsubishi UFJ Securities International does not necessarily agree with or endorse the content of this thesis.
1. GENERAL INTRODUCTION, CHANGING VOLATILITY MODELS AND EUROPEAN OPTIONS PRICING

1.1 General Introduction to the Thesis

This thesis of my Ph.D work in financial mathematics mainly focuses on financial modeling with non-constant volatility and the pricing of financial derivatives. Financial modelling with non-constant volatility has been a widely studied topic for more than 20 years (cf. [2],[128],[129], etc.) and has become more so in this volatile market environment we are experiencing, since the financial meltdown in late 2007. How to accurately and effectively price and risk manage the derivative products has posted an ever challenging task for academics and practitioners alike. In this thesis, my research begins with the introduction of changing volatility models, which are special cases of local volatility models introduced by Dupire (cf.[130]), Derman and Kani (cf. [131]). The introduction of these models has got its own economic meanings, and subsequently, analytic formulae for European option prices under the model settings are obtained. These specific models and the pricing of European options have not been studied before, to the author’s best knowledge, and the model implementation can be quickly put into practice. At the end of the chapter, a simple addition of incomplete information on the volatility term extends the model setting to another category of non-constant volatility models: stochastic volatility models and more general, local-stochastic volatility models, in which the level of asset volatility not only depends on the level of asset price but also is driven by its own stochastic process.

Local-stochastic volatility models, while their set-up has been discussed in [14], the analytical (or semi-analytical) formulae for European option prices have only been obtained for several specific forms, e.g. SABR model ([30]), Zhou Model ([28]), among others. Chapter 2 mainly discusses the model set-up of stochastic volatility and local-stochastic volatility models, from the general framework (follows Romano and Touzi’s setting) to the summary of recent works of different models. At the end of the chapter the author derives the adjustments to the greeks’ calculation in the setting of non-constant implied volatility. These adjust-
ments are crucial for the implementation of local-stochastic volatility models and management of volatility risk.

My major work in this thesis begins from chapter 3, where the first part introduces a general form of foreign-exchange (FX) rate modeling with local-stochastic volatility and two stochastic bond price processes. This local-stochastic volatility process is general that it encompasses all the local-stochastic volatility models summarized in chapter 2, such that the specific form of the model is up to the user’s preference. Then the pricing of FX vanilla options under this general setting is examined through the use of an asymptotic expansion method, based on Watanabe-Yoshida theory (see, [95]). This semi-analytical formula is accurate for the pricing of short and medium expiry options and easy to implement, as shown in the appendix.

Later in chapter 2 another model with stochastic interest rates, stochastic volatility and jump process is proposed and the pricing of FX vanilla option can be derived through Fourier Transform approach. This modelling approach is also general that it includes several forms of stochastic volatility models, jump process models and stochastic interest rate models. The pricing formula is accurate even for long maturity options. Then the model calibration and implementation are applied on a complex structured product on FX rate, namely the PRDC.

The last part of chapter 2 discusses the perfect/partial hedging issues of FX options in the presence of local stochastic volatility and stochastic interest rates, as well as the hedging error analysis in the partial hedging process. Also, the impact of stochastic volatility on the pricing of FX-IR structured products (PRDCs) is examined. These two general models are new at the time of writing, and the model calibration, implementation as well as the extensive discussion on hedging process of FX options are first time seen in literature.

In the financial practice, Monte-Carlo simulation has gained more and more importance in the valuation and risk-management of derivatives, especially the complex structured products. How to effectively simulate asset price process with stochastic volatility has been a widely studied area in financial engineering. In chapter 3 a new method of non-biased Monte Carlo simulation for a popular stochastic volatility model (Heston Model) is proposed by the author, by the use of the powerful Saddle point method borrowed from statistics.

The last chapter studies the LIBOR/swap market model with stochastic volatility and jump process, as well as the pricing of interest rate options under the model.
Here a new model of bond price is proposed, with stochastic volatility and a general jump process (marked point process), and subsequently the LIBOR forward rate model and swap rate model are derived, by the use of change of measure and various approximation techniques. Finally the pricing formulae of vanilla options in forward rate markets and swap rate markets are derived through Fourier Transform and approximation methods.

1.2 Introduction to Changing Volatility Models

In the financial markets, the volatility of financial assets is changing rather than keeping constant. A simple and realistic case is that volatility depends on the state of the asset and/or its movement path, which will produce incomplete information for the pricing and hedging of contingent claims (cf. [17]). A few special cases will be studied in this chapter including the changing volatility models of the log-normal and Ornstein-Uhlenbeck types. The pricing and hedging results for European options will be provided.

1.3 Model Completeness and European Option Pricing

Consider a probability space $(\Omega, \mathcal{P}, \mathcal{F}_T)$ over a finite time interval $[0, T]$. The market model consists of a risk-free asset $B(t)$ and a risky asset $S(t)$ for time $t \geq 0$.

We consider a model given by:

$$
\begin{align*}
    dS(t) &= \mu S(t)dt + \sigma(t)S(t)dW_s(t) \quad (1.1) \\
    dB(t) &= rB(t)dt \quad (1.2)
\end{align*}
$$

under the real world measure $\mathcal{P}$.

If we define the volatility process $t \rightarrow \sigma(t) \in \mathcal{L}^2$ as:

$$
\sigma(t) = \sigma_1 1_{0 \leq t < T_0} + \left\{ \sigma_2 1_{T_0 \leq t < T} 1_{E} + \sigma_1 1_{T_0 \leq t < T} 1_{\Omega \setminus E} \right\} \quad (1.3)
$$

with $\sigma_1, \sigma_2$ are constant value and known at time 0, $T_0$ is fixed and $T_0 \in \mathcal{F}_0$, the probability set $E$ is also pre-specified.

We consider the case in which $E := \{ \omega \in \Omega; S_{T_0}(\omega) \geq B \} \in \mathcal{F}$ with barrier $B \in \mathcal{F}_0$

**Remark** This model has the ability to change the log-normal volatility depending on the state of the spot price at the future deterministic time $T_0$. This can be
seen as a realistic extension of the Black Scholes model([12]), as observed in the practice, future volatility sometimes depends on the asset price level (or the trend).

In the volatility changing case (1.3), we can write the asset price process as:

\[
S_t = S_0 + \int_0^{T_0 \wedge t} \mu_s ds + \int_0^{T_0 \wedge t} \sigma_{1s} dW_s + \sigma_1 \int_0^{T_0 \wedge t} S_s dW_s + \int_{T_0 \wedge t}^{T} \mu_s ds + \int_{T_0 \wedge t}^{T} \sigma_{2s} S_s dW_s + \int_{T_0 \wedge t}^{T} \sigma_1 \Omega_s E S_s dW_s
\]

for \(0 < t < T\). Through Ito’s lemma, we have:

\[
\log S_t = \log S_0 + \mu t + \sigma_1 W_{T_0 \wedge t} + (W_t - W_{T_0 \wedge t})(\sigma_2 I_E + \sigma_1 \Omega_{1 \wedge t})
- \frac{1}{2} \left[ \sigma_1^2 (T_0 \wedge t) + (\sigma_2^2 I_E + \sigma_1^2 \Omega) (t - T_0 \wedge t) \right]
= \log S_0 + \mu t + Z_t - \frac{1}{2} <Z> + Z_t
\]

where \((Z_t)\) is defined as:

\[
\sigma_1 W_{T_0 \wedge t} + (W_t - W_{T_0 \wedge t})(\sigma_2 I_E + \sigma_1 \Omega_{1 \wedge t})
\]

and \(<Z>_t\) is the cross variation of \(Z_t\).

Thus, we can write price process \((S_t)\) in an explicit form:

\[
S_t = S_0 e^{\mu t - \frac{1}{2} <Z>_t + Z_t}
\]

We see that \((Z_t)\) is a (local) martingale: When letting \(0 \leq s < t < T\), there are two cases: First if \(T_0 \leq s < t\)

\[
E[Z_t | \mathcal{F}_s] = E[\sigma_1 W_{T_0} + (W_t - W_{T_0})(\sigma_2 I_E + \sigma_1 \Omega_{1 \wedge t}) | \mathcal{F}_s]
= \sigma_1 W_{T_0} + (W_s - W_{T_0})(\sigma_2 I_E + \sigma_1 \Omega_{1 \wedge t})
= Z_s
\]

Second, when \(s \leq t < T_0\):

\[
E[Z_t | \mathcal{F}_s] = E[\sigma_1 W_t | \mathcal{F}_s]
= \sigma_1 W_s = Z_s
\]

We then show in the economy (1.1)-(1.2), and in a more general case with only one Brownian motion with state dependent volatility, the market is complete.
Theorem 1.1 If the volatility process \( \sigma_t \) is time \( t \) and state \( S_t \) dependent, \( \sigma := \sigma(t, S_t) \), then the market is complete.

Consider the market given by:

\[
\begin{align*}
    dB_t &= rB_t dt \\
    dS_t &= S_t \mu(t, S_t) dt + S_t \sigma(t, S_t) dW_t
\end{align*}
\]  

(1.4)

and a contingent claim \( \chi = \phi(S_T) \).

Define \( F \) as a solution to the PDE:

\[
F_t + r s F_s + \frac{1}{2} \sigma(t, s)^2 s^2 F_{ss} = r F \\
F(T, s) = \phi(s)
\]

and assuming \( F_s \sigma(t, s) s \) is in \( \mathcal{L}^2 \).

Then \( \chi \) can be replicated by the relative portfolio:

\[
\begin{align*}
    u^0(t) &= \frac{F(t, S(t)) - S(t) F_s(t, S(t))}{F(t, S(t))} \\
    u^1(t) &= \frac{S(t) F_s(t, S(t))}{F(t, S(t))}
\end{align*}
\]  

(1.5)

and the portfolio value process \( V^u(t) \) is given by:

\[
V^u(t) = F(t, S(t))
\]

(1.6)

Proof: Apply Ito’s lemma to \( V(t) \) defined by (1.6), \(^1\)

\[
\begin{align*}
    dV(t) &= (F_t + \mu(t, s)s F_s + \frac{1}{2} \sigma(t, s)^2 s^2 F_{ss}) dt + s \sigma(t, s) F_s dW(t) \\
    &= V \left( \frac{F_t + \mu(t, s)s F_s + \frac{1}{2} \sigma(t, s)^2 s^2 F_{ss}}{V} \right) dt + V \left( \frac{s \sigma(t, s) F_s}{V} \right) dW(t)
\end{align*}
\]

\(^1\) Some technical assumptions have to be imposed here: since \( S(t) \) is semi-martingale, we assume \( \phi(S_T) \) is a simple claim (sufficient for the European option case we consider here) such that \( F_t(s) \) is a smooth, \( C^{1,2} \) function, for \( 0 \leq t < T \). And \( \forall s, \int_0^T |\mu(t, s)| dt < \infty, \int_0^T \sigma^2(t, s) dt < \infty \) a.e. \( \mathcal{P} \).
According to the definition of $u^0, u^1$ in (1.5), we see $u^0(t) + u^1(t) = 1$, and we have:

$$dV = Vu^0(t)rdt + V\mu(t,s)u^1(t)dt + Vu^1(t)\sigma(t,s)dW(t)$$

$$= Vu^0(t)rdt + Vu^1(t)(\mu(t,s)dt + \sigma(t,s)dW(t))$$

$$= \left(\frac{Vu^0(t)}{B(t)}\right)dB(t) + \left(\frac{Vu^1(t)}{S(t)}\right)dS(t)$$

(1.7)

(1.7) shows the portfolio is indeed self-financing

Since we have:

$$F_t + rsF_s + \frac{1}{2}\sigma(t,s)^2 s^2 F_{ss} - rF = 0$$

$$F(T,s) = \phi(s) \text{ for all } s \geq 0$$

this shows $V^u(t) = F(t,S(t))$ does replicate the claim $\chi = \phi(S_T)$ for all $S_T \geq 0$ while the hedging strategy at time $t$ is given by holding $\frac{V(t)u^0(t)}{B(t)}$ units of bonds and $\frac{V(t)u^1(t)}{S(t)}$ units of risky assets. Where $(u^0(t), u^1(t))$ are given in (1.5).

Now we have constructed a hedging portfolio $V^h(t)$ which perfectly replicates the value of $\chi$, according to [29], since every contingent claim $\chi$ is attainable, the market defined in (1.4) is complete.

**Remark:** Actually, we can prove the completeness of the market (1.4) from the point of view of change of measure: Considering the discounted asset price process: $\tilde{S}_t \equiv S_t/B_t$, from (1.4), we see the process of $\tilde{S}_t$ as:

$$\frac{d\tilde{S}_t}{\tilde{S}_t} = (\mu(t,S_t) - r)dt + \sigma(t,S_t)dW_t$$

Then apply a change of measure from real measure $P$ to a new measure $Q$ with the Doleans exponential (see [15]) given by $^2$:

$$\varepsilon(t) = \exp\{-\frac{1}{2}\int_0^t \left(\frac{\mu(u,S_u) - r}{\sigma(u,S_u)}\right)^2 du - \int_0^t \left(\frac{\mu(u,S_u) - r}{\sigma(u,S_u)}\right)dW_u\}$$

Then, via the Girsanov theorem, the new Brownian motion under $Q$ becomes:

$^2$ Assuming here $\int_0^T \left(\frac{\mu(u,S_u) - r}{\sigma(u,S_u)}\right)^2 du < \infty \text{ a.e. } P.$
1. General Introduction, Changing Volatility Models and European Options Pricing

\( W^Q(t) = W(t) + \int_0^t \left( \frac{\mu(u,S_u)}{\sigma(u,S_u)} - r \right) du \). Following on from this, under the new measure \( Q \), \( \tilde{S}_t \) is given as:

\[
\frac{d\tilde{S}_t}{S_t} = \sigma(t,S_t)dW^Q_t
\]

and becomes a martingale. And \( Q \) can be referred to as the risk-neutral measure. Since we only have one volatility parameter \( \sigma(t,s) \) here, according to Theorem 2.2 of ([132]), which is listed in Appendix, we see the Equivalent Martingale Measure (EMM) \( Q \) is unique, then according to [29], the market defined by (1.4) is complete.

According to (1.8), the Doleans exponential in the volatility changing case (1.3) is given as:

\[
\varepsilon(t) = \exp\left\{ -\frac{1}{2} \int_0^t \left( \frac{\mu - r}{H_u^2} \right)^2 \, du - \int_0^t \left( \frac{\mu - r}{H_u^2} \right) dW_u \right\}
\]

where \( H_t \) is given as:

\[
H_t^2 := < Z >_t = \sigma_1^2 (T_0 \wedge t) + (\sigma_1^2 \mathbb{1}_{\Omega} + \sigma_2^2 \mathbb{1}_{E}) (t - (T_0 \wedge t))
\]

Since the market is complete, perfect replication is achievable, so we can consider the pricing and hedging of a vanilla European call option \( C(S_t, \sigma(t,S_t), K, T, r) \) under the risk-neutral measure \( Q \).

1.4 Single Period Volatility Changing Problems

1.4.1 Fixed Volatility Changing Time with Barrier \( B \)

**Proposition 1.1** In the case given in (1.1)-(1.2), the call option price at time \( t = 0 \) defined as \( e^{-rT} \mathbb{E}[S_T - K]^+ \), with strike \( K \), maturity \( T \) and risk-free rate \( r \), is given by:
\[ C(S_0, \sigma, K, T, r) = \text{BS}(S_0, \sigma_1, K, T, r) \]
\[ + \ S_0 e^{-\frac{1}{2} \sigma_1^2 T_0 + \frac{1}{2} A^2} N_2 \left( x - A, \frac{a_1 A + B_1}{\sqrt{1 + a_1^2}}; \frac{-a_1}{\sqrt{1 + a_1^2}} \right) \]
\[ - \ K e^{-r T} N_2 \left( x, \frac{B_2}{\sqrt{1 + a_2^2}}; \frac{-a_2}{\sqrt{1 + a_2^2}} \right) \]
\[ - \ S_0 e^{-\frac{1}{2} \sigma_1^2 T_0 + \frac{1}{2} A^2} N_2 \left( x - A, \frac{a_3 A + B_3}{\sqrt{1 + a_3^2}}; -\frac{a_3}{\sqrt{1 + a_3^2}} \right) \]
\[ + \ K e^{-r T} N_2 \left( x, \frac{B_4}{\sqrt{1 + a_4^2}}; \frac{-a_4}{\sqrt{1 + a_4^2}} \right) \]

with

\[ x = \frac{\log(S_0/B) + r T_0 - \frac{1}{2} \sigma_1^2 T_0}{\sigma_1 \sqrt{T_0}} \]
\[ A = -\sigma_1 \sqrt{T_0} \]
\[ a_1 = a_2 = -\frac{\sigma_1 \sqrt{T_0}}{\sigma_2 \sqrt{T - T_0}} \]
\[ a_3 = a_4 = -\sqrt{\frac{T_0}{T - T_0}} \]
\[ B_1 = \frac{\log(S_0/B) + r T - 1/2 \sigma_1^2 T_0 + 1/2 \sigma_2^2 (T - T_0)}{\sigma_2 \sqrt{T - T_0}} \]
\[ B_2 = \frac{\log(S_0/B) + r T - 1/2 \sigma_1^2 T_0 - 1/2 \sigma_2^2 (T - T_0)}{\sigma_2 \sqrt{T - T_0}} \]
\[ B_3 = \frac{\log(S_0/B) + r T + 1/2 \sigma_1^2 (T - 2T_0)}{\sigma_1 \sqrt{T - T_0}} \]
\[ B_4 = \frac{\log(S_0/B) + (r - 1/2 \sigma_1^2) T}{\sigma_1 \sqrt{T - T_0}} \]

Here, \( N_2 \) is defined as (cf. [127]):

\[ N_2(x, y; \rho) \triangleq \int_{-\infty}^{x} \int_{-\infty}^{y} n_2(u, v; \rho) dudv \]
\[ n_2(x, y; \rho) \triangleq \frac{1}{2\pi \sqrt{1 - \rho^2}} e^{-\frac{x^2 - 2\rho xy + y^2}{2(1 - \rho^2)}} \]
Proof. See Appendix 1.7.1.

Apart from the log-normal model given in (1.1)-(1.2), we now consider this changing volatility problem with an Ornstein-Uhlenbeck process, which is given as (under the risk-neutral pricing measure $Q$):

\[ dF_t = \theta(\bar{F} - F_t)dt + \sigma_t dW_t \quad (1.9) \]
\[ dB_t = rB_t dt \quad (1.10) \]

Defining the mean value of $F$ at time $T$ as $M_{t,T}(F) := \text{Mean}(F_T \mid F_t)$, and variance $V_{t,T}(F) := \text{Variance}(F_T \mid F_t)$ by standard manipulation[78], and assuming the volatility $\sigma$ is constant throughout, the distribution of $F_t$ is given by:

\[ M_{t,T}(F) = e^{-\theta(T-t)}F_t + \bar{F}(1 - e^{-\theta(T-t)}) \quad (1.11) \]
\[ V_{t,T}(F) = \frac{\sigma^2}{2\theta}(1 - e^{-2\theta(T-t)}) \quad (1.12) \]

We see the distribution of $F_T \mid F_t$ is Gaussian, and its density function $q(x)$ is given by:

\[ q(x) = \frac{1}{\sqrt{2\pi V_{t,T}(F)}} e^{-\frac{(x-M_{t,T}(F))^2}{2V_{t,T}(F)}} \quad (1.13) \]

**Proposition 1.2** In the cases given in (1.9)-(1.10), assuming $\sigma$ is constant throughout $t \in [0,T]$, the call option price at time $t$ defined as $e^{-r(T-t)}E[S_T - K]^+$ with strike $K$, maturity $T - t$ and risk-free rate $r$, is given as:

\[ C^{OU}(F(t), \sigma, K, T, r) = e^{-r(T-t)} \int_K^{\infty} (x - K) \frac{1}{\sqrt{2\pi V}} e^{-\frac{(x-M)^2}{2V}} dx \]
\[ = e^{-r(T-t)} \left\{ \sqrt{\frac{V}{2\pi}} e^{-\frac{1}{2} \left( \frac{M-K}{\sqrt{V}} \right)^2} + (M - K)N \left( -\frac{M-K}{\sqrt{V}} \right) \right\} \]
\[ = e^{-r(T-t)} \sqrt{V} \left( n(y) + yN(y) \right) \quad (1.14) \]

with $M, V$ denoting $M_{t,T}$ and $V_{t,T}$ which are given in (1.11)-(1.12), $y = \frac{M-K}{\sqrt{V}}$.

Furthermore, the delta-hedging strategy follows as: (for $0 \leq t < T$)

\[ \Delta^{OU}(t) = e^{-(r+\theta)(T-t)}N(y) \quad (1.15) \]

**Proof:** See Appendix 1.7.2.

---

3 If $Q$ is a risk-neutral measure and $F_t$ a forward price process, then $\bar{F}$ needs to be time-dependent in order to ensure $F_t$ is a martingale.
We now extend our constant $\sigma$ case in Proposition 1.2 to the changing volatility (here, I name the diffusion parameter $\sigma$ as the volatility, which is not the effective volatility parameter as shown in (1.12)) case as in (1.3). Again, we specify $\mathbb{E} := \{\omega \in \Omega; S_{T_0}(\omega) \geq B\} \in \mathcal{F}$ with hitting barrier $B \in \mathcal{F}_0$

**Proposition 1.3** In this changing volatility case, the call option value defined as $e^{-rT}\mathbb{E}[F_T - K]^+$ at time 0, with strike $K$, maturity $T$, volatility changing time $T_0$ and risk-free rate $r$, is given by:

$$
C^{OU}(F_0, \sigma_1, K, T, r) + \frac{e^{-rT}}{\sqrt{V_{0T_0}}} \{I_1 + I_2 + I_3\} \tag{1.16}
$$

with

$$
I_1 = \sqrt{\frac{V_{T_0}^2}{D_{12}}} e^{-\frac{1}{2} E_2 n(C_{12})} \mathcal{N}\left( -\frac{B V_{0T_0} - M_{0T_0} V_{T_0}^2}{\sqrt{D_{12}}} \right)
$$

$$
I_2 = -\sqrt{V_{0T_0}} A N_2 \left( q, \frac{D_{22}}{\sqrt{1 + a_{22}^2}}; -a_{22} \frac{1}{1 + a_{22}^2} \right)
$$

$$
I_3 = -e^{-\theta(T-T_0)} \sqrt{V_{0T_0}} \left\{ -n(q) \mathcal{N}(a_{22} q + D_{22}) + \frac{n(D_{22}) \mathcal{N}(F_2)}{\sqrt{1 + a_{22}^2}} \right\}
$$

$$
+ e^{-\theta(T-T_0)} M_{0T_0} N_2 \left( q, \frac{D_{22}}{\sqrt{1 + a_{22}^2}}; -a_{22} \frac{1}{\sqrt{1 + a_{22}^2}} \right)
$$

$$
+ e^{-\theta(T-T_0)} \sqrt{V_{0T_0}} \left\{ -n(q) \mathcal{N}(a_{21} q + D_{21}) + \frac{n(D_{21}) \mathcal{N}(F_1)}{\sqrt{1 + a_{21}^2}} \right\}
$$

$$
- e^{-\theta(T-T_0)} M_{0T_0} N_2 \left( q, \frac{D_{21}}{\sqrt{1 + a_{21}^2}}; -a_{21} \frac{1}{\sqrt{1 + a_{21}^2}} \right)
$$
where: \( q = \frac{M_{0T_0} - B}{\sqrt{V_{0T_0}}} \), \( A = \bar{F}(1 - e^{-\theta(T-T_0)}) - K \), \( \bar{B} = B + A e^{-\theta(T-T_0)} \). And

\[
V_{T_0 T}^{1,2} = \frac{\sigma_{1,2}^2}{2\theta}(1 - e^{-2\theta(T-T_0)})
\]

\[
D_{11} = e^{-2\theta(T-T_0)}V_{0T_0} + V_{1T_0}^1
\]

\[
D_{12} = e^{-2\theta(T-T_0)}V_{0T_0} + V_{2T_0}^2
\]

\[
D_{21} = \frac{A + M_{0T_0}e^{-\theta(T-T_0)}}{\sqrt{V_{1T_0}^1}}
\]

\[
D_{22} = \frac{A + M_{0T_0}e^{-\theta(T-T_0)}}{\sqrt{V_{2T_0}^2}}
\]

\[
C_{11} = \frac{Ae^{-\theta(T-T_0)}V_{0T_0} - M_{0T_0}V_{1T_0}^1}{\sqrt{D_{11}}}
\]

\[
C_{12} = \frac{Ae^{-\theta(T-T_0)}V_{0T_0} - M_{0T_0}V_{2T_0}^2}{\sqrt{D_{12}}}
\]

\[
E_{1,2} = A^2V_{0T_0} + (M_{0T_0})^2V_{1,2T_0}^{1,2}
\]

\[
a_{21} = -\sqrt{V_{1T_0}^1} e^{-\theta(T-T_0)}
\]

\[
a_{22} = -\sqrt{V_{2T_0}^2} e^{-\theta(T-T_0)}
\]

\[
F_1 = (1 + a_{21}^2)q + a_{21}B_{21}
\]

\[
F_2 = (1 + a_{22}^2)q + a_{22}B_{22}
\]

and \( C^{OU}(\cdot, r, K, T, \sigma) \) is given by (1.14). The delta-hedging strategy for \( 0 \leq t < T_0 \) is given in Appendix 1.7.3, while \( t \geq T_0, \Delta(t) \) is given by (1.15).

**Proof:** See Appendix 1.7.3.

**Remark** This approach can be extended to any asset price process with an analytical or semi-analytical density distribution function. Take the square root process as an example:

\[
dF_t = \kappa(\theta - F_t)dt + \eta\sqrt{F_t}dW_F(t)
\]

\[
F_0 = F
\]

\[
 dB_t = rB_t dt
\]
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Under the same volatility changing criterion as in Proposition 1.3, the option pricing formulae in Propositions 1.2 and 1.3 apply to the square root process, but only if we change the density function \( q(x) \) to a semi-analytical form via Fourier inversion:

\[
p(T, y | F) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iu\phi_T(iu)} du
\]

where \( \phi_T(u) \) is the characteristic function defined as: \( E[e^{uF_T}|F] \), and it is given as a standard result (cf. [91]):

\[
\phi_T(u) = \exp\left(\frac{uV}{1 - \frac{u^2}{2}} - \frac{u^2}{2}\right)
\]

(1.17)

\[
c = 2\kappa \frac{1 - e^{-\kappa T}}{\eta}\]

\[
b = 2\kappa V_t \left(1 - e^{-\kappa T}\right)\eta^2
\]

1.4.2 Random Volatility Changing Time with a Hitting Barrier

Now we extend our changing volatility case to \( \text{random changing time} \ T_0 = \inf\{0 \leq t < T, S(t) = B\} \), with \( B \in F_0 \) is the hitting barrier and the volatility term \( \sigma(t) \) will change at \( T_0 \), i.e. \( \sigma(T_0-) = \sigma_1, \sigma(T_0) = \sigma_2 \). In this case, firstly assuming that asset price \( S(t) \) follows (1.1)-(1.2) under a risk-neutral measure, we give the following proposition:

**Proposition 1.4** In the random volatility changing time case, where \( T_0 = \inf\{0 \leq t < T, S(t) = B\} \), the European call option, \( C(S_0, \sigma_1, K, T, r) \) is given as:

\[
BS(S_0, \sigma_1, K, T, r) + \int_0^T e^{-rt} BS(B, \sigma_2, \sigma_1, K, T - t, r) f_{\tau_B}(t; S_0, B) dt
\]

where \( f_{\tau_B}(t; S_0, B) \) is the first hitting time density function:

\[
f_{\tau_B}(t; S_0, B) = \frac{|\log(S_0/B)|}{\sqrt{2\pi}\sigma_1 t^{3/2}} e^{-|\log(S_0/B)| + (r - \frac{1}{2}\sigma_1^2)t^2/2\sigma_1^2 t}
\]

In addition, delta hedging for \( 0 \leq t < \tau_B \) is given in Appendix, for \( \tau_B \leq t < T \), delta is given by: \( N(d_3), d_3 = \frac{\log(S_t/K) + (r - \frac{1}{2}\sigma_2^2)(T-t)}{\sigma_2 \sqrt{T-t}} \).

**Proof:** See Appendix 1.7.4.

**Remark:** The above models a realistic case, that if the asset price ”triggers” a critical psychological level (\( B \) here), a large selling off/buying will follow, which
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changes its volatility to a new level ($\sigma_2$ here).

### 1.5 Multi-Period Volatility Changing Problems

It is interesting to extend the volatility changing problem with the log-normal process considered in Proposition 1.1 to a multi-period volatility changing case. Firstly, let us consider a three-period volatility changing problem with an additional changing time $T_1 \in (T_0, T)$ with changing barrier $B_1$. We then specify that if $S(T_1) \geq B_1$ the volatility term $\sigma(t) \to \sigma_3$; otherwise, it will stay at the previous level through time $t \in (T_1, T)$. The volatility changing condition at time $T_0$ remains the same as in Proposition 1.1, in other words if $S(T_0) \geq B_0$, then $\sigma(t) = \sigma_2$; otherwise, $\sigma(t) = \sigma_1$ for $t \in [T_0, T_1)$.

In this case, we give the following proposition:

**Proposition 1.5** In the three-period volatility changing case considered above, assuming $t = 0$, the European call option price $e^{-rT}E[S_T - K]_+$ is given as:

\[
C(x_0, \sigma(t), K, T, r) = \Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4
\]

\[
\Lambda_1 = S_0 N_3(d_{1,+}^{(1)}, d_{2,+}^{(1)}, d_{3,+}^{(1)}; \rho_{12}^{(1)}, \rho_{13}^{(1)}, \rho_{23}^{(1)})
- e^{-rT}K N_3(d_{1,-}^{(1)}, d_{2,-}^{(1)}, d_{3,-}^{(1)}; \rho_{12}^{(1)}, \rho_{13}^{(1)}, \rho_{23}^{(1)})
\]

\[
\Lambda_2 = S_0 N_3(d_{1,+}^{(2)}, -d_{2,+}^{(2)}, d_{3,+}^{(2)}; -\rho_{12}^{(2)}, \rho_{13}^{(2)}, -\rho_{23}^{(2)})
- e^{-rT}K N_3(d_{1,-}^{(2)}, -d_{2,-}^{(2)}, -d_{3,-}^{(2)}; -\rho_{12}^{(2)}, \rho_{13}^{(2)}, -\rho_{23}^{(2)})
\]

\[
\Lambda_3 = S_0 N_3(-d_{1,+}^{(3)}, -d_{2,+}^{(3)}, d_{3,+}^{(3)}; -\rho_{12}^{(3)}, -\rho_{13}^{(3)}, \rho_{23}^{(3)})
- e^{-rT}K N_3(-d_{1,-}^{(3)}, -d_{2,-}^{(3)}, d_{3,-}^{(3)}; -\rho_{12}^{(3)}, -\rho_{13}^{(3)}, \rho_{23}^{(3)})
\]

\[
\Lambda_4 = S_0 N_3(-d_{1,+}^{(4)}, -d_{2,+}^{(4)}, d_{3,+}^{(4)}; \rho_{12}^{(4)}, -\rho_{13}^{(4)}, -\rho_{23}^{(4)})
- e^{-rT}K N_3(-d_{1,-}^{(4)}, -d_{2,-}^{(4)}, d_{3,-}^{(4)}; \rho_{12}^{(4)}, -\rho_{13}^{(4)}, -\rho_{23}^{(4)})
\]

(1.18)
where for \( n \in \{1, 2, 3, 4\} \)

\[
d_{1, \pm}^{(n)} = \frac{\log(S_0/B_0) + rT_0 \pm \frac{1}{2} V_{0T_0}^{(n)}}{\sqrt{V_{0T_0}^{(n)}}}
\]

\[
d_{2, \pm}^{(n)} = \frac{\log(S_0/B_1) + rT_1 \pm \frac{1}{2} V_{0T_1}^{(n)}}{\sqrt{V_{0T_1}^{(n)}}}
\]

\[
d_{3, \pm}^{(n)} = \frac{\log(S_0/K) + rT \pm \frac{1}{2} V_{0T}^{(n)}}{\sqrt{V_{0T}^{(n)}}}
\]

\[
\rho_{12}^{(n)} = \sqrt{\frac{V_{0T_0}^{(n)}}{V_{0T_1}^{(n)}}}, \quad \rho_{13}^{(n)} = \sqrt{\frac{V_{0T_0}^{(n)}}{V_{0T_1}^{(n)} V_{0T}^{(n)}}}, \quad \rho_{23}^{(n)} = \sqrt{\frac{V_{0T_1}^{(n)}}{V_{0T}^{(n)}}}.
\]

The variance terms are:

\[
V_{0T_0}^{(1)} = \sigma_1^2 T_0, \quad V_{0T_1}^{(1)} = \sigma_1^2 T_0 + \sigma_2^2 (T_1 - T_0), \quad V_{0T_0}^{(2)} = \sigma_1^2 T_0 + \sigma_2^2 (T_1 - T_0) + \sigma_3^2 (T - T_1);
\]

\[
V_{0T_0}^{(2)} = \sigma_1^2 T_0, \quad V_{0T_1}^{(2)} = \sigma_1^2 T_0 + \sigma_2^2 (T_1 - T_0), \quad V_{0T_0}^{(3)} = \sigma_2^2 T_0 + \sigma_3^2 (T - T_1);
\]

\[
V_{0T_0}^{(3)} = \sigma_1^2 T_0, \quad V_{0T_1}^{(3)} = \sigma_1^2 T_0 + \sigma_2^2 (T_1 - T_0), \quad V_{0T_0}^{(4)} = \sigma_1^2 T_0 + \sigma_3^2 (T - T_1) ;
\]

\[
V_{0T_1}^{(4)} = \sigma_1^2 T_0, \quad V_{0T_1}^{(4)} = \sigma_1^2 T_1, \quad V_{0T}^{(4)} = \sigma_1^2 T_0 + \sigma_2^2 (T - T_1).
\]

The cumulative trivariate Normal distribution function \( \mathcal{N}_3(X_{1, \pm}, X_{2, \pm}, X_{3, \pm}; \rho_{12}, \rho_{13}, \rho_{23}) \) is defined as:

\[
\int_{-\infty}^{X_{1, \pm}} \int_{-\infty}^{X_{2, \pm}} \int_{-\infty}^{X_{3, \pm}} \frac{\exp\left(-\frac{1}{2} \mathbf{Y}^T \mathbf{C}^{-1} \mathbf{Y} \right)}{\sqrt{(2\pi)^3 \det \mathbf{C}}} \, d\mathbf{Y}
\]

with \( \mathbf{Y} = (Y_{1, \pm}, Y_{2, \pm}, Y_{3, \pm}) \):

\[
Y_{1, \pm} = \frac{\log(S_T/S_0) \pm \frac{1}{2} V_{0T_0}}{\sqrt{V_{0T_0}}}
\]

\[
Y_{2, \pm} = \frac{\log(S_T/S_0) \pm \frac{1}{2} V_{0T_1}}{\sqrt{V_{0T_1}}}
\]

\[
Y_{3, \pm} = \frac{\log(S_T/S_0) \pm \frac{1}{2} V_{0T}}{\sqrt{V_{0T}}}
\]

while the correlation matrix \( \mathbf{C} \) is:

\[
\begin{pmatrix}
1 & \rho_{12} & \rho_{13} \\
\rho_{12} & 1 & \rho_{23} \\
\rho_{13} & \rho_{23} & 1
\end{pmatrix}
\]

**Proof:** See Appendix 1.7.5.
Remark This result is applicable also to the O-U process.

As we extend the models to more volatility changing periods, say $n$-period with $n-1$ changing time $\{T_0, T_1, ..., T_{n-2}\}$, the rationale is the same. Firstly, we fix the future volatility path as:

$$S(t) \xrightarrow{\sigma_1} S(T_0) \xrightarrow{\sigma_1/\sigma_2} S(T_1) \xrightarrow{\sigma_1/\sigma_2/\sigma_3} S(T_2) \rightarrow \cdots \rightarrow S(T)$$

which will give us the analytical density function at each time interval with pre-specified volatility. Next, we can compute the European option value as a conditional expectation of the final pay-off as a $n$-dimensional integral. For the $n$-period problem, we have $2^n-1$ volatility paths, associated with the asset price level at each volatility changing time, which will give us the integration ranges. Thus, we can write the European option value as:

$$C(S_t, \sigma(t), K, T, r) = e^{-rT} \sum_{i=1}^{2^n-1} \mathbb{E}^Q[f(S_T)\mid \omega_i]$$

with $f(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ as the pay-off function and $\omega_i \in \mathcal{F}$ the pre-specified $i$-th asset price path with a corresponding volatility path. Writing in integral form, the option value reads:

$$e^{-rT} \sum_{i=1}^{2^n-1} \int_{\Omega_i} p(x_t, x_1, \sigma_{[t, T_0]}, T_0 - t) \int_{\Omega_{i+1}} p(x_1, x_2, \sigma_{[T_0, T_1]}, T_1 - T_0) \int_{\Omega_{i+2}} p(x_n-1, x_n, \sigma_{[T_{n-2}, T_{n-1}]}, T - T_{n-2}) f(x_T) dx_1 dx_2 \cdots dx_n$$

with the integration ranges $\Omega_i$ being pre-specified and associated with the $i$th volatility path.

We can also extend the random volatility changing time problem, whose one-period case was considered in Proposition 1.4, to multi-changing times. The volatilities $\sigma_n \in \mathcal{F}_0$, $n = \{1, 2, ..., N\}$ are pre-specified constant values in $\mathbb{R}_+$, known at the starting time. This changes from $\sigma_i$ to $\sigma_{i+1}$, $i \in \{1, 2, .., N-1\}$ when asset price $S(t)$ hits the pre-specified constant barrier $B_i$. We firstly extend the result in Proposition 1.4 to a two-volatility changing barrier case, with one more barrier $B_2$ which can only be hit after $B_1$ has been hit. When hitting occurs, volatility $\sigma(t)$ changes from $\sigma_2$ to $\sigma_3$.

**Proposition 1.6** In the two-barrier volatility changing problem, under the log-normal asset price process $dS = rSdt + \sigma(t)SdW_t$ in the risk-neutral measure, the European option price $e^{-rT} \mathbb{E}^Q[S_T - K]_+$ at time 0 is given as:
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\[ \text{BS}(S_0, \sigma_1, K, T, r) + \int_0^T e^{-rt} \left[ C(B_1, B_2, \sigma(t), K, T - t, r) - \text{BS}(B_1, \sigma_1, K, T - t, r) \right] f_{\tau_{B_1}}(t) \, dt \]

with the hitting time density function \( f_{\tau_{B_1}}(t) \) given in Proposition 1.4 and \( C(B_1, B_2, \sigma(t), K, T - t, r) \) is given as:

\[ \text{BS}(B_1, \sigma_2, K, T - t, r) + \int_t^T e^{-(s-t)r} \text{BS}(B_2, \sigma_3, \sigma_2, K, T - s, r) f_{B_1,B_2}(s) \, ds \]

\( f_{B_1,B_2}(\cdot) \) is the density function of the first hitting time starting from \( B_1 \) to hit \( B_2 \). It has the same form as \( f_{\tau_{B_1}}(\cdot) \) when replacing \( B_1, S_0, \sigma_1 \) by \( B_2, B_1, \sigma_2 \), respectively.

**Proof**: See Appendix 1.7.6.

**Remark** This barrier-hitting volatility changing problem can also be extended to the \( n \)-barrier \((n > 2)\) case, with the constraint that barrier \( B_n \) can only be hit after \( B_{n-1} \) has been hit. The approach is to work backwards, based on the idea of loss function triggered by volatility changes. Firstly, we compute the option price when the last barrier is hit, then the value conditional on the penultimate barrier is hit until we arrive at the option value at the inception time.

We see that the density function of the first hitting time plays a crucial role in this pricing problem, which is not limited to the log-normal asset price process. For an asset price following an O-U process or square root(CIR) process, the first hitting time density function possesses an expansion form, as given in [116].

### 1.6 Extension to Incomplete Market

#### 1.6.1 A Simple Random Volatility Changing Model – Extension to Stochastic Volatility Model

A simple addition to the volatility changing case considered in Proposition 1.1 will make the market consisting of \((S, B)\) incomplete. To view this, let the volatility process \( \sigma(t) \) be:

\[ \sigma(t) = \sigma_1 1_{0 \leq t < T_0} + \sigma_2 1_{T_0 \leq t < T} \]

for \( 0 \leq t < T \). Now, random volatility \( \sigma_2 \in \mathcal{F}_{T_0} \) has the probability density function: \( p(x) : \mathbb{R}_+ \to \mathbb{R}_+ \). In this case, as we see from Section 1.2, especially (1.8), the martingale measure \( Q \) is not unique anymore, since \( \sigma_2 \in \mathcal{F}_{T_0} \), \( \sigma_2 \) itself
is a non-predictable random variable. Under the martingale measure $Q$, again by the use of the loss function approach, we can easily see that the European option price $C(S_0, \sigma(t), K, T, r) := e^{-rT} E^Q [S_T - K]_+$ is given as:

$$BS[e^{x_0}, \sigma_1, K, T, r] + e^{-rT_0} E^Q [L(x_{T_0}, \sigma_1, \sigma_2, K, T - T_0, r)]$$

with $x_t := \log(S_t)$ and the loss function $L(x_0, \sigma_1, \sigma_2, K, T - T_0, r)$ is defined as before: $BS(e^{x_{T_0}}, \sigma_2, K, T - T_0, r) - BS(e^{x_{T_0}}, \sigma_1, K, T - T_0, r)$. Now, $E^Q [L(x_{T_0}, \sigma_1, \sigma_2, K, T - T_0, r)]$ is given below by a two-dimensional integral, with $x_{T_0}, \sigma_2 \in \mathcal{F}_{T_0}$:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} L(x_{T_0}, \sigma_1, \sigma_2, K, T - T_0, r) d\mu(x_{T_0}) p(\sigma_2) d\sigma_2$$

where $\mu(x_{T_0})$ is the distribution law of $x_{T_0}$ and is given as a Gaussian distribution with mean $\log S_0 + rT_0 - \frac{1}{2} \sigma_1^2 T_0$ and variance $\sigma_2^2 T_0$.

**Remark** Although we can calculate the European option price for this simple random volatility case, we cannot construct a portfolio with $(S_t, B_t)$ to perfectly replicate it at time $t < T_0$, simply because here $\sigma_2 \in \mathcal{F}_{T_0}$ and $\sigma_2$ in the above integral is not a tradeable asset.

### 1.6.2 Future Research

Some special cases of changing volatility problems are considered here, which can be extended to other volatility changing criteria, as well as applied to other asset/volatility processes.

### 1.7 Appendix: Proof

#### 1.7.1 Proof of Proposition 1.1

**Approach 1.**

A direct way is through the computation of conditional expectation, exemplified by two scenarios: if $S(T_0) \geq B$, then

$$\sigma(s) = \begin{cases} \sigma_1 & 0 \leq s < T_0 \\ \sigma_2 & T_0 \leq s < T \end{cases}$$

otherwise, $\sigma(s) = \sigma_1$ throughout the option’s maturity. Then, from a distribution point of view, the European call option value is given as:

$$C(S_0, \sigma, K, T, r) = e^{-rT} E^Q [(e^{x_T} - e^k)_+ | x_{T_0} \geq \log B] + e^{-rT} E^Q [(e^{x_T} - e^k)_+ | x_{T_0} < \log B]$$
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where \( x_t = \log S_t \), \( k = \log K \).

As we know, the density function \( p(x, y, \sigma, \tau, r) \) of \( x \), being at \( y \) after time \( \tau \) with volatility \( \sigma \) under measure \( Q \), is given as:

\[
p(x, y, \sigma, \tau, r) = \frac{1}{\sqrt{2\pi \sigma^2 \tau}} e^{-\frac{(y + \frac{1}{2} \sigma^2 \tau - x - r \tau)^2}{2\sigma^2 \tau}}
\]

These two conditional expectations can be written as:

\[
E_Q[(e^{x_T} - e^k)_{+}|x_{T_0} \geq \log B] = \int_{\log B}^{\infty} p(x_0, x, \sigma_1, T_0, r) \left\{ \int_{\log B}^{\infty} p(x, y, \sigma_2, T - T_0, r)(e^y - e^k) dy \right\} dx
\]

\[
E_Q[(e^{x_T} - e^k)_{+}|x_{T_0} < \log B] = \int_{-\infty}^{\log B} p(x_0, x, \sigma_1, T_0, r) \left\{ \int_{k}^{\infty} p(x, y, \sigma_1, T - T_0, r)(e^y - e^k) dy \right\} dx
\]

with \( BS[S, \sigma, K, \tau, r] \) denoting the well-known Black-Scholes formula:

\[
BS[S, \sigma, K, \tau, r] = SN(d_1) - e^{-r\tau} KN(d_2)
\]

\[
d_{1,2} = \frac{\log(S/K) + (r \pm 1/2\sigma^2)\tau}{\sigma\sqrt{\tau}}
\]

and \( N(\cdot) \) is the standard normal cumulative distribution function.

**Approach 2.**

Instead of the direct computation of the conditional expectation of the option value, we can see this problem from a hedging point of view. Firstly, assuming at time \( T_0 \) that the volatility change is not triggered, that is \( S_{T_0} < B \) or equivalently \( x_{T_0} < \log B \), then the European call option price is given by the Black-Scholes formula: \( BS(e^{x_0}, \sigma_1, K, T, r) \) which is given in Approach 1. However, if volatility changes, a loss\(^4\) will be triggered. We define a loss function at time \( t \) for the change of our option value: \( L(x_t, \sigma_1, \sigma_2, k, T - t, r) \) and since the volatility for the remaining maturity \( T - t \) is fixed, we see

\[
L(x_t, \sigma_1, \sigma_2, k, T - t, r) = BS(e^{x_t}, \sigma_2, e^k, T - t, r) - BS(e^{x_t}, \sigma_1, e^k, T - t, r) =: BS(e^{x_t}, \sigma_1, \sigma_2, e^k, T - t, r)
\]

\(^4\) can be either positive or negative
Thus, at time $t$, the option value can be decomposed to:

$$C(S_0, \sigma, K, T, r) = e^{-rT}E^Q[e^{xT} - e^{k}] + BS(e^{xT}, \sigma_1, K, T, r) + e^{-rT_0}E^Q[L(x_{T_0}, \sigma_1, \sigma_2, k, T - T_0, r)]$$

Since the volatility changing time $T_0$ is fixed, we now need the distribution of $x_{T_0}$, which is a Gaussian variable:

$$x_{T_0} \sim N\left(x_0 - \frac{1}{2}\sigma_1^2T_0 + rT_0, \sigma_1^2T_0\right)$$

and its density function:

$$p_t(y) = \frac{1}{\sqrt{2\pi\sigma_1^2T_0}} e^{-\frac{(y + \frac{1}{2}\sigma_1^2T_0 - x_0 - rT_0)^2}{2\sigma_1^2T_0}}$$

(1.19)

The call option price follows:

$$C(S_0, \sigma, K, T, r) = BS(S_0, \sigma_1, K, T, r) + e^{-rT_0} \int_{\log B}^{\infty} L(x, \sigma_1, \sigma_2, k, T - T_0, r)p(x)dx$$

and delta $\Delta(t) = \frac{\partial C}{\partial S}(S, \sigma(t), K, T, r)$ follows directly thereafter.

Further computation shows that both approaches give the same result.

Actually, we can simplify the option price formula further with the use of bivariate standard normal distribution $N_2$ with the following identities:

$$\int_{-\infty}^{x} N(az + B)n(z)dz = N_2\left(x, \frac{B}{\sqrt{1 + a^2}}; -\frac{a}{\sqrt{1 + a^2}}\right)$$

(1.20)

$$\int_{-\infty}^{x} e^{Az}N(az + B)n(z)dz = e^{A^2/2}N_2\left(x - A, \frac{aA + B}{\sqrt{1 + a^2}}; -\frac{a}{\sqrt{1 + a^2}}\right)$$

(1.21)

where

$$N(x) \triangleq \int_{-\infty}^{x} n(y)dy$$

$$N_2(x, y; \rho) \triangleq \frac{1}{2\pi\sqrt{1 - \rho^2}} \int_{-\infty}^{x} \int_{-\infty}^{y} \exp\left(-\frac{X^2 - 2\rho XY + Y^2}{2(1 - \rho^2)}\right)dXdY$$

A probabilistic proof of equalities (1.20)-(1.21) can be found in [126], the simplified results are shown in Proposition 1.1.
The delta hedging at time 0 is given by:

\[ \Delta(0) = N(d_1) + e^{-rT_0} \int_{\log B}^{\infty} \tilde{B}\mathcal{S}(e^x, \sigma_2, \sigma_1, K, T - T_0, r) \frac{f(x)}{e^x} \, dx \]

with \( f(x) \) given as:

\[ f(x) = \frac{1}{\sqrt{2\pi} \sigma_1^2 T_0} e^{-\frac{(x-x_0+\frac{1}{2}\sigma_1^2 T_0-rT_0)^2}{2\sigma_1^2 T_0}} \]

At time \( t \geq T_0 \), we then apply standard Black-Scholes delta hedging strategy with:

\[ \Delta(t) = N(d_1) \]

\( d_1 \) is given as:

\[ \frac{(x_0-\log K)+\frac{1}{2}\sigma_1^2(T-t)}{\sigma_1\sqrt{T-t}} \]

1.7.2 Proof of Proposition 1.2

As under the pricing measure \( F_t \) follows the O-U process, its probability density function at option expiry time \( T \) is known in closed form as: (1.11)-(1.12). In this case, the European call option price is given as:

\[ C(F_t, \sigma, K, T - T, r) := e^{-r(T-t)} \mathbb{E}^Q[F_T - K]_+ \]

\[ = e^{-r(T-t)} \int_{K}^{\infty} (x - K) p(x) \, dx \]

\[ = e^{-r(T-t)} \left\{ \int_{K}^{\infty} \frac{x}{\sqrt{2\pi V}} e^{-\frac{(x-M)^2}{2V}} \, dx - K \int_{K}^{\infty} \frac{1}{\sqrt{2\pi V}} e^{-\frac{(x-M)^2}{2V}} \, dx \right\} \]

With a change of variable \( y = \frac{x-M}{\sqrt{V}} \), the two terms in the bracket are given as:

\[ \sqrt{V} \int_{\frac{K-M}{\sqrt{V}}}^{\infty} \frac{y}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \, dy + (M - K) \int_{\frac{K-M}{\sqrt{V}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \, dy \]

which leads to (1.14). The delta hedging strategy is given by the direct differentiation: \( \Delta^{OU}(t) = \frac{\partial C}{\partial F}(F, \sigma, K, T - t, r) \) and is given in (1.15) through direct algebra.
1.7.3 Proof of Proposition 1.3

Following the ideas in Proposition 1.1 for the log-normal process, we define a loss function if volatility is changed as $L(F_t, \sigma_1, \sigma_2, K, T - t, r)$. Here, it is given as $C^{OU}(F_t, \sigma_2, K, T - t, r) - C^{OU}(F_t, \sigma_1, K, T - t, r)$, $C^{OU}$ is the call option value at time $t$, given in Proposition 1.2. Following on from this, the call option price is decomposed to:

$$C(F_t, \sigma(t), K, T - t, r) := e^{-rT} E^Q[F_T - K]_+$$

$$= C^{OU}(F_t, \sigma_1, K, T - t, r) + e^{-rT_0} E^Q[L(F_{T_0}, \sigma_1, \sigma_2, K, T - T_0, r)]$$

The density function $q(x)$ of $F_{T_0}$ is given in (1.13) with volatility $\sigma_1$ and the option value at time 0:

$$C(F_0, \sigma(t), K, T, r) = C^{OU}(F_0, \sigma_1, K, T, r)$$

$$+ e^{-rT_0} \int_B^\infty L(x, \sigma_1, \sigma_2, K, T - T_0, r)q(x)dx$$

$$= C^{OU}(F(0), \sigma_1, K, T, r)$$

$$+ e^{-rT_0} \int_B^\infty [C^{OU}(x, \sigma_2, K, T - T_0, r) - C^{OU}(x, \sigma_1, K, T - T_0, r)]q(x)dx$$

(1.22)

with $q(x)$ as the density function of $F_{T_0}$, which is given by:

$$q(x) = \frac{1}{\sqrt{2\pi V_{T_0}(F)}} e^{-\frac{(x - M_{T_0}(F))^2}{2V_{T_0}(F)}}$$

(1.23)

Using the results of Proposition 1.2, we can rewrite the second term at the RHS of (1.22) as:

$$e^{-rT_0} \int_B^\infty \left[ \sqrt{V_{T_0}^2(\frac{M_{T_0}T - K}{\sqrt{V_{T_0}^2}})} + (M_{T_0}T - K)N\left(\frac{M_{T_0}T - K}{\sqrt{V_{T_0}^2T}}\right) 

- \sqrt{V_{T_0}^1Tn}\left(\frac{M_{T_0}T - K}{\sqrt{V_{T_0}^1T}}\right) - (M_{T_0}T - K)N\left(\frac{M_{T_0}T - K}{\sqrt{V_{T_0}^1T}}\right) \right]q(x)dx$$

(1.24)

where $V_{T_0}^{1,2}$ represent the variance terms with volatility terms $\sigma_1$ and $\sigma_2$, respectively:

$$V_{T_0}^{1,2} = \frac{\sigma_{1,2}^2}{2\theta} (1 - e^{-2\theta(T-T_0)})$$

and mean $M_{T_0}$ is given as:

$$M_{T_0} = e^{-\theta(T-T_0)}x + \bar{F}(1 - e^{-\theta(T-T_0)})$$
Further tedious Gaussian calculations with the repeated use of (1.20)-(1.21) give the above (1.24) as:

\[ e^{-rT} \sqrt{V_0 T_0} \{ I_1 + I_2 + I_3 \} \]

as shown in Proposition 1.3.

The delta hedge at time \( t \in [0, T_0) \) is given by direct computation:

\[
\Delta(t) = \Delta^{OU}(t) + \int_B^\infty \left[ C^{OU}(x, r, K, T - T_0) - C^{OU}(x, r, K, T_0) \right] q(x) \left( \frac{M_t T_0}{V_t T_0} \right) e^{-\theta T_0} dx
\]

where \( \Delta^{OU}(t) \) is given in Proposition 1.2. For \( t \in [T_0, T) \), since \( \sigma \) is fixed, the delta hedging strategy is the same as (1.15).

### 1.7.4 Proof of Proposition 1.4

Under the log-normal process, when barrier \( B \) is not triggered, the option price is given by a standard Black-Scholes formula with volatility \( \sigma_1 \). Otherwise, let us define a loss function: \( L(B, \sigma_1, \sigma_2, K, T - \tau_B, r) \), given by \( BS(B, \sigma_2, K, T - \tau_B, r) - BS(B, \sigma_1, K, T - \tau_B, r) = BS(B, \sigma_2, K, T - \tau_B, r) \). \( BS(\cdot) \), is the Black-Scholes formula at hitting time \( \tau_B \) with the asset price level at \( B \). In this scenario, the option price at time 0 is decomposed as:

\[
C(S_0, \sigma(t), K, T, r) := e^{-RT} \mathbb{E}[e^{\tau_B} L(B, \sigma_1, \sigma_2, K, T - \tau_B, r)] = BS(S_0, \sigma_1, K, T, r) + \mathbb{E}[e^{-\tau_B} L(B, \sigma_1, \sigma_2, K, T - \tau_B, r)]
\]

Since in our loss function only the first hitting time \( \tau_B \) is a random variable, we need to acquire its distribution density function. Via the reflection principle\((15)\) and a change of measure, the distribution becomes a standard result in the pricing of barrier options, e.g.\((92)\), which is given as:

\[
f_{\tau_B}(t; S_0, B) = \frac{\left| \log \left( \frac{S_0}{B} \right) \right|}{\sqrt{2\pi \sigma_1^2 t}} e^{-\left( \log(S_0/B) + (r - \frac{1}{2} \sigma_1^2) t \right)^2/2\sigma_1^2 t}
\]

Thus,

\[
\mathbb{E}[e^{-\tau_B} L(B, \sigma_1, \sigma_2, K, T - \tau_B, r)] = \int_0^T e^{-rt} L(B, \sigma_1, \sigma_2, K, T - t, r) f_{\tau_B}(t) dt
\]

and the option price follows as:

\[
BS(S_0, \sigma_1, K, T, r) + \int_0^T e^{-rt} \bar{BS}(B, \sigma_2, \sigma_1, K, T - t, r) f_{\tau_B}(t) dt
\]
Delta hedging at time $t \in [0, \tau_B]$ is given by direct partial differentiation with respect to $S_t$ (assuming $S_0 < B$):

$$
\Delta(t) = N(d_3) + \int_0^T e^{-rt} \widetilde{BS}(B, \sigma_2, \sigma_1, K, T - t, r) \frac{\partial f_{\tau_B}}{\partial S}(t) dt
$$

with

$$
\frac{\partial f_{\tau_B}}{\partial S}(t) = \frac{-e^{-\frac{d_3^2}{2\sigma_1^2t}}}{S_t \sigma_1 t^{3/2} \sqrt{2\pi}} \left\{ 1 + \log \left( \frac{B}{S_t} \right) \frac{d_1}{\sigma_1^2 t} \right\}
$$

$$
d_1 = \log \frac{S_t}{B} + \left( r - \frac{1}{2} \sigma_1^2 \right) t
$$

$$
d_3 = \log \left( \frac{S_t/K}{\sigma_1 \sqrt{T - t}} \right) + \frac{r + \frac{1}{2} \sigma_1^2 \sqrt{T - t}}{\sigma_1}
$$

Furthermore, $n(x)$ is a standard normal density function: $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

#### 1.7.5 Proof of Proposition 1.5

**Proof** We follow the rationale in Approach 2 in order to prove Proposition 1.1. At this point, we now have $2^2 = 4$ scenarios (volatility paths):

1. $S(T_0) \geq B_0, S(T_1) \geq B_1$ with associated volatility path: $\sigma_1 \rightarrow \sigma_2 \rightarrow \sigma_3$
2. $S(T_0) \geq B_0, S(T_1) < B_1$ with associated volatility path: $\sigma_1 \rightarrow \sigma_2 \rightarrow \sigma_2$
3. $S(T_0) < B_0, S(T_1) \geq B_1$ with associated volatility path: $\sigma_1 \rightarrow \sigma_1 \rightarrow \sigma_3$
4. $S(T_0) < B_0, S(T_1) < B_1$ with associated volatility path: $\sigma_1 \rightarrow \sigma_1 \rightarrow \sigma_1$

The European call option value (the put option value can be found via put-call parity), as an conditional $\mathcal{F}_t$ expectation, is now given by four components:

$$
C(x_t, \sigma(t), K, T, r) = e^{-rT} \mathbb{E}_Q [(e^{x_T} - e^k)^+ | x_T \geq \log B_0, x_T \geq \log B_1] + e^{-rT} \mathbb{E}_Q [(e^{x_T} - e^k)^+ | x_T \geq \log B_0, x_T < \log B_1] + e^{-rT} \mathbb{E}_Q [(e^{x_T} - e^k)^+ | x_T < \log B_0, x_T \geq \log B_1] + e^{-rT} \mathbb{E}_Q [(e^{x_T} - e^k)^+ | x_T < \log B_0, x_T < \log B_1] = \Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4
$$

(1.25)
with $x(t) = \log S(t)$, $k = \log K$. These four conditional expectations are given as multidimensional integrals (denoted as $\Lambda$s) depending on the asset price path:

$$\Lambda_1 = e^{-rT} \int_{\log B_0}^\infty p(x_t, x, \sigma_1, T_0 - t) \int_{\log B_1}^\infty p(x, y, \sigma_2, (T_1 - T_0)) \left[ \int_k^\infty p(y, z, \sigma_3, T - T_1)(e^z - e^k)dz \right] dy dx$$

$$= e^{-rT_1} \int_{\log B_0}^\infty p(x_t, x, \sigma_1, T_0 - t) \cdot \left\{ \int_{\log B_1}^\infty p(x, y, \sigma_2, (T_1 - T_0))BS[e^y, \sigma_3, K, T - T_1]dy \right\} dx$$

$$\Lambda_2 = e^{-rT} \int_{\log B_0}^\infty p(x_t, x, \sigma_1, T_0 - t) \int_{-\infty}^\infty p(x, y, \sigma_2, (T_1 - T_0)) \left[ \int_k^\infty p(y, z, \sigma_2, T - T_1)(e^z - e^k)dz \right] dy dx$$

$$= e^{-rT_1} \int_{\log B_0}^\infty p(x_t, x, \sigma_1, T_0 - t) \cdot \left\{ \int_{-\infty}^\infty p(x, y, \sigma_2, (T_1 - T_0))BS[e^y, \sigma_2, K, T - T_1]dy \right\} dx$$

$$\Lambda_3 = e^{-rT} \int_{\log B_0}^\infty p(x_t, x, \sigma_1, T_0 - t) \int_{\log B_1}^\infty p(x, y, \sigma_1, (T_1 - T_0)) \left[ \int_k^\infty p(y, z, \sigma_3, T - T_1)(e^z - e^k)dz \right] dy dx$$

$$= e^{-rT_1} \int_{-\infty}^\infty p(x_t, x, \sigma_1, T_0 - t) \cdot \left\{ \int_{\log B_1}^\infty p(x, y, \sigma_1, (T_1 - T_0))BS[e^y, \sigma_3, K, T - T_1]dy \right\} dx$$

$$\Lambda_4 = e^{-rT} \int_{\log B_0}^\infty p(x_t, x, \sigma_1, T_0 - t) \int_{\log B_1}^\infty p(x, y, \sigma_1, (T_1 - T_0)) \left[ \int_k^\infty p(y, z, \sigma_1, T - T_1)(e^z - e^k)dz \right] dy dx$$

$$= e^{-rT_1} \int_{-\infty}^\infty p(x_t, x, \sigma_1, T_0 - t) \cdot \left\{ \int_{\log B_1}^\infty p(x, y, \sigma_1, (T_1 - T_0))BS[e^y, \sigma_1, K, T - T_1]dy \right\} dx$$
where \( p(x, y, \sigma, \tau) \) is the density function of \( x_t \) given as:

\[
p(x, y, \sigma, \tau) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\exp \left\{-\frac{(y - x + \frac{1}{2}\sigma^2\tau)^2}{2\sigma^2\tau}\right\}}
\]

In the log-normal process with deterministic volatility, \( \int_k^\infty p(x, y, \sigma, \tau)(e^y - e^k)dy \) is given by the Black-Scholes formula:

\[\text{BS}[e^x, \sigma, e^k, \tau, r].\]

Further tedious Gaussian calculations with the use of cumulative tri-variate normal distribution function:

\[
N_3(X_{1,\pm}, X_{2,\pm}, X_{3,\pm}; \rho_{12}, \rho_{13}, \rho_{23}) := \int_{-\infty}^{X_{1,\pm}} \int_{-\infty}^{X_{2,\pm}} \int_{-\infty}^{X_{3,\pm}} e^{\exp(-\frac{1}{2}C^{-1}Y\cdot Y^T)} \sqrt{(2\pi)^3detC} dY
\]

give us the simplified form of \( \Lambda_s \), as shown in Proposition 1.5. The corresponding cumulative variance terms \( V_{r,m}^{T,n} \), \( n = \{0, 1, 2\}, m = \{1, 2, 3, 4\}, T_2 = T \) are derived according to corresponding volatility paths.

### 1.7.6 Proof of Proposition 1.6

Here, we work backwards. If barrier \( B_1 \) is hit, the volatility changes: \( \sigma_1 \rightarrow \sigma_2 \), then the option value conditional on \( B_1 \) is hit: \( C(B_1, B_2, \sigma(t), K, T - \tau_{B_1}, r) \) at hitting time \( \tau_{B_1} \) is given as:

\[
\text{BS}(B_1, \sigma_2, K, T - \tau_{B_1}, r) + \int_{\tau_{B_1}}^T \text{BS}(B_2, \sigma_3, \sigma_2, K, T - s, r) f_{B_1, B_2}(s) ds
\]

This is proved in Proposition 1.4, as in the single period case. Additionally, \( f_{B_1, B_2}(\cdot) \) is the density function of the first hitting time from \( B_1 \) to hit \( B_2 \) with volatility \( \sigma_2 \), and possesses the same form as \( f_{\tau_{B_1}}(\cdot) \) given in Proposition 1.4 when replacing \( S_0, B_1, \sigma_1 \) by \( B_1, B_2, \sigma_2 \), respectively.

We again work backwards by defining a loss function \( L(B_1, \sigma_1, \sigma_2, K, T - \tau_{B_1}, r) \) as:

\[C(B_1, B_2, \sigma(t), K, T - \tau_{B_1}, r) - \text{BS}(B_1, \sigma_1, K, T - \tau_{B_1}, r)\]

at time \( \tau_{B_1} \), which represents a loss (either positive or negative) triggered by the volatility change. If we apply the standard Black-Scholes pricing and hedging procedure with constant volatility \( \sigma_1 \), then the option value at time 0 should read:

\[
\text{BS}(S_0, \sigma_1, K, T, r) + \int_0^T e^{-rt} L(B_1, \sigma_1, \sigma_2, K, T - t, r) f_{\tau_{B_1}}(t) dt
\]

with the density function \( f_{\tau_{B_1}}(t) \) of the first hitting time for barrier \( B_1 \) given in Proposition 1.4. Substituting \( L(B_1, \sigma_1, \sigma_2, K, T - \tau_{B_1}, r) = C(B_1, B_2, \sigma(t), K, T - \tau_{B_1}, r) - \text{BS}(B_1, \sigma_1, K, T - \tau_{B_1}, r) \) into the above formula gives Proposition 1.6.
1.7.7 Theorem 2.2 of [132]: Uniqueness of the Equivalent Martingale Measure

Let $\Sigma : \tau \times \Omega \to \mathbb{R}^{d \times d}$ be defined as the $d \times d$ volatility matrix:

$$\Sigma(t, \omega) = \begin{bmatrix} \sigma_{11}(t, \omega) & \cdots & \sigma_{1d}(t, \omega) \\ \vdots & \ddots & \vdots \\ \sigma_{d1}(t, \omega) & \cdots & \sigma_{dd}(t, \omega) \end{bmatrix}$$

Suppose there exists an equivalent probability measure $\tilde{P}$ on $(\Omega, \mathcal{F})$ making $S_i(t) : t \in \tau$ $\tilde{P}$-martingales with respect to $(\mathcal{F}_t : t \in \tau)$ for $i = 1, \ldots, d$. Then $\tilde{P}$ is unique if and only if $P(\Sigma(t) \text{ is nonsingular a.e.}) = 1$. 
2. INTRODUCTION TO STOCHASTIC VOLATILITY AND LOCAL STOCHASTIC VOLATILITY MODELS

2.1 Stochastic Volatility Models – a General Set-Up

2.1.1 Model Set-Up

Here, we define the general framework for a stochastic volatility model\(^1\), give a review of Romano and Touzi’s work ([14]):

\[
\begin{align*}
\frac{dS_t}{S_t} &= \mu(t, S_t, Y_t)dt + \sigma(Y_t)\sqrt{1 - \rho^2(t, S_t, Y_t)}dW^1_t + \sigma(Y_t)\rho(t, S_t, Y_t)dW^2_t \\
        &+ \sigma(Y_t)\rho(t, S_t, Y_t)dW^1_t + \sigma(Y_t)\rho(t, S_t, Y_t)dW^2_t. \\
\end{align*}
\]

Under the real world measure \(\mathcal{P}\), \(\{S_t, 0 \leq t \leq T\}\) is the asset price process paying no dividends, \(\{\sigma_t = \sigma(Y_t), 0 \leq t \leq T\}\) is the stochastic volatility process and \(\{\rho_t = \rho(t, S_t, Y_t), 0 \leq t \leq T\}\) is the correlation between the log asset changes and stochastic volatility changes, taking values in \([-1, 1]\). \(W = (W^1, W^2)_T\) is a standard Brownian motion on a probability space \((\Omega, \mathcal{P}, \mathcal{F})\) and \(\{\mathcal{F}_t, 0 \leq t \leq T\}\) is the \(\mathcal{P}\)

- \(\sigma : \mathbb{R} \rightarrow \mathbb{R}_+\) is Lipschitz, \(C^1\)-diffeomorphism and is bounded by \(\sigma_1, \sigma_2 > 0\) such that \(\forall y \in \mathbb{R}, \sigma_1 \leq \sigma(y) \leq \sigma_2\).
- \(\exists \epsilon > 0\), such that \(\forall (t, x, y) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}, \min\{1 - \rho^2(t, e^x, y), \gamma(t, e^x, y)\} \geq \epsilon\).
- The functions \((t, x, y) \rightarrow \mu(t, e^x, y), (t, x, y) \rightarrow \rho(t, e^x, y), (t, x, y) \rightarrow \eta(t, e^x, y), (t, x, y) \rightarrow \gamma(t, e^x, y)\) are bounded, Lipschitz in \((x, y)\) and uniformly in \(t\).
- The function \((t, x, y) \rightarrow \eta(t, e^x, y)\) is locally Lipschitz in \((t, x, y)\).

These conditions guarantee a strong SDE solution (2.1)-(2.2), see ([15]).

---

\(^1\)This model includes the local stochastic volatility model, in which the stochastic volatility term \(\sigma(\cdot)\) is also a function of the asset price: \(S_t\).
2. Introduction to Stochastic Volatility and Local Stochastic Volatility Models

2.1.2 Change of Measure and Model Incompleteness

We denote the instantaneous deterministic interest rate by \( r_t : [0, T] \rightarrow \mathbb{R}^2 \). For the purpose of derivatives pricing, we need to find an equivalent martingale measure \( Q (\sim \mathcal{P}) \), under which the risk-free rate discounted asset price is a \( \mathcal{F}_t \)-adapted martingale.

We write down the Radon-Nikodym derivative as:

\[
\frac{dQ}{dP} \bigg|_{\mathcal{F}_t} = \exp \left( - \int_0^t \lambda_u dW^1_u - \frac{1}{2} \int_0^t \lambda^2_u du - \int_0^t \nu_u dW^2_u - \frac{1}{2} \int_0^t \nu^2_u du \right), \tag{2.3}
\]

where \((\lambda, \nu)^t\) is adapted to \(\{\mathcal{F}_t\}\) and square integrable for \( t \in [0, T] \):

\[
\int_0^T \lambda^2_u du < \infty \quad \text{and} \quad \int_0^T \nu^2_u du < \infty, \ a.s.
\]

By Girsanov’s theorem:

\[
dW^Q_1(t) = dW^1(t) + \lambda(t) dt, \tag{2.4}
\]

\[
dW^Q_2(t) = dW^2(t) + \nu(t) dt.
\]

In order to assure the discounted asset price process is a martingale under new measure \( Q \), we need the following equality:

\[
\mu_t - r_t = \lambda_t \sqrt{1 - \rho_t^2} \sigma_t + \nu_t \rho_t \sigma_t, \tag{2.5}
\]

for \( t \in [0, T] \).

As a result, the stochastic volatility (SV) model (2.1)-(2.2) becomes:

\[
\frac{dS(t)}{S(t)} = r_t dt + \sigma(Y_t) \left( \sqrt{1 - \rho^2(t, S_t, Y_t)} dW^Q_1(t) + \rho(t, S_t, Y_t) dW^Q_2(t) \right), \tag{2.6}
\]

\[
dY(t) = \left( \eta(t, S_t, Y_t) - \nu_t \gamma(t, S_t, Y_t) \right) dt + \gamma(t, S_t, Y_t) dW^Q_2(t). \tag{2.7}
\]

We see for each choice of "volatility risk premium"\(^\text{3}\): \( \nu_t \), the "asset risk premium" \( \lambda_t \), \( t \in [0, T] \), is uniquely determined by (2.5), as is the equivalent martingale measure \( Q \), which we rewrite as \( Q(\nu) \) to show its dependence on \( \nu \). Under \( Q(\nu) \), the no arbitrage price of option \( O(t, S_t; T) \) at time \( t \) with maturity \( T \) is given by:

\[
O(t, S_t; T) = e^{-\int_t^T r_s ds} \mathbb{E}_t^{Q(\nu)}[\phi(S_T)], \tag{2.8}
\]

\(^2\) The positivity constraint of interest rates is alleviated here because occasional negative short-term rates are observed, e.g., in Japan.

\(^3\) Or "the market price of volatility risk"
where \( \phi(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R} \) is the pay-off function on the maturity.

Since \( \nu \) is an unobservable quantity in the market, we have a range of \( \mathcal{Q}(\nu) \), which offers no arbitrage option prices. Since the EMM is non-unique, we can say that this model is incomplete (see [29]).

**Remark** Actually, the incompleteness associated with stochastic volatility models is one of their major "drawbacks". Local volatility models (in which the volatility term is a function of \( t \) and state variable \( S_t \) only) bypass this problem by "stuffing" all the information of asset price movement into one random source (Brownian motion) in order to retain the model’s completeness, thus resulting in "perfect hedgeability" with the underlying asset.

### 2.2 Making the Stochastic Volatility Economy Complete

From the hedging point of view, to see why we cannot achieve perfect hedging under stochastic volatility\(^4\), we express the option price process in a PDE form in order to gain a more intuitive view of the hedging process. In order to do that, we need the following two restrictions from ([14]):

**Assumption 2.2.1** The terminal pay-off function \( \phi(\cdot) \) is continuous and satisfies the logarithmic growth condition:

\[
| \phi(x) | \leq K(1 + (\log x)^\theta), \quad x \in \mathbb{R}_+
\]

for some positive constants \( K \) and \( \theta \).

**Assumption 2.2.2** The volatility risk premium depends only on the contemporaneous values of the state variables \( \nu_t = \nu(t, S_t, Y_t), \quad 0 \leq t \leq T \). Moreover, \( \nu \) is bounded and the function \( (t, x, y) \rightarrow \nu(t, e^x, y) \gamma(t, e^x, y) \) is locally Lipschitz in \( (t, x, y) \).

These assumptions preserve the Markov properties of the diffusion process \((S_t, Y_t)\) and guarantee the technical conditions for applying the Feynman-Kac theorem[19]). In this case, we obtain:

\[
\begin{align*}
\mathcal{L}O(t, s, y) &= 0, \quad \forall (t, x, y) \in ([0, T] \times \mathbb{R}_+ \times \mathbb{R}) \quad (2.9) \\
O(T, s, y) &= \phi(s), \quad \forall (s, y) \in (\mathbb{R}_+ \times \mathbb{R}) \quad (2.10)
\end{align*}
\]

\(^4\) Which is equivalent to stating "a contingent claim in a stochastic volatility economy cannot be attained perfectly", thus, due to ([29]), the market is incomplete.
where

\[ \mathcal{L}O = rO - \frac{\partial O}{\partial t} - rs\frac{\partial O}{\partial s} - \beta(t,s,y) \frac{\partial O}{\partial y} - \frac{1}{2}s^2\sigma^2(y) \frac{\partial^2 O}{\partial s^2} - \frac{1}{2}\gamma^2(t,s,y) \frac{\partial^2 O}{\partial y^2} - \rho(t,s,y)\gamma(t,s,y)\sigma(y)s \frac{\partial^2 O}{\partial s\partial y} \] (2.11)

and \( \beta(t,s,y) = \eta(t,s,y) - \nu t \) is the risk-neutralised drift term of the process of \( Y_t \). Using the Feynman-Kac formula, the unique \( C^{1,2}([0,T],R_+ \times R) \) solution is given by the expectation form (2.8).

In order to hedge the volatility risk, as ([14]) shows, we can introduce a new contingent claim:

**Definition 2.2.3** Let \( U(t,S_t,Y_t), 0 \leq t \leq T \) be a \( C^{1,2}([0,T],R_+ \times R) \) function with \( \forall (t,s,y) \in [0,T) \times R_+ \times R, U_y(t,s,y) = \frac{\partial U}{\partial y}(t,s,y) \neq 0 \). In addition, under the risk-neutral measure \( Q \), the discounted contingent claim price \( \{e^{-\int_0^t r_s ds} U(t,S_t,Y_t), 0 \leq t \leq T \} \) is a martingale.

We now want to set up a portfolio consisting of \( \alpha_t, \beta_t, \chi_t : [0,T] \to R \) units of stock \( S_t \), risk-free bond \( B_t \) and newly introduced contingent claim \( U_t \), respectively, to dynamically replicate one unit of option \( O_t \) at time \( t \).

The portfolio is given as:

\[ \Pi_t = O(t,S_t,Y_t) - \alpha_t S_t - \beta_t B_t - \chi_t U(t,S_t,Y_t) \]

with 0 as the initial cost.

Next, we have the portfolio value change at time \( t \):

\[ d\Pi_t = dO(t,S_t,Y_t) - \alpha_t dS_t - \beta_t dB_t - \chi_t dU(t,S_t,Y_t) \] (2.12)

Applying Ito’s lemma and setting \( dW_1 \) and \( dW_2 \) diffusion terms to zero, which is equivalent to making our hedging portfolio riskless, we obtain:

\[ \chi_t = \frac{O_y(t)}{U_y(t)}; \] (2.13)

\[ \alpha_t = O_s(t) - \frac{O_y(t)}{U_y(t)} U_s(t); \] (2.14)

---

5 This means that the new contingent claim \( U \) "incorporates" the volatility risk.

6 The same argument appears in ([6]) and several other works. Note here that there is a bond part (a martingale term guarantees self-financing) missing in Gatheral’s derivation.
By setting up this portfolio with zero initial cost, thus:

\[
\beta_t = \frac{O(t) - (O_s(t) - \frac{O_s(t)}{U_y(t)} U_s(t)) S(t) - \frac{O_s(t)}{U_y(t)} U(t)}{B_t}
\]  

(2.15)

where \( O_z(t) = \frac{\partial O}{\partial z}(t, s, y), \ U_z(t) = \frac{\partial U}{\partial z}(t, s, y), \ z \in \{s, y\}. \)

Substituting (2.13)-(2.15) into (2.12) and keeping \( d\Pi_t = 0, \) after some algebra (for details see [16] or [5]), we see that the market price of volatility risk \( \nu_t \) is given as:

\[
\nu_t = \frac{\eta(t, s, y) - \hat{\nu}(t, s, y)}{\gamma(t, s, y)}
\]

\[
\hat{\nu} = \frac{\partial U}{\partial y}.
\]  

(2.16)

**Remark** From the above, we see indeed that \( \nu_t \) possesses a functional form as \( \nu(t, s, y) \), which depends on the asset price and stochastic volatility process. Further, the hedging instrument \( U(t, s, y) \) can be any contingent claim satisfying Definition 2.2.3, which complicates the estimation and computation of the "volatility risk premium". Thus, in practice, people usually bypass this problem and assume that the stochastic volatility model being used is given as volatility risk-neutral.

### 2.3 European Option Price

Under the model set-up (2.6)-(2.7), we make two further assumptions (see ([14])):

**Assumption 2.3.1** The correlation coefficient \( \rho \) as well as volatility of volatility coefficient \( \gamma \) do not depend on the asset price, i.e. \( \rho_t = \rho(t, Y_t) \) and \( \gamma_t = \gamma(t, Y_t) \).

**Assumption 2.3.2** The volatility risk premium \( \nu \) is such that \( \eta(t, S_t, Y_t) - \nu \gamma(t, S_t, Y_t) \) under \( Q \) does not depend on the asset price, i.e. \( \beta_t = \beta(t, Y_t) \).

With this in mind, we can give the European call option price (see Proposition 4.1 of ([14])) under the risk-neutral measure \( Q \), which is dependent on the choice of volatility risk premia \( \nu \):
2. Introduction to Stochastic Volatility and Local Stochastic Volatility Models

\[ C_t = \mathbb{E}_t^Q \left[ C_{t}^{BS} \left( S_t e^{Z_{t,T}}; \int_t^T (1 - \rho_u^2) \sigma_u^2 du / (T - t) \right) \right] \]

\[ = S_t \mathbb{E}_t^Q \left[ e^{Z_{t,T}} N \left( \frac{x_t + Z_{t,T}}{V_{t,T}} + \frac{V_{t,T}}{2} \right) - e^{-x_t} N \left( \frac{x_t + Z_{t,T}}{V_{t,T}} - \frac{V_{t,T}}{2} \right) \right]. \]

(2.17)

where \( N(\cdot) \) is the standard normal cumulative distribution function and \( C_{t}^{BS} \) the classical Black-Scholes formula, with:

\[ x_t = \log \left( \frac{S_t}{K e^{r(T-t)}} \right), \]

\[ Z_{t,T} = \int_t^T \rho_u \sigma_u dW^Q(u) - \frac{1}{2} \int_t^T \rho_u^2 \sigma_u^2 du, \]

\[ V_{t,T} = \sqrt{\int_t^T (1 - \rho_u^2) \sigma_u^2 du}. \]

(2.18)

**Proof** (cf. Proposition 4.1 of [14]) We sketch the proof as follows. First, we can write:

\[ S_T = S_t e^{Z_{t,T}} \exp \left( r(T - t) - \frac{1}{2} \int_t^T \sigma_u^2 (1 - \rho_u^2) du + \int_t^T \sigma_u \sqrt{1 - \rho_u^2} dW^Q(u) \right) \]

Based on Assumptions 2.3.1 and 2.3.2, the correlation and volatility are independent of \( \{W^Q_1(u), 0 \leq u \leq T\} \). Additionally, the European call option can be written as the expectation of a Black-Scholes formula with \( S_t \) and total variance being replaced by \( S_t e^{Z_{t,T}}, V_{t,T}^2 \), respectively.

The European put option price can be written in a similar form or obtained by put-call parity.

**Remark** Although this looks elegant, the expression here does not buy us much for practical option pricing. However, if we expand our filtration \( \mathcal{F}_t = \sigma(S_u, Y_u; 0 \leq u \leq t), t \in [0, T] \) to \( \bar{\mathcal{F}}_t = \mathcal{F}_t \vee \sigma(Y_u; t \leq u \leq T) \), with the future information of the volatility (path) being known, we can arrive at the Black-Scholes formula directly with the asset price as \( S_t e^{Z_{t,T}} \) and \( V_{t,T}^2 \) as the total variance.
2.4 Local Stochastic Volatility Models: an Introduction

In this section we focus on the forward asset price process\(^7\) under the forward measure \(Q^T\), where it is a martingale:

\[
\frac{dF_t}{F_t} = \Gamma(t, F_t, V_t)dW_s(t) \tag{2.19}
\]

Here, \(\Gamma : ([0, T], \mathbb{R}_+, \mathbb{R}_+) \rightarrow \mathbb{R}_+\) is Lipschitz and bounded. In the local stochastic volatility model (2.19), the volatility term can be the combination of a stochastic volatility term and a local volatility term which is state and time-dependent:

\[
\Gamma(t, F_t, V_t) = \psi(t, F_t)f(V_t)
\]

Here, \(F_t\) is the forward asset price at time \(t\), \(\psi : ([0, T), \mathbb{R}_+) \rightarrow \mathbb{R}_+\) the local volatility component, satisfying certain regularity conditions and \(f : \mathbb{R}_+ \rightarrow \mathbb{R}\) the stochastic volatility component. In practice, the local volatility component \(\psi(t, F_t)\) often is parameterised\(^8\) to: \(\alpha(t)\varphi\left(\frac{F_t}{L_t}\right), L : [0, T] \rightarrow \mathbb{R}_+,\) which is a scaling factor used to make the model calibration more intuitive and convenient; \(\alpha(t)\) is the time-dependent deterministic volatility term which generally controls the level of volatility; \(\varphi(x_t)\) is a state-dependent volatility term usually specified either as a \(CEV\) form:

\[
x_t^\beta
\]

or a \(displaced\ \text{diffusion}\)\(^9\) form:

\[
\beta_t x_t + \gamma_t
\]

where \(\beta_t\) and \(\gamma_t\) are time-dependent deterministic functions;

or even a quadratic form:

\[
\theta_t x_t^2 + \beta_t x_t + \gamma_t
\]

Popular choices for the functional form of \(f(\cdot)\) are:

\[
f(V_t) = V_t^\beta
\]

with \(\beta = 1.0\) or \(0.5\). When \(\beta = 0.5\), \(V_t\) is called a \(stochastic\ \text{variance}\) process.

---

\(^7\) This can also be a discounted asset price under the risk-neutral measure, a forward rate under the \(T\)-forward measure or a swap rate under the annuity measure.

\(^8\) Here, we do not consider the general non-parametric form which can only be calibrated from the market option prices given the stochastic volatility component. See [121]

\(^9\) Also called "shifted log-normal".
Alternatively, it can be of an exponential form:

\[ f(V_t) = e^{V_t} \]

where \( V_t \) is a stochastic volatility (or variance) process, given as:

\[ dV_t = a(t, V_t)dt + b(t, V_t)dW_v(t) \]

\( a : ([0, T), \mathbb{R}_+) \rightarrow \mathbb{R}, \ b : ([0, T), \mathbb{R}_+) \rightarrow \mathbb{R}_+ \) satisfy the regularity conditions presented in Section 2.1. Brownian motions \( dW_s \) and \( dW_v \) are correlated as:

\[ \mathbb{E}^Q_T[dW_s(t)dW_v(t)] = \rho(t)dt. \]

**Examples** Some popular local stochastic volatility models are\(^{10}\):

- Andersen-Brotherton-Ratcliffe model(2001, cf. [51]):

  \[
  \psi(t, x) = \sigma(t)\psi(x) \\
  f(v) = \sqrt{v} \\
  a(t, v) = \kappa(\theta - v) \\
  b(t, v) = \epsilon\phi(v) \\
  \rho(t) = 0
  \]

  \( \gamma(\cdot), \alpha(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R} \) are general level dependent functions;

- SABR(Stochastic Alpha, Beta, Rho) model(2002, cf. [30]):

  \[
  \psi(t, x) = \alpha x^\beta \\
  f(v) = v \\
  a(t, v) = 0 \\
  b(t, v) = \eta v \\
  \rho(t) = \rho_{SV}
  \]

  with \( \alpha, \eta \in \mathbb{R}_+ \) have constant value;

- Zhou’s model(2003, cf. [28]):

  \[
  \psi(t, x) = \sigma\gamma(x) \\
  f(v) = v \\
  a(t, v) = 0 \\
  b(t, v) = b\alpha(v) \\
  \rho(t) = \rho_{SV}
  \]

  with \( \gamma(\cdot), \alpha(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R} \) are general level dependent functions;

\(^{10}\) Since this is a fast growing area, here I only present some of the published models up to the writing time of June, 2007.
2. Introduction to Stochastic Volatility and Local Stochastic Volatility Models

- Piterbarg’s model\(^{11}\) (2005, cf. [53])
  \[
  \psi(t, x) = \sigma(t)(\beta(t)x(t) + (1 - \beta(t))x(0))
  \]
  \[
  f(v) = \sqrt{v}
  \]
  \[
  a(t, v) = \theta(v(0) - v(t))
  \]
  \[
  b(t, v) = \gamma(t)\sqrt{v(t)}
  \]
  \[
  \rho(t) = 0
  \]

- Blacher’s model\(^{12}\) (2001, cf. [122])
  \[
  \psi(t, x) = \sigma(1 + \alpha(x - x_0) + \beta(x - x_0)^2)
  \]
  \[
  f(v) = v
  \]
  \[
  a(t, v) = \kappa(\theta - v(t))
  \]
  \[
  b(t, v) = \epsilon v(t)
  \]
  \[
  \rho(t) = \rho_{SV}
  \]

**Remark** As argued by Blacher\([122]\), pure stochastic volatility models (with local volatility component being time-dependent only) can only generate volatility smiles with no systematic change of the shape when underlying value \((F_t)\) moves, which is contrary to observations in the financial markets and will incur hedging (delta, vega, etc.) discrepancies. Besides, in our practice, we found pure stochastic volatility models (e.g. Heston model, [4]) are not able to generate strong (or, downward steep) enough implied volatility skews for options with relatively short maturities. Furthermore, unrealistically large values, especially for the volatility of volatility and correlation parameters, have to be imposed to reproduce the market-observed implied volatility skew/smile. The introduction of a local volatility component makes the model capable of generating strong enough volatility skews, because CEV or displaced diffusion-type models can produce monotonic downward volatility skews by themselves.

Hence, for a better fitting to market volatility smiles and skews, and for generating more realistic smile dynamics, a combination of these two types of models seems desirable. A local volatility function is used to produce the level and slope\(^{12}\) of the volatility skew, and then a stochastic volatility component is added to fine-tune the skew and generate the desired smile ("curvature"), thus achieving a more flexible modelling approach.

---

\(^{11}\) This is the Andersen-Andreasen model (2002) with time-dependent parameters

\(^{12}\) Or, as argued in [30], the "backbone", i.e. the deterministic part of the smile/skew, depending on strikes.
Due to the complexity of the functional form of the volatility term (diffusion term) involved, usually an analytical option pricing formula (or even semi-analytical formula through the Fourier transform method) is not available, especially for a model with a CEV-type local volatility component. Normally, asymptotic expansion (for the Zhou and Andersen-Brotherton-Ratcliffe model) and perturbation methods (for SABR model) are employed to obtain approximate European option prices and associated implied volatilities. In the next chapter, I will present my results on a general local stochastic volatility model with stochastic interest rates, which can be seen as a generalisation of all the models mentioned above.

2.5 Adjustment to the Calculation of Greeks in a Non-Constant Implied Volatility Model

Under the local stochastic volatility model, our implied volatility is defined as:

\[
\text{Opt}(S, K, r, \sigma(t, S, v), t, T) = \text{BS}(S, K, r, I, t, T)
\]

Here, \( \text{Opt} \) is the option price under the local stochastic volatility model defined in (2.19), while \( \sigma \) is the local-stochastic volatility function depending on \( S \), stochastic volatility variable \( v \) and valuation time \( t \). \( \text{BS} \) is the price from the Black-Scholes model with implied volatility \( I \), such that they return the same option value.

Assume our implied volatility (denoted as \( I \)) now has a functional form (while complicated) as:

\[
I := I(t, S, \sigma) : C^{1,2}([0, T), \mathbb{R}_+ \times \mathbb{R}_+) \rightarrow \mathbb{R}_+
\]

Then, as delta is defined as:

\[
\frac{\partial \text{Opt}}{\partial S}
\]

this needs some adjustment from B-S delta: \( \Delta^\text{BS} := \frac{\partial \text{BS}}{\partial S}(S, K, r, I, t, T) \):

\[
\Delta = \frac{\partial \text{Opt}}{\partial S} = \frac{\partial \text{BS}}{\partial S} + \frac{\partial \text{BS}}{\partial I} \frac{\partial I}{\partial S} = \Delta^\text{BS} + \text{vega} \left( \frac{\partial I}{\partial S} \sigma + \frac{\partial I}{\partial \sigma} \frac{\partial \sigma}{\partial S} \right)
\]
Analogously, for theta (the partial derivative respect to time $t$):

$$
\Theta = \frac{\partial O_{pt}}{\partial t} = \frac{\partial B_S}{\partial t} + \frac{\partial B_S \partial I}{\partial I \partial t} = \Theta^{BS} + \text{vega} \left( \frac{\partial I}{\partial t} |_\sigma + \frac{\partial I}{\partial \sigma} \frac{\partial \sigma}{\partial t} \right)
$$

while gamma is defined as:

$$
\frac{\partial^2 O_{pt}}{\partial S^2} = \frac{\partial \Delta}{\partial S}
$$

we have an adjustment from B-S gamma: $\Gamma^{BS} := \frac{\partial^2 B_S}{\partial S^2}(S,K,r,I,t,T)$:

$$
\Gamma = \frac{\partial^2 O_{pt}}{\partial S^2} = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 B_S}{\partial S^2} + \frac{\partial \Delta^{BS}}{\partial I} \frac{\partial I}{\partial S} + \frac{\partial \text{vega}}{\partial I} \frac{\partial I}{\partial S} + \frac{\partial \text{vega}}{\partial S} \left( \frac{\partial I}{\partial S} \right)^2 + \frac{\partial \text{vega}}{\partial I} \left( \frac{\partial I}{\partial S} \right)^2 + \text{vega} \frac{\partial^2 I}{\partial I^2} + \text{vega} \left( \frac{\partial I}{\partial S} \right)^2
$$

where $\frac{\partial I}{\partial S}$ and $\frac{\partial^2 I}{\partial S^2}$ are given with adjustments:

$$
\frac{\partial I}{\partial S} = \frac{\partial I}{\partial S} |_\sigma + \frac{\partial I}{\partial \sigma} \frac{\partial \sigma}{\partial S}
$$

$$
\frac{\partial^2 I}{\partial S^2} = \frac{\partial^2 I}{\partial S^2} |_\sigma + \frac{\partial^2 I}{\partial S \partial \sigma} \frac{\partial \sigma}{\partial S} + \frac{\partial^2 I}{\partial \sigma \partial S} \frac{\partial \sigma}{\partial S} + \frac{\partial \sigma}{\partial \sigma} \left( \frac{\partial I}{\partial S} \right)^2 + \frac{\partial \sigma}{\partial \sigma} \left( \frac{\partial I}{\partial \sigma} \right)^2
$$
3. FOREIGN EXCHANGE OPTIONS WITH LOCAL STOCHASTIC VOLATILITY AND STOCHASTIC INTEREST RATES

3.1 Introduction

The past years have seen fast development in financial derivatives, notably in structured products and hybrid derivatives\(^1\), which have introduced new challenges to modelling techniques, especially in the presence of market-implied volatility skews and smiles across different maturities (the so-called "volatility surface"). In order to capture this important market information of implied volatilities, a flexible modelling approach – the local stochastic volatility model – will be introduced here. Unfortunately, because of the complexity of the functional form, an analytical formula is usually not available, thus some numerical methods (e.g. Monte Carlo simulation, finite difference method, etc.) need to be employed even for computing the prices of vanilla European options, mainly for the purpose of model calibration. In order to facilitate this calibration procedure, which normally involves the computation of hundreds if not thousands of European option prices, an approximation formula is desired. Some recent works addressing this issue on the pricing of foreign exchange options with local volatility and (or) stochastic volatility, as well as stochastic interest rates, include the following approaches:

An approximation method called \textit{Markovian Projection}, based on Gyongy’s work[11], is utilised to attack this problem. This method was pioneered by Piterbarg’s work [39], which starts from a Hull-White short rate process and a local volatility(CEV) process for the evolution of FX rate. By using the “Markovian Projection” method, Piterbarg successfully derived an \textit{approximately equivalent}\(^2\) displaced-diffusion model\(^3\), which allows a simple Black-Scholes-type option pricing formula. Later, Antonov and Misirpashaev achieved similar results in [40],

\(^1\) Financial derivatives whose underlying consists of multi-asset classes, i.e. a combination of interest rate, equity, credit, FX, etc.

\(^2\) Which allows for the same terminal distribution, thus returning the same European option prices.

\(^3\) For CEV and displaced-diffusion models, see Chapter 2 for an introduction.
 Apart from this approach, another method of asymptotic expansion, based on Watanabe-Yoshida’s theory in the Malliavin calculus (cf.[95]), includes the work [25] of Kawai and Jackel where they examined a LIBOR market model with a displaced diffusion process, as well as a displaced diffusion process for FX rates, via an expansion on ”small volatility”. This asymptotic expansion method has been proven flexible and powerful, and this extension provides the capability to capture both the volatility skews (slopes) in FX and interest rate markets. Using this asymptotic expansion approach, Osajima [96] extended the FX rate process to a SABR-type model (the so-called stochastic alpha, beta, rho model, see [30]), which captures both the FX volatility skew and smile (the curvature), plus a Hull-White interest rate process.

My work here consists of two modelling approaches. The first is a general local stochastic volatility process (not limited to any specific form) for FX rate evolution, with the stochastic interest rates process being the Hull-White short rate model or LIBOR market model. The analytical formulae of option prices and implied volatility are derived through the asymptotic expansion method based on Watanabe-Yoshida’s theory. Another approach employs the Fourier transform method to obtain semi-analytical results, for a different model set up which consists of stochastic volatility, stochastic interest rates and jump process. Later, a complete analysis on the perfect/partial hedging issue of FX options will be presented, as well as the impact of stochastic volatility on the pricing of FX-IR structure products (PRDCs).

In the next section, I will present the first modelling approach, in which the model setup will start from a domestic risk-neutral measure, then change to a domestic T-forward measure; In Section 3.3, the asymptotic expansion method is applied to arrive at a European call option pricing formula and, further, the formula of implied volatility; In Section 3.4, the model implementation will be given on both the Hull-White short rate and LIBOR market models; Section 3.5 will present the second modelling approach via Fourier transform method; Sections 3.6-3.8 will discuss how to perfectly and partially hedge FX options in the presence of stochastic interest rates and local stochastic volatility; Section 3.9 will study the impact of stochastic volatility on the valuation of exotic derivative products. Finally, some comments are given, followed by a conclusion.
3.2 The FX-IR Hybrid Model

Consider a probability space \((\Omega, P, \mathcal{F}_T)\) over a finite time interval \([0, T]\). A two-country economy is considered with "domestic" and "foreign" currencies. \(P_{d,f}(t, T)\) are domestic and foreign zero-coupon bonds with maturity \(T\), seen at time \(t\), in their respective currencies. Let \(r_{d,f}(t)\) be the short interest rates in these two countries and \(S(t)\) the spot exchange rate. Under the domestic risk-neutral measure \(Q^d\), the arbitrage-free processes for \(S(t), P_{d,f}(t, T)\) are given as:

\[
\frac{dS(t)}{S(t)} = (r_d(t) - r_f(t))dt + \Gamma(t, S(t), \sigma(t))dW_s(t)
\]

\[
\frac{dP_d(t, T)}{P_d(t, T)} = r_d(t)dt + \sigma_d(t, T)dW_d(t)
\]

\[
\frac{dP_f(t, T)}{P_f(t, T)} = (r_f(t) - \rho_{fs}(t)\sigma_f(t, T)\Gamma(t, S(t), \sigma(t)))dt + \sigma_f(t, T)dW_f(t)
\]

(3.1)

The \(-\rho_{fs}(t)\sigma_f(t, T)\Gamma(t, S(t), \sigma(t))\) term in the drift of \(P_f(t, T)\) process is known as the "quanto-adjustment", which comes from the change of measure from a foreign risk-neutral measure \(Q^f\) to a domestic risk-neutral measure \(Q^d\) (see p.311 of [92] for reference).

The volatility function \(\Gamma : ([0, T), \mathbb{R}_+, \mathbb{R}_+) \to \mathbb{R}_+\) is assumed to be:

\[
\Gamma(t, S(t), \sigma(t)) = \alpha(t)\sigma(t)\gamma\left(\frac{S(t)}{L(t)}\right),
\]

(3.2)

\(\alpha : [0, T) \to \mathbb{R}_+\) is the deterministic time-dependent volatility, \(\sigma(t)\) is square-integrable stochastic volatility processes, \(\gamma : \mathbb{R}_+ \to \mathbb{R}_+\), the local volatility function, is smooth and has derivatives of any order bounded and \(L : [0, T) \to \mathbb{R}_+\) is a deterministic scaling factor, introduced in [39], for the convenience of model calibration. Three popular choices of this local volatility function are (also mentioned in Chapter 2) the:

CEV type (with \(\beta\) as a so-called "CEV-parameter"):

\[
\gamma(x) = x^{\beta-1},
\]

(3.3)

Displaced diffusion type (with \(\beta\) as the so-called "skew parameter")

\[
\gamma(x) = \beta x + (1 - \beta),
\]

(3.4)
In both cases, we have $\gamma(1) = 1$.
Alternatively, we can have a Quadratic type (popular in the FX market):

$$\gamma(x) = \alpha x^2 + \beta x + C,$$

The general (Markovian) stochastic volatility process here is defined as:

$$d\sigma(t) = a(t, \sigma(t))dt + b(t, \sigma(t))dW_\sigma(t),$$

where functions $a : ([0,T], \mathbb{R}_+) \to \mathbb{R}$ and $b : ([0,T], \mathbb{R}_+) \to \mathbb{R}_+$ are smooth, and the first and second order derivatives are bounded.

**Remark** This choice of stochastic volatility process in my model is very general, as different volatility processes represent different views on the dynamics of the actual evolution of instantaneous volatility. This implies that path-dependent and other exotic options’ prices depend on the joint evolution of underlying and its volatility (see [112]). In addition, as discussed in Chapter 2, due to the existence and complexity of the volatility risk premium, a well specified stochastic volatility model under the risk-neutral pricing measure (assuming we fix a volatility risk premium) will possess a complicated drift term. The specific model choice in implementation is up to the reader’s preference, while the approach here is general.

The Brownian motions vector $(W_S(t), W_d(t), W_f(t), W_\sigma(t))$ under measure $Q_d$ has a non-negative definite symmetric correlation matrix (assuming no correlation between the driving factors of stochastic volatility and interest rates, which is quite natural):

$$
\begin{pmatrix}
1 & \rho_{ds} & \rho_{fs} & \rho_{\sigma s} \\
\rho_{ds} & 1 & \rho_{df} & 0 \\
\rho_{fs} & \rho_{df} & 1 & 0 \\
\rho_{\sigma s} & 0 & 0 & 1
\end{pmatrix}
$$

(3.7)

Now, the forward FX rate is defined as:

$$F(t, T) = S(t) \frac{P_f(t, T)}{P_d(t, T)},$$

and the change of measure from domestic risk-neutral measure to domestic-T forward measure:

$$
\frac{dQ^T_d}{dQ_d}|_{\mathcal{F}_t} = \frac{P_d(t, T)/P_d(0, T)}{B_d(t)},
$$

(3.9)

where $B_d(t)$ is the domestic bank account starting from one unit of currency and growing at a risk-free domestic interest rate.
Using Girsanov’s theorem, under the new measure, we have:

\[ dW^T_S(t) = dW_S(t) + \rho_d(t)dW_d(t), \quad (3.10) \]
\[ dW^T_d(t) = dW_d(t) + \rho_d(t)dt, \quad (3.11) \]
\[ dW^T_f(t) = dW_f(t) + \rho_f(t)dW_f(t), \quad (3.12) \]
\[ dW^T_\sigma(t) = dW_\sigma(t), \quad (3.13) \]

Now, substituting \( S(t) \) with \( F(t) \), we obtain the drift-less FX forward rate process:

\[ \frac{dF(t,T)}{F(t,T)} = \Gamma(t, \sigma(t), S(t))dW^T_S(t) - \sigma_d(t,T)dW^T_d(t) + \sigma_f(t,T)dW^T_f(t), \quad (3.14) \]

Further, define the scaling factor \( L(t) = F(0,t) \) and \( X(t) \) as (later we will mainly work on \( X(t) \)):

\[ X(t) = \frac{F(t,T)}{F(0,T)}, \quad (3.15) \]

The SDE for \( X(t) \) is then given by:

\[ \frac{dX(t)}{X(t)} = \alpha(t)\sigma(t)\gamma(X(t)(1 + \hat{D}(t,T)))dW^T_S(t) - dD(t,T), \quad (3.16) \]
\[ d\sigma(t) = \alpha(t, \sigma(t))dt + b(t, \sigma(t))dW^T_\sigma(t), \quad (3.17) \]

with \( X(0) = 1, \gamma(1) = 1 \). For notational simplicity, \( D(t,T), \hat{D}(t,T) \) are defined as:

\[ D(t,T) = \int_0^t \sigma_d(s,T)dW^T_f(s) - \int_0^t \sigma_f(s,T)dW^T_d(s), \quad (3.18) \]
\[ \hat{D}(t,T) = D(t,T) - (\int_0^t \sigma_d(s,t)dW^T_f(s) - \int_0^t \sigma_f(s,t)dW^T_d(s)), \]

Without loss of generality, we can assume \( \sigma(0) = 1 \) (by letting \( \alpha(t) \) act as the scaling factor). In order to enlighten the role of the so-called "volvol" (volatility of volatility) in the option pricing formula, we assume \( b(t, \sigma(t)) = \eta(t)\tilde{b}(\sigma(t)) \), and a popular choice will be the power function form: \( \tilde{b}(\sigma) = \sigma^\varsigma \). For a more general \( b(t, \sigma) \), we can treat it in a similar way as we will do on \( a(t, \sigma) \).

From this model set-up we will look at the pricing of European options and obtain the associated Black-Scholes implied volatilities, for the purpose of model calibration to the vanilla option market.
3. Asymptotic Expansion

3.3 A Brief Introduction

An asymptotic expansion is a series of functions in which truncating the series for a finite number of terms will supply an approximation to a given (or target) function. A formal definition is given as:

**Definition 3.1 Asymptotic Expansion** (Kevorkian & Cole, [125] ) A sum of the terms of the form \( \sum_{n=1}^{N} a_n(x)\phi_n(\epsilon) \) is called an asymptotic expansion of the function \( f(x,\epsilon) \) to \( N \) terms (\( N \) may be infinite) as \( \epsilon \to \epsilon_0 \) with respect to the sequence \( \phi_n(\epsilon) \) if:

\[
 f(x,\epsilon) - \sum_{n=1}^{M} a_n(x)\phi_n(\epsilon) = o(\phi_M) \quad \text{as} \quad \epsilon \to \epsilon_0
\]

for each \( M = 1, 2, ..., N \). Or, equivalently,

\[
 f(x,\epsilon) - \sum_{n=1}^{M-1} a_n(x)\phi_n(\epsilon) = O(\phi_M) \quad \text{as} \quad \epsilon \to \epsilon_0
\]

for each \( M = 2, ..., N \).

For the asymptotic expansion method used here, we start from a general model set up by assigning small model parameters (normally \( \epsilon \ll 1 \)). Then we derive the European option value as the sum of a sequence. By truncating the series of \( n \) terms, a desired accuracy will be achieved, and the remaining part of the magnitude of \( \epsilon^{n+1} \) will be neglected.

3.3.2 European Option Pricing and Implied Volatility

As usual with asymptotic expansions, we insert the "small parameter" \( \epsilon \) into (3.16)-(3.17).

\[
\frac{dX^\epsilon(t)}{X^\epsilon(t)} = \epsilon \sigma^\epsilon(t)\alpha(t)\gamma(X^\epsilon(t)(1 + \epsilon^2\tilde{D}(t,T)))dW^T_S(t) - \epsilon^2 dD(t,T), \quad (3.19)
\]

\[
\frac{d\sigma^\epsilon(t)}{\sigma^\epsilon(t)} = \epsilon a(t,\sigma^\epsilon(t))dt + \epsilon\eta(t)\tilde{b}(\sigma^\epsilon(t))dW^T_\sigma(t), \quad (3.20)
\]

\[\text{Footnotes:} \quad ^4 \text{Here we choose the "small parameters": local volatility } \alpha(t), \text{bond volatilities } D(t,T) \text{ and } \tilde{D}(t,T), \text{volatility drift } a(t,\sigma(t)) \text{ and volatility of volatility } \tilde{b}(\sigma(t)) \text{ to do asymptotic expansions on.} \]

\[\text{Footnotes:} \quad ^5 \text{Normally, in practice, bond volatilities are of } 1\% \text{ magnitude while FX(JPY/USD) volatility is of } 10\% \text{ magnitude, all annualized.} \]
Starting from this model setting, we will derive the European call option price formula by the use of Watanabe-Yoshida’s theory in Malliavin calculus (cf. [95]) as well as Nualart et al’s results on the moments of multiple Wiener-Ito integral (cf. [97]), which will give us an asymptotic expansion form as below (the corresponding put option price can be obtained via put-call parity).

**Theorem 3.3.1** For $y = \frac{K - F(0, T)}{\epsilon F(0, T) \Sigma_{fwd}}$, $\epsilon \in (0, 1]$, the European call option price on FX rate with maturity $T$, strike $K$ is given by:

$$\text{Call}(F(0, T), T, K) = \epsilon P_d(0, T) F(0, T) \Sigma \{ \Upsilon_1 + \Upsilon_2 + \Upsilon_3 + \Upsilon_4 \} + \mathcal{O}(\epsilon^4) \quad (3.21)$$

with:

- $\Upsilon_1 = G(y)$, \hspace{1cm} (3.22)
- $\Upsilon_2 = \epsilon (C_{11} y + C_{12} + C_{21}) \phi(y)$, \hspace{1cm} (3.23)
- $\Upsilon_3 = \epsilon^2 \{ C_{31} (y^2 - 1) + C_{32} y + C_{41} + C_{33} + C_{34} \} \phi(y)$, \hspace{1cm} (3.24)
- $\Upsilon_4 = \frac{\epsilon^2}{2} \{ C_{51} (y^4 - 6y^2 + 3) + C_{52} (y^3 - 3y)
+ C_{53} (y^2 - 1) + C_{54} y + C_{55} + C_{61} (y^2 - 1) + C_{62}
+ C_{71} (y^3 - 3y) + C_{72} (y^2 - 1) + C_{73} y + C_{74} \} \phi(y)$ \hspace{1cm} (3.25)

where:

$$G(x) = x \Phi(x) + \phi(x)$$
3. Foreign Exchange Options with Local Stochastic Volatility and Stochastic Interest Rates

\[
C_{11} = \frac{1 + \gamma_1}{2} \Sigma + \frac{1}{\Sigma^3} \int_0^T \alpha^2(t) \left\{ \int_0^t \eta(s) \alpha(s) \rho_{\alpha\sigma}(s) ds \right\} dt,
\]

\[
C_{12} = \frac{1}{\Sigma^2} \int_0^T \alpha^2(t) \left\{ \int_0^t a(s, \sigma(0)) ds \right\} dt,
\]

\[
C_{21} = \frac{1}{\Sigma^2} \left\{ \int_0^T \sigma_f(t, T) \alpha(t) \rho_{fs}(t) dt - \int_0^T \sigma_d(t, T) \alpha(t) \rho_{ds}(t) dt \right\},
\]

\[
C_{31} = \frac{1}{\Sigma^4} \left\{ \frac{\gamma_{b1}}{2} \int_0^T \alpha^2(t) \Pi_{\eta\alpha}(t) dt \right. \\
+ (1 + \gamma_1) \int_0^T \alpha^2(t) \int_0^t \alpha^2(s) \Pi_{\eta\sigma}(s) ds dt \\
+ \left. \frac{(1 + \gamma_1)^2}{6} \Sigma^6 \right.
\]

\[
C_{32} = \frac{1}{\Sigma^3} \left\{ \int_0^T \alpha^2(t) \left( \int_0^t \gamma_{a1}(s) \Pi_{\eta\sigma}(s) ds \right) dt \\
+ \int_0^T \alpha^2(t) \left( \int_0^t \gamma_{b1}(s) \eta(s) \alpha(s) \rho_{\sigma\alpha}(s) \Pi_{\alpha}(s) ds \right) dt \\
+ (1 + \gamma_1) \int_0^T \alpha^2(t) \left( \int_0^t \Pi_{\alpha}(s) ds \right) dt \\
+ (1 + \gamma_1) \int_0^T \alpha^2(t) \Pi_{\alpha\alpha}(t) \Pi_{\alpha\alpha}(t) dt \right\},
\]

\[
C_{33} = \frac{1}{\Sigma^2} \left\{ \frac{(\gamma_2}{2} + \gamma_1) \Sigma^4}{2} + (1 + \gamma_1) \int_0^T \alpha^2(t) \Pi_{\eta\alpha}(t) dt \right\},
\]

\[
C_{34} = \frac{1}{\Sigma^2} \left\{ \int_0^T \alpha^2(t) \left( \int_0^t \gamma_{a1}(s) \left( \int_0^s a(u, 1) du \right) ds \right) dt \right\},
\]

\[
C_{41} = \frac{1}{\Sigma^3} \left\{ \int_0^T \alpha^2(t) \Pi_{f\alpha}(t) dt - \int_0^T \alpha^2(t) \Pi_{d\alpha}(t) dt \right. \\
+ \gamma_1 \int_0^T \alpha^2(t) \left( \int_0^t \sigma_f(s, t) \alpha(s) \rho_{fs}(s) ds \right) dt \\
- \gamma_1 \int_0^T \alpha^2(t) \left( \int_0^t \sigma_d(s, t) \alpha(s) \rho_{ds}(s) ds \right) dt \\
+ \int_0^T \alpha(t) \Pi_{\alpha\alpha}(t) \sigma_f(t, T) \rho_{fs}(t) dt \\
- \int_0^T \alpha(t) \Pi_{\alpha\alpha}(t) \sigma_d(t, T) \rho_{ds}(t) dt \right\},
\]
\[ C_{51} = \frac{1}{\Sigma^2} \left\{ \left( \int_0^T \alpha^2(t) \Pi_{\eta\alpha}(t) dt \right)^2 \right. \\
+ \frac{(1 + \gamma_1)^2}{4} \Sigma^4 \\
+ (1 + \gamma_1) \Sigma^4 \int_0^T \alpha^2(t) \Pi_{\eta\alpha}(t) dt \right\} \]

\[ C_{52} = \frac{1}{\Sigma^2} \left\{ 2 \left( \int_0^T \alpha^2(t) \Pi_{\eta\alpha}(t) dt \right) \left( \int_0^T \alpha^2(t) \Pi_a(t) dt \right) \\
+ (1 + \gamma_1) \Sigma^4 \left( \int_0^T \alpha^2(t) \Pi_a(t) dt \right) \right\} \]

\[ C_{53} = \frac{1}{\Sigma^2} \left\{ 2 \left( \int_0^T \alpha^2(t) \left( \int_0^t \alpha^2(s) \Pi_{\eta\eta}(s) ds \right) dt \right) \\
+ 2 \int_0^T \alpha^2(t) \Pi_{\eta\eta}(t) dt \\
+ (1 + \gamma_1) \Sigma^4 \right\} 
\]

\[ C_{54} = \frac{2}{\Sigma^2} \left\{ \int_0^T \alpha^2(t) \Pi_a(t) \Pi_{\eta\alpha}(t) dt \\
+ \int_0^T \alpha^2(t) \left( \int_0^t \eta(s) \alpha(s) \rho_{\eta\alpha}(s) \Pi_a(s) ds \right) dt \\
+ (1 + \gamma_1) \int_0^T \alpha^2(t) \Pi_a(t) \Pi_{\eta\alpha}(t) dt \\
+ (1 + \gamma_1) \int_0^T \alpha^2(t) \left( \int_0^t \alpha^2(s) \Pi_a(s) ds \right) dt \right\} \]

\[ C_{55} = \frac{1}{\Sigma^2} \left\{ \int_0^T \alpha^2(t) \Pi_{\eta\eta}(t) dt \\
+ \frac{(1 + \gamma_1)^2}{2} \Sigma^4 \\
+ \int_0^T \alpha^2(t) \Pi_{\eta\eta}^2(t) dt \\
+ 2(1 + \gamma_1) \int_0^T \alpha^2(t) \Pi_{\eta\alpha}(t) dt \right\} \]
\[ C_{61} = \frac{1}{\Sigma_4} \left\{ \Pi_{d\alpha}^2(T) + \Pi_{f\alpha}^2(T) - 2\Pi_{d\alpha}(T)\Pi_{f\alpha}(T) \right\} \]

\[ C_{62} = \frac{1}{\Sigma_2} \left\{ \int_0^T \sigma_d^2(t, T)dt + \int_0^T \sigma_f^2(t, T)dt \right\} \]

\[ C_{71} = \frac{2}{\Sigma_5} \left\{ \left( \int_0^T \alpha^2(t)\Pi_{\eta\alpha}(t)dt + \frac{1 + \gamma_1}{2} \Sigma^4 \right) \left( \Pi_{f\alpha}(T) - \Pi_{d\alpha}(T) \right) \right\} \]

\[ C_{72} = \frac{2}{\Sigma_3} \left( \int_0^T \alpha^2(t)\Pi_{\alpha}(t)dt \left( \Pi_{f\alpha}(T) - \Pi_{f\alpha}(T) \right) \right) \]

\[ C_{73} = \frac{2}{\Sigma_3} \left\{ \int_0^T \alpha(t)\sigma_f(t, T)\rho_{sf}(t)\Pi_{\eta\alpha}(t)dt \right\} \]

\[ C_{74} = \frac{2}{\Sigma_2} \left\{ \int_0^T \alpha(t)\sigma_f(t, T)\rho_{sf}(t)\Pi_{\alpha}(t)dt \right\} \]
with:

\[
\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},
\]

(3.26)

\[
\Phi(x) = \int_{-\infty}^{x} \phi(x) dx,
\]

(3.27)

\[
\Sigma = \sigma(0) \gamma(X(0)) (\int_{0}^{T} \alpha^2(t) dt)^{\frac{1}{2}},
\]

(3.28)

\[
\Pi_a(t) = \int_{0}^{t} a(s, \sigma(0)) ds,
\]

(3.29)

\[
\Pi_{\eta\alpha}(t) = \int_{0}^{t} \eta(s) \alpha(s) \rho_{s\alpha}(s) ds,
\]

(3.30)

\[
\Pi_{\alpha\alpha}(t) = \int_{0}^{t} \alpha^2(s) ds,
\]

(3.31)

\[
\Pi_{\eta\eta}(t) = \int_{0}^{t} \eta^2(s) ds,
\]

(3.32)

\[
\Pi_{f\alpha}(t) = \int_{0}^{t} \sigma_f(s, T) \alpha(s) \rho_{fs}(s) ds,
\]

(3.33)

\[
\Pi_{d\alpha}(t) = \int_{0}^{t} \sigma_d(s, T) \alpha(s) \rho_{ds}(s) ds,
\]

(3.34)

\[
\gamma_1 = \gamma'(X(0)), \gamma_2 = \gamma''(X(0)), \gamma_{a1}(t) = a'(t, \sigma(0)), \gamma_{a2}(t) = a''(t, \sigma(0)), \gamma_{b1}(t) = \tilde{b}'(\sigma(0)), \gamma_{b2}(t) = \tilde{b}''(\sigma(0)),
\]

Proof: See Appendix.

Remark: As easily can be seen from expression (3.21), the leading term, \( T_1 \), corresponds to the solution from a simple Normal model, where the diffusion term (including the asset price level) keeps constant.

Having obtained the call option price formula, we need to arrive at an implied Black-Scholes volatility formula such that the implied volatility can be computed directly.
Theorem 3.3.2 Let strike \( K^c = F(0,T)(1 + \epsilon \Sigma y) \), \( \epsilon \in (0,1] \). The implied Black volatility is given by:

\[
\sigma_B(T, K^c) = \frac{\epsilon \Sigma}{\sqrt{T}} \{ 1 + \epsilon (\Xi_1 - \frac{y \Sigma}{2}) + \epsilon^2 (\Xi_2 + \frac{\Sigma^2}{24}) \\
+ \epsilon^2 y (\Xi_3 - \frac{\Sigma \Xi_1}{2}) + \epsilon^2 y^2 (\Xi_4 + \frac{\Sigma^2}{3}) + \epsilon^2 y^3 \Xi_5 \\
+ \epsilon^2 y^4 \Xi_6 \} + \mathcal{O}(\epsilon^4)
\] (3.35)

with:

\[
\Xi_1 = C_{11} y + (C_{12} + C_{21}), \\
\Xi_2 = -C_{31} + C_{33} + C_{34} + \frac{1}{2} \{ 3C_{51} - C_{53} + C_{55} - C_{61} + C_{62} \\
- C_{72} + C_{74} \}, \\
\Xi_3 = C_{32} + C_{41} - \frac{1}{2} \{ 3C_{52} - C_{54} + 3C_{71} - C_{73} \}, \\
\Xi_4 = C_{31} - 3C_{51} + \frac{1}{2} \{ C_{53} + C_{61} + C_{72} - C_{12} - C_{21} - 2C_{12}C_{21} \}, \\
\Xi_5 = \frac{C_{52}}{2} + \frac{C_{71}}{2} - C_{11}C_{12} - C_{11}C_{21}, \\
\Xi_6 = \frac{C_{51}}{2} - \frac{C_{11}^2}{2},
\]

\((C_{11} - C_{74} \text{ are given as above}).

Proof: Here, we borrow the idea of implied Normal volatility from Hagan et al. [38]. First, we consider the normal model:

\[
d\tilde{F}(t) = \sigma d\tilde{W}(t),
\] (3.36)

with \( \tilde{F}(0) = F_0 \), the European call option value is known as:

\[
\text{Call}(T, K) = \tilde{G}(K - F_0, \sigma^2 T).
\] (3.37)

where \( \tilde{G} : \mathbb{R} \to \mathbb{R}_+ \) is given by:

\[
\tilde{G}(x, v) = \int_{x}^{\infty} (y - x) n(y, v) dy, x \in \mathbb{R}.
\]

\[
= x N\left(\frac{x}{\sqrt{v}}\right) + \sqrt{\frac{v}{2\pi}} e^{-\frac{x^2}{2v}}.
\] (3.38)

and \( n(y, v) \) is the normal distribution density function of \( y \) with \( \text{Mean} = 0 \) and \( \text{Variance} = v \). It is simple to prove that the European call option price
is increasing strictly in $\sigma$, thus there is a unique $\sigma_N$ satisfying: $\text{Call}(T, K) = \text{Call}(T, K; \sigma_N^2 T)$. This unique $\sigma_N$ is called implied normal volatility. The term $\sigma_N^2 T$ can be recognised as the total normal variance.

For later use, we denote:

$$\tilde{G}_x = \mathcal{N}\left(\frac{x}{\sqrt{v}}\right), \quad (3.39)$$

$$\tilde{G}_v = \frac{1}{2} n(x, v), \quad (3.40)$$

$$\tilde{G}_{vv} = \frac{x^2 - v}{4v^2} n(x, v) \quad (3.41)$$

Now, we rewrite the call option price formula (3.21) as:

$$\frac{\text{Call}(T, K_\epsilon)}{P_d(0, T)} = \tilde{G}(x, (\epsilon \Sigma F_0)^2) + (\epsilon \Sigma F_0) (\Upsilon_2 + \Upsilon_3 + \Upsilon_4).$$

where $x = K - F(0, T)$.

In addition, we write the Taylor expansion of $\text{Call}(T, K; \sigma_N^2 T)$ around its initial total normal variance term $(\epsilon \Sigma F_0)^2$ as:

$$\frac{\text{Call}(T, K; \sigma_N^2 T)}{P_d(0, T)} = \tilde{G}(x, (\epsilon \Sigma F_0)^2) + \frac{\partial \tilde{G}}{\partial v} \bigg|_{v=(\epsilon \Sigma F_0)^2} (\sigma_N^2 T - (\epsilon \Sigma F_0)^2)$$

$$+ \frac{1}{2} \frac{\partial^2 \tilde{G}}{\partial v^2} \bigg|_{v=(\epsilon \Sigma F_0)^2} (\sigma_N^2 T - (\epsilon \Sigma F_0)^2)^2 + ...$$

Compare these two equalities and use (3.40)-(3.41). Equating the corresponding terms, and using the Taylor expansion again, we arrive at:

$$\sigma_N(T, K) = \frac{\epsilon \Sigma F_0}{\sqrt{T}} \left\{ 1 + \frac{1}{n(y)} (\Upsilon_2 + \Upsilon_3 + \Upsilon_4) \right\}$$

$$- \frac{1}{2} \frac{y^2}{n^2(y)} (\Upsilon_2 + \Upsilon_3 + \Upsilon_4)^2 \quad (3.42)$$

On the other hand, we apply the same asymptotic expansion method to the Black-Scholes model, which is a simplified case of this local stochastic volatility model (with volvol: $\eta(t) = 0$, $a(t, \sigma) = 0$, $\alpha(t) = \alpha_0$, $D(t, T) = D(t, T) = 0$, $\gamma(x) = 1$.) The total normal variance is then given by:

$$\sigma_B^2 T = F_0^2 \sigma_B^2 T \left\{ 1 + (y \epsilon \Sigma) + \frac{1}{12} (y \epsilon \Sigma)^2 - \frac{1}{12} \sigma_B^2 T \right\}$$
which leads to:

$$\sigma_B(T, K) = \frac{\sigma_N}{F_0} \left\{ 1 - \frac{1}{2} (y\epsilon\Sigma) + \frac{1}{3} (y\epsilon\Sigma)^2 + \frac{1}{24} (\epsilon\Sigma)^2 \right\}$$  (3.43)

where $\sigma_B$ denotes the Black-Scholes volatility.

Plugging the formula (3.42) for $\sigma_N$ into (3.43), we arrive at (3.35) in Theorem 3.3.2.

### 3.4 Model Implementation and Numerical Results

At this point we want to check the accuracy of the formula (3.35) we derived. Firstly, some numerical results will be shown in the deterministic interest rate world (setting the bond volatilities to 0). This setting will compare the implied volatility formula given in (3.35) with another method based on WKB perturbation (known as the SABR model, for which the implied BS volatility formula is outlined in [30]). Other comparisons will be made with different numerical methods, namely the Tree method and Monte Carlo method, in order to check the accuracy of the implied volatility formula for the local stochastic volatility model (without stochastic interest rates). All implementations are carried out in C++ and code is available upon request.

![Figure 3.1 Here](image1.png)

![Figure 3.2 Here](image2.png)

We see from the comparison above that our asymptotic expansion formula returns very close results with the SABR model for not-too-far-from-the-money implied volatilities. It also achieves high accuracy when compared to the Monte Carlo/Tree methods.

Secondly, we will compare the BS-implied volatility smile/skew generated by formula (3.35) to those obtained from the Monte Carlo method with both local stochastic volatility and stochastic interest rates. The interest rate model used here is the Hull-White model(1994)(see [45] for a complete reference), which, in the asymptotic expansion sense, is defined as $^6$:

$$dr^*_d = (\theta_d(t) - \lambda_d(t)r^*_d(t))dt + \epsilon^2\sigma_d(t)dW_d(t),$$  (3.44)

$$dr^*_f = (\theta_f(t) - \lambda_f(t)r^*_f(t))dt + \epsilon^2\sigma_f(t)dW_d(t),$$  (3.45)

$^6$ As mentioned in (3.19), since the magnitude for interest rate volatilities $\sigma_d(t), \sigma_f(t)$ is around 1% level, it is reasonable to assign an "ultra small" parameter $\epsilon^2$ to them.
under the respective risk-neutral measure.

In this model, the bond volatilities are given as:

$$\sigma_{d,f}(t,T) = \sigma_{d,f}(t) \int_t^T e^{-\int_u^T \lambda_{d,f}(u)du} ds$$  \hspace{1cm} (3.46)

In our implementation, for the sake of simplicity, we assume the mean-reversion rate $\lambda_{d,f}(t)$ to be constant throughout time. In this case:

$$\sigma_{d,f}(t,T) = \sigma_{d,f}(t) e^{-\lambda_{d,f}(T-t)} - 1 \over \lambda_{d,f}.$$  \hspace{1cm} (3.47)

Again, as we see from the comparison results in the environment of local stochastic volatility and stochastic interest rates, the expansion formula returns very close results with those from the Monte Carlo simulation, even for the long maturity of 10 years. The figures are listed below:

Figure 3.3, 3.4 Here  
Figure 3.5, 3.6 Here

Furthermore, as the LIBOR interest rate market model has gained popularity in the financial industry, it is worth looking at this hybrid model implementation via LIBOR rates. Considering that the dynamics of the LIBOR rates involved here are much more complicated than those of short rate models, we start from a simple LIBOR market model (we mean no local, stochastic volatility or jump diffusion modifications which are to be shown in Chapter 5) and make some approximations.

The LIBOR market model in asymptotic expansion form considered here is defined as (the notations are in line with [80]):

$$\frac{dL_i^d(t)}{L_i^d(t)} = \epsilon^2 \mu_i^d \, dt + \epsilon^2 \gamma_i^d(t) \cdot dW_t^d,$$

$$\mu_i^d = - \sum_{k=i+1}^{N-1} \frac{L_k^d(t) \delta_k^d}{1 + L_k^d(t) \delta_k^d} \gamma_k^d(t) \gamma_i^d(t) \mu_{k,i}(t).$$

$$\frac{dL_i^f(t)}{L_i^f(t)} = \epsilon^2 \mu_i^f \, dt + \epsilon^2 \gamma_i^f(t) \cdot dW_t^f,$$

$$\mu_i^f = - \sum_{k=i+1}^{N-1} \frac{L_k^f(t) \delta_k^f}{1 + L_k^f(t) \delta_k^f} \gamma_k^f(t) \gamma_i^f(t) \mu_{k,i}(t).$$  \hspace{1cm} (3.48)
under the respective $T_N$ forward measure, with $P_d(t, T_N)$ and $P_f(t, T_N)$ as numeraire, respectively.\footnote{Same argument applies here as we set up the Hull-White model for the use of asymptotic expansion in (3.44)-(3.45): since the drift terms and volatility terms in the LIBOR market model are of small magnitude, we assign $\epsilon^2$ to them.}

We note from the option price formula in Theorem 3.3.1 that only bond volatilities enter into the valuation. Thus, in order to implement our model, we do not need a full asymptotic expansion on the LIBOR market model – but only these terms are of relevance:

- $\sigma_{d,f}(t, T_N)$
- $D(t, T)$
- $\int_0^T \alpha(t) \rho_{f,s}(t) \sigma_f(t, T_N) dt - \int_0^T \alpha(t) \rho_{d,s}(t) \sigma_d(t, T_N) dt$
- $\int_0^T \sigma_{d,f}^2(t, T_N) dt$
- $\int_0^T \sigma_d(t, T_N) \sigma_f(t, T_N) \rho_{d,f}(t) dt$

In the model set-up (3.48), by the use of the approximation method – the ”freezing coefficient” – these are approximated as:

$$\sigma_{d,f}(t, T_N) \approx - \sum_{k=\eta(t)}^{T_N-1} \frac{L_{d,f}^k(0) \delta_{d,f}^k(t)}{1 + L_{d,f}^k(0) \delta_{d,f}^k(t)} \gamma_f(t),$$  \hspace{1cm} (3.49)

$$D(t, T) \approx - \int_0^T \sum_{k=\eta(s)}^{T_N-1} \frac{L_{d}^k(0) \delta_{d}^k(s) \gamma_d(s)}{1 + L_{d}^k(0) \delta_{d}^k(s)} \rho_{s,L_d}^f(s) \cdot dW_d^T(s) + \int_0^T \sum_{k=\eta(s)}^{T_N-1} \frac{L_{f}^k(0) \delta_{f}^k(s) \gamma_f(s)}{1 + L_{f}^k(0) \delta_{f}^k(s)} \rho_{s,L_f}^d(s) dW_f^T(s) \hspace{1cm} (3.50)$$

$$\int_0^T \alpha(t) \rho_{f,s}(t) \sigma_f(t, T_N) dt - \int_0^T \alpha(t) \rho_{d,s}(t) \sigma_d(t, T_N) dt \approx$$

$$- \int_0^T \alpha(t) \sum_{k=\eta(t)}^{T_N-1} \frac{L_{f}^k(0) \delta_{f}^k(t) \gamma_f(t)}{1 + L_{f}^k(0) \delta_{f}^k(t)} \rho_{s,L_f}^d(t) dt + \int_0^T \alpha(t) \sum_{k=\eta(t)}^{T_N-1} \frac{L_{d}^k(0) \delta_{d}^k(t) \gamma_d(t)}{1 + L_{d}^k(0) \delta_{d}^k(t)} \rho_{s,L_d}^f(t) dt \hspace{1cm} (3.51)$$
\[
\int_0^T \sigma_{d,f}^2(t, T_N) dt \approx \int_0^T \sigma_d(t, T_N) \sigma_f(t, T_N) \rho_{d,f}(t) dt
\]

where \( \sigma(t) = \text{inf}\{i : T_i \geq t\} \). For time-dependent LIBOR volatilities in the above formulae, we follow Rebonato’s approach [80] to choose the Nelson-Siegel form:

\[
\gamma_{d,f}^{i,j}(t) = C_{d,f}^{i,j} (a_{d,f} + b_{d,f} (T_i - t)) e^{-c_{d,f} (T_i - t)} + d_{d,f}, \quad (3.52)
\]

where \( C_{d,f}, a, b, c, d \) are constant scaling factors fitted to the observed LIBOR rates’ volatilities term structure.

Additionally, the correlation structure of LIBOR rates (on the same currency) is given by:

\[
\rho_{d,f}^{i,j}(t) = e^{-g_{d,f} |(T_i - t) - (T_j - t)|}, \quad \beta_1
\]

where \( g_{d,f} \) are constant and the real value \( \beta_1 \) is normally chosen as 0.5 or 1.

We assume the correlation structure between the FX rate and LIBOR rates are given by:

\[
\rho_{S,L}^{d,f}(t) = q_{d,f} e^{-p_{d,f} |(T_i - t) - (T_j - t)|}, \quad (3.54)
\]

where \( q_{d,f}, p_{d,f} \) and \( \beta_2 \) are constant real values.

We assume the correlation between the LIBOR rates in different currencies \( \rho_{L_i^{d},L_j^{f}}(t) = 0 \) for \( t \in [0, T_N] \).

The LIBOR model we use here is a basic log normal variety. A number of extensions including stochastic volatility and jump diffusion can be found in Chapter 5.

Finally, we calibrate our model to the FX options market. For the stochastic interest rate we choose the Hull-White short rate model, the model parameters are calibrated from interest rate derivatives. The FX options data we use comes
from the USD-JPY market, which exhibits stronger volatility skews than other FX pairings. We find the volatility skew/smile generated from our formula indeed fit the market skew/smile very well, for different maturities.

### 3.5 FX Option Pricing via Fourier Transform under Stochastic Interest Rates, Stochastic Volatility and the Jump Process

#### 3.5.1 The Multi-Factor Model

As we have already seen, the expansion method for the general local stochastic volatility model with stochastic interest rates works well and gives accurate results for the implied volatilities of vanilla options with maturities up to 10 years or more. However, due to the assumptions made under the asymptotic expansion method, accuracy deteriorates with increasing maturities of more than 15 years. With a growing market of long-dated FX options, we need accurate analytical or semi-analytical formulae for option prices (or, equivalently, implied volatilities) of maturities of up to 30 years or more. The Fourier transform method, regarded as a semi-analytical formula, now comes into play.

Now we present a multi-factor model with a different setting of the FX spot rate process from (3.1) under the domestic risk-neutral measure $\mathcal{Q}_d$. \(^9\):

\[
\frac{dS(t)}{S(t)} = (r_d(t) - r_f(t))dt + \alpha_1(t)dW_s(t) + \alpha_2(t)\sigma(t)dW_S(t) + \text{JumpTerm}
\]

\[
\frac{dP_d(t, T)}{P_d(t, T)} = r_d(t)dt + \sigma_d(t, T)dW_d(t)
\]

\[
\frac{dP_f(t, T)}{P_f(t, T)} = (r_f(t) - \rho_{sf}(t)\sigma_f(t, T)\alpha_1(t))dt + \sigma_f(t, T)dW_f(t)
\]

(3.55)

with stochastic volatility $\sigma(t)$ evolves as:

\[
d\sigma(t) = a(t, \sigma(t))dt + b(t, \sigma(t))dW_\sigma(t)
\]

and we choose a log-normal type (Merton 76) Poisson jump process:

\[
\text{JumpTerm} = -\lambda(t)\overline{\mu}(t)dt + (e^{J(t)} - 1)d\eta(t)
\]

\(^8\) This work was undertaken in 2005/2006 with Mitsubishi UFJ Securities International. Jesper Andreasen [35] proposed another multi-factor model based on the Fourier transform method with a different model set-up around the same time.

\(^9\) This will de-correlate the stochastic interest rates and stochastic volatility parts in order to analytically calculate their characteristic functions separately.
with \( \lambda(t) \) as the deterministic jump intensity at time \( t \), \( J(t) \) is the Gaussian variable with mean \( a(t) \) and standard deviation \( \delta(t) \) and \( q(t) \) is the Possion process:

\[
dq(t) = \begin{cases} 
0 & \text{with probability } 1 - \lambda(t)dt \\
1 & \text{with probability } \lambda(t)dt 
\end{cases}
\]

In order to guarantee the jump term as a martingale, \( \mu(t) \) should be defined as:

\[
E[e^{J(t)} - 1] = e^{a(t)} + \frac{1}{2} \delta^2(t) - 1.
\]

For further simplicity, we assume the jump part is independent of the Brownian diffusion part.

The correlation matrix in this multi-factor model is specially chosen to decorrelate the stochastic interest rates and stochastic volatility parts, in order to write their respective characteristic functions separately, with a benefit that will become clear later. This is defined as:

\[
\begin{pmatrix}
    dW_d & dW_f & dW_S & dW_\sigma & d\overline{W}_S \\
    dW_d & 1 & \rho_{df} & \rho_{ds} & 0 & 0 \\
    dW_f & \rho_{df} & 1 & \rho_{fs} & 0 & 0 \\
    dW_S & \rho_{ds} & \rho_{fs} & 1 & 0 & 0 \\
    dW_\sigma & 0 & 0 & 0 & 1 & \rho_{sS} \\
    d\overline{W}_S & 0 & 0 & 0 & \rho_{sS} & 1 
\end{pmatrix}
\]

\[ (3.56) \]

### 3.5.2 Change of Measure and Option Pricing

We now change the measure from a domestic risk-neutral measure \( Q_d \) to a domestic \( T \)-forward measure \( Q^T_d \). Similar to the manipulation in last section, we have:

\[
\begin{align*}
    dW^T_S(t) &= dW_S(t) + \rho_{ds}(t)\sigma_d(t,T)\alpha_1(t)dt, \\
    dW^T_d(t) &= dW_d(t) + \sigma_d(t,T)dt, \\
    dW^T_f(t) &= dW_f(t) + \rho_{df}(t)\sigma_d(t,T)\sigma_f(t,T)dt.
\end{align*}
\]

With \( dW_\sigma(t), d\overline{W}_S(t) \) and the jump term are unchanged because of independence to \( dW_d(t) \). The FX forward rate is defined as:

\[
F(t,T) = S(t) \frac{P_f(t,T)}{P_d(t,T)}
\]

Using Ito’s lemma with Brownian motions under the new measure, we give the process of the forward FX rate as:

\[
\frac{dF(t,T)}{F(t,T)} = \alpha_1(t) dW^T_S(t) - \sigma_d(t,T) dW^T_d(t) + \sigma_f(t,T) dW^T_f(t) + \alpha_2(t) \sigma(t) d\overline{W}_S(t) + \text{JumpTerm}
\]

3. Foreign Exchange Options with Local Stochastic Volatility and Stochastic Interest Rates

In order to apply the Fourier transform method, we need to calculate the characteristic function of \( \log\left(\frac{F(T,T)}{F(0,T)}\right) \), which is defined as:

\[
CF(u) := \mathbb{E}^{QT}[e^{iu\log\left(\frac{F(T,T)}{F(0,T)}\right)}]
\]

by using Ito’s lemma we obtain:

\[
\log\left(\frac{F(T,T)}{F(0,T)}\right) = -\frac{1}{2} \int_0^T \Lambda^2(t) dt + \int_0^T \Lambda(t)dW^T(t)
\]

\[
-\frac{1}{2} \int_0^T \alpha_2^2(t) dt + \int_0^T \alpha_2(t)\rho_{\sigma S}(t)dW_\sigma(t)
\]

\[
+ \int_0^T \alpha_2(t)\sigma(t)\sqrt{1 - \rho_{\sigma S}^2(t)}dW_\sigma(t)
\]

\[
- \int_0^T \lambda(t)\mu(t) dt + \int_0^T J(t) dq(t)
\]

with \( dW_S(t) \) written as \( \rho_{\sigma S}(t)dW_\sigma(t) + \sqrt{1 - \rho_{\sigma S}^2(t)}dW_\sigma(t) \), \( dW_\sigma(t) \) and \( dW_\sigma(t) \) are independent. \( \Lambda(t) \) is given as:

\[
\Lambda(t)dW^T_\Lambda(t) = \alpha_1(t)dW^T_S(t) - \sigma_d(t,T)dW^T_d(t) + \sigma_f(t,T)dW^T_f(t)
\]

and

\[
\Lambda^2(t) = \alpha_1^2(t) + \alpha_2^2(t) + \sigma_f^2(t,T) - 2\alpha_1(t)\sigma_d(t,T)\rho_{\sigma d}(t)
\]

\[
+ 2\alpha_1(t)\sigma_f(t,T)\rho_{\sigma f}(t) - 2\sigma_d(t,T)\sigma_f(t,T)\rho_{d f}(t)
\]

Thus, the characteristic function can be decomposed to three independent parts given their mutual independence:

\[
CF(u) = CF_{IR}(u) \cdot CF_{SV}(u) \cdot CF_{Jump}(u)
\]

with

\[
CF_{IR}(u) = e^{-\frac{1}{2}u^2} \int_0^T \Lambda^2(t) dt
\]

\( CF_{Jump}(u) \) is given by Levy-Khintchine Representation([47]):

\[
CF_{Jump}(u) = \exp\left\{-iu \int_0^T \lambda(t)(e^{u(t)} + \frac{1}{2}\delta(t)^2) dt
\right\}
\]

\[
+ \int_0^T \lambda(t) \left\{ e^{iu(t)} - 1 \right\} - \frac{1}{2\pi} \int_0^\infty \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}u^2} du dt
\]

\[
= \exp\left\{-iu \int_0^T \lambda(t)(e^{u(t)} + \frac{1}{2}\delta(t)^2) dt + \int_0^T \lambda(t)(e^{-\frac{u^2\delta(t)^2}{4} + iu(t)} - 1) dt\right\}
\]
3. Foreign Exchange Options with Local Stochastic Volatility and Stochastic Interest Rates

For the stochastic volatility part, we may adopt two different models with analytical characteristic functions, namely the Heston model ([4]) and Schobel & Zhu ([21]) model. Both possess a form of characteristic function:

\[ CF_{SV}(u) = e^{A(u,T)+B(u,T)\sigma_0+C(u,T)V_0} \]

With variance \( V(t) := \sigma^2(t) \). \( A(u,T) \), \( B(u,T) \) and \( C(u,T) \) can be found in ([59]), while \( B(u,T) = 0 \) in the Heston model.

Having obtained the characteristic function, the call option price with strike \( K \) and maturity \( T \) is given as:

\[
C(F(0,T),K,T) = P_d(0,T) \left\{ F(0,T) - \frac{F(0,T)}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-iuk+\alpha k}CF(u+(\alpha-1)i)}{(u+\alpha i)(u+(\alpha-1)i)} \, du \right\}
\]

with \( \alpha \) the "optimal dampening factor". For completeness, the proof of this call option formula is in Appendix 3.11.5.

3.5.3 Model Implementation

As we see this multi-factor model with stochastic interest rates, stochastic volatility and jump process is quite general and can be viewed as a "Building Block Model" with three main blocks:

- Stochastic interest rates. As what we need is bond volatilities, this can be implemented via short rate models such as the Vasicek or Hull-White models (see [45]), the HJM forward rate model or the LIBOR market model, as stated and implemented in the last section.

- Stochastic volatility. We can choose either the Schobel and Zhu model (see [21]) for an O-U type stochastic volatility process with piece-wise constant volatility of volatility \( \eta(t) \):

\[
d\sigma = \kappa(\theta - \sigma)dt + \eta(t)dW_\sigma(t)
\]

with characteristic function as in [59];

or the Heston ([4]) model with a square root stochastic variance process with piece-wise constant volatility of variance \( \zeta(t) \):

\[
dV = \kappa(\theta - V)dt + \zeta(t)\sqrt{V}dW_\sigma(t)
\]

\( V = \sigma^2 \) is the variance process, and its characteristic function \( e^{A(u,T)+C(u,T)V_0} \) is given in Chapter 5. More generally, we can use other models with an affine form of the characteristic function, as stated in [76].
Jump process. We choose a Merton-type log-normal jump process here with a time-dependent jump size mean/standard deviation and jump intensity. We use this here because of its simplicity and intuitive meaning. Of course, other more complicated jump process/Levy processes such as double exponential jumps (see [119]) or simultaneous jumps in asset price and volatility (see [120]) can also be applied here, as long as they possess an analytical characteristic function. For a detailed treatment on jump processes, we refer to ([47]).

Note now in this multi-factor model that the correlation between the FX spot return and two bond price returns differentiate from the input values \( \rho_{dS} \) and \( \rho_{fS} \). In fact, according to the definition of correlation:

\[
\rho(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}
\]

\( (Var(\cdot)) \) denotes the variance

we can derive the instantaneous "effective correlation" (the true correlation) between the FX spot return and bond price returns at time \( t \) as:

\[
\tilde{\rho}_{dS}(t) = \frac{\alpha_1(t)\rho_{dS}(t)}{\sqrt{\alpha_1^2(t) + \alpha_2^2(t)\sigma^2(t)}}
\]

\[
\tilde{\rho}_{fS}(t) = \frac{\alpha_1(t)\rho_{fS}(t)}{\sqrt{\alpha_1^2(t) + \alpha_2^2(t)\sigma^2(t)}}
\]

which are scaled by a factor: \( \frac{\alpha_1(t)}{\sqrt{\alpha_1^2(t) + \alpha_2^2(t)\sigma^2(t)}} \) at time \( t \). Normally, in the model implementation we may firstly assess effective correlations \( \tilde{\rho}s \) from historical data, after that we need to calibrate the correlation factors \( \rho_{dS} \) and \( \rho_{fS} \) given the scaling factor.

Since this "Building Block" approach is quite general, as long as the "blocks" are independent of each other (such that we can obtain the characteristic function as the product of each of their own), we can apply the Fourier transform, and European option prices can be obtained in a second. A similar approach appears as "Modular Option Pricing" in [37] on equity modelling, but these two approaches are different in that Zhu models the spot price process under a risk-neutral measure, while here we consider the forward exchange rate process under a domestic forward measure; in Zhu’s work there is only one stochastic interest rate independent of the stock price process, but here we consider two stochastic interest rates that are mutually dependent with the exchange rate process with a three additional correlation matrix, which complicates the problem and, of course, means that more factors are required. Finally, all coefficients in our model are time-dependent (piece-wise constant) rather than constant as in Zhu’s model.
3. Foreign Exchange Options with Local Stochastic Volatility and Stochastic Interest Rates

3.5.4 Calibration Results for the USD/JPY Market

We calibrate our model to the USD/JPY market volatility surface with maturities from 1 to 30 years, observed in April, 2007 (see Figure 3.7). Consequently, we see from the calibration results in Figure 3.8 that the SIR+SV model (without jumps) alone fits well to the volatility surface, especially for medium and long maturities, with most differences below 20 bps for 5-year and longer maturities. However, for short maturities we see some extreme parameters (see Table 3.1), especially for volatility of variance (volvol), which can be as high as 270%.

In order to improve the volatility surface fitting results for short maturities, as well as to "dampen" the extreme model parameters for short maturities, we use the more complicated SIR+SV+Jumps model. The case for one-year volatility smile is shown in Figure 3.9, which shows that the introduction of jumps indeed improves the fitting result, and the volvol goes down to 120%, which can be regarded as "reasonable", at the expense of calibrating three more parameters: jump intensity, jump size mean and jump size standard deviation.

<table>
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<th>Maturity</th>
<th>Vol(%)</th>
<th>Correlation</th>
<th>Volvol</th>
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<td>-0.65</td>
<td>2.70</td>
</tr>
<tr>
<td>2.0</td>
<td>11.00</td>
<td>-0.88</td>
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<td>-0.67</td>
<td>0.45</td>
</tr>
<tr>
<td>30.0</td>
<td>29.20</td>
<td>-0.65</td>
<td>0.43</td>
</tr>
</tbody>
</table>

Tab. 3.1: Calibration Results: Model Parameters
3. Foreign Exchange Options with Local Stochastic Volatility and Stochastic Interest Rates

3.6 Perfect Hedging with Stochastic Interest Rates and Local Stochastic Volatility

We now consider the hedging of contingent claims in the general model (3.1) with correlation matrix (3.7), given in Section 3.2. A contingent claim \( C(t) \) on FX rate \( S(t) \), in the world of stochastic interest rates and local stochastic volatility, follows a SDE via Ito’s lemma (with the subscript denotes a partial derivative and \( <\cdot,\cdot> \) is the cross variation):

\[
dC(t) = C_t dt + C_S dS(t) + C_\sigma d\sigma(t) + C_{r_d} dr_d(t) + C_{r_f} dr_f(t) \\
+ \frac{1}{2} \left\{ C_{SS} d\langle S, S \rangle_t + C_{\sigma\sigma} d\langle \sigma, \sigma \rangle_t + C_{r_d r_d} d\langle r_d, r_d \rangle_t + C_{r_f r_f} d\langle r_f, r_f \rangle_t \right\} \\
+ \left\{ C_{S\sigma} d\langle S, \sigma \rangle_t + C_{S r_d} d\langle S, r_d \rangle_t + C_{S r_f} d\langle S, r_f \rangle_t + C_{\sigma r_d} d\langle \sigma, r_d \rangle_t + C_{\sigma r_f} d\langle \sigma, r_f \rangle_t \right\} dt \\
+ \left\{ C_{r_d\sigma} d\langle r_d, \sigma \rangle_t + C_{r_d r_f} d\langle r_d, r_f \rangle_t + C_{r_f \sigma} d\langle r_f, \sigma \rangle_t + C_{r_d r_f} d\langle r_d, r_f \rangle_t \right\} d\sigma(t)
\]

(3.57)

Firstly, we consider a hedging portfolio \( \Pi(t) \) to complete the market. Different hedging portfolios and their implications will be proposed.

3.6.1 Hedging with Options

Analogous to the hedging argument in Section 2.2, in order to dynamically hedge the contingent claim \( C(S,T,K) \)\(^{10}\), we set up our hedging portfolio at time \( t \in [0,T] \) which consists of \( \Delta(t) \) units of FX spot rate \( S(t) \) and three other vanilla options: \( \Lambda(t) \) units of \( D(t) \), \( \Omega(t) \) units of \( E(t) \) and \( \Phi(t) \) units of \( F(t) \) on the same FX rate\(^{11}\), satisfying: \( U_\sigma(t) \neq 0, U_{r_d}(t) \neq 0, U_{r_f}(t) \neq 0, U \in \{D,E,F\}, t \in [0,T], T \) is the maturity of option \( C \).

Thus, our portfolio becomes:

\[
\Pi(t) = C(t) - \Delta(t)S(t) - \Lambda(t)D(t) - \Omega(t)E(t) - \Phi(t)F(t)
\]

and from (3.57) we have:

\[
d\Pi(t) = \{ \mathcal{L}C - \Lambda(t)\mathcal{L}D - \Omega(t)\mathcal{L}E - \Phi(t)\mathcal{L}F - \Delta(t)S(t)r_f(t) \} dt \\
+ \{ C_S - \Delta(t)S_C + \Delta(t)D_S - \Omega(t)E_S - \Phi(t)F_S \} dS(t) \\
+ \{ C_{r_d} - \Delta(t)D_{r_d} - \Omega(t)E_{r_d} - \Phi(t)F_{r_d} \} dr_d(t) \\
+ \{ C_{r_f} - \Delta(t)D_{r_f} - \Omega(t)E_{r_f} - \Phi(t)F_{r_f} \} dr_f(t) \\
+ \{ C_{\sigma} - \Delta(t)D_{\sigma} - \Omega(t)E_{\sigma} - \Phi(t)F_{\sigma} \} d\sigma(t)
\]

(3.58)

\(^{10}\) Here, \( C \) can be an exotic option or a structured product, therefore not limited to the vanilla option.

\(^{11}\) However, with different strikes or maturities
where $\mathcal{L}$ is an operator (denoting the time-deterministic terms):

\[
\mathcal{L} \cdot dt = \frac{\partial}{\partial t} dt + \frac{1}{2} \left\{ \frac{\partial^2}{\partial S^2} d < S >_t + \frac{\partial^2}{\partial \sigma^2} d < \sigma >_t + \frac{\partial^2}{\partial r_d^2} d < r_d >_t + \frac{\partial^2}{\partial r_f^2} d < r_f >_t \right\} \\
+ \left\{ \frac{\partial^2}{\partial S \partial \sigma} d < \sigma, S >_t + \frac{\partial^2}{\partial S \partial r_d} d < S, r_d >_t + \frac{\partial^2}{\partial S \partial r_f} d < S, r_f >_t + \frac{\partial^2}{\partial r_d \partial r_f} d < r_d, r_f >_t \right\}
\]

(3.59)

In order to make our hedging portfolio instantaneously risk-free, we want to eliminate the $dS(t)$, $dr_d(t)$, $dr_f(t)$ and $d\sigma(t)$ terms, which yields:

\[
\Delta(t) = C_S - \Lambda(t) D_S - \Omega(t) E_S - \Phi(t) F_S \\
0 = C_{r_d} - \Lambda(t) D_{r_d} - \Omega(t) E_{r_d} - \Phi(t) F_{r_d} \\
0 = C_{r_f} - \Lambda(t) D_{r_f} - \Omega(t) E_{r_f} - \Phi(t) F_{r_f} \\
0 = C_\sigma - \Lambda(t) D_\sigma - \Omega(t) E_\sigma - \Phi(t) F_\sigma
\]

(3.60)

This set of linear equations is trivial to solve for $(\Delta(t), \Lambda(t), \Omega(t), \Phi(t))$ for the dynamic hedge at time $t$.

Now, since our hedging portfolio is risk-free, $\Pi(t)$ grows at risk-free rate $r_d(t)$:

\[
d\Pi(t) = \Pi(t) r_d(t) dt \\
= \{ \mathcal{L} C - \Lambda(t) \mathcal{L} D - \Omega(t) \mathcal{L} E - \Phi(t) \mathcal{L} F - \Delta(t) S(t) r_f(t) \} dt
\]

(3.61)

(3.61) suggests:

\[
\Pi(t) r_d(t) = \{ C(t) - \Delta(t) S(t) - \Lambda(t) D(t) - \Omega(t) E(t) - \Phi(t) F(t) \} r_d(t) \\
= \{ \mathcal{L} C - \Lambda(t) \mathcal{L} D - \Omega(t) \mathcal{L} E - \Phi(t) \mathcal{L} F - \Delta(t) S(t) r_f(t) \} dt
\]

Substituting the equation for $\Delta(t)$ in (3.60) into above equality gives:

\[
\{ C r_d(t) - C_S S(t) (r_d(t) - r_f(t)) \} - \mathcal{L} C = \Lambda(t) \{ D r_d(t) - D_S S(t) (r_d(t) - r_f(t)) \} - \mathcal{L} D \\
+ \Omega(t) \{ E r_d(t) - E_S S(t) (r_d(t) - r_f(t)) \} - \mathcal{L} E \\
+ \Phi(t) \{ F r_d(t) - F_S S(t) (r_d(t) - r_f(t)) \} - \mathcal{L} F
\]

(3.62)

If we denote $\tilde{\mathcal{L}}$ as:

\[
\tilde{\mathcal{L}} = r_d(t) - \frac{\partial}{\partial S} S(t) (r_d(t) - r_f(t)) - \mathcal{L}
\]

(3.63)
we arrive at:
\[
\hat{L}C = \Lambda(t)\hat{L}D + \Omega(t)\hat{L}E + \Phi(t)\hat{L}F
\] (3.64)

**Example 1. Deterministic IRs and Stochastic Volatility**

It is interesting to see some simpler cases – say, for instance, interest rates are deterministic and only the volatility is stochastic – in which we can only use one option \(D(t)\) instead of three for the hedging process (that is, \(\Omega(t) = \Phi(t) = 0\)). Then \(U_{rd} = U_{rf} = 0\), \(U \in \{C, D\}\) and (3.60) gives us:

\[
\begin{align*}
\Lambda(t) &= \frac{C_\sigma}{D_\sigma} \\
\Delta(t) &= C_S - \Lambda(t)D_S
\end{align*}
\]

now (3.64) becomes\(^{12}\):

\[
\frac{\hat{L}C}{C_\sigma} = \frac{\hat{L}D}{D_\sigma}
\]

Denoting each term above as a function \(\lambda(t, S_t, \sigma_t) : ([0, T], R_+, R_+) \rightarrow R\), we have:

\[
\hat{L}C = \lambda(t, S_t, \sigma_t)C_\sigma
\]

with terminal condition \(C(S, K, T) = f(S_T, K, T), f : (R_+, R, R_+) \rightarrow R\) the pay-off function. Thus, \(\lambda(t, S_t, \sigma_t)\) has economic meaning as the risk-neutralised drift of the volatility process. Indeed, it depends on \(S(t)\) and \(\sigma(t)\), and it can be determined by any option on spot \(S(t)\).

**Example 2. One Stochastic IR and Stochastic Volatility**

When we take one step further, for instance, \(r_d(t)\) is stochastic while \(r_f(t)\) keeps deterministic and \(\sigma(t)\) is stochastic. For the purpose of perfect hedging, we then need FX spot \(S(t)\), as well as two other options – \(\Lambda(t)\) units of \(D(t)\) and \(\Omega(t)\) units of \(E(t)\) – to hedge the domestic interest rate risk and FX volatility risk, respectively. Analogous to the logic in Example 1, we calculate the number of units of hedging instruments at time \(t\) as\(^{13}\):

\[
\begin{align*}
\Omega(t) &= \frac{C_{rd}D_\sigma - C_\sigma D_{rd}}{E_{rd}D_\sigma - E_\sigma D_{rd}} \\
\Lambda(t) &= \frac{C_\sigma - \Omega(t)E_\sigma}{D_\sigma} \\
\Delta(t) &= C_S - \Lambda(t)D_S - \Omega(t)E_S
\end{align*}
\] (3.65, 3.66, 3.67)

\(^{12}\) Note that now the partial derivatives with respect to \(r_d\) and \(r_f\) become 0 in operator \(\hat{L}\)

\(^{13}\) At this point all of the partial derivatives with respect to \(r_f\) become zero.
Substituting (3.65)-(3.67) into (3.64), we now have:

\[ \tilde{L}C(D_\sigma E_{rd} - E_\sigma D_{rd}) = \tilde{L}D(C_\sigma E_{rd} - E_\sigma C_{rd}) + \tilde{L}E(D_\sigma C_{rd} - C_\sigma D_{rd}) \]  

(3.68)

Now, \( \tilde{L}U \) depends on \( U_\sigma \) and \( U_{rd} \) (\( U \in \{C, D, E\} \)). If we assume:

\[ \tilde{L}U = p_U U_\sigma + q_U U_{rd} \]

and substitute it into (3.68), collecting and equating \( U_{rd}, U_\sigma \) terms, we find:

\[ p_C = p_D = p_E \]

\[ q_C = q_D = q_E \]

Thus, if \( p \) and \( q \) are determined by any other two options on the same \( S(t) \), denoting them as \( A \) and \( B \), with \( U_\sigma \neq 0, U_{rd} \neq 0 \) and \( U \in \{A, B\} \), through simple algebra we give:

\[ q = \frac{\tilde{L}A B_\sigma - \tilde{L}B A_\sigma}{A_{rd} B_\sigma - A_\sigma B_{rd}} \]

\[ p = \frac{\tilde{L}A B_{rd} - \tilde{L}B A_{rd}}{A_\sigma B_{rd} - A_{rd} B_\sigma} \]  

(3.69)

Now we see that \( p \) and \( q \) depend on \( S(t), \sigma(t), r_d(t) \) with

\[ \tilde{L}U = p(t, S(t), \sigma(t), r_d(t))U_\sigma + q(t, S(t), \sigma(t), r_d(t))U_{rd} \]

This results in the risk-neutralised drift terms of stochastic volatility \( \sigma(t) \) and stochastic interest rate \( r_d(t) \) as being \( p(t, S(t), \sigma(t), r_d(t)) \) and \( q(t, S(t), \sigma(t), r_d(t)) \), respectively, which are determined by any two options on the spot \( S(t) \).

**Remark** Actually, this general hedging approach can be extended to the hedging of other hybrid derivatives. As long as there are \( n \) risk factors, we may choose a portfolio consisting of underlying spot (or forward) and \( n - 1 \) vanilla options on the same underlying, but with different strikes or maturities to make our hedging risk-free. This generally will result in a set of linear equations which is trivial to solve for the number of units of hedging instruments.

Although it seems promising, perfect hedging with a number of options can be expensive in practice due to the associated large bid-offer spreads and transaction costs. Thus, alternatively, we also want to consider other hedging approaches.
3. Foreign Exchange Options with Local Stochastic Volatility and Stochastic Interest Rates

3.6.2 Hedging with Options and Bonds

If, instead, we use both domestic and foreign zero-coupon bonds $P_d, P_f$ to hedge the interest rates risk\textsuperscript{14} and use another option $D$ on the same FX spot rate to hedge the volatility risk, the hedging portfolio becomes:

$$\Pi(t) = C(t) - \Delta(t)S(t) - \Lambda(t)D(t) - \Omega(t)P_d(t) - \Phi(t)(P_f(t)S(t))$$  (3.70)

Here, we hold $\Delta(t)$ units of FX spot rate, $\Lambda(t)$ units of another option $D$, $\Omega(t)$ shares of domestic zero-coupon bond $P_d$ and $\Phi(t)$ shares of foreign zero-coupon bond $P_f$, the value of which is seen in domestic as $P_f(t)S(t)$ at time $t$. Then, via Ito’s lemma, the instantaneous change in our portfolio is:

$$d\Pi(t) = dC(t) - (\Delta(t)dS(t) + \Delta(t)S(t)r_f(t)dt) - \Lambda(t)dD(t) - \Omega(t)dP_d(t) - \Phi(t)d(P_f(t)S(t))$$

$$= \mathcal{L}Cdt - \Lambda(t)\mathcal{LD}dt - \Delta(t)S(t)r_f(t)dt - \Omega(t)(\frac{\partial P_d}{\partial t}dt + \frac{1}{2}\frac{\partial^2 P_d}{\partial r_d^2}d < r_d >_t)$$

$$- \Phi(t)S(t)(\frac{\partial P_f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 P_f}{\partial r_f^2}d < r_f >_t) - \Phi(t)d < S, P_f >_t$$

$$+ \left\{ \frac{\partial C}{\partial S}(t) - \Delta(t) - \Lambda(t)\frac{\partial D}{\partial S}(t) - \Phi(t)P_f(t) \right\} dS(t)$$

$$+ \left\{ \frac{\partial C}{\partial \sigma}(t) - \Lambda(t)\frac{\partial D}{\partial \sigma}(t) \right\} d\sigma(t)$$

$$+ \left\{ \frac{\partial C}{\partial r_d}(t) - \Lambda(t)\frac{\partial D}{\partial r_d}(t) - \Omega(t)\frac{\partial P_d}{\partial r_d}(t) \right\} dr_d(t)$$

$$+ \left\{ \frac{\partial C}{\partial r_f}(t) - \Lambda(t)\frac{\partial D}{\partial r_f}(t) - \Phi(t)S(t)\frac{\partial P_f}{\partial r_f}(t) \right\} dr_f(t)$$  (3.71)

At this point I assume $P_d, P_f$ are independent of the FX volatility change\textsuperscript{15} and $\mathcal{L}$ is the operator defined in (3.59). As usual, to make this portfolio risk-free (equating $dS(t), d\sigma(t), dr_d(t), dr_f(t)$ terms to 0), we calculate the value of

\textsuperscript{14} This approach is actually a common industry practice – Bakshi, Cao and Chen (see [98]) applied it to the hedging of equity options where there was only one stochastic interest rate.

\textsuperscript{15} As the evidence of strong correlation between bond prices change and FX volatility change is not found yet.
3. Foreign Exchange Options with Local Stochastic Volatility and Stochastic Interest Rates

\[ \Lambda(t), \Phi(t), \Omega(t), \Delta(t) \text{ as:} \]

\[
\begin{align*}
\Lambda(t) &= C_\sigma(t)/D_\sigma(t) \\
\Phi(t) &= C_{rf}(t) - \Lambda(t)D_{rf}(t) \\
\Omega(t) &= C_{rd}(t) - \Lambda(t)D_{rd}(t) \\
\Delta(t) &= C_S(t) - \Lambda(t)D_S(t) - \Phi(t)P_f(t)
\end{align*}
\]

with subscripts denoting partial derivatives.

Note that in "Affine Term-Structure Models" (see [45]) the bond price can be written in the form: \( P(t, T) = A(t, T)e^{-B(t, T)r(t)} \), with A and B deterministic functions of time. For example, in Vasicek’s (1977) (cf. [124]) model:

\[
\begin{align*}
\frac{dr(t)}{dt} &= \kappa(\theta - r(t))dt + \sigma dW(t) \\
r(0) &= r_0
\end{align*}
\] (3.72)

\( A(t, T), B(t, T) \) are given as:

\[
\begin{align*}
A(t, T) &= \exp \left\{ (\theta - \frac{\sigma^2}{2\kappa^2})[B(t, T) + t - T] - \frac{\sigma^2}{4\kappa}B(t, T)^2 \right\} \\
B(t, T) &= \frac{1 - e^{-\kappa(T-t)}}{\kappa}
\end{align*}
\]

while in CIR (1985) model:

\[
\begin{align*}
\frac{dr(t)}{dt} &= \kappa(\theta - r(t))dt + \sigma \sqrt{r(t)}dW(t) \\
r(0) &= r_0
\end{align*}
\] (3.73)

\( A(t, T), B(t, T) \) are given as:

\[
\begin{align*}
A(t, T) &= \left\{ \frac{2he^{\frac{1}{2}(k+h)(T-t)}}{2h + (k + h)(e^{h(T-t)} - 1)} \right\}^{2e\theta/\sigma^2} \\
B(t, T) &= \frac{2(e^{h(T-t)} - 1)}{2h + (k + h)(e^{h(T-t)} - 1)} \\
h &= \sqrt{k^2 + 2\sigma^2}
\end{align*}
\]

Thus, in Affine Term Structure models, the partial derivative of the bond price with respect to \( r(t) \) is given by: \( P_r(t) = -B(t, T)P(t, T) \). This creates a practical way of implementing the above hedging strategy.
3. Foreign Exchange Options with Local Stochastic Volatility and Stochastic Interest Rates

3.7 Partial Hedging with Hedging Error Analysis

In reality, perfect hedging may be a luxury. It is not uncommon that hedging instruments, like other options, can be hardly available from the market, or expensive due to bid/ask spread; also, the increasing number of hedging instruments can make the hedging process more complicated (or bring in new risks). In such circumstances, a trader may prefer to perform a "partial hedge" with a lower number of hedging instruments to offset part of the risk; hence, it is important to analyse the hedging error in this case. This section will consider two partial hedging processes, namely using another option to hedge only the volatility risk with the exposure to the change of interest rates un-hedged, and using another option to hedge the interest rate risk only without covering the volatility risk exposure.

3.7.1 Hedging with One Option For the Volatility Risk

We now use another option \( D \), FX spot \( S \) to construct our hedging portfolio \( \Pi \), with FX volatility risk hedged by option \( D \). Furthermore, we assume that only the domestic interest rate \( r_d(t) \) is stochastic, with the foreign interest rate \( r_f(t) \) deterministic.

\[
\Pi(t) = C(t) - \Delta(t)S(t) - \Lambda(t)D(t)
\]

Apply Ito’s lemma:

\[
d\Pi(t) = dC(t) - (\Delta(t)dS(t) + \Delta(t)S(t)r_f(t)dt) - \Lambda(t)dD(t)
\]

\[
= \mathcal{L}C dt - \Lambda(t)\mathcal{L}D dt + (C_S(t) - \Delta(t) - \Lambda(t)D_S(t))dS(t) - \Delta(t)S(t)r_f dt
\]

\[
+ (C_\sigma(t) - \Lambda(t)D_\sigma(t))d\sigma(t) + (C_{r_d}(t) - \Lambda(t)D_{r_d}(t))dr_d(t)
\]

we choose \( \Lambda(t) = C_\sigma(t)/D_\sigma(t) \) to hedge the volatility risk (vega risk) and choose \( \Delta(t) = C_S(t) - \Lambda(t)D_S(t) \) to make the \( dS(t) \) term vanish (for delta risk). Then:

\[
d\Pi(t) = (\mathcal{L}C - \Lambda(t)\mathcal{L}D)dt - (C_S(t) - \Lambda(t)D_S(t))S(t)r_f dt + (C_{r_d}(t) - \Lambda(t)D_{r_d}(t))dr_d(t)
\]

In Example 2, we have arrived at \( \tilde{\mathcal{L}}U = p_UU_\sigma + q_UU_{r_d}, U \in \{C, D\} \), i.e. the risk-neutralised drift terms of stochastic volatility process \( \sigma(t) \) and stochastic domestic interest rate process \( r_d(t) \) depend on the two options \( C \) and \( D \). In addition, they are determined by (3.69). Furthermore, by using the equality (3.63), we acquire the equality for operator \( \mathcal{L} \):

\[
\mathcal{L}U dt = U r_d(t) dt - U_S(t)S(t)(r_d(t) - r_f(t)) dt - A^*(r_d(t))U_{r_d}(t) dt
\]

\[
- a^*(\sigma(t))U_\sigma(t) dt
\]

U \in \{C, D\}
Here, $A^\ast(r_d(t))$ and $a^\ast(\sigma(t))$ denote the risk-neutralised drift terms of the domestic interest rate process and stochastic volatility process, respectively. Substituting operator $\mathcal{L}$ into (3.74), we have:

$$d\Pi(t) = (\mathcal{L}C - \Lambda(t)LD)dt - (C_S(t) - \Lambda(t)D_S(t))S(t)r_f dt + (C_{r_d}(t) - \Lambda(t)D_{r_d}(t))dr_d(t)$$

$$= (C(t) - C_S(t)S(t))r_d(t)dt - \frac{C_\sigma(t)}{D_\sigma(t)}(D(t) - D_S(t)S(t))r_d(t)dt$$

$$+ (C_{r_d}(t) - \frac{C_\sigma(t)}{D_\sigma(t)}D_{r_d}(t))dr_d(t)$$

$$- (C_{r_d}(t) - \frac{C_\sigma(t)}{D_\sigma(t)}D_{r_d}(t))A^\ast(r_d(t))dt$$

(3.75)

Assume the domestic interest rate evolves as:

$$dr_d(t) = A(r_d(t))dt + B(r_d(t))dW_{r_d}(t)$$

with $A(r_d(t))$ as the drift term in the real world. According to the relation $A(r_d(t)) = A^\ast(r_d(t)) + \lambda^r(t)B(r_d(t))$, where $\lambda^r$ is the so-called interest rate risk premium (or market price of the interest rate risk), we substitute this into (3.75) and finally have:

$$d\Pi(t) = \left\{ (C(t) - C_S(t)S(t)) - \frac{C_\sigma(t)}{D_\sigma(t)}(D(t) - D_S(t)S(t)) \right\} r_d(t)dt$$

$$+ \left\{ C_{r_d}(t) - \frac{C_\sigma(t)}{D_\sigma(t)}D_{r_d}(t) \right\} \lambda^r(t)B(r_d(t))dt$$

$$+ \left\{ C_{r_d}(t) - \frac{C_\sigma(t)}{D_\sigma(t)}D_{r_d}(t) \right\} B(r_d(t))dW_{r_d}(t)$$

The expectation of an instantaneous hedging error under the real world measure (due to the incompleteness of the market, a risk-neutral measure is not applicable here) is given by (with Brownian motion terms vanishing):

$$\mathbb{E}[d\Pi(t)] = \left\{ (C(t) - C_S(t)S(t)) - \frac{C_\sigma(t)}{D_\sigma(t)}(D(t) - D_S(t)S(t)) \right\} r_d(t)dt$$

$$+ \left\{ C_{r_d}(t) - \frac{C_\sigma(t)}{D_\sigma(t)}D_{r_d}(t) \right\} \lambda^r(t)B(r_d(t))dt$$

while the variance of an instantaneous hedging error is:

$$\text{Var}[d\Pi(t)] = \left\{ C_{r_d}(t) - \frac{C_\sigma(t)}{D_\sigma(t)}D_{r_d}(t) \right\}^2 B^2(r_d(t))dt$$
3. Foreign Exchange Options with Local Stochastic Volatility and Stochastic Interest Rates

3.7.2 Hedging with One Option for the Interest Rate Risk

Suppose the trader thinks the stochastic interest rate risk is more significant\(^{16}\), she may want to use another option \(D\) to hedge it. In this scenario, our hedging portfolio becomes:

\[
\Pi(t) = C(t) - \Delta(t)S(t) - \Lambda(t)D(t)
\]

and

\[
d\Pi(t) = \mathcal{L}Cd t - \Lambda(t) \mathcal{L}Ddt + (C_S(t) - \Delta(t) - \Lambda(t)D_S(t))dS(t) - \Delta(t)S(t)\rho d t
\]

\[
+ (\sigma(t) - \Lambda(t)\sigma_d(t))d\sigma(t) + (C_{rd}(t) - \Lambda(t)D_{rd}(t))dr_d(t)
\]

which is the same as in the last subsection. However, now we choose \(\Lambda(t) = C_{rd}(t)/D_{rd}(t)\) for option \(D\) to make \(dr_d(t)\) terms vanish (hedge the domestic rho risk). \(\Delta(t) = C_S(t) - \Lambda(t)D_S(t)\) is applied to hedge the delta risk, so we now have:

\[
d\Pi(t) = (\mathcal{L}C - \Lambda(t)\mathcal{L}D)dt - (C_S(t) - \Lambda(t)D_S(t))S(t)\rho d t + (\sigma(t) - \Lambda(t)\sigma_d(t))d\sigma(t)
\]

Assume stochastic volatility evolves as:

\[
d\sigma(t) = a(\sigma(t))dt + b(\sigma(t))dW_\sigma(t)
\]

Its risk-neutralised drift term \(a^*(\sigma(t))\) is determined by options \(C\) and \(D\). Denoting \(\lambda^*\) as the \textit{volatility risk premium} (or market price of volatility risk) \(a(\sigma(t)) = a^*(\sigma(t)) + \lambda^*b(\sigma(t))\), and following the same steps as in the last subsection, the hedging error is:

\[
d\Pi(t) = \left\{(C(t) - C_S(t)S(t)) - \frac{C_{rd}(t)}{D_{rd}(t)}(D(t) - D_S(t)S(t))\right\}r_d(t)dt
\]

\[
+ \left\{C_\sigma(t) - \frac{C_{rd}(t)}{D_{rd}(t)}\sigma_d(t)\right\}\lambda^*b(\sigma(t))dt
\]

\[
+ \left\{C_\sigma(t) - \frac{C_{rd}(t)}{D_{rd}(t)}\sigma_d(t)\right\}b(\sigma(t))dW_\sigma(t)
\]

The expectation and variance then follow:

\[
\mathbb{E}[d\Pi(t)] = \left\{(C(t) - C_S(t)S(t)) - \frac{C_{rd}(t)}{D_{rd}(t)}(D(t) - D_S(t)S(t))\right\}r_d(t)dt
\]

\[
+ \left\{C_\sigma(t) - \frac{C_{rd}(t)}{D_{rd}(t)}\sigma_d(t)\right\}\lambda^*b(\sigma(t))dt
\]

\[
\text{Var}[d\Pi(t)] = \left\{C_\sigma(t) - \frac{C_{rd}(t)}{D_{rd}(t)}\sigma_d(t)\right\}^2b^2(\sigma(t))dt
\]

\(^{16}\) Especially for long-dated FX options, which have large interest rate exposure
3. Foreign Exchange Options with Local Stochastic Volatility and Stochastic Interest Rates

3.8 Model Mis-specification and Hedging Error Analysis

3.8.1 Delta Hedging Difference between the CEV and CEV-SV Models

For non-constant volatility models we see the most important greek – delta (a partial derivative with respect to the spot/forward price) – which has a relationship with the greeks from the Black-Scholes model (by the chain rule):

\[
\begin{align*}
\frac{\partial C}{\partial S} &= \frac{\partial C^{BS}}{\partial S} + \frac{\partial C^{BS}}{\partial \sigma^{BS}} \frac{\partial \sigma^{BS}}{\partial S} \\
\frac{\partial C}{\partial K} &= \frac{\partial C^{BS}}{\partial K} + \frac{\partial C^{BS}}{\partial \sigma^{BS}} \frac{\partial \sigma^{BS}}{\partial K}
\end{align*}
\] (3.76)

Here, \( C^{BS} \) is the option price given by the Black-Scholes formula; \( \sigma^{BS} \), which is the B-S implied volatility defined as: \( C(S,T,K) = C^{BS}(S,T,K,\sigma^{BS}(T,S,K)) \). Since \( C \) and \( C^{BS} \) are homogeneous of degree one in \( (S,K) \) (cf. [63]), we have:

\[ S \frac{\partial \sigma^{BS}}{\partial S} + K \frac{\partial \sigma^{BS}}{\partial K} = 0 \]

and this gives:

\[
\frac{\partial C}{\partial S} = \frac{\partial C^{BS}}{\partial S} - \frac{\partial C^{BS}}{\partial \sigma^{BS}} \frac{\sigma^{BS}}{S} \frac{\partial \sigma^{BS}}{\partial K}
\] (3.77)

Remark: Some financial meanings are as follows: \( \frac{\partial C}{\partial S} \) is the “true” delta given by a non-Black Scholes model, \( \frac{\partial C^{BS}}{\partial S} \) is the Black-Scholes delta, \( \frac{\partial C^{BS}}{\partial \sigma^{BS}} \) is the Black-Scholes vega and \( \frac{\partial \sigma^{BS}}{\partial K} \) is the skew/slope defined in implied volatility smiles.

To produce more practical results, we make use of the asymptotic expansion results of the Black-Scholes implied volatility given in (3.35) and truncate the first two orders of \( \epsilon \):

\[ \sigma^{BS}(T,K) \approx \frac{\Sigma}{\sqrt{T}} \{ 1 + (C_{11}y + C_{12} + C_{21} - \frac{\eta\Sigma}{2}) \} + O(\epsilon^3) \] (3.79)

where \( C_{11}, C_{12}, \) and \( C_{21} \) are given in Theorem 2.3.1.

To take an example, we choose specific models – the CEV (local volatility model) and CEV-Stochastic volatility model (local stochastic volatility model) – commonly used in practice, to compare the results of the true delta. In the general model set-up (3.16)-(3.17), we choose volatility function \( \gamma(x) = x^{\beta-1} \) for the CEV model, while we keep instantaneous volatility \( \alpha(t) \equiv \alpha \), vol-vol parameter \( \eta(t) \equiv \eta \), correlation parameter between forward and volatility changes \( \rho(t) \equiv \rho \), and initial
stochastic volatility $\sigma(0) = 1$ for the stochastic volatility process. Substitute $C_{11}$, $C_{12}$, and $C_{21}$ in (3.79) and differentiate w.r.t. strike $K$:

$$
\frac{\partial \sigma_{BS}^{CEV}}{\partial K}(T, K) \approx -\frac{(1 - \beta)\alpha}{2F}
$$

$$
\frac{\partial \sigma_{BS}^{CEV+SV}}{\partial K}(T, K) \approx -\frac{1}{2F}\left\{(1 - \beta)\alpha - \eta\rho T^{3/2}\right\}
$$

$F$ is today’s forward price. Consequently, through the use of the relationship between $\frac{\partial \sigma_{BS}}{\partial S}$ and $\frac{\partial \sigma_{BS}}{\partial K}$, we have the true deltas given in (3.78):

$$
\Delta_{CEV} \approx \Delta_{BS} + \nu_{BS}\left\{\frac{\alpha K}{2F^2}(1 - \beta)\right\} \quad (3.80)
$$

$$
\Delta_{CEV+SV} \approx \Delta_{BS} + \nu_{BS}\left\{\frac{K}{2F^2}\right\}\left\{(1 - \beta)\alpha - \eta\rho T^{3/2}\right\} \quad (3.81)
$$

where $\Delta_{BS}$ and $\nu_{BS}$ are Black-Scholes delta and vega, it is notable that when calibrated to the same volatility surface, the model parameters in the CEV and CEV-SV models are normally different, as we denote $\tilde{\alpha}$ and $\tilde{\beta}$ for the values of $\alpha$ and $\beta$ in the CEV model, with notation unchanged in the CEV-SV model. In this case, we have:

$$
\Delta_{CEV} - \Delta_{CEV+SV} \approx \nu_{BS}\frac{K}{2F^2}\{\tilde{\alpha}(1 - \tilde{\beta}) - \alpha(1 - \beta) + \eta\rho T^{3/2}\} \quad (3.82)
$$

This is the approximate difference in delta hedging between using the CEV model and CEV-Stochastic volatility model. The vol-vol ($\eta$) correlation ($\rho$) parameters of the volatility process play roles here, as we expected.

### 3.8.2 Model Mis-specification: The Importance of Stochastic Interest Rates

Here, I will show the importance of stochastic interest rates in the hedging of foreign exchange derivatives with long maturities, which have greater exposure to the movement of interest rates than those with shorter maturities.

We start this analysis from a model mis-specification.\footnote{Notably the Power-Reverse-Dual-Currency notes, which normally have maturities up to 30 years. These will be discussed in detail in the next section.} Assume the real market evolution of exchange rate $S(t)$, the associated interest rates $r_d(t)$, $r_f(t)$ and the

\footnote{Galluccio and Di Graziano (see [68]) conducted an analysis of model mis-specification on stochastic volatility models; here, I extend their idea to this more complicated case, in which mutually correlated stochastic interest rates and FX rate are involved.}
stochastic volatility $\sigma(t)$ are governed by respective diffusion processes under the domestic risk-neutral measure $\mathcal{Q}^d$:

\[
\begin{align*}
\frac{dS(t)}{S(t)} &= (r_d(t) - r_f(t))dt + \Gamma(t, S(t), \sigma(t))dW_S(t) \\
\frac{d\sigma(t)}{\sigma(t)} &= \tilde{a}(t, \sigma(t))dt + \tilde{b}(t, \sigma(t))dW_{\sigma}(t) \\
\frac{dr_{d,f}}{r_{d,f}} &= A_{d,f}(r_{d,f}(t))dt + B_{d,f}(r_{d,f}(t))dW_{d,f}(t)
\end{align*}
\]  

(3.83)

with communal deterministic time-dependent correlations $dW_i(t)dW_j(t) = \rho_{i,j}(t)dt$, $i, j \in \{S, \sigma, r_d, r_f\}$. Here, without loss of generality, we assume $dW_{\sigma}(t)dW_{d,f}(t) = 0$, which means the change in stochastic volatility is independent of changes in interest rates.

On the other hand, assume the model we are using is the one of stochastic volatility, but without stochastic interest rates (only constant interest rates are used). The model we use is thus given as:

\[
\begin{align*}
\frac{dS(t)}{S(t)} &= (\bar{r}_d - \bar{r}_f)dt + \gamma(t, S(t), \sigma(t))dW_S(t) \\
\frac{d\sigma(t)}{\sigma(t)} &= \bar{a}(t, \sigma(t))dt + \bar{b}(t, \sigma(t))dW_{\sigma}(t) \\
\frac{dr_{d,f}}{r_{d,f}} &= \bar{A}_{d,f}(r_{d,f}(t))dt + \bar{B}_{d,f}(r_{d,f}(t))dW_{d,f}(t)
\end{align*}
\]  

(3.84)

where $\bar{r}_d$ and $\bar{r}_f \in \mathbb{R}$ are constant values. Further, we assume the model we use is fully calibrated to the current market perfectly\textsuperscript{19}.

Since we ”believe” the interest rates will remain constant in our model, we only introduce another option $D$ to hedge the stochastic volatility risk. Then, when we construct a hedging portfolio of $\Delta(t)$ units of exchange rate spot and $\Lambda(t)$ units of another option $D(t)$ to hedge our option $C(t)$, the hedging error process is defined as:

\[
d\Pi(t) = dC(t) - \Delta(t)dS(t) - \Lambda(t)dD(t)
\]

Now we follow the approaches in sections 3.6 and 3.7. In order to avoid repetition, some derivations will be simplified here. The real process for $\Pi(t)$ is given as:

\[
d\Pi(t) = dC(t) - \Delta(t)dS(t) - \Lambda(t)dD(t)
\]

\[
= (\Lambda(t)D_S - C_S)S(t)r_f(t)dt + (C_{r_d} - \frac{C_{\sigma}}{D_\sigma}D_{r_d})dr_d(t) + (C_{r_f} - \frac{C_{\sigma}}{D_\sigma}D_{r_f})dr_f(t)
\]

\[+(LC - \Lambda(t)LD)dt
\]

(3.85)

Here, we choose $\Lambda(t) = \frac{C_{\sigma}}{D_\sigma}$ and $\Delta(t) = C_S - \Lambda(t)D_S$ (as before) to eliminate the $d\sigma(t)$ and $dS(t)$ terms (volatility and the delta risk).

\textsuperscript{19} Which means the model we use can reproduce the current implied volatility surface well.
3. Foreign Exchange Options with Local Stochastic Volatility and Stochastic Interest Rates

For the model in (3.84), via Feynmann-Kac, we have the PDEs:

\[
C_{\bar{r}_d} = C_t + \frac{1}{2}\gamma^2 S^2 C_{SS} + \frac{1}{2}\bar{b}^2 C_{\bar{r}r} + \rho_{\sigma\gamma} \bar{S} \bar{b} C_{S\bar{r}} + \bar{a} C_v
\]

\[
D_{\bar{r}_d} = D_t + \frac{1}{2}\gamma^2 S^2 D_{SS} + \frac{1}{2}\bar{b}^2 D_{\bar{r}r} + \rho_{\sigma\gamma} \bar{S} \bar{b} D_{S\bar{r}} + \bar{a} D_v
\]

With terminal conditions \( C(T, S(T)) = \phi_1(S_T), D(T, S(T)) = \phi_2(S_T), \phi_1(\cdot), \phi_2(\cdot) \) are pay-off functions.

By plugging the above equations into (3.85) and replacing the \( C_t \) and \( D_t \) terms, we can rewrite (3.85) as:

\[
d\Pi(t) = \left\{ (C_{\bar{r}_d} - \frac{C_{\sigma}}{D_{\sigma}} D_{\bar{r}_d}) \bar{r}_d + (C_{r_f} - \frac{C_{\sigma}}{D_{\sigma}} D_{r_f}) \bar{r}_f \right\} dt + (C_{\bar{r}_d} dt - C_{S} S \bar{r}_d dt) + (C_{S} S \bar{r}_d dt - C_{S} S r_f dt) - \Lambda_t(D_{\bar{r}_d} dt - D_{S} S \bar{r}_d dt)
\]

\[
- \Lambda_t(D_{S} S \bar{r}_f dt - D_{S} S r_f dt) - \frac{1}{2}\left\{ \gamma^2 S^2 C_{SS} + 2 \rho_{S\gamma} \bar{S} \bar{b} C_{S\bar{r}} \right\} dt + \frac{1}{2}\bar{G} C dt
\]

\[
+ \frac{1}{2}\left\{ \gamma^2 S^2 D_{SS} + 2 \rho_{S\gamma} \bar{S} \bar{b} D_{S\bar{r}} \right\} dt + \frac{1}{2}\bar{G} D dt
\]

Where operator \( \bar{G} \) is defined as the cross-variational terms:

\[
\bar{G} U dt = U_{SS} d < S >_t + U_{\sigma r} d < \sigma >_t + U_{r_{\bar{r}d}} d < r_d >_t + U_{r_{\bar{r}f}} d < r_f >_t
\]

\[
+ 2 U_{S\sigma} d < \sigma, S >_t + 2 U_{Sr_d} d < S, r_d >_t + 2 U_{Sr_f} d < S, r_f >_t + 2 U_{r_{\bar{r}r}} d < r_{\bar{r}}, r_f >_t
\]

\[\begin{align*}
&U \in \{ C, D \}
\end{align*}\]

Under the setup of the real market model (3.83), we can rewrite the above equation as:

\[
\bar{G} U = U_{S\sigma} \gamma^2 S^2 + U_{\sigma r} b^2 + U_{r_{\bar{r}d}} B_{\bar{r}}^2 + U_{r_{\bar{r}f}} B_{\bar{f}}^2
\]

\[
+ 2 U_{S\sigma} \Gamma S b \rho_{S\sigma} + 2 U_{Sr_d} \Gamma S B_d \rho_{Sr_d} + 2 U_{Sr_f} \Gamma S B_f \rho_{Sr_f} + 2 U_{r_{\bar{r}r}} B_{\bar{r}} B_f \rho_{r_{\bar{r}}f}
\]

\[\begin{align*}
&U \in \{ C, D \}
\end{align*}\]

Putting (3.87) into (3.86), we now see as \( \Pi(T) = \Pi(0) + \int_0^T d\Pi(t) dt \), the total hedging error due to the assumption of constant interest rates can be significant for large maturity \( T \), depending on the significance of model parameters. This shows the importance of incorporating stochastic interest rates into the modelling and pricing of our long-dated FX derivatives.
3. Application to Power-Reverse-Dual-Currency Notes

3.9.1 PRDC-TARN: The Structured Product

After calibrating our stochastic volatility and stochastic interest rates model to the volatility surface of the FX market, we may use the calibrated model with time-dependent parameters to price some exotic/hybrid structured products. One of the most popular products in the FX/Interest rate markets is the so-called "Power-Reverse-Dual-Currency" (referred to as "PRDC" hereafter). To define briefly the basic structure, we follow [39]:

Given a tenor structure (suppose today is time 0 and $T_N$ is maturity):

$$0 < T_1 < T_2 < T_3 < \cdots < T_N$$

with tenor $\tau_n = T_{n+1} - T_n$.

A PRDC is essentially a swap with payments such as (during the period $[T_n, T_{n+1}]$, $n = 1, 2 \cdots N - 1$):

- a funding leg. This is normally a LIBOR rate at time $T_n$ plus a fixed spread (can be positive or negative): $\tau_n(L(T_n) + S_p(T_n))$.

- a structured leg with an exotic coupon payment on the FX rate at time $T_n$: $\tau_nC_n(S(T_n))$

where

$$C_n(S) = \min\left(\max\left(\frac{g_f S(T_n)}{s} - g_d, b_l\right), b_u\right)$$

$g_f, g_d$ are the foreign and domestic coupons and $b_l$ and $b_u$ the floor and cap. In the basic structure, $b_l = 0$ and $b_u = +\infty$. $s$ is the scaling factor, often set as the initial FX forward rate $F(0, T_n)$. Parameters can vary depending on the deal and $n$, $n = 1, 2 \cdots N - 1$. A normally large initial fixed coupon payment is made at time 0.

When $b_l = 0$, $b_u = +\infty$, the structured leg can be seen as a strip of FX call options:

$$C_n(S) = d_n(S(T_n) - K_n)_+$$

with $d_n = \frac{g_f}{s}$, $K_n = \frac{g_d}{g_f}$. That is why we need a model fully calibrated to the vanilla option prices of different maturities and strikes (hence, the volatility surface).

---

20 At the time of writing in 2006/2007.
A variety of PRDC products possess the Target-Redemption-Note (TARN hereafter) feature. Here, we focus mainly on the valuation of this product due to its popularity.

The TARN feature\(^{21}\) is a swap which pays the investor (contract buyer) a fixed coupon \(C_0\) at \(T_0\), while at the remaining time of the contract \(\{T_n\}, n = 1, 2 \cdots N - 1\) it pays:

\[
\tau_n C_n 1_{A_n < Tar}
\]

where \(Tar\) is the target of the coupon payment and \(A_n\) is the aggregated coupon paid up to time \(T_{n-1}\):

\[
A_n = \sum_{i=1}^{n-1} \tau_i C_i
\]

At expiration time \(T_N\) the investor is paid the remaining coupon payment given the aggregated coupon is less than the target coupon payment:

\[
(Tar - A_{N-1}) 1_{A_{N-1} < Tar}
\]

Conversely, the issuer (seller) of TARN products will be paid LIBOR rates:

\[
\tau_i L_i(T_i)
\]

for \(i = 0, 1 \cdots N - 1\).

We see this feature is highly path-dependent. In the next subsection we will employ a Monte Carlo simulation to implement our fully-calibrated models (log-normal, CEV, stochastic volatility, etc.) to price this structured product, in order to see the effect of stochastic volatility on the valuation.

### 3.9.2 Smile Impact on PRDC-TARN Product Valuation

The Monte Carlo simulation of the FX rate process consists of the simulation of interest rates (we use the Hull-White model here), stochastic volatility (we employ the popular Heston-type model) and the FX process itself. For the Hull-White model we use the method in [106]. The simulation method for the Heston model will be discussed in detail in the next chapter. For the log-normal model we use a simple Black-Scholes model with time-dependent volatility and stochastic interest rates (Hull-White model), while for the CEV model we use Piterbarg’s model (see [39]), which is a CEV local volatility model (without a stochastic volatility component) plus stochastic interest rates (Hull-White model).

\(^{21}\) There are many varieties of TARN, but we define the most basic one here.
We now use these three models to value a set of PRDC-TARN contracts. We select three deals with JPY/USD and different underlying contract parameters:

<table>
<thead>
<tr>
<th>Deal</th>
<th>TARN-Tar</th>
<th>LIBOR-SP</th>
<th>PRD-Cap</th>
<th>PRD-Floor</th>
<th>Maturity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deal1</td>
<td>8%</td>
<td>-12.5bps</td>
<td>4%</td>
<td>0.01%</td>
<td>30yr</td>
</tr>
<tr>
<td>Deal2</td>
<td>12%</td>
<td>-10bps</td>
<td>6%</td>
<td>0%</td>
<td>30yr</td>
</tr>
<tr>
<td>Deal3</td>
<td>16%</td>
<td>-12bps</td>
<td>4%</td>
<td>0.01%</td>
<td>30yr</td>
</tr>
</tbody>
</table>

Tab. 3.2: PRD-TARN Deals

The valuation results from these three models are given at Figure 3.10, which shows the prices of three PRDC-TARN deals with different contract parameters as percentages of the notional value. For each deal, the fully calibrated log-normal, CEV and stochastic volatility models are employed for the valuation.

Figure 3.10 Here

We see that the prices of these path-dependent FX/IR hybrid products given by different models (which give different assumptions of the FX rate dynamics) have non-negligible differences, sometimes even significant (as in Deal 3, we see an almost 30% difference between prices from log-normal and stochastic volatility models!). Since we view stochastic volatility and stochastic interest rates as a more general and realistic modelling approach capable of reproducing market volatility smiles/skews, we arrive at the conclusion that volatility smile has a notable effect on the valuation of this hybrid derivative product (as well as on many other products that are volatility smile-sensitive).

3.10 Conclusion and Future Research

This chapter serves as the main part of my Ph.D research in financial mathematics. The full coverage of modelling, implementation, calibration, pricing and hedging issues on FX and FX-IR options contributes greatly to the existing literature in this area, with most of it has produced new research results.

In this study we have presented the asymptotic expansion formulae for both the vanilla European option price and B-S implied volatility. Numerical results show high accuracy has been achieved through this approximation method for a wide range of options maturities. Thus, we are confident in employing these formulae in model calibration and other fast pricing procedures, as well as in better options hedging. As we also note, this expansion method is general and flexible, and can be applied to a wide range of financial math/modelling problems. On the
other hand, another modelling approach with jump processes, through the use of the Fourier transform method, has been utilised mainly for the pricing/hedging of long-dated FX options. This semi-analytical method is also quite general and can be extended to incorporate other risk factors. The hedging approach for FX derivatives proposed in this chapter, as well as the analysis of partial hedging and model mis-specification, are also general, which can be extended easily to the analysis of hedging of other complicated hybrid derivatives in which multi-risk factors are involved.

We also see the significant impact of stochastic volatility on the valuation of FX/IR hybrid path-dependent derivatives, namely the PRDC-TARN. Comparison with other simpler modelling approaches, i.e. the log-normal model, the CEV model is valuable from both the pricing and risk management point of view. We establish the importance of seeking more realistic, flexible and general modelling approaches, and thus the importance of quants:

\[ X^\epsilon(T) = X(0) + \epsilon \int_0^T \sigma^\epsilon(t) \alpha(t) X^\epsilon(t) \gamma(X^\epsilon(t)(1 + \epsilon^2 \tilde{D}(t, T))) dW^T(t) \]

\[ -\epsilon^2 \int_0^T X^\epsilon(t) dD(t, T) \]

\[ \sigma^\epsilon(t) = \sigma(0) + \epsilon \int_0^t a(s, \sigma^\epsilon(s)) ds + \epsilon \int_0^t \eta(s) \tilde{b}(\sigma^\epsilon(s)) dW^T(s). \]

We then start from the stochastic volatility process. Firstly, we expand \( \sigma^\epsilon(t) \) using the Taylor expansion in small parameter \( \epsilon \) to the \( O(\epsilon^3) \) term:

\[ \sigma^\epsilon(t) = \sigma(0) + \epsilon \frac{\partial \sigma^\epsilon}{\partial \epsilon} \bigg|_{\epsilon=0} + \frac{1}{2} \epsilon^2 \frac{\partial^2 \sigma^\epsilon}{\partial \epsilon^2} \bigg|_{\epsilon=0} + O(\epsilon^3). \]
After expanding (3.89) and matching the terms of order $\epsilon^0, \epsilon, \epsilon^2$ with (3.90), respectively, we obtain:

$$\frac{\partial \sigma^\epsilon}{\partial \epsilon} \bigg|_{\epsilon=0} = \int_0^t a(s, \sigma(0)) ds + \int_0^t \eta(s) \tilde{b}(\sigma(0)) dW_T^\sigma(s), \quad (3.91)$$

$$\frac{\partial^2 \sigma^\epsilon}{\partial \epsilon^2} \bigg|_{\epsilon=0} = 2 \int_0^t \gamma_1(s) \int_0^s a(u, 1) du ds + 2 \int_0^t \gamma_1(s) \int_0^s \eta(u) dW_T^\sigma(u) ds$$

$$\quad + 2 \int_0^t \eta(s) \gamma_3 a(u, 1) dW_T^\sigma(s) + 2 \int_0^t \eta(s) \gamma_3 \int_0^s \eta(u) dW_T^\sigma(u) dW_T^\sigma(s) \quad (3.92)$$

then plug the expansion formula of $\sigma^\epsilon(t)$ (3.90)-(3.92) into (3.88). In the same way as we treat $\sigma^\epsilon(t)$, we apply the Taylor expansion in small parameter $\epsilon$ to $X^\epsilon(T)$ up to the third order:

$$X^\epsilon(T) = X(0) + \epsilon \frac{\partial X^\epsilon}{\partial \epsilon} \bigg|_{\epsilon=0} + \frac{\epsilon^2}{2} \frac{\partial^2 X^\epsilon}{\partial \epsilon^2} \bigg|_{\epsilon=0} + \frac{\epsilon^3}{6} \frac{\partial^3 X^\epsilon}{\partial \epsilon^3} \bigg|_{\epsilon=0} + O(\epsilon^4).$$

After expanding (3.88) and matching the corresponding terms $\epsilon^0, \epsilon, \epsilon^2$, we arrive at:
\[
\frac{\partial X^\epsilon}{\partial \epsilon} \bigg|_{\epsilon=0} = \int_0^T \alpha(t) dW_s^T(t), \\
\frac{\partial^2 X^\epsilon}{\partial \epsilon^2} \bigg|_{\epsilon=0} = 2 \int_0^T \alpha(t) \left( \int_0^t a(s,1)ds \right) dW_s^T(t) \\
+ 2 \int_0^T \alpha(t) \left( \int_0^t \eta(s) dW_\sigma^T(s) \right) dW_s^T(t) \\
+ 2(1 + \gamma_1) \int_0^T \alpha(t) \left( \int_0^t \alpha(s) dW_s^T(s) \right) dW_s^T(t) \\
- 2D(T,T), \\
\frac{\partial^3 X^\epsilon}{\partial \epsilon^3} \bigg|_{\epsilon=0} = 6 \int_0^T \alpha(t) \left\{ \int_0^t \gamma_{a1}(s) \left( \int_0^s a(u,1)du \right) ds \right\} dW_s^T(t) \\
+ 6 \int_0^T \alpha(t) \left\{ \int_0^t \gamma_{a1}(s) \left( \int_0^s \eta(u) dW_\sigma^T(u) \right) ds \right\} dW_s^T(t) \\
+ 6\gamma_{b1} \int_0^T \alpha(t) \left\{ \int_0^t \eta(s) \left( \int_0^s \left( \int_0^s a(u,1)du \right) dW_\sigma^T(s) \right) ds \right\} dW_s^T(t) \\
+ 6\gamma_{b1} \int_0^T \alpha(t) \left\{ \int_0^t \eta(s) \left( \int_0^s \eta(u) dW_\sigma^T(u) \right) dW_s^T(s) \right\} dW_s^T(t) \\
+ 6(1 + \gamma_1) \int_0^T \alpha(t) \left\{ \int_0^t \alpha(s) \left( \int_0^s a(u,1)du \right) dW_s^T(s) \right\} dW_s^T(t) \\
+ 6(1 + \gamma_1) \int_0^T \alpha(t) \left\{ \int_0^t \alpha(s) \left( \int_0^s \eta(u) dW_\sigma^T(u) \right) dW_s^T(s) \right\} dW_s^T(t) \\
+ 6(1 + \gamma_1)^2 \int_0^T \alpha(t) \left\{ \int_0^t \alpha(s) \left( \int_0^s \alpha(u) dW_\sigma^T(u) \right) dW_s^T(s) \right\} dW_s^T(t) \\
- 6(1 + \gamma_1) \int_0^T \alpha(t) D(t,T) dW_s^T(t) \\
+ (3\gamma_2 + 6\gamma_1) \int_0^T \alpha(t) \left( \int_0^t \alpha(s) dW_s^T(s) \right)^2 dW_s^T(s) \\
+ 6\gamma_1 \int_0^T \alpha(t) \tilde{D}(t,T) dW_s^T(t) \\
+ 6(1 + \gamma_1) \int_0^T \alpha(t) \left( \int_0^t a(s,1)ds \right) \left( \int_0^t \alpha(s) dW_s^T(s) \right) dW_s^T(t) \\
+ 6(1 + \gamma_1) \int_0^T \alpha(t) \left( \int_0^t \eta(s) dW_\sigma^T(s) \right) \left( \int_0^t \alpha(s) dW_\sigma^T(s) \right) dW_s^T(t) \\
- 6 \int_0^T \left( \int_0^t \alpha(s) dW_\sigma^T(s) \right) dD(t,T) \\
\]
Having expanded $X^\epsilon(T)$, we can write $X^\epsilon(T)$ in a compact form:

$$X^\epsilon(T) = X(0) + \sum \{ \epsilon G_1(t) + \epsilon^2 G_2(t) + \epsilon^3 G_3(t) + \epsilon^2 H_1(t) + \epsilon^3 H_2(t) \} \quad (3.93)$$

where $H_1, H_2$ are the terms containing bond volatilities, given as:

$$H_1(t) = \frac{1}{\Sigma} \left\{ \int_0^T \sigma_f(t,T) dW^T_f(t) - \int_0^T \sigma_d(t,T) dW^T_d(t) \right\}, \quad (3.94)$$

$$H_2(t) = \frac{1}{\Sigma} \left\{ - \int_0^T \alpha(t) D(t,T) dW^T_s(t) - \gamma_1 \int_0^T \alpha(t) D(t,t) dW^T_s(t) - \int_0^T \Sigma G_1(t) dD(t,T) \right\} \quad (3.95)$$

and $G_1(t)$ is a scaled random variable performing the standard normal distribution:

$$G_1(T) = \frac{\int_0^T \alpha(t) dW^T_s(t)}{\sqrt{\int_0^T \alpha^2(t) dt}} \sim N(0,1). \quad (3.96)$$

for which $G_2, G_3$ are the remaining scaled terms of $\epsilon^2, \epsilon^3$ order, respectively. Using a martingale pricing formula we now define the European call option price at time 0 with maturity $T$ and strike $K$ as:

$$\text{Call}(T,K) = P_d(0,T) E_T^0 \left[ X^\epsilon(T) F(0,T) - K \right]^+ \quad (3.97)$$

where $E_T^0[\cdot]$ denotes expectation under the domestic T-Forward measure $Q_T^d$ (using $P_d(0,T)$ as numeraire) filtrated to $\mathcal{F}_0$. (3.97) can be expanded further (through the use of (3.93)) as:

$$\begin{align*}
\text{Call}(T,K) &= P_d(0,T) F(0,T) E_T^0 \left[ (X(0) - \bar{K}) + \epsilon \Sigma \{ G_1(T) + \epsilon G_2(T) + H_1(T) \} \\
&\quad + \epsilon^2 (G_3(T) + H_2(T)) \right]^+ \quad (3.98)
\end{align*}$$

Defining function $f : \mathbb{R} \to \mathbb{R}$ as $f(x) = max(x,0)$, it is well known that $f'(x-y) = H(x-y)$, $f''(x-y) = \delta(x-y)$, where $y$ is a constant $y \in \mathbb{R}$ and $H : \mathbb{R} \to \{0,1\}$ is the Heaviside equation, defined as:

$$H(x-y) = \begin{cases} 
1 & \text{for } x > y \\
0 & \text{for } x \leq y 
\end{cases}$$

and $\delta(x-y)$, the Dirac-Delta function possesses the following property, which will be used later:

$$\int_{-\infty}^{\infty} f(x) \delta(x-y) dx = f(y). \quad (3.99)$$
Using the results from Appendix 3.11.2, which is an application of the lemma

Weiner-Ito integral:

We now need to calculate the following conditional expectations of the multiple

\[ E \int_0^T [f \left( G_1(t) - \frac{\bar{K} - 1}{\epsilon \Sigma} \right) + \epsilon (G_2(t) + H_1(t)) \]

\[ + \epsilon^2 (G_3(t) + H_2(t)) \right] ] \]

(3.100)

\[ = P_d(0, T) F(0, T) \epsilon \Sigma \{ \mathbb{E}^T_0 [G_1(T) - \frac{\bar{K} - 1}{\epsilon \Sigma}] \} + \epsilon \mathbb{E}^T_0 [H(x - y) G_2(T) + H_1(T)] \mid G_1(T) = x \]

\[ + \epsilon^2 \mathbb{E}^T_0 [H(x - y) (G_3(T) + H_2(T))] \mid G_1(T) = x \]

\[ + \frac{\epsilon^2}{2} \mathbb{E}^T_0 [\delta(x - y)(G_2(T) + H_1(T))^2] \mid G_1(T) = x \} + ... \]  

(3.101)

\[ = P_d(0, T) F(0, T) \epsilon \Sigma \{ G(y) + \epsilon \int_y^{\infty} \mathbb{E}^T_0 [G_2(T) + H_1(T)] \mid G_1(T) = x \} n(x) dx \]

\[ + \epsilon^2 \int_y^{\infty} \mathbb{E}^T_0 [G_3(T) + H_2(T)] \mid G_1(T) = x \} n(x) dx \]

\[ + \frac{\epsilon^2}{2} \int_{-\infty}^{\infty} \delta(x - y) \mathbb{E}^T_0 [G_2^2(T) + H_1^2(T) + 2G_2(T)H_1(T)] \mid G_1(T) = x \]

\[ n(x) dx \} \]

(3.102)

where \( \bar{K} = \frac{K}{F(0, T)} \). From (3.100)\(\rightarrow\) (3.101) we use the result of Theorem 2.3 of
[95], listed in Appendix 3.11.3, for completeness.

We now need to calculate the following conditional expectations of the multiple
Weiner-Ito integral: \( \mathbb{E}^T_0 [G_2(T) \mid G_1(T) = x] \), \( \mathbb{E}^T_0 [H_1(T) \mid G_1(T) = x] \),
\( \mathbb{E}^T_0 [G_3(T) \mid G_1(T) = x] \), \( \mathbb{E}^T_0 [H_2(T) \mid G_1(T) = x] \), \( \mathbb{E}^T_0 [G_2(T) \mid G_1(T) = x] \),
\( \mathbb{E}^T_0 [H_1^2(T) \mid G_1(T) = x] \), \( \mathbb{E}^T_0 [2G_2(T)H_1(T) \mid G_1(T) = x] \).

Using the results from Appendix 3.11.2, which is an application of the lemma of
[97], we obtain the conditional expectations:

\[ \mathbb{E}^T_0 [G_2(T) \mid G_1(T) = x] = C_{11} (x^2 - 1) + C_{12} x, \]

(3.103)

\[ \mathbb{E}^T_0 [H_1(T) \mid G_1(T) = x] = C_{21} x, \]

(3.104)

\[ \mathbb{E}^T_0 [G_3(T) \mid G_1(T) = x] = C_{31} (x^3 - 3x) + C_{32} (x^2 - 1) + C_{33} x + C_{34}, \]

(3.105)

\[ \mathbb{E}^T_0 [H_2(T) \mid G_1(T) = x] = C_{41} (x^2 - 1), \]

(3.106)

\[ \mathbb{E}^T_0 [G_2^2(T) \mid G_1(T) = x] = C_{51} (x^4 - 6x^2 + 3) + C_{52} (x^3 - 3x) + C_{53} (x^2 - 1) + C_{54} x + C_{55}, \]

(3.107)

\[ \mathbb{E}^T_0 [H_1^2(T) \mid G_1(T) = x] = C_{61} (x^2 - 1) + C_{62}, \]

(3.108)

\[ \mathbb{E}^T_0 [2H_1(T)G_2(T) \mid G_1(T) = x] = C_{71} (x^3 - 3x) + C_{72} (x^2 - 1) + C_{73} x + C_{74}. \]

(3.109)
$C_{11} - C_{74}$ are given in Theorem 3.3.1. Substituting (3.103)-(3.109) into call option price formula (3.102) and using the following simple results (as well as (3.99)):

\[
\int_{y}^{\infty} x^n dx = n(y), 
\]
(3.110)

\[
\int_{y}^{\infty} x^2 n(x) dx = yn(y) + N(-y), 
\]
(3.111)

\[
\int_{y}^{\infty} x^3 n(x) dx = (y^2 + 2)n(y). 
\]
(3.112)

we arrive at (3.21) Theorem 3.3.1.

### 3.11.2 Conditional Expectations of the Multiple Weiner-Ito Integral

**Lemma C.1** [97] Let $T = [0, \infty)$, $f(u_1, ..., u_p) \in L^2(T^p)$, and let $I_p(f)$ denote the multiple Weiner-Ito integral of the kernel $f$. Let $h \in L^2(T)$ and consider $h^{\otimes p}$; the $p$-th tensor product of $h(h^{\otimes p}(u_1, ..., u_p) = h(u_1) \cdot h(u_2) \cdots h(u_p))$. In this case we have:

\[
\mathbb{E}(I_p(f) \mid I_1(h)) = \frac{<f, h^{\otimes p} >_{L^2(T^p)}}{(\| h \|^2_{L^2(T)})^p} I_p(h^{\otimes p}), 
\]
(3.113)

The following lemma is an application of Lemma C.1:

**Lemma C.2** Let $(W_i(t), i = 0, 1, 2, 3; t \in [0, T])$ be a four-dimensional Brownian motion with correlation $\rho_{ij}(t) : [0, T] \rightarrow [-1, 1]$:

\[
\mathbb{E}[dW_i(t)dW_j(t)] = \rho_{ij}(t)dt. 
\]
(3.114)

Let $q, q_1, q_2, q_3, q_4 : [0, T] \rightarrow \mathbb{R}$ be deterministic functions. For

\[
\int_{0}^{T} q^2(t)dt = 1, 
\]
(3.115)
In this instance, conditional expectations of a multiple Weiner-Ito integral are given as:

\[
\mathbb{E}\left[ \int_0^T q_1(t) dW_1(t) \mid \int_0^T q(t) dW_0(t) = x \right] = A_1 x,
\]

\[
\mathbb{E}\left[ \int_0^T \left( \int_0^t q_2(s) dW_2(s) \right) q_1(t) dW_1(t) \mid \int_0^T q(t) dW_0(t) = x \right] = A_2 (x^2 - 1),
\]

\[
\mathbb{E}\left[ \int_0^T \left( \int_0^t \left( \int_0^s q_3(u) dW_3(u) \right) q_2(s) dW_2(s) \right) q_1(t) dW_1(t) \mid \int_0^T q(t) dW_0(t) = x \right] = A_3 (x^3 - 3x),
\]

\[
\mathbb{E}\left[ \left( \int_0^T q_1(t) dW_1(t) \right) \left( \int_0^T q_2(t) dW_2(t) \right) \mid \int_0^T q(t) dW_0(t) = x \right] = A_4 (x^2 - 1) + A_5,
\]

\[
\mathbb{E}\left[ \left( \int_0^T q_2(s) dW_2(s) \right) q_1(t) dW_1(t) \mid \int_0^T q(t) dW_0(t) = x \right]
= A_6 (x^2 - 6x^2 + 3) + A_7 (x^2 - 1) + A_8,
\]

\[
\mathbb{E}\left[ \left( \int_0^T q_2(s) dW_2(s) \right) q_1(s) dW_1(s) \left( \int_0^T q_3(t) dW_3(t) \right) \mid \int_0^T q(t) dW_0(t) = x \right]
= A_9 (x^3 - 3x) + A_{10} x,
\]

\[
\mathbb{E}\left[ \left( \int_0^T q_2(s) dW_2(s) \right) q_1(t) dW_1(t) \left( \int_0^T q_3(s) dW_3(s) \right) \mid \int_0^T q(t) dW_0(t) = x \right]
= A_{11} (x^3 - 3x) + A_{12} x,
\]

\[
\mathbb{E}\left[ \left( \int_0^T q_2(s) dW_2(s) \right) q_1(t) dW_1(t) \left( \int_0^T q_4(s) dW_4(s) \right) q_3(t) dW_3(t) \mid \int_0^T q(t) dW_0(t) = x \right]
= A_{13} (x^4 - 6x^2 + 3) + A_{14} (x^2 - 1) + A_{15},
\]
where:

\[
A_1 = \int_0^T q_1(t)q(t)\rho_{01}(t)dt, \\
A_2 = \int_0^T \left( \int_0^t q_2(s)q(s)\rho_{02}(s)ds \right) q_1(t)q(t)\rho_{01}(t)dt, \\
A_3 = \int_0^T \left( \int_0^t \left( \int_0^s q_3(u)q(u)\rho_{03}(u)du \right) q_2(s)q(s)\rho_{02}(s)ds \right) q_1(t)q(t)\rho_{01}(t)dt, \\
A_4 = \left( \int_0^T q_1(t)q(t)\rho_{01}(t)dt \right) \left( \int_0^T q_2(t)q(t)\rho_{02}(t)dt \right), \\
A_5 = \int_0^T q_1(t)q_2(t)\rho_{12}(t)dt, \\
A_6 = \left( \int_0^T \left( \int_0^t q_2(s)q(s)\rho_{02}(s)ds \right) q_1(t)q(t)\rho_{01}(t)dt \right)^2, \\
A_7 = 2 \int_0^T \left( \int_0^t \left( \int_0^s q_2(u)q(u)^2du \right) q_1(s)q(s)\rho_{01}(s)ds \right) q_1(t)q(t)\rho_{01}(t)dt \\
+ 2 \int_0^T \left( \int_0^t \left( \int_0^s q_2(u)q(u)\rho_{02}(u)du \right) q_1(s)q_2(s)\rho_{12}(s)ds \right) q_1(t)q(t)\rho_{01}(t)dt \\
+ 2 \int_0^T \left( \int_0^t \left( \int_0^s q_2(u)q(u)\rho_{02}(u)du \right) q_2(s)q(s)\rho_{02}(s)ds \right) q_1(t)^2dt, \\
A_8 = \int_0^T \left( \int_0^t q_2(s)^2ds \right) q_1(t)^2dt, \\
A_9 = \left( \int_0^T \left( \int_0^t q_2(s)q(s)\rho_{02}(s)ds \right) q_1(t)q(t)\rho_{01}(t)dt \right) \left( \int_0^T q_3(t)q(t)\rho_{03}(t)dt \right) \\
+ \int_0^T \left( \int_0^t q_2(s)q(s)\rho_{02}(s)ds \right) q_1(t)q_3(t)\rho_{13}(t)dt + \int_0^T \left( \int_0^t q_2(s)q_3(s)\rho_{23}(s)ds \right) q_1(t)q(t)\rho_{01}(t)dt, \\
A_{10} = \int_0^T \left( \int_0^t q_2(s)q(s)\rho_{02}(s)ds \right) q_1(t)q_3(t)\rho_{13}(t)dt + \int_0^T \left( \int_0^t q_2(s)q_3(s)\rho_{23}(s)ds \right) q_1(t)q(t)\rho_{01}(t)dt, \\
A_{11} = \int_0^T \left( \int_0^t q_2(s)q(s)\rho_{02}(s)ds \right) \left( \int_0^t q_3(s)q(s)\rho_{03}(s)ds \right) q_1(t)q(t)\rho_{01}(t)dt, \\
A_{12} = \int_0^T \left( \int_0^t q_2(s)q_3(s)\rho_{23}ds \right) q_1(t)q(t)\rho_{01}(t)dt, \\
A_{13} = \left( \int_0^T \left( \int_0^t q_2(s)q(s)\rho_{02}(s)ds \right) q_1(t)q(t)\rho_{01}(t)dt \right) \left( \int_0^T \left( \int_0^t q_4(s)q(s)\rho_{04}(s)ds \right) q_3(t)q(t)\rho_{03}(t)dt \right) \
\]
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\[ A_{14} = \int_0^T \left( \int_0^t \int_0^s q_4(u)q_2(u)\rho_{24}(u)du \right) q_5(s)q(s)\rho_{03}ds q_1(t)q(t)\rho_{01}(t) dt \]
\[ + \int_0^T \left( \int_0^t \int_0^s q_4(u)q_2(u)\rho_{24}(u)du \right) q_1(s)q(s)\rho_{01}ds q_3(t)q(t)\rho_{03}(t) dt \]
\[ + \int_0^T \left( \int_0^t \int_0^s q_4(u)q(u)\rho_{02}(u)du \right) q_4(s)q(s)\rho_{14}ds q_3(t)q(t)\rho_{03}(t) dt \]
\[ + \int_0^T \left( \int_0^t \int_0^s q_4(u)q(u)\rho_{02}(u)du \right) q_3(s)q(s)\rho_{23}ds q_1(t)q(t)\rho_{01}(t) dt \]
\[ + \int_0^T \left( \int_0^t \int_0^s q_4(u)q(u)\rho_{02}(u)du \right) q_3(s)q(s)\rho_{04}ds q_1(t)q_3(t)\rho_{13}(dt) \]
\[ + \int_0^T \left( \int_0^t \int_0^s q_4(u)q(u)\rho_{02}(u)du \right) q_3(s)q(s)\rho_{02}ds q_1(t)q_3(t)\rho_{13}(dt) \]
\[ A_{15} = \int_0^T \left( \int_0^t q_2(s)q_4(s)\rho_{24}(s)ds \right) q_1(t)q_3(t)\rho_{13}(t) dt. \]

### 3.11.3 Watanabe Theorem

**Theorem 2.3** of [95]: Let \( F(\epsilon, w) \in D^\infty(R^d) \) and \( \epsilon \in (0, 1] \) satisfy the assumption, i.e. it is uniformly non-degenerate and has the asymptotic expansion \( F(\epsilon, w) \sim f_0 + \epsilon f_1 + \cdots \) in \( D^\infty(R^d) \) as \( \epsilon \downarrow 0 \). Then, for every \( T \in F(R^d) \), \( T(F(\epsilon, w)) \in D^{\infty} \) (defined for \( \epsilon \in [0, 1] \)) has the asymptotic expansion in \( D^{\infty} \) (and a fortiori in \( D^{\infty} \)):

\[ T(F(\epsilon, w)) \sim \Phi_0 + \epsilon \Phi_1 + \cdots \]

in \( D^{\infty} \) as \( \epsilon \downarrow 0 \), and \( \Phi_0, \Phi_1, \ldots \in D^{\infty} \) are determined by the formal Taylor expansion:

\[ T(f_0 + [\epsilon f_1 + \epsilon^2 f_2 + \cdots]) = \sum_n \frac{1}{n!} D^n T(f_0)[\epsilon f_1 + \epsilon^2 f_2 + \cdots]^n \]
\[ = \Phi_0 + \epsilon \Phi_1 + \cdots, \]

where \( n = (n_1, \ldots, n_d) \) is a multi-index, \( n! = n_1! \cdots n_d! \), \( a^n = a_1^{n_1} \cdots a_d^{n_d} \) for
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In particular, denoting \( \partial^i = \partial / \partial x^i \),

\[
\Phi_0 = T(f_0),
\]

\[
\Phi_1 = \sum_{i=1}^{d} f_i \partial^i T(f_0),
\]

\[
\Phi_2 = \sum_{i=1}^{d} f_i^2 \partial^i T(f_0) + \frac{1}{2!} \sum_{i,j=1}^{d} f_i f_j \partial^i \partial^j T(f_0),
\]

\[
\Phi_3 = \sum_{i=1}^{d} f_i^3 \partial^i T(f_0) + \frac{2}{2!} \sum_{i,j=1}^{d} f_i f_j^2 \partial^i \partial^j T(f_0) + \frac{1}{3!} \sum_{i,j,k=1}^{d} f_i f_j f_k \partial^i \partial^j \partial^k T(f_0), \ldots
\]

**Proof** See [95].

3.11.4 The European Option Formula from the Fourier Transform Method

In addition, see ([6]). Instead of pricing the call option directly, we consider a covered call \( \text{Cov}(x,k,T) := \min[S e^{x_T} , S e^{k}] \), where \( x_T = \log(S_T/S), x = x_0 = 0 \) and \( k = \log(K/S) \). Thus, \( \text{Call}(x_T,k,T) := S_T - \text{Cov}(x,k,T) \).

We apply the Fourier transform to this covered call:

\[
\tilde{\text{Cov}}(x,u,T) = \int_{-\infty}^{\infty} e^{i u k} \text{Cov}(x,k,T) dk
\]

\[
= S \int_{-\infty}^{\infty} e^{i u k} E[\min[e^{x_T} , e^{k}]] dk
\]

\[
= S E[\int_{-\infty}^{\infty} e^{i u k} \min[e^{x_T} , e^{k}] dk | x_0 = 0]
\]

\[
= S E[\int_{-\infty}^{x_T} e^{i u k} e^{k} dk + \int_{x_T}^{\infty} e^{i u k} e^{x_T} dk | x_0 = 0]
\]

\[
= S E[e^{(1+iu) x_T} - e^{(1+iu) x_T} | x_0 = 0]
\]

\[
= S \frac{1}{u(u-i)} E[e^{(1+iu) x_T} | x_0 = 0]
\]

\[
= S \phi_T(u-i) \frac{u}{u(u-i)}
\]

where the characteristic function \( \phi_T(u) := E[e^{iu x_T}] \). Furthermore, the regularity and positivity of price conditions on (3.116) and (3.117) impose \( 0 < u_i < 1 \). \( u_i \) is
the imaginary part of complex value \( u \in \mathbb{C} \).

As a result, the call option price becomes:

\[
C(x, k, T) = \mathbb{E}[S_T - Cov(x, k, T)] \\
= S - \mathbb{E}[Cov(x, k, T)] \\
= S - \frac{S}{2\pi} \int_{-\infty}^{\infty} \frac{\phi_T(u - i)e^{-iuk}}{u(u - i)} du
\]

(3.118)

Since the integrand exhibits singularity at \( u = 0 \) and \( i \), we do not want the integration contour to pass these two points. Considering an integration contour shift along \( u_i = \alpha \), \( 0 < \alpha < 1 \):

\[
C(x, k, T) = S - \frac{S}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-iuk}\phi_T(u - i)}{u(u - i)} du \\
= S - \frac{S}{2\pi} \int_{-\infty}^{\infty} \text{Re} \left\{ \frac{e^{-iuk+\alpha k}\phi_T(u + (\alpha - 1)i)}{(u + \alpha i)(u + (\alpha - 1)i)} \right\} du
\]

Due to the oscillating nature of the integrand functions, we need to find an optimal integration contour shift \( \alpha^* \in (0, 1) \), a method is proposed in [59].
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Fig. 3.1: Expansion Comparison with the SABR model. The Parameters we used here are forward $F(0, T) = 98.03$, forward CEV power $\beta_1 = 1.0$, SV CEV power $\beta_2 = 1.0$, instantaneous volatility $\alpha = 0.4$, $volvol = 0.2$, correlation $\rho_{SV} = -0.2$ and maturity $T = 5.0$.

Fig. 3.2: Expansion Comparison with the SABR and Tree Method. The parameters we used here are forward $F(0, T) = 112.235$, forward CEV power $\beta_1 = 1.0$, SV CEV power $\beta_2 = 1.0$, instantaneous volatility $\alpha = 0.4$, $volvol = 0.2$, correlation $\rho_{SV} = -0.6$ and maturity $T = 1.0$. 
Fig. 3.3: Expansion Comparison with Monte Carlo: $T = 1.0$. The parameters we used here are forward $F(0,T) = 112.24$, forward CEV power $\beta_1 = 1.0$, SV CEV power $\beta_2 = 1.0$, instantaneous volatility $\alpha = 0.18$, $\text{volvol} = 0.2$, correlations $\rho_{SV} = -0.15$, $\rho_{ds} = 0.1816$, $\rho_{fs} = 0.3812$, $\rho_{df} = 0.2799$, domestic interest rate vol $\sigma_d = 0.6\%$, foreign interest rate vol $\sigma_f = 0.9\%$, domestic interest rate mean reversion rate $\theta_d = 0.01$, foreign interest rate mean reversion rate $\theta_f = 0.03$ and maturity $T = 1.0$.

Fig. 3.4: Expansion Comparison with Monte Carlo: $T = 3.0$. The parameters we used here are forward $F(0,T) = 104.48$ and maturity $T = 3.0$. The remaining parameters are the same as in $T = 1.0$. 
3. Foreign Exchange Options with Local Stochastic Volatility and Stochastic Interest Rates

Fig. 3.5: Expansion Comparison with Monte Carlo: $T = 5.0$. The parameters we used here are forward $F(0, T) = 98.03$ and maturity $T = 5.0$. The remaining parameters are the same as in $T = 1.0$.

Fig. 3.6: Expansion Comparison with Monte Carlo: $T = 10.0$. The parameters we used here are: forward $F(0, T) = 85.37$ and maturity $T = 10.0$. The remaining parameters are the same as in $T = 1.0$. 
Fig. 3.7: Volatility Surface for the USD/JPY Market, Apr, 2007
Fig. 3.8: SV+SIR calibration error, USD/JPY Vol Surface, 20th/April, 2007. The model parameters are mean reversion rates for domestic and foreign interest rates $\kappa_d = 0.01, \kappa_f = 0.03$ and interest rates volatilities $\sigma_d = 0.5\%, \sigma_f = 1\%; \alpha_1 = 0.5\%$, $\text{Variance}(0) = \theta = 1.0, \kappa = 0.03$. The stochastic volatility parameters (here given as constant instead of piece-wise constant) are shown in Table 3.1
Fig. 3.9: SV+SIR+Jumps calibration result, USD/JPY Vol Smile of a one-year maturity, April, 2007. The new model parameters (different from those above) are time-dependent volatility $\delta = 5.85\%$, SV correlation $\rho = -0.8$, volatility of variance $\epsilon = 1.2$, jump intensity $\lambda = 0.1$, jump size mean $\alpha = -20\%$ and jump size s.d. $\delta = 0.3$. 
3. Foreign Exchange Options with Local Stochastic Volatility and Stochastic Interest Rates

Fig. 3.10: PRDC-TARN Valuation Comparison, LN-CEV-SV Models
4. NON-BIASED MONTE CARLO SIMULATION FOR A HESTON-TYPE STOCHASTIC VOLATILITY MODEL

4.1 Introduction

Stochastic volatility models are gaining popularity in the pricing and hedging of financial derivatives; however, because of their analytical complexity, some exotic options (e.g. path-dependent options) do not possess closed form pricing formulae, and some numerical methods such as the finite difference and lattice method become tedious or unstable in this kind of models because of their multi-dimensionality. The Monte Carlo simulation, with its simplicity and flexibility in the model implementation, turns out to be the first choice for pricing exotic products with stochastic volatility.

However, 20 years since the first stochastic volatility model ([2]) was proposed, little research has been done on the efficient and accurate Monte Carlo simulation applied in this area, especially for one of the most popular models, namely the Heston-type stochastic volatility model([4]). The main problem is that some traditional discretisation schemes used in the Monte Carlo simulation, e.g. the Euler scheme, do not guarantee the positivity of the variance in the square root process with certain model parameters quite common in practical use. In [22], Broadie and Kaya conducted an exact simulation of the Heston stochastic volatility model and found that while elegant and non-biased, the implementation is complicated and the computation load is too large for practical use. Other approaches, e.g. [24], propose an implicit Milstein scheme for the square root process; nonetheless, while it works well for certain cases, the variance process can still become negative under extreme model parameters. Lord, Koekkoek and Dijk [23] consider the Euler Scheme with certain rules to ensure the variance process being positive, but this would incur bias, and, as Andersen([27]) points out, the bias can be high in some cases.

Andersen([27]) proposes an efficient method in his paper by approximating the

1 A simple stochastic volatility model by itself is two-dimensional – asset price and its volatility process.
non-central $\chi^2$ distribution of the variance process with some simple functions, based on the moment-matching technique. The method seems to be efficient and easy to implement, but it makes no use of the analytically available characteristic function of the square root process, which is largely relevant to the distribution of the stochastic variance. In this chapter, I will show how to make use of it in sampling the non-biased variance process, based on the saddle point approximation method borrowed from statistics. The computation speed is largely improved compared to the Broadie and Kaya method ([22]). Further simulation towards asset price simulation is shown through the use of the moment matching technique.

4.2 Properties of the Square Root Process

We consider the original Heston model defined as:

4.1

\[
\frac{dF_t}{F_t} = \sqrt{V_t} dW_S(t)
\]

4.2

\[
dV_t = \kappa(\theta - V_t) dt + \eta \sqrt{V_t} dW_V(t)
\]

\[
dW_S(t)dW_V(t) = \rho dt
\]

The square root process (4.2) of the variance was firstly introduced in finance by Cox, Ingersoll and Ross ([31]) when they used it to model the interest rate process. Heston([4]) used this process to model the stochastic variance process. In [73], Dufresne comprehensively researches this process as well as the integrated square process.

Several well known properties of the square root process and Heston model are given as below:

Proposition 4.2.1 Given $V(0) > 0$, the process for $V$ can never reach 0 if $2\kappa\theta \geq \eta^2$, if $2\kappa\theta < \eta^2$, as the origin is accessible and strongly reflecting.

Remark While in practice, the condition $2\kappa\theta \geq \eta^2$ is often violated, and the variance process can indeed reach 0, which causes problems for simulation.

Proposition 4.2.2 The transition law of $V_T$ given $V_t(T > t)$ is expressed as:

\[
V_T|V_t = \frac{\eta^2(1 - e^{-\kappa(T-t)})}{4\kappa} \chi^2_d \left( \frac{4\kappa e^{-\kappa(T-t)}}{\eta^2(1 - e^{-\kappa(T-t)})} V_t \right)
\]

where $d = \frac{4\theta\kappa}{\eta^2}$, $\chi^2_d(\lambda)$ is non-central chi-squared distributed with $d$ degrees of freedom and non-centrality parameter $\lambda$. 
Its distribution:

\[
\Prb[V_T < y| V_t] = F_{\chi^2_d}( \frac{yd}{\theta(1 - e^{-\kappa(T-t)}) : \theta(1 - e^{-\kappa(T-t)})})
\]

where \(F_{\chi^2_d}(z, \lambda)\) is the cumulative distribution function of the non-central chi-square distribution with \(d\) degrees of freedom and \(\lambda\) the non-centrality parameter, and is given as:

\[
F_{\chi^2_d}(z, \lambda) = e^{-\frac{\lambda}{2}} \sum_{i=0}^{\infty} \frac{(\lambda/2)^i}{i! 2^{d/2+i} \Gamma(d/2 + i)} \int_0^z x^{d/2+i-1} e^{-x/2} dx.
\]

(4.4)

Remark Broadie and Kaya ([22]) directly sample \(V_{t+\Delta t}\) given \(V_t\) from the above formula with the parameter of the degree of freedom being determined by a draw from the Poisson distribution. As we see, this exact sampling scheme, while bias-free, can be numerically expensive.

**Proposition 4.2.3** \(V_T\)'s first two moments are given as (conditional on \(V_t\)):

\[
E[V_T| V_t] = \theta + (V_t - \theta)e^{-\kappa(T-t)},
\]

(4.5)

\[
Var[V_T| V_t] = \frac{V_t \eta^2 e^{-\kappa(T-t)}}{\kappa}(1 - e^{-\kappa(T-t)}) + \frac{\theta \eta^2}{2\kappa}(1 - e^{-\kappa(T-t)})^2.
\]

(4.6)

**Proposition 4.2.4** The characteristic function of \(V_T\) conditional on \(V_t\), defined as \(\phi_{T-t}(u) := E[e^{iuV_T}| V_t]\), is given as:

\[
\phi_T(u) = \frac{\exp(uV_t + \frac{bu}{2\kappa})}{(1 - \frac{u}{c})^{\frac{2\kappa}{\eta^2}}}
\]

(4.7)

\[
c = \frac{2\kappa}{(1 - e^{-\kappa T})\eta^2}
\]

\[
b = \frac{2\kappa V_t}{e^{\kappa T}(1 - e^{-\kappa T})\eta^2}
\]

Remark One of the reasons for the popularity of the square root process used in finance is that its characteristic function is analytically available, from which the transition density function can be obtained through (numerically) integral inversion. In other words (cf. [69]):

\[
p(T-t,y| V_t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iuy} \phi_{T-t}(iu) du
\]

(4.8)

Rather than applying a computationally expensive numerical inversion to obtain the transition density of the variance process, we choose the saddle point approximation method to obtain the cumulative distribution function, which is vital in sampling for the Monte Carlo simulation.
4. Non-Biased Monte Carlo Simulation for a Heston-Type Stochastic Volatility Model

4.3 Simulation of $V_t$: Application of the Saddle Point Method

The saddle point method, first introduced to statistics by Daniel([67]) in 1954, has been widely and successfully used on many topics. In [64], Rogers and Zane applied it to option pricing with the distribution functions of the asset price approximated by the saddle point method, given the characteristic function was known explicitly. Since in the case of the square root process this is readily available in (4.7), we can utilise the power of this accurate approximation.

Firstly, we define the cumulant generating function of $V_T$ as $K_T(u) := \log \phi_T(u)$, so for the square root process we see:

$$K_T(u) = (uV_t + \frac{bu}{c - u}) - (q + 1)\log(1 - \frac{u}{c})$$

(4.9)

with $b, c$ given in (4.7).

Now we rewrite (4.8) as:

$$p(T - t, y|V_t) = \frac{1}{2\pi i} \int_{u^* - i\infty}^{u^* + i\infty} e^{-uy\phi_T(u)}du$$

(4.10)

where $u^*$ is the optimal integration contour shift in the saddle point method, defined as:

$$\frac{dK_T(iu)}{du}|_{u=u^*} = iy;$$

Lugannani and Rice ([66]) offered an approximation formula for the cumulative distribution function:

$$Q_T(y) := \text{Prob}[V_T > y|V_t]$$

$$= \frac{1}{2} \text{erfc}(\sqrt{-f_0}) + \sum_{n=0}^{\infty} (A_n - B_n)$$

where $f_0 = K_T(iu^*) - iu^*y$, $A_n$, $B_n$ and $n = 0, 1, 2$ were given in ([66]). We list them in the Appendix for completeness.

Finally, we summarise:

For sample $V_T$ given $V_t$, define $\tilde{Q}_T(y) := \text{Prob}[V_T \leq y|V_t] = 1 - Q_T(y)$.

• Firstly, generate a uniform random variable $U \in [0, 1]$.

• Search $y$, s.t. $\tilde{Q}_T(y) = U$. 
The second step could be done numerically (e.g. the bisection method), since the calculation of \( \tilde{Q}_T(y) \) only involves a few function evaluations, this requires little computational effort.

4.4 Simulation of \( \int_t^{t+\Delta} V_s ds \) given \( V_t \) and \( V_{t+\Delta} \): Moment Matching Technique

Firstly, let us re-write the Heston model (4.1-4.2):

\[
\log F(t + \Delta t) = \log F(t) + \frac{\rho}{\eta} (V(t + \Delta t) - V(t) - \kappa \theta \Delta t) + (\frac{\kappa \rho}{\eta} - \frac{1}{2}) \int_t^{t+\Delta t} V_s ds + \sqrt{1 - \rho^2} \int_t^{t+\Delta t} \sqrt{V(s)} dW(s) \tag{4.11}
\]

\[
E[\int_t^{t+\Delta t} V(s) ds | V_t] = \theta \Delta t - \frac{\theta - V_t}{\kappa} (e^{-\kappa t} - e^{-\kappa(t+\Delta t)}) \tag{4.14}
\]

Having sampled \( V_{t+\Delta} \) given \( V_t \), we now move on to the term \( \int_t^{t+\Delta t} V_s ds \) given \( V_t \) and \( V_{t+\Delta} \). Broadie and Kaya ([22]) derived its characteristic function and used a numerical Fourier transform inversion to obtain the cumulative distribution function of \( \int_t^{t+\Delta t} V_s ds \), conditional on \( V_t \) and \( V_{t+\Delta} \). However, at least three complexities are involved here: the first is that the characteristic function contains two modified Bessel functions of the first kind, which complicates computation; secondly, the numerical integration on the infinite domain requires a large computational load; and thirdly, to sample the value of \( \int_t^{t+\Delta t} V_s ds \) one has to sort by numerical search (bisection or Newton method); thus, one sampling requires a computational load that is the product of these three processes. Additionally, because of the complexity of the characteristic function, it seems formidable to derive its high order derivatives, which precludes the use of the saddle point method we presented in the last section. Here, we adopt a moment matching method.

We can write:

\[
\int_t^{t+\Delta t} V(s) ds \approx [\alpha_1 V(t) + \alpha_2 V(t + \Delta t)] \Delta t \tag{4.13}
\]

The Euler setting gives \( \alpha_1 = 1, \alpha_2 = 0 \). Andersen ([27]) sets \( \alpha_1 = \alpha_2 = 0.5 \), which is a central discretisation. Indeed, this can be improved by moment matching (as suggested in the conclusion of [27]):
4. Non-Biased Monte Carlo Simulation for a Heston-Type Stochastic Volatility Model

The second moment of $\int_t^{t+\Delta t} V(s)ds$ is given by Dufresne: [73]. Then an Ito integral $\int_t^{t+\Delta t} V(s)dW(s)$ is easily sampled as:

$$\int_t^{t+\Delta t} V(s)dW(s) \sim \mathcal{N}(0, \int_t^{t+\Delta t} V(s)ds)$$

4.5 Simulation of $F_{t+\Delta t}$ given $F_t$

Finally, from (4.11), given $\log F(t)$, $V(t)$, $V(t+\Delta t)$, $\int_t^{t+\Delta t} V(s)ds$, we arrive at:

$$\log F(t+\Delta t) = \log F(t) + \frac{\rho}{\eta} (V(t+\Delta t) - \kappa \theta \Delta t - V(t)) + \left(\frac{\kappa \rho}{\eta} - \frac{1}{2}\right) \int_t^{t+\Delta t} V(s)ds$$

$$+ \epsilon \sqrt{1 - \rho^2} \int_t^{t+\Delta t} V(s)ds \quad (4.15)$$

where $\epsilon$ is a standard normal variable.

4.6 Conclusion and Future Research

Accurate Monte Carlo methods for stochastic volatility models are currently being actively studied. In the future research, a further improvement to the simulation of $\int_t^{t+\Delta t} V(s)ds$ may be given.

4.7 Appendix: Proof

4.7.1 Characteristic Function and Moments of the Square Root Process

According to (4.2), the variance process starts from $V(0) = V$:

$$V(t) = V + \kappa \int_0^t (\theta - V(s))ds + \int_0^t \eta \sqrt{V(s)}dW_V(s)$$

and its conditional expectation follows:

$$\mathbf{E}[V(t)|V] = V + \kappa \int_0^t (\theta - \mathbf{E}[V(s)|V])ds$$

$$= V + \kappa \theta t - \kappa \int_0^t \mathbf{E}[V(s)|V]ds \quad (4.16)$$

Denoting $\bar{V}(t) := \mathbf{E}[V(t)|V]$, we have:
\[ \dot{V}(t) = \kappa \theta - \kappa V(t) \]
\[ V(0) = V \]

This is solved trivially:
\[ \bar{V}(t) = \theta + (V - \theta)e^{-\kappa t} \]

and (4.5) is produced.

Now we move to the second moment of the variance process, by applying Ito’s lemma to \( V^2(t) \):
\[
dV^2(t) = \{2\kappa \theta + \eta^2\} V(t) - 2\kappa V^2(t) \} dt + 2\eta V^{3/2}(t) dW_V(t) \]
\[ V^2(0) = V^2 \]

Thus,
\[
\mathbb{E}[V^2(t)|V] = (2\kappa \theta + \eta^2) \int_0^t \mathbb{E}[V(s)|V] ds - 2\kappa \int_0^t \mathbb{E}[V^2(s)|V] ds
\]

Plugging \( \mathbb{E}[V^2(t)|V] = (\mathbb{E}[V(t)|V])^2 + \text{Var}(V(t)|V) \) into above equation, with \( \mathbb{E}[V(t)|V] \) given and denoting \( Z(t) := \text{Var}(V(t)|V) \), we arrive at:
\[
Z'(t) = (2\kappa \theta + \eta^2)[\theta + (V - \theta)e^{-\kappa t}] - 2\kappa[\theta + (V - \theta)e^{-\kappa t}]^2 - 2\kappa Z(t)
\]
\[ Z(0) = 0 \]

which gives the variance of the variance process \( Z(t) \) as (4.6).

More generally, we may consider a Moment Generating Function (MFG) or Laplace transform of \( V(t) \):
\[ \phi_T(u) := \mathbb{E}[e^{uV(T)}|V] \]

This is analogous to the pricing of a zero coupon bond using the CIR interest rate model (see [31]). We outline the steps here:
\[
\phi_T(u) = \mathbb{E}[\exp\{uV + uk \int_0^T (\theta - V(t)) dt + u\eta \int_0^T \sqrt{V(t)} dW_V(t)\}]
\]
\[
= \mathbb{E}[\exp\{uV + uk \int_0^T (\theta - V(t)) dt + \frac{1}{2}u^2\eta^2 \int_0^T V(t) dt\}]
\]
\[ = e^{A(T) + B(T)V} \]

(4.18)
which can be represented in an affine form. Through the use of Feynman-Kac ([15]), \( \phi_T(u) \) satisfies:

\[
\frac{\partial \cdot}{\partial \tau} = \kappa(\theta - V) \frac{\partial \cdot}{\partial V} + \frac{1}{2} \eta^2 V \frac{\partial^2 \cdot}{\partial V^2};
\]

\[
A(0) = 0;
\]

\[
B(0) = u.
\]

where \( \tau = T - t \) is the maturity. Thus, we have a set of Riccati-type ODEs:

\[
B'(\tau) = -\kappa B(\tau) + \frac{1}{2} \eta^2 B^2(\tau);
\]

\[
A'(\tau) = \kappa \theta B(\tau) \quad (4.19)
\]

subject to IC \( A(0) = 0, B(0) = u \). The Riccati ODE can be transformed to a Bernoulli-type ODE (see [92]) which possesses an analytical solution:

\[
B(T) = \frac{2\kappa e^{-\kappa T}}{\eta^2 u(e^{-\kappa T} - 1) - 2\kappa};
\]

\[
A(T) = -\frac{2\kappa \theta}{\eta^2} \log \left\{ \frac{\eta^2 u(e^{-\kappa T} - 1) - 2\kappa}{2\kappa} \right\}
\]

and \( \phi_T(u) \) is given in (4.7). Any higher moments of variance process \( V(t) \) can be derived via MGF.

4.7.2 The Lugannani and Rice Formula for the Cumulative Distribution Function

See [66].

The cumulative distribution function:

\[
Q_T(y) := \text{Prob}[X_T > y] = \frac{1}{2} \text{erfc}(\sqrt{-f_0}) + \sum_{n=0}^{\infty} (A_n - B_n)
\]

\[
f_0 = K_T(iu_0) - iu_0 y
\]

\( K_T(\cdot) \) is the cumulant generating function, while \( u_0 \) is the optimal integration contour shift.
The \( A_n, B_n, n = 0, 1, 2 \) are given as:

\[ A_0 = \frac{\mu}{\sqrt{2\pi}} e^{f_0}, \]
\[ A_1 = -3A_0 \left[ \frac{1}{3}\mu^2 + \mu \theta_3 + \frac{1}{2}(5\theta_3^2 - 2\theta_4) \right], \]
\[ A_2 = 15A_0 \left[ \frac{1}{5} \mu^4 + \mu^3 \theta_3 + \frac{1}{2} \mu^2 (7\theta^2 - 2\theta_4) \right. \]
\[ \left. + \frac{1}{4} \mu(42\theta_3^3 - 28\theta_3\theta_4 + 4\theta_5) \right] \]
\[ \left. + \frac{1}{8} (231\theta_3^4 - 252\theta_3^2\theta_4 + 56\theta_3\theta_5 + 28\theta_4^2 - 8\theta_6) \right] \]
\[ B_0 = \frac{e^{f_0}}{2\sqrt{-\pi f_0}}, \]
\[ B_1 = \frac{e^{f_0}}{4\sqrt{-\pi f_0} f_0}, \]
\[ B_2 = \frac{3e^{f_0}}{8\sqrt{-\pi f_0} f_0^2} \]

where \( \mu = \frac{1}{f_0 \sqrt{\phi(2)}} \), and \( \theta_j = \frac{\phi(j)}{j! \sqrt{\phi(2)}} \).
5. THE LIBOR MARKET MODEL WITH STOCHASTIC VOLATILITY AND JUMP PROCESSES

5.1 Introduction

The LIBOR market model (also known as the BGM/J model) has gained importance in both financial industry and academia since it was introduced by Brace et al. (1997) ([3]), Jamshidian (1997) ([43]) and Miltersen et al. (1997) ([90]). Three notable virtues contribute to its popularity. First, a Black-Scholes-type formula is admitted for the pricing of both caplets and swaptions, although some necessary approximations have to be made. This simplifies the market practice of quotation and allows better accordance to other options market, e.g. the equity and FX markets. Secondly, the underlying variables (forward rate, swap rate, etc.) are directly observable and the LIBOR market has been well developed. Last, but not least, this general model allows for a flexible term structure of volatility and correlation, as well as various extensions, e.g. local/stochastic volatility.

In the presence of implied volatility skews/smiles in the interest rate derivatives market, in order to be able to fit (or, regenerate) the market volatility surface, several model extensions have been made mainly through three approaches: (1) The local volatility function has been incorporated mainly through a CEV/displaced diffusion type: Andersen and Andreasen ([49]) and Kawai ([50]) added a CEV type volatility process, which is able to generate a monotonic downward volatility skew; since CEV type models are not able to generate volatility smiles (curvature), which are commonly seen in the market, stochastic volatility models later came into the play. Andersen and Brotherton-Ractliffe ([51]) introduced a local stochastic volatility type process and obtained caplet/swaption prices through an asymptotic expansion method; in Andersen and Andreasen ([52]), a displaced-diffusion type local volatility coupled with a Heston type stochastic volatility model is studied and analytical option pricing formulae are obtained via the Fourier transform method. Later on, Piterbarg ([54]) extended his model to a dynamic one in which the model parameters were time-dependent. This new type of model allows simultaneous calibration to the vanilla options of different maturities and is consistent with the whole volatility surface; also Wu and Zhang ([44]) proposed a Heston-type stochas-
tic volatility, with non-zero correlations between LIBOR rates, and analytical formulae were obtained via Fourier transform. Joshi and Rebonato ([81]) proposed a displace-diffusion type LIBOR market model with stochastic coefficients in the volatility parameterisation form. While their model allows for joint evolution of swap rates and implied volatilities, there are no analytical formulae for the option prices, which makes the calibration procedure burdensome.

Parallel to the local/stochastic volatility approach, the jump-diffusion/Levy processes are also considered to be able to cope with the volatility skew/smile problem. Pioneered by Glasserman and Kou([82]), and two subsequent papers by Glasserman and Merener([83],[84]), the LIBOR market model with jump-diffusions (marked point process) was proposed and discussed regarding numerical computation. As the more general Levy processes have gained broad interest in recent years, Eberlein and Ozkan([85]) developed a LIBOR forward rate model on the Levy process and further, in ([86]), a swap rate market model was developed and a swaption price obtained. However, the Levy process models so far are still mainly of academic interest, as there are still numerous difficulties in the computation and calibration procedures.

In this chapter, I will present a general setup for LIBOR modelling, including both stochastic volatility and jump processes, with time-dependent model parameters. This model setup is general and necessary because, as Chen and Scott([87]), Jarrow, Li and Zhao ([88]) point out, both stochastic volatility and jumps are present in the interest rate markets. Their models combined stochastic volatility and jumps for the short-term futures rate process and LIBOR forward rate process, respectively, but shared a limitation only on the pricing of forward rate derivatives (interest rate futures options, caps/floors, etc.), without extending to the swap rate market, which is important and more complicated. In this chapter I will develop a consistent and general modelling framework with stochastic volatility and general jumps for the bond price process and LIBOR forward rate/swap rate processes, along with the pricing formulae for interest rate options. Both caplet/floorlet and swaption prices are derived under this framework.

This chapter will be organised as follows: In Section 5.2 we will derive the LIBOR forward rate model from the assumption of the bond price process with stochastic volatility and a marked point jump process; In Section 5.3 we will present the LIBOR swap rate model; Subsequently, the caplet/swaption pricing formula will be given via Fourier transform method in Section 5.4 and finally we conclude in

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1 Chen and Scott found evidence of stochastic volatility and jumps in short-term interest rate markets in four major countries: US, UK, Germany and Japan.
Section 5.5.

5.2 The LIBOR Forward Rate Model

5.2.1 Risk-Neutral Measure

Since the definition of interest rates normally is derived from the market-observable zero-coupon bond (ZCB) prices, we start from the process of ZCB: \( P(t, T) \). Under the risk-neutral measure \( Q \), we assume \( P(t, T) \) has a stochastic variance added in its diffusion term as well as a jump diffusion term:

\[
\frac{dP(t, T)}{P(t, T)} = r(t)dt + \sigma(t, T) \cdot dW^Q(t) + \int_F F(t, x, T) \tilde{\mu}(dt, dx) \\
F(t, x, T) = e^D(t, x, T) - 1 \\
\tilde{\mu}(dt, dx) = \mu dt, dx - \lambda^Q(t, dx)dt
\] (5.1)

Then we use the relationship between LIBOR forward rates and zero-coupon bond prices as our starting point:

\[
L_j(t) = \frac{1}{\delta_j} \left( \frac{P(t, T_j)}{P(t, T_{j+1})} - 1 \right)
\] (5.4)

From (5.4), we use Ito’s lemma (Ito’s lemma for Jump processes is listed in Appendix):

\[
dL_j(t) = \frac{1}{\delta_j} d \left( \frac{P(t, T_j)}{P(t, T_{j+1})} \right) \\
= \frac{1}{\delta_j} \left( 1 + \delta_j L_j(t) \right) \left( \left[ \sigma^*(t, T_j) - \sigma^*(t, T_{j+1}) \right] \cdot dW^Q(t) \\
+ V(t) \sigma^*(t, T_{j+1}) [\sigma^*(t, T_{j+1})^T - \sigma^*(t, T_j)^T] dt \\
+ \int_F e^{D(t, x, T_j)} [e^{D(t, x, T_{j+1})} - 1] \lambda^Q(t, dx) dt \\
+ \int_F \left[ e^{\int_{T_j}^{T_{j+1}} h(t, x, s) ds} - 1 \right] \mu(dt, dx) \right)
\] (5.5)

We make an approximation above:

\[
e^{D(t, x, T_j)} + e^{-D(t, x, T_{j+1})} - 2 \approx e^{\int_{T_j}^{T_{j+1}} h(t, x, s) ds} - 1
\] (5.6)

where \( D(t, x, T_j) \) is defined as:

\[
D(t, x, T_j) = - \int_t^{T_j} h(t, x, s) ds
\] (5.7)
As noted, \(d \frac{\delta_j L_j(t)}{L_j(t_-)}\) has a Brownian motion diffusion term \([\sigma^*(t, T_j) - \sigma^*(t, T_{j+1})]\), so assume \(d \frac{\delta_j L_j(t)}{L_j(t_-)}\) has a diffusion term \(\gamma_j(t)\), in which case we have:

\[
[\sigma^*(t, T_j) - \sigma^*(t, T_{j+1})] = \frac{\delta_j L_j(t) \gamma_j(t)}{1 + \delta_j L_j(t_-)},
\]

(5.8)

Furthermore, following [3], we assume \(\sigma^*(t, T_{n(t)}) = 0\), where \(\eta(t)\) denotes the smallest integer such that \(T_{\eta(t)} \geq t\). In this case we obtain:

\[
\sigma^*(t, T_{j+1}) = - \sum_{k=\eta(t)}^j \frac{\delta_k L_k(t) \gamma_k(t)}{1 + \delta_k L_k(t_-)}
\]

(5.9)

In (5.5), we choose \(h_i, i = 1, 2, \ldots, r\), such that:

\[
\int_{T_n}^{T_n+1} h(t, x, s) ds = \log \left[ \frac{1 + \delta_n L_n(t_-)(1 + H_n(x))}{1 + \delta_n L_n(t_-)} \right]
\]

(5.10)

where \(H_n(\cdot)\) is defined as the ”jump size” of \(\frac{dL_n(t_-)}{L_n(t_-)}\). We now rewrite (5.5) with the use of (5.10), and add a stochastic variance factor \(V(t)\):

\[
\frac{dL_j(t)}{L_j(t_-)} = \sqrt{V(t)\gamma_j(t)} \cdot dW^Q(t) + V(t) \sum_{k=\eta(t)}^j \frac{\delta_k L_k(t) \gamma_k(t) \gamma_j(t)T}{1 + \delta_k L_k(t_-)} dt + \frac{1 + \delta_j L_j(t_-)}{\delta_j L_j(t_-)} \cdot \text{JumpTerm}
\]

(5.11)

where the jump term is given by:

\[
\int_E \left( \prod_{k=\eta(t)}^j \frac{1 + \delta_k L_k(t_-)}{1 + \delta_k L_k(t_-)(1 + H_k(x))} - \prod_{k=\eta(t)}^{j-1} \frac{1 + \delta_k L_k(t_-)}{1 + \delta_k L_k(t_-)(1 + H_k(x))} \right) \lambda^Q(dx, t) dt
\]

(5.12)

Thus:

\[
\frac{dL_j(t)}{L_j(t_-)} = \sqrt{V(t)\gamma_j(t)} \cdot dW^Q(t) + V(t) \sum_{k=\eta(t)}^j \frac{\delta_k L_k(t) \gamma_k(t) \gamma_j(t)T}{1 + \delta_k L_k(t_-)} dt + \int_E H_j(x) \{ \mu(dx, dt) - \prod_{k=\eta(t)}^j \frac{1 + \delta_k L_k(t_-)}{1 + \delta_k L_k(t_-)(1 + H_k(x))} \lambda^Q(dx, t) dt \}
\]

(5.13)
Where the added stochastic variance process is assumed to be a Heston(1993)(cf: [4]) type:
\[
dV = \kappa(\theta - V(t))dt + \eta \sqrt{V(t)}dZ^Q(t)
\] (5.14)
and correlations \( \rho_k(t) : [0, T] \rightarrow [-1, 1] \) between the stochastic variance and the LIBOR forward rates are considered as:
\[
E^Q[\mathbf{d}Z^Q(t) \cdot \left( \frac{\gamma_k(t)}{\|\gamma_k(t)\|} \cdot \mathbf{d}W^Q(t) \right)] = \rho_k(t)dt,
\]
under the risk-neutral measure \( Q \).

### 5.2.2 Change of Measure

In order to price the caplet, which is a European-type call option on the LIBOR forward rate \( L_j(T_j) \), we are interested in the forward rate dynamics under the \( P_j^{j+1} \) forward measure, under which the caplet and ZCB price ratio \( \frac{C_{pl}(t,T_j;K)}{P(t,T_j+1)} \) is a martingale.

Under the \( P_j^{j+1} \) measure, ZCB \( P(t,T_j+1) \) is used as the numeraire, so using the ZCB process given in (5.1), we give the lemma:

**Lemma 5.1.** Under the \( P_j^{j+1} \) measure, the LIBOR forward rate process in (5.13) is given as:
\[
\frac{dL_j(t)}{L_j(t_-)} = \sqrt{V(t)} \gamma_j(t) \cdot \mathbf{d}W^{j+1}(t) + \int_E H_j(x) \left[ \mu(dx,dt) - \lambda^{j+1}(dx,t)dt \right]
\] (5.15)
with:
\[
dV(t) = \kappa(\theta - \eta^{j+1}(t)V(t))dt + \eta \sqrt{V(t)}dZ^{j+1}(t)
\] (5.16)
where:
\[
dW^{j+1}(t) = dW^Q(t) - \sqrt{V(t)} \sigma^+(t,T_{j+1})dt;
\] (5.17)
\[
dZ^{j+1}(t) = dZ^Q(t) + \frac{j}{1 + \delta_{kL_k(t)} \rho_k(t)dt}
\] (5.18)
and:

\[ \eta^{j+1}(t) = 1 + \frac{\eta}{\kappa} \tilde{\eta}_j(t), \quad (5.19) \]

\[ \tilde{\eta}_j(t) \approx \sum_{k=\eta(t)}^{j} \frac{\delta_k L_k(0) \gamma_k(t)}{1 + \delta_k L_k(0)} \rho_k(t), \quad (5.20) \]

\[ \lambda^{j+1}(dx,t) = e^{D(t,x,T_{j+1})} \lambda^Q(dx,t) \]

\[ = \prod_{k=\eta(t)}^{j} \frac{1 + \delta_k L_k(t_-)}{1 + \delta_k L_k(t_-)(1 + H_k(x))} \lambda^Q(dx,t) \quad (5.21) \]

(5.21) is due to the use of Girsanov theorem on the jump process, which is given in Appendix.

**Proof:** See Appendix 5.6.1.

### 5.3 The LIBOR Swap Rate Model

#### 5.3.1 The Swap Market

The swap rate for the period \((T_n, T_m), n < m\), is defined as:

\[ S_{n,m}(t) = \frac{P(t, T_n) - P(t, T_m)}{B_{n,m}(t)} \quad (5.22) \]

with *annuity* \(B_{n,m}(t)\) defined as:

\[ B_{n,m}(t) = \sum_{j=n}^{m-1} \delta_j P(t, T_{j+1}) \quad (5.23) \]

and the discounted pay-off of swaption at time \(T_n\) given by:

\[ B_{n,m}(T_n) [S_{n,m}(T_n) - K]_+ \quad (5.24) \]

#### 5.3.2 Change of Measure

In order to price the swaption, we note that the swap rate is a martingale under the *annuity measure* \(\mathcal{P}^{n,m}\), with \(B_{n,m}(t)\) being the numeraire (cf: [43]).
Thus we need to apply a change of measure from $Q$ to $\mathcal{P}^{n,m}$ first. This is given below:

**Proposition 5.2.** Under the annuity measure $\mathcal{P}^{n,m}$, the changes of the Brownian motions $W^Q, Z^Q$ are given as:

\[ dW^{n,m}(t) = dW^Q(t) - \sum_{k=n}^{m-1} b_{k+1}(t) \sqrt{V(t)} \sigma^*(t, T_{k+1}) dt \]  \hspace{1cm} (5.25)\]
\[ dZ^{n,m}(t) = dZ^Q(t) + \sqrt{V(t)} \eta^{n,m}(t) dt \]  \hspace{1cm} (5.26)\]

the jump intensity under measure $\mathcal{P}^{n,m}$ is given as:

\[ \lambda^{n,m}(dx, t) = \sum_{k=n}^{m-1} b_{k+1}(t) \prod_{j=\eta(t)}^{k} \frac{1 + \delta_j L_j(t_\cdot)}{1 + \delta_j L_j(t_\cdot)(1 + H_j(x))} \lambda^Q(dx, t) \]  \hspace{1cm} (5.27)\]

and the stochastic variance process under $\mathcal{P}^{n,m}$ is:

\[ dV(t) = \kappa[\theta - \tilde{\eta}^{n,m}(t)V(t)] dt + \eta \sqrt{V(t)} dZ^{n,m}(t) \]  \hspace{1cm} (5.28)\]

with:

\[ \eta^{n,m}(t) = \sum_{k=n}^{m-1} b_{k+1}(t) \sum_{j=\eta(t)}^{k} \frac{\delta_j L_j(t_\cdot) \gamma_j(t) \rho_j(t)}{1 + \delta_j L_j(t)} \]
\[ \tilde{\eta}^{n,m}(t) = 1 + \frac{\eta}{\kappa} \eta^{n,m}(t) \]
\[ \approx 1 + \frac{\eta}{\kappa} \left( \sum_{k=n}^{m-1} b_{k+1}(0) \sum_{j=\eta(t)}^{k} \frac{\delta_j L_j(0) \gamma_j(t) \rho_j(t)}{1 + \delta_j L_j(0)} \right) \]  \hspace{1cm} (5.29)\]
\[ b_{k+1}(t) = \frac{\delta_k P(t, T_{k+1})}{\sum_{j=n}^{m-1} \delta_j P(t, T_{j+1})} = \frac{\delta_k P(t, T_{k+1})}{B_{n,m}(t)} \]  \hspace{1cm} (5.30)\]

**Proof:** See Appendix 5.6.2.

We now have the process of LIBOR forward rate under measure $\mathcal{P}^{n,m}$, expressed by Brownian motions $W^{n,m}(t), Z^{n,m}(t)$ given in (5.25)-(5.26) and the jump inten-
in (5.22) and (5.4), respectively, it is easy to obtain:

By using the definitions of the swap rate and LIBOR forward rate, which are given in (5.27), we arrive at the following proposition:

**Proposition 5.3** Under the $P^{n,m}$ annuity measure, the swap rate $S_{n,m}$ process is a martingale and given by:

$$dS_{n,m}(t) = \sqrt{V(t)}dW^{n,m}(t)$$

$$+ \frac{P(t, T_n)}{B_{n,m}(t)} \left\{ \int_E \frac{1}{\sum_{k=n}^{m-1} b_{k+1}(t) e^{D(T_n, x, T_{k+1})}} \lambda^{n,m}(dx, t) dt \right\}$$

$$+ \frac{P(t, T_m)}{B_{n,m}(t)} \left\{ \int_E \frac{1}{\sum_{k=n}^{m-1} b_{k+1}(t) e^{-D(T_{k+1}, x, T_m)}} \lambda^{n,m}(dx, t) dt \right\}$$

$$+ \frac{1}{P(t, T_n)} \int_E \frac{1}{\sum_{k=n}^{m-1} b_{k+1}(t) e^{D(T_{k+1}, x, T_m)}} e^{D(t, x, T_m)} - 2\mu(dx, dt)$$

$$+ \frac{1}{P(t, T_m)} \int_E \frac{1}{\sum_{k=n}^{m-1} b_{k+1}(t) e^{-D(T_k, x, T_{k+1})}} e^{D(t, x, T_m)} - 2\mu(dx, dt)$$

(5.33)

where $\sigma_{n,m}(t)$ is given by:

$$\sigma_{n,m}(t) = \sum_{k=n}^{m-1} b_{k+1}(t) \sum_{j=n}^{k} \frac{\delta_j L_j(t) \gamma_j(t)}{1 + \delta_j L_j(t)} + \frac{P(t, T_n)}{P(t, T_m) - P(t, T_m)} \frac{\delta_k L_k(t) \gamma_k(t)}{1 + \delta_k L_k(t)}$$

(5.34)

**Proof:** See Appendix 5.6.3.
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Nevertheless, the formula in Proposition 5.3 is still too complicated for practical implementation, since the LIBOR forward rates involved are also stochastic, so we adopt an approximation method, which starts from:

\[ dS_{n,m}(t) = \sum_{j=n}^{m-1} \frac{\partial S_{n,m}(t)}{\partial L_j(t)} dL_j(t) \] (5.35)

This is given by Euler’s theorem. Proceeding from (5.35), we give:

Proposition 5.4. Under the \( P^{n,m} \) annuity measure, the approximated swap rate \( S_{n,m} \) process is given by:

\[ \frac{dS_{n,m}(t)}{S_{n,m}(t^{-})} \approx \sqrt{V(t)} \sum_{k=n}^{m-1} \frac{\partial S_{n,m}(0)}{\partial L_k(0)} \frac{L_k(0)}{S_{n,m}(0)} \gamma_k(t) \cdot dW^{n,m}(t) + \sum_{k=n}^{m-1} \frac{\partial S_{n,m}(0)}{\partial L_k(0)} \frac{L_k(0)}{S_{n,m}(0)} \cdot \text{JumpTerms} \] (5.36)

where

\[ \frac{\partial S_{n,m}(0)}{\partial L_k(0)} = b_{k+1}(0) + \frac{\delta_k}{1 + \delta_k L_k(0)} \cdot \{ \sum_{j=n}^{k-1} b_{j+1}(0)[L_j(0) - S_{n,m}(0)] \} \] (5.37)

and the jump terms are given as:

\[ \int_E H_k(x) \mu(dx, dt) - \frac{1 + \delta_k L_k(0)}{1 + \delta_k L_k(0)[1 + H_k(x)]} \lambda^{n,m}(dx, t) dt \]

\[ = \int_E H_k(x) \mu(dx, dt) - \lambda^{n,m}_k(dx, t) dt \]

Proof: See Appendix 5.6.4.

5.4 Caplet and Swaption Pricing Via Fourier Transform

5.4.1 Caplet Pricing

Caplet, the European-type call option on the LIBOR forward rate, is defined as:

\[ \text{Capl}(0, L_j(0), T_j, K) = P(0, T_{j+1}) \delta_{j} E_{j+1}^{j}[ (L_j(T_j) - K)_+ ] \] (5.38)
under the $P^{j+1}$ measure. Moreover, $\frac{Capl(t, L_j(t), T_j, K)}{P(t, T_j+1)}$ is a martingale. We need to calculate the distribution of $L_j(T_j)$ under this forward measure from its process given in (5.15), filtrated to $\mathcal{F}_0$.

As the general jump term of the marked point process for $\frac{dL_j(t)}{L_j(t_-)}$ is given as:

$$\int_E H_j(x) \left[ \mu(dx, dt) - \lambda^{j+1}(dx, t) dt \right]$$

we now choose a special (and simple) case – a Poisson jump diffusion with a log-normal jump size to attain analytical tractability. This jump diffusion in the option pricing was first introduced by Merton ([89]) in 1976 and has been widely applied in the equity and FX derivatives markets (see chapter 3) where jumps are more common. From this point we can rewrite (5.15) as:

$$\frac{dL_j(t)}{L_j(t_-)} = -\lambda^j(t) \bar{\mu}^{j+1}(t) dt + \sqrt{V(t)} \gamma_j(t) \cdot dW^{j+1}(t)$$

$$+ (e^{Q(t)} - 1) dJ(t)$$

$$dV(t) = \kappa(\theta - \eta^{j+1}(t)V(t)) dt + \eta \sqrt{V(t)} dZ^{j+1}(t)$$

where

$$Q(t) \sim \mathcal{N}(\alpha(t), \delta(t));$$

$$dJ(t) = \begin{cases} 
0 & \text{with probability } 1 - \lambda^j(t) dt \\
1 & \text{with probability } \lambda^j(t) dt 
\end{cases}$$

and

$$\bar{\mu}^{j+1}(t) = \mathbb{E}_0^{j+1}[e^{Q(t)} - 1]$$

$$= e^{\alpha(t) + \frac{1}{2} \delta^2(t)} - 1$$

Note that $\lambda^j(t)$ is the jump intensity for the LIBOR forward rates at time $t$. $e^{Q}$, the log-normal jump process with $Q$ being the jump size, performs a normal distribution with a time-dependent mean $\alpha(t)$ and standard deviation $\delta(t)$ at time $t$.

In order to calculate the distribution of $L_j(T_j)$, we apply the Fourier transform method. Firstly, we need to calculate the characteristic function that "contains" the probability distribution information, which we define as:

$$\phi^{j+1}(u) = \mathbb{E}_0^{j+1}[e^{iu\log(L_j(T_j)/L_j(t_-))}]$$
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With the assumption that \( dZ, dW \) are independent of \( dJ \), so we can write our characteristic function \( \phi^{j+1}(u) \) as \( \phi^{j+1}(u) \phi^{j+1}(u) \): the product of a continuous part (a Brownian motion diffusion term) and jump part (the remaining drift and jump terms). As derived in the Appendix:

\[
\begin{align*}
\phi^{j+1}(u) &\equiv \phi^{j+1}(u)\phi^{j+1}(u) & (5.46) \\
\phi^{j+1}(u) &\equiv \exp[A(0) + B(0)V_0] & (5.47) \\
\phi^{j+1}(u) &\equiv \exp\left\{ \sum_{i=\eta(t)}^j \left[ -iu\lambda^i_j\left( e^{\alpha^i_j + \frac{1}{2}\delta^2_i} - 1 \right) + \lambda^i_j\left( e^{iu\alpha^i_j - \frac{1}{2}u^2\delta^2_i} - 1 \right) \right] \right\} & (5.48)
\end{align*}
\]

Where \( A(t), B(t) \) are given in Appendix 5.6.7. In addition, \( A(0) \) and \( B(0) \) can be obtained recursively, as \( \alpha, \delta \) and \( \lambda \) are the time piece-wise constant jump size mean, standard deviation and jump intensity under measure \( P_{j+1} \), respectively.

We now arrive at the caplet price:

\[
\text{Capl}(0, L_j(0), T_j, K) = P(0, T_{j+1})\delta_j\left\{ L_j(0) - \frac{L_j(0)}{2\pi} \right\} \int_{-\infty}^{\infty} \text{Re}\left[ e^{i\log(K)} + \alpha_k \phi^{j+1}(u + (\alpha - 1)i) \right] du \}
\]

where \( \alpha \) is the optimal shift of integral contour and can be found by the method described in [59].

5.4.2 Swaption Pricing

Swaption (here we refer to the payer swaption), the European-type option on the swap rate, is defined as:

\[
\text{Swpt}(0, S_{n,m}(0), T_n, K) = B_{n,m}(0)E_0^{n,m}\left[ (S_{n,m}(T_n) - K)_+ \right] (5.50)
\]

under the annuity measure \( P^{n,m} \). Here, \( \frac{\text{Swpt}(t, S_{n,m}(t), T_n, K)}{B_{n,m}(t)} \) is a martingale. The approximated swap rate process is given in Proposition 5.4. It is easily noted that the Brownian motion part (the stochastic volatility part) in (5.36) is similar to the volatility term in (5.40). Thus, the corresponding characteristic function \( \phi^{n,m}_C \) possesses the same form as that for the LIBOR forward rates:

\[
\phi^{n,m}_C(u) = \exp\{ \tilde{A}(0) + \tilde{B}(0)V_0 \}
\]

through replacing \( \|\gamma_j(t)\| \) by \( \sum_{k=n}^{m-1} \left\{ \frac{\partial S_{n,m}(0)}{\partial L_k(0)} \frac{L_k(0)}{S_{n,m}(0)} \|\gamma_k(t)\| \right\} \), and \( \eta^{j+1}(t) \) by \( \tilde{\eta}^{n,m}(t) \).
For the jump part, firstly from Proposition 5.4 we see the jump intensity \( \tilde{\lambda}_{k}^{n,m}(t) = \lambda^{k+1}(t) \). This equality also has an intuitive meaning – since the swap rate is approximately a linear combination of LIBOR forward rates, its jump occurs when the forward rate jumps and no other jumps exist.

From (5.36), and using Ito’s lemma for the jump processes, we obtain the jump part of \( \log \left( \frac{S_{n,m}(T_{n})}{S_{n,m}(0)} \right) \) as:

\[
- \int_{0}^{T_{n}} \int_{E} \sum_{k=n}^{m-1} \frac{\partial S_{n,m}(0)}{\partial L_{k}(0)} \frac{L_{k}(0)}{S_{n,m}(0)} H_{k}(x) \tilde{\lambda}_{k}^{n,m}(t)(dx,t) dt \\
+ \int_{0}^{T_{n}} \int_{E} \log \left( 1 + \sum_{k=n}^{m-1} \frac{\partial S_{n,m}(0)}{\partial L_{k}(0)} \frac{L_{k}(0)}{S_{n,m}(0)} H_{k}(x) \right) \mu(dx,dt)
\]

(5.51)

In the Poisson jump with log-normal jump magnitude (Merton jump), similar to the assumption we made on caplet pricing, we assume:

\[
\int_{E} H_{k}(x) \mu(dx,dt) = (e^{Q_{k}^{n,m}(t)} - 1) dJ_{k}^{n,m}(t)
\]

where:

\[
dJ_{k}^{n,m}(t) = \begin{cases} 
0 & \text{with probability } 1 - \lambda_{k}^{n,m}(t) dt \\
1 & \text{with probability } \lambda_{k}^{n,m}(t) dt
\end{cases}
\]

and

\[
Q_{k}^{n,m}(t) \sim \mathcal{N}(\alpha_{k}^{n,m}(t), \delta_{k}^{n,m}(t));
\]

(5.52)

Still, the second part of (5.51) does not possess an analytical expression, so we need to approximate it further.

**Proposition 5.5** The characteristic function of the jump part \( \phi_{j}^{n,m}(u) \) is given by:

\[
\phi_{j}^{n,m}(u) = \exp \left[ -iu \int_{0}^{T_{n}} \sum_{k=n}^{m-1} \frac{\partial S_{n,m}(0)}{\partial L_{k}(0)} \frac{L_{k}(0)}{S_{n,m}(0)} \left[ e^{\alpha_{k}(t)} + \frac{1}{2} \delta_{k}(t)^{2} - 1 \right] \tilde{\lambda}_{k}^{n,m}(t) dt \right] \\
\quad \cdot \exp \left[ \int_{0}^{T_{n}} \lambda^{n,m}(t) \left[ e^{iu\tilde{\alpha}(t)} - \frac{1}{2} u^{2} \delta(t)^{2} - 1 \right] dt \right]
\]

(5.53)
where:

\begin{align*}
\bar{\alpha}(t) & = \log(1 + \bar{\mu}(t)) - \frac{1}{2} \bar{\delta}(t)^2, \\
\bar{\delta}(t)^2 & = \log \left\{ \Gamma_{n,m}(t) + (1 + 2\bar{\mu}(t)) \right\} \\
\Gamma_{n,m}(t) & = \sum_{i=n}^{m-1} \sum_{j=n}^{m-1} \frac{\partial S_{n,m}(0)}{\partial L_i(0)} \frac{\partial S_{n,m}(0)}{\partial L_j(0)} \frac{L_i(0)}{S_{n,m}(0)} \\
& \cdot \left( e^{2\alpha_l(t)+2\delta_l(t)^2} - 2e^{\alpha_l(t)+\frac{1}{2}\delta_l(t)^2} + 1 \right) \\
l & = \max \{i,j\}, \\
\bar{\mu}(t) & = \sum_{k=n}^{m-1} \frac{\partial S_{n,m}(0)}{\partial L_k(0)} \frac{L_k(0)}{S_{n,m}(0)} \left( e^{\alpha_k(t)+\frac{1}{2}\delta_k(t)^2} - 1 \right)
\end{align*}

(5.54)

**Proof:** See Appendix 5.6.7.

Assuming the jump process is independent with the Brownian motion (the continuous part), we can write characteristic function \( \phi^{n,m}(u) \) as a product of \( \phi^{n,m}_C(u) \) and \( \phi^{n,m}_J(u) \):

\[ \phi^{n,m}(u) = \phi^{n,m}_C(u) \cdot \phi^{n,m}_J(u) \]

Along with the caplet pricing formula, the swaption price follows:

\[
Swapt(0, S_{n,m}(0), T_n, K) = B_{n,m}(0) \left\{ S_{n,m}(0) - \frac{S_{n,m}(0)}{2\pi} \int_{-\infty}^{\infty} \text{Re} \left[ e^{i\log(S_{n,m}(0)) + \alpha k} \phi^{n,m}(u + (\alpha - 1)i) \right] du \right\}
\]

(5.55)

where \( \alpha \) is the optimal shift of the integral contour.

### 5.5 Conclusion and Future Research

Various extensions, especially the introduction of local/stochastic volatility and jump processes to the LIBOR market model, offer an active research area due to the flexibility and wide use of LIBOR model in the financial industry. This chapter presents a framework for the systematic and consistent modelling of the bond price process, LIBOR forward rate process and swap rate process, with stochastic
volatility and jump processes. Since all the parameters in the model are time-dependent, it allows for simultaneous calibration to interest rate options of different maturities, with the flexibility of fitting the short-term volatility skew/smile well (thanks to the jump process). Future research areas may include finding a more robust approximation method in the model transition from forward rates to swap rates.

5.6 Appendix: Proof

5.6.1 Proof of Lemma 5.1

ZCB price $P(t, T + \delta)$ is chosen as the numeraire under this new measure $\mathcal{P}^{T+\delta}$. Thus, the Radon-Nikodym derivative with respect to the risk-neutral measure $\mathcal{Q}$ is given by:

$$
\frac{d\mathcal{P}^{T+\delta}}{d\mathcal{Q}} \bigg|_{F_t} = \frac{P(t, T + \delta)}{P(0, T + \delta)}e^{\int_0^t r(s)ds} = M(t, T + \delta) \quad (5.56)
$$

With the ZCB price process given in (5.1), we have the process for $M(t, T + \delta)$:

$$
dM(t, T + \delta) = \sqrt{V(t)}\sigma^*(t, T + \delta) \cdot dW^Q(t) + \int_E (e^{D(t,x,T+\delta)} - 1)(\mu^Q(dx, dt) - \lambda^Q(dx, t)dt) \quad (5.57)
$$

By using the Girsanov theorem, or more conveniently the ”Change of Numeraire Toolkit” (see Chapter 2 of [45]), we arrive at the Brownian motions under the new measure $\mathcal{P}^{T+\delta}$:

$$
dW^{T+\delta}(t) = dW^Q(t) - <dW^Q(t), \frac{dM(t, T + \delta)}{M(t_-, T + \delta)}> = dW^Q(t) - \sqrt{V(t)}\sigma^*(t, T + \delta)dt \quad (5.58)
$$

$$
dZ^{T+\delta}(t) = dZ^Q(t) - <dZ^Q(t), \frac{dM(t, T + \delta)}{M(t_-, T + \delta)}> = dZ^Q(t) - \sqrt{V(t)}\sigma^*(t, T + \delta) \cdot dW^Q(t) > = dZ^Q(t) + \sqrt{V(t)} \sum_{k=\eta(t)}^T \frac{\delta_k L_k(t)\gamma_k(t)}{1 + \delta_k L_k(t)} \rho_k(t)dt \quad (5.59)
$$

According to Girsanov theorem on jump processes (see Appendix 5.6.6), the new jump intensity $\lambda^{T+\delta}$ under $\mathcal{P}^{T+\delta}$ is given by:

$$
\lambda^{T+\delta}(dx, t) = e^{D(t,x,T+\delta)}\lambda^Q(dx, t) \quad (5.60)
$$
It is notable that (5.59) involves the LIBOR forward rates, which are also stochastic, thus complicates model implementation. In order to make the model analytically tractable, we make an approximation here:

$$dZ^{T+\delta}(t) \approx dZ^Q(t) + \sqrt{V(t)} \sum_{k=m(t)}^T \frac{\delta_k L_k(0) \gamma_k(t)}{1 + \delta_k L_k(0)} \rho_k(t) dt$$  \hspace{1cm} (5.61)$$

This approximation method is called "freezing coefficient", which will be used repeatedly in the manipulation of LIBOR market model.

Writing $T + \delta$ as $T_{j+1}$, and plugging (5.58)-(5.60) into (5.13), we have our assertion.

### 5.6.2 Proof of Proposition 5.2.

Annuity $B_{n,m}(t)$ is the numeraire under the annuity measure $\mathcal{P}^{n,m}$. Similar to the proof of Lemma 5.1, we see the Radon-Nikodym derivative with respect to measure $Q$ as:

$$\frac{d\mathcal{P}^{n,m}}{dQ} |_{F_t} = \frac{B_{n,m}(t)}{B_{n,m}(0) e^{\int_0^t \rho(s) ds}} = M_{n,m}(t)$$  \hspace{1cm} (5.62)$$

With the ZCB price process given in (5.1), the $M^{n,m}(t)$ process is given by:

$$\frac{dM^{n,m}(t)}{M^{n,m}(t_-)} = \sum_{k=n}^{m-1} b_{k+1}(t) \left\{ \sqrt{V(t)} \sigma^*(t, T_{k+1}) dW^Q(t) + \int_E (e^{D(t,x,T_{k+1})} - 1)(\mu(dx, dt) - \lambda^Q(dx, t) dt) \right\}$$  \hspace{1cm} (5.63)$$

Based on the assumptions we made for jump terms in (5.7) and (5.10), (5.63) becomes:

$$\frac{dM^{n,m}(t)}{M^{n,m}(t_-)} = \sum_{k=n}^{m-1} b_{k+1}(t) \sqrt{V(t)} \sigma^*(t, T_{k+1}) dW^Q(t) + \int_E \left( \sum_{k=n}^{m-1} b_{k+1}(t) \prod_{j=n(t)}^{k} \frac{1 + \delta_j L_j(t_-)}{1 + \delta_j L_j(t_-)(1 + H_j(x))} - 1 \right) \cdot (\mu(dx, dt) - \lambda^Q(dx, t) dt)$$  \hspace{1cm} (5.64)$$
By using the Girsanov theorem:

\[ dW_{n,m}(t) = dW^Q(t) - \frac{dM_{n,m}(t,T+\delta)}{M_{n,m}(t_,T+\delta)} > \]

\[ = dW^Q(t) - \sum_{k=n}^{m-1} b_{k+1}(t) \sqrt{V(t)} \sigma^*(t,T_{k+1}) dt \] (5.65)

\[ dZ_{n,m}(t) = dZ^Q(t) - \frac{dM(t,T+\delta)}{M(t_,T+\delta)} > \]

\[ = dZ^Q(t) - \sum_{k=n}^{m-1} b_{k+1}(t) \sqrt{V(t)} \sigma^*(t,T_{k+1}) \rho_k(t) dt \] (5.66)

Further approximation (using the “freezing coefficient” again) is made to arrive at (5.28).

By using the Girsanov theorem for the jump process, we obtain the following equality from (5.64):

\[ \lambda_{n,m}(dx,t) = \sum_{k=n}^{m-1} b_{k+1}(t) \prod_{j=\eta(t)}^{k} \frac{1 + \delta_j L_j(t_-)}{1 + \delta_j L_j(t_-)(1 + H_j(x))} \lambda^Q(dx,t) \] (5.67)

5.6.3 Proof of Proposition 5.3.

According to the definition of the swap rate in (5.22), firstly we write:

\[ dS_{n,m}(t) = d \left( \frac{P(t,T_n)}{B_{n,m}(t)} \right) - d \left( \frac{P(t,T_m)}{B_{n,m}(t)} \right) \] (5.68)

The process of annuity \( B_{n,M}(t) \) under risk-neutral measure \( Q \) is easily given as:

\[ dB_{n,m}(t) = \sum_{j=n}^{m-1} \delta_j dP(t,T_{j+1}) \]

\[ = B_{n,m}(t) r(t) dt + \sqrt{V(t)B_{n,m}(t)} \sum_{j=n}^{m-1} b_{j+1}(t) \sigma^*(t,T_{j+1}) \cdot dW^Q(t) \]

\[ + B_{n,m}(t_-) \sum_{j=n}^{m-1} b_{j+1}(t) \int_E \left( e^{D(t,x,T_{j+1})} - 1 \right) \]

\[ (\mu(dx,dt) - \lambda^Q(dx,t)dt) \] (5.69)
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Moving on from this, we change the measure from $Q$ to $P^{n,m}$ through the use of (5.65) and (5.67), and arrive at the process of $\frac{P(t,T_j)}{B(t)}$, $(j \in [n, m])$ under $P^{n,m}$ as:

$$
\frac{d \left( \frac{P(t,T_j)}{B(t)} \right)}{\frac{P(t,T_j)}{B(t)}} = \sqrt{V(t)} \left( \sigma^*(t, T_{j+1}) - \sum_{k=n}^{m-1} b_{k+1}(t) \sigma^*(t, T_{k+1}) \right) \cdot dW^{n,m}(t) \\
+ \int E \left( \frac{e^{D(t, x, T_j)}}{\sum_{k=n}^{m-1} b_{k+1}(t)e^{D(t, x, T_{k+1})}} - 1 \right) \lambda^{n,m}(dx, t)dt \\
+ \int E \left( \frac{1}{\sum_{k=n}^{m-1} b_{k+1}(t)e^{D(t, x, T_{k+1})}} + e^{D(t, x, T_j)} - 2 \right) \mu(dx, dt)
$$

(5.70)

The process of $S_{n,m}(t)$ in (5.33) is then obtained from (5.68) and (5.70), by writing the volatility of ZCB in the form of LIBOR forward rates: $(j \geq \eta(t))$

$$
\sigma^*(t, T_{j+1}) = - \sum_{k=\eta(t)}^{j} \frac{\delta_k L_k(t) \gamma_k(t)}{1 + \delta_k L_k(t)}.
$$

(5.71)

After some simplification, we obtain (5.34).

5.6.4 Proof of Proposition 5.4.

Generally speaking, LIBOR forward rates are not martingale under the $P^{n,m}$ measure. As proved in Proposition 5.3, if $S_{n,m}$ is a martingale under $P^{n,m}$, then the drift terms of the LIBOR forward rates are irrelevant for the swap rate process in our scenario.

From (5.35), and through the use of Ito’s lemma, we have:

$$
dS_{n,m}(t) = \sum_{j=n}^{m-1} \frac{\partial S_{n,m}(t)}{\partial L_j(t)} \sqrt{V(t)} L_j(t) \gamma_j(t)dW^{n,m}(t) + \text{JumpTerms} \tag{5.72}
$$

Consider $\frac{\partial S_{n,m}(t)}{\partial L_j(t)}$ and borrow Wu and Zhang’s results [44] here:

$$
\frac{\partial S_{n,m}(t)}{\partial L_j(t)} = \frac{\partial (\sum_{k=n}^{m-1} b_{k+1}(t)L_k(t))}{\partial L_j(t)} = b_{j+1}(t) + \sum_{k=n}^{m-1} \frac{\partial b_{k+1}(t)}{\partial L_j(t)} L_k(t) \tag{5.73}
$$
From Proposition 3.2 of [44], we have:

\[
\frac{\partial S_{n,m}(t)}{\partial L_j(t)} = b_{j+1}(t) + \sum_{k=n}^{m-1} \frac{\partial b_{k+1}(t)}{\partial L_j(t)} L_k(t)
\]

\[
= b_{j+1}(t) + \frac{\delta_j}{1 + \delta_j L_j(t)} \left( \sum_{k=n}^{j-1} b_{k+1}(t)[L_k(t) - S_{n,m}(t)] \right) \tag{5.74}
\]

for \( j \in [n, m] \).

Then, by using the LIBOR forward rate process (5.31) under \( P_{n,m} \), we have:

\[
dS_{n,m}(t) = \sum_{j=n}^{m-1} b_{j+1}(t) dL_j(t)
\]

\[
+ \sum_{j=n}^{m-1} \frac{\delta_j dL_j(t)}{1 + \delta_j L_j(t)} \sum_{k=n}^{j} b_{k+1}(t)[L_k(t) - S_{n,m}(t)] \tag{5.75}
\]

This expression is still complicated, as it involves LIBOR forward rates. Alternatively, we can assume a log-normal approximation for the swap rate in order to obtain the analytical pricing formula for swaption:

\[
\frac{dS_{n,m}(t)}{S_{n,m}(t)} = \sqrt{V(t)} \left( \sum_{k=n}^{m-1} \frac{\partial S_{n,m}(0)}{\partial L_k(0)} \frac{L_k(0)}{S_{n,m}(0)} \gamma_k(t) \right) dW_{n,m}(t)
\]

\[
+ \text{JumpTerms} \tag{5.76}
\]

where \( \frac{\partial S_{n,m}(0)}{\partial L_k(0)} \) is given in (5.74) and the jump terms are approximated as:

\[
\sum_{k=n}^{m-1} \frac{\partial S_{n,m}(0)}{\partial L_k(0)} \frac{L_k(0)}{S_{n,m}(0)} .
\]

\[
\int_\mathcal{E} \left\{ H_k(x) \mu(dx, dt) - \frac{\prod_{i=\eta(t)}^{k-1} \frac{1 + \delta_i L_i(0)}{1 + \delta_i L_i(0)(1 + H_i(x))} \lambda_{n,m}^n(dx, t) dt} \sum_{i=n}^{m-1} b_{i+1}(0) \prod_{j=\eta(t)}^{k-1} \frac{1 + \delta_j L_j(0)}{1 + \delta_j L_j(0)(1 + H_j(x))} \right\}
\]

5.6.5 The Marked Point Process

A marked point process on \((\Omega, \mathcal{F}, \mathcal{P})\) is a sequence \((T_n, X_n)_{n \geq 1}\) where:

- \((T_n)_{n \geq 1}\) is an increasing sequence of non-anticipating random times with \(T_n \to \infty\) a.s. as \(n \to \infty\).
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• \((X_n)_{n \geq 1}\) is a sequence of random variables taking values in \(E\).

• The value of \(X_n\) is revealed at \(T_n\): \(X_n\) is \(\mathcal{F}_{T_n}\) measurable.

Define a random counting measure \(\mu(dt \times dx)\) by:

\[
\mu([0,T] \times A) = \sum_{n \geq 1} 1_{T_n \leq T, X_n \in A}, \quad \forall (t, A) \in [0,T] \times E.
\] (5.77)

and let \(\lambda\) be such that:

1. For every \((\omega, t) \in \Omega \times (0,T]\), the set function \(\lambda_t(\omega, \cdot)\) is a finite Borel measure on \(E\).
2. For every \(A \in E\), the process \(\lambda_t(A)\) is \(\mathcal{P}\)-measurable.

If the equation \(E[\int_0^T Y_s \mu(ds \times A)] = E[\int_0^T Y_s \lambda_t(A)ds]\) holds for every \(A \in E\) for any non-negative \(\mathcal{P}\) measurable process \(Y\), then it is said that the marked point process \(\mu(dt, dx)\) has the \(\mathcal{P}\)-intensity \(\lambda_t(dx)\).

**Integration Theorem** (for proof, see [48]). Let \(\mu(dt, dx)\) be a \(E\)-marked point process with \(\mathcal{P}\)-intensity kernel \(\lambda_t(dx)\). Let \(\Gamma\) be a \(\mathcal{P} \otimes E\)-measurable process. It then follows that:

1. If the integrability condition \(E[\int_0^T \int_E |\Gamma_s(x)| \lambda_s(dx)ds] < \infty\) holds, then the process \(\int_0^T \int_E \Gamma_t(x) \mu(dt, dx) - \lambda_t(t, dx)dt\) is a \(\mathcal{P}\)-martingale.

2. If \(\Gamma \in \mathcal{L}(\lambda_t(dx))\), then the process \(\int_0^T \int_E \Gamma_t(x) \mu(dt, dx) - \lambda_t(t, dx)dt\) is a local \(\mathcal{P}\)-martingale.

5.6.6 The Girsanov Theorem and Ito’s Lemma on Jump Processes

**The Girsanov Theorem for Jump Processes**

(cf:[46]) 1. Let \(\Gamma\) be a predictable process and \(\Phi = \Phi(\omega, t, x)\) a strictly positive \(\mathcal{P}\)-measurable function such that for finite \(t\):

\[
\int_0^t |\Gamma_s|^2 ds < \infty,
\]

\[
\int_0^t \int_E |\Phi(s, x)| \lambda(s, dx)ds < \infty.
\] (5.78)
Define the process $L$ by:

$$
\log L_t = \int_0^t \Gamma_s dW_s - \frac{1}{2} \int_0^t |\Gamma|^2 dW_s + \int_0^t \int_E \log \Phi(s,x) \mu(ds,dx) + \int_0^t \int_E (1 - \Phi(s,x)) \nu(ds,dx) \tag{5.79}
$$

or, equivalently, by:

$$
dL_t = L_t \Gamma_t dW_t + L_t \int_E (\Phi(t,x) - 1) \mu(dt,dx) - \nu(dt,dx),
L_0 = 1.
\tag{5.80}
$$

and suppose that for all finite $t$, $E^F[L_t] = 1$, then there exists a probability measure $Q$ on $\mathcal{F}$, locally equivalent to $P$ with the Radon-Nikodym derivative:

$$
dQ \, dP|_{\mathcal{F}_t} = L_t, \tag{5.81}
$$

such that:

1. We have:
   $$
dW(t) = \Gamma_t dt + d\tilde{W}(t), \tag{5.82}
$$
   where $\tilde{W}$ is a $Q$-Wiener process.

2. The point process $\mu$ has a $Q$-intensity:
   $$
   \lambda_Q(t,dx) = \Phi(t,x) \lambda(t,dx). \tag{5.83}
   $$

2. Every probability measure $Q$ locally equivalent to $P$ has the structure above.

**Ito’s formula for Jump-Diffusion Processes**

(cf: [47]) Let $X$ be a diffusion process with jumps, defined as the sum of a drift term, a Brownian stochastic integral and a compound Poisson process:

$$
X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + \sum_{i=1}^{N_t} \triangle X_i, \tag{5.84}
$$

where $b_t$ and $\sigma_t$ are continuous non-anticipating processes with:

$$
E[\int_0^T \sigma_t^2 dt] < \infty. \tag{5.85}
$$
Then, for any $C^{1,2}$ function $f : ([0, T] \times \mathbb{R}) \rightarrow \mathbb{R}$, the process $Y_t = f(t, X_t)$ can be represented as:

$$f(t, X_t) - f(0, X_0) = \int_0^t \left( \frac{\partial f}{\partial s}(s, X_s) + \frac{\partial f}{\partial x}(s, X_s) b_s \right) ds + \frac{1}{2} \int_0^t \sigma_s^2 \frac{\partial^2 f}{\partial x^2}(s, X_s) ds + \int_0^t \frac{\partial f}{\partial x}(s, X_s) \sigma_s dW_s + \sum_{i \geq 1, T_i \leq t} [f(X_{T_i} + \triangle X_i) - f(X_{T_i})].$$

(5.86)

In differential notation:

$$dY_t = \frac{\partial f}{\partial t}(t, X_t) dt + \frac{\partial f}{\partial x}(t, X_t) b_t dt + \frac{1}{2} \sigma_t^2 \frac{\partial^2 f}{\partial x^2}(t, X_t) dt + \frac{\partial f}{\partial x}(t, X_t) \sigma_t dW_t + [f(X_{t-} + \triangle X_t) - f(X_{t-})].$$

(5.87)

A more generalized form will be Ito's formula for Levy processes: finite variation jumps:

Let $X$ be a finite variation Levy process with characteristic exponent:

$$\psi_X(u) = iubu + \int_{-\infty}^{\infty} (e^{iuy} - 1) \nu(dy),$$

where the Levy measure $\nu$ verifies $\int |y| \nu(dy) < \infty$. Then for any $C^{1,1}$ function $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$,

$$f(t, X_t) - f(0, X_0) = \int_0^t \left[ \frac{\partial f}{\partial s}(s, X_{s-}) + b \frac{\partial f}{\partial x}(s, X_{s-}) \right] ds + \sum_{0 \leq s \leq t} [f(X_{s-} + \Delta X_s) - f(X_{s-})].$$

If $f$ and its first derivative in $x$ are bounded, then $Y_t = f(t, Y_t)$ is the sum of a martingale part given by

$$\int_0^t \int_{-\infty}^{\infty} [f(s, X_{s-} + y) - f(s, X_{s-})] \tilde{J}_X(ds, dy)$$

and a "drift" part given by:

$$\int_0^t \left[ \frac{\partial f}{\partial s}(s, X_{s-}) + b \frac{\partial f}{\partial x}(s, X_{s-}) \right] ds + \int_0^t ds \int_{\mathbb{R}} \nu(dy) [f(s, X_{s-} + y) - f(s, X_{s-})].$$
5.6.7 The Derivation of Characteristic Functions

Caplet

In order to calculate \( \phi_{j+1}^{+1}(u) \), we need to consider \( \log L_j \) under the \( \mathcal{P}^{j+1} \) measure first. Using Ito’s lemma on jump processes:

\[
d\log L_j(t) = - \left[ \int_E H_j(x) \lambda^{j+1}(dx, t) + \frac{1}{2} V(t) \gamma_j^2(t) \right] dt + \sqrt{V(t)} \gamma_j(t) \cdot dW^{j+1}(t) + \int_E \log(1 + H_j(x)) \mu(dt, dx)
\]

it follows that:

\[
\log \left( \frac{L_j(t)}{L_j(0)} \right) = - \int_0^t \left[ \int_E H_j(x) \lambda^{j+1}(dx, s) + \frac{1}{2} V(s) \gamma_j^2(s) \right] ds + \int_0^t \sqrt{V(s)} \gamma_j(s) \cdot dW^{j+1}(s) + \int_0^t \int_E \log(1 + H_j(x)) \mu(ds, dx)
\]

We now write \( \phi_{j+1}(u) = \phi_{C,j+1}^{+1}(u) \phi_{j+1}^{+1}(u) \), where \( \phi_{C,j+1}^{+1}(u) \) is the characteristic function (CF) of the stochastic volatility (continuous) part:

\[
\phi_{C,j+1}^{+1}(u) = \mathbb{E}_0^{j+1} \left[ e^{-iu \frac{1}{2} \int_0^T V(t) \gamma_j(t) dt + iu \int_0^T \sqrt{V(t)} \gamma_j(t) dW^{j+1}(t)} \right]
\]

with

\[
V(t) = V(0) + \kappa \int_0^t [\theta - \eta^{j+1}(s)V(s)] ds + \eta \int_0^t \sqrt{V(u)} dZ^{j+1}
\]

Writing \( dW^{j+1}(t) = \rho_j(t) dZ^{j+1}(t) + \sqrt{1 - \rho_j(t)^2} d\tilde{W}^{j+1}(t) \) and using the Feynman-Kac formula, we can express the expectation in (5.90) as:

\[
-s_1 V \phi = \frac{\partial \phi}{\partial t} + \kappa(\theta - \eta^{j+1} V) \frac{\partial \phi}{\partial V} + \frac{1}{2} \eta^2 V \frac{\partial^2 \phi}{\partial V^2}
\]

with \( v = V(0) \). Assuming its solution is given by an exponential affine form \( e^{A(t) + B(t)V(t)} \), with \( A(T) = B(T) = 0 \) substituting into the above PDE, we obtain a set of ODEs:

\[
A'(t) = -\kappa \theta B(t),
\]

\[
B'(t) = -\frac{1}{2} \eta^2 B^2(t) + B(t)(\eta^{j+1} \kappa - iu \rho_j \eta \gamma) + \frac{1}{2} \gamma_j^2 (iu + u^2)
\]
For constant coefficients, Riccati-type ODEs can be solved explicitly (cf: [92]), while for time-dependent coefficients these can be numerically solved via the Runge-Kutta scheme (cf: [94]). Indeed, we can still obtain an analytical solution if the coefficients are piece-wise constant, which is given below: (see also Proposition 2.2 of [44])

\[
A(t) = A(T_k) + \frac{\kappa \theta}{\eta^2} \left\{ (a_k + d_k)(T_k - t) - 2\log\left[ \frac{1 - g_k e^{d_k(T_k-t)}}{1 - g_k} \right] \right\} \\
B(t) = B(T_k) + \frac{(a_k + d_k - \eta^2 B(T_k))(1 - e^{d_k(T_k-t)})}{\eta^2(1 - g_k e^{d_k(T_k-t)})} 
\]

(5.95)

for \( t \in [T_{k-1}, T_k) \), \( k = j, ..., 1 \) with:

\[
a_k = \kappa \eta^{j+1} - iu \rho_j(T_k) \eta \| \gamma_j(T_k) \| , \\
d_k = \sqrt{a^2 + \| \gamma_j(T_k) \|^2 \eta^2 (u^2 + iu)} , \\
g_k = \frac{a_k + d_k - \eta^2 B(T_k)}{a_k - d_k - \eta^2 B(T_k)} 
\]

(5.96)

For the jump-diffusion part, as we see from (5.89), we have:

\[
\phi_j^{j+1}(u) = E^{j+1}_0 \left[ \exp(-iu \int_0^{T_j} \int_E H_j(x) \lambda^{j+1}(dx, s) ds + iu \int_0^{T_j} \int_E \log(1 + H_j(x)) \mu(ds, dx)) \right] 
\]

(5.97)

In the case of the Poisson jump process with a log-normal jump magnitude, as given in (5.40), \( \phi_j^{j+1}(u) \) becomes:

\[
\phi_j^{j+1}(u) = \exp\left[ -iu \int_0^{T_j} \lambda_j(t) \left( e^{a(t) + \frac{1}{2} \delta(t)^2} - 1 \right) dt \right] \cdot \exp\left[ \int_0^{T_j} \lambda_j(t) \left( e^{iu \alpha(t) - \frac{1}{2} u^2 \delta(t)^2} - 1 \right) dt \right] 
\]

(5.98)

Here, we use the exponential formula for Poisson random measures from Chapter 3 of ([47]), which states:

Let \( M \) be a Poisson random measure with intensity parameter \( \mu \). As a result,
the following formula holds for every measurable set $B$ such that $\mu(B) < \infty$ and for all functions $f$ such that $\int_B e^{f(x)} \mu(dx) < \infty$:

$$
E[e^{\int_B f(x) \mu(dx)}] = e^{\int_B (e^{f(x)} - 1) \mu(dx)}
$$

This is how we arrive at the second part in the RHS of (5.98).

The assumption of piece-wise constant coefficients is commonly used in practice. In the case of piece-wise constant $\lambda_j(t)$, $\alpha(t)$, $\delta(t)$, (5.98) becomes:

$$
\phi_{j+1}(u) = \exp \{ -iu \sum_{i=1}^j \delta_i \lambda_i^J (e^{\alpha_i + \frac{1}{2} \delta_i^2} - 1) + \sum_{i=1}^j \delta_i \lambda_i^J [e^{iu \alpha_i - \frac{1}{2} u^2 \delta_i^2} - 1] \} \tag{5.99}
$$

where $X(t) = X_i$ for $t \in (T_{i-1}, T_i]$, $X \in \{\lambda, \alpha, \delta\}$. Beware that in the above formula the first $\delta$ denotes the LIBOR tenor while the one in the exponent is the standard deviation of the log-normal jump size.

**Swaption**

**Proof of Proposition 5.5:** From (5.51) with the Merton jump process, it is easily seen that:

$$
\phi_{j}^{n,m}(u) = \exp \left[ -iu \int_0^{T_n} \sum_{k=1}^{m-1} \frac{\partial S_{n,m}(0)}{\partial L_k(0)} \frac{L_k(0)}{S_{n,m}(0)} \left[ e^{\alpha_k^{n,m}(t)} + \frac{1}{2} \delta_k^{n,m}(t)^2 - 1 \right] \lambda_k^{n,m}(t) dt \right] \cdot E_{0}^{n,m} \left\{ \exp[iu \int_0^{T_n} \log(1 + \sum_{k=1}^{m-1} \frac{\partial S_{n,m}(0)}{\partial L_k(0)} \frac{L_k(0)}{S_{n,m}(0)} (e^{Q_k(t)} - 1))dJ_k(t)] \right\} \tag{5.100}
$$

Since the expectation part is too complicated to find an analytical expression, we need further approximation using the technique of *Moment Matching*. The idea is that we want to find an effective jump process:

$$
(e^{Q(t)} - 1)dJ(t) \tag{5.101}
$$

where:

$$
\tilde{Q}(t) \sim N(\tilde{\alpha}(t), \tilde{\delta}(t))
$$
to approximate the jump sum:

$$\sum_{k=n}^{m-1} \frac{\partial S_{n,m}(0)}{\partial L_k(0)} \frac{L_k(0)}{S_{n,m}(0)} (e^{Q_k(t)} - 1)dJ_k(t) \tag{5.102}$$

We consider matching the first two moments of jump size when the jump occurs and the jump intensity remains unchanged:

$$E^{n,m}[e^{\bar{Q}(\tau)} - 1 \mid \tau, S_{n,m}(\tau_-)] =$$

$$E^{n,m} \left[ \sum_{k=n}^{m-1} \frac{\partial S_{n,m}(0)}{\partial L_k(0)} \frac{L_k(0)}{S_{n,m}(0)} (e^{Q_k(\tau)} - 1) \mid \tau, S_{n,m}(\tau_-) \right]$$

and:

$$E^{n,m} \left[ \left( e^{Q(\tau)} - 1 \right)^2 \mid \tau, S_{n,m}(\tau_-) \right] =$$

$$E^{n,m} \left[ \left( \sum_{k=n}^{m-1} \frac{\partial S_{n,m}(0)}{\partial L_k(0)} \frac{L_k(0)}{S_{n,m}(0)} (e^{Q_k(\tau)} - 1) \right)^2 \mid \tau, S_{n,m}(\tau_-) \right]$$

where $\tau$ denotes the jump time.

This gives us:

$$\bar{\mu}(t) \equiv E^{n,m}[e^{\bar{Q}(t)} - 1]$$

$$= e^{\bar{\alpha}(t) + \frac{1}{2}\bar{\delta}(t)^2} - 1$$

$$= \sum_{k=n}^{m-1} \frac{\partial S_{n,m}(0)}{\partial L_k(0)} \frac{L_k(0)}{S_{n,m}(0)} \left( e^{\alpha_i(t) + \frac{1}{2}\delta_i(t)^2} - 1 \right)$$

and:

$$e^{2\bar{\alpha}(t) + 2\bar{\delta}(t)^2} - 2e^{\bar{\alpha}(t) + \frac{1}{2}\bar{\delta}(t)^2} + 1 =$$

$$\sum_{i=n}^{m-1} \sum_{j=n}^{m-1} \frac{\partial S_{n,m}(0)}{\partial L_i(0)} \frac{L_i(0)}{S_{n,m}(0)} \frac{\partial S_{n,m}(0)}{\partial L_j(0)} \frac{L_j(0)}{S_{n,m}(0)}$$

$$\cdot \left( e^{2\alpha_i(t) + 2\delta_i(t)^2} - 2e^{\alpha_i(t) + \frac{1}{2}\delta_i(t)^2} + 1 \right)$$

$$l = \max\{i, j\},$$

Here, we assume at jump time $e^{Q_i(\tau)} - 1 = e^{Q_j(\tau)} - 1$ for $i < j$, which means that when a forward rate jumps, all the forward rates maturing earlier also jump, with the same jump-size mean and standard deviation.

Consequently, $\bar{\alpha}(t)$ and $\bar{\delta}(t)$ follow from the above two equalities.
BIBLIOGRAPHY


