A method for non-parametric identification of non-linear vibration systems with asymmetric restoring forces from a resonant decay response

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Abstract
A method for non-parametric identification of systems with asymmetric non-linear restoring forces is proposed in this paper. The method, named the zero-crossing method for systems with asymmetric restoring forces (ZCA), is an extension of zero-crossing methods and allows identification of backbones, damping curves and restoring elastic and dissipative forces from a resonant decay response. The validity of the proposed method is firstly demonstrated on three simulated resonant decay responses of the systems with off-centre clearance, bilinear and quadratic stiffness. Then, the method is applied to experimental data from a micro-electro-mechanical resonator in order to quantify its non-linear damping and stiffness effects. Throughout the paper the proposed method is also compared with the Hilbert vibration decomposition to demonstrate that the ZCA yields more accurate results with much less effort.

Keywords: Asymmetric restoring forces, Zero-crossing method for asymmetric restoring forces, Hilbert vibration decomposition, Micro-electro-mechanical system, Non-linear system identification

Highlights
• Zero-crossing method for systems with asymmetric restoring forces (ZCA) is proposed
• Backbones, damping curves, and asymmetric non-linear restoring forces can be identified
• The proposed method is applied to both simulated and experimental data
• A detailed comparison with the Hilbert vibration decomposition is made
• The proposed method exceeds the Hilbert vibration decomposition in many aspects

Nomenclature

\begin{tabular}{ll}
Symbol & Description \\
\hline
a, \( f \) & Instantaneous amplitude and frequency \\
\( f_{nl} \) & Non-linear restoring force \\
\( f_u, a_u \) & Frequency and amplitude from the upper part of a signal \\
\( f_l, a_l \) & Frequency and amplitude from the lower part of a signal \\
m, c, k & Mass, damping, stiffness of a linear system \\
\end{tabular}

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1. Introduction

As non-linear behaviour becomes increasingly important in structural dynamics, so does an urgent need for experimental identification of non-linear systems. Numerous methods for non-linear system identification have been proposed [1–3]. A considerable number of studies has focused on non-linear system identification via time-frequency analysis, including the short-time Fourier transform [4] and wavelet transform [5]. More recently, methods based on instantaneous frequency (IF) and amplitude (IA), such as the Hilbert-Huang transform [6, 7] and Hilbert transform [8, 9], have been developed and used in structural dynamics for parametric and non-parametric identification [10–12]. To further improve the performance of these methods, some alternative methods for the estimation of the IF and IA, including zero-crossing methods [13, 14], have been introduced.

A vast majority of non-linear system identification methods is limited to systems with symmetric restoring forces. Little research has been conducted on the problem of non-parametric identification of asymmetric restoring forces. The measured signals may however exhibit asymmetry with respect to time axis caused by different stiffness or dissipative forces for the positive and negative part of the motion. Such signals can originate from systems with bilinear, piecewise or off-set stiffness, which are not uncommon in engineering structures. For instance, gaps, end stops and pre-stress effects can exhibit asymmetric restoring forces.

The Hilbert vibration decomposition (HVD) [8, 9, 15–17] is to the authors’ knowledge the only time-frequency analysis method able to cope with a lack of symmetry in the restoring forces. The HVD for systems with asymmetric restoring forces is loosely based on the idea that each signal branch (the lower and upper part of the signal with respect to time axis) is defined on its half-plane only. So practically it is enough to identify matching instantaneous characteristics of each signal branch [16]. The HVD as used in [16] can determine not only the system’s backbone and damping curves, but also the initial asymmetric non-linear elastic and dissipative restoring forces. However, the HVD involves extensive signal processing, may be sensitive to measured noise, and may suffer from numerical issues of the Hilbert transform. The HVD will be used as a reference method throughout the paper and it is therefore described in more details in Appendix A.

This paper proposes a new method, which avoids some of the problems experienced with the HVD, but which still allows non-parameter identification of the same class of systems. As the proposed method does not use the Hilbert transform it does not require sophisticated signal processing, it is easy to implement and is less sensitive to measured noise. The method is underlined by the same idea as the HVD for identification of asymmetric systems, i.e. the instantaneous characteristics of each signal branch are identified separately. Therefore, it provides equivalent results in terms of backbones, damping curves, elastic and dissipative

\[ k_{nl}, c_{nl} \quad \text{Quadratic stiffness and non-linear damping coefficient of a MEMS} \]
\[ k_1, k_2 \quad \text{Bilinear stiffness coefficients} \]
\[ t_i, t_u, t_l \quad \text{Times of zero-crossings, maxima, and minima} \]
\[ x, \dot{x}, \ddot{x} \quad \text{Displacement, velocity, and acceleration} \]
\[ x_1, x_2 \quad \text{Thresholds of off-centre clearance} \]
\[ y \quad \text{Smoothed displacement} \]
\[ F_{el}, F_{dd} \quad \text{Elastic and dissipative restoring force} \]
\[ \beta, \gamma \quad \text{Off-centre clearance and quadratic stiffness coefficients} \]
\[ \delta_u, \delta_l \quad \text{Damping rates estimated from the upper and lower part of a signal} \]
\[ \lambda \quad \text{Smoothing parameter} \]
\[ HT \quad \text{Hilbert transform} \]
\[ HVD \quad \text{Hilbert vibration decomposition} \]
\[ IF, IA \quad \text{Instantaneous frequency and amplitude} \]
\[ SDOF \quad \text{Single-degree-of-freedom} \]
\[ MDOF \quad \text{Multi-degree-of-freedom} \]
\[ SNR \quad \text{Signal-to-noise ratio} \]
\[ ZC \quad \text{Zero-crossing method} \]
\[ ZCA \quad \text{Zero-crossing method for systems with asymmetric restoring forces} \]
restoring forces, while not requiring sophisticated signal processing. The method proposed identifies the instantaneous characteristics directly using modified zero-crossing approach from a resonant decay response and is named zero-crossing method for systems with asymmetric restoring forces (ZCA). Throughout the paper several sets of numerically and experimentally acquired data are used to validate the ZCA and a detailed comparison with the HVD is also made.

The paper is organised as follows: the zero-crossing method for asymmetric systems is proposed in section 2. In section 3 the ZCA is applied to three simulated cases, namely to the systems with bilinear stiffness, off-centre clearance (backlash) and quadratic stiffness. Section 4 then shows the application of the proposed ZCA as well as HVD to the data obtained experimentally from a micro-electro-mechanical beam. Section 5 discusses limitations and a range of applicability of the proposed method. The paper is complemented by two appendixes - a detailed description of the HVD is given in Appendix A and the Whittaker smoother, which is used in the paper to process noisy data, is described in Appendix B.

2. A zero-crossing method for systems with asymmetric restoring forces

This section introduces zero-crossing methods and subsequently the zero-crossing method for systems with asymmetric restoring forces is proposed. Both methods are applicable to a resonant decay response, so the way how such a response can be obtained is firstly described.

2.1. Obtaining a resonant decay response

A resonant decay response is a free decay that consists of a single mode of vibration. Every free decay of a single-degree-of-freedom (SDOF) system is therefore a resonant decay response. For multi-degree-of-freedom (MDOF) systems, the resonant decay response can be measured as well. This can be achieved by exciting the system using a single-frequency harmonic excitation at the frequency of the mode of interest and turning off the excitation suddenly when the system has reached the required amplitude. After removing the excitation, the system decays on a single mode of vibration, resulting in a resonant decay response. This process is commonly known as phase resonance testing or force appropriation [18–20].

Any resonant decay response (either from an SDOF system or one mode of vibration of an MDOF system) can be formally obtained from the following equation

$$\ddot{x} + 2\delta(\dot{x})\dot{x} + \omega_0^2 x = 0,$$

where $\omega_0(x)$ is the amplitude-dependent angular frequency of vibration, $\delta(\dot{x})$ is the amplitude-dependent damping rate, $x, \dot{x}, \ddot{x}$ are displacement, velocity and acceleration, and $F_{el}$ and $F_d$ are the elastic and dissipative restoring forces, respectively. When the frequency and damping are not amplitude-dependent, the system is linear so the restoring forces linearly increase with the vibration amplitude. On the other hand, if the system is non-linear, the restoring forces can be described by any functional form. It can be either symmetric or asymmetric with respect to the equilibrium point ($x = 0$).

For the identification of systems with symmetric restoring forces, the well-known Freevib algorithm [12] which is based on the Hilbert transform has been used for a long time. However, there are several numerical problems while using the Hilbert transform so the alternative procedures based on zero-crossing methods were developed in [13, 21–23]. The key difference between the Freevib and zero-crossing methods is that the zero-crossing times are used to obtain the instantaneous amplitude and frequency instead of the Hilbert transform. For the considered type of signals, no sophisticated zero-crossing detection method is required. The zero-crossing times can be obtained with sufficient accuracy using linear interpolation in vicinity of $x(t) = 0$. Only two points, one immediately before and one immediately after the crossing, are needed for the interpolation. If the amount of noise is significant, it may be suggested to smooth out the signals around the zero-crossing points using a moving average filter [13]. Because the proposed zero-crossing method for systems with asymmetric restoring forces is an extension of the zero-crossing methods, they are introduced in the next section.
2.2. Introduction to the zero-crossing methods

The zero-crossing method (ZC) is a straightforward method for the estimation of the instantaneous frequency (IF) and instantaneous amplitude (IA) from a resonant decay response. The IF and IA can then be used for the computation of non-linear restoring forces. Several similar versions of this method can be found in literature [13, 14, 22]. The basic zero-crossing method from [13] is illustrated in Fig. 1. The IF is determined from the inverse of the period over one complete vibration cycle and is assigned to the crossing time at the centre of this cycle

\[ \omega(t_i) = 2\pi \frac{(t_{i+1} - t_{i-1})^{-1}}{T_i} = 2\pi T_i^{-1}. \]  

The IA is found using the first-order polynomial interpolation of the absolute extrema of the signal. The values of these polynomials are evaluated at the zero-crossing points \( t_i \). Thus, a set of discrete values \([\omega(t_i), a(t_i)]\) is obtained. This set does not characterise frequency and amplitude locally, but with one cycle accuracy.

A slightly modified version of the ZC method has been recently presented in [14, 24]. In this modification a resonant decay response is divided into \( N \) intervals and the vibration period is estimated as

\[ T_k = 2(t_{k,n_k} - t_{k,1})/(n_k - 1), \]

where \( t_{k,i} \) is the \( i \)th zero-crossing point and \( n_k \) is the number of zero-crossing points within the \( k \)th interval. This method essentially averages the periods, assuming that the frequency does not change significantly within the interval. The instantaneous amplitude is found in a similar manner, i.e. absolute extrema of the signal are averaged over short intervals. Due to the averaging, this version of the zero-crossing method provides even less instantaneous results than the ZC, but it should be much less sensitive to measured noise.

According to the Freevib algorithm [12], the estimated IF and IA are used to compute the natural frequency and viscous damping rate using

\[ \omega_0^2 = \omega^2 - \frac{\dot{a}}{a} + \frac{2\delta^2}{a^2} + \frac{\dot{\omega}}{a\omega}, \quad \delta = -\frac{\dot{a}}{a} - \frac{\dot{\omega}}{2\omega}, \]  

respectively. The non-linear elastic \( F_{el} \) and dissipative \( F_d \) restoring forces can then be determined using

\[ F_{el} = \begin{cases} \omega_0^2 a, & x(t) \geq 0 \\ -\omega_0^2 a, & x(t) < 0 \end{cases} \quad F_d = \begin{cases} 2\delta a_x, & \dot{x}(t) \geq 0 \\ -2\delta a_x, & \dot{x}(t) < 0 \end{cases}, \]

respectively, where \( a_x \) is the instantaneous amplitude (envelope) estimated from the velocity signal.

The zero-crossing methods are intuitive, straightforward, and easy to implement [13, 14, 22]. They provide a reliable means of estimating the backbones which correspond well with the backbones predicted using analytical and numerical approaches [13, 14, 25]. Although the zero-crossing methods remove the need to use the Hilbert transform, they cannot be utilised for the analysis of systems with asymmetric non-linearities since they do not take the difference in the upper and lower part of the signal into account.

\[ \text{Figure 1: Zero-crossing method for systems with symmetric restoring forces} \]
2.3. Zero-crossing method for systems with asymmetric restoring forces

A straightforward, intuitive and easy to implement method, hereafter referred to as zero-crossing method for systems with asymmetric restoring forces (ZCA), is proposed in this section. The ZCA allows non-parametric identification of non-linear vibration systems with asymmetric non-linearities from a resonant decay response. The proposed method is an extension of the ZC approaches and follows the same underlying idea as the HVD [16], i.e. by identifying the instantaneous frequency and amplitude of the upper (positive) and lower (negative) part of the signal individually, the vibration characteristics of the system (backbones and damping curves, and elastic and dissipative restoring forces) can be obtained even if the non-linear restoring forces in Eq. (1) are asymmetric.

In line with the principal idea, the upper (in the following marked by subscript u) and lower (subscript l) parts of the signal are treated separately as indicated in Fig. 2 where \( t_u \) and \( t_l \) stand for the times of maxima and minima of the upper and lower part, respectively, and \( t_i \) marks the zero-crossing points. The IF of the upper and lower part is estimated as inverse values of the periods \( T_u \) and \( T_l \) from Fig. 2. The values of the IF are then assigned to the time of maxima or minima and marked as \( f_u(t_u) \) and \( f_l(t_l) \), respectively. The IF of the upper \( a_u(t_u) \) and lower \( a_l(t_l) \) part is given by the maxima and minima as indicated in Fig. 2. The procedure does not yield fully instantaneous results, i.e. \( f_u(t_u) \), \( f_l(t_l) \), \( a_u(t_u) \) and \( a_l(t_l) \) are not estimated for all time, but only at the times of maxima and minima. This resolution is, however, sufficient for reliable identification as demonstrated in section 3 and section 4.

Two sets of values \([f_u, a_u]\) and \([f_l, a_l]\) (the time dependency is dropped for brevity) can be recast into modal properties. Similarly to the evaluation of modal properties in the Freevib algorithm [12] (Eq. (4)), the angular modal frequency can be evaluated separately for the upper and lower part as

\[
\begin{align*}
\omega_u(t) &= \frac{\omega^2}{a_u} - \frac{\ddot{a}_u}{a_u^3} + \frac{\dot{a}_u \dot{\omega}_u}{a\omega_u}, & \omega^2(t) &= \omega^2 - \frac{\ddot{a}_l}{a_l^3} + \frac{2\dot{a}_l^2}{a_l^3} + \frac{\dot{a}_l \dot{\omega}_l}{a_l\omega_l}
\end{align*}
\]

with \( \omega_u = 2\pi f_u \) and \( \omega_l = 2\pi f_l \). However, the angular modal frequency of the upper and lower parts can be often reduced [8] to

\[
\omega_u \approx 2\pi f_u, \quad \omega_l \approx 2\pi f_l,
\]

since the influence of the other terms is small. In the following, the subscript 0 is dropped and all frequencies are assumed to be modal. The modal damping rate of the upper and lower parts is given by

\[
\delta_u(t_u) = -\frac{\dot{a}_u}{a_u} \frac{\dot{\omega}_u}{2\omega_u}, \quad \delta_l(t_l) = -\frac{\dot{a}_l}{a_l} - \frac{\dot{\omega}_l}{2\omega_l},
\]

where the first-order numerical derivatives must be evaluated numerically.

Having estimated modal frequency and modal damping, the elastic \( F_{el} \) and dissipative \( F_d \) restoring forces can be calculated using

\[
F_{el} = \begin{cases} 
4\pi^2 f_u^2 a_u, & x(t) \geq 0 \\
4\pi^2 f_l^2 a_l, & x(t) < 0 
\end{cases}, \quad F_d = \begin{cases} 
2\delta_u a_{yu}, & x(t) \geq 0 \\
2\delta_l a_{yl}, & x(t) < 0
\end{cases}
\]

Figure 2: Zero-crossing method for non-linear vibration systems with asymmetric restoring forces
where \( a_{xu} \) and \( a_{xl} \) are the maxima and minima of velocity. Equation (9) is generally valid for all common types of non-linearities and its apparent simplicity neither prevents, nor limits the identification of the original restoring forces.

All results obtained by the proposed method closely correspond to those obtained by the HVD, because the HVD and ZCA share the same underlying idea. In addition, the identification of restoring forces is governed by the same equations. The key difference is that the IF and IA from the upper and lower part used in Eq. (9) are replaced by positive and negative congruent characteristics in Eq. (A.5). These two methods are compared in the following sections to demonstrate that the ZCA often provides more accurate results while requiring less sophisticated signal processing.

3. Application to simulated data

To demonstrate the validity of the ZCA, it is applied to three numerical examples in this section, namely to the systems with bilinear stiffness, off-centre clearance (backlash), and quadratic stiffness. The corresponding non-linear restoring forces are drawn in Fig. 3. These three cases are described here to show the capabilities of the ZCA and allow its detailed comparison with the HVD.

In each case, the resonant decay response was simulated by the numerical integration. For the piece-wise characteristics, i.e. bilinear stiffness and off-centre clearance, a special integration scheme had to be implemented [26]. A sampling frequency was fifty times higher than the expected natural frequency. This value of sampling frequency is in line with the recommendations given in [8, 9] for the HVD which requires the sampling frequency to be twenty to eighty times higher than the highest frequency of interest. By using this high sampling frequency, some of the possible signal processing problems in the HVD were prevented and a reliable comparison with the ZCA therefore allowed.

Before applying the ZCA and HVD, the response was polluted by white Gaussian noise with signal-to-noise ratio 25 dB to make the data more realistic. Although some amount of noise in the data does not prevent either of the methods to estimate the IF and IA, it was decided to apply to the Whittaker smoother before applying the methods. Without this initial smoothing, the results obtained by the HVD and modal damping obtained by the ZCA according to Eq. (8) were very distorted. As a consequence, no reliable comparison of the methods was possible.

3.1. A system with bilinear stiffness

The system is governed by

\[
m\ddot{x}(t) + c\dot{x}(t) + f_{nl}(x) = 0, \quad f_{nl}(x) = \begin{cases} \kappa_1, & x(t) \leq 0 \\ \kappa_2, & x(t) > 0, \end{cases}
\]

Figure 3: Asymmetric non-linear restoring forces of simulated systems: (a) bilinear stiffness, (b) off-centre clearance, and (c) quadratic stiffness

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\]
where \( m = 1 \text{ kg} \) and \( c = 0.2 \text{ N s m}^{-1} \). The non-linear elastic restoring force \( f_{nl}(x) \) is shown in Fig. 3(a) and its coefficients are \( k_1 = (2\pi)^2 \approx 157.9 \text{ N m}^{-1} \) and \( k_2 = (2\pi 4)^2 \approx 631.7 \text{ N m}^{-1} \). These stiffness values were chosen so that the modal frequencies of the upper and lower parts are 2 Hz and 4 Hz, respectively. The initial conditions were \( x(0) = 5 \text{ m} \) and \( \dot{x}(0) = 0 \text{ m s}^{-1} \), sampling frequency \( f_s = 200 \text{ Hz} \), and the response was simulated for time \( t = 0 - 25 \text{ s} \). The large initial displacement was selected to emphasise the effect of the non-linearity. The displacement of 5 m is not realistic, but the validity of the results is not corrupted by this choice. These parameters and the initial conditions were chosen to allow reliable demonstration of the proposed method and its comparison with the HVD.

The noisy resonant decay response (polluted by white Gaussian noise with SNR 25 dB) was processed using the Whittaker smoother. The smoothing parameter was automatically determined by a leave-one-out strategy (see Appendix B). The original and smoothed signals are shown in Fig. 4(a). It can be seen that despite a significant level of noise, especially in a region of lower amplitudes, the Whittaker smoother was able to achieve reasonably smooth and accurate approximation of the original signal. However, some inaccuracies in extreme amplitudes can still be observed in the region of lower amplitudes as seen in the right inset in Fig. 4(a). It is clear that the resonant decay response is not symmetric with respect to the time axis. Although the initial displacement was \( x(0) = 5 \text{ m} \), the minimum value of displacement is almost \( 10 \text{ m} \). This increase in the absolute amplitude is given by the significantly lower stiffness \( k_1 \) for \( x \leq 0 \).

The ZCA and HVD were applied to the smoothed signal. The resulting IA is displayed in Fig. 4(a). The IA estimated using the ZCA corresponds, by definition, to the maxima (minima) of the signal. The HVD did not initially produce the smooth estimate of the IA. However, after additional low-pass filtering the IA was also obtained with good accuracy. The IA is relatively smooth and encloses the signal closely.

The IF estimated using the ZCA and HVD is shown in Fig. 4(b). These frequencies are the modal frequencies defined by Eq. (A.4) and Eq. (7). The two estimated frequencies are in line with Eq. (10) which, if treated piece-wise, describes two linear systems with different natural frequencies of 2 Hz and 4 Hz. The

![Figure 4: Bilinear stiffness: (a) resonant decay response with instantaneous amplitude and (b) instantaneous frequency](image-url)
ZCA was able to estimate both of these frequencies correctly. Some small discrepancies can be observed close to the end of the signal \((t > 20 \text{s})\) where the smoothing of the original noisy signal is not as good as at the beginning (see the insets in Fig. 4(a)). The HVD produced two fluctuating frequencies so they had to be again smoothed using a low-pass filter and the ends excluded to obtain the desired estimates.

The estimated backbones of the system are shown in Fig. 5(a). For the sake of clarity, the raw (without smoothing) results of the HVD are not shown. Two clearly separated backbones centred around 2 Hz and 4 Hz can be seen. The difference between the ZCA and the smoothed HVD results is minimal. Both backbones are only slightly influenced by residual noise. On the other hand, the distortion of the estimated damping curves in Fig. 5(b) is more significant despite the additional smoothing. This highlights the problems with the numerical differentiation in Eq. (7). Due to residual noise, the estimated amplitudes are not perfectly smooth decreasing functions so the numerical derivative cannot be estimated accurately. The ZCA is less influenced because it uses fewer amplitude points. Nevertheless, despite this distortion the values of the estimated damping rate are centred around the correct value of 0.1 s\(^{-1}\).

The estimated elastic restoring forces are shown in Fig. 6(a). Similarly to the backbones, the qualitative difference between the two methods is minimal. Both methods led to visually smooth elastic restoring forces.
that correspond to the original restoring force given by Eq. (10). The estimated dissipative restoring forces shown in Fig. 6(b) correspond to the original one as well. However, similarly to the damping curves, they are more influenced by residual noise than the elastic restoring force.

The detection and characterisation of non-linearity are possible using the backbones and damping curves as well as the restoring forces. The two backbone curves in Fig. 5(a) are straight lines, but on different frequencies. This is a clear indication that the dynamics of the system is different for the upper and lower part of the signal. Based on this finding the bilinear stiffness model can be deduced. In contrast, the damping curves in Fig. 5(b) are straight, but on the same amplitude. Therefore, no non-linearity in damping is indicated.

The estimated restoring forces in Fig. 6 describe directly the non-linear phenomena of the system. Therefore, an appropriate model can be easily selected and fitted to these forces. The results are summarised in Tab. 1 in which the relative errors of the estimated coefficients are written in parentheses. The bilinear stiffness coefficients were estimated by the ZCA very accurately, having the error less than 0.5%. The HVD also estimated the stiffness coefficients quite well, but with slightly lower accuracy than the ZCA with the largest error being close to 10%. The damping coefficients were estimated by both methods accurately.

Based on the results presented so far it can be stated that the ZCA can obtain the equivalent results with similar or better accuracy compared to the HVD. It is important to note that the HVD required more sophisticated signal processing and additional filtering. Therefore, the ZCA generally requires less effort to achieve the same results. In addition, the detection and characterisation may be easier while using the ZCA, because measured noise does not influence the results as much as in the HVD.

### 3.2. A system with off-centre clearance

The system is governed by

\[ m\ddot{x}(t) + c\dot{x}(t) + kx(t) + f_{nl}(x) = 0, \]  

(11)

where \( m = 1 \) kg, \( c = 0.8 \text{ N s m}^{-1} \), and \( k = (2\pi)^2 \approx 39.5 \text{ N m}^{-1} \). The off-centre clearance \( f_{nl}(x) \) is shown in Fig. 3(b) and it is described by

\[
f_{nl}(x) = \begin{cases} 
\beta(x(t) + x_1), & x(t) < -x_1 \\
0, & -x_1 \leq x(t) \leq x_2 \\
\beta(x(t) - x_2), & x > x_2,
\end{cases}
\]  

(12)

where the off-centre clearance stiffness coefficient was \( \beta = 6000 \text{ N m}^{-1} \) and the thresholds of the clearance described by the constants \( x_1 \) and \( x_2 \) were chosen to be 0.02 m and 0.05 m, respectively. The initial conditions were \( x(0) = 0.1 \) m and \( \dot{x}(0) = 0 \text{ m s}^{-1} \), sampling frequency \( f_s = 500 \text{ Hz} \) and the response was simulated for time \( t = 0 \sim 10 \) s. The computed response was polluted by white Gaussian noise with SNR 25 dB to simulate more realistic data.

The computed response was processed using the Whittaker smoother. As the signal is significantly different at the beginning and the end of the time interval, the first (\( t \leq 4.5 \) s) and second parts (\( t > 4.5 \) s) were smoothed separately to improve the performance of the smoother. The smoothing parameter was determined for each part individually using the leave-one-out validation strategy. The performance of the smoother was very similar to the case of bilinear stiffness so only the smoothed signals are shown in Fig. 7(a).

<table>
<thead>
<tr>
<th>Method</th>
<th>( k_1 ) [N m(^{-1})]</th>
<th>( k_2 ) [N m(^{-1})]</th>
<th>( c ) [N s m(^{-1})]</th>
</tr>
</thead>
<tbody>
<tr>
<td>original</td>
<td>157.9</td>
<td>631.7</td>
<td>0.2</td>
</tr>
<tr>
<td>ZCA</td>
<td>158.30 (0.24%)</td>
<td>629.79 (−0.29%)</td>
<td>0.19 (−3.26%)</td>
</tr>
<tr>
<td>HVD</td>
<td>143.75 (−9.85%)</td>
<td>624.75 (−1.11%)</td>
<td>0.20 (1.05%)</td>
</tr>
</tbody>
</table>

Table 1: The estimated coefficients of the system with bilinear stiffness (the values in parentheses are the relative errors (in %) between the estimated and original coefficients)
Figure 7: Off-centre clearance stiffness: (a) resonant decay response with instantaneous amplitude and (b) instantaneous frequency

It can be seen that the signal is not only asymmetric, but also significantly different in frequencies at the beginning and the end. The frequency at the beginning is very high due to the presence of the high stiffness caused by the off-centre clearance. In contrast, the frequency at the end of the free decay is very low due to the low stiffness of the underlying linear system.

The ZCA and HVD were applied to the smoothed signal. The resulting IA is shown in Fig. 7(a). As for the system with bilinear stiffness, the HVD did not initially produce the smooth estimate of the IA. Therefore, additional smoothing was needed and the end effects had to be excluded. Despite this smoothing, the envelope was badly estimated around the thresholds of the clearance between 4 and 5 s. On the other hand, the ZCA led to the accurate estimates of the IA.

The estimated modal frequency is shown in Fig. 7(b). It can be seen that two different frequencies were estimated in the first half of the time interval. These frequencies are higher at the beginning and merge into a single frequency for lower amplitudes. Unlike for the system with bilinear stiffness, significant differences in the IF occurred between the ZCA and HVD results. Overall, the latter is not so smooth despite the additional smoothing. Nevertheless, both methods were able to estimate the natural frequency of the linear system in the second half of the time interval (t > 5 s) correctly.

The estimated backbones of the system are shown in Fig. 8(a). For the sake of clarity, the initial (without smoothing) results of the HVD are not included. It can be seen that two backbones were estimated by each method. They both start on the natural frequency of the underlying linear system and bend to the right at higher amplitude. The HVD did not estimate the region between linear and non-linear behaviour (for amplitudes \(a \in (0.02, 0.05)\)m) as accurately as the ZCA. The thresholds of the off-centre clearance can be clearly determined from the backbones estimated by the ZCA whereas they are indistinguishable for the HVD. In addition, the backbone for the negative part estimated using the HVD has significantly different slope at higher amplitudes.

The damping curves are shown in Fig. 8(b). They are not so smooth as the backbones due to the numerical derivative used in Eq. (8). As for the system with bilinear stiffness, the ZCA appears to estimate smoother characteristics. The damping curves estimated by the ZCA are much closer to the correct value of 0.4 s\(^{-1}\) than those estimated by the HVD.

The non-linear elastic restoring force is shown in Fig. 9(a). As can be seen, both methods follow the
original characteristics qualitatively quite well. Three piece-wise linear regions can be clearly distinguished. From these non-linear restoring forces, the thresholds of non-linearities can be easily established. On the other hand, neither of the methods was able to estimate the upper part of clearance stiffness correctly.

The estimated dissipative restoring forces are shown in Fig. 9(b). Again, both methods seem to follow the original force quantitatively very well. However, in order to obtain these results by the HVD the end effects were removed.

The detection and characterisation of non-linearity are possible using the backbones and damping curves as well as the restoring forces. The two backbone curves in Fig. 8(a) are straight for low amplitudes and bend to the right at higher amplitude. Moreover, each of them deviate from the straight line at a different amplitude. This is a clear indication that the dynamics of the system is different for the upper and lower part of the signal. Moreover, piece-wise behaviour of hardening type can be deduced from these backbones. The damping curves in Fig. 8(b) are not so smooth as the backbones, but it is still possible to deduce that they are essentially straight lines. Therefore, there is no non-linearity in damping. The detection and characterisation may be generally clearer while using the ZCA, because the measured noise does not influence the results as much as in the HVD. For example, from the damping curves it could be wrongly assume that the results of the HVD indicate non-linear behaviour in the damping.
The estimated restoring forces in Fig. 9 describe directly the non-linear phenomena of the system. Therefore, an appropriate model can be easily selected and fitted to these forces. By doing so, the system is fully qualified. The results are summarised in Tab. 2 in which the relative errors of the estimated coefficients are written in parentheses. To avoid significant errors, points around the threshold (corresponding to the

<table>
<thead>
<tr>
<th>Method</th>
<th>( \beta^{(l)} ,[\text{N m}^{-1}] )</th>
<th>( \beta^{(u)} ,[\text{N m}^{-1}] )</th>
<th>( k ,[\text{N m}^{-1}] )</th>
<th>( c ,[\text{Ns m}^{-1}] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>original</td>
<td>6000</td>
<td>6000</td>
<td>39.5</td>
<td>0.8</td>
</tr>
<tr>
<td>ZCA</td>
<td>5795.16 (-3.53)</td>
<td>5194.66 (-15.50)</td>
<td>39.58 (-0.31)</td>
<td>0.76 (-4.83)</td>
</tr>
<tr>
<td>HVD</td>
<td>8079.09 (25.73)</td>
<td>5349.13 (-12.17)</td>
<td>86.64 (54.43)</td>
<td>0.75 (-6.25)</td>
</tr>
</tbody>
</table>

Table 2: Identified coefficients of the off-centre clearance (the values in parentheses are the relative errors between the estimated and original coefficients)

amplitude \( a \in (-0.3, -0.1) \cup (0.4, 0.6) \) were excluded from the fitting. The upper and lower parts were fitted separately, yielding \( \beta^{(l)} \) and \( \beta^{(u)} \), respectively. In almost all cases the ZCA led to more accurate estimates. The largest error of 15\% was obtained for the non-linear stiffness coefficients of the upper part. In contrast, the largest errors of the HVD was more than 50\%. The ZCA also yielded slightly better estimates of damping. As for the system with bilinear stiffness, the results indicate that the ZCA can obtain the same results with similar or better accuracy than the HVD while requiring less effort for signal processing and additional smoothing.

3.3. A system with quadratic stiffness

This testing case is described here not only to show the application of the ZCA, but also to allow better evaluation of the experimental findings in section 4. The system with quadratic stiffness is governed by

\[
m\ddot{x}(t) + c\dot{x}(t) + kx(t) + \gamma x^2(t) = 0,
\]

where \( m = 1 \text{ kg} \), \( c = 0.1 \text{ Ns m}^{-1} \), and \( k = (2\pi)^2 \approx 39.5 \text{ N m}^{-1} \). The non-linear restoring force is shown in Fig. 3(c) and its coefficient is \( \gamma = 3 \text{ N m}^{-2} \). The initial conditions were \( x(0) = 5 \text{ m} \) and \( \dot{x}(0) = 0 \text{ m s}^{-1} \), sampling frequency \( f_s = 50 \text{ Hz} \) and the response was simulated for time \( t = 0 - 50 \text{ s} \). The large initial displacement has been chosen to allow reliable demonstration of the proposed method and its comparison with the HVD. The choice of the system parameters does not corrupt the validity of the results. The computed resonant decay response was again polluted by white Gaussian noise (SNR 25 dB) to simulate more realistic measured data.

The noisy free decay was processed using the Whittaker smoother with the smoothing parameter evaluated by the leave-one-out procedure. The performance of the smoother was very similar to the case of bilinear stiffness so only the smoothed signal is shown in Fig. 10(a). It can be seen that the nature of this signal is different than in the two previous cases. This resonant response does not have any regions of significantly higher frequencies or significantly different amplitudes of the upper and lower parts. The asymmetry is not so obvious so the signal overall appears to be a free decay of a linear SDOF system. In addition, the non-linearity in Eq. (13) is smooth whereas the previous non-linearities were piece-wise linear.

The ZCA and HVD were applied to the smoothed signal. The resulting IA is shown in Fig. 10(a). It can be seen that the IA estimated using the HVD did not require so strong additional smoothing, but the end effects had to be again removed.

The instantaneous frequencies estimated using the ZCA and HVD are shown in Fig. 10(b). It can be seen that at the beginning of the signal, two different frequencies were estimated using both methods. Towards the end \( (t > 25) \) these two frequencies merge for the HVD and stay very close to each other in case of the ZCA. The value of the frequency towards the end of the time interval corresponds to the natural frequency of the underlying linear system. One of the frequencies at the beginning is higher than the natural frequency and the other one is lower, indicating hardening and softening behaviour, respectively. This is in line with the quadratic restoring forces in Fig. 3(c).
Figure 10: Quadratic stiffness: (a) resonant decay response with instantaneous amplitude and (b) instantaneous frequency

Figure 11: Quadratic stiffness: (a) backbones and (b) damping curve

The backbones are shown in Fig. 11(a). For the sake of clarity, only the smoothed results of the HVD are shown. Two backbones were estimated, corresponding to the softening and hardening behaviour. The results of both methods match very well at higher amplitudes ($a > 2$ m). In contrast, a single value of the backbones, indicating linear behaviour, was estimated using the HVD at low amplitudes, while the ZCA estimated two separated backbones, albeit slightly distorted and close to each other, at this amplitude level. This finding indicates that the ZCA might be able to qualitatively identify weaker non-linear behaviour better than the HVD.

The damping curves are shown in Fig. 11(b). In this case, both methods were able to produce good estimates although the residual noise still influences the smoothness of the results. Both damping curves are close to the correct value of $0.05 \, \text{s}^{-1}$.

The elastic and dissipative restoring forces are shown in Fig. 12. It can be seen that the difference between
the methods is minimal. Neither of them was however able to estimate the elastic force very accurately. Although the elastic force is estimated qualitatively quite well, the absolute values, especially for lower part of the signal, are different.

Detection and characterisation of non-linearity are possible using the backbones and damping curves as well as the restoring forces. The two backbone curves in Fig. 11(a) are not straight lines, which immediately eliminates the possibility of the linear system. Moreover, one of them indicates softening and the other hardening behaviour. From these findings, the quadratic stiffness can be deduced. The damping curves in Fig. 11(b) are straight lines centred around one value. Therefore, there is no non-linearity in damping.

The coefficients of the quadratic $\gamma$ and linear $k$ stiffness, and damping $c$ were found using the fitting of the elastic and dissipative restoring forces. Their determined values are tabulated in Tab. 3 with their relative errors in the parentheses. It can be seen from the presented results that both methods were able to estimate the qualitative characteristics very well. In this testing case the accuracy of the ZCA and HVD was almost identical. However, it appears that the ZCA was able to correctly describe non-linear behaviour at lower frequency where the HVD misleadingly identified linear behaviour.

3.4. Summary of the application to simulated data

The results obtained by the proposed ZCA have been compared with the results estimated by the HVD. Although the results varied for each simulated case, the proposed ZCA generally estimated the coefficients of non-linearities with higher accuracy. For the system with the bilinear stiffness the ZCA estimated the coefficients with less than 1% relative error whereas the HVD obtained the same coefficients with the relative error of almost 10%. In the case of the off-centre stiffness the difference was even more dramatic. While the ZCA estimated the non-linear coefficients with the maximum relative error of 10%, the coefficients estimated by the HVD were up to 50% higher than the correct values. In the third case, in which quadratic stiffness was used, the accuracy of both methods was almost the same. However, the ZCA was able to qualitatively

<table>
<thead>
<tr>
<th>Method</th>
<th>$\gamma$ [N m$^{-2}$]</th>
<th>$k$ [N m$^{-1}$]</th>
<th>$c$ [N s m$^{-1}$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>original</td>
<td>3</td>
<td>39.5</td>
<td>0.1</td>
</tr>
<tr>
<td>ZCA</td>
<td>2.54 (-17.92)</td>
<td>39.43 (-0.11)</td>
<td>0.095 (-4.88)</td>
</tr>
<tr>
<td>HVD</td>
<td>2.57 (-16.69)</td>
<td>39.65 (0.43)</td>
<td>0.098 (-1.26)</td>
</tr>
</tbody>
</table>

Table 3: Identified coefficients of the quadratic stiffness (the values in parentheses are the relative errors between the estimated and original coefficients)
identify non-linear behaviour even at low amplitudes where the HVD misleadingly indicated linear behaviour (see Fig. 11(a)). Based on the investigated simulated cases it can be concluded that the ZCA can identify the coefficients of asymmetric restoring forces with higher accuracy. It may also qualitatively describe weaker non-linearity better than the HVD. Overall, with regards to sophisticated signal processing and the need for additional smoothing in the HVD, the ZCA seems to provide at least the same or better results with much less effort.

4. Application to experimental data from a micro-electro-mechanical system

To demonstrate the capabilities of the ZCA, it is used for the identification of elastic and dissipative non-linearities in a double-anchored double-ended-tuning-fork micro-electro-mechanical (MEMS) resonator. The resonator was developed and tested, and the data were kindly provided by Stanford University [14, 21, 24, 27]. The resonator, which was designed for time-keeping applications, consists of two micro-mechanical beams (200 µm long, 6 µm wide and 40 µm thick) that are connected on both ends to perforated masses, which are attached to the base. In order to obtain the resonant decay response (ring-down) data, the phase resonance testing was used. The system was driven into the non-linear resonant regime and, once the system reached its steady-state response at a specific amplitude and frequency, the external forcing was turned off and the transient ring-down response recorded. More information about the micro-electro-mechanical resonator and measurement set-up can be found in [14, 21, 24, 27].

The voltage linearly proportional to the displacement was measured and the resonant decay acquired with the sampling frequency of $f_s = 50$ MHz for three levels of excitation, leading to maximum initial amplitudes of $a_0 = 125$ mV, 165 mV, and 225 mV. The application of the ZCA is shown in detail for $a_0 = 165$ mV and only the most significant results of the two other cases are presented. The data are significantly different than the simulated cases in terms of time and frequency scales. The simulated cases had their modal frequencies in order of several hertz whilst the frequency of the MEMS resonant is in order of megahertz. However, it will be shown that the ZCA can be reliably applied in this case too.

To allow a reliable comparison of the ZCA with the HVD, the Whittaker smoother was used to reduce measured noise. The resulting resonant decay response shown in Fig. 13(a) appears to be almost symmetric with respect to the time axis. This apparent symmetry led to the assumption of symmetric restoring forces and the development of a non-linear vibration model for this MEMS resonator in [14, 21, 24]. However, after a closer examination of the signal in Fig. 13(a) and the other two cases for the different initial amplitudes (not shown here), it was observed that some asymmetry exists in the data. This observed asymmetry suggested that the restoring forces might be asymmetric. The presence of the asymmetry was also confirmed by the restoring force surface method in [23]. This asymmetry is more significant at higher amplitudes and almost disappears at lower amplitudes of vibration. The fact that the asymmetry disappears at low amplitudes indicates that the system approaches linear behaviour.

The ZCA and HVD have been applied to the signal and the IA is shown in Fig. 13(a). Both ZCA and HVD were influenced by residual noise, especially at lower amplitudes. The HVD also suffered from strong end-effects, which were eventually removed and additional smoothing had to be used to obtain the results shown.

It can be seen in Fig. 13(b) that two clearly separated modal frequencies are present at high amplitudes at the beginning of the signal. These two frequencies then slowly merge towards the end of the time interval, indicating linear behaviour at lower amplitudes. The results of both methods correspond qualitatively to each other very well, thereby highlighting the reliability of the proposed ZCA as well as demonstrating that after some careful signal processing the HVD can be used with experimental data. One of the frequencies is lower than the natural frequency, the other one is higher. This indicates that there is hardening behaviour for positive amplitudes and softening behaviour for the negative ones.

Despite the same global qualitative appearance of the modal frequencies estimated by the ZCA and HVD there are a few differences which are worth mentioning. Firstly, the HVD indicates slightly stronger non-linear behaviour. Secondly, the branches of the IF estimated by the ZCA appear to be smoother than those from the HVD, even though the additional smoothing was not used. Lastly, the HVD produced an
Figure 13: Micro-electro-mechanical resonator: (a) ring-down response with instantaneous amplitude and (b) instantaneous frequency.

An unexplained discontinuity in the IF from the negative part of the free decay (see Fig. 13(b) for $t < 0.002$ s), whereas the ZCA estimated a smooth frequency in this region.

The extracted backbones are shown in Fig. 14(a). In order to visualise the results clearly, only every 50th point estimated by the ZCA was plotted. Two separated backbones were clearly obtained by both methods and they exhibit similar discrepancies as the modal frequency. Qualitatively, the results are the same, while quantitatively they are slightly different since the HVD indicates stronger non-linear behaviour. Both sets...
of results, however, approach the same linear natural frequency at lower amplitudes.

The estimated damping curves are shown in Fig. 14(b). Again, to allow a better visualisation, only every 50th point estimated by the ZCA was plotted. The both sets of results are highly distorted due to residual measured noise, especially at lower amplitudes. Even after additional filtering, the HVD produced much more distorted results than the ZCA. The detection of non-linearity could be difficult using the HVD results, because one could argue that this damping curve is a straight, highly distorted line. On the other hand, while considering the damping curve estimated by the ZCA, the non-linear behaviour is more evident. Moreover, this behaviour is of a hardening type because the damping rate increases with the increasing amplitude.

The estimated elastic and dissipative restoring forces are shown in Fig. 15. While it can be seen that

![Figure 15: Micro-electro-mechanical resonator: (a) elastic and (b) dissipative restoring forces. It should be noted that due to the significantly different values on the x- and y-axis, the non-linear restoring forces appear as straight lines.](image)

the difference between the methods is not significant, the non-linear nature of the restoring forces cannot be readily observed since both restoring forces appear to be almost straight lines. This is probably caused by significantly different orders of magnitude on the x- and y-axes. Therefore, unlike in the simulated cases, detection and characterisation by visual inspection are only possible from the backbones and damping curves shown in Fig. 14.

The comparison of the experimental results with the results for the system with the quadratic stiffness suggests that the quadratic stiffness could be present in the MEMS resonator. In particular, Fig. 10 and Fig. 13 exhibit the same features - a signal which is asymmetric at the beginning, but becomes symmetric towards the end and the two different frequencies which merge at low vibration amplitudes. The backbones in Fig. 11(a) and Fig. 14(a) are also qualitatively the same, having a V-shape created by the estimates from the upper and lower parts. Both backbones suggest the presence of hardening behaviour for the positive amplitudes and softening behaviour for the negative amplitudes. Therefore, it can be concluded that the elastic restoring force of the MEMS can be modelled using quadratic stiffness. This observation was also confirmed in [23] by the use of the restoring force surface (RFS) method. As discussed in [24] many different sources of non-linearities exist in MEMS - mechanical (kinematic or material), electrostatic and thermal. Moreover, they can interact with each other, resulting in non-trivial responses. Therefore, the origin of the identified quadratic stiffness has not been explained theoretically, but since three non-parametric methods (ZCA, HVD and RFS) led to the same conclusion, it is likely that this type of non-linearity can be used to adequately model the MEMS in the region of recorded amplitudes.

The damping curves indicate that the damping exhibits non-linear behaviour which increases with the increasing amplitude. This finding is also supported by the theoretical explanation of the possible origin of the non-linear dissipation. Based on the microscopic theory of dissipation, the non-linear dissipative phenomenon can be explained using the non-linear interaction of the primary resonant mode with photons.
as detailed in [14]. Therefore, it is possible to conclude that the dissipative behaviour can be modelled using a cubic damping model.

All results which have been presented so far were obtained for the initial amplitude \( a_0 = 165 \text{ mV} \). To illustrate that the results are consistent for all three available initial amplitudes, the backbones and damping curves for all three cases estimated by the ZCA are shown together in Fig. 16. For the clarity of presentation, several periods were averaged (as described in section 2) and the HVD results are not shown. It is clear that the backbones for all initial amplitudes are very close to each other, having the matching V-shape with different amplitudes. All of them also approach the same natural linear frequency at low amplitudes. The damping curves also overlay well and all of them indicate non-linear dissipative behaviour.

The previous detection and characterisation have been done in a non-parametric manner and led to the following model

\[
\ddot{x}(t) + 2\delta \dot{x}(t) + c_{nl} \dot{x}^3(t) + 4\pi^2 f_0^2 x(t) + k_{nl} x^2(t) = 0,
\]

where \( x(t) \) is a voltage proportional to the displacement, \( \delta \) is a damping rate, \( c_{nl} \) is a cubic hardening damping coefficient, \( f_0 \) is a natural frequency, and \( k_{nl} \) is a quadratic stiffness coefficient. Since the mass of the MEMS resonator does not have to be considered, the restoring forces can be evaluated in units of \( \text{V s}^{-2} \).

To quantify all parameters, the restoring forces in Fig. 15 were fitted by the expressions outlined in Eq. (14). The resulting coefficients are summarised in Tab. 4.

<table>
<thead>
<tr>
<th>Method</th>
<th>( a_0 ) [mV]</th>
<th>( k_{nl} ) [( \text{V}^{-1} \text{s}^{-2} )]</th>
<th>( f_0 ) [Hz]</th>
<th>( c_{nl} ) [( \text{V}^{-2} \text{s}^{-1} )]</th>
<th>( \delta ) [s(^{-1})]</th>
</tr>
</thead>
<tbody>
<tr>
<td>ZCA</td>
<td>125</td>
<td>( 2.81 \times 10^{13} )</td>
<td>( 1.217 \times 10^6 )</td>
<td>( 3.13 \times 10^{-11} )</td>
<td>112.72</td>
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<tr>
<td></td>
<td>165</td>
<td>( 2.82 \times 10^{13} )</td>
<td>( 1.217 \times 10^6 )</td>
<td>( 3.35 \times 10^{-11} )</td>
<td>112.17</td>
</tr>
<tr>
<td></td>
<td>225</td>
<td>( 2.97 \times 10^{13} )</td>
<td>( 1.217 \times 10^6 )</td>
<td>( 5.36 \times 10^{-11} )</td>
<td>108.01</td>
</tr>
<tr>
<td>HVD</td>
<td>125</td>
<td>( 3.46 \times 10^{13} )</td>
<td>( 1.217 \times 10^6 )</td>
<td>( 2.64 \times 10^{-11} )</td>
<td>131.34</td>
</tr>
<tr>
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<td>121.36</td>
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<tr>
<td></td>
<td>225</td>
<td>( 3.74 \times 10^{13} )</td>
<td>( 1.220 \times 10^6 )</td>
<td>( 2.63 \times 10^{-10} )</td>
<td>160.26</td>
</tr>
</tbody>
</table>

Table 4: Identified coefficients of the micro-electro-mechanical resonator

The coefficients of the quadratic stiffness \( k_{nl} \) seems to be increasing with increasing initial amplitude for both methods. The coefficients are not exactly the same for all cases, but they have the same order of

\[0.1 \times 10^{13}, 0.2 \times 10^{13}, 0.3 \times 10^{13}\]
magnitude. They also correspond to the quadratic stiffness coefficient estimated by the RFS in [23]. The coefficients identified using the ZCA are slightly lower than those identified by the HVD, which is in line with the previous observation of weaker non-linear behaviour estimated by the ZCA in Fig. 13(a) and Fig. 14(a).

It can be seen that the natural frequency $f_0$ is more or less the same for all initial amplitudes and both methods. It corresponds to the natural frequency that was estimated using the ZC in [14, 24] and by the restoring force method in [23]. The same natural frequency can also be visually identified in Fig. 13(b), Fig. 14(a) and Fig. 16(a).

The non-linear cubic damping coefficients $c_{nl}$ are very small. This is caused by the significantly different orders of magnitude of the velocity amplitude and dissipative force (see Fig. 15(b)). Unfortunately, a comparison with other studies is not possible because none of them has used the same model of damping. Nonetheless, most of the coefficients have the same order of magnitude and the results of both methods are quite close to each other. There is a difference between the ZCA and HVD for the highest initial amplitude ($a_0 = 225$ mV). This difference can, however, be explained by the strong distortion of the dissipative forces that have been fitted.

The damping rates $\delta$ estimated by the ZCA are almost the same for all investigated cases. Moreover, they match very well to the damping rates estimated in [14, 24]. On the other hand, the damping rates estimated by the HVD are higher and quite scattered. This, however, is not surprising given the amount of distortion in Fig. 14(b) and Fig. 15(b).

Overall, with regards to the simulated cases for which the accuracy of the ZCA was often better and due to extensive signal processing used in the HVD, it is likely that the coefficients estimated by the ZCA are more accurate. The comparison of the estimated coefficients with other studies shows that the accuracy of the ZCA is the same as that of the zero-crossing method [14, 24] and restoring force surface method [23]. Based on the presented findings it can be concluded that the newly developed ZCA method can be used reliably for the identification of non-linear systems with asymmetric restoring forces from an experimentally obtained resonant decay response.

5. Discussion

The zero-crossing method for systems with asymmetric non-linearities (ZCA) proposed in this paper is an extension of the zero-crossing methods. It allows non-parametric identification of backbones, damping curves, and elastic and dissipative restoring forces from a resonant decay response without any a priori knowledge of the signal or system’s parameters. The ZCA is based on the idea that since each signal branch is defined on its half-plane only, it is practically enough to identify matching instantaneous characteristics of each signal branch. To observe the capability of the ZCA and compare it with the Hilbert vibration decomposition in detail, these methods have been applied to three simulated resonant decay responses from the systems with bilinear, off-centre clearance and quadratic stiffness. Before applying the methods, the resonant decay responses obtained by the numerical integration were polluted by white Gaussian noise with signal-to-noise ratio 25 dB to make the data more realistic. Although some noise in the data does not generally prevent either method from estimating the IF and IA, the use of the Whittaker smoother (see Appendix B) led to very good noise removal from the vibration signal in all investigated cases.

In order to perform the relevant comparison of the ZCA with the HVD, some known issues of the HVD were avoided beforehand by selecting a relatively high sampling frequency. Generally, the HVD requires the sampling frequency to be twenty to eighty times higher than the highest frequency of interest [8, 9]. In addition, extra smoothing and filtering had to be used in the HVD to obtain the presented results. In contrast, the ZCA does not require the additional smoothing. There is no requirement on the sampling frequency either. As long as the zero-crossing points, minima and maxima are estimated, the rest of the signal is not important.

To further demonstrate the capabilities of the ZCA as well as the HVD whose application to the experimental data has rarely been reported in literature [8], both methods have been used to investigate the experimentally acquired ring-down response of a micro-electro-mechanical resonator [14, 21, 24, 27]. Despite the need for additional smoothing and end-effects removing in the HVD, both methods were eventually able
to estimate reasonable results. Although some differences may be observed, they cannot be readily explained based on the data. However, it may be argued that since the ZCA did not require extensive signal processing, was not influenced by any end-effects, and the results were overall smoother, it is likely that the results estimated by the ZCA are more accurate than those obtained by the HVD.

Although the two methods can theoretically reach the same accuracy, the results of the ZCA are obtained with much less effort in contrast to the HVD. There is no complicated decomposition or filtering required so the method is straightforward to implement. Moreover, the ZCA should be less sensitive to measured noise due to the use of maxima, minima and zero-crossing points only. As long as these points are well estimated, the noise in the rest of the signal does not have any effect on the estimated results. In order to estimate zero-crossing points of a noisy signal, a local smoothing [13] or interpolation [24] around zero-crossing points may be used. Furthermore, similarly to the zero-crossing method proposed in [14] which is described in section 2.2, a sensitivity of the ZCA to measured noise may be improved by averaging of the frequencies and amplitudes over the intervals in which negligible changes of the IF and IA are assumed. Although some information may be lost due to the averaging, the results are much smoother, thereby allowing better visualisation and interpretation. This averaging capability was shown in section 4 for the experimental data. In general, a range of standard tools for de-noising of the signal may be used, one of which, the Whittaker smoother described in Appendix B, has also been used in this paper. The use of the Whittaker smoother was mainly required to allow a reliable comparison between the ZCA and HVD.

The zero-crossing methods [13, 14, 22] as well as the proposed ZCA have several limitations. Due to the fact that these methods do not yield instantaneous results, their application to very short, strongly damped resonant decay responses is limited. For such signals very few zero-crossing points would be available to construct the backbone and damping curves, and elastic and dissipative restoring forces, thereby making detection, characterisation and quantification of non-linearities unreliable. For this type of signals the HVD is also unlikely to provide accurate results due to the end effects. In addition, the ZC and ZCA do not allow non-linear system identification that requires considering sub- and super harmonics since only a single frequency is extracted for each vibration period. In contrast, the sub- and super harmonics can be identified using the HVD as shown in [8, 16]. Moreover, due to the same reason the ZC and ZCA do not provide any information about intra-wave frequency modulation for either symmetric or asymmetric signals. If the instantaneous results are needed to study the variance of the instantaneous frequency within a vibration period, the Hilbert transform [8], direct quadrature [22] or other methods, some of which are reviewed in [23], must be used instead for the zero-crossing approaches.

The modal frequency, modal damping and restoring forces are obtained in a non-parametric manner, i.e. no specific model of the structure is needed and only a general governing equation (Eq. (1)) is assumed. If the proposed method is applied to a system with symmetric restoring forces, the identified characteristics for the upper and lower parts will be the same and they will correspond to the results obtained by zero-crossing methods [23]. The shape of the backbones, damping curves and restoring forces can be used for detection and characterisation of non-linearities. If these backbones and damping curves are straight lines and the same for the upper and lower parts of the signal, the system is linear. If there is any significant deviation from the constant values and/or the characteristics are not the same for the upper and lower part, the system is non-linear. Furthermore, the shape of backbones, damping curves and restoring forces is unique for common types of non-linearities as evidenced in section 3. Therefore, the shape can help to deduce the type and the mathematical expression of the non-linearity. This expression can then be fitted to restoring forces to obtain the coefficients of non-linearities.

The ZCA is a non-parametric method which can be applied to any resonant decay response. Such a resonant decay response can be either measured from an SDOF system or using phase resonance testing [18–20] for an MDOF system. If the resonant decay response has been obtained from an SDOF system, the model in Eq. (1) with quantified coefficients describes the system completely. Theoretically, such a model can be used to predict the response of the system for any range of loading conditions. On the other hand, for a resonant decay response of an MDOF system measured by the phase resonance testing, the model is only valid for the given mode of vibration. Therefore, this model should be used to compute the response of the system only in a close proximity of this mode.
6. Conclusion

This paper proposed a non-parametric method for the identification of systems with asymmetric restoring forces from a resonant decay response. Since the method is based on the zero-crossing method, it is termed zero-crossing methods for systems with asymmetric restoring forces (ZCA). Although the ZCA does not provide instantaneous frequency and amplitude, it was demonstrated on the simulated and experimental data that it can provide results of the same or higher accuracy than the Hilbert vibration decomposition. At the same time, however, it is more robust against measured noise and requires less sophisticated signal processing.

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References

Appendix A. The Hilbert vibration decomposition

The Hilbert vibration decomposition (HVD) [8, 9, 15–17, 28] is a method dedicated to the analysis of quasi-periodic or almost periodic oscillating signals, such as free decays of non-linear systems. It is an extension of the Hilbert transform that allows to estimate the instantaneous frequency (IF) and amplitude (IA) of a multi-component, non-stationary and non-linear vibration signal. It is similar to the Hilbert-Huang transformation (HHT) [6], but it does not use the empirical sifting process based on the cubic splines but rather the Hilbert vibration decomposition (HVD) assumes that (i) the signal is a superposition of quasi-harmonic components, (ii) the envelopes of each component differ, and (iii) several longest periods of the corresponding slowest components are included in

$$x(t) = \sum_{i=1}^{N_{\text{HVD}}} x_i(t) = \sum_{i=1}^{N_{\text{HVD}}} a_i(t) \cos \left( \int \omega_i(t)dt \right),$$

where $a_i(t)$ is the IA and $\omega_i(t)$ is the IF of the $i$-th component. The decomposition is performed by iterating the following steps [28]: (1) the IF $\omega_i(t)$ of the largest energy component is extracted by low-pass filtering of the IF of the original signal (which does not have physical meaning) obtained by the Hilbert transform, (2) the synchronous detection, which is also based on the Hilbert transform and low-pass filtering, then extracts the amplitude $a_i(t)$ of the vibration component with the previously estimated frequency, (3) having estimated the IF and IA of the largest energy component, it can be subtracted from the original signal and

References

the process repeated until all \( N_{\text{HVD}} \) components are extracted. The decomposition ends when the standard deviation of two subsequent components is smaller than a defined tolerance or the selected total number of components \( N_{\text{HVD}} \) has been reached. The obtained components are not generally equal to intrinsic mode functions obtained by the empirical mode decomposition (EMD) in the HHT because the means of decomposition is different. The HVD does not use the shifting process so the potentially problematic cubic spline fitting is avoided and the frequency resolution of the HVD can be better [8]. On the other hand, the low-pass filtering and the Hilbert transform are extensively used, both of which are influenced by numerical problems. The properties of the low-pass filter, especially the cut-off frequency, must be carefully selected because the results of the decomposition can vary significantly due to the setting of the filter. However, there is not general guidelines as to how to select these values.

The Hilbert vibration decomposition for asymmetric systems is based on the idea that each signal branch (the lower and upper part of the signal with respect to time axis) is defined on its half-plane only. Therefore, in order to recover the initial restoring forces, these parts must be analysed separately. The applicability of the HVD for asymmetric systems is limited to resonant decay responses for which the extracted components \( x_i \) represent the primary and higher harmonics.

In order to identify asymmetric restoring forces, a congruent amplitude and frequency must be defined as [16]

\[
a_c(t) = \sum_{i=1}^{N_{\text{HVD}}} a_i(t) \cos(\phi_i(t)),
\]

and

\[
\omega_c(t) = \sum_{i=1}^{N_{\text{HVD}}} \omega_i(t) \cos(\phi_{\omega i}(t)),
\]

where \( \phi_i(t) \) and \( \phi_{\omega i}(t) \) are phase angles between the amplitude and frequency of the primary and \( i \)-th harmonic, respectively. It should be noted that for asymmetric resonant decay responses, the trend (aperiodic slowly varying component) must be excluded. The congruent amplitude is an envelope of the amplitude (also called the envelope of the envelope in [16]). By defining the congruent amplitude as the envelope of the envelope (and similarly the congruent frequency as the envelope of the frequency), the correct instantaneous characteristics for the positive \( a_{cp} (\omega_{cp}) \) and negative \( a_{cn} (\omega_{cn}) \) part of the original signal can be obtained as

\[
a_c(t) = \begin{cases} a_{cp}(t), & x_i(t) \geq 0 \\ a_{cn}(t), & x_i(t) < 0 \end{cases} \quad \omega_c = \begin{cases} \omega_{cp}(t), & x_i(t) \geq 0 \\ \omega_{cn}(t), & x_i(t) < 0 \end{cases}.
\]

The congruent functions allow to estimate the original restoring forces separately for the positive and negative parts of a signal as

\[
F_{el} = \begin{cases} \omega_{cp}^2 a_{cp}, & x(t) \geq 0 \\ \omega_{cn}^2 a_{cn}, & x(t) < 0 \end{cases}, \quad F_d = \begin{cases} 2\delta_{cp} a_{\omega cp}, & x(t) \geq 0 \\ 2\delta_{cn} a_{\omega cn}, & x(t) < 0 \end{cases},
\]

where \( \delta_{cp} \) and \( \delta_{cn} \) are congruent damping rates estimated using Eq. (8) from the positive and negative part, respectively, and \( a_{\omega cp} \) and \( a_{\omega cn} \) are congruent amplitude of the positive and negative parts of the velocity, respectively. Although the damping was not considered in [16] it is shown in section 3 and section 4 that it can be estimated using the HVD as well.

The Hilbert vibration decomposition determines the congruent backbones, damping curves, elastic and dissipative asymmetric non-linear restoring forces. However, it involves extensive signal processing which may be sensitive to measured noise and suffer from numerical issues of the Hilbert transform. The HVD used throughout this paper as a reference method was computed by the code distributed alongside of [8]\(^1\).

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\(^1\)Available at [http://ht.net.technion.ac.il/](http://ht.net.technion.ac.il/) (cited on 5 July 2017)
Appendix B. Whittaker smoother

The Whittaker smoother [29] is a data smoothing method that fits the data using a penalty least square method by balancing the fidelity of the data and their roughness:

\[
\min_y \left[ \sum_i (x_i - y_i)^2 + \lambda \sum_i (y_i - 2y_{i-1} + y_{i-2})^2 \right]. \tag{B.1}
\]

where \( x_i = x(t_i) \) is an initial (noisy) signal, \( y_i = y(t_i) \) is a smoothed signal and \( \lambda \) is a smoothing parameter. The smoothing parameter weights the fidelity and the smoothness of the data. The requirement of the smoothness is stronger for larger values of \( \lambda \). Therefore, while \( y \) becomes smoother for larger \( \lambda \), the fit of the original data \( x \) becomes worse. The appropriate \( \lambda \) might be chosen by tuning it until the resulting series \( y \) is visually satisfying. However, this parameter can be also determined automatically (and more objectively) based on a leave-one-out cross validation strategy combined with minimum search optimisation.

The leave-one-out cross validation strategy is one of many cross validation procedures used in a machine learning community [30, 31]. In case of the Whittaker smoother, this strategy consists of removing one element of the series \( x \), smoothing the remaining data, and evaluating the error of the prediction for the removed element. After repeating the process for all elements in \( x \), the total standard error can be calculated [29, 30]. Then, the smoothing parameter \( \lambda \) for which this total standard error is at its minimum is taken as the optimal value.

The Whittaker smoother was originally proposed in chemical engineering [29], but it has many attractive properties for the non-linear system identification as well:

- It is very quick even if the cross validation procedure is used.
- It handles missing data and adapts to boundaries automatically, so it does not suffer from any end effects.
- The smoothness is controlled using a single parameter which can be selected automatically.
- It does not assume any particular form (polynomial or sinusoidal) of the signal [32].
- Since the smoothness of the derivative is required, the methods that use this derivative, such as the zero-crossing method for systems with asymmetric restoring forces (ZCA) proposed in this paper, can benefit from the use of the Whittaker smoother.

The Whittaker smoother used in this paper has been implemented based on the pseudo-code provided in [29] and complemented by an optimisation procedure to determine the smoothing parameter \( \lambda \) automatically using the leave-one-out cross validation strategy.