Optimal path-tracking of virtual race-cars using gain-scheduled preview control

Mark P. Thommyppillai

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Abstract

In the search for a more capable minimum-lap-time-prediction program, the presence of an alternative solution has been introduced, which requires the development of a high-quality path-tracking controller. Preview Discrete Linear Quadratic Regulator (DLQR) theory has been used to generate optimal tracking control gains for a given car model. The calculation of such gains are performed off-line, reducing the computational burden during simulated tracking trials. A simple car model was used to develop limit-tracking control strategies, first for an understeering and then for an oversteering car, travelling at a constant forward speed. Adaptation in the controller, with respect to front-/rear-lateral-slip ratio, facilitated superior tracking performance over the non-adaptive counterpart in a number of challenging tracking manoeuvres.

Once complete, development work was focused on the control of a complex car model. Such a model required an extension to the preview DLQR theory, to allow variable speed, two-channel $(x,y)$ optimal path tracking. Significant benefits were observed when using an adaptive control strategy, firstly scheduling with respect to forward ground speed and then including adaptation with respect to mean front-lateral-slip ratio. A variable weighting strategy was used to suppress oscillations in the tracking controller when operating near the limit of the car. Such a strategy places a higher cost on control effort expenditure, relative to tracking error, as the car approaches the limit of the front axle. Further oscillatory behaviour, due to the presence of lightly-damped eigenmodes, was suppressed by increasing the car’s suspension stiffness and damping parameters.

The tracking controller, that has resulted from the work documented by this thesis, has demonstrated high-quality tracking when operating in a number of different scenarios, including lateral limit tracking. Variable speed limit tracking is suggested as the next development step, which will then allow the controller to be implemented in initial learning trials. Successful development of the speed and path optimisers in such trials will complete the development of a novel solution to the minimum lap-time problem.
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<td>$A, B, C, E$</td>
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<tr>
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<tr>
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<tr>
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<td>Cost placed on lateral tracking error</td>
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<td>Cost/performance allocation matrix</td>
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<td>Augmented optimal control gain vector</td>
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<td>Trimmed value of steering wheel</td>
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<td>Optimal steering wheel angle control input</td>
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<tr>
<td>$\theta_{K_1}$</td>
<td>Feedback contribution of optimal steering control input</td>
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<td>$\theta_{K_2}$</td>
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<td>Elements of lateral road displacements measured in global reference frame</td>
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<td>$\tilde{y}'_{ri}$</td>
<td>Elements of lateral road displacements measured in intermediate reference frame</td>
</tr>
<tr>
<td>$\tilde{y}''_{r}$</td>
<td>Vector of lateral road displacements measured in local (car) reference frame</td>
</tr>
<tr>
<td>$\tilde{y}''_{ri}$</td>
<td>Elements of lateral road displacements measured in local (car) reference frame</td>
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<td>$(2 \times 2)$ zero matrix</td>
</tr>
<tr>
<td>$I_2$</td>
<td>$(2 \times 2)$ identity matrix</td>
</tr>
<tr>
<td>$\alpha_{car}$</td>
<td>Car lateral slip angle</td>
</tr>
<tr>
<td>$r_a$</td>
<td>Radius of arc describing future trajectory of constant-speed, constant latacc, trim-state</td>
</tr>
<tr>
<td>$\text{arc}_{xi}$</td>
<td>$x$-coordinates of future trajectory of car</td>
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<td>$\text{arc}_{yi}$</td>
<td>$y$-coordinates of future trajectory of car</td>
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<tr>
<td>$R_\alpha$</td>
<td>Rotation matrix for car slip angle adjustment</td>
</tr>
<tr>
<td>$\text{arc}'_{xi}$</td>
<td>$x$-coordinates of future trajectory of car after adjustment for car slip angle</td>
</tr>
</tbody>
</table>
Symbol | Parameter
--- | ---
\( \text{arc}_yi' \) | \( y \)-coordinates of future trajectory of car after adjustment for car slip angle
\( \tilde{xy}_r \) | Global vector of path coordinates
\( \tau \) | Throttle/brake demand
\( K_{2\theta_x} \) | Steering wheel gains relating to longitudinal tracking errors
\( K_{2\theta_y} \) | Steering wheel gains relating to lateral tracking errors
\( K_{2\tau_x} \) | Throttle/Brake gains relating to longitudinal tracking errors
\( K_{2\tau_y} \) | Throttle/Brake gains relating to lateral tracking errors
\( R \) | Cost matrix for control input energy in two-channel tracking problem
\( \alpha_{mf_{lar}} \) | Mean front-lateral-slip ratio
\( \psi \) | Control tightness multiple
\( A_{ts}, B_{ts}, C_{ts} \) | Control tightness multiple shaping-factors

Complex car parameters

\( M_s \) | Sprung mass of complex car
\( I_s \) | Matrix of rotational inertia for sprung mass
\( I_{whly} \) | Rotational inertia of each wheel
\( k_{tyr} \) | Vertical tyre stiffness
\( c_{tyr} \) | Tyre damping coefficient
\( k_{fsus} \) | Front suspension stiffness
\( k_{rsus} \) | Rear suspension stiffness
\( c_{fsus} \) | Front suspension damper coefficient
\( c_{rsus} \) | Rear suspension damper coefficient
\( F_d \) | Aerodynamic drag force
\( C_d \) | Drag coefficient
\( \rho \) | Air density
\( A \) | Frontal cross-sectional area
\( F_{zf} \) | Front lift force
\( F_{zr} \) | Rear lift force

– continued on next page
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Parameter</th>
</tr>
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<tbody>
<tr>
<td>$C_{lf}$</td>
<td>Front lift coefficient</td>
</tr>
<tr>
<td>$C_{lr}$</td>
<td>Rear lift coefficient</td>
</tr>
<tr>
<td>$\Omega_n$</td>
<td>Low-pass filter cut-off frequency</td>
</tr>
<tr>
<td>$k_{st}$</td>
<td>Torsional stiffness of steering column</td>
</tr>
<tr>
<td>$\tau^*$</td>
<td>Delayed throttle/braking signal</td>
</tr>
<tr>
<td>$\chi$</td>
<td>Engine crank-shaft torque</td>
</tr>
<tr>
<td>$B_t, E_t$</td>
<td>Throttle opening-shaping parameters</td>
</tr>
<tr>
<td>$\omega$</td>
<td>Rotational speed of engine</td>
</tr>
<tr>
<td>$B_s, C_s, D_s, E_s$</td>
<td>Engine-speed-shaping parameters</td>
</tr>
<tr>
<td>$\varphi$</td>
<td>Braking torque</td>
</tr>
<tr>
<td>$k_{brf}$</td>
<td>Front brake constant</td>
</tr>
<tr>
<td>$k_{brr}$</td>
<td>Rear brake constant</td>
</tr>
<tr>
<td>$B_c$</td>
<td>Brake shaping factor</td>
</tr>
</tbody>
</table>

subscripts:  f, front;  l, left;  r, rear or right

1, feedback; 2, preview; dem, demand; eq, equilibrium
Chapter 1

Introduction

In recent years, budget restrictions, as well as on-track testing limitations, have motivated the motorsport industry to use simulation software to quantify the performance of race-cars. Several software packages exist which model specific features of a race-car, such as aero- and engine-dynamics. However, there is a growing need for software which calculates the end result of the improvements in each area, to judge the car’s overall performance. A commonly used parameter which can quantify a race-car’s performance is its minimum lap-time, the minimum time taken for it to traverse a given circuit whilst remaining within the constraints of the track. In the highly competitive environment of Formula 1, for example, an increase of 0.6 s in the minimum lap-time can mean the difference between pole-position and 10th place on the starting-grid [1]. It is no surprise, then, that race-car engineers are interested in making design and setup changes in an attempt to reduce the minimum lap-time by 1/1000th’s of a second. A good minimum lap-time program should be able to work at this resolution in quantifying how such changes affect the race-car’s performance. These simulations are also required to be quick as setup and design changes are usually time-critical in the motorsport environment.

The job of finding the minimum lap-time is usually performed by a human racing-driver. However, this is an expensive way of testing, both in time and money. Not only does a dedicated testing-team need to support the driver and car at the track, most drivers simply can’t deliver the car’s absolute minimum lap-time, accurate to 1/1000th of a second, consistently lap-after-lap. Computational methods have therefore been developed to simulate what a real driver does in finding the minimum lap-time. The problem can be solved using a number of different strategies:

- By simplifying the problem and assuming the driver takes the car to, and keeps it at, its physical limits at each instant in time. Steady-State (SS) and Quasi-Steady-State (QSS) methods use this technique.
• Casting the problem as a pure optimisation. After specifying the car’s equations of motion and the constraints placed on the system, an optimal control problem is solved using direct or indirect methods.

• Calculating the minimum lap-time by using non-linear model-predictive/receding-horizon optimal control.

Each strategy will be presented more thoroughly in the literature review that follows this chapter, where the advantages and disadvantages, whether it is speed of convergence or accuracy of simulation, will be highlighted. From the analysis of current techniques, an alternative solution to the problem of finding a race-car’s minimum lap-time is thought to exist.

The method starts by tracking an approximation of the racing-line, the trajectory taken by the car to yield its minimum lap-time, for a particular track. Preview of the track ahead of the car is used to ensure high-quality tracking of the path. The speed profile of the car along this path is optimised so that the car completes the lap in the minimum time. The path that is tracked is then perturbed and the optimal speed profile for this new approximation is found. After successive iterations, the method is hoped to converge to the true racing-line together with the matching speed profile and control inputs that yield the minimum lap-time.

Using this method, the main problem is decomposed into three sub-problems: tracking a given path, optimising the speed along it and perturbing the path to obtain the minimum lap-time. This should reduce the total number of optimisation parameters which need to be used, when compared to the alternative methods presented. Hence, this method should lend itself to finding a faster solution to the minimum lap-time problem when compared to the methods, presented earlier, which give the most accurate results. Similarly, the proposed method should produce a more accurate solution when compared to methods where the lap-time is computed rapidly. As the method essentially tries to optimise trajectories, it can be initialised with a rough approximation of the racing-line, which is usually known for most tracks. By avoiding the need to start from scratch and converge to this approximation, the computer is then used only to find the essential unknown parameters. This should lend itself to faster convergence to a solution.

The author believes this method is also more representative of how a human driver optimises the lap-time, although an investigation into such learning behaviour is yet to be conducted formally. When learning to drive a race-car quickly around an unknown race-track, the driver would initially try to follow a safe approximation of the racing-line, which is not likely to challenge them or the car. As the driver gains confidence, he/she optimises the speed along this path by braking later and accelerating harder, relying on his/her knowledge of the car to correct any errors when pushing
too hard. This optimisation usually takes several laps, with the best drivers able to zone-in on the minimum lap-time very quickly. Because of the parallel to human driver learning, this technology could also be useful to commercial road-vehicle designers. The software could potentially be used in this area to study what factors affect driver learning and how it can be improved.

Another advantage of the proposed technique is how it converges to a solution. Pure optimisation strategies may only return a final solution to the problem without an indication of the optimisation path taken in obtaining it, which may itself be obscure due to the manner in which the software optimises the lap-time. As the proposed optimisation deals with the trajectory of the car, it could offer a visual insight into how the program learns to drive the car quicker on each lap iteration. For example, if the setup of the car is such that it understeers at a critical turn, the evolution of the trajectories at the corner could show the problem developing. Design changes could then be made early on, if engineers identify such patterns/problems between iterations, so that time is not wasted converging to a solution which is not ideal.

The purpose of this thesis is to document work on the first stage of the proposed alternative solution to finding the minimum lap-time: developing a combined longitudinal and lateral optimal tracking controller. Several technologies were brought together in a modular fashion so that the final tracking controller could be progressively built-up from each component. Chapter 3 presents the first car model, the simple car, which was used to test the basic principles of the tracking controller. The process of linearising the system, to make it applicable for the linear optimal control techniques that will be used, is also presented.

Chapter 4 introduces Discrete Linear Quadratic Regulator (DLQR) theory, which forms the basis of the optimal tracking controller that was developed. The theory is then extended to preview DLQR techniques, which use information of the road ahead to improve tracking quality. The concept of gain-scheduling/adaptation is introduced in the interest of keeping the control inputs optimal for the full extent of the car’s capabilities. The details of how such a process can be implemented are also presented in the same chapter.

The simple car model presented in chapter 3 and the gain-scheduled controller described in chapter 4 are brought together in the tracking trials in chapters 5 and 6. Following the modular approach of the research, tracking trials were first conducted for moderate manoeuvres before studying limit tracking. Control strategies for the limit tracking of two distinctly different setups of the simple car are presented. Chapter 5 tackles front-axle limited cornering, commonly known as understeering, whilst chapter 6 deals with rear-axle limited cornering, commonly known as oversteering.

Chapter 7 then presents the detailed complex car model, which was used as a more realistic
model on which to develop the controller. The use of key non-linearities such as combined-slip
 tyres, aero-dynamic- and engine-maps are presented in this chapter. Chapter 8 demonstrates the
 use of the controller, designed for the complex car, for longitudinal and lateral tracking trials.
 Chapter 9 will then provide the conclusions followed by a description of future work.
Chapter 2

Literature review

This chapter has two purposes. Firstly, it documents the research and development which has led to the state-of-the-art in the area of minimum lap-time optimisation and identifies the opportunity for an alternative approach to solving the problem. Secondly, it presents the development of the preview DLQR techniques which are used in the tracking controller as part of this alternative solution. In this pursuit, the first section of this literature review will introduce the main techniques used in minimum lap-time optimisation in order of method complexity.

2.1 Current approaches

2.1.1 Steady-State (SS) approach

Arguably the simplest solution to the minimum lap-time problem is the steady-state approach. A steady-state is when a system is in equilibrium and the time-dependent variables are constant [14]. The technique assumes that a given lap can be decomposed into sections of either pure lateral or pure longitudinal acceleration. This corresponds to constant speed, constant radius cornering or sections where the car is accelerating in a straight line. Rapid computation of the lap-time using this technique comes from the very fact that this assumption over-simplifies the problem. In reality, the car experiences a combination of lateral and longitudinal acceleration when cornering. The inclusion of this detail leads to a more accurate theoretical lap-time [59].

2.1.2 Quasi-Steady-State (QSS) approach

The Quasi-Steady-State approach attempts to improve on the inaccuracy of the SS method mentioned previously. A map is built-up of limit steady-states, each state being a combination of lateral and longitudinal acceleration, for different speeds. It is therefore commonly known as a
G-G speed map, a typical example of which can be seen in Figure 2.1.

Figure 2.1: Diagrammatic representation of a G-G speed map for a race-car with aerodynamic down-force.

QSS techniques assume that a lap can be constructed by moving instantly from one steady-state to another. The minimum lap-time for a given racing-line is found by firstly locating the apex of each turn, the point where the car is assumed to be turning at constant speed. This is usually done by identifying points along the path where the radius of curvature is a minimum. The G-G speed map is then used to identify the maximum forward speed the car is capable of executing whilst sustaining the lateral acceleration attributed to the radius of the apex. Knowing the control inputs that give rise to this steady-state, the algorithm then marches forward and backward from the apex, for each instant in time. The curvature of the track will decrease in either direction, as does, by implication, the lateral acceleration demanded at each new point. The spare capacity of the tyres at each new point can therefore be used to accelerate or decelerate the car longitudinally. This marching strategy is employed at each corner until the acceleration zone of one meets the deceleration zone of the neighbouring corner, the next braking-point. The entire control history for the track is built-up by stitching together the control histories of individual corners.

An early attempt at implementing this technique was made by Griffiths [23]. The concept of a tyre friction-circle was used to judge the combination of the car’s lateral and longitudinal
capabilities and obtain its minimum lap-time. Later, Gadola et al. [21] used a similar technique, using a G-G speed map together with a genetic algorithm, to find the optimal combination of key control decisions for a corner: the braking point, the coasting point and the acceleration point. Recently, Brayshaw and Harrison [11] documented the use of the Matlab\textsuperscript{TM} \texttt{fmincon} function to quickly compute a car’s G-G speed map. The predicted lap-time using their QSS method was 2.19 s slower than the lap-time predicted by the transient techniques employed by Casanova [12] using the same car model, which, despite the authors’ claim of a good agreement, in the context of race-car lap-times, is a substantial difference. It is also worth noting that the QSS method predicts a slower lap-time than the transient technique. This goes against the theory that QSS methods should produce a faster lap-time as they omit transient dynamics, which are attributed to increasing the lap-time.

All three contributions to QSS methods admit that the main short-coming of the approach is the assumption that the car can move instantly from one steady-state to another. This critically omits transient dynamics from the problem. Important factors, such as suspension dynamics and yaw moment of inertia, which are known to be key parameters which influence the minimum lap-time, are neglected. These omissions can lead to large discrepancies between QSS methods and transient techniques, as seen in [11]. Despite this problem, QSS techniques are used due to their ability to rapidly compute lap-times, with scaling being employed to get agreement between the predicted lap-time and the real-life equivalent. However, the need for more accurate modelling, which captures the transient behaviour of the car, still exists.

2.1.3 Inverse dynamics approach

Inverse dynamics techniques were considered in an initial attempt to include transient dynamics into minimum lap-time optimisation. As the name implies, these methods found the control history required to track a predefined path. Metz and Williams [35] employed this technique to find the optimal speed and control history for a pre-defined path. A moderately detailed car model was used, although the tyre force variation due to loading effects was not considered. Later, La Joie [32] used a simple 3 Degree Of Freedom (DOF) car model in combination with a quasi-Newton penalty method to find the optimal state trajectories for predefined manoeuvres. Soon afterwards, Jennings [28] presented a software suite which was developed to model transient non-linear vehicle dynamics. A driver model was presented which also used inverse dynamics to optimally control the car along a prescribed path. It used a technique which previewed the change in curvature of the path at a point ahead of the car. The distance of this point from the car, the \textit{curvature transition distance}, was changed dynamically to compensate for the lag between the control demand and
the response of the vehicle, due to inertial effects. The software also used a predefined parameter which limited the rate at which the tyre forces are allowed to change for a given change in steering angle. The value of this parameter is not presented, nor is the process of finding its correct value, but it looks likely that it will play a significant role in tracking performance. Additionally, the contributions from Jennings and from Metz and Williams do not mention the computational burden of the respective techniques that are used. Such information is crucial to making an informed decision on the viability of such methods, especially when comparing them to alternative solutions to the path tracking problem.

It must also be noted that the methods presented above require the path to be predefined, which is seen as an oversimplification. In reality, the path which needs to be followed, which results in an optimal lap-time, will need to be part of the optimisation. In light of the problems, in accuracy of steady-state methods and the limitations of inverse transient approaches, that have been presented, many have tried to find the minimum lap-time by casting it as a pure optimisation.

2.1.4 Non-Linear Optimal Control

An alternative solution to the minimum lap-time problem comes from defining the system’s dynamics and constraints and transforming it into an optimal control problem. Once defined, the non-linear optimal control problem can be solved in two ways: indirect or direct methods.

Indirect methods

Indirect methods typically start with an approximation of the solution and solve a two-point boundary-value problem. Optimality conditions, usually specified by Pontryagin’s minimum principle, are used to explicitly guide the next approximation closer to the true optimal solution. The first notable attempt at implementing indirect methods on the minimum-time-to-maneuuvre problem was by Fujioka and Kimura [20]. A Sequential Conjugate Gradient Restoration Algorithm (SCGRA) [66], which is usually used to find optimal trajectories of satellites, was used to optimally manoeuvre a car between two predefined points of a hair-pin corner. Two sequential phases are executed at each iteration when converging to a solution. The first tries to find a solution which minimises the cost function. The second then drives the solution closer to satisfying the system’s constraints, to a certain tolerance. In Fujioka’s work, the system parameters at the start and end of the manoeuvre were specifically chosen so as to avoid the use of constraints during the manoeuvre. This is seen as an oversimplification as a real-life control problem will be subject to a number of constraints which need to be specified explicitly.
The main development in using indirect methods on minimum-time-to-maneuuvre problems came from Hendrikx et al. [26]. A 3 DOF car model was used together with a gradient-descent algorithm to find the optimal control for a given lane-change. The gradient-descent algorithm, although robust against poor initial approximations of the solution, has been criticised by Kirk [30] for its slow rate of convergence to a solution. Cossalter et al. [13] later used an alternative relaxation algorithm, documented in Press et al. [41], to solve the two-point boundary-value problem for the optimal control of motorbikes.

Several difficulties exist when using indirect methods to solve the minimum lap-time problem. The methods require the symbolic differentiation of the equations that govern the optimal control problem. This can be difficult, especially when dealing with the complex non-linear systems present in race-cars. The control history is also required to be smooth, which again can be difficult to satisfy in a lap-time simulation problem, for example, when modelling the abrupt transition from full acceleration to maximum braking. Finally, a good initial approximation of the optimal solution and the co-states, measures of the sensitivity of the performance/cost function to changes in the system states, are required to get good convergence properties. These approximations are difficult to specify accurately. In light of the difficulties that exist with indirect methods, many have tried to solve the problem using direct methods, which will be discussed next.

Direct methods

Direct methods model the continuous optimal control solution of a problem with a discrete approximation. By discretising the control, the problem can be cast into a non-linear programming task. This task assumes that the control inputs can only be varied at each discretisation point, known as a node. It is important to note that this grid of nodes is independent of the grid used to integrate the system equations. Computational algorithms are employed to progressively improve the discrete control input decisions and converge to an optimal control sequence. A close fit to the original continuous problem is then found by interpolating between the nodes of this discrete control sequence. The interpolation acts to also smooth-out discontinuities, such as the transition from maximum acceleration to maximum braking in the minimum lap-time problem, which would hinder indirect methods. Three general types of direct optimisation exist, each characterised by the choice of optimisation parameters at each node [27], and can be visualised by the diagram in Figure 2.2:
Figure 2.2: Diagrammatic representation of alternative direct methods for solving optimal control problems.
Direct Single-Shooting

In a direct single-shooting algorithm, only the control inputs at each node are varied at each optimisation iteration. The state trajectory, and the value of the performance/cost function, can be derived explicitly from this sequence of discrete control inputs. Hence, for each optimisation iteration, changes to the discrete control sequence are trialled by performing a one-pass integration, shooting, over the optimisation horizon to determine the effect on the performance/cost function. The information on how the performance/cost function was affected allows the optimiser to use a better guess of the discrete control sequence at the next iteration and converge to a solution.

An example of the direct single-shooting method can be seen in Allen [2]. A direct single-shooting algorithm, developed by Kraft [31], was used in conjunction with Sequential Quadratic Programming (SQP) to solve a minimum-time-to-manoeuvre problem. SQP essentially performs an optimisation of the parameters dictated by the direct method being used. In the case of the direct single-shooting method, the parameters are the sequence of control inputs at each node. SQP relies on the fact that the Lagrangian, a function that facilitates the solution of constrained minimisation problems, can be approximated by a quadratic function [10]. At each iteration, a new quadratic approximation of the Lagrangian is found and the parameters altered so that the solution is driven to a minimum. Allen’s work was limited to small sections of track and a simple 3 DOF car model. It was admitted that the calibration of the system parameters and optimisation variables was critical to the stability of the method and the convergence to a solution. This is typical of direct single-shooting methods. If long optimisation horizons are used, constraint violations for the later parts of the optimisation are critically dependent on early control decisions, so called journey sensitivity. This can lead to convergence issues, so the user needs to keep the optimisation length short and carefully choose optimisation parameters, as observed in Allen’s work.

Direct Collocation

As previously mentioned, the direct single-shooting method uses the minimal set of parameters that are optimised, the control input sequence at each node. Using less parameters reduces the flexibility of the optimisation which, depending on the problem, can result in a slower rate of convergence [7]. On the other hand, with direct collocation methods, the values of the control inputs and states, which satisfy the appropriate constraint and system equations, are found at every node as part of the optimisation process. More parameters generally allow more flexibility in the optimisation which, again depending on the problem, can lead to faster convergence properties [12].

The obvious short-coming is that, for problems that are initially large, using direct-collocation
increases the size of the problem further still, which can be computationally demanding. The speed of the technique can be increased by using relaxation. Relaxation algorithms work by substituting the differential equations associated with the problem with a grid of linked finite-difference equations. The state trajectory, at each node, for each optimisation iteration is found after executing a number of sub-iterations. At the start of each sub-iteration, the conditions placed on the system are said to be relaxed and the states at each node are not required to comply with the finite-difference equations or constraints. However, during each sub-iteration, an algorithm then works to ensure the solution of the main iteration satisfies all system equations and constraints. Due to the relaxation on the possible solutions that are allowed within each iteration, for certain problems, relaxation can be more efficient than shooting methods [41]. Relaxation methods are however not ideal when the solution is highly oscillatory, where a large number of nodes would be required to capture this information. Alternative direct methods may be better in these cases.

Direct Multi-/Parallel-shooting

From the descriptions in the preceding sections, it was noted that direct single-shooting methods suffer from slow convergence and journey sensitivity issues, while direct-collocation methods can increase problem size considerably. A compromise can be found in the form of direct multi-shooting methods. Here, the optimisation horizon is broken-up into smaller sections, with a guess being made of the states at the start of each section. Using this method, the problem can essentially be converted into a sequence of smaller single-shooting sub-problems. The trajectory, and final value, of the states within each sub-problem are found by shooting from the start to the end of each section. As part of the optimisation, in order to ensure continuity of state values, the error between the states at the end of one section and the guessed states at the start of the following section is driven to zero [9]. The structure of the resulting task makes it a good candidate for parallelised computing in the interest of reducing computation time [8]. This allows a balance between the faster rate of convergence, afforded by collocation, whilst keeping the problem size small, as in direct single-shooting. By breaking up the optimisation problem into smaller sections and optimising across each, the problem of journey sensitivity is also minimised.

Casanova [12] provided a complete lap-time optimisation package which used direct multi/parallel-shooting methods in conjunction with SQP. He used three computational grids to break up the problem: the track section grid, the control implementation grid and the grid for the integration of the equations of motion. A detailed race-car was modelled which included aerodynamics, a realistic engine-map and a limited-slip differential. His approach shared the same caveats as other direct methods. Convergence to a solution was not guaranteed. If a solution did exist, it was not
guaranteed to be a global minimum. The process of using three computational grids, although avoiding the problem of journey sensitivity, was also very computationally intensive.

An alternative way of overcoming the journey sensitivity problem, which is associated with long optimisation horizon problems, such as finding the minimum lap-time, is to use receding-horizon optimal control. Here, a smaller optimal control problem is solved at each instant. The process is fully documented in the following section.

2.1.5 Non-linear Model Predictive Control (NMPC) / Receding-horizon optimal control

Non-linear Model Predictive Control (NMPC), or receding-horizon control, works by solving an open-loop discrete-time optimal control problem on-line over a finite, and usually short, preview horizon under a set of constraints [18]. Only the first control demand of the proposed control trajectory is used and the system is marched forward in time, see Figure 2.3. At the next time instant, a new optimal control problem is solved and the process repeated. This method facilitates the use of shorter optimisation horizons than the techniques discussed in the previous section. However, it can be wasteful as only the first control of each solution is used at each time instant. In light of this, the method can be computationally intensive, as noted by Gerdts et al. [22]. Hence, when using this method, a balance must be found in simplifying the problem, to increase the speed of computation, and retaining the essential details, so that the solution remains relevant to the original problem.

One way of simplifying the problem is to adjust the control parameters which are used to calculate an optimal solution. Shortening the preview horizon is a possibility, but the performance attributed to such methods can be poor, potentially inducing instability [18,19]. The computational burden can also be reduced by increasing the time between control implementations. However, as mentioned, a balance must be found between the possible speed-up offered by this approach and the degradation in system performance that is associated with re-setting the control less frequently. Finally, the control horizon, the length of time for which an optimal control sequence must be found, can be made shorter than the preview horizon. A shorter optimal control sequence affords faster computation of the solution at each time instant. However, the user must decide how the period of time beyond the control horizon, up to the preview horizon, contributes to the overall performance/cost function. One possibility is to assume the control input is held constant beyond the control horizon, as seen in Figure 2.3, and calculate the cost over the whole preview horizon as usual. Another option is to add a terminal cost term, to the main cost function, which attempts to account for the cost attributed to the control past the control horizon.
Figure 2.3: Diagrammatic representation of NMPC/receding-horizon method. The control is assumed to be held constant after the end of the control horizon, up to the preview horizon.
An alternative simplification is to linearise the plant model. Keen and Cole [29] linearise a car with a non-linear tyre model for different permutations of front and rear tyre lateral-slip values off-line. Although the tracking performance is acceptable using a simple car model, the process is likely to become more complicated if a more general car model is used, where more non-linearities are present. In work documented by Falcone et al. [17], the linearisation of the car model is done on-line. The trajectory of the linearised model is required to stay close to the true trajectory of the car for the calculated controls to remain optimal. The length of the preview horizon is therefore compromised and must be kept short enough to satisfy this requirement. Artificial constraints are also necessary to keep the tyres predominantly in their linear region of operation to minimise discrepancies between the real car model and its linear approximation. The authors admit that the removal of these constraints may lead to instability in the controller. The tracking results presented appear poor but could be improved by tuning parameters such as the length of the optimisation horizon, the time step and the constraints placed on the tyres. This tuning process is likely to be tedious and may be manoeuvre dependent, which is not ideal.

In conclusion to this section, a number of alternative methods, aimed at solving the minimum lap-time problem as one of pure optimal control, have been presented. Problems with each method have been highlighted which include speed of computation, guarantees of convergence and model complexity. From the analysis of these methods, the author believes that there exists a novel opportunity to address these problems in solving the minimum lap-time problem.

2.2 Optimal DLQR preview control

As mentioned earlier, the alternative solution to the minimum lap-time problem is comprised of three sub-problems: optimal tracking of a given path, optimising the speed along this path and perturbing the path to obtain the minimum lap-time. This section will present the development of the preview Discrete Linear Quadratic Regulator (DLQR) control technique which is used in the tracking controller in chapters 5 to 6.

It is widely accepted that previewing the path ahead leads to high-quality tracking control [24, 34, 40]. From a practical viewpoint, tracking without preview can be likened to driving a car looking at the road through a hole in the car floor rather than through the front windscreen. It is intuitive to see why using information of the road ahead to plan, prepare and execute control decisions in advance leads to improved tracking performance. Sheridan [58] presented the early idea of preview control in a mathematical framework. It was proposed that preview gains could be applied to the error between the future path of a plant, based on an internal model, and the
desired path. Tomizuka and Whitney [65] later presented the concept of diminishing returns in preview control, when longer preview horizons are used, and how preview information helped to smooth-out control decisions when tracking complex paths.

The framework of DLQR preview control delivers the behaviour and control characteristics proposed by Sheridan and Tomizuka. A thorough mathematical introduction of this theory is given in chapter 4, but a summary of the typical approach is presented as follows. Where appropriate, the system is linearised and discretised and, through the use of a quadratic cost function that penalises the departure of the system states from their equilibrium point, the problem can be formulated to obtain the optimal feedback gains for each state variable. This corresponds to the standard DLQR control problem. Recursive techniques can then be used to obtain the preview gains required for DLQR preview control, as documented in Prokop and Sharp [42].

DLQR preview control has been used in a number of practical applications. In the area of preview suspension control, following on from work conducted by Bender [6] and Tomizuka [64], Louam et al. [33] used front suspension deflection to act as a preview of the road so that control could be applied to the rear suspension to improve ride qualities. A performance index was constructed which highlighted the improvements to the system through the use of preview. Later, Prokop and Sharp [42] also conducted work on active suspension systems. The relationship between the suspension actuator’s bandwidth and the preview gains based on it was studied. The preview gains were found to adapt themselves to the eigenmodes of the suspension system, trying to excite high frequency eigenmodes over short preview lengths and lower frequency eigenmodes over longer preview lengths.

A considerable amount of research has been conducted applying DLQR preview control specifically to optimal path-tracking, which is of particular interest for the work in this thesis. Worthy of mention is Sharp and Valtetsiotis’ [57] presentation of the invariance of optimal control preview gains to the transformation from global to local reference-frames. This important development extended the use of DLQR preview control from tracking small perturbations about a fixed global-axis to tracking more general paths using a local reference-frame. Progress has also been made in both longitudinal [47, 49] and lateral [48] tracking control. The quality of the tracking controller derived from DLQR preview methods have been demonstrated in the control of both two-wheeled [46, 50] and four-wheeled [45] vehicles. The tracking controller used in these contributions relies on the off-line linearisation of the vehicle model for an equilibrium/trim-state, usually straight-running at a constant forward speed. The solution to the DLQR preview problem is found and the preview gains stored pre-simulation. During the tracking simulation, the gains are used on the respective errors between the predicted path, associated with the trim-state the gains were designed for, and
the intended path of the car to calculate and apply the optimal control input at each time instant.

The current state of DLQR preview control methods was criticised in work conducted by Hazell [25], where a general framework for the preview control of vehicles was proposed using a full information $H_2$ preview controller. The work argued that although good tracking qualities have been demonstrated with DLQR preview control, the control gains are designed for only one trim-state. Hence, this limits the conditions for which the controls, which are calculated from these gains, are optimal. The practice of separating the longitudinal and lateral dynamics of the vehicle, in order to compute the optimal control gains, was also criticised as it prevented the controller from capitalising on the coupling between such dynamics.

In an attempt to overcome the limitations identified by Hazell, the author has firstly identified the opportunity of using DLQR preview control over the full extent of the car’s capability by linearising the non-linear model for a number of different trim-states. It is assumed that driving an optimal lap involves smooth and progressive changes in control input and that the car is always close to one of these equilibrium/trim-states. Hence, the optimal control gains for each trim-state are calculated and stored pre-simulation. Gain-scheduling and interpolation are employed during the tracking simulations to match and retrieve the most appropriate trim at each time instant, and install the optimal gain-set based on it. It must be noted that an erratic driving style, especially when driving near the car’s limit, would cause the car to deviate from steady-state behaviour. Any controls that are installed when the car is driven in this style would not be guaranteed to be optimal. However, as this driving style typically produces a poor lap-time, it does not need to be considered in the development of the tracking controller.

The proposed method has the inherent benefit that the optimal control gains are based on the car setup rather than the path being followed. This can prove useful when conducting trials using the same car design/setup on a number of different paths or manoeuvres. Here, a one-off calculation of optimal control gains can be performed, with the gains being stored and reused to find the optimal tracking control for the different paths. This is in contrast with other optimisation processes which would require the repetition of, often lengthy, optimisation calculations for each track.

In addition to extending the work on adaptive DLQR preview control, the author also notes the opportunity to combine the previously decoupled problems of longitudinal and lateral tracking into a combined two-channel $(x,y)$ tracking controller, which was noted as a key requirement for effective tracking by Hazell. If previous limitations of DLQR preview control are addressed, the technology can be made applicable for the first stage of the novel solution to the minimum lap-time problem. Having identified an opportunity to develop the proposed novel solution, the next stage
is to define the car model that will be used to formulate the problem. Details of the *simple* car model, which was used during the initial stage of controller development, will be presented in the next chapter.
Chapter 3

Simple Car Model

As mentioned in the previous chapter, the preview path-tracking controller is designed around, and is therefore specific to, the car model that is used. It is therefore essential to establish the car model which will be used in the initial stages of controller development. The model is required to be simple enough to be useful, modelling only the essential features of a real race-car at this stage, the simplicity of the model allowing a clearer investigation of the controller during its early development. The design philosophies of real race-cars allow a number of modelling simplifications to be made whilst other details can be excluded at this level of investigation. This chapter firstly explains the simplifying assumptions that were made when defining the simple car model. Two distinctly different setups of the simple car, which are worthy of investigation, will then be presented. Finally, the chapter will deal with the need for linearisation and document how trim-states are obtained to facilitate the linearisation process.

The simple car is the result of modelling decisions based on different aspects of a real race-car. Each aspect will now be discussed in detail.

3.1 Chassis

The chassis of a real race-car is required to be a rigid platform for its suspension [36]. It is reasonable to assume it is a single rigid body with a total mass, $M$, with a yaw moment of inertia, $I_z$, about the car’s centre of mass. Each of the car’s four wheels is attached to this rigid body, typically in a dual-track setup, as seen in figure 3.1. The translational velocity of each tyre’s contact point with the road is used to define its respective lateral-slip ratio. However, this velocity, as with any point on the car, can be considered to be dominated by the longitudinal velocity of the whole car. Hence, the velocity of each contact point is approximately the same as its velocity if each tyre were located at the centre of its axle. This justifies the use of a more convenient single-track
bicycle model which is used in several publications [16, 29, 36]. The conversion from a dual-track vehicle model to a single-track bicycle model, where the tyres are lumped at the centre of each axle, can be seen in figure 3.1.

Figure 3.1: Diagrammatic representation of forces and dimensions of the simple car model.

3.2 Suspension

The suspension system allows the handling characteristics of a car to be tuned for the specific application/track. Circuit race-cars often employ aerodynamics to generate down-force, which can improve the car’s performance, as seen in figure 2.1. The performance attributed to the aerodynamics is usually very sensitive to the ride-height, pitch and roll of the car. A stiff suspension setup is usually employed to keep these parameters at the required level throughout a lap. This implies that the car’s vertical displacement, as well as its movement in pitch and roll, is very small. It is therefore reasonable, when modelling the chassis, to exclude these directions from the original six degrees of freedom. This leaves the chassis with only three degrees of freedom: translation in $x$ and $y$ and rotation in yaw, $\psi$. These degrees of freedom can be measured in either a global, ground-fixed, reference frame or a local, car, reference frame as seen in figure 3.2.
3.3 Aerodynamics

As mentioned previously, modern circuit race-cars utilise aerodynamics to increase vertical tyre-load. The shear forces generated by the tyres, as well as the car’s performance, are generally increased as a result. Aerodynamic drag is also present and will affect the performance of the car. However, in the present context of the simple car, aerodynamics will not be modelled at this stage.

3.4 Steering

The steering system of a race-car is required to be as responsive as possible. In modelling such a system, the steering of the simple car consists of a direct link between the steering wheel angle, $\theta_{sw}$, and the steering of the front lumped tyre system, $\theta$, such that

$$\theta = \frac{\theta_{sw}}{G}$$

where $G$ is the steering gear ratio.

Hence, the lateral-slip ratio of the front lumped tyre system, $\alpha_f$, is defined as
\[
\alpha_f = \frac{u \sin \delta - \left( v + a\dot{\psi} \right) \cos \delta}{u \cos \delta + \left( v + a\dot{\psi} \right) \sin \delta}
\]

whilst the lateral-slip ratio of the rear lumped tyre system is

\[
\alpha_r = \frac{b\dot{\psi} - v}{u}
\]

At this level of simulation, the modelling of the wheels is kept relatively simple. They are considered to be massless, possessing no moment of inertia, and do not have any degrees of freedom.

### 3.5 Driving/Braking force

The engine is usually the source of the driving force that propels a race-car. This force is usually transmitted through a complex drive-train, which includes a gearbox and differential gear assembly, before being applied to the driven wheels. When decelerating the car, standard brakes, engine-braking and aerodynamic drag usually provide the necessary retardation force. Although the longitudinal dynamics which result from these driving-/braking-forces should be considered, at this stage of investigation, only the lateral dynamics of the simple car will be studied. In light of this decision, the forward speed of the simple car will be kept constant. Due to the exclusion of aerodynamic drag, the driving force is required to only balance the drag force on the car. This drag force, resulting from resolving the lateral forces generated by the front tyres \(F_{yf}\) along the car’s longitudinal direction, will be very small. The modelling of a complex drive-train assembly to produce such magnitudes of force is seen as unnecessary. The driving force is therefore supplied instead by a force, \(F_x\), applied at the car’s centre of mass. A PID speed controller is used to vary this force and keep the forward speed of the car constant.

### 3.6 Tyres

Attention will now shift to the essential interface between a race-car and the track, the tyres. Each tyre typically generates a longitudinal-/lateral-force as well as an aligning moment [39]. The main factors affecting these forces/moments are: the tyre’s vertical load, its longitudinal slip and its lateral slip.

In constructing a tyre model for the simple car, the aligning moment will be excluded at this level of investigation. Additionally, the driving force of the car will be provided by the longitudinal force, \(F_x\), as mentioned previously. Hence, the tyre is not required to provide a longitudinal force.
and this characteristic is also excluded from the tyre model. The tyre model is therefore only required to model the generation of lateral tyre force.

With regard to the factors which serve as inputs to the tyre model, the constant forward-speed condition combined with the exclusion of aerodynamic drag results in the longitudinal slip of the tyres being negligible. Modelling the effect of tyre-load variation on lateral tyre-force is likely to provide an unnecessary complexity to the simple car model at this stage. As a compromise, an approximation is found which averages the variation of lateral tyre force with respect to tyre load. This can be done by averaging the value of each tyre parameter across a range of operating tyre loads.

From these simplifying assumptions, it is clear that the tyre model of the simple car is only required to model the relationship between the lateral slip of the tyre and the lateral force it generates. The modelling of such a relationship is given by the Pacejka Magic Formula [39]

\[
F_y = 2D_m \sin \left( C_m \arctan \left( B_m \alpha - E_m \left( B_m \alpha - \arctan \left( B_m \alpha \right) \right) \right) \right)
\]

with the lateral-slip ratio, \( \alpha \), being \( \alpha_f \) for the front lumped tyre system or \( \alpha_r \) for the rear. \( B_m, C_m, D_m \) and \( E_m \) are tyre-force-shaping factors. As previously mentioned, they are the result of averaging coefficients of the Pacejka Magic Formula, presented in Bakker et al. [5], for a range of tyre loads and expressing them in SI units. The values of these factors are presented in table 3.1 along with the remaining parameters of the simple car model. The resulting lateral force vs. lateral-slip ratio curve can be seen in Figure 3.3, which clearly shows the saturating nature of the tyre force.

![Figure 3.3: Lateral force as a function of lateral-slip ratio for each tyre.](image)

The car model is built using the multi-body modelling program, VehicleSim®, formerly called AutoSim [37,43,53]. The program uses a LISP input-file to describe the structure of the system.
The model is required to include the absolute lateral displacement of the car amongst its outputs. This parameter is used to follow the road, which itself is a sequence of lateral displacement demands, in the tracking trials and is therefore required when formulating the control problem. Due to the simplicity of the model, the kinematics of the car can be expressed easily as follows:

\[ \dot{x} = u \cos \psi - v \sin \psi \]
\[ \dot{y} = v \cos \psi + u \sin \psi \]

The dynamic equations are:

\[ M \left( \dot{u} - v \dot{\psi} \right) = F_x - F_{yf} \sin \theta \]
\[ M \left( \dot{v} + u \dot{\psi} \right) = F_{yr} + F_{yf} \cos \theta \]
\[ I_z \ddot{\psi} = aF_{yf} \cos \theta - bF_{yr} \]

By defining the longitudinal position of the car’s centre of mass along the chassis, the user can specify two contrasting car setups which are of interest. The Society of Automotive Engineers (SAE) procedure for characterising the cornering behaviour of a car [15] is used to classify these two setups. The procedure finds the steering wheel angle required to maintain the car on a path with a constant radius of curvature, for a range of forward speeds. Figure 3.4 shows how the steering wheel angle must vary as the forward speed, and therefore the lateral acceleration, is increased for the first setup case, where the centre of gravity is forward of the chassis centre (using the values of \( a = 0.92 \) m and \( b = 1.38 \) m, originally presented in work conducted by Sharp and Valtetsiotis [57]). The gradient of the plot is used to qualify the characteristic of the car, with a positive gradient attributed to an understeering car. The setup is such that the lateral force at the front axle will always limit before the rear axle and the car tends to plough ahead when trying to execute a tighter corner. The maximum lateral acceleration of such a setup is found by first considering the free-body diagram of the simple car in such a scenario, as seen in figure 3.5.

Taking moments about the centre of mass, the following statement must hold for the car to be in equilibrium

\[ F_{yf} a = F_{yr} b \]  \hspace{1cm} (3.3)
Figure 3.4: Plots of steering wheel angle required to maintain the simple car on a path with a constant radius of curvature, of 133 m, for a range of lateral accelerations. Three car setups are represented: understeering (solid blue line, $a = 0.92$ m, $b = 1.38$ m), neutral-steering (dashed red line, $a = 1.15$ m, $b = 1.15$ m) and oversteering (dot-dashed green line, $a = 1.38$ m, $b = 0.92$ m). The critical acceleration point of the oversteering setup, where the steering angle becomes negative, and the setup is known to become divergently unstable, is also highlighted.

Figure 3.5: Top view free-body diagram of understeering simple car setup as the front axle limits (considering only lateral dynamics and using small angle approximations).
Additionally, the lateral forces must equal the lateral acceleration of the car, such that

\[ F_{yf} + F_{yr} = M a_{lim} \]  \hspace{1cm} (3.4)

where \( a_{lim} \) is the lateral acceleration of the car as the front axle limits. Substituting equation 3.3 into 3.4 and expressing in terms of the front, limiting, axle lateral force, \( F_{yf} \), results in

\[ a_{lim} = \left( \frac{b}{a} + 1 \right) \frac{F_{yf}}{M} \]  \hspace{1cm} (3.5)

From the tyre characteristics presented earlier, the maximum lateral force available at each axle is 7800 N, which, when substituted into equation 3.5 indicates that the understeering setup is capable of front-axle-limited cornering at 12.38 m/s\(^2\) lateral acceleration.

The simple car’s cornering characteristics are changed significantly by relocating the centre of gravity rearward of the centre of the chassis \((a = 1.38 \text{ m}, b = 0.92 \text{ m})\). Following the same procedure conducted on the understeering setup, a plot of the steering angle required to keep the car on the same constant radius path, for varying values of forward speed, is found and presented in figure 3.4. The resulting plot has a negative gradient throughout the range of operation. Hence, the car is qualified as having an oversteering setup in this case. For the oversteering case, the car becomes divergently unstable once past a critical speed/lateral acceleration, indicated in figure 3.4, where the required steering angle becomes negative. This point will become significant when considering the control of the oversteering simple car model in the vicinity of this region. Such a setup is theoretically capable of rear-axle-limited cornering, employing the same procedure presented above for the understeering car, again at 12.38 m/s\(^2\) lateral acceleration. However, the instability associated with such a setup limits the configuration to a realistic limit of 9.88 m/s\(^2\) lateral acceleration. The characteristics of a neutral steering setup, where the centre of mass is coincident with the centre of the chassis \((a = 1.15 \text{ m}, b = 1.15 \text{ m})\), is also included in figure 3.4 for completeness. Such a setup requires a constant steering angle to keep on the same path, with a constant radius of 133 m, as forward speed is increased.

Chapters 5 and 6 will document how the optimal control strategy must be adapted to account for the distinctly different chassis setups of the understeering and oversteering simple car.

Although the simple car model represents a significant reduction in detail when modelling a real race-car, it still possesses non-linear features, such as saturating tyre-forces. The equation
Table 3.1: Parameters, symbols and values for the car and tyres for understeering and oversteering simple car setups. Parameter values are based on figures originally presented by Sharp and Valtetsiotis [57].

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Understeerer</th>
<th>Oversteerer</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass</td>
<td>$M$</td>
<td>1050 kg</td>
<td>same</td>
</tr>
<tr>
<td>Yaw inertia</td>
<td>$I_z$</td>
<td>1500 kgm$^2$</td>
<td>same</td>
</tr>
<tr>
<td>Mass centre to front axle distance</td>
<td>$a$</td>
<td>0.92 m</td>
<td>1.38 m</td>
</tr>
<tr>
<td>Mass centre to rear axle distance</td>
<td>$b$</td>
<td>1.38 m</td>
<td>0.92 m</td>
</tr>
<tr>
<td>Steering gear ratio</td>
<td>$G$</td>
<td>17</td>
<td>same</td>
</tr>
<tr>
<td>Stiffness factor (per tyre)</td>
<td>$B_m$</td>
<td>17.5</td>
<td>same</td>
</tr>
<tr>
<td>Shape factor (per tyre)</td>
<td>$C_m$</td>
<td>1.68</td>
<td>same</td>
</tr>
<tr>
<td>Peak factor (per tyre)</td>
<td>$D_m$</td>
<td>3900 N</td>
<td>same</td>
</tr>
<tr>
<td>Curvature factor (per tyre)</td>
<td>$E_m$</td>
<td>0.6</td>
<td>same</td>
</tr>
</tbody>
</table>

which describes the non-linear evolution of the car’s states, $\mathbf{x}$, can be expressed generally as

$$\dot{\mathbf{x}} = f(\mathbf{x}, \theta_{sw})$$ (3.6)

In order to be compatible with the linear optimal control methods detailed in the next chapter, the car model must be linearised. Equation 3.6 can be rewritten as a perturbation in the system states, $\mathbf{x}_p$, and input, $\dot{\theta}_{sw}$, from an arbitrary state ($\mathbf{x}_{eq}, \theta_{eq}$) such that

$$\dot{\mathbf{x}} = f(\mathbf{x}_{eq} + \mathbf{x}_p, \theta_{eq} + \dot{\theta}_{sw})$$ (3.7)

Using a Taylor expansion and excluding second- and high-order terms, equation 3.7 is simplified to its linear form

$$\dot{\mathbf{x}} = f(\mathbf{x}_{eq}, \theta_{eq}) + \left. \frac{\partial f}{\partial \mathbf{x}} \right|_{\mathbf{x}=\mathbf{x}_{eq}} \mathbf{x}_p + \left. \frac{\partial f}{\partial \theta_{sw}} \right|_{\mathbf{x}=\mathbf{x}_{eq}, \theta_{sw}=\theta_{eq}} \dot{\theta}_{sw}$$ (3.8)

If the state were chosen such that $f(\mathbf{x}_{eq}, \theta_{eq}) = 0$, i.e. an equilibrium state, for small perturbations, the linearised system can be represented in general state-space form as

$$\dot{\mathbf{x}} = A_c \mathbf{x}_p + B_c \theta_{sw}$$ (3.9)
where $A_c = \frac{\partial f}{\partial x} \bigg|_{x=x_{eq} \theta_{sw}=\theta_{eq}}$ and $B_c = \frac{\partial f}{\partial \theta} \bigg|_{x=x_{eq} \theta_{sw}=\theta_{eq}}$. For the simple car, $x_p$ represents a state error vector of the form

$$x_p = \begin{bmatrix} y - y_{eq} \\ \psi - \psi_{eq} \\ \dot{y} - \dot{y}_{eq} \\ \dot{\psi} - \dot{\psi}_{eq} \end{bmatrix} \quad (3.10)$$

where each element represents a small perturbation of the current state from its value at equilibrium, denoted by the subscript $eq$.

The VehicleSim® software package is able to perform such a linearisation, of a model defined in its LISP file format, provided that the non-linear equations are continuous. The resulting MATLAB® M-file uses symbolic code to create the linearised state-space matrices, $A_c$ and $B_c$, using the equilibrium-/trim-values of $x_{eq}$ and $\theta_{eq}$ as inputs. In order to be compatible with the DLQR process presented in the next chapter, the system in equation 3.9 must be discretised, resulting in

$$x_p(k+1) = A_p x_p(k) + B_p \theta_{sw}^T(k) \quad (3.11)$$

with the discrete time-counter, $k$.

The trim-state used to generate the state-space equation matrices in equation 3.11 is usually collected offline i.e. prior to the tracking simulations. This is done by installing PID controllers at each control input. In the case of the simple car, a front/rear lateral-slip ratio controller actuates the steering wheel angle whilst a speed controller actuates the forward driving force, $F_x$. The demanded values of these controllers are set at the beginning of the trim collection simulation. The simulation is then run, with the value of the states and input being collected once all transients have decayed to zero. These values are fed into the M-file generated by VehicleSim to create a linearisation of the car for the specific equilibrium-/trim-state. The MATLAB® $c2dm$ function is then used to discretise the model using a zero-order hold and a time-step of $T$, resulting in the discrete state-space equations necessary for the optimal control calculations.

In conclusion, this chapter has presented the assumptions used to define the simple car model.
Two distinctly different chassis setups were specified: an understeering and oversteering setup. Finally, the process of linearising the car model, so that it is compatible with the linear methods in the next chapter, has also been presented. As the modelling details of the car have now been finalised, attention will now shift to the controller which will be used to optimally control the vehicle.
Chapter 4

The DLQR preview tracking controller

This chapter aims to introduce the reader to the preview DLQR tracking controller, which is seen as a vital component of the proposed solution to the minimum lap-time problem. The chapter is divided into three sections. The first section will present the underlying theory behind preview DLQR control, summarising work that has already been conducted in the field. The use of standard DLQR theory, to solve a general tracking problem, will be presented initially. The theory will then be extended to preview DLQR control before it is applied to the problem of optimal path tracking. Observations of the resulting optimal control scheme will be made at this stage. The second section will detail how the optimal control scheme, derived from the theory in section I, is typically implemented in tracking simulations to provide optimal tracking control. Whilst the preceding sections serve to summarise previous work in the field of DLQR preview control, the third section will be devoted to documenting novel research conducted by the author and how it has contributed to the development of advanced tracking controllers. These contributions are: implementing an adaptive path tracking controller, through the use of gain-scheduling, and the development of a two-channel, variable speed, path tracking controller.

4.1 DLQR preview control theory

A standard tracking problem can be set up by first defining the plant which is to be controlled, in this case, one which is linear and time-invariant. The plant is discretised using a zero-order-hold and a time step of $T$. This ensures compatibility with the shift register process, introduced in the later part of this section, which is represented more straightforwardly in discrete form. The
state-space equations of such a plant are

\[ x_p(k + 1) = A_p x_p(k) + B_p \eta(k) \]

\[ y_p(k) = C_p x_p(k) \]

with discrete-time counter \( k \), plant state vector \( x_p \) and single control input \( \eta \).

Consider the tracking problem where the states of the plant, \( x_p \), are required to track a state signal, \( \tilde{x}_p \). A quadratic cost function

\[ J_p = \lim_{n \to \infty} \sum_{k=0}^{n} \{ (x_p(k) - \tilde{x}_p(k))^T Q_p (x_p(k) - \tilde{x}_p(k)) + \eta^2(k) R_p \} \]  

(4.1)

can be set up such that the matrix \( Q_p \) places weightings on the tracking performance over the optimisation horizon, whilst the scalar \( R_p \) places a cost on the expenditure of control input energy [38]. If the pair \( (A_p, B_p) \) is stabilizable, standard Discrete Linear Quadratic Regulator theory can be used to find the time-invariant optimal control, \( \eta^* \), which minimises the cost function [3].

\[ \eta^*(k) = -K_1 (x_p(k) - \tilde{x}_p(k)) = -K_1 x_p(k) + K_1 \tilde{x}_p(k) \]  

(4.2)

where

\[ K_1 = (R_p + B_p^T P_{11} B_p)^{-1} B_p^T P_{11} A_p \]  

(4.3)

given that \( P_{11} \) satisfies the matrix-difference-Riccati equation:

\[ P_{11} = A_p^T P_{11} A_p - A_p^T P_{11} B_p (R_p + B_p^T P_{11} B_p)^{-1} B_p^T P_{11} A_p + Q_p \]

The significance of the structure of equation 4.2 is described by Athans and Falb [4]. The first term is a multiplication of the full state-feedback gain vector, \( K_1 \), to the state vector of the linear system, \( x_p \). As the linear plant represents a perturbation from an equilibrium-/trim-state, this will serve to stabilise the system about this state. The second term serves to counterbalance the feedback and facilitates the tracking of the demand signal. Once formulated, computer software
such as the MATLAB\textsuperscript{TM} \textit{dlqr} function can be used to obtain the optimal control gains for the tracking problem.

The performance of the tracking controller can be improved if information of the demand signal is known ahead of time, using so-called preview [6,24,64]. The dynamics of such a process can be captured by the form

\[
\tilde{y}_r(k + 1) = A_r \tilde{y}_r(k) + B_r \tilde{y}_{rn}(k) \tag{4.4}
\]

with scalar $\tilde{y}_{rn}$, now representing the demand signal to be tracked, entering the preview process at time instant $k$ and $\tilde{y}_r$ being the vector of demands captured by the preview. For $n$ previewed demands, $\tilde{y}_r$ is $(n \times 1)$. $A_r$ is a $(n \times n)$ matrix of the form:

\[
A_r = \begin{bmatrix}
0_1 & I_1 & 0_1 & \ldots & 0_1 \\
0_1 & 0_1 & I_1 & \ldots & 0_1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0_1 & 0_1 & 0_1 & \ldots & I_1 \\
0_1 & 0_1 & 0_1 & \ldots & 0_1 \\
\end{bmatrix} \tag{4.5}
\]

which represents a shift register process. When tracking a single demand signal, $0_1$ is a $(1 \times 1)$ zero matrix while $I_1$ is a $(1 \times 1)$ identity matrix. At the next time instant, $(k + 1)$, the first element of $\tilde{y}_r$ is lost. The second element at time instant $k$ becomes the first element at time instant $(k + 1)$, the third element at time instant $k$ becomes the second at time instant $(k + 1)$, and so on. Finally, the demand which was just outside the preview vector at time instant $k$, $\tilde{y}_{rn}(k)$, now becomes the last element of the preview vector at time instant $(k + 1)$.

The plant dynamics can be augmented with the dynamics of the preview process [33,42,64] to yield the equation

\[
\begin{bmatrix}
x_p(k + 1) \\
\tilde{y}_r(k + 1)
\end{bmatrix} = \begin{bmatrix}
A_p & 0 \\
0 & A_r
\end{bmatrix} \begin{bmatrix}
x_p(k) \\
\tilde{y}_r(k)
\end{bmatrix} + \begin{bmatrix}
B_p \\
0
\end{bmatrix} \eta(k) + \begin{bmatrix}
0 \\
B_r
\end{bmatrix} \tilde{y}_{rn}(k) \tag{4.6}
\]
which takes the standard discrete-time form:

\[ z(k + 1) = Az(k) + B\eta(k) + E\tilde{y}_{rn}(k) \]  

(4.7)

\[ y(k) = Cz(k) \]  

(4.8)

where \( z(k) \) is the full system state vector. A new cost function is set up for the augmented system which is analogous to equation 4.1.

\[ J = \lim_{n \to \infty} \sum_{k=0}^{n} \{z^T(k)Qz(k) + \eta^2(k)R\} \]  

(4.9)

\( Q \) is now of the form \( Q = C^T q C \), where \( q \) is a diagonal weighting matrix, \( \text{diag}[q_1, q_2, \ldots] \), with terms corresponding to the number of performance aspects contributing to the cost function. \( C \) is chosen to define the performance aspects, such that the quadratic term \( z^T(k)Qz(k) \) places a cost on the sum of the squares of the errors, over the optimisation horizon.

If \( \tilde{y}_{rn} \) is a sample from a white-noise random sequence, the pair \((A, B)\) is stabilizable and the pair \((A, C)\) is detectable, the optimal control which minimises the cost function of the augmented system is now

\[ \eta^*(k) = -Kz(k) \]  

(4.10)

where \( K = (R + B^TPB)^{-1} B^TPA = [K_1 \quad K_2] \).

Here, \( K_1 \) is the original feedback matrix corresponding to the plant without preview. \( K_2 \) is vector of optimal shift register state-feedback gains, or preview gains, which are multiplied to the vector of previewed demands, \( \tilde{y}_r \).

The full system can be passed to the MATLAB\textsuperscript{TM} \textit{dlqr} function, as before, to obtain a numerical solution for \( K \). However, an alternative, potentially faster, method exists which has been documented by Louam et al. [33]. The process first finds the solution to the discrete algebraic Ricatti equation of the system without preview, \( P_{11} \). This result forms part of the solution to the full preview case

\[ P = A^TPA - A^TPB (R + B^TPB)^{-1} B^TPA + Q \]
where

\[
P = \begin{bmatrix}
P_{11} & P_{12} \\
P_{21} & P_{22}
\end{bmatrix}
\]  

(4.11)

Elements of this symmetric matrix solution can be systematically constructed from the recursive relationship

\[
P_{12} = [\Phi_c^T, (\Phi_c^2)^T, \ldots, (\Phi_c^n)^T]P_{11}
\]  

(4.12)

where \( \Phi_c \) is the closed-loop matrix for the system without preview and is equal to \( A - BK_1 \). Only the sub-matrices \( P_{11} \) and \( P_{12} \) are required to calculate the optimal control gains, with \( K_1 \) found using equation 4.3 whilst \( K_2 \) is found from the relationship

\[
K_2 = (R + B^TP_{11}B)^{-1}B^TP_{12}A_r
\]  

(4.13)

Having set up and solved a general tracking problem using preview DLQR theory, a final step remains in making the theory relevant to the current problem: applying the process to the task of optimal path tracking. In this context, the plant is the simple car model linearised for a specific trim (chapter 3, equation 3.9), with the car’s steering wheel angle, \( \theta_{sw} \), representing the single control input. The tracking problem requires the car to follow a path at a constant forward speed, \( u \). The demand signal, \( \tilde{y}_{rn} \), steps through each element of a sequence of discrete lateral displacements, \([\tilde{y}_{r0} \ \tilde{y}_{r1} \ \tilde{y}_{r2} \ldots]\), which describe the path in a global reference frame. The path tracking problem is diagrammatically represented in figure 4.1.

In the context of lateral path tracking, the performance matrix, \( Q \), is designed to place a cost on a single performance factor, lateral tracking error. The ratio between the single diagonal element, \( q_2 \), and \( R \) in equation 4.9 represents the balance in importance between reducing lateral path tracking error and reducing control effort at the steering wheel. When more importance is placed on reducing tracking error, typically resulting in higher control input activity, the control scheme is said to be tight. Conversely, when more importance is placed on reducing control input activity, typically resulting in larger tracking errors, the control scheme is said to be loose. The effect of choosing different weighting strategies on the resulting preview gains can be seen in figure 4.2. The
Figure 4.1: Diagrammatic representation of tracking problem with the path, measured in a global reference frame, followed at a constant speed, $u$.

$K_2$ preview gains of three control tightness strategies are presented: tight ($q_2=75$, $R=1$), medium ($q_2=25$, $R=1$) and loose ($q_2=1$, $R=1$). The straight-running configuration of the understeering simple car setup, presented in chapter 3, is used for this demonstration.

The preview gains for each control strategy in figure 4.2 exhibit the common feature of diminishing returns when increasing the number of preview points, initially observed in work by Tomizuka and Whitney [65]. It is clear that after a certain number of preview points are used, a further increase will have a negligible effect on control decisions. The controller is said to have full preview at this point. A typical procedure used to approximate the full preview length, the so-called 99% rule [45], is to continue to increase the preview length until 99% of the area under the total gain curve has been enclosed. As observed in figure 4.2, using a loose control scheme requires a longer full preview length, requiring 1.76 s under the 99% rule. This is a consequence of placing a greater cost on control input activity. More information of the path ahead is required to keep the control optimal and compensate for this restriction. In contrast, a tighter control scheme is characterised by higher gain magnitudes concentrated close to the car. This results in a sharper control action being employed, which can improve tracking performance. Hence, the tighter control scheme typically requires a shorter full preview length to achieve optimal control, requiring only 0.84 s under the 99% rule, as observed in figure 4.2.

The closed-loop control system frequency-responses for the three exemplary control strategies are presented in figure 4.3. The input to each system is a series of lateral displacements, $\tilde{y}_{rn}$, applied at the end of the preview window whilst the output is the lateral displacement of the car’s
Figure 4.2: $K_2$ preview gains for three control strategies: tight (solid blue line, $q_2=75$, $R=1$, requiring 0.84 s under the 99% preview rule), medium (dashed red line, $q_2=25$, $R=1$, requiring 0.90 s under the 99% preview rule) and loose (dot-dashed green line, $q_2=1$, $R=1$, requiring 1.76 s under the 99% preview rule). Each strategy is designed for the understeering simple car setup in a straight-running trim configuration, using a time step of 0.01 s.
centre of mass. By exciting the input using a sinusoidal path with a range of frequencies, the closed-loop control system bandwidth for each control strategy can be observed. As the output is responding to an excitation at a finite distance ahead of it, there is an inherent phase lag between it and the input, termed as the transport delay, which can be seen in the phase plot. The plots in figure 4.3 show the typical result where a tighter control scheme has a wider bandwidth by virtue of expending more control input energy. As lively control action is penalised when using a loose control strategy, the bandwidth associated with it is appreciably lower.

![Bode plot of control system frequency response](image)

Figure 4.3: Closed-loop control system frequency response in Bode plot form corresponding to control strategies presented in figure 4.2: tight (solid blue line, $q_2=75$, $R=1$), medium (dashed red line, $q_2=25$, $R=1$) and loose (dot-dashed green line, $q_2=1$, $R=1$). The input is an absolute lateral path displacement, $\tilde{y}_{rn}$, applied at the end of the preview window. The output is the lateral displacement of the car’s centre of mass. The phase lag, or transport delay, is the consequence of the input being a finite distance ahead of the output. Each control strategy is designed for the understeering simple car setup in a straight-running trim configuration.

As previously mentioned, the solution provided by preview DLQR theory assumes the demand signal is a sample of a white-noise random sequence. When applied to the context of path tracking, the demand signal, which now represents lateral path displacements, has more of a resemblance to low-pass filtered white-noise. Prior research [45, 65] has found that the optimal control scheme based on tracking white-noise is still optimal when tracking a low-pass filtered version as long as full preview is utilised. Hence, the optimal control gains derived from the methods described earlier remain relevant to the problem of tracking typical path profiles.
In summary to this section, the theory behind DLQR control has been introduced and extended to preview DLQR control. The process of obtaining the optimal control gains has been described and the relevance to lateral path tracking has been established. The focus will now shift to the practical implementation of these control gains in the tracking controller, in order to provide optimal control.

4.2 Implementation of DLQR preview control

This section sets out to specify the optimal control calculations performed by the controller at each time step of the constant speed tracking simulations. The specification is then used to highlight the consequences of taking measurements in a local reference frame.

Equation 4.10 will be used as a basis for determining how the optimal steering control input is calculated at each time instant, \( k \). The optimal control term, \( \eta^* \), now referred to as \( \theta^*_sw \) in the context of lateral path tracking, represents a perturbation from the value of the steering wheel angle at trim, \( \theta_{eq} \). This implies that the absolute value of the optimal steering control input, \( \theta_{sw} \), is the sum of: the trimmed value of the control input (\( \theta_{eq} \)), the feedback contribution (\( \theta_{K1} \)) and the preview contribution (\( \theta_{K2} \)). In short

\[
\theta_{sw} = \theta_{eq} + \theta^*_sw = \theta_{eq} + \theta_{K1} + \theta_{K2} \tag{4.14}
\]

Details of each contribution to the final steering input signal will now be considered.

When applied to the problem of constant speed lateral tracking, the reference-/trim-state typically used is one of straight-running. The value of the steering input corresponding to this trim is clearly \( \theta_{eq} = 0 \).

The feedback contribution is the result of a multiplication between the feedback gains, \( K_1 \), and the states of a linear system, in this case, the simple linear car model. Equation 3.10 has already shown how the states of the car model can be represented as a state error vector

\[
x_p = \begin{bmatrix}
y - y_{eq} \\
\psi - \psi_{eq} \\
y' - y_{eq} \\
\psi' - \psi_{eq}
\end{bmatrix}
\]

where each element represents a perturbation of the current state from its value at equilibrium, denoted by the subscript \( eq \). As each state is measured in a global reference frame, \( y_{eq} \) and \( \psi_{eq} \)
can be set to zero when using a straight-running trim-state. $\dot{y}_{eq}$ and $\dot{\psi}_{eq}$ can also be set to zero whilst maintaining generality.

The feedback contribution in equation 4.14 can therefore be expressed as

$$\theta_{K_1} = -K_1 x_p = -K_1(1)y - K_1(2)\psi - K_1(3)\dot{y} - K_1(4)\dot{\psi} - ...$$  \hspace{1cm} (4.15)$$

The preview contribution is the result of a multiplication between the preview gains, $K_2$, and the vector of preview errors, $\tilde{y}_r$. In the current case, the preview errors, which are also measured in a global reference frame, represent absolute lateral displacements of the previewed path. When expanded, this contribution takes the form

$$\theta_{K_2} = -K_2 \tilde{y}_r = -K_2(1)\tilde{y}_r0 - K_2(2)\tilde{y}_r1 - K_2(3)\tilde{y}_r2 - ...$$  \hspace{1cm} (4.16)$$

The calculation of the optimal steering wheel angle at each time step can therefore be expressed as

$$\theta_{sw} = \theta_{eq} - K_1(1)y - K_1(2)\psi - K_1(3)\dot{y} - K_1(4)\dot{\psi} - ...$$

$$- K_2(1)\tilde{y}_r0 - K_2(2)\tilde{y}_r1 - K_2(3)\tilde{y}_r2 - ...$$  \hspace{1cm} (4.17)$$

The tracking control scheme can also be expressed diagrammatically, as in figure 4.4.

As mentioned earlier, the system, together with the lateral displacements which define the path, is defined in a global reference frame. Although this keeps the specification of the shift register process in equation 4.4 simple, a more natural representation comes from using the local reference frame of the car, as seen in figure 4.5.

The process of moving from a global to a local reference frame can be performed in two stages. The first, measures lateral displacements, $\tilde{y}'_r$, in an intermediate reference frame, see figure 4.5, centred on the car such that

$$\tilde{y}'_r = \tilde{y}_r - y$$  \hspace{1cm} (4.18)$$
Figure 4.4: Diagrammatic representation of a constant speed lateral path-tracking controller. The path sample values pass through a serial-in, parallel-out shift register operation at each time step. \( \tilde{y}_{rn} \) is the previewable lateral displacement signal. \( K_1 \) represents the full car-state feedback control, while \( K_2 \) represents the preview control, in the form of feedback of the previewed path. Finally, \( \theta_{eq} \) represents the value of the steering wheel corresponding to the trim-state.
where \( y \) is the global lateral displacement of the car and subscript \( i \) is the index of previewed path points, \( i = [0 1 2 ... n] \).

The relationship between the lateral displacements of the path points when measured in a local reference frame, \( \tilde{y}'_{ri} \), and those in the intermediate reference frame, \( \tilde{y}'_{ri} \), is

\[
\tilde{y}'''_{ri} = \tilde{y}'_{ri} \cos \psi - iuT \sin \psi \tag{4.19}
\]

which simplifies under small angle approximations to

\[
\tilde{y}'''_{ri} = \tilde{y}'_{ri} - iuT \psi \tag{4.20}
\]

Substituting equation 4.18 into 4.20 results in the following relationship between the lateral displacements of the path points measured in a local reference frame, \( \tilde{y}'_{ri} \), and those in a global frame, \( \tilde{y}_{ri} \)

\[
\tilde{y}_{ri} = \tilde{y}'_{ri} + y + iuT \psi \tag{4.21}
\]
Using a similar line of reasoning, the relationship between the lateral velocity in a global, \( \dot{y} \), and local frame, \( \dot{y}'' \), is

\[
\dot{y} = \dot{y}'' + u\psi
\]  
\[ (4.22) \]

When equations 4.21 and 4.22 are substituted into the original equation used to calculate the optimal control in a global reference frame, equation 4.17 becomes

\[
\theta''_{sw} = \theta_{eq} - K_1(1)y - K_1(2)\psi - K_1(3)(\dot{y}'' + u\psi) - K_1(4)\dot{\psi}'' - ... \\
- K_2(1)(\ddot{y}_{r0} + y) \\
- K_2(2)(\ddot{y}_{r1} + y + uT\psi) \\
- K_2(3)(\ddot{y}_{r2} + y + 2uT\psi) - ... 
\]  
\[ (4.23) \]

which, by collating terms, simplifies to

\[
\theta''_{sw} = \theta_{eq} - (K_1(1) + \sum_{j=1}^{n+1} K_2(j))y \\
- (K_1(2) + uK_1(3) + \sum_{j=1}^{n+1} (j - 1)uT K_2(j))\psi \\
- K_1(3)\ddot{y}'' + K_1(4)\dot{\psi}'' - ... \\
- K_2(1)\ddot{y}'_{r0} \\
- K_2(2)\ddot{y}'_{r1} \\
- K_2(3)\ddot{y}'_{r2} - ... 
\]  
\[ (4.24) \]

which is exactly the equation for control decisions if the car’s local reference frame in figure 4.5 were used

\[
\theta''_{sw} = \theta_{eq} - K_1(3)\ddot{y}'' - K_1(4)\dot{\psi}'' - K_2(1)\ddot{y}'_{r0} - K_2(2)\ddot{y}'_{r1} - K_2(3)\ddot{y}'_{r2} - ... 
\]  
\[ (4.25) \]
if the following conditions are met

\[ K_1(1) = - \sum_{j=1}^{n+1} K_2(j) \]  

\[ K_1(2) = -uK_1(3) - \sum_{j=1}^{n+1} (j-1)uTK_2(j) \]  

Numerical checks performed by Sharp and Valtetsiotis [57] have shown that these conditions are satisfied if full preview is utilised, with the implication that the optimal control calculations are invariant to the use of either a global or local reference frame.

At this stage it is also worth noting the following points:

- The local states \( y'' \) and \( \psi'' \) do not appear in the calculation of the optimal control input in equation 4.25 as they are zero. Hence, the gains \( K_1(1) \) and \( K_1(2) \), which are multiplied to these states respectively, are typically set to zero when performing the calculation out of convenience.

- The discrete points along the local x-axis in figure 4.5, from which the lateral displacements are effectively measured, coincidentally define the car’s future path if it continued in its straight-running trim-state. This fact will become significant when we consider more interesting cornering trim-states, for which we will need to adapt how the lateral displacements are measured.

### 4.3 Research contributions to DLQR preview control

The details presented in the first two sections have been based on previous research which has already been conducted. This section will use the background theory to detail work conducted by the author in developing advanced DLQR tracking controllers.

The first of these developments addresses the fact that the tracking controller has so far been based on a linearisation about a single trim-state. A straight-running, constant speed, trim-state has typically been used for this purpose. A linearisation serves as a good approximation to a non-linear system, for small perturbations about a trim-state. It follows then, that the optimal control scheme based on such a linearisation is strictly only optimal when applied to conditions which represent small perturbations from the straight-running trim. This is seen as a limitation on the running conditions of the car and the implied paths that can be tracked. During path
tracking, the car is likely to be close to numerous other trims. If a range of these trims can be collected off-line, optimal control schemes based on them can be calculated and stored too. Then, during tracking simulations, an adaptive control scheme can be used to install a trim, and matching optimal control scheme, which is close to the car’s running state at each time step. Hence, the installed controls are likely to be optimal for a wider range of conditions whilst tracking.

The collection of trim-states, and the calculation of optimal controls based on them, need only be repeated when there is a change in either the car model or control scheme parameters. However, these tasks are likely to take a sizeable portion of time and should be included when calculating the total time taken to compute a solution to the minimum lap-time problem. This will give a fairer representation of the computation time of the proposed solution when it is compared to alternative methods. It is therefore imperative that the processes of trim collection and gain calculation are carried out as rapidly and efficiently as possible. With this mind, a minimal set of trim-states is collected within the operating envelope of the car. A minimal set of scheduling parameters is then used to uniquely identify each stored trim-state and corresponding optimal control scheme. These scheduling parameters are then monitored during tracking simulations to install the reference trim-state closest to the car’s running state at each time step. In the case of constant speed tracking using the simple car model, two scheduling parameters are used: front (or rear) axle-lateral-slip ratio and forward speed.

The process of collecting multiple trim-states builds on the procedures introduced in chapter 3, which are designed to collect single trim-states. PID controllers are employed to track demands in the values of scheduling parameters. In the case of the simple car, a front (or rear) axle-lateral-slip ratio PID controller is used to control the steering wheel angle whilst the speed is regulated by a PID speed controller actuating the forward driving force, \( F_x \). Several trim collection simulations are run in an ordered manner. In each simulation, one scheduling parameter is swept through a range of values whilst the remaining scheduling parameters are held constant. The scheduling parameter is varied slowly enough that the car is assumed to be close to a trim-state at each time instant. Hence, optimal control schemes can be calculated from snap-shots, of the car’s states and inputs, at any instant in the simulation. The control schemes and their respective trims are stored in an ordered grid off-line, ready to be used in tracking simulations.

As mentioned in the second section, the future trajectory of a given trim is used as a reference to calculate the vector of preview errors as part of the optimal control calculation. The process of calculating the discrete future trajectory for constant speed, straight-running, trims in previous work has been trivial. In collecting a range of trims, which include constant-speed constant-lateral-cornering, a more general way of calculating the future trajectory efficiently was required. The
discrete future trajectory of a stored cornering trim can be constructed efficiently using three variables corresponding to it: the absolute velocity \(v_{\text{abs}}\), the yaw rate \(\dot{\psi}\), and the car’s side-slip angle \(\alpha_{\text{car}}\).

The radius of the arc which describes the future path of a particular constant-speed, constant latacc, trim, \(r_a\), is equal to \(\frac{v_{\text{abs}}}{\dot{\psi}}\). The radius is stored off-line, with its respective trim. Then, during tracking simulations, the following equations are used to construct the discrete future trajectory of the car for the relevant trim-state. For the case where the car has no side-slip \((\alpha_{\text{car}} = 0)\)

\[
\begin{align*}
\text{arc}_{x_i} &= |r_a| \sin \left( \frac{iv_{\text{abs}}T}{|r_a|} \right) \\
\text{arc}_{y_i} &= r_a \left( 1 - \cos \left( \frac{iv_{\text{abs}}T}{|r_a|} \right) \right)
\end{align*}
\]

where subscript \(i\) signifies the index of the discrete future points, \(i = [0 \ 1 \ 2 \ \ldots \ n]\). Subscripts \(x\) and \(y\) signify the longitudinal and lateral coordinate of each discrete point, measured in the car’s own local reference frame.

In the more general case, the car will experience side-slip. The future arc of the car should therefore start from the car’s centre of mass and be aligned with the velocity vector associated with that trim. A rotation matrix is applied to the coordinates of the arc in equations 4.28 and 4.29 to achieve this, such that

\[
\begin{bmatrix}
\text{arc}'_{x_i} \\
\text{arc}'_{y_i}
\end{bmatrix} = R_\alpha \begin{bmatrix}
\text{arc}_{x_i} \\
\text{arc}_{y_i}
\end{bmatrix}
\]

where \(R_\alpha\) has the form of a clockwise rotation matrix for a two-dimensional Cartesian system

\[
R_\alpha = \begin{bmatrix}
\cos(\alpha_{\text{car}}) & \sin(\alpha_{\text{car}}) \\
-\sin(\alpha_{\text{car}}) & \cos(\alpha_{\text{car}})
\end{bmatrix}
\]

Figure 4.6 shows a diagrammatic representation of the resulting future trajectories of two different cornering trims: high latacc and low latacc. The figure also presents the convention of measuring displacements of the path from the future cornering trajectories, in each case.

The preview errors can also be calculated using an alternative method, which is presented diagrammatically in figure 4.6 for the high latacc case. Such a procedure uses several separate
Figure 4.6: Diagrammatic representation of future trajectory of two trims, (a) low latacc and (b) high latacc, using convention of measuring preview errors using the local car reference frame in each case. Case (c) represents the alternative method of measuring preview errors, where the reference frame is aligned with the future orientation of the car at each time step.
reference frames, each aligned with the projected orientation of the car at each future time step, to calculate the respective preview errors. Employing such an algorithm to calculate the path errors is likely to increase the computational burden of the preview error calculations. Preliminary calculations suggest that employing such a scheme results in a negligible change in the preview contribution to the optimal control signal, when compared to the strategy of using a local (car) reference frame. For instance, adopting the alternative scheme to calculate the control decisions 10 s into the simulation for the tuned complex car setup in figure 8.28 results in a change of 0.0235 rad in the steering wheel angle and 0.0047 in the throttle position magnitudes. This signifies a relatively small change to the absolute values of the control decisions, a change of 1.6% and 2.8% respectively, whilst the car is executing a taxing manoeuvre with a lateral acceleration of 6.33 m/s². Such small changes to the control input signals, when alternating between the two previewing methods, are a consequence of the fact that the significant preview gains are multiplied to preview errors concentrated close to the car, where there is a smaller discrepancy between the two methods when measuring preview errors. With both previewing schemes producing predominantly the same control input signal, a single fixed local reference frame, centred and aligned with the car, was used in the interest of keeping the computational burden of the preview error measurements to a minimum. The difference between the control decisions obtained from this method and the proposed method is likely to depend on factors such as the control weighting strategy employed and the consequent effect on the length of the preview window. A full investigation, to identify the range of operating conditions for which the current strategy is a good representation of the alternative method, is recommended but has not been conducted in the current research effort.

The theory presented in this chapter so far has been limited to tracking a single-channel demand signal. This has limited previous work to lateral tracking constrained to a constant forward speed or longitudinal tracking constrained to straight-running conditions. The remainder of this chapter details how the tracking controller can be extended to follow a dual-channel demand signal which facilitates variable speed (x, y) tracking.

To proceed with this development, the description of the path which is tracked must first be revised. The path now consists of both x and y coordinates defined at discrete time intervals, T, so that the speed of the car at each time instant is implied. Consider the path vector,

\[ \tilde{x}_y_r = [\tilde{x}_{r0} \ \tilde{y}_{r0} \ \tilde{x}_{r1} \ \tilde{y}_{r1} \ \tilde{x}_{r2} \ \tilde{y}_{r2} \ ...]^T \]  

(4.32)

where values (\tilde{x}_{rk}, \tilde{y}_{rk}) represent the coordinates of each path point at time instant k.
The shift register presented in equation 4.5 can be revised to accommodate the new definition of the path so that $A_r$ is now a $(2n \times 2n)$ matrix of the form:

$$A_r = \begin{bmatrix}
0_2 & I_2 & 0_2 & \ldots & 0_2 \\
0_2 & 0_2 & I_2 & \ldots & 0_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0_2 & 0_2 & 0_2 & \ldots & I_2 \\
0_2 & 0_2 & 0_2 & \ldots & 0_2
\end{bmatrix} \tag{4.33}$$

where $0_2$ is a $(2 \times 2)$ zero matrix while $I_2$ is a $(2 \times 2)$ identity matrix. In this form, the shift register performs the usual job of marching each path point closer to the car at each time step.

Given the new requirement to track a variable speed profile, a complex car, with a realistic drive-train and engine-map, is seen as a more appropriate model to use for tracking such a path. The full details of the model are left until chapter 7 but for the purposes of this section, it is sufficient to mention that the model uses two control inputs: the steering wheel angle ($\theta_{sw}$) and the throttle/brake demand ($\tau$). Tracking a two-channel path description using a model with two control inputs requires a revision of the preview matrix obtained using the recursive method in equation 4.12. The resulting preview gain matrix is now of size $(2 \times 2n)$ and of the form

$$K_2 = \begin{bmatrix}
K_{2rx}(1) & K_{2ry}(1) & K_{2rx}(2) & K_{2ry}(2) & K_{2rx}(3) & K_{2ry}(3) & \ldots \\
K_{2\theta x}(1) & K_{2\theta y}(1) & K_{2\theta x}(2) & K_{2\theta y}(2) & K_{2\theta x}(3) & K_{2\theta y}(3) & \ldots
\end{bmatrix} \tag{4.34}$$

which, when collated, leads to the construction of four preview matrices, $K_{2\theta x}$, $K_{2\theta y}$, $K_{2rx}$ and $K_{2ry}$.

Here, $K_{2\theta x}$ and $K_{2rx}$ correspond to preview gains used to calculate the steering wheel and throttle/brake control inputs based on the vector of longitudinal preview errors, $[\tilde{x}_{r0} \ \tilde{x}_{r1} \ \tilde{x}_{r2} \ldots]$, respectively. Analogously, $K_{2\theta y}$ and $K_{2ry}$ correspond to preview gains used to calculate the steering wheel and throttle/brake control inputs based on the vector of lateral preview errors, $[\tilde{y}_{r0} \ \tilde{y}_{r1} \ \tilde{y}_{r2} \ldots]$, respectively. An alternative to using the MATLAB\textsuperscript{TM} $dlqr$ function exists in Hazell’s MATLAB\textsuperscript{TM} control toolbox [25]. The toolbox requires the setting up of the car’s state-space matrices and control weightings to obtain the necessary optimal control gains. However, it has been noted that the toolbox was unable to generate the necessary control gains when the car
approaches its limit, where it is known to be particularly unstable. Hence, the combination of the MATLAB™ dlqr function and the recursive method has been adopted throughout the research to generate the necessary optimal control gains.

The calculations performed at each time step, when tracking a dual-channel input with two control inputs and using a global reference frame, are now of the general form

\[
\theta_{sw} = \theta_{eq} - K_1\theta(1)(x - x_{eq}) - K_1\theta(2)(y - y_{eq}) - K_1\theta(3)(\psi - \psi_{eq})
\]
\[
- K_1\theta(4)(\dot{x} - \dot{x}_{eq}) - K_1\theta(5)(\dot{y} - \dot{y}_{eq}) - K_1\theta(6)(\dot{\psi} - \dot{\psi}_{eq}) - ...
\]
\[
- K_2\theta x(1)\bar{x}_r0 - K_2\theta x(2)\bar{x}_r1 - K_2\theta x(3)\bar{x}_r2 - ...
\]
\[
- K_2\theta y(1)\bar{y}_r0 - K_2\theta y(2)\bar{y}_r1 - K_2\theta y(3)\bar{y}_r2 - ...
\]

(4.35)

\[
\tau = \tau_{eq} - K_1\tau(1)(x - x_{eq}) - K_1\tau(2)(y - y_{eq}) - K_1\tau(3)(\psi - \psi_{eq})
\]
\[
- K_1\tau(4)(\dot{x} - \dot{x}_{eq}) - K_1\tau(5)(\dot{y} - \dot{y}_{eq}) - K_1\tau(6)(\dot{\psi} - \dot{\psi}_{eq}) - ...
\]
\[
- K_2\tau x(1)\bar{x}_r0 - K_2\tau x(2)\bar{x}_r1 - K_2\tau x(3)\bar{x}_r2 - ...
\]
\[
- K_2\tau y(1)\bar{y}_r0 - K_2\tau y(2)\bar{y}_r1 - K_2\tau y(3)\bar{y}_r2 - ...
\]

(4.36)

The proof provided in the second section, showing the invariance of control decisions to a change from global to local reference frames, can be extended to the dual-input dual-channel tracking problem. The necessary proof follows the same arguments made in the second section and is detailed in Appendix A

An interesting feature of the control scheme becomes apparent when the system is linearised for a straight-running trim. The longitudinal and lateral dynamics become decoupled for this special case, resulting in the gains \(K_{2\theta x}\) and \(K_{2\tau y}\) being zero. This implies that under straight-running conditions, the longitudinal error can only be influenced by the throttle/brake whilst the lateral error can only be influenced by the steering wheel. More details of the control scheme will be presented in chapter 8, following a formal definition of the complex car model.

### 4.4 Conclusions

This concludes the chapter detailing the DLQR preview tracking controller. The chapter initially presented the culmination of prior work in the area of DLQR preview control, introducing the
theory and working details of a typical controller in the first two sections. The third section then presented the author’s novel contribution to the field which increases the capability of the technology. The collection of several cornering trim-states has been proposed for use in a novel gain-scheduled DLQR preview tracking controller, together with a description of how the previewing strategy should vary when using such cornering trims. Such an adaptive controller is intended to offer superior tracking performance when compared to the fixed gain, non-adaptive, controllers that are used currently in this field. Additionally, the working details of a novel two-channel variable speed tracking controller developed by the author, which combines the previously decoupled problems of longitudinal and lateral tracking, has also been introduced. This chapter serves as a prelude to the tracking results, where such controllers are finally used, which are presented in the forthcoming chapters.
Chapter 5

Tracking results of simple understeering car model

The novel solution to the minimum lap-time problem is intended to be an iterative process consisting of three phases: optimal tracking of a given path, optimising the speed along the path and perturbing the path to obtain the minimum lap-time. Chapter 4 has presented the DLQR techniques which allow optimal path-tracking, required for the first phase. During the optimisation process, the remaining phases may suggest trajectories which will push the car to the limit of its capability, and possibly beyond it. In the latter case, a less ambitious speed/path combination should be generated at the next optimisation iteration. For the software to recognise that the current trajectory was too ambitious, the tracking controller must do its best to track the current path, keeping the car close to its limit and ensuring the tracking phase completes satisfactorily. Failure of the controller in such a scenario will result in spurious information being passed back to the speed-/path-optimiser phases. In the best case, this problem will result in the optimiser requiring more iterations to converge to a solution. In the worst case, it will not converge to a solution at all. Neither option is seen as acceptable.

The behaviour of the controller, when the car is asked to follow trajectories that are beyond its capability, is therefore of particular interest to the current research effort. This chapter investigates the strategy required to optimally control the understeering simple car setup, presented in chapter 3, in a limit tracking situation. Previous work has relied on tracking controllers designed for only one trim-state, typically straight-running. Such controllers require that the car stays close to this trim-state for the control to remain optimal. These controllers lack knowledge of the non-linearity of the car, the tyres in particular, which is likely to result in sub-optimal tracking, and even controller instability, when operating at the limit of the car’s capabilities. A gain-scheduled,
adaptive, tracking controller, detailed in chapter 4, has the potential to overcome such problems afflicting fixed gain controllers.

The remainder of the chapter will present the details of a novel adaptive control scheme that was developed for the understeering simple car model. Comparative tracking tests, between the adaptive and non-adaptive controllers, are presented for a number of different path-tracking problems. Finally, conclusions will be drawn on the benefits of using an adaptive control scheme. The results presented in this chapter are based on the author’s novel published work [61] regarding the control of a front-limited simple car model.

5.1 Control scheme

The concept of using an adaptive controller has already been introduced in chapter 4, together with its theoretical benefits. By collecting a range of trim-states and corresponding optimal control gains off-line, the controller can switch to a stored trim-state which is close to the car’s running state at each time step of the tracking simulation, thus making the controls optimal for a wider range of manoeuvres. The choice of the scheduling parameters, used to index and retrieve the trim-states/gains, is seen as vital to the efficiency of this process.

Initial trials investigated the use of the car’s lateral acceleration for this purpose. However, for high values of lateral acceleration, there are two trim-states corresponding to one value of lateral acceleration: one being the desired one and the other corresponding to a scenario where the tyres are beyond saturation. Referencing each stored trim uniquely with this parameter therefore becomes difficult. Additionally, lateral acceleration is seen as an indirect measure of the utilisation of the front tyres, which are critical to the car’s understeering behaviour. This problem is exacerbated by the car’s infinite-bandwidth steering system, which is able to change the state of these tyres instantly, with the measured lateral acceleration potentially only responding to these changes after several time steps. This is problematic as the adaptation/scheduling needs to be highly sensitive to the state of the tyres to remain useful. If the adaptation is not exactly matched with the changing state of the tyres, the controller is at risk of becoming sub-optimal, or even unstable.

The sensitivity issue is overcome by using the front-axle-lateral-slip ratio, $\alpha_f$, as the scheduling parameter. This parameter is the only active variable that affects the lateral force generated by the front axle, as seen in equation 3.2. If trims are scheduled based on such a parameter, the optimal control derived from them will be matched to the state of the front axle, regardless of how vigorous the steering control becomes.
A time step of 0.01 s was chosen for the discretisation of the system. This value represents a good balance between updating the control input very frequently and reducing the computational burden associated with such a decision. The understeering simple car setup is accommodating of a reasonably tight control scheme. $q_2$, the weighting placed on lateral tracking error, was set to 100 whilst $R$, the weighting placed on control effort, was set to 1. Using the trim-collection process documented in chapter 4, optimal control gains are collected for a range of cornering trims. These gains are: $K_1$, the vector of car-state-feedback gains and $K_2$, the vector of preview gains. Figure 5.1 displays the preview gains for a variety of cornering trims with a constant forward speed of 30 m/s. A preview horizon of 6 s, corresponding to 600 preview points, is seen as sufficient to ensure that full preview is used in all cases. A selection of feedback gains relating to lateral velocity and yaw rate, for the same speed of 30 m/s, is shown in table 5.1. Two-dimensional slices of figure 5.1, corresponding to the same front-axle-slip ratios presented in table 5.1, are shown in figure 5.2.

Figure 5.1 demonstrates how the preview gains vary with cornering intensity. As the cornering-effort is increased, and control authority decreases, the magnitude of the preview gains is reduced whilst the full preview horizon is lengthened. This behaviour is similar in effect to loosening the control scheme, the loss of control authority corresponding to the need to place more importance/cost on the remaining control capacity. The magnitude of the preview gains continues to fall as cornering intensity is increased, with the gains being close to zero at the point where maximum front-axle tyre-force is generated. Past this point, the preview gain magnitudes start to increase but have a change in sign. The consequence of this feature being that, below saturation, the preview contribution uses the tyres to meet the tracking needs. However, once past the saturation point, the preview provides an opposing contribution to guide the tyres back to their peak-force point. In short, the controller ensures the tyres are used up to their peak-force generation point, but not past it, which is exactly the behaviour required when tracking ambitious paths.

A similar sign reversal can be seen in the $K_1$ gains presented in table 5.1. In contrast with the preview gains, the gain magnitudes increase as the tyres approach their saturation point, signifying a greater effort in stabilising the system about the respective trim-state, as control authority is progressively lost.

The closed-loop control system frequency responses of the four exemplary trim-states in table 5.1 are plotted in figure 5.3. The input to each system is a lateral displacement applied at the end of the preview window, whilst the output is the lateral displacement of the car’s centre of mass. By exciting the input using a sinusoidal path with a range of frequencies, the closed-loop control system bandwidth for each control strategy can be observed. As the output is responding
Figure 5.1: Three-dimensional plot of $K_2$ preview gain sequences, for the understeering simple car setup, as functions of front-axle-lateral-slip ratio, representing the cornering-effort level, for a speed of 30 m/s.

Table 5.1: Car-state-feedback gains relating to lateral velocity and yaw rate for a car speed of 30 m/s and four different trim-states defined by front-axle-slip ratio.

<table>
<thead>
<tr>
<th>Front-axle-slip ratio ((\pm))</th>
<th>0</th>
<th>0.04</th>
<th>0.06</th>
<th>0.40</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lateral velocity (rad s m(^{-1}))</td>
<td>0.5971</td>
<td>1.3639</td>
<td>2.4931</td>
<td>-5.7844</td>
</tr>
<tr>
<td>Yaw rate (s)</td>
<td>0.9767</td>
<td>1.9513</td>
<td>3.1738</td>
<td>-5.7633</td>
</tr>
</tbody>
</table>
to an excitation at a finite distance ahead of it, there is an inherent phase lag between it and the
input, termed as the transport delay, which can be seen in the phase plot. The plots show how the
system bandwidth decreases as a result of an increase in cornering intensity. As more tyre force is
used, the sensitivity of the tyre reduces. This result has a bearing on how a real driver needs to
drive just short of the car’s limit, in the interest of retaining some control authority. The better
the driver, the smaller the required safety margin. This safety margin becomes significant in the
control of the oversteering setup, presented in the following chapter, where the penalty for pushing
over the limit is potentially greater.

5.2 Tracking Simulation Results

The tracking ability of the adaptive controller, built around the collection of trim-states depicted
in figure 5.1, will now be demonstrated. The results will be compared and contrasted with that of
a fixed-gain, non-adaptive, controller when tracking the same paths.

As mentioned earlier, the value of the front-axle-lateral-slip ratio is measured at each time step
and used to install the stored trim-state closest to the car’s running state. This trim-state is used as
a reference from which to calculate the optimal tracking control contributions, seen in equation 4.25.
This extends from using it to calculate the state error vector, for the feedback contribution, to
calculating the future trajectory of the car, when determining the preview contribution. At each
time step, the adaptive controller can switch to another trim, which is closer to the car’s running
state, with the new trim now being the reference for the control decisions. The switching from

Figure 5.2: Two dimensional plot of $K_2$ preview gains, for the four trim-states, defined by front-
axle-slip ratios, presented in table 5.1.
Figure 5.3: Frequency responses in Bode plot form for driver-controlled car at 30 m/s speed, for the four trim-states, defined by front-axle-slip ratios, presented in table 5.1. The input is an absolute lateral road displacement, applied at the end of the preview window, while the output is the lateral displacement of the car’s centre of mass.
one reference trim to another, together with the implied switching between corresponding optimal control gains, can, in some situations, cause the control to become oscillatory unstable. The fact that the driver has an infinite bandwidth, meaning the changes are instantaneous at a switching point, contributes to this problem.

The instability can be prevented by installing a rate limit on the steering wheel motion. A value of $9 \text{ rad/s}$ is seen as a reasonable limit in the context of real-world driving [55]. The value of the limit itself is seen as non-critical as similar tracking results were observed for a limit as low as $3 \text{ rad/s}$. The chosen value is large enough to provide stabilisation without impeding the steering controller in tracking the path. Such a rate limit was used in both the adaptive and non-adaptive controllers to maintain consistency.

Three challenging paths were chosen to test the control schemes presented in this chapter: a hair-pin bend, a lane-change and a double s-bend, see figure 5.4. When tracked at a constant speed, each path essentially represents a lateral position target sequence appearing in the car’s local reference frame. The paths are designed such that, at a speed of $30 \text{ m/s}$, perfect tracking requires slightly more lateral force from the front, limiting, axle than is available. Hence, the behaviour of both adaptive and non-adaptive controllers can be scrutinised for limit tracking scenarios. At the start of each manoeuvre, the reference trim-state is that of straight-running, which remains the case when non-adaptive control is employed. When the control is adaptive, the reference trim-state changes using the process described earlier. A total of 110 individual, equi-spaced, cornering trims are stored and used in the adaptive controller.

The hair-pin starts with a straight section, $72.6 \text{ m}$ long. The curvature of the path then changes uniformly with distance, in the form of a clothoid curve, to a radius of $72.6 \text{ m}$. The curvature is then held constant over an arc of $0.75\pi \text{ rad}$ before reducing uniformly, using another clothoid curve, back to zero, to form another straight section.

The lane change consists of a sinusoidal spline bridging the gap between two parallel straight sections, each $300 \text{ m}$ in length. The spline is required to form a smooth transition between the lateral ($10 \text{ m}$) and longitudinal ($50 \text{ m}$) offsets between the end of the first straight and the beginning of the second.

The double s-bend starts with a straight section $330 \text{ m}$ in length. This leads into a right-handed semi-circular section with a radius of $80 \text{ m}$. A short $30 \text{ m}$ straight section follows before leading into a left-handed semi-circular section, of the same radius as the first. Another $30 \text{ m}$ straight section forms the link to the second s-bend. This starts with another right-handed arc, now with a radius of $75 \text{ m}$. A short $30 \text{ m}$ straight section precedes the final arc, extended over an angle of $0.9\pi \text{ rad}$. A straight section, tangential to the final arc, then completes the path.
Tracking errors for the adaptive and non-adaptive controls when tracking the hair-pin, lane-change and double s-bend are plotted in figures 5.5, 5.7 and 5.9 respectively. Corresponding time histories for front-axle lateral-slip and steering wheel angle are shown in figures 5.6, 5.8 and 5.10.

In the hair-pin tracking simulation, both controllers bring the front-axle side force close to its peak, at about 5 s into the simulation. The non-adaptive controller, having no knowledge of the saturating nature of the tyres, demands more steering angle in an attempt to reduce the steadily increasing lateral tracking error. This behaviour continues until the steering reaches full-lock, set at 9 rad in the simulation, with the car veering off-course as the simulation progresses. In contrast, the adaptive controller adopts a more measured approach. Although the car is travelling too quickly to track the path perfectly, the controller does the best it can by keeping the front-axle lateral-slip at the value that generates the highest lateral force. The controller is then able to realign the car with the path when it becomes possible to do so, at approximately 12 s.

A similar situation is observed during the tracking of the lane-change manoeuvre. Both controllers behave in a similar manner in the first 10 seconds of the simulation, the distinction between the two occurring when the front tyres are close to saturating. The non-adaptive controller again demands larger steering wheel angles in an attempt to reduce the growing tracking error. The steering lock is reached shortly after 12 s. At this point, the steering exhibits a slow oscillatory behaviour, moving between each full-lock position, the controller clearly having lost control by this point. The adaptive controller repeats the measured approach demonstrated on the hair-pin, with
Figure 5.5: Lateral tracking errors for understeering car in simulated hair-pin manoeuvre at 30 m/s speed with non-adaptive (fixed-gain) or adaptive (gain-scheduled) controls.

Figure 5.6: Front-axle-lateral-slip and steering wheel angle histories for understeering car in hair-pin manoeuvre of figure 5.4 with non-adaptive (fixed-gain) or adaptive (gain-scheduled) controls.
the controller knowing to reduce the steering demand once the tyres have saturated. This decision helps the controller maintain control of the car whilst keeping the tracking error to a minimum. The controller eventually realigns the car with the path, when it is possible to do so, shortly after 12 s.

Figure 5.7: Lateral tracking errors for understeering car in simulated lane-change manoeuvre at 30 m/s speed with non-adaptive (fixed-gain) or adaptive (gain-scheduled) controls.

Figure 5.8: Front-axle-lateral-slip and steering wheel angle histories for understeering car in lane-change manoeuvre of figure 5.4 with non-adaptive (fixed-gain) or adaptive (gain-scheduled) controls.
When tracking the first of the two s-bends, the non-adaptive controller manages to maintain control of the car. However, the tracking performance in the period between 11 and 27 s is noticeably poorer than with its adaptive counterpart. The mismatch between the straight-running reference trim, used by the non-adaptive controller, and the car’s running state inevitably results in the sub-optimal control, ultimately leading to the poor tracking performance being observed. On entry into the second s-bend, the non-adaptive controller demands more from the front axle, which is already close to saturation, subsequently losing control of the car at 30 s. As in the other two trials, the adaptive controller keeps the tyres at their peak force capacity and maintains control at the critical stage of the simulation, returning the car back to the path when it is able to do so.

Figure 5.9: Lateral tracking errors for understeering car in simulated double s-bend manoeuvre at 30 m/s speed with non-adaptive (fixed-gain) or adaptive (gain-scheduled) controls.

5.3 Conclusions

This chapter aimed to present the control scheme required to optimally control an understeering simple car setup in limit tracking scenarios. A gain-scheduled tracking controller, using adaptation with respect to the front-axle-lateral-slip level, has been developed for this purpose. The adaptive control scheme produces excellent tracking results for the three challenging scenarios presented, whilst the non-adaptive control scheme produced poor tracking and was unstable in each case. This is seen as a consequence of the mismatch, in the non-adaptive controller, between the reference trim-state and the car’s running state throughout most of the manoeuvres. The adaptive control scheme presented in this chapter is therefore recommended for the control of an understeering simple car setup during the tracking phase of the minimum lap-time solution. A rate limiter was required to prevent instability caused by switching noise, generated when moving from one reference
Figure 5.10: Front-axle-lateral-slip and steering wheel angle histories for understeering car in double s-bend manoeuvre of figure 5.4 with non-adaptive (fixed-gain) or adaptive (gain-scheduled) controls.
trim-state to another. An alternative solution, which will be investigated in the next chapter, is to use interpolation to smooth the transition between the discrete trim-states/gain-sequences.

In summary, a novel adaptive DLQR preview tracking controller has been implemented in several path tracking scenarios. Such a controller has been demonstrated to be superior to the fixed gain (non-adaptive) controllers that represent the previous state-of-the-art in DLQR preview tracking controllers. The novel controller was also able to demonstrate lateral limit control of an understeering simple car model, whilst maintaining excellent tracking ability, where previous non-adaptive DQLR technology has failed. This further underlines the capability of the adaptive controller that has been developed by the author, which is thought to be a significant contribution to the field. The next chapter aims to investigate whether similar benefits can be derived in applying such a control scheme to an oversteering setup.
Chapter 6

Tracking results of simple oversteering car model

In the previous chapter, the benefits of using an adaptive, gain-scheduled, control scheme, for the limit tracking of an understeering simple car setup, were established. An obvious question has been posed, of whether a similar strategy can be used to control an oversteering setup, presented in chapter 3. In such a setup, the rear axle force limits before the front axle, causing the car to spin, once past the limit case. This divergent instability is particularly challenging to control. This chapter aims to present the adaptive control scheme designed to control the car near such a limit scenario. Tracking trials are then used to demonstrate the effectiveness of this control scheme, comparing and contrasting it with the performance of a non-adaptive, fixed-gain, controller. Finally, conclusions will be drawn on the benefits of using an adaptive controller in the oversteering problem. This will complete the development work on adaptive tracking controllers for the simple car model. The results presented in this chapter are based on the author’s novel published work [63] regarding the control of a rear-limited simple car model.

6.1 Control scheme

The choice of the scheduling parameter used for the adaptation of the tracking controller is seen as key to the success of the controller, as was the case in the understeering setup. The use of a general parameter, such as the car’s lateral acceleration, was again investigated for this purpose. However, the parameter is afflicted with the same problems which excluded it from the understeering scenario. The scheduling parameter is still required to be highly sensitive to the limiting feature of the car, in this case, the rear axle lateral force. Initial trials confirmed
that neither the lateral acceleration, nor the front-axle-lateral-slip ratio, used as the scheduling parameter for the understeering problem, were appropriate for this reason. In contrast, the rear-axle-lateral-slip ratio has a direct effect on the limiting rear axle force, and was therefore chosen as the scheduling parameter for the current problem. Figure 6.1 illustrates how key trim-state parameters vary with the chosen scheduling parameter for increasing cornering intensity.

Figure 6.1: Exemplary trim-states for oversteering simple car setup, for a speed of 30 m/s and a full range of rear-axle-lateral-slip ratios.

A time step of 0.01 s was again chosen for the discretisation of the system, for the reasons explained in the previous chapter. The oversteering setup requires a relatively loose control scheme \((q_2=0.5, R=1)\), in contrast with the tight control adopted in the understeering problem \((q_2=100, R=1)\). A tighter control strategy is more favourable due to the associated increase in closed-loop control system bandwidth, observed in figure 4.3. However, figure 3.4 has already established that above a critical speed/lateral acceleration the oversteering car setup becomes particularly unstable and difficult to control. A tight control strategy has been noted to compound this problem, resulting in a loss of control near this critical region. In light of this observation, a looser control strategy was chosen. Figure 6.2 displays the variation of the preview gain vector, \(K_2\), with
cornering intensity, for a constant speed of 30 m/s. The figure illustrates how a looser control scheme requires a longer full preview horizon to remain optimal. A horizon of 20 s, corresponding to 2000 preview points, is required to capture full preview for the majority of the cornering trim-states presented. The feedback gains, $K_1$, vary in a more complex manner, with respect to cornering intensity, than observed in the understeering problem. A selection of feedback gains relating to lateral velocity and yaw rate is shown in table 6.1. A collection of two-dimensional slices of the preview gain sequences of figure 6.2, corresponding to four trim-states, equi-spaced in rear-axle-lateral-slip ratio, presented in table 6.1, are shown in figure 6.3.

![Figure 6.2: Three-dimensional plot of $K_2$ preview gain sequences, for the oversteering simple car setup, as functions of rear-axle-lateral-slip ratio, representing the cornering-effort level, for a speed of 30 m/s.](image)

The preview gains in figure 6.2 display the common characteristic, of reducing in magnitude and requiring a longer full preview horizon, as cornering intensity is increased. The gains switch sign once past the critical value of rear-axle lateral-slip, where the rear tyres generate the maximum lateral force. This is similar in behaviour to the gains presented in figure 5.1 for the understeering problem. Closed-loop control system frequency responses, using the exemplary trim-states from figure 6.3, are plotted in figure 6.4.
Figure 6.3: Two dimensional plot of $K_2$ preview gains, for four of the trim-states, defined by rear-axle-slip ratios, presented in table 6.1.

Figure 6.4: Frequency responses in Bode plot form for driver-controlled car at 30 m/s speed, for four of the trim-states, defined by rear-axle-slip ratios, presented in table 6.1. The input is an absolute lateral road displacement, applied at the end of the preview window, while the output is the lateral displacement of the car’s centre of mass.
Table 6.1: Car-state-feedback gains relating to lateral velocity and yaw rate for a car speed of 30 m/s and eight different trim-states defined by rear-axle-slip ratio.

<table>
<thead>
<tr>
<th>Rear-axle-slip ratio (-)</th>
<th>0</th>
<th>0.020</th>
<th>0.040</th>
<th>0.080</th>
<th>0.102</th>
<th>0.103</th>
<th>0.120</th>
<th>0.133</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lat. vel. (rad m$^{-1}$)</td>
<td>-0.022</td>
<td>-0.068</td>
<td>-0.217</td>
<td>0.184</td>
<td>6.331</td>
<td>-11.18</td>
<td>-3.432</td>
<td>-3.131</td>
</tr>
<tr>
<td>Yaw rate (s)</td>
<td>0.537</td>
<td>0.741</td>
<td>1.428</td>
<td>1.801</td>
<td>-4.520</td>
<td>13.61</td>
<td>5.70</td>
<td>5.412</td>
</tr>
</tbody>
</table>

The input to each system in the figure is a lateral displacement applied at the end of the preview window, whilst the output is the lateral displacement of the car’s centre of mass. By exciting the input using a sinusoidal path with a range of frequencies, the closed-loop control system bandwidth for each control strategy can be observed. As the output is responding to an excitation at a finite distance ahead of it, there is an inherent phase lag between it and the input, termed as the transport delay, which can be seen in the phase plot. The plots show a distinct drop in controller bandwidth for the trim-states close to the limit of the rear axle force. The poorer tracking performance, in comparison with the understeering case, is attributed to the looser control strategy that is required by the oversteering problem.

With the tracking performance already likely to be poor when operating near limit of the rear axle, attention must focus on the divergent instability of the car in this region instead. Trials have shown that the controller is unable to regain control after an onset of such behaviour, with the program itself potentially crashing as a result, which is unacceptable. To ensure that the car is kept sufficiently clear from its region of instability, it is recommended that the oversteerer is driven short of the absolute limit of the rear tyres, typically 80% of the critical rear-axle-lateral-slip value. The speed and path optimisation phases of the minimum lap-time solution must ensure that suggested trajectories do not force the oversteering setup past this recommended limit. Leaving such a safety margin is unlikely to affect the minimum lap-time significantly, as it corresponds to losing only 1.1% of the maximum force that the rear axle is capable of generating.

An adaptive tracking controller which switches between the trim-states, and corresponding optimal control gains, that have been presented is likely to suffer from the same switching instability observed in the understeering problem. A rate-limiter was used in the previous case to avoid such instability without impeding the tracking performance. The use of quadratic interpolation is seen as a better solution. The scheduling process takes the same form as that documented in the understeering problem, with the rear-axle lateral-slip now serving as the scheduling parameter. However, interpolation now takes place between the trim-state closest to the car’s running state and its two neighbouring stored trim-states. At each simulation time step, quadratic functions...
are used to form smooth transitions between the respective values of parameters in each of the three trim-states. Interpolation is then used to obtain values of: the trim-state that is used as a reference trim, the optimal control gain vectors \( (K_1 \text{ and } K_2) \), and the parameters used to generate the future trim trajectory of the car, detailed in chapter 4. This results in a smoother transition between discrete trim conditions during tracking simulations and avoids switching instabilities, which the rate-limiter was designed to suppress. By smoothing across a sparser set of discrete trim-states, interpolation reduces the number of cornering trim-states that need to be collected to provide the same level of tracking performance. This would be favourable as collecting fewer trim-states, and calculating fewer optimal control gains based on them, reduces the time taken to complete the off-line phase of the minimum lap-time solution.

### 6.2 Tracking Simulation results

The tracking performance of the adaptive control scheme, presented in the previous section, will now be demonstrated for two challenging paths. The results will be compared and contrasted with the performance of its non-adaptive counterpart.

Two challenging paths were chosen to demonstrate the tracking ability of the adaptive control scheme designed for the oversteering simple car model: a *hair-pin* bend and a lane-change manoeuvre, see figure 6.5.

When tracked at a constant speed, each path essentially represents a lateral position target sequence appearing in the car’s local reference frame. The paths are designed such that, at a speed of 30 m/s, perfect tracking requires slightly more lateral force from the rear, limiting, axle than is available. Hence, the behaviour of both adaptive and non-adaptive controllers can be scrutinised for limit tracking scenarios. At the start of each manoeuvre, the reference trim-state is that of straight-running, which remains the case when non-adaptive control is employed. When the control is adaptive, the reference trim-state changes using the interpolated scheduling process described earlier. A total of 36 individual, equi-spaced, cornering trims are stored and used in the adaptive controller.

The *hair-pin* is identical to the *hair-pin* used in the previous chapter. A clothoid section provides a uniform increase in the path’s curvature, from a straight section to an arc, 72.6 m in radius, which itself is extended over an angle of 0.75\( \pi \) rad. The curvature of the path then decreases uniformly, using another clothoid section, to form another straight section, as seen in figure 6.5.

The lane-change is similar to the manoeuvre presented in the previous chapter. The path starts and ends with two straight sections, 300 m in length, which are parallel to each other. A straight
section is now used to bridge the lateral (35.9 m) and longitudinal (82.2 m) offsets between the end of the first straight and the beginning of the second. The discontinuity in curvature between the straight sections has been deliberately chosen to investigate how the presence of such features affects the behaviour of the controller in an already taxing manoeuvre.

Tracking errors for the adaptive and non-adaptive controls when tracking the hair-pin and lane-change are plotted in figures 6.6 and 6.8 respectively. Corresponding time histories for rear-axle lateral-slip and steering wheel angle are shown in figures 6.7 and 6.9.

At the start of the hair-pin simulation, both adaptive and non-adaptive controllers behave in a similar manner. Approximately 4.5 s into the simulation, both controllers start to apply a steering angle opposite to the direction of the turn. This behaviour mirrors how a real driver may react when the rear of the car starts to saturate. The application of an opposite steering angle attempts to reduce the build up of yaw rate, which can quickly evolve into a spin if not corrected. At 5 s, each controller has brought the rear-axle lateral-slip close to 80% of the critical value which generates the maximum rear-axle force. As mentioned earlier, perfect tracking of the path would require more lateral force than the rear axle is capable of generating. After 5 s, the non-adaptive controller, knowing only of the behaviour of the tyres in the straight-running trim it was designed for, demands more steering angle in an attempt to reduce the growing lateral tracking
error. The rear tyres are consequently pushed over their saturation point and the controller loses control shortly after this point, with the steering oscillating between both steering lock limits, set at $\pm 9\text{rad}$. In contrast, the adaptive controller, having knowledge of the saturating nature of the rear axle forces, knows that some tracking performance must be forfeited in order to maintain controllability/stability at this crucial stage of the simulation. The controller stabilises the car at this point and keeps the steering angle approximately constant in the period between 5 to 9 s, whilst the tracking error continues to grow. Then, when it is required to do so, the controller provides the necessary steering action, facilitated by the force capacity remaining in the rear axle, to realign the car with the path. The magnitude of the tracking errors associated with the adaptive controller are noticeably larger than those observed when tracking the same path with the adaptive controller in the understeering problem. The larger errors stem from the looser control adopted in the oversteering problem, with tracking performance being forfeited in the interest of maintaining controller stability.

![Figure 6.6: Lateral tracking errors for oversteering car in simulated hair-pin manoeuvre at 30 m/s speed with non-adaptive (fixed-gain) or adaptive (gain-scheduled) controls.](image)

Similar controller behaviour is observed in the tracking of the lane-change manoeuvre. Both controllers again bring the rear-axle lateral slip close to 80% of the critical value, approximately 9.5 s into the simulation, each controller employing a subtly different steering demand to do so. The non-adaptive controller then increases the steering demand, which forces the rear axle into saturation. The steering angle demanded by the controller continues to increase and starts to oscillate between the two steering lock positions. With the car continuing to travel at a constant speed of 30 m/s, the controller is unable to recover control, nor is it able to align it with the path after the lane-change. In contrast, the adaptive controller provides a more measured control action which prevents the rear axle from overshooting the critical lateral-slip value. Although tracking
Figure 6.7: Rear-axle-lateral-slip and steering wheel angle histories for oversteering car in *hair-pin* manoeuvre of figure 6.5 with non-adaptive (fixed-gain) or adaptive (gain-scheduled) controls.
performance is degraded due to this reason, the lateral error is kept within respectable levels whilst maintaining control of the car throughout the simulation. When it is possible to do so, having negotiated the lane change itself, the controller returns the car to the path at 15.5 s.

![Lateral tracking errors for oversteering car in simulated lane-change manoeuvre at 30 m/s speed with non-adaptive (fixed-gain) or adaptive (gain-scheduled) controls.](image)

**Figure 6.8:** Lateral tracking errors for oversteering car in simulated lane-change manoeuvre at 30 m/s speed with non-adaptive (fixed-gain) or adaptive (gain-scheduled) controls.

### 6.3 Conclusions

In conclusion, an adaptive control scheme has been developed which controls an oversteering simple car model, which is distinctly different in setup to the understeering problem presented in the previous chapter. A controller which adopts a relatively loose control tightness has been presented, along with the interpolation process used to improve the controller performance. The tracking performance of the adaptive controller has been demonstrated on two challenging paths, with a comparison made against its non-adaptive counterpart. The results of such trials show that the adaptive controller, having knowledge of the non-linearity of the car model, uses a more modest control action when operating near the car’s limit, placing less emphasis on tracking performance when the state of the car is critical. This decision proves advantageous in both tracking trials where the adaptive controller retains control of the car, and is able to realign it with the path, whilst the non-adaptive controller loses control. The slight degradation in tracking performance during these phases of the simulation is seen as a tolerable compromise in guarding against controller instability.

Recommendations have been made on the ambition of the trajectories the oversteerer is required to track, the maximum demand in rear-axle-lateral-slip ratio curtailed to 80% of its critical value. This is in contrast with trajectories suggested for the understeering setup which can push the front, limiting, tyres to their absolute limit. The safety margin suggested for the oversteering problem
Figure 6.9: Rear-axle-lateral-slip and steering wheel angle histories for oversteering car in lane-change manoeuvre of figure 6.5 with non-adaptive (fixed-gain) or adaptive (gain-scheduled) controls.
ensures that the car is kept clear of a possible spin situation, which is seen to have a greater penalty than the corresponding limit scenario in the understeering problem. Calculations suggest that the minimum lap-time should not to be significantly affected when enforcing such a margin.

In summary, the novel adaptive controller which has been developed by the author has been successfully applied to the difficult problem of controlling an oversteering setup near its lateral limit. Such a scenario has not been investigated previously using a DLQR preview controller due to the inherent limitations of the fixed gain controllers that represented the state-of-the-art in DLQR preview controller design. The use of controller adaptation, via gain-scheduling, has been demonstrated to be key to designing high-quality tracking controllers for more relevant and interesting tracking problems, such as the oversteering control problem which has been presented.

The research detailed in this chapter concludes work on the limit control of the simple car model. Controllers have been developed for two distinctly different setups of such a model: the understeerer and oversteerer. The model has fulfilled its purpose, of including only primary features of a real race-car, during the initial stage of controller development. The next stage of research will tackle variable speed tracking along a trajectory. This provides an opportune moment to revise the modelling details of the car. The next chapter will specify the characteristics of a complex car model, aimed at including more modelling details which are relevant to a real road car, the control of which is a step closer to the ultimate goal of the research: the control of a realistic race-car model.
Chapter 7

Complex Car Model

Chapters 5 and 6 have documented how adaptive controllers can successfully control a simple car model for the purposes of optimal path tracking. Whilst the simplicity of the model facilitated the rapid development of such controllers, they are limited by the underlying modelling assumptions that are employed, such as the constraint on the forward speed of the car. Developing more relevant controllers requires a revision of the car model by including such details as longitudinal dynamics and the influence of tyre load on tyre forces. The first section of this chapter specifies the next evolution of the model, the complex car. Each aspect of the model will be presented in a similar format to that of chapter 3. The second section will then provide a description of the scheduling, trim-collection and linearisation processes associated with the complex car model.

7.1 Modelling details

7.1.1 Chassis

The chassis of the complex car model is represented by a sprung mass, $M_s$, with an inertia matrix, $I_s$, about the centre of mass. To facilitate the accurate modelling of tyre load variations, the chassis is modelled with the full six degrees of freedom: translation in $x$, $y$ and $z$ and rotation in yaw, pitch and roll. Figure 7.1 illustrates the relationship between the chassis, labelled $s$, and other components of the car, together with their respective degrees of freedom.

7.1.2 Suspension

The chassis incorporates four suspension assemblies, one for each wheel, distributed in a dual-track setup. Each assembly consists of a transverse swing arm, one end of which is connected to a pre-defined point on the chassis whilst the other is attached to a hub-carrier. The front hub-carriers
Figure 7.1: Bodies and freedoms included in complex car model. Low-pass filters associated with throttle and steering actuators, to represent driver response delays, are also shown. mb means “massless body”. hc means “hub carrier” and whl means “wheel”. The steering wheel is linked to the pinion by a torsion rod, which, in turn, is linked to the front hub carriers by a rack/pinion assembly. l means “left” and r means “right”. have an additional motion in yaw to allow them to be steered. Each hub-carrier is connected to a wheel, with rotational inertia, $I_{whly}$, which is allowed to rotate about its local $y$-axis. The radial compliance of each tyre is captured by a parallel spring and damper assembly, with the respective coefficients $k_{tyr}$ and $c_{tyr}$, located between each wheel and the road. The road is treated as perfectly flat and level.

The location of the pivot axis of each swing arm can be chosen such that small perturbations of arbitrary suspension configurations can be modelled. The suspension geometry is chosen such that the suspension roll-centres [15] are located at the ground level in the nominal state, as seen in figure 7.2. This setup represents a special case which has been implemented previously in production cars such as the Citroën 2CV and the front suspension of the Volkswagen Beetle. A parallel spring and damper system is used in each suspension assembly to control the vertical movement of the respective hub-carrier. The dynamics of the front and rear suspension systems can be specified independently using their respective damping, $c_{f\text{sus}}/c_{r\text{sus}}$, and stiffness values, $k_{f\text{sus}}/k_{r\text{sus}}$. Front and rear anti-roll bar effects are also modelled to complete the modelling of a realistic suspension setup.
Figure 7.2: Diagrammatic view of car suspension from rear. The instantaneous centre of rotation, associated with suspension displacement, of each wheel is at ground level on the car’s centre-line, which represents a special case in suspension setup that has been employed in production cars such as the Citroën 2CV and the Volkswagen Beetle.
7.1.3 Aerodynamics

Both aerodynamic drag and lift forces are now included in the modelling detail. Aerodynamic drag, \( F_d \), acting at the chassis’ centre of mass, is modelled by the equation

\[
F_d = 0.5 C_d \rho A u^2
\]  \hspace{1cm} (7.1)

where \( C_d \) is the drag coefficient, \( \rho \) is the air density, \( A \) is the car’s frontal cross-sectional area and \( u \) is the car’s forward speed. In a similar fashion, the lift forces acting at either the front, \( F_{zf} \), or rear axle, \( F_{zr} \), are defined as

\[
F_{zf} = 0.5 C_{lf} \rho A u^2
\]  \hspace{1cm} (7.2)

\[
F_{zr} = 0.5 C_{lr} \rho A u^2
\]  \hspace{1cm} (7.3)

with \( C_{lf} \) and \( C_{lr} \) being the lift coefficients for the front and rear axle respectively.

Sections 7.1.4 and 7.1.5 will detail the two control inputs of the complex car model: steering wheel angle and the throttle/brake position. Low-pass filters are placed between each output of the tracking controller and the control inputs of the car to represent the delay in control application. The cut-off frequencies of the filters, \( \Omega_n \), have been chosen to represent a reasonable estimate to the delay of a real driver in implementing control decisions.

7.1.4 Steering

The steering system, depicted in figure 7.3, consists of a steering wheel which is attached to a steering column, with a torsional stiffness of \( k_{st} \). The column is connected, via a rack and pinion gearing system, to the rotational motion of the front hub-carriers. Torque generated in the system, when the steering wheel is turned relative to the pinion gear, is used to steer the front wheels. Each front wheel is rotated about a vertical axis which has an offset, the mechanical trail, forward of the wheel’s centre. Bump steer is modelled by explicitly defining the contribution that suspension deflections make to the lateral slip ratio of the relevant tyre.
Figure 7.3: Top view of steering system in the complex car model.
7.1.5 Driving/Braking force

The throttle/brake signal, $\tau^\star$, which is the output of the low-pass filter that was described earlier, actuates either the engine or braking system depending on its sign. When the sign is positive, the signal is interpreted as a throttle opening demand which operates the engine. The drive-shaft torque, $\chi$, is constructed by multiplying the outputs of three functions: the throttle-opening function

$$\frac{1}{\omega} \sin \left[ \arctan \left( B_t \tau^\star - E_t \left( B_t \tau^\star - \arctan \left( B_t \tau^\star \right) \right) \right) \right]$$  \hspace{1cm} (7.4)

where $B_t$ and $E_t$ are throttle opening-shaping parameters, the engine-speed map

$$D_s \sin \left[ C_s \arctan \left( B_s \omega - E_s \left( B_s \omega - \arctan \left( B_s \omega \right) \right) \right) \right]$$  \hspace{1cm} (7.5)

where $B_s$, $C_s$, $D_s$ and $E_s$ are engine-speed-shaping parameters, $\omega$ being the engine speed in rad/s, and the gear-ratio map

$$3 - 2 \sin \left( \arctan \left( 0.2 \left( u - 15 \right) \right) \right)$$  \hspace{1cm} (7.6)

Figure 7.4 illustrates how engine torque, which is limited to positive values, varies with engine speed and throttle opening, $\tau^\star$, while figure 7.5 demonstrates how the gear-ratio map varies with the car’s forward speed, $u$. This torque is transmitted through a longitudinal drive-shaft to a viscous, limit-slip, differential gearbox to drive the rear wheels. The inertial effect of the transmission components on the engine torque, as the gear-ratio is varied, is not considered at this level of investigation.

If the throttle/brake signal is negative, braking torque determined by the equation

$$\varphi = k_{br} \tau^\star \arctan \left( B_r \Omega_w \right)$$  \hspace{1cm} (7.7)

is applied to each wheel. Equation 7.7 ensures that in the case where the wheel speed, $\Omega_w$, drops sufficiently low, the braking torque does not drive the wheels backward. The front and rear braking parameters, $k_{brf}$ and $k_{brr}$ respectively, are chosen to apply a proportional braking torque to each wheel in the ratio of 74% front and 26% rear.
Figure 7.4: Engine torque output as a function of engine speed and throttle opening. Parameter values are $B_s=0.0014$, $C_s=1.6$, $D_s=100000$, $E_s=-8$, $B_t=1.8$, $E_t=-12$.

Figure 7.5: Engine gear-ratio as a function of car speed, see equation 7.6.
7.1.6 Tyres

A detailed tyre model is now used. This incorporates the four significant variables which influence tyre force: vertical tyre load, camber angle, longitudinal slip and lateral slip. The tyre’s shear forces and aligning moment are represented by a combined-slip model which is a combination of Pacejka’s *Magic Formula* [39] and the normalisation process developed by Sharp and Bettella [44, 51, 52]. Such a model represents a realistic tyre under general running conditions, the pure-slip longitudinal and lateral force plots of which are illustrated in figure 7.6.

![Figure 7.6: Longitudinal (left) and lateral (right) tyre forces associated with variations in load and longitudinal or lateral slip ratio respectively, for zero camber angle. Tyre parameters used come directly from [52].](image)

Both car and tyre parameters are presented in Table 7.1. The values have been chosen to represent a typical European family saloon and are based on work originally conducted by Sharp and Fernández [54]. Figure 7.7 demonstrates the car’s dynamic characteristics by plotting the steering angle required to follow three different constant radius arcs (150 m, 200 m and 300 m), for a range of lateral accelerations. Employing the same process of qualification used in chapter 3, the positive gradient of the plots suggest the complex car’s setup has a predominantly *understeering* characteristic. A VehicleSim LISP input-file is again used to describe the structure of the system. The model is now required to include both the absolute lateral and longitudinal displacements of the car amongst its outputs. These parameters are used to follow the road in the tracking trials and are therefore required when formulating the control problem, as seen in chapter 4.
Figure 7.7: Plot of steering wheel angle required to maintain a course with a constant radius of curvature of: 150 m, 200 m and 300 m using the complex car model, for a range of lateral accelerations.

Table 7.1: Symbols and parameter values of the complex car.
The values presented have been derived from work originally conducted by Sharp and Fernández [54].

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<th>Symbol</th>
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</table>

7.2 Operational details

The choice of the scheduling parameter for the simple car model was crucial to the success of the relevant adaptive controller. The front-axle-lateral-slip ratio was chosen for the understeering problem due to its high sensitivity to the limiting feature of the setup, the saturating front axle forces. A similar argument was used to choose the first scheduling parameter of the complex car model. The mean value of the front-tyre-lateral-slip ratio shows good sensitivity to the state of the complex car’s, limiting, front axle. Averaging the lateral slip of both front tyres, avoids the need to distinguish between left- and right-hand tyres when scheduling. To reflect the new requirement to follow speed profiles, forward ground speed was chosen as the second scheduling parameter for the complex car.

As with the simple car model, the process of collecting trim-states can be specified once the scheduling parameters have been chosen. Several trim collection simulations are run in an ordered manner to collect a uniform grid of trims within the operating envelope of the complex car. A PID speed controller, which actuates the throttle/brake demand, is used to hold a constant target speed throughout each simulation. For each speed, a second PID controller actuates the steering wheel to follow the ramped demand in the mean front-lateral-slip ratio. The demand is ramped slowly enough that the car can be assumed to be close to an equilibrium-/trim-state at each time step of the simulation. Once collected and indexed using the scheduling parameters, the trim-states can be used to linearise the car model.

Care must be taken when linearising the complex car model, as required by the DLQR techniques presented in chapter 4, as the modelling of complex components introduces discontinuities which were not present in the simple car. For example, the tyre model is discontinuous between the phases before and after saturation [44]. The transition from acceleration to braking in the new model is another example. The simple car model, having no such discontinuities, requires the gen-
eration of a single VehicleSim@generated MATLAB™ M-file, which, when given the appropriate trim-state data, generates a locally linearised model. The linearisation of the complex car involves the generation of multiple M-files, each only capable of linearising the model for a sub-set of running conditions. After the trim-states have been collected, the program matches each trim-state to an appropriate linearising file to correctly linearise the model. The system is then discretised, augmented with the shift register process and passed to the dlqr function to obtain the optimal control gains, as described in chapter 4.

In conclusion, the modelling details of the complex car, which is to be used in the forthcoming tracking trials, have been presented. The model includes a higher level of modelling detail than the simple car, including features such as tyre force load dependence, suspension- and longitudinal-dynamics. Changes to the scheduling, trim-collection and linearisation processes, which are necessary for the new car model, have also been highlighted. The following chapter will describe the development of advanced tracking controllers based on the complex car model.
Chapter 8

Complex car tracking results

Chapters 5 and 6 have presented the development of tracking controllers used to optimally control a simple car in limit tracking situations. The additional complexity and detail of the complex car model, presented in the preceding chapter, requires more advanced controllers. This chapter will document the incremental development of such controllers, as the tracking task is made progressively more difficult. In this pursuit, a fixed-gain variable-speed tracking controller, which implements the basic theory introduced in chapter 4, will first be presented. Modest paths and speed profiles will be used to verify the correct operation of the controller. These results are based on the author’s published work [60, 62] in the area. Next, the use of adaptation, using forward ground speed and the mean front-lateral-slip ratio as scheduling parameters, will be investigated with a view to improving tracking performance during more severe manoeuvres. Sharp et al. [56] presents results based on the basic principles which will be discussed. Finally, the special case of path tracking near the lateral limit of the car will be investigated, with a presentation of the difficulties that are associated with such a scenario. Control strategies which overcome such difficulties will be recommended, with a demonstration of their benefits. The tracking controller that results from the development work detailed in this chapter can subsequently be used to develop the speed and path optimisation phases of the novel solution to the minimum lap-time problem.

8.1 Fixed-gain, variable-speed, tracking controller

The literature review in chapter 2 has already highlighted the fact that whilst preview DLQR controllers have been designed for the separate cases of longitudinal and lateral tracking, a combined variable speed \((x,y)\) tracking controller using the same theory is yet to be developed. Chapter 4 has outlined the method used to construct such a controller. This section will serve to present, and demonstrate the correct operation of, this controller.
A relatively tight control strategy will be used in the forthcoming tracking trials. The diagonal matrices which place a cost on longitudinal/lateral tracking error, $q$, and on control effort, $R$, are set to

$$q = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix} = \begin{bmatrix} 300 & 0 \\ 0 & 300 \end{bmatrix} \quad R = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix}$$

(8.1)

In its present form, equation 8.1 places an equal cost on longitudinal- and lateral-tracking errors. The relative difference in the diagonal elements of the matrix $R$ accounts for the difference in the ranges of operation between the steering wheel angle, which varies between $\pm 10\,\text{rad}$, and the throttle/brake position, which is allowed to vary between $\pm 1$. Such a matrix places the same cost on using one control input relative to the other. Figure 8.1 presents an exemplary set of optimal preview gains, $K_{2\theta x}$, $K_{2\theta y}$, $K_{2\tau x}$ and $K_{2\tau y}$, which correspond to the control strategy presented above, for a straight-running-trim state at a forward speed of 35 m/s. A time step of 0.01 s was again found to be adequate for the discrete implementation of the optimal controls.

The figure suggests that 400 preview points, corresponding to a preview length of 4 s, can be used to capture full preview for the system. In the special case where the car is linearised for a straight-running trim, the longitudinal and lateral dynamics of the car become decoupled, which is reflected in the optimal control gains that are based on them. In this case, the controller is aware that only steering control decisions will affect the lateral dynamics of the car, which are consequently used to reduce lateral-tracking errors. The preview gains relating the throttle/brake demand to lateral-tracking errors are zero. Conversely, only the throttle/brake demand will influence the longitudinal dynamics, which are consequently used to reduce longitudinal-tracking error. Hence, the preview gains relating the steering wheel angle to longitudinal errors are also zero. Table 8.1 presents the feedback gains corresponding to the trim-state employed in figure 8.1. The table highlights how, having knowledge of the car’s dynamics, the optimal control scheme establishes varying levels of feedback between the errors of relevant car states and each control input. For example, the throttle/brake signal, which is known to only influence the longitudinal dynamics in this case, is predominantly affected by errors in the states which are related to these dynamics, such as the forward speed of the car. Similarly, the steering wheel angle, which is known to only influence the lateral dynamics in this case, is predominantly affected by errors in the states which are related to these dynamics, such as the car’s roll-angle, lateral speed and yaw rate. Due to the decoupling of the car’s dynamics, feedback gains which affect the longitudinal dynamics
have a negligible influence on the control input affecting the lateral dynamics, the steering wheel angle, and vice versa. The fact that the car is not perfectly symmetric, due to the presence of the longitudinal drive-shaft, through which torque is applied from the engine to the differential gearbox, means these *out-of-plane* feedback gains are, very close to but, not quite zero.

![Figure 8.1: Preview gains for a straight-running trim at a forward speed of 35 m/s, using $q = [300 0; 0 300]$ and $R = [10 0; 0 1]$.](image)

Two tracking trials are used to demonstrate the application of the control scheme that has been presented: a lane change and a hair-pin manoeuvre, see figure 8.2. The severity of each manoeuvre has been curtailed in recognition of the limitations of the non-adaptive linear control scheme that is employed.

The car is required to maintain a constant speed of 35 m/s when tracking the lane change. The start of the manoeuvre consists of a straight section, 175 m in length. This is followed by another straight section, 210 m in length, which makes a $5^\circ$ direction change from the $x$-axis. The direction change is reversed at the end of the second straight, leading into a final straight, 175 m long, which is again aligned with the $x$-axis.

Time histories of the steering wheel angle and throttle-controller demands, when tracking the lane change, are plotted in figure 8.3, together with the corresponding output of the low-pass filter that represents the driver delay in both controls. Lateral- and longitudinal-tracking errors for the manoeuvre are presented in figure 8.4, together with the car’s deviation from the constant-speed
Table 8.1: Trimmed values of control inputs, presented with significant feedback gains for a complex car, designed for a straight-running-trim state at a forward speed of 35 m/s.

<table>
<thead>
<tr>
<th>State</th>
<th>Throttle/Brake feedback gain</th>
<th>Steering feedback gain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Throttle trimmed values</td>
<td>0.0933</td>
<td>0 rad</td>
</tr>
<tr>
<td>State</td>
<td>Throttle/Brake feedback gain</td>
<td>Steering feedback gain</td>
</tr>
<tr>
<td>Pitch angle</td>
<td>-2.7964 rad⁻¹</td>
<td>0.0436</td>
</tr>
<tr>
<td>Gas pedal response</td>
<td>0.8678</td>
<td>-0.0004 rad⁻¹</td>
</tr>
<tr>
<td>Longitudinal speed</td>
<td>2.0175 s m⁻¹</td>
<td>0.0008 rad s m⁻¹</td>
</tr>
<tr>
<td>Pitch rate</td>
<td>0.1241 s rad⁻¹</td>
<td>-0.0099 s</td>
</tr>
<tr>
<td>Roll angle</td>
<td>-0.0065 rad⁻¹</td>
<td>3.2857 s</td>
</tr>
<tr>
<td>Steering angle response</td>
<td>0.0000 rad⁻¹</td>
<td>2.1059</td>
</tr>
<tr>
<td>Lateral speed</td>
<td>0.0002 s m⁻¹</td>
<td>2.9544 rad s m⁻¹</td>
</tr>
<tr>
<td>Yaw rate</td>
<td>0.0000 s rad⁻¹</td>
<td>5.1725 s</td>
</tr>
</tbody>
</table>

Figure 8.2: Path demands for 2 manoeuvre simulations showing points at 2 s intervals along them. (a) A lane change. (b) A hair-pin.
The control histories presented in figure 8.3 illustrate how the controller provides smooth control actions to follow both, relatively sharp, direction changes. As mentioned previously, the control inputs are decoupled due to the straight-running trim that is used. The control inputs therefore perform independently of one another, with the throttle responding to longitudinal errors developing as a result of steering control decisions, and vice versa. The steering control signal varies between $\pm 1.3$ rad, whilst the throttle is actuated between 0.05 and 0.32 of its full range. The oscillatory nature of the steering control signal prior to the major direction changes is a typical feature of the preview controller, where it excites the eigendynamics of the car to reduce control effort [45, 46, 48]. The controller displays good tracking ability, with the magnitudes of lateral and longitudinal errors being kept below 0.22 m and 0.05 m respectively.

The discrete set of coordinates defining the hair-pin bend are spaced such that the car must
Figure 8.4: Time histories of lateral-/longitudinal-tracking-errors and speed deviation for lane-change manoeuvre of figure 8.2 with non-adaptive (fixed-gain) controller.
follow a varying speed profile whilst tracking the bend. The car starts at the origin and is required to initially follow a straight section along the $x$-axis. Ten seconds into the simulation, the car is required to decelerate in a straight-line, from its initial speed of 35 m/s, to 25 m/s in 6 s in preparation for the corner. The car must then follow a constant radius arc, with a radius of 183 m, whilst continuing to decrease its speed to 20 m/s. Once the target speed is reached, it must be maintained whilst tracking the remainder of the first arc section, which extends over $0.5\pi$ rad. The car is then required to accelerate back to its starting speed over a period of 15 s. During this period, the car must follow a second arc section, with a radius of 200 m extended over $0.5\pi$ rad, after which it accelerates, back to 35 m/s, in a straight line. Various aspects of the simulation results are shown in figures 8.5 - 8.9.

![Graph](image)

**Figure 8.5:** Time histories of steering wheel angle demand (and the delayed driver response to this demand), and delayed throttle and brake responses, for *hair-pin* manoeuvre of figure 8.2 with non-adaptive (fixed-gain) controller.

As with the lane-change manoeuvre, the steering control signal displays oscillatory behaviour in regions where the curvature of the path is discontinuous. Overall, the steering control signal
Figure 8.6: Time histories of lateral-/longitudinal-tracking-errors for hair-pin manoeuvre of figure 8.2 with non-adaptive (fixed-gain) controller.

Figure 8.7: Times histories of car speed, lateral acceleration, body pitch and roll-angles for hair-pin manoeuvre of figure 8.2 with non-adaptive (fixed-gain) controller.
Figure 8.8: Time histories of tyre loads for *hair-pin* manoeuvre of figure 8.2 with non-adaptive (fixed-gain) controller.
Figure 8.9: Time histories of tyre shear forces for hair-pin manoeuvre of figure 8.2 with non-adaptive (fixed-gain) controller.
is well behaved and allows the car to negotiate the bend whilst keeping the magnitude of lateral and longitudinal errors below 0.15 m and 0.12 m respectively. This result is especially significant given that the controller maintains good speed tracking throughout the manoeuvre. The actuation of the brakes between 10 and 20 s allows the car to decelerate to the slower speed of 20 m/s. A constant throttle position, approximately 0.03 of full range, after this phase allows this speed to be maintained before accelerating back to the initial speed. The oscillations observed in the throttle signal shortly after 40 s are again a result of the throttle reacting to steering decisions, a consequence of the decoupled nature of the trim-state that is employed. Lateral acceleration grows steadily as the car accelerates out of the corner, following the second arc, with a short burst to higher values as the steering control prepares the car for a return to straight-running. The car displays noticeable pitching, of ±0.03 rad, during points of maximum acceleration and deceleration, whilst its roll-angle drops to −0.1 rad when tracking the second arc section, where the speed, and consequently lateral acceleration, is steadily built-up. Figure 8.8, illustrates how load is transferred to the front axle under braking, between 10 and 15 s. This fact, combined with the braking bias that is used, results in the front tyres generating the majority of the retardation force in figure 8.9. As the car prepares to enter the corner, at 16 s, the left-front tyre generates the highest lateral force of the four tyres, owing to the lateral and longitudinal load transfer to it. For the same reason, the right-rear tyre, having the lowest tyre load at this point, generates the lowest lateral tyre force. As the car starts to accelerate, load is transferred rearward, assisting the rear wheels in generating the longitudinal driving force required to return the car to its starting speed. The left-rear tyre generates more longitudinal driving force than in the right-rear as a result of a higher loading on this tyre and the effect of the limited-slip differential gearbox.

8.1.1 Conclusions

The results presented in this section demonstrate the successful implementation of the novel two-channel, variable speed, DLQR preview tracking controller developed by the author. Such trials lay the foundation work for two-channel (longitudinal/lateral) tracking, which has not been addressed by previous work in the field of DLQR preview control. This widens the scope of applications for the technology, which was previously limited to either path tracking at a constant forward speed or variable speed tracking trials along a straight line. As mentioned previously, the severity of the manoeuvres have been limited in light of the linear nature of the controller. The tracking of more intense paths can be accomplished by employing gain-scheduling/adaptation.
8.2 Adaptive tracking controllers

The tracking performance of adaptive controllers have been proven to be superior to that of their non-adaptive counterparts when used to track aggressive paths with the simple car. A similar advantage is expected when using an adaptive controller with the complex car. The choice of scheduling parameters for such a controller has been discussed in the previous chapter, with two being chosen: the car’s mean front-lateral-slip ratio and forward ground speed. Figures 8.10 to 8.13 illustrate how each, of the set of four, optimal preview gains vary with the car’s mean front-lateral-slip ratio. The plots highlight the fact that, as longitudinal and lateral dynamics are now coupled for cornering trims, the controller will use a combination of both control inputs to reduce longitudinal and lateral errors individually. The optimal preview gain surfaces display a similar characteristic to the preview gains of the simple car, figures 5.1 and 6.2: as the lateral limit of the car is approached, the preview length required to capture full preview increases. In addition, ridge-like features are present in the majority of the preview gain surfaces, at a cornering trim corresponding to a value of $\sim0.19$ in the mean front-lateral-slip ratio. This value represents the point where the front, limiting, axle force saturates.

Figures 8.14 and 8.15 illustrate how the significant optimal preview gains of a straight-running-trim state vary with the remaining scheduling parameter, the car’s forward ground speed. The optimal preview gain surface in figure 8.14, relating lateral tracking error to steering control decisions, only varies marginally as the car speed is increased. This reflects the fact that the lateral dynamics of the car, linearised about the straight-running-trim state, do not vary significantly as speed is increased. This situation is likely to change if features such as the aerodynamic downforce generated by a race-car, which affects the lateral dynamics of the car and varies significantly with speed, are captured by the model. In contrast to figure 8.14, figure 8.15 shows a greater variation in preview gains, which relate the longitudinal tracking error to throttle/brake decisions, as speed is increased. This is a consequence of the engine’s performance, and its effect on the longitudinal dynamics of the car, varying significantly with speed.

A hair-pin bend, similar in shape to the hair-pin in figure 8.2, with dimensions and speed profile chosen to represent a more aggressive manoeuvre, is used to compare and contrast the performance of adaptive and non-adaptive tracking controllers. The manoeuvre again starts with the car following a straight section at a constant speed of 35 m/s, leading into an arc with a constant radius of 48.3 m. Ten seconds into the simulation, the car must start decelerating, slowing to 13 m/s in another 10 s and maintaining this speed for the remainder of the first arc, which extends for $0.5\pi$ rad. The car is then given 18.5 s to accelerate back to its initial speed. Whilst accelerating,
Figure 8.10: Three dimensional plot of $K_{2\tau x}$ preview gain sequences as functions of mean front-tyre-lateral-slip ratio, for a speed of 30 m/s. These gains relate the throttle-/brake-pedal position to longitudinal-tracking-errors.
Figure 8.11: Three dimensional plot of $K_{2\tau y}$ preview gain sequences as functions of mean front-tyre-lateral-slip ratio, for a speed of 30 m/s. These gains relate the throttle-/brake-pedal position to lateral-tracking-errors.
Figure 8.12: Three dimensional plot of $K_{26x}$ preview gain sequences as functions of mean front-tire-lateral-slip ratio, for a speed of $30 \text{ m/s}$. These gains relate the steering wheel angle to longitudinal-tracking-errors.
Figure 8.13: Three dimensional plot of $K_{20y}$ preview gain sequences as functions of mean front-tyre-lateral-slip ratio, for a speed of 30 m/s. These gains relate the steering wheel angle to lateral-tracking-errors.
Figure 8.14: Three dimensional plot of $K_{2\theta y}$ preview gain sequences, for straight-running trims, as functions of car speed. These gains relate the steering wheel angle to lateral-tracking-errors.
Figure 8.15: Three dimensional plot of $K_{2\tau x}$ preview gain sequences, for straight-running trims, as functions of car speed. These gains relate the throttle-/brake-pedal position to longitudinal-tracking-errors.
it must firstly follow a second arc with a constant radius of 200 m, again extended over \(0.5\pi\) rad, which leads into the final straight section, where the car accelerates back to 35 m/s. The *hair-pin* path which is to be tracked is presented in figure 8.16, together with its corresponding speed profile.

![Diagram of the hair-pin path](image)

Figure 8.16: Path, showing points at 2 s intervals along it, and speed demand for aggressive *hair-pin* manoeuvre simulation.

At the start of the manoeuvre, the reference trim-state for both controllers is that of straight-running at a speed of 35 m/s, used in the previous simulations. This remains the same for the non-adaptive controller. The adaptive controller is able to select a reference trim from a stored set which has been collected off-line, using the process described in chapter 7. The set of trims consists of 24 speeds, with a separation of 1 m/s from one another, with a corresponding grid of 81 cornering trims for each speed. To avoid the instability in the controls, when switching from one discrete trim to another, quadratic interpolation is used. The process allows the controller to operate smoothly between the discrete set of stored trims to find the most appropriate trim to install. When 1-dimensional (1D) adaptation is employed, the controller is allowed to search the stored set for a straight-running-trim state that has a speed closest to the car’s current state.
Once found, straight-running trims which correspond to speeds just below and above this speed are located and are used as the boundaries for the interpolation process. Quadratic functions are fitted using the respective values of each parameter in each of these three trim-states. Interpolation is then used to obtain values of: the trim-state that is used as a reference trim, the optimal control gain vectors and the parameters used to generate the future trim trajectory of the car, detailed in chapter 4. When 2-dimensional (2D) adaptation is employed, the controller is allowed to search the stored set for the closest match in both speed and mean front-lateral-slip ratio. Once located, the interpolation is executed in two phases, which starts with a grid of the nine trims (3-by-3) closest to the car’s running state. The first phase takes place in the direction where only the speed of the trim varies, as before. This results in a set of three interpolated trims which form the interpolation points for the next phase. The second phase takes place along the direction where only the mean front-lateral-slip ratio is varied, using the three interpolated trims to perform the final interpolation.

Three control strategies were used to track the aggressive hair-pin bend: the non-adaptive controller used in the previous section, the 1D adaptive controller and the 2D adaptive controller. The time histories of the steering, throttle and brake control inputs for each control strategy are presented in figure 8.17. Corresponding time histories of the lateral-, longitudinal- and speed-errors are plotted in figure 8.18.

Only a marginal difference is observed between the throttle control employed by each controller in figure 8.17. The respective braking signals of each controller show a greater variation in the period between 16 and 20 s. The ability of the 1D controller, to search for a straight-running trim with a closer match to the instantaneous speed of the car, allows it to install a more appropriate control scheme, and corresponding future trajectory, than the non-adaptive case. An improvement in tracking performance is observed with the 1D controller employing a combination of different steering and braking demands to reduce both the lateral- and longitudinal-tracking errors relative to those of its non-adaptive counterpart. The 2D adaptive controller represents a further step forward, in being able to search for a trim which matches the car’s current speed and cornering intensity. Such a scheme allows the controller to adopt a different combination of steering and braking strategies to produce a further reduction in lateral- and longitudinal-tracking errors, requiring a momentary drop in speed tracking performance to execute the manoeuvre. The tyres are worked harder to provide this increase in lateral- and longitudinal-tracking performance, with the heavily loaded left-front tyre generating the highest portion of the cornering force during the most taxing period of the simulation, at approximately 16.5 s. At this time instant, the normalised slip of this critical tyre rises to a maximum of 67% of the normalised slip which generates the
Figure 8.17: Time histories of delayed steering wheel angle, throttle and brake responses, for hair-pin manoeuvre of figure 8.16 with: fixed-gain (non-adaptive), scheduling in speed only (1D adaptive) and scheduling in both speed and mean front-lateral-slip ratio (2D adaptive).
Figure 8.18: Time histories of lateral-, longitudinal- and speed-errors for simulated hair-pin manoeuvre of figure 8.16 using: fixed-gain (non-adaptive), scheduling in speed only (1D adaptive) and scheduling in both speed and mean front-lateral-slip ratio (2D adaptive).
tyre’s maximum force capacity. This indicates that the path/speed profile combination, although representing an aggressive manoeuvre, does not push the car to its absolute limit.

Having established the fact that a poor match between the stored trim-state and the car’s running state results in poorer tracking performance, it is worth noting that although the tracking errors are relatively low, the most significant longitudinal errors, occurring at 15 and 35 s, are likely to result from a remaining mismatch between the trims and the running state of the car. This is a consequence of the trim-states currently being collected under constant speed conditions, whilst the running state of the car involves moderate longitudinal deceleration and acceleration, as is the case at 15 and 35 s. This mismatch is likely to result in the poorer tracking performance of the controller at these instances. The collection of a set of trims which capture longitudinal acceleration, and their subsequent use in the tracking trials, is likely to improve the tracking performance at these instances. However, the efficient calculation of the future trajectory of such trims is not straightforward. The improvement in tracking performance when employing such a scheme may not be sufficient to justify the extra complexity in the controller, and the additional trims that will need to be collected. This may be significant given the fact that the original 2D adaptive control strategy may provide an equal improvement in performance if the control strategy is tightened accordingly. The improvement in tracking performance associated with each of these suggested methods will not be investigated by the current research effort.

8.2.1 Conclusions

In conclusion, the benefits of using controller adaptation has been demonstrated on an aggressive tracking manoeuvre using a complex car model. Improvements in tracking performance have been noted in moving from a non-adaptive controller to 1D adaptation in forward speed and then to 2D adaptation in forward speed and cornering intensity. Such improvements are a result of installing trims, and the respective optimal control schemes, which are closer to the car’s running state at each instant. Such tracking trials represent a valuable contribution to the field of DLQR preview tracking control as they combine the two technologies of two-channel variable speed tracking control and adaptive gain-scheduling, which are themselves novel contributions to the field. Such adaptive controllers provide the necessary tools to attempt high-quality tracking of more ambitious manoeuvres, with the knowledge that the controller will do its best to control the car. The optimal tracking of such manoeuvres has previously not been possible due to the limitations of fixed-gain, single-channel, controllers that were employed. The novelty of the results from this section will therefore be of interest to the field in understanding the behaviour of a more capable controller, controlling a detailed car model, during more realistic manoeuvres. Limit behaviour
whilst tracking an ambitious path will now be investigated. As mentioned previously, this scenario is of interest in the wider context of the novel solution to the minimum lap-time. The stability of the controller when tracking such cases is crucial to ensuring the speed and path optimisation phases produce a minimum lap-time solution efficiently.

8.3 Lateral limit tracking

In the previous section, adaptive controllers have facilitated the optimal control of the complex car during intense tracking manoeuvres whilst keeping tracking errors to a minimum. The final stage of investigation involves adaptive control of the, predominantly understeering, car near its limit. Only lateral limit behaviour will be investigated, with the investigation of the longitudinal limit behaviour being left to future work.

An ambitious clothoid path is used to investigate the behaviour of the controller near the car’s lateral limit. The path is similar in construction to those used in chapters 5 and 6. To keep the analysis straightforward, the controller is required to keep the car at a constant speed of $30 \text{ m/s}$ throughout the manoeuvre. The path starts with a straight section, $186 \text{ m}$ in length. To avoid the discontinuities in curvature, which were present in previous paths, that consequently cause oscillations in the controller, a clothoid section is used to provide a uniform variation in path curvature from the straight section to an arc. The arc has a constant radius of $124 \text{ m}$ and extends over $0.75\pi \text{ rad}$. At the end of this arc, another clothoid allows a uniform decrease in curvature back to a straight section. Perfect tracking of the path, plotted in figure 8.19, requires the car to sustain lateral accelerations up to $7.26 \text{ m/s}^2$.

The time histories of the control inputs are presented in figure 8.20, while the car’s mean front-lateral-slip ratio is plotted in figure 8.21. Due to the load transfer during the manoeuvre, the left-front tyre has the highest contribution to the lateral force generated by the car. The saturation of lateral forces generated by this tyre is therefore seen as critical to the manoeuvre. Hence, the utilisation of the left-front tyre is included in figure 8.21. The tyre’s utilisation is a percentage value calculated from the ratio of its instantaneous normalised-slip and the critical normalised-slip when the tyre is operating at its maximum capacity, $l_{barp}$. Using such a variable significantly reduces the number of parameters needed to monitor the overall state of each tyre. Such a parameter can be used in both speed and path optimisation phases of the minimum lap-time solution, where the state of the tyres must be monitored accurately to judge the effect of optimisation iterations.

The behaviour of the controller when tracking the \textit{hair-pin} manoeuvre in figure 8.19 is noted to be somewhat poor, with the control inputs exhibiting high frequency oscillations when the
Figure 8.19: Path which represents a severe manoeuvre, showing points at 2s intervals along it.
Figure 8.20: Time histories of delayed steering wheel angle and throttle position responses, for severe \textit{hair-pin} manoeuvre of figure 8.19 with: constant and variable weighting controllers.
Figure 8.21: Time histories of mean front-lateral-slip ratio and left-front tyre utilisation for severe hair-pin manoeuvre using: constant and variable weighting controllers.
Figure 8.22: Time histories of lateral-, longitudinal- and speed-errors for severe hair-pin manoeuvre using: constant and variable weighting controllers.
limit of the front axle is approached. Both lateral- and longitudinal-tracking errors, presented in figure 8.22, get progressively larger as the simulation progresses. Nineteen seconds into the simulation, the errors and oscillations grow to a point where the controller loses control of the car, causing the tracking simulation to crash. Such controller instability, and resulting program crash, is unacceptable in the context of the minimum lap-time prediction program.

The oscillatory behaviour of the controller is initiated when reference trims close to the lateral limit of the front axle are installed. Ridge-like features in the preview gain surfaces are known to exist in this region, especially in figures 8.12 and 8.13. Previous adaptive controllers, owing to the simplicity of the car model that was used, have a smoother transition at the critical point of the front axle, with the gain magnitudes falling to zero before reversing sign. In the case of the complex car, the gains either side of the critical point do not fall to zero, causing the ridge-like features that are observed. The use of trims from this region, where the preview gain surface changes dramatically, is thought to cause the high frequency oscillation about the lateral limit of the front axle, observed in the tracking results in figure 8.20.

To prevent such instability near the lateral limit of the car, attention was focused on the control strategy used to obtain the optimal controls, which previously used a fixed weighting strategy for all trim-states. It can be argued that when the car is operating near the limit of its capabilities, less importance should be placed on tracking error, shifting instead to providing a smoother control signal to avoid the onset of instability at this critical stage. The looser control strategy that is implied results in more stable controller behaviour near the car’s limit, as observed in the oversteering problem of the simple car. To maintain tracking performance when operating at a sufficient distance from the car’s limit, a variable control weighting scheme was proposed. A multiplier, $\Psi$, which varies with the mean front-lateral-slip ratio, an indicator of the cornering intensity, is applied to the matrix placing a cost on tracking error, $q$. The cost matrix, which is now specific to the cornering intensity of the trim being used, is then used in the standard procedure of obtaining the optimal control gains, as discussed in chapter 4. Equation 8.2 presents the function describing this multiplier.

$$\Psi = C_{ts} + (1 - C_{ts}) \sin^2(A_{ts}(\alpha_{mflsr} - B_{ts}))$$

where $A_{ts}$, $B_{ts}$ and $C_{ts}$ are shaping factors while $\alpha_{mflsr}$ is the mean front-lateral-slip ratio. Figure 8.23 illustrates the variation of this multiplier with the mean front-lateral-slip ratio. Using such a scheme retains the original tight control strategy when operating close to straight-running while
the control gets progressively looser as the front axle approaches saturation. Hence, the control is required to be smoother to prevent lively control action, which can result in the instabilities that have been observed previously, when the car is close to its limit.

![Graph](image.png)

Figure 8.23: Variation of weighting multiplier, $\Psi$, against mean front-lateral-slip ratio, using $A_{ts} = 8$, $B_{ts} = 0.1915$ and $C_{ts} = 0.5e^{-7}$.

The consequence of employing such a variable control scheme, on the trims used to generate figures 8.10 to 8.13, can be seen in the corresponding figures 8.24 to 8.27. The looser control strategy near the limit of the front axle will typically require a longer full preview horizon. However, the control strategy is made loose enough near the limit that the complementary feature, of the reduction in the gain magnitudes, plays a more dominant role in shaping the gains. The magnitude of the preview gains fall to sufficiently low levels to be considered negligible, offsetting the need for a longer preview length. The variation in the control tightness, therefore, has the effect of flattening the ridge-like features near the limit of the front axle, which are thought to cause the high frequency oscillations that result in controller instability.

The corresponding control actions, tyre utilisation and tracking errors time-histories, when using a variable tightness scheme, are plotted in figures 8.20, 8.21 and 8.22 respectively. The high-frequency content of the control signals employing a variable control tightness, when operating near the lateral limit of the car, is significantly lower than in the constant weighting case. The throttle response exhibits a sharp variation at approximately 19 s, in preparation for the return to straight-running shortly afterward. In a manner typical of a looser control strategy, the lateral and longitudinal tracking performance is forfeited in an effort to stabilise the car, but is kept within respectable levels. Lateral-tracking errors are kept below 0.1 m whilst the longitudinal errors are kept below 0.15 m. The speed of the car is well regulated, with the speed error kept between -0.25
Figure 8.24: Three dimensional plot of $K_{2\tau x}$ preview gain sequences as functions of mean front-tyre-lateral-slip ratio, for a speed of 30 m/s. These gains are a result of using the multiplier defined in equation 8.2 and relate the throttle-/brake-pedal position to longitudinal-tracking-errors.
Figure 8.25: Three dimensional plot of $K_{2\tau y}$ preview gain sequences as functions of mean front-tyre-lateral-slip ratio, for a speed of 30 m/s. These gains are a result of using the multiplier defined in equation 8.2 and relate the throttle-/brake-pedal position to lateral-tracking-errors.
Figure 8.26: Three dimensional plot of $K_{20x}$ preview gain sequences as functions of mean front-tyre-lateral-slip ratio, for a speed of 30 m/s. These gains are a result of using the multiplier defined in equation 8.2 and relate the steering wheel angle to longitudinal-tracking-errors.
Figure 8.27: Three dimensional plot of $K_{2\tau y}$ preview gain sequences as functions of mean front-tyre-lateral-slip ratio, for a speed of 30 m/s. These gains are a result of using the multiplier defined in equation 8.2 and relate the steering wheel angle to lateral-tracking-errors.
and 0.12 m/s. The absence of high frequency oscillations ensures the controller is able to maintain control of the car during the critical phase and successfully complete the tracking trial, unlike its constant weighting counterpart. The mean front-lateral-slip ratio is kept close to the critical value whilst the saturation level of the left-front tyre is maintained at approximately 80% during the critical phase. It is recommended that the car is not forced far beyond this point during tracking simulations, in the interest of maintaining controller stability. A similar margin has been suggested when dealing with the instability in the oversteering simple car problem.

8.3.1 Conclusions

The novel work conducted in this section has provided three main contributions to the field. Firstly, a study of the tracking performance near the lateral limit of a detailed, non-linear, complex car model has been conducted. This has been made possible by employing the adaptive controllers that have been developed by the author in previous sections. Secondly, the stability of the tracking controller has been analysed close to the lateral limit of the complex car. Finally, a variable weighting scheme has allowed the car to be brought close to its lateral limit during tracking whilst reducing the oscillatory controller behaviour which was previously observed in this region. The development of such a variable weighting strategy is a novel contribution to the field of DLQR tracking control, as previous work only utilised constant weighting strategies. The extra flexibility afforded by a variable weighting scheme in this case ensures a good balance between high-quality tracking, when the car is sufficiently clear of limit behaviour, and providing smoother control when the limit is approached. The ability to vary the control strategy adds an extra degree of flexibility in controller design which could potentially help overcome controller issues in the future.

8.4 Controller behaviour near the lateral limit

In studying the limit tracking behaviour of the variable tightness controller, clothoid paths similar in shape to the path in figure 8.19, but with different minimum radii, were tracked. It was noted that the tracking of clothoids with particular radii increased the oscillatory behaviour of the variable tightness controller. A strategy was sought to minimise the occurrence of this behaviour which is known to potentially cause instability in the controller and subsequent program crashes. Figure 8.28 demonstrates the oscillatory behaviour of the controller when tracking a clothoid track with a minimum radius of 126 m. Tracking such a path should place a lower requirement on the controller than the clothoid in figure 8.19, with the car using trims further away from the critical region of front axle force saturation. Hence, it was concluded that the oscillations were not a result
of the unstable limit behaviour observed previously.

Eigenmodes of the closed-loop system were scrutinised just before the onset of oscillations, at 11 s. The analysis indicated the presence of lightly-damped modes which, given the right conditions, could initiate the oscillations that were observed. Three lightly-damped eigenmodes of the closed-loop control system, which were identified as problematic, are presented in figure 8.31. By increasing the damping and stiffness of the suspension components, the lightly-damped modes were moved sufficiently clear of the imaginary axis. The tuning of the suspension parameters is presented in table 8.2, with the car now representing a sports-tuned version of the standard family saloon which was modeled previously. The resulting combination of controller and car setup was used to track the problematic clothoid path, with a minimum radius of 126 m, with the various aspects of the simulation included in figures 8.28, 8.29 and 8.30. The tuning of the suspension parameters offers a clear reduction in controller oscillations, see figure 8.28, which in turn results in more consistent tracking behaviour, as seen in figure 8.30.

8.4.1 Conclusions

This section has studied the behaviour of the closed-loop controller-car system at a particular operating condition which causes the onset of controller oscillation. Eigenmode analysis has been used to identify the cause of such oscillations and to suggest setup changes which minimise them. Such a case study is novel as the limitations of fixed gain controllers have previously prevented studies of controllers operating car models near their lateral limit, where such oscillations may arise. The work which has been completed is potentially beneficial to the field in highlighting the possible problems which result when controlling such detailed car models near their lateral limit. Documenting the process of identifying the cause of the oscillations, and their subsequent suppression, may provide a useful reference to help overcome similar oscillatory controller behaviour near the car’s lateral limit in future trials.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Symbol</th>
<th>Standard</th>
<th>Sports-tuned</th>
</tr>
</thead>
<tbody>
<tr>
<td>Front suspension damping (Nm$^{-1}$s)</td>
<td>$c_{fsus}$</td>
<td>1500</td>
<td>3000</td>
</tr>
<tr>
<td>Rear suspension damping (Nm$^{-1}$s)</td>
<td>$c_{rsus}$</td>
<td>1500</td>
<td>3000</td>
</tr>
<tr>
<td>Front suspension stiffness (Nm$^{-1}$)</td>
<td>$k_{fsus}$</td>
<td>19480</td>
<td>22000</td>
</tr>
<tr>
<td>Rear suspension stiffness (Nm$^{-1}$)</td>
<td>$k_{rsus}$</td>
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<td>18000</td>
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<td>$k_{fsb}$</td>
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<td>18000</td>
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<tr>
<td>Rear sway bar stiffness (Nrad$^{-1}$)</td>
<td>$k_{rsb}$</td>
<td>4000</td>
<td>6000</td>
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</tbody>
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In conclusion to this chapter, a set of adaptive controllers has been developed to control the
Figure 8.28: Time histories of delayed steering wheel angle and throttle position for *hair-pin* manoeuvre, with a minimum radius of 126 m, using : standard and sports-tuned suspension setups.
Figure 8.29: Time histories of mean front-lateral-slip ratio and left-front tyre utilisation for hairpin manoeuvre, with a minimum radius of 126 m, using: standard and sports-tuned suspension setups.
Figure 8.30: Time histories of lateral-, longitudinal- and speed-errors for simulated hair-pin manoeuvre, with a minimum radius of 126 m, using: standard and sports-tuned suspension setups.
Figure 8.31: Plot of three lightly-damped eigenmodes of the closed-loop controlled car system, for a variety of cornering trims, for two suspension setups: standard (black triangles) and sports-tuned (blue circles). Red X’s indicate the eigenvalues of each of the modes, for the standard suspension setup, at the onset of oscillations in figure 8.28, where a trim corresponding to a front-lateral-slip-ratio of 0.17 is used. Green +’s indicate the eigenvalues of each of the modes for the same problematic trim, but with a sports-tuned suspension setup.
complex car. Tracking trials have been presented at each development step to demonstrate the benefits of each improvement. This includes the use of adaptation in tyre slip and speed, using a variable weighting scheme and finally using a tuned suspension setup. This concludes work on developing an adaptive tracking controller for the, reasonably realistic, complex car model. The resulting controller can now be implemented inside a wider optimisation loop which provides speed and path optimisation. Although the development of the controller has been satisfactory, there remain a number of issues which are outside the scope of the present research. These will be highlighted in the following chapter, where possible solutions will be presented. The importance of the work documented in this thesis will also be highlighted, together with some concluding remarks on the remaining work required to form a complete alternative solution to the minimum lap-time problem.
Chapter 9

Conclusions and future work

The chapters preceding the current chapter have presented the development work on an optimal tracking controller, intended for the novel solution to the minimum lap-time problem. This chapter will serve two purposes. It will firstly summarise the work that has been conducted, presenting the main accomplishments of the research, including the hurdles that have been cleared. This will be followed by a critique of the work together with the author’s views on how current shortcomings could be surmounted. Focus will then shift to the future of the technology, specifying a road-map of the next stages of research that are required to provide a complete solution to the minimum lap-time problem.

The novel solution to the minimum lap-time problem consists of three phases: optimal path-tracking, optimisation of the speed profile along the path and the perturbation of the path itself. The successive iteration of these phases should allow the program to converge to a solution. The current research effort has been focused on the first phase. The behaviour of the controller when presented with a trajectory which is beyond the limit of the car has been of particular interest as this mimics the situation the controller will be presented with in the search for the minimum lap-time. Sustaining high-quality tracking whilst maintaining stability in such a scenario is seen as crucial to the success of the novel solution. A modular development path was taken in the lead-up to the ultimate goal: the optimal control of a realistic race-car model. A simple car model was used during initial development work. Once sufficient information had been gleaned from the control of the simple car, the modelling detail of the car was increased, resulting in a complex car model. The remainder of this section will summarise the details of the controllers that were developed for each of these car models.

The tracking problem for the simple car was focused on understanding, and solving, the key problems associated with lateral limit tracking-control. To keep the analysis straightforward, the
longitudinal speed of the car was kept fixed. The control of two distinctly different car setups was investigated, each requiring a different control strategy when operating at the limit. The front axle limited understeering setup accommodated a relatively tight control strategy \((q_2=100, R=1)\). The adaptation of the controls, in relation to the front axle’s lateral-slip-ratio, facilitated high-quality tracking and control when compared to its non-adaptive counterpart. When presented with paths which were beyond the capability of the car, the adaptive controller was able to take the front, limiting, axle up to its limit but ensured that it was not forced past it, which is exactly the behaviour required when presented with the same scenario during the speed/path optimisation phases. A rate limiter was required to suppress the effects of switching noise which was observed when moving from one reference trim to another during the adaptation process. Using such a solution was seen not to limit the tracking ability of the controller. In hindsight, it is expected that an interpolated adaptive scheme, which was found to benefit the oversteering car, would provide similar benefits to the understeering car and remove the need for a steering rate limit.

The divergent instability associated with the oversteering simple car required a different control strategy. A looser control strategy \((q_2=0.5, R=1)\) was chosen in light of the fact that tight control resulted in instability near the critical speed/acceleration of the oversteering setup. Adaptation of the controller, now with respect to the rear axle lateral-slip-ratio, allowed it to maintain control during critical stages of limit tracking simulations, with the non-adaptive controller losing control due to its limited knowledge of the non-linearity of the tyres. The use of quadratic interpolation in the adaptation process successfully removed the switching noise that was a feature of the understeering controller. To prevent the car from operating too close to its region of instability, it was recommended that the rear axle lateral-slip-ratio is taken to only 80% of the critical value where the rear axle generates the maximum lateral force. This leaves a margin of 1.1% from the maximum force which the rear tyres can generate, which is seen as a tolerable compromise to avoid loss of control. Having exhausted the information that could be derived from a controller based on the simple car, focus was shifted to the control of a more detailed car model, which presented its own difficulties.

The complex car model was able to capture a number of interesting features, with a higher level of detail than the simple car. The model included a drive-train and braking system, which facilitated the modelling of realistic longitudinal dynamics. The more relevant problem, of variable speed \((x,y)\) tracking, could therefore be addressed when controlling such a model. Adaptation, firstly in the car’s forward speed and then including the mean front-lateral-slip-ratio, was seen to systematically improve the tracking performance of the car. Superior tracking performance, in relation to the non-adaptive controller, was demonstrated by virtue of installing reference trims
which were progressively closer to the running state of the car, ensuring that the controls based on these trims were optimal. Oscillatory behaviour was noted in the controller when operating near the limit of the front axle, which was considered as suboptimal behaviour. A variable control tightness strategy was employed to suppress this problem. The controller was made progressively looser as the limit of the front axle was approached, in recognition of the fact that smoother control actions are required at this critical stage. The loosening of the controls had the additional benefit of removing the ridge-like features present in the optimal preview gain surfaces, which were thought to contribute to the oscillatory behaviour of the controller. Employing a variable weighting control scheme ensures that the controller provides high-quality tracking, owing to tighter controls, during normal operation whilst avoiding loss of control when operating at the limit. The setup of the car was noted to cause oscillatory controller behaviour when tracking less demanding paths, with the presence of lightly-damped eigenmodes thought to contribute to the problem. By increasing the suspension stiffness and damping parameters of the car, the oscillations were suppressed, resulting in more consistent controller behaviour in the same scenario.

9.1 Research contributions

The work that has been conducted can be summarised in the context of new contributions to the research area, building on the state of previous work in the field of DLQR preview control. The contribution of the author’s work addresses the two main limitations with the previous state of the technology: the implied limitations on optimality from using a single trim-state for controller design and the decoupling of longitudinal and lateral dynamics when formulating the tracking problem.

9.1.1 Adaptive DLQR preview tracking controllers

As highlighted in section 2.2 of the literature review, previous work conducted in the field of DLQR control has been limited to the design of controllers for a single trim-state, typically straight-running at a constant forward speed. The author has conducted extensive research into the use of adaptive controllers with a view to improving tracking performance. Associated work on trim-state collection, interpolation between such states and the details of the previewing strategy for cornering trims, which form the basis of such controllers, has been completed. The resulting controller has been observed to be superior to the previous fixed gain controllers, in terms of tracking performance and controller stability. The provision of such controllers has allowed the author to conduct a number of novel studies on the limit behaviour of both understeering and
oversteering simple car setups. The novel work represents a significant advance from previous efforts in the field, where such trials were not possible due to the limitations of the non-adaptive controllers. By virtue of using a more capable controller, control strategies for the understeering and oversteering case have been established. Later, the control of a more detailed complex car model was also studied near its lateral limit, demonstrating the capabilities and advantages of using the novel adaptive controller. This work again represents a novel contribution to the field, with such trials not being conducted previously.

9.1.2 Two-channel \((x, y)\) variable speed tracking controllers

Section 2.2 of the literature review has previously mentioned how work conducted by Hazell criticised the state of DQLR preview control due to its treatment of longitudinal and lateral control problems as separate issues. The author has conducted work on extending the single channel longitudinal/lateral controller technology to a novel two-channel controller in order to overcome such a limitation. Such a controller facilitates the tracking of path and speed profile combinations, which represents a new capability for DLQR preview control technology, one which is thought to be highly useful. Successful tracking trials, using a complex car model, have demonstrated the application of such a controller on a variety of challenging scenarios. Novel substudies have been facilitated using such a controller. These include: the study of controller behaviour near the complex car's lateral limit, which required the use of a variable weighting strategy, and the study of oscillatory controller behaviour near the limit of the complex car. The former case allowed the author to exercise the new concept of varying the control weighting strategy based on operating conditions. Such technology is seen as advantageous when compared to the previous practice of using a fixed weighting strategy. Additional tracking trials conducted near the limit of the complex car model, where oscillatory behaviour was again observed, also serves to document the closed-loop stability of such models near the limit. Such information is seen as valuable to the field as it can be used as a reference if similar conditions are encountered in future development trials.

9.2 Recommended analysis

This chapter has so far highlighted how highly capable tracking controllers have been developed for two different car models. Each controller has demonstrated high-quality tracking of severe manoeuvres whilst maintaining closed-loop stability close to the lateral limit of the car. The author believes that such results are close to satisfying the original aims of the research, which was to progress the state of DLQR preview technology to a point where it can be used to generate
highly capable tracking controllers, which are to be used in the first phase of the novel solution to the minimum lap-time problem. The controllers can be made more capable if issues relating to their operation are addressed. These issues will now be presented, together with the author’s recommendations on solutions.

9.2.1 Tuning of control weightings

The first issue concerns the control weightings that are used in the controller. It has already been established that the correct choice of these weightings plays an important role in the success of the tracking controller. Controls that are set too tight have been known to cause closed-loop instability while a controller which is set too loose sacrifices tracking performance unnecessarily. A general indication of the weightings necessary for each of the cases covered in this research has been presented. Whilst these values represent the right trend in the weightings that should be used, it is recommended that a more rigorous study of the ideal controller weightings is conducted. This is likely to involve a trial-and-error process in testing a number of different controller weighting combinations. This task is especially tedious given the number of weighting terms involved, and the corresponding interactions between every term, in the complex car model. This research has not been concerned with such issues but the tuning of these values is likely to result in an improvement in tracking performance, which is seen as beneficial. Obtaining controls that are as tight as possible will ensure that the car is worked harder at each optimisation iteration, when compared to a looser control strategy, which could potentially reduce the number of iterations required to bring the car to its limit.

9.2.2 Consideration of upper tyre utilisation limit

A second issue, which is closely related to the previous point, is the necessity to impose an upper limit on the utilisation of critical tyres to ensure controller stability, such as in the oversteering simple car and the understeering complex car. An arbitrary value of 80% utilisation of the critical tyre(s) has been recommended. Although the benefit of increasing this limit diminishes due to the non-linearity of the system, a higher level of tyre utilisation would still be preferred given the requirement to extract the absolute maximum from a given car setup. The control strategy has been shown to be key to dictating the balance between maximising performance and the prevention of controller instability. It is anticipated that a thorough investigation into the tuning of the control weightings will allow the current recommended limit of 80% tyre utilisation to be increased.
9.2.3 Improvements to variable weighting strategy

Staying on the topic of control strategy, the use of a variable weighting control scheme has been noted to improve the limit control of the complex car model, allowing the controller to retain control in such scenarios. The selection of shaping parameters allows the variable weighting control scheme to remove the ridge-like features in the preview gain sequences, which are thought to induce oscillatory controller behaviour and the subsequent loss of control. The user is responsible for selecting the appropriate shaping parameters to suit the behaviour of the gains for a given speed and car setup, matching the point where the front axle force saturates with the point where the variable tightness is loosest. If the speed or car setup is altered, potentially changing the point where the front axle saturates, the user must select different shaping parameters accordingly, which is seen as tedious. The variation of control tightness shaping parameters to such changes can be automated in one of two ways. The first option is to run trim collection simulations for each speed/car setup, sweeping through varying intensities of cornering, allowing the program to determine exactly where the front axle saturates at each instant and set the control tightness accordingly. This would effectively automate the process which the user is currently required to perform manually. The second solution involves a more theoretical approach whereby the gradient information, of how the force generated at the front axle varies as cornering intensity is increased, is found and used to mathematically set the variation in control weighting. The use of either automated process will ensure the proper adjustment to the variable control strategy, as the car/trim parameters are changed, without requiring user intervention.

9.2.4 Collection and storage of trim-states

Focus will now shift from the issues related to controller weighting strategies to the need to ensure a good match between reference trim-states and the car’s running state during tracking simulations. A good match is required between these states to ensure the controls remain optimal throughout. In the best case, a poor match will result in poor tracking performance. In the worst case, it can result in complete loss of control. Neither case is seen as acceptable. To ensure a good match, a large number of trim-states, which represent various cornering conditions, can be collected off-line. However, increasing the number of trim-states that are collected must be balanced against the increase in computational burden, and the extra time taken to collect them. A uniform grid has currently been used to store reference trims, with the regularity of the grid being exploited to enable efficient interpolation between them. The use of interpolation reduces the need to have a dense grid of trims as it serves to smooth the gap between them. However, the number of trims
that are required is likely to increase if: the car model complexity is increased or the number of scheduling parameters is increased. In order to keep the number of trim-states to a minimum in such cases, it is recommended that a grid of irregularly spaced trims is collected instead, with radial-basis functions being used to efficiently interpolate across such a set of trims. The spacing between trims can be varied so that more trims are collected near the limit of the car, where more resolution is required, whilst a sparser set is collected for less intense operating conditions, where the variation in parameters, between neighbouring trims, is likely to be smaller. Such a process is recommended under the assumption that the burden, placed on computation and on time, of using a more complex interpolation process on a smaller set of trims is lower than the burden of employing a simpler process to collect and store a significantly larger set of trims.

9.2.5 Scheduling with longitudinal acceleration

Another issue, stemming from the need for a good match between the car’s running state and reference trim, is the fact that trims are currently collected for a constant forward speed. It has been noted that, in the search for the minimum lap-time, the car will be required to perform manoeuvres which require substantial longitudinal acceleration and deceleration, which are not represented by the current set of trims that are collected off-line. The poor match between these conditions is thought to give rise to the poor longitudinal tracking performance during heavy braking and acceleration in the variable speed tracking trials involving the complex car model. Longitudinal acceleration could be added to the list of scheduling parameters used for the adaptation of the controller. The current trim collection process will need to be adjusted to collect trims which represent *snap-shots* of the car during acceleration/deceleration. Such trims are strictly not equilibrium conditions but they can be approximated to be if D’Alembert type inertial forces are included in the model to represent the influence of acceleration and deceleration. In each trim collection simulation, such an artificial horizontal force, aligned with the forward ground speed of the car, which corresponds to an arbitrary level of longitudinal acceleration, can be applied. The application of either the throttle or brake pedal must be used to ensure the car remains in a steady-state, required for the linearisation of the system. By collecting trims associated with varying levels of longitudinal acceleration, a better match between the reference trim and the car’s running state during tracking simulations can be provided. Although the collection of such trims is possible, the efficient calculation of the future path associated with each trim is not straightforward. Such a calculation does not share the same ease of computation as the cornering case without longitudinal acceleration, which takes the form of a constant radius arc. The efficient calculation of the future path is seen as key to the viability of using longitudinal acceleration as an
additional scheduling parameter and must be investigated further. The author also notes that care
must be taken when considering the implications of increasing the number of scheduling param-
eters, and therefore trim-states that are collected, with a view to improving the match between
stored trim-states and running states. A re-evaluation of the novel method is required if the trim
collection process becomes computational large in order to maintain optimality.

9.2.6 Investigation into an alternative previewing strategy

Section III in chapter 4 introduced an alternative procedure for the calculation of preview errors
where several separate reference frames are used, each aligned with the projected orientation of
the car at each future time step. Initial calculations have suggested such a technique would only
result in a small change to the control signals, when compared to the current method of using
the car’s local reference frame. A more thorough investigation would be beneficial to the research
effort in ascertaining the range of operating conditions for which these two previewing methods
can be considered close approximations to one another.

9.2.7 Requirements of model linearisation

The use of linear optimal control theory requires the linearisation of the complex car model, a
process which has been performed successfully throughout the course of the research. Linearisation
does however pose some difficulties when trying to model relevant features. Firstly, the model must
use continuous, differentiable, functions. The modelling of interesting features in real race-cars
may break this requirement. Methods exist which overcome this problem to an extent, such as
the use $arctan$ functions, which have been shaped accordingly, to model features which involve
abrupt switching. However, this method can only be used in selective situations and requires
skill in implementation. A different approach must be taken for the modelling of components which
display a more complex behaviour, such as the nature of the tyre forces before/after saturation or
the transition between the application of engine-/braking-torque. Here, the current linearisation
process requires the user to manually generate several linear models, each capturing the nature
of the model in each sub-set of operating conditions. The management of such linear models can
become tedious if the complexity of the model is increased. The use of a more capable modelling
software package, which is able to automatically perform smoothing between discontinuous regions
and manage the linearisation of the car, across its entire operating envelope, would be preferable
in such cases.

Having summarised the main achievements of the research, it can be concluded that good
progress has been made in developing a versatile high-quality tracking controller. The modular development strategy that has been employed has given considerable insight into the function of the controller at each stage. The results that have been presented are encouraging for the future development of the controller and its intended use as part of a minimum lap-time solution. Important issues relating to the controller have been identified by the author, with a suggestion of possible solutions and analysis to improve its capability. The final section aims to present the author’s view of the development road-map for the controller and the novel solution to the minimum lap-time solution. Details of the possible structure of the learning controller, required at the next stage of development, will be presented. Finally, additional sub-studies and applications which will serve to add value to the research will be presented.

9.3 Future work

9.3.1 Remaining analysis of the tracking controller

The analysis of the lateral limit tracking behaviour of the complex car has already been conducted under constant speed conditions, with promising results. The controller will now need to be verified during variable-speed limit-tracking trials, with a potential study on longitudinal limit-tracking. The use of an additional scheduling parameter, longitudinal acceleration, has already been suggested for such a process. Once final trials on the limit behaviour of the car have been completed, attention should then move to the development of learning controllers which optimise both the speed and path that are tracked by the controller. Mirroring the modular development that has been executed on the controller so far, it is recommended that the learning controller is limited initially to a simple manoeuvre, such as a single hair-pin manoeuvre.

9.3.2 Speed profile parametrisation and optimisation

The learning controller could monitor each tyre’s utilisation history for manoeuvres required by each optimisation iteration. The critical tyre, which is the most influential to the manoeuvre, could be used to judge the ambition of the trajectory at the next iteration. In converging to an optimal speed profile, the speed optimisation phase could alter key decisions of the profile, such as the selection of the time instances when the car is required to decelerate and accelerate in the corner. Algorithms could be developed to allow these points to be moved independently along the time-history of the corner, depending on the utilisation of the tyre in the previous iteration, as seen in figure 9.1. The figure presents a sequence of three hypothetical speed profiles which are designed to demonstrate how a speed-learning controller would operate for a straight-running manoeuvre. The
first iteration clearly shows that the critical, rear, tyre is well below 100\% utilisation throughout the manoeuvre. Hence, the second iteration chooses a more aggressive strategy. In doing so, the optimiser requires the car to brake later, and asks the car to start its acceleration earlier, than the previous iteration. The optimiser also experiments by requiring a higher acceleration after 10 s in recognition of the fact that the tyres weren’t used to their full potential. The tyre utilisation from the simulation shows that the iteration was too eager, causing the tyre to saturate between 5 and 10 s, and that the speed ambition should be curtailed in this region. The third iteration offers a more reasonable speed profile, where the deceleration is made less aggressive but the acceleration is made more severe to push the tyres closer to their maximum potential after 10 s.

Figure 9.1: Example of speed optimisation, highlighting the speed ambition of optimisation iterations and its effect on rear-tyre utilisation, for a straight-running manoeuvre. Three hypothetical iterations are presented: 1st/unambitious (blue dot-dashed-line), 2nd/too ambitious (red dashed-line) and 3rd/close to optimal (green solid-line). Circles and X’s mark the time instants where the car is required to accelerate and decelerate respectively.
9.3.3 Path parametrisation and optimisation

Once the speed optimisation phase is complete, with the optimum speed profile for a given path being found, control is then passed to the path optimiser. It is recommended that the path is parametrised using variable lengths of straight-sections, clothoids and constant radius arcs, so that a minimum set of variables can be used to vary the shape of the path quickly. The length of each section, together with the variation of arc radii, can be optimised depending on the tyre utilisation in each section. An example of such a path optimisation process can be seen in figure 9.2. Three hypothetical path profiles have been designed to demonstrate how a path optimising controller could operate. Each path consists of a typical sequence: a straight section, a clothoid transition to a constant radius arc, followed by another clothoid section back to the final straight. The first iteration is judged to be too conservative, with the utilisation of the critical, left-front, tyre only reaching a maximum of 60% during the manoeuvre. The second iteration suggests a more aggressive path, with the car making a more rapid turn into a tighter corner, with a shorter constant radius arc. This has the effect of pushing the tyre beyond its capability and is likely to prompt the speed optimiser to reduce the car’s speed unnecessarily at the next iteration. To avoid such a situation, the path optimiser produces a moderate turn which allows the critical tyre to be brought close to the recommended target of 80% utilisation. In this iteration, the path optimiser experiments by increasing the length of the constant radius arc whilst making the final clothoid section shorter, to work the tyres harder at the end of the manoeuvre. The successive iteration, of tracking trials based on suggested paths and speed profiles, should allow the program to converge to a solution of the minimum-time-to-manoeuvre problem. It must be noted that during the path optimisation phase of the program, the ambition of the paths must be restricted to the boundaries of the track. This constraint is required to avoid the path optimiser suggesting paths which clearly violate the boundaries of the track, which would be deemed illegal in a racing situation.

Although procedures have been suggested to give a general indication of how both the speed-profile- and path-parametrisation phases of the novel solution to the minimum lap-time are conducted, the development of each stage is anticipated to provide its own difficulties once research work commences. The author is wary that issues which serve to increase the computation time of these stages, or insurmountable problems associated with them, will require a re-evaluation of the feasibility of the novel solution with respect to the current methods that are employed.
Figure 9.2: Example of path parametrisation, illustrating the path required at each optimisation iteration and its effect on left-front tyre utilisation. Three hypothetical iterations are presented: 1st/unambitious (blue dot-dashed-line), 2nd/too ambitious (red dashed-line) and 3rd/close to optimal (green solid-line). Markers are used to identify the transition to different sections of the path in both plots.
9.3.4 Upgrading the car model

Once the learning controller has been successfully demonstrated to optimise the speed and path for a simple cornering manoeuvre, it is recommended that the car model is upgraded to capture the dynamics of a realistic race-car, such as a contemporary Formula 1 race-car. The features of such a model are likely to require a further development step of the tracking controller, hence the manoeuvre is required to be kept simple during this development phase. Once a minimum-time-to-manoeuvre program has been successfully developed and demonstrated on a single corner, for such a detailed car model, the process can finally be extended to a complete race-course, with the software required to find the minimum lap-time of a particular combination of car setup and track. At this stage, a complete minimum lap-time solution is present and can be compared against existing methods to demonstrate the expected benefits of the novel method.

Once a basic minimum lap-time package has been developed, a number of interesting sub-studies can be conducted.

9.3.5 Track undulations

The software currently excludes track undulations and their effect on lap-time. The modelling of such features is seen as a necessary step in improving the accuracy of the program in predicting a car’s real minimum lap-time. This is especially the case when modelling tracks which involve large changes in height, where there is likely to be a large influence of vertical profiling on minimum lap-time. As control decisions for the remainder of the lap are closely linked to such regions, small discrepancies in the optimal control of such regions may have a cascading effect on other sections of the lap and have a larger effect on the overall lap-time. For example, the final speed of a race-car travelling up a particularly steep incline will be considerably different to the speed computed by the program if the track were treated as flat. The resulting speeds at the top of the hill would then be used in the calculation of the manoeuvres that follow which may lead to an altogether different solution of the optimal lap in each case. An investigation into whether the inclusion of road undulations results in an improvement in the accuracy of predicting the minimum lap-time would be beneficial.

The requirement for the path optimisation phase to produce paths within the boundary of the track could be relaxed if track undulations are considered. The constraint which requires the car to remain on the track could be avoided by modelling kerb features at the corners. Real race-car drivers tend to use such features as part of the track in an attempt to minimise the lap-time. However, this causes excitation in the suspension, and if used too much, can potentially cause the
tyres to lose contact with the ground, losing traction and resulting in a counter-productive effect on the lap-time. The modelling of such kerb features would be interesting, with attention being paid to whether the program, as with a real driver, limits how much of the kerb to use. This would provide an indirect constraint to avoid the controller from corner-cutting. The adoption of different suspension setups to ride such kerb features, with minimal disruption to traction, is a relevant problem in real circuit-racing and forms a potentially interesting sub-study.

9.3.6 Modelling fuel economy

In recent years, fuel economy has become an ever more concerning problem, in the general automotive environment as well as motorsport. In addition to the weighting placed on tracking error and control effort expended, a cost could be place on fuel consumption. A study could then be conducted to assess how placing an additional cost on fuel consumption affects the minimum lap-time and the control strategy that must be used. The results may provide engineers some insight into driver behaviour when more frugal racing is required, which in turn may help design more fuel-efficient race-cars which compete in events such as the *Le Mans 24 hour race* and long-distance rallying, where fuel consumption is an important factor of the racing.

9.3.7 Testing of system to improve fuel efficiency

Closely linked to the issue of fuel economy, the systems which improve fuel efficiency are becoming more prevalent in both the automotive and motorsport industry. Kinetic Energy Recovery Systems (KERS) were demonstrated in the 2009 Formula 1 season with good effect. The optimal control of such systems is key to their successful adoption in both industries. The minimum lap-time program could be used to quickly trial different control strategies for such systems and monitor the result on car performance. A study into this process could yield an insight into how best to deploy these systems for a variety of different conditions.

9.3.8 Optimisation of key car parameters

The current proposition for the minimum lap-time program produces solutions for a combination of car setup and track. The important variable in such a combination is the car setup, which is usually tuned by a race-engineer. A potential extension to the program could include an additional optimisation loop, which varies important car parameters such as ride-height and suspension stiffness. Such a program could potentially suggest starting values for key parameters which could save race-teams time in converging to an optimal setup for each track. Such a program is not
necessarily limited to motorsport applications. Automotive designers may be able to use the same technology to quickly tune important parameters based on requirements in handling qualities and comfort, which can be explicitly defined in the cost functions used to derive the optimal control gains.

Finally it is hoped that if the minimum lap-time software becomes sufficiently accurate, and quick, at predicting the true potential of a race-car, then it can be integrated into current race-strategy software packages. Decisions on pit-stops, tyre management and car setup changes could potentially be made, using the program to accurately predict the variation in car performance as a result of these strategy changes during the race simulation.
Appendix A

Invariance of controls to transition from global to local reference frames

In specifying the DLQR problem in chapter 4, a global reference frame was chosen when employing the shift register process in equation 4.5. The chapter then proceeded to show how, for the simple case of constant speed lateral tracking, the optimal control calculations were invariant to the use of either global or local reference frames. The proof will now be extended to the more general case of variable speed \((x,y)\) tracking, using two control inputs, the steering wheel angle and the throttle/brake pedal. Figure A.1 diagramatically represents how a global reference frame is used to track a path consisting of road points, \(r_i = [r_0, r_1, r_2...]\), using a straight-running trim-state.

Figure A.1: Measurement of errors in a global reference frame using a straight-running trim-state.
Equation A.1 provides a general expression for the series of points, defined in a global reference frame, from which the respective path displacements are measured.

\[ l_i = \begin{bmatrix} l_{ix} \\ l_{iy} \end{bmatrix} = \begin{bmatrix} i \dot{x}_{eq} T \\ i \dot{y}_{eq} T \end{bmatrix} \]  

(A.1)

where \( i = [0 \ 1 \ 2 \ \ldots \ n] \), is the preview point index and \( \dot{x}_{eq} \) and \( \dot{y}_{eq} \) are longitudinal and lateral speeds corresponding to the trim state, respectively. In this framework, the optimal steering wheel angle, \( \theta^*_{sw} \), is constructed by applying perturbations from \( \theta_{K_1} \) state feedback and \( \theta_{K_2} \) preview error contributions to the control’s trim position, \( \theta_{eq} \).

The optimal control of the steering wheel that results is

\[
\theta_{sw} = \theta_{eq} - K_{1\theta}(1)(x - x_{eq}) - K_{1\theta}(2)(y - y_{eq}) - K_{1\theta}(3)(\psi - \psi_{eq}) \\
- K_{1\theta}(4)(\dot{x} - \dot{x}_{eq}) - K_{1\theta}(5)(\dot{y} - \dot{y}_{eq}) - K_{1\theta}(6)(\dot{\psi} - \dot{\psi}_{eq}) - ... \\
- K_{2\theta x}(1)\tilde{x}_{r0} - K_{2\theta x}(2)\tilde{x}_{r1} - K_{2\theta x}(3)\tilde{x}_{r2} - ... \\
- K_{2\theta y}(1)\tilde{y}_{r0} - K_{2\theta y}(2)\tilde{y}_{r1} - K_{2\theta y}(3)\tilde{y}_{r2} - ... 
\]  

(A.2)

Variables with subscript \( eq \) denote values of state variables at equilibrium. \( x_{eq}, y_{eq}, \psi_{eq} \) can be set to zero without loss of generality. Additionally, for a straight-running case, \( \theta_{eq}, y_{eq} \) and \( \dot{\psi}_{eq} \) become zero.

Control decisions can also be made by using the car’s local reference frame by translating the original reference frame along \( x \) and \( y \) and then rotating by \( \psi \), as seen in figure A.2.

The transformation which describes the relationship between the errors measured in the global reference frame, \( \tilde{x}_{ri} \) and \( \tilde{y}_{ri} \), to errors measured in the local one, \( \tilde{x}_{ri}' \) and \( \tilde{y}_{ri}' \), can be derived as follows.

The vector of errors, \( x_{riG} \) and \( y_{riG} \), between the discrete future trajectory measured in the car’s local frame and the respective preview point on the track, when measured in the global reference frame, can be expressed as

\[
\begin{bmatrix} x_{riG} \\ y_{riG} \end{bmatrix} = \begin{bmatrix} r_{ix} \\ r_{iy} \end{bmatrix} - R^{-1} \begin{bmatrix} i \dot{x}_{eq} T \\ i \dot{y}_{eq} T \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix} 
\]  

(A.3)
where the global road coordinates, $r_{ix}$ and $r_{iy}$, can be expressed as

$$
\begin{bmatrix}
  r_{ix} \\
  r_{iy}
\end{bmatrix} =
\begin{bmatrix}
  l_{ix} \\
  l_{iy}
\end{bmatrix} +
\begin{bmatrix}
  \tilde{x}_{ri} \\
  \tilde{y}_{ri}
\end{bmatrix}
$$

and $R$ is the general clockwise rotation matrix for a 2 dimensional cartesian system. Under small angle approximations

$$R^{-1} = \begin{bmatrix}
  1 & -\psi \\
  \psi & 1
\end{bmatrix}$$

Expressing the preview errors derived in equation A.3 in the car’s local reference frame gives

$$
\begin{bmatrix}
  \tilde{x}_{ri}'' \\
  \tilde{y}_{ri}''
\end{bmatrix} = R
\begin{bmatrix}
  x_{riG} \\
  y_{riG}
\end{bmatrix}
$$

(A.4)

Substituting Equation A.3 into Equation A.4, rearranging and cancelling second-order terms, gives the relationship between the errors measured in the global reference frame, $\tilde{x}_{ri}$ and $\tilde{y}_{ri}$, to errors
measured in the local one, \( \tilde{x}_{ri} \) and \( \tilde{y}_{ri} \)

\[
\tilde{x}_{ri} = \tilde{x}^{''}_{ri} + x - \psi \cdot \dot{y}_{eq} T \\
\tilde{y}_{ri} = \tilde{y}^{''}_{ri} + y + \psi \cdot \dot{x}_{eq} T 
\]

By substituting into equation A.2, the optimal control of the steering wheel becomes

\[
\theta_{sw} = \theta_{eq} - (K_1 \theta(1) + \sum_{j=1}^{n+1} K_{2\theta_x}(j))x \\
- (K_1 \theta(2) + \sum_{j=1}^{n+1} K_{2\theta_y}(j))y \\
- (K_1 \theta(3) - \sum_{j=1}^{n+1} (j-1) \dot{y}_{eq} T K_{2\theta_x}(j) + \sum_{j=1}^{n+1} (j-1) \dot{x}_{eq} T K_{2\theta_y}(j)) \psi \\
- K_1 \theta(4)(\dot{x} - \dot{x}_{eq}) - K_1 \theta(5)(\dot{y} - \dot{y}_{eq}) - K_1 \theta(6)(\dot{\psi} - \dot{\psi}_{eq}) - ... \\
- K_{2\theta_x}(1)\tilde{x}_{r0}^{''} - K_{2\theta_x}(2)\tilde{x}_{r1}^{''} - K_{2\theta_x}(3)\tilde{x}_{r2}^{''} - ... \\
- K_{2\theta_y}(1)\tilde{y}_{r0}^{''} - K_{2\theta_y}(2)\tilde{y}_{r1}^{''} - K_{2\theta_y}(3)\tilde{y}_{r2}^{''} - ... 
\]

which simplifies, by collating terms, to

\[
\theta_{sw} = \theta_{eq} - K_1 \theta(4)(\dot{x} - \dot{x}_{eq}) - K_1 \theta(5)(\dot{y} - \dot{y}_{eq}) - K_1 \theta(6)(\dot{\psi} - \dot{\psi}_{eq}) - ... \\
- K_{2\theta_x}(1)\tilde{x}_{r0}^{''} - K_{2\theta_x}(2)\tilde{x}_{r1}^{''} - K_{2\theta_x}(3)\tilde{x}_{r2}^{''} - ... \\
- K_{2\theta_y}(1)\tilde{y}_{r0}^{''} - K_{2\theta_y}(2)\tilde{y}_{r1}^{''} - K_{2\theta_y}(3)\tilde{y}_{r2}^{''} - ... 
\]

which is exactly the equation for the control decision if the car’s local frame in Figure A.2 is used, if the following conditions are met
\begin{align*}
K_{1\theta}(1) &= - \sum_{j=1}^{n+1} K_{2\theta x}(j) \quad (A.7) \\
K_{1\theta}(2) &= - \sum_{j=1}^{n+1} K_{2\theta y}(j) \quad (A.8) \\
K_{1\theta}(3) &= \sum_{j=1}^{n+1} (j - 1) \dot{y}_{eq} T K_{2\theta x}(j) - \sum_{j=1}^{n+1} (j - 1) \dot{x}_{eq} T K_{2\theta y}(j) \quad (A.9)
\end{align*}

Numerical analysis on the optimal control gains shows that these conditions are satisfied when full preview is employed, which compliments initial tests carried out by Valtetsiotis [57]. Hence, the optimal control is invariant with translation in \( x \) and \( y \), and rotation in \( \psi \), required to move from a global to local car reference frame.

Similarly, when considering the throttle control, the optimal control contributions in the local reference frame are

\[
\tau = \tau_{eq} - K_{1\tau}(4)(\dot{x} - \dot{x}_{eq}) - K_{1\tau}(5)(\dot{y} - \dot{y}_{eq}) - K_{1\tau}(6)(\dot{\psi} - \dot{\psi}_{eq}) - ... \\
- K_{2\tau x}(1)\ddot{x}_{r0}' - K_{2\tau x}(2)\ddot{x}_{r1}' - K_{2\tau x}(3)\ddot{x}_{r2}' - ... \\
- K_{2\tau y}(1)\ddot{y}_{r0}' - K_{2\tau y}(2)\ddot{y}_{r1}' - K_{2\tau y}(3)\ddot{y}_{r2}' - ... \quad (A.10)
\]

Invariance of the throttle control before and after repositioning of the reference frame demands that

\begin{align*}
K_{1\tau}(1) &= - \sum_{j=1}^{n+1} K_{2\tau x}(j) \quad (A.11) \\
K_{1\tau}(2) &= - \sum_{j=1}^{n+1} K_{2\tau y}(j) \quad (A.12) \\
K_{1\tau}(3) &= \sum_{j=1}^{n+1} (j - 1) \dot{y}_{eq} T K_{2\tau x}(j) - \sum_{j=1}^{n+1} (j - 1) \dot{x}_{eq} T K_{2\tau y}(j) \quad (A.13)
\end{align*}

For cornering trim states, equations A.6 and A.10 must now include non-zero values of \( \theta_{eq} \), \( \dot{y}_{eq} \) and
While the diagram in figure A.2 is now transformed into figure A.3. The coordinates used to plot the future trajectory of the car in the local reference frame, $l_i''$, describe a constant radius arc, from which preview errors are measured. The trajectory should start from the car and be tangent to the car's absolute velocity. This ensures that as the car experiences more aggressive cornering trims, the discrete points continue to accurately represent the future trajectory of the car, as seen in figure A.3. $\tilde{x}_{r1}''$ and $\tilde{y}_{r1}''$ are calculated with respect to the local (car) $x$ and $y$ axes. The choice of these axes is thought to be the most appropriate.

The transition from equation A.2 to A.6 still holds for the cornering case, as we move from the global reference frame to the local reference frame. Hence equations A.7 to A.9 and A.11 to A.13, which are required for this invariance in reference frame repositioning, still hold in the cornering case.

![Figure A.3: Measurement of errors in a local (car) reference frame using a cornering trim-state.](image-url)
Bibliography


