Higher genus hyperelliptic reductions of the Benney equations

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Abstract. It was shown by Gibbons and Tsarev (1996 Phys. Lett. A 211 19, 1999 Phys. Lett. A 258 263) that $N$-parameter reductions of the Benney equations correspond to $N$-parameter families of conformal maps. Here, we consider a specific set of these, the hyperelliptic reductions. The mapping function for this is calculated explicitly by inverting a second-kind Abelian integral on the stratum $\Theta_1$ of the Jacobi variety of a genus $g \geq 3$ hyperelliptic curve. This is done using a method based on the result of Jorgenson (1992 Israel Journal of Mathematics 77 273).

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1. Introduction

1.1. Reductions of the Benney Moment Equations

The Benney equations [3] are an example of an infinite system of hydrodynamic type. These can be written as a Vlasov equation [7], [15]

$$\frac{\partial f}{\partial t_2} + p \frac{\partial f}{\partial x} - \frac{\partial A^0}{\partial x} \frac{\partial f}{\partial p} = 0.$$  

Here $f = f(x, p, t)$ is a distribution function and the moments are defined by

$$A^n = \int_{-\infty}^{\infty} p^n f dp.$$  

Benney showed that this system has infinitely many conserved densities, polynomial in the moments $A^n$.

Following [14] and [1], we will now consider reductions of the moment equations; that is the case where only a finite number, $N$, of the $A^n$ are independent. Here, the moment equations can be reduced to a diagonal system of hydrodynamic type with $N$ Riemann invariants, $\hat{\lambda}_i$ say, dependent on $N$ characteristic speeds, $\hat{p}_i$. We will assume that the characteristic speeds are real and distinct.

It was shown by Tsarev and one of the authors that in such a case the reductions correspond to $N$–parameter families of conformal mappings of slit domains. For details
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of the properties of these maps and the general construction of such a domain see [8] and [9]. We will now consider a specific set of these reductions which we will call the hyperelliptic reductions.

1.2. Hyperelliptic reductions

For this set of reductions the conformal mapping \( \lambda(p) : \Gamma_1 \rightarrow \Gamma_2 \) is defined as follows. Let \( \Gamma_1 \) be the upper half \( p \)-plane with \( 3n \) real points marked on it, \( p_i (i = 1, \ldots, 2n) \) and the set of characteristic speeds \( \hat{p}_j (j = 1, \ldots, n) \). These satisfy

\[
p_1 < \hat{p}_2 < p_3 < p_4 < \hat{p}_3 < p_5 < \cdots < p_{2n-1} < \hat{p}_n < p_{2n}.
\]

The domain \( \Gamma_2 \) is the upper half \( \lambda \)-plane with \( n \) vertical slits going from the fixed real points \( \lambda_0^i \) to the variable points \( \hat{\lambda}_i (i = 1, \ldots, n) \). Here, \( \hat{\lambda}_i \) is the Riemann invariant associated with the characteristic speed \( \hat{p}_i \) and it satisfies the relation

\[
\text{Re} \left( \hat{\lambda}_i \right) = \lambda_0^i.
\]

We now impose the conditions

\[
\lambda(p) = p + O \left( \frac{1}{p} \right) \quad \text{as} \quad p \rightarrow \infty
\]

and

\[
\lambda(p_{2i-1}) = \lambda(p_{2i}) = \lambda_0^i \quad (i = 1, \ldots, n).
\]

It follows that \( \lambda(p) \) is a function of \( n \) independent parameters which may be taken to be \( \text{Im}(\hat{\lambda}_i) (i = 1, \ldots, n) \), the varying heights of the slits \( \dagger \) and that \( \Gamma_2 \) is a polygonal domain. The map \( p \rightarrow \lambda(p) \) is thus of Schwarz-Christoffel type:

\[
\lambda(p) = p + \int_{\infty}^{p} \left[ \varphi(p') - 1 \right] \, dp'
\]

where \( \varphi(p) \) is given by

\[
\varphi(p) = \frac{\prod_{i=1}^{n} (p - \hat{p}_i)}{\sqrt{\prod_{i=1}^{2n} (p - p_i)}}.
\]

\[
\begin{align*}
\hat{p}_1 & \quad p_1 \\
\hat{p}_2 & \quad p_2 \\
\hat{p}_3 & \quad p_3 \\
\hat{p}_4 & \quad p_4 \\
\vdots & \quad \vdots \\
\hat{p}_n & \quad p_{2n-1} \\
p_{2n} & \quad p_{2n}
\end{align*}
\]

Figure 1. (The \( n \) parameter reduction) The \( p \)-plane with \( n \) branch cuts.

\( \dagger \) Note that since \( \text{Im}(\lambda) \geq 0 \, \forall \, \lambda \) and the distribution function \( f = -\pi \text{Im}(\lambda) \), the distribution function is negative.
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One of the conditions in (1) and (2) may be replaced by the constraint that the residue of \( \varphi(p) \), as \( p \to \infty \) on either sheet, is zero. This provides a relation between the set of points \( p_i \) and the set of characteristic speeds \( \hat{p_j} \). Rewriting

\[
\varphi(p) = \frac{p^n - \alpha_{n-1} p^{n-1} - \alpha_{n-2} p^{n-2} - \cdots - \alpha_1 p - \alpha_0}{\sqrt{\prod_{i=1}^{2n} (p - p_i)}},
\]

we find that the expansion of \( \varphi(p) \) near infinity is

\[
1 + \left(\frac{1}{2} \sum_{i=1}^{2n} p_i - \frac{\alpha_{n-1}}{p}\right) + O\left(\frac{1}{p^2}\right).
\]

The condition on the residue is therefore satisfied when

\[
\alpha_{n-1} = \frac{1}{2} \sum_{i=1}^{2n} p_i,
\]

that is,

\[
\sum_{i=1}^{n} \hat{p_i} = \frac{1}{2} \sum_{i=1}^{2n} p_i. \tag{4}
\]

It follows that \( \varphi(p) \, dp \) is a second kind Abelian differential on the Riemann surface

\[
R_g = \left\{ (p, v) : v^2 = \prod_{i=1}^{2n} (p - p_i) \right\}
\]

where \( g = n - 1 \). That is, the differential 1-form \( \varphi(p) \, dp \) is meromorphic on \( R_g \) with zero residue at each singular point.

This surface may be constructed from two copies of the complex \( p- \)plane joined along the closed intervals

\[
[p_{2i-1}, p_{2i}] \quad (i = 1, 2, \ldots, g + 1).
\]

A homology basis \((a_1, a_2, \ldots, a_g; b_1, b_2, \ldots, b_g)\) for \( R_g \) is given in figure 3.

The first three examples of these maps, \( g = 0, 1, 2 \), have been worked out in detail. For \( g = 0 \) the mapping may be calculated directly. The case of the \( n = 2 \) elliptic reduction was evaluated in [14] by Yu and Gibbons. The \( n = 3 \) genus 2 hyperelliptic reduction was studied in [1] by the authors. We now consider the case for \( g \geq 3 \). All such maps, once known explicitly, correspond to reductions of Benney’s equations to systems

Figure 2. The \( \lambda \)-plane associated with figure 1.
of hydrodynamic type with finitely many Riemann invariants. Tsarev’s generalised
hodograph transformation [13] leads to solutions of these, in terms of the solution of an
over-determined system of linear equations. The construction of \( n \)-parameter families
of such maps is thus an important step towards understanding the solutions of these
equations.

2. Transformation of the integral

Following [1], the integral we need to evaluate is (3):

\[
\lambda(p) = p + \int_{-\infty}^{\infty} \left[ \frac{\prod_{i=1}^{g+1} (p' - \hat{p}_i)}{\sqrt{\prod_{i=1}^{2g^2+2} (p' - p_i)}} - 1 \right] \, dp'.
\]

Setting \( p = p_{2g+2} - 1/t \) in the integrand \( (\varphi(p) - 1) \, dp \), we find

\[
(\varphi(p) - 1) \, dp = \frac{\left( A_{g+1} t^{g+1} + A_g t^g + \cdots + A_2 t^2 + A_1 t + (-1)^{g+1} \right)}{\sqrt{\prod_{i=1}^{2g^2+2} [(p_{2g+2} - p_i) t - 1]}} \, \frac{dt}{t^2}
\]

for some constants \( A_i \) \((i = 1, 2, \ldots, g + 1)\). We note here that

\[
A_1 = (-1)^g \sum_{i=1}^{g+1} (p_{2g+2} - \hat{p}_i).
\]

This may be expressed in terms of just the \( p_i \) using identity (4):

\[
A_1 = \frac{(-1)^g}{2} \sum_{i=1}^{2g+1} (p_{2g+2} - p_i).
\]

If we now remove the constant imaginary factor

\[
k = \left( \frac{-4}{\prod_{i=1}^{2g+1} (p_{2g+2} - p_i)} \right)^{1/2}
\]
from (5), then we obtain a standardized form for the irrational denominator,
\[
\varphi(p) \, dp = k \left( \frac{A_{g+1} t^{g+1} + A_g t^g + \cdots + A_2 t^2 + A_1 t + (-1)^{g+1}}{s} \right) \frac{dt}{t^2}
\]
\[
= k \left( A_{g+1} t^{g-1} + A_g t^{g-2} \cdots + A_2 + \frac{A_1}{t} + \frac{(-1)^{g+1}}{t^2} \right) \frac{dt}{s}
\]
(7)

where
\[
s^2 = -k^2 + k^2 \sum_{i=1}^{2g+1} (p_{2g+2} - p_i) t + \cdots + \mu_{2g} t^{2g} + 4t^{2g+1}
\]
\[
= \mu_0 + \mu_1 t + \cdots + \mu_{2g} t^{2g} + 4t^{2g+1}.
\]
(8)

The term
\[
\varphi_1(p) \, dp = k \left( A_{g+1} t^{g-1} + A_g t^{g-2} + \cdots + A_2 \right) \frac{dt}{s}
\]

in (7) may be evaluated directly since the set
\[
du_i = t^{i-1} \frac{dt}{s} \quad (i = 1, 2, \ldots, g)
\]
forms a basis of holomorphic Abelian differentials. The last two terms in \(\varphi(p) \, dp\) can be rewritten using (6) and the definitions of \(\mu_0\) and \(\mu_1\) in (8). We have
\[
\varphi_2(p) \, dp = k \left[ \frac{(-1)^{g+1}}{t^2} + \frac{A_1}{t} \right] \frac{dt}{s}
\]
\[
= (-1)^{g+1} k \left[ \frac{1}{t^2} - \frac{1}{2} \left( \sum_{i=1}^{2g+1} (p_{2g+2} - p_i) \right) \frac{1}{t} \right] \frac{dt}{s}
\]
\[
= (-1)^{g+1} k \left[ \frac{1}{t^2} + \frac{1}{2} \frac{\mu_1 \mu_0}{\mu_0} \right] \frac{dt}{s}.
\]
(9)

This is a second kind differential on \(R_g\). As in the genus 2 case, we can evaluate \(\varphi_2(p) \, dp\) using a restriction of the Jacobi inversion theorem to a one complex dimensional subspace of the Jacobi variety, the one-dimensional stratum of the theta divisor, \(\Theta_1\).

3. The \(\Theta\) divisor

Following Enolski \[4, 5\], let \(R_g(s, t)\) be the hyperelliptic curve where \(s\) and \(t\) satisfy
\[
s^2 = 4 \prod_{i=1}^{2g+1} (t - t_i) = \sum_{i=0}^{2g} \mu_i t^i + 4t^{2g+1}.
\]

We define a set of holomorphic and their associated set of second kind differentials on \(R_g\) to be, respectively,
\[
du_i = t^{i-1} \frac{dt}{s} \quad (i = 1, 2, \ldots, g)
\]
(10)

and
\[
dr_i = \sum_{k=i}^{2g+1-i} (1 + k - i) \mu_{1+i+k} \frac{t^k dt}{4s} \quad (i = 1, 2, \ldots, g).
\]
(11)
From the period integrals of these differentials we form the matrices $\omega, \omega', \eta, \eta'$:

$$
2\omega = \left( \oint_{a_i} du_j \right) \quad 2\omega' = \left( \oint_{b_i} du_j \right)
$$

$$
2\eta = \left( -\oint_{a_i} dr_j \right) \quad 2\eta' = \left( -\oint_{b_i} dr_j \right) \quad (i, j = 1, 2, \ldots, g).
$$

These matrices satisfy the generalized Legendre relation

$$
\left( \begin{array}{cc}
\omega & \omega' \\
\eta & \eta'
\end{array} \right) \left( \begin{array}{cc}
0 & -I_g \\
I_g & 0
\end{array} \right) \left( \begin{array}{cc}
\omega & \omega' \\
\eta & \eta'
\end{array} \right)^T = -\frac{i\pi}{2} \left( \begin{array}{cc}
0 & -I_g \\
I_g & 0
\end{array} \right),
$$

where $I_g$ is the $g \times g$ identity matrix.

Letting $\Lambda = 2\omega \oplus 2\omega'$ be the lattice generated by the periods of the holomorphic differentials, the Jacobi variety, $\text{Jac}(R_g)$, is the $g$-dimensional complex torus $\mathbb{C}^g/\Lambda$. The Jacobi variety can be subdivided into $k$-dimensional strata, $\Theta_k$, defined by

$$
\Theta_k = \sum_{i=1}^{k} \int_{(t_i,s_i)}^{(t_0,s_0)} du + 2\omega K_{(t_0,s_0)} \quad (k = 1, 2, \ldots, g)
$$

where $K_{(t_0,s_0)}$ is the vector of Riemann constants with base point $(t_0,s_0)$. These have the structure $\text{Jac}(R_g) = \Theta_g \supset \Theta_{g-1} \supset \cdots \Theta_2 \supset \Theta_1$. Such stratifications have been studied by Ōnishi [12] and others.

The Abel map, $\mathfrak{A} : R_g \to \text{Jac}(R_g)$, is given by $u(z)$:

$$
\mathfrak{A}(D) = \sum_{i=1}^{M} n_i \int_{z_0}^{z} du_i, \quad (i = 1, 2, \ldots, g)
$$

where the $u_i(z)$ are taken modulo $\Lambda$ and the base point $z_0 = (t_0,s_0)$ is any fixed point in $R_g$. These create a one-dimensional image of the hyperelliptic curve in the Jacobi variety. For the inversion theorem we require an extension of this map to a set of points.

**Definition 3.1** A divisor $\mathcal{D}$ on the Riemann surface $R_g$ is defined by the finite formal sum

$$
\mathcal{D} = \sum_{i=1}^{M} n_i z_i
$$

where $n_i \in \mathbb{Z}$ and $z_i = (s_i, t_i) \in R_g$.

We define the Abel mapping of $\mathcal{D}$ into $\text{Jac}(R_g)$ by

$$
\mathfrak{A}(\mathcal{D}) = \sum_{i=1}^{M} n_i \int_{z_0}^{z_i} du \mod \Lambda.
$$

The lower limit of integration, here the point $z_0$, is called the base point of the Abel map. From now we shall set this to be $(\infty, \infty)$. 
3.1. Hyperelliptic functions

**Definition 3.2** The theta function is defined by the Fourier series
\[
\theta((2\omega)^{-1}u) = \sum_{m \in \mathbb{Z}} \exp \left\{ i\pi \left[ m^T \tau m + m^T (\omega^{-1} u) \right] \right\},
\]
where \( \tau = \omega^{-1} \omega' \) is a symmetric matrix with positive definite imaginary part.

One important property of this function is that it is zero when \( u = 2\omega K' \), the vector of Riemann constants associated with the point \((\infty, \infty)\). For further properties see [4].

From the \( \theta \)-function we define the Kleinian \( \sigma \)-function of the curve \( R_g \) to be
\[
\sigma(u) = C \exp (u^T \chi u) \theta((2\omega)^{-1}u - K)
\]
where
\[
C = \sqrt{\frac{\pi^3}{\det 2\omega} \left( \prod_{1 \leq i < j \leq 2g+1} (t_i - t_j) \right)}^{\frac{1}{4}}
\]
and \( \chi = \eta (2\omega)^{-1} \) is a symmetric matrix.

In analogy to the Weierstrass \( \wp \)-function, the Kleinian \( \wp \)-function is defined as [4]
\[
\varphi_{ij} = -\frac{\partial^2}{\partial u_i \partial u_j} \ln [\sigma(u)] = \left( \sigma_i \sigma_j - \sigma_{ij} \sigma \right) (u)
\]
where

\[
\sigma_i = \frac{\partial}{\partial u_i} \sigma(u), \quad \sigma_{ij} = -\frac{\partial^2}{\partial u_j \partial u_i} \sigma(u).
\]

Higher logarithmic derivatives of \( \sigma \) are expressed similarly. For example
\[
\varphi_{ijkl} = -\frac{\partial^4}{\partial u_i \partial u_j \partial u_k \partial u_l} \ln [\sigma(u)].
\]

3.2. Jacobi Inversion formula

**Theorem 1** (Jacobi inversion theorem) [4] The Abel preimage of the point \( u \in \text{Jac}(R_g) \) is given by the set \( S = \{(t_1, s_1), (t_2, s_2), \ldots, (t_g, s_g)\} \in (R_g)^g \), where \( t_k \) are the zeros of the polynomial
\[
P(t; u) = t^g - t^{g-1} \varphi_{g,g}(u) - t^{g-2} \varphi_{g,g-1}(u) - \ldots - \varphi_{g,1}(u)
\]
and the \( s_k \) are given by
\[
s_k = -\frac{\partial P(t; u)}{\partial u_g} \bigg|_{t = t_k}.
\]

For the integral of the differential (9), we need the preimage of \( u \) when the points \( t_i \to \infty \) \( (i = 2, \ldots, g) \). That is, for the case when \( S = \{(t_1, s_1)\} \) and so \( u \in \Theta_1 \):
\[
\mathcal{A}(S) = \int_{t_1}^{\infty} du.
\]
This relation has been calculated from the results of Jorgenson [11] by Enolski (see Appendix A). We obtain
\[ t_1 = -\frac{\sigma_1(u)}{\sigma_2(u)} \bigg|_{u \in \Theta_1} \tag{12} \]
where the one-dimensional stratum \( \Theta_1 \) may be defined as
\[ \Theta_1 = \{ u : \sigma_1(u) = 0, \sigma_k(u) = 0 \ (k = 3, \ldots, g) \} \].
This useful result (12) was first given by Grant in [10].

4. Evaluation of the integral
We now further transform the integrand \((\varphi_1(p) + \varphi_2(p)) \, dp\) using the substitution
\[ t = (-\sigma_1/\sigma_2)(u) \] (12) and the definitions of the holomorphic differentials, \(du_i (i = 1, 2, \ldots, g)\) (10).

| Table 1. A list of branch points \((p_i)\) and poles \((\infty_\pm)\) of \(\lambda(p)\) with the corresponding points in the \(t\) and \(u\) variables. |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \((p)\)         | \(p_1\)         | \(p_2\)         | \(p_{2g+1}\)    | \(p_{2g+2}\)    | \(\infty_\pm\)  |
| \((t)\)         | \(t_1\)         | \(t_2\)         | \(t_{2g+1}\)    | \(\infty\)      | \(0_\pm\)       |
| \((u)\)         | \(u_1\)         | \(u_2\)         | \(u_{2g+1}\)    | \(0\)           | \(\pm u_0\)     |

Lemma 1 Let \( t = (-\sigma_1/\sigma_2)(u) \) where \( u \in \Theta_1 \) and define \(du_i = t^{-1}dt/s\), a set of holomorphic differentials on \(R_g\). Then
\[ \varphi(p) \, dp = k \left( A^T \cdot du \right) + (-1)^{g+1} k \left( \frac{\sigma_2^2}{\sigma_1^2}(u) - \frac{1}{2} \frac{\mu_1}{\mu_0} \frac{\sigma_2}{\sigma_1}(u) \right) \frac{dt}{s} \]
where \(A^T = (A_2, A_3, \ldots, A_{g+1})\).

The term
\[ \varphi_2(u) \, du_1 = \left( \frac{\sigma_2^2}{\sigma_1^2}(u) - \frac{1}{2} \frac{\mu_1}{\mu_0} \frac{\sigma_2}{\sigma_1}(u) \right) du_1 \]
is a second kind differential with a pole of order 2 at \( u = \pm u_0 \) (see Table 1). This can be verified as follows.

Since \( u_0 \) is a regular point on the hyperelliptic curve \(R_g\), we can evaluate the expansion of \(\varphi_2\) near \(u_0\) in terms of the local parameter \(t\). Setting \(v_k = e_k^T \cdot (u - u_0)\) where \((e_k)_j = \delta_{kj}\), we have
\[ v_k = \int_{-\infty}^{t} du_k - \int_{-\infty}^{0} du_k \]
\[ = \int_{0}^{t} \int_{0}^{t} \frac{t^k - 1}{\sqrt{4t^{2g+1} + \mu_{2g}t^{2g} + \ldots + \mu_1 t + \mu_0}} \, dt.\]
These expansions may be simplified by using the substitutions for $\sigma$ (using (13)). The higher genus hyperelliptic reductions of the Benney equations

This gives

$$v_k = \left( \frac{1}{k} \right)^2 \left( \frac{1}{\sqrt{\mu_0}} \right)^2 t^k - \left( \frac{1}{2(k + 1)} \right) \frac{\mu_1}{\mu_0^2} t^{k+1} + O(t^{k+2}) \quad (k = 1, 2, \ldots, g)$$

and so for $k > 1$

$$v_k = \left( \frac{1}{k} \mu_0^{(k-1)/2} \right) v_1^k + O(v_1^{k+1}). \quad (13)$$

The Taylor series of $\varphi_2$ near $u_0$ can thus be expressed in terms of the single parameter $v_1 = e_1^T \cdot (u - u_0)$. We have

$$\frac{\sigma_2}{\sigma_1^2} (u_0 - (u_0 - u)) = \frac{(\sigma_2) + (\sigma_{12}) v_1 + \cdots}{(\sigma_{11}) v_1 + \cdots} = \left( \frac{\sigma_2}{\sigma_{11}} \right) v_1^{-1} + O(1)$$

and

$$\frac{\sigma_2^2}{\sigma_1^2} (u_0 - (u_0 - u)) = \frac{\sigma_2^2 + (2 \sigma_{12} v_1 + \cdots)}{\sigma_{11}^2 v_1^2 + (\sigma_{11} \sigma_{111}) v_1^3 + (2 \sigma_{11} \sigma_{12}) v_1 v_2 + \cdots}$$

$$= \left( \frac{\sigma_2^2}{\sigma_{11}^2} \right) v_1^{-2} + \left( \frac{2 \sigma_{12} \sigma_{111}}{\sigma_{11}^3} - \frac{2 \sigma_2^2 \sigma_{11}}{\sigma_{11}^3} - \sqrt{\mu_0} \frac{\sigma_2^2 \sigma_{12}}{\sigma_{11}^3} \right) v_1^{-1} + O(1)$$

(using (13)).

These expansions may be simplified by using the substitutions for $\sigma_{11}(u_0)$ and $\sigma_{111}(u_0)$ calculated in Appendix B. This gives

$$\left( \frac{\sigma_2^2}{\sigma_{11}^2} - \frac{1}{2} \frac{\mu_1}{\mu_0} \frac{\sigma_2}{\sigma_{11}} \right) (u_0 - (u_0 - u)) = \left( \frac{1}{\mu_0} \right) v_1^{-2} + O(1) \quad (\forall g \geq 3). \quad (14)$$

In analogy to the genus 2 case, we now consider the function

$$\Psi(u) = -\frac{1}{\mu_0} \frac{\sigma_{11}}{\sigma_1^2} (u)$$

for $u \in \Theta_1$. Since $d u_i = (-\sigma_i / \sigma_2) (i-1) du_1$, the derivative of $\Psi$ with respect to $u_1$ along $\Theta_1 = \{ u : \sigma = 0, \sigma_k = 0 \ (k = 3, \ldots, g) \}$ is

$$\psi = \frac{d}{du_1} \left[ -\frac{1}{\mu_0} \frac{\sigma_{11}}{\sigma_1} \right]$$

$$= -\frac{1}{\mu_0} \sum_{i=1}^g (-1)^{i-1} \left( \frac{\sigma_1}{\sigma_2} \right)^{i-1} \left( \frac{\sigma_{11} \sigma_{111} \sigma_{12}}{\sigma_1^3} - \frac{\sigma_{11} \sigma_{111}}{\sigma_1^2} \right). \quad (15)$$

This function is only singular when $\sigma_1(u) = 0$, that is when $u = \pm u_0$.

We calculate the Taylor series of $\psi$ near the singular point $u_0$ as follows. Since just the first three terms in the sum contain negative powers of $\sigma_1$ we will rewrite $\psi(u)$ as

$$\psi = -\frac{1}{\mu_0} \left[ (-\sigma_1^2) \frac{1}{\sigma_1^2} + \left( \frac{\sigma_{11} \sigma_{111} \sigma_{12}}{\sigma_1^3} \right) \frac{1}{\sigma_1} + O(1) \right] \quad (\forall g \geq 3)$$

for $u$ near $u_0$. If we now take the limit $u \rightarrow u_0 \Leftrightarrow p \rightarrow \infty$, we obtain

$$\lim_{u \rightarrow u_0} \left[ \frac{1}{\mu_0} \frac{\sigma_{11}^2}{\sigma_1^2} \right] = \lim_{v_1 \rightarrow 0} \left[ \frac{1}{\mu_0} \frac{\sigma_{11}^2 (v_1^2 + (2 \sigma_{11} \sigma_{111}) v_1 + \cdots)}{\sigma_{11}^2 v_1^2 + (\sigma_{11} \sigma_{111}) v_1^3 + (2 \sigma_{11} \sigma_{11} \sigma_{12}) v_1 v_2 + \cdots} \right]$$

$$= \lim_{v_1 \rightarrow 0} \left[ \left( \frac{1}{\mu_0} \right) v_1^{-2} + \left( \frac{1}{\mu_0} \frac{\sigma_{11} \sigma_{111} \sigma_{12}}{\sigma_{11}^3} - \frac{1}{\sqrt{\mu_0} \sigma_1} \right) v_1^{-1} + O(1) \right]$$
and
\[
\lim_{u \to u_0} \left[ -\frac{1}{\mu_0} \left( \frac{\sigma_{111}}{\sigma_1} + \frac{\sigma_{11} \sigma_{12}}{\sigma_2 \sigma_1} \right) \right] = \lim_{v_1 \to 0} \left[ \frac{-\left(\sigma_{111} \sigma_2 + \sigma_{11} \sigma_{12}\right)}{\mu_0 \sigma_2 \sigma_{11}} v_1 + \cdots \right]
\]
\[
= \lim_{v_1 \to 0} \left[ \left( -\frac{1}{\mu_0} \frac{\sigma_{111}}{\sigma_{11}} - \frac{1}{\mu_0} \frac{\sigma_{12}}{\sigma_2} \right) v_1^{-1} + O(1) \right].
\]
Combining these gives
\[
\lim_{u \to u_0} \psi(u) = \lim_{v_1 \to 0} \left[ \left( \frac{1}{\mu_0} \right) v_1^{-2} + \left( \frac{1}{\sqrt{\mu_0}} \frac{\sigma_{12}}{\sigma_{11}} (u_0) - \frac{1}{\mu_0} \frac{\sigma_{12}}{\sigma_2} (u_0) \right) v_1^{-1} + O(1) \right]
\]
\[
= \left( \frac{1}{\mu_0} \right) v_1^{-2} + O(1) \quad (\forall g \geq 3) \quad (16)
\]
(using substitution (B.1)).

From the expansion of \(\varphi_2 (14)\) and \(\psi (16)\) near their singular points, it follows that \((\varphi_2 (u) - \psi (u))\) is a holomorphic function on \(R_g\). We thus have that
\[
(-1)^{g+1} \varphi_2 (u) du_1 + A^T \cdot du = (-1)^{g+1} \psi (u) du_1 + B^T \cdot du
\]
for some \(g\)-vector of constants \(B = (B_2, B_3, \ldots, B_{g+1})^T\).

5. Evaluation of the vector \(B\).

Following [2], let \(f\) be a function on the Riemann surface \(R_g\). The divisor of \(f\), \((f)\), is defined as
\[
(f) = \sum n_i Z_i - \sum m_i P_i \quad n_i, m_i \in \mathbb{Z}^+
\]
where \(Z_i\) is a zero of \(f\) of degree \(n_i\) and \(P_i\) is a pole of \(f\) of order \(m_i\). The degree of the divisor of \(f\) is
\[
\deg (f) = \sum n_i - \sum m_i.
\]
For any function \(f\) and Abelian differential \(dv\) the following hold:
\[
\begin{align*}
\deg (f) &= 0; \quad (18) \\
\deg (dv) &= 2g - 2.
\end{align*}
\]
We will now consider the Abelian differential
\[
(-1)^{g+1} \left[ \varphi_2 (u) - \psi (u) \right] du_1.
\]
By construction, \(du_1\) is a first kind Abelian differential. It therefore has no poles on \(R_g\) and zeros of total degree \((2g - 2)\). From section 4, we know that the hyperelliptic function \((\varphi_2 - \psi)\) has no poles and so, by (18), it cannot have any zeros. Hence, for some constant \(C_0\), we have
\[
C_0 \ du_1 = (-1)^{g+1} \left[ \varphi_2 (u) - \psi (u) \right] du_1.
\]
Rewriting this using identity (17) gives
\[
C_0 \ du_1 = (B - A)^T \cdot du
\]
\[
\Rightarrow C_0 \ \frac{dt}{s} = \left[ (B_2 - A_2) + (B_3 - A_3) t + \cdots + (B_{g+1} - A_{g+1}) t^{g-1} \right] \ \frac{dt}{s}.
\]
Matching coefficients of $t$, we see

$$C_0 = B_2 - A_2$$

and so

$$B_i = A_i \quad (i = 3, \ldots, g + 1).$$

The value of $B_2$ may be found by evaluating $(\varphi_2(u) - \psi(u))$ at a specific point. If, for example, we take $u = u_0$, then we obtain

$$C_0 = \lim_{u \to u_0} [\varphi_2(u) - \psi(u)] = \left( \frac{1}{\sqrt{\mu_0} \sigma_2^3(u_0)} + \frac{2}{\mu_0} \frac{\sigma_{112}}{\sigma_2} (u_0) - \frac{2}{\mu_0} \frac{\sigma_{12}^2}{\sigma_2^2} (u_0) \right) + O(v_1)$$

(using substitutions $(B.1), (B.2)$ and $(B.3)$ from Appendix B). From this we have

$$B_2 = A_2 - (-1)^{g+1} \left( \frac{1}{\sqrt{\mu_0} \sigma_2^3(u_0)} + \frac{2}{\mu_0} \frac{\sigma_{112}}{\sigma_2} (u_0) - \frac{2}{\mu_0} \frac{\sigma_{12}^2}{\sigma_2^2} (u_0) \right).$$

It would be possible to rewrite $\sigma_{112}(u_0)$ in terms of lower order $\sigma-$derivatives using the following procedure. For each $g \geq 1$ there exists a set of PDE of the form

$$\varphi_{ijkl} - f(\mu_0, \ldots, \mu_{2g+1}; \varphi_{mn}) = 0,$$

where $1 \leq i \leq j \leq k \leq l \leq g$ and $1 \leq m \leq n \leq g$ (see [4]). If we expand (19) for $u$ near $u_0$, then we get Taylor series equal to zero. The relations between the $\sigma-$derivatives at the point $u_0 \in \Theta_1$ are then found by setting $\sigma(u_0) = \sigma_1(u_0) = \sigma_k(u_0) = 0 \quad (k = 3, \ldots, g)$ and equating each coefficient with zero. This process, however, cannot easily be generalized for all $g \geq 3$.

6. Result

Setting

$$k = \pm \sqrt{\mu_0} = \pm \left( \frac{4}{\prod_{i=1}^{2g+1} (p_{2g+2} - \mu_i)} \right)^{\frac{1}{2}},$$

$$\tilde{B}_2 = (-1)^{g+1} \left( \frac{1}{\sqrt{\mu_0} \sigma_2^3(u_0)} + \frac{2}{\mu_0} \frac{\sigma_{112}}{\sigma_2} (u_0) - \frac{2}{\mu_0} \frac{\sigma_{12}^2}{\sigma_2^2} (u_0) \right)$$

and substituting

$$p = p_{2g+2} - \frac{1}{t} = p_{2g+2} + \frac{\sigma_2}{\sigma_1}(u)$$

into (3) we have

$$\lambda(p) = p + \int_{-\infty}^{p} [\varphi(p') - 1] \, dp'$$

$$= \left( p_{2g+2} + \frac{\sigma_2}{\sigma_1}(u) \right) + \int_{0}^{1} \left[ k \, A^T \cdot d\mathbf{u} + k \, \tilde{B}_2 \, d\mathbf{u}_1 + (-1)^{g+1} k \left( \frac{d}{du_1} \Psi(u) \right) d\mathbf{u}_1 - \frac{dt}{t^2} \right]$$

$$= \left( p_{2g+2} + \frac{\sigma_2}{\sigma_1}(u) \right) + \left[ k \, (A + \tilde{B}_2 \, e_1)^T \cdot \mathbf{u} + (-1)^g k \, \frac{\sigma_{11}}{\mu_0} \frac{\sigma_1}{\sigma_1}(u) \right] + \tilde{C}. $$
The value of the constant $\tilde{C}$ can be found by considering the limit of $(\lambda(p) - p)$ as $p \to \infty \iff u \to +u_0$. Since

$$\lim_{p \to \infty} [\lambda(p) - p] = 0,$$

we have that

$$\tilde{C} = -k \left( A + \tilde{B}_2 e_1 \right)^T u_0 + \lim_{u \to u_0} \left[ (-1)^{g+1} \frac{k}{\mu_0} \frac{\sigma_{11}}{\sigma_1} (u) + \frac{\sigma_2}{\sigma_1} (u) \right].$$

Expanding the terms in this limit we obtain

$$\lim_{u \to u_0} \left[ (-1)^{g+1} \frac{k}{\mu_0} \frac{\sigma_{11}}{\sigma_1} \right] = (-1)^{g+1} \left( \frac{k}{\mu_0} \right) \lim_{v_1 \to 0} \left[ \frac{(\sigma_{11}) + (\sigma_{111}) v_1 + \cdots}{(\sigma_{11}) v_1 + \left( \frac{1}{2} \sigma_{111} \right) v_1^2 + (\sigma_{12}) v_2 + \cdots} \right]$$

$$= (-1)^{g+1} \left( \frac{k}{\mu_0} \right) \lim_{v_1 \to 0} \left[ v_1^{-1} + \left( \frac{1}{2} \sigma_{111} - \frac{\sqrt{\mu_0} \sigma_{12}}{2} \right) v_2 + \sigma_{12} \right] + O(v_1)$$

$$= (-1)^{g+1} \left( \frac{k}{\mu_0} \right) \lim_{v_1 \to 0} \left[ v_1^{-1} + \left( \frac{1}{2} \sigma_{111} - \frac{\sqrt{\mu_0} \sigma_{12}}{2} \right) v_2 + \sigma_{12} \right] + O(v_1)$$

and

$$\lim_{u \to u_0} \left[ \frac{\sigma_2}{\sigma_1} \right] = \lim_{v_1 \to 0} \left[ \frac{(\sigma_2) + (\sigma_{12}) v_1 + \cdots}{(\sigma_{11}) v_1 + \left( \frac{1}{2} \sigma_{111} \right) v_1^2 + (\sigma_{12}) v_2 + \cdots} \right]$$

$$= \lim_{v_1 \to 0} \left[ v_1^{-1} + \left( \frac{1}{2} \sigma_{111} - \frac{\sqrt{\mu_0} \sigma_{12}}{2} \right) v_2 + \sigma_{12} \right] + O(v_1)$$

$$= \lim_{v_1 \to 0} \left[ v_1^{-1} + \left( \frac{1}{2} \sigma_{111} - \frac{\sqrt{\mu_0} \sigma_{12}}{2} \right) v_2 + \sigma_{12} \right] + O(v_1).$$

Since $\tilde{C}$ is constant we set $k = (-1)^{g+1} \sqrt{\mu_0}$ and hence

$$\tilde{C} = (-1)^{g} \sqrt{\mu_0} \left( A + \tilde{B}_2 e_1 \right)^T u_0 + \frac{2}{\sqrt{\mu_0}} \frac{\sigma_{12}}{\sigma_2} (u_0) + \frac{1}{2} \frac{\mu_1}{\mu_0}.$$

This gives the following result.

**Theorem 2** Let

$$\lambda(p) = p + \int_{p}^{\infty} \frac{\prod_{i=1}^{g+1} (p' - \hat{p}_i)}{\sqrt{\prod_{i=1}^{2g+2} (p' - p_i)}} \, dp',$$

$$k = (-1)^{g+1} \left( \frac{4}{\prod_{i=1}^{2g+2} (p_{2g+2} - p_i)} \right)^{\frac{1}{2}},$$

$$\tilde{B}_2 = (-1)^{g+1} \left( \frac{1}{\sqrt{\mu_0}} \frac{\sigma_{22}}{\sigma_2} (u_0) + \frac{2}{\mu_0} \frac{\sigma_{112}}{\sigma_2} (u_0) - \frac{2}{\mu_0} \frac{\sigma_{12}^2}{\sigma_2^2} (u_0) \right),$$

and $A^T = (A_2, A_3, \ldots, A_{g+1})$ where the $A_i$ are defined as

$$\sum_{i=0}^{g+1} A_i t^i = \prod_{i=1}^{g+1} [(p_{2g+2} - \hat{p}_i) t - 1].$$

Then, if we set

$$p = p_{2g+2} + \frac{\sigma_2}{\sigma_1} (u),$$

we can find $\lambda(p)$ using the above results.
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with \( \mathbf{u}, \mathbf{u}_0 \in \Theta_1 \) and \( \sigma_1(\mathbf{u}_0) = 0 \), we have

\[
\lambda(p) = (-1)^{g+1} \sqrt{\mu_0} \left( \mathbf{A} + \mathbf{B} \mathbf{e}_1 \right)^T (\mathbf{u} - \mathbf{u}_0) - \frac{1}{\sqrt{\mu_0}} \sigma_1(\mathbf{u})
+ \frac{p_{2g+2}}{\sqrt{\mu_0}} \sigma_2(\mathbf{u}_0) + \frac{1}{2\mu_0} \mu_1
\]

on sheet \( R_g^+ \) of the Riemann surface

\[
R_g = \left\{ (v, p) \in \mathbb{C}^g : v^2 = \prod_{i=1}^{2g+2} (p - p_i) \right\}
\]

associated with the relation \( p \to \infty_+ \iff \mathbf{u} \to +\mathbf{u}_0 \).

We note that in the \( g = 2 \) case the analogous solution to (20) could be rewritten using the relation

\[
\frac{\sigma_{11}}{\sigma_1}(\mathbf{u}) = \frac{\sigma_1}{\sigma}(\mathbf{u} + \mathbf{u}_0) + \frac{\sigma_1}{\sigma}(\mathbf{u} - \mathbf{u}_0) = \zeta_1(\mathbf{u} + \mathbf{u}_0) + \zeta_1(\mathbf{u} - \mathbf{u}_0)
\]

for \( \mathbf{u} \in \Theta_1 \). In the case of higher genus reductions this is not possible since \( (\mathbf{u} \pm \mathbf{u}_0) \in \Theta_2 \) and \( \zeta_1 \) is singular everywhere on \( \Theta_2 \).

The formula (20) seems a little more complicated than the analogous results in genus 1 and 2; the reason for this is the difficulty of expanding the terms involving \( \mathbf{u}_0 \) in the general case. However, we consider it remarkable that essentially the same formula is valid for any genus.

Acknowledgments

We would like to thank V Z Enolski for bringing [11] to our attention and for the result given in Appendix A.

Appendix A. Reduction of the Inversion theorem to \( \Theta_1 \).

Following Enolski and Previato [6], we begin by rewriting the main result of [11] in terms of first derivatives of the \( \sigma \)-function.

**Theorem 3** Let \( \mathbf{K}_P \) be the vector of Riemann constants associated with the point \( P, \{P_1,P_2,\ldots,P_{g-1}\} \) be a set of points on \( R_g \) and let \( \mathbf{a} = (a_1,a_2,\ldots,a_g)^T \), \( \mathbf{b} = (b_1,b_2,\ldots,b_g)^T \in \mathbb{C}^g \) be any nonzero vectors. Then the following identity holds

\[
\frac{\sum_{j=1}^g \sigma_j(\mathbf{u}) a_j}{\sum_{j=1}^g \sigma_j(\mathbf{u}) b_j} = \frac{\det [\mathbf{a} | \mathbf{d}(P_1)| \cdots | \mathbf{d}(P_{g-1})]}{\det [\mathbf{b} | \mathbf{d}(P_1)| \cdots | \mathbf{d}(P_{g-1})]}
\]

where the point \( \mathbf{u} \) is given by

\[
\mathbf{u} = \sum_{k=1}^{g-1} \int_{P_k}^{P} \mathbf{d}u + 2 \omega \mathbf{K}_P.
\]
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Here, we take the $du_i$ to be the holomorphic differentials defined above:

$$du_i = \frac{t^{i-1}}{s} \, dt \quad (i = 1, \ldots, g).$$

**Corollary 3.1** Let the points $P_1, P_2, \ldots, P_g$ coalesce to a point $P$. Then we obtain by L'Hôpital's rule

$$\sum_{j=1}^g \sigma_j (2\omega K_P) a_j \sigma_j (2\omega K_P) b_j = \frac{\det \left[ a | du(P) | du(P)^{(1)} | \cdots | du(P)^{(g-2)} \right]}{\det [b] \, du(P) | du(P)^{(1)} | \cdots | du(P)^{(g-2)}]} \quad (A.1)$$

where $du(P)^{(k)}$ denotes the column of $k^{th}$ derivatives of the holomorphic differentials $du(P)$.

Expanding the RHS of (A.1) we find that the numerator is the determinant of the matrix

$$C = \begin{bmatrix}
  a_1 & 1 & 0 & 0 & \cdots & 0 & 0 \\
  a_2 & t & 0 & 0 & \cdots & 0 & 0 \\
  a_3 & t^2 & 0 & 0 & \cdots & 0 & 1 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  a_{g-1} & t^{g-2} & 0 & 1 & \cdots & 0 & 0 \\
  a_g & t^{g-1} & 1 & 0 & \cdots & 0 & 0 \\
\end{bmatrix}$$

for some constant $C$. The matrix in the denominator of the RHS is of the same form, but with $b_i$ instead of $a_i (i = 1, \ldots, g)$. It follows that (A.1) can be written as

$$\sum_{j=1}^g \sigma_j (2\omega K_P) a_j \sigma_j (2\omega K_P) b_j = \frac{a_1 t - a_2}{b_1 t - b_2} \quad (A.2)$$

To evaluate $t$ in terms of the $\sigma_j$ we can therefore set $a = (1,0,\ldots,0)^T$ and $b = (0,1,0,\ldots,0)^T$. This gives

$$\frac{\sigma_1}{\sigma_2} (u) = -t$$

for $u \in \Theta_1$. Further, since only $a_1, a_2$ and $b_1, b_2$ appear in the RHS of (A.2), we obtain the following definition for $\Theta_1$:

$$\Theta_1 = \{ u : \sigma(u) = 0, \sigma_k(u) = 0 \quad (k = 3, \ldots, g) \}.$$

**Appendix B. Differential relations holding at $u = u_0$.**

For any $u$ in $\Theta_1$ we have $\sigma(u) = 0$. Expanding this identity near $u_0$ we obtain a Taylor series in $v_k = e_k^T \cdot (u - u_0)$ equal to zero:

$$0 = \sigma(u_0 - (u_0 - u)) = \left[ \frac{1}{2} \sigma_{11}(u_0) \right] v_1^2 + [\sigma_2(u_0)] v_2 + [\sigma_{12}(u_0)] v_1 v_2 + \left[ \frac{1}{6} \sigma_{111}(u_0) \right] v_1^3 + \cdots$$

(since $\sigma(u_0) = \sigma_1(u_0) = \sigma_3(u_0) = 0$). If we now substitute relations (13)

$$v_k = \left( \frac{1}{k} \mu_0^{(k-1)/2} \right) v_1^k + O (v_1^{k+1}) \quad (k = 2, 3, \ldots, g)$$

then we obtain

$$0 = \left[ \frac{1}{2} \sigma_{11}(u_0) \right] v_1^2 + \left[ \frac{1}{6} \sigma_{111}(u_0) \right] v_1^3 + \cdots$$

and compare coefficients.
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into this expansion, then for \( g \geq 3 \) we have
\[
0 = \left[ \frac{1}{2} \sigma_{11}(u_0) + \frac{1}{2} \sqrt{-\mu_0} \sigma_2(u_0) \right] v_1^2 + \left[ \frac{1}{6} \sigma_{111}(u_0) + \frac{1}{12} \mu_1 \sigma_2(u_0) + \frac{1}{2} \sqrt{-\mu_0} \sigma_{12}(u_0) \right] v_1^3 + O(v_1^4).
\]
Setting each coefficient to zero, we find
\[
\sigma_{11}(u_0) = -\sqrt{-\mu_0} \sigma_2(u_0) \tag{B.1}
\]
and
\[
\sigma_{111}(u_0) = -\frac{1}{2} \mu_1 \sigma_2(u_0) - 3 \sqrt{-\mu_0} \sigma_{12}(u_0) \tag{B.2}
\]
for \( u_0 \in \Theta_1 \) with \( \sigma_1(u_0) = 0 \) and for \( \forall g \geq 3 \).

If we repeat the above procedure for the identity \( \sigma_3(u) = 0 \ (\forall u \in \Theta_1) \), then we obtain the following expansion
\[
0 = \sigma_3(u_0 - (u_0 - u))
\]
\[
= [\sigma_{13}(u_0)] v_1 + [\sigma_{23}(u_0)] v_2 + \left[ \frac{1}{2} \sigma_{113}(u_0) \right] v_1^2 + \cdots
\]
\[
= [\sigma_{13}(u_0)] v_1 + O(v_1^2).
\]
This gives the identity
\[
\sigma_{13}(u_0) = 0 \quad \text{for} \ g \geq 3. \tag{B.3}
\]

[14] L Yu and J Gibbons 2000 The initial value problem for reductions of Lax equations Inverse Problems 16 605