Standard filter approximations for low power Continuous Wavelet Transforms

Alexander J. Casson and Esther Rodriguez-Villegas

***Abstract*— Analogue domain implementations of the Continuous Wavelet Transform (CWT) have proved popular in recent years as they can be implemented at very low power consumption levels. This is essential for use in wearable, long term physiological monitoring systems. Present analogue CWT implementations rely on taking mathematical a approximation of the wanted mother wavelet function to give a filter transfer function that is suitable for circuit implementation. This paper investigates the use of standard filter approximations (Butterworth, Chebyshev, Bessel) as an alternative wavelet approximation technique. This extends the number of approximation techniques available for generating analogue CWT filters. An example ECG analysis shows that signal information can be successfully extracted using these CWT approximations.**

I. INTRODUCTION

Analogue domain implementations of the Continuous Wavelet Transform (CWT) have become popular approaches for biomedical signal processing: they have been used for electrocardiogram (ECG) analysis [1], electroencephalogram (EEG) analysis [2], and signal de-noising [3]. The CWT is defined as

$$W(a,b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(t) \psi^* \left(\frac{t-b}{a} \right) dt \tag{1}$$

where $f(t)$ is the signal to be transformed, $\psi(t)^*$ is the complex conjugate of the *mother wavelet function*, a is the *analysis scale*, and b is the time at which the transform is taken. For real mother wavelets (1) is equivalent to the convolution of signal $f(t)$ with an impulse response

$$h(t) = \frac{1}{\sqrt{a}} \psi \left(\frac{-t}{a} \right). \tag{2}$$

The CWT at a fixed scale can thus be carried out by passing $f(t)$ through a system which has impulse response $h(t)$. Given the properties of the CWT the required system corresponds to a bandpass filter [4] and: the scale a sets the filter centre frequency, and the mother wavelet $\psi(t)$ determines the filter shape and roll-off. Multiple filters can be used in parallel (as a filter bank) to extract time–frequency information at different scales, and the filters can be implemented in either the analogue (s) or digital (z) domain.

Analogue implementations are potentially of significant use in wearable bio-sensors for prolonged physiological monitoring. For comfort and user acceptance these sensors must be as physically small as possible, operating from small batteries. This means that the average power draw must be no more than a few hundred micro-Watts [2]. In general, recorded physiological signals are analogue in nature. By implementing the CWT filter in the analogue domain power intensive analogue-to-digital converters can be removed from parts of the system [1], or the specifications relaxed. Also, for low Signal-to-Noise Ratio (SNR) applications, below around 40 dB, analogue filters are potentially much more power efficient than their digital counterparts [5]. This highly motivates the use of analogue CWT implementations in power constrained situations. Analogue CWT filters with power consumptions ranging down to nano-Watts have been previously reported [1].

Unfortunately, not all mother wavelet functions are suitable for implementation as analogue bandpass filters. To allow implementation a mathematical approximation stage is ordinarily carried out (Section II). This paper presents the use of Butterworth, Chebyshev and Bessel bandpass filters as an alternative, non-mathematical, approximation procedure (Section III). The use of the filters is demonstrated (Section IV) through the CWT analysis of ECG data and the results are commented upon. The bandpass filters used correspond to the *standard filter approximations* from classical filter design. The new approximation method thus extends the range of approximation options available, and intrinsically gives filters for which the circuit level design is possible and well documented previously.

II. MATHEMATICAL FILTER APPROXIMATION

Despite the utility of analogue CWT filters, they are not able to implement all mother wavelet functions exactly. For example, the Mexican hat mother wavelet function is defined as the second derivative of a Gaussian function as

$$\psi(t) = \frac{2}{\pi^{1/4}\sqrt{3}} \left(1 - t^2\right) e^{-t^2/2}. \tag{3}$$

Substituting (3) into (2), the required impulse response is symmetrical around $t = 0$ and is non-causal. This does not present an issue in digital implementations where arbitrary delays can be introduced. In all analogue domain filters, however, this results in poles in the right hand complex plane and the filter being unstable. To implement the analogue CWT filter an approximation procedure is required to give a finite order and stable filter.

Padé and L_2 [1] and Maclaurin [4] approximation procedures have been reported previously. The Maclaurin method, termed the LPCWT, delays the ideal impulse response by

A. J. Casson and E. Rodriguez-Villegas are with the Department of Electrical and Electronic Engineering, Imperial College London, UK. Email: {acasson,e.rodriguez}@imperial.ac.uk.

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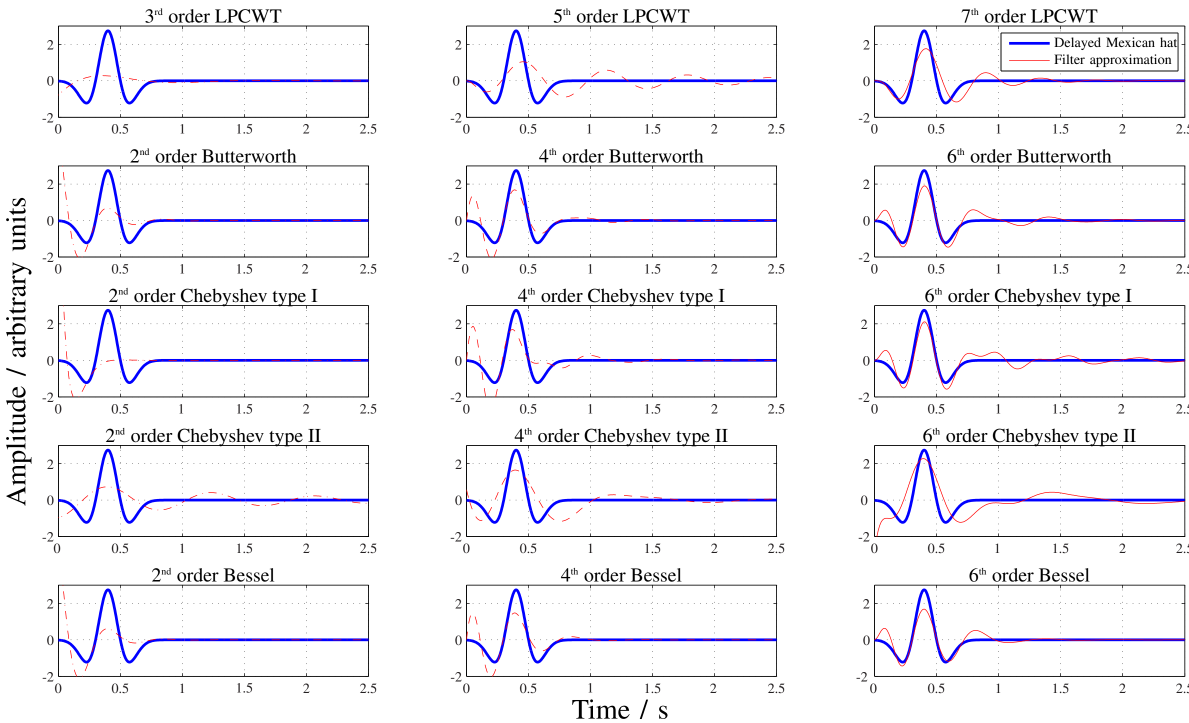

Fig. 1. Impulse responses of Mexican hat filter approximations for different approximation methods and filter orders. Filter parameters are in Table I.

TABLE I

LOW PASS PROTOTYPE FUNCTIONS FOR THE STANDARD FILTER TYPES AND PARAMETERS USED TO RESEMBLE THE MEXICAN HAT. COEFFICIENTS FOR THESE FILTERS CAN BE FOUND IN TABLES, [6], OR FROM THE MATLAB `BUTTER`, `CHEBY1`, `CHEBY2` AND `BESSELF` FUNCTIONS.

Filter	Prototype	Parameters	Notes
Butterworth	$\lvert H(j\omega)\rvert^2 = \frac{1}{1+\omega^{2n}}$	$\omega_c = 15$ rads^{-1}, $Q = 1.5$	n is the filter order
Chebyshev type I	$\lvert H(j\omega)\rvert^2 = \frac{1}{1+\epsilon^2 C_n^2(\omega)}$	$\omega_c = 15$ rads^{-1}, $Q = 1.35$, $\epsilon = 0.8$	$C_n(\omega) = \cos\big(n\cos^{-1}\omega\big)$ for $\lvert\omega\rvert \leq 1$
Chebyshev type II	$\lvert H(j\omega)\rvert^2 = \frac{\epsilon^2 C_n^2(1/\omega)}{1+\epsilon^2 C_n^2(1/\omega)}$	$\omega_c = 7.5$ rads^{-1}, $Q = 0.55$, $\epsilon = 20$	$C_n(\omega) = \cosh\big(n\cosh^{-1}\omega\big)$ for $\lvert\omega\rvert > 1$
Bessel	$\lvert H(j\omega)\rvert^2 = \left\lvert\frac{B_n(0)}{B_n(j\omega)}\right\rvert^2$	$\omega_c = 15$ rads^{-1}, $Q = 1.35$	B_n is the n^{th} order Bessel function

an amount T: $h(t) \to h(t-T)$ so that the response is not symmetrical around $t = 0$ [4]. Taking the Fourier Transform, the required filter transfer function is then given by

$$H(s) = -\pi^{1/4}\sqrt{8a^5/3}\,s^2 \Big/ e^{sT-a^2s^2/2}. \tag{4}$$

A Maclaurin series expansion of the denominator can be taken to give

$$H(s) = \frac{-\pi^{1/4}\sqrt{8a^5/3}\,s^2}{1+sT+\left(\frac{T^2}{2}-\frac{a^2}{2}\right)s^2+\left(\frac{T^3}{6}-\frac{Ta^2}{2}\right)s^3+\cdots} \tag{5}$$

which is then truncated to give the desired LPCWT filter order. Values of a and T can be chosen to ensure that the wanted CWT filter is stable.

LPCWT impulse responses are shown in Fig. 1 for $a = 0.1$, giving a 2.1 Hz centre frequency filter, and $T = 0.4$. Third, fifth and seventh order approximations as used in [4] are shown. For comparison, the ideal Mexican hat at this scale, delayed by amount T, is also shown in Fig. 1. The resemblance, which improves with LPCWT order, can be seen. The frequency domain magnitude responses of these filters are shown in Fig. 2.

III. STANDARD FILTER APPROXIMATIONS

Ordinarily Butterworth, Chebyshev and Bessel filters are designed in the frequency domain. Instead here we are concerned with the time domain: the aim is to have a filter impulse response that resembles the Mexican hat. In this work we adjust the free parameters of these bandpass filters in order to tailor the impulse response shape to resemble the wanted mother wavelet.

To generate the CWT filter approximations the design procedure begins with a prototype function $H(j\omega)$ which describes a low pass Butterworth, Chebyshev or Bessel filter. These prototype functions are given in Table I, [6]. The prototype is converted to a bandpass filter with centre frequency ω_c by performing the low pass to bandpass transform

$$j\omega \to Q(j\omega/\omega_c + \omega_c/j\omega) \tag{6}$$

where Q is the filter quality factor. This transform doubles the order from the prototype, and so only even order bandpass filters are possible.

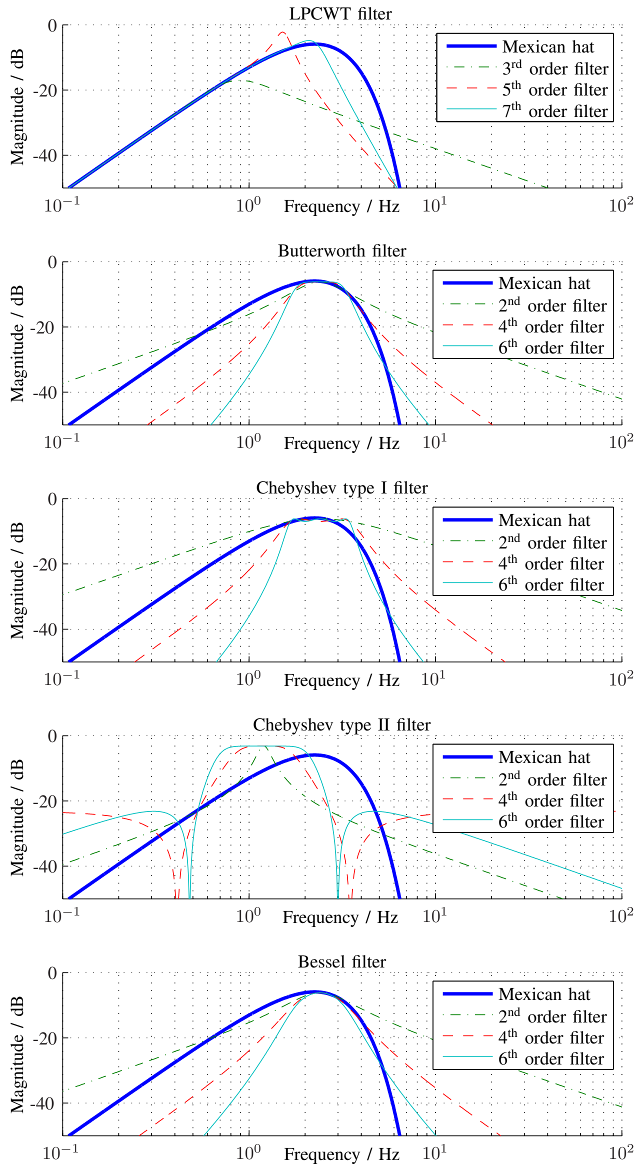

Fig. 2. Bode magnitude responses of Mexican hat filter approximations.

To perform a time–frequency analysis multiple bandpass filters are required in parallel to form a filter bank. From (2) the centre frequency gain of each filter depends on the centre frequency. To form a CWT filter bank the filter response of each bandpass filter is thus modified to

$$H(j\omega) \rightarrow \sqrt{3.63/\omega_c} \times H(j\omega). \quad (7)$$

Here the 3.63 factor is introduced so that the gain at $\omega_c = 1$ rads^{-1} matches that of the idealised Mexican hat.

Three degrees of freedom are available for the Butterworth and Bessel filters: the order n, centre frequency ω_c and the Q. For the Chebyshev filters the ripple factor ϵ provides a fourth degree of freedom. To form a filter bank for each approximation only w_c is changed, all other parameters are constant. The parameters used here to give impulse responses that resemble the Mexican hat at scale $a = 0.1$ are given in Table I.

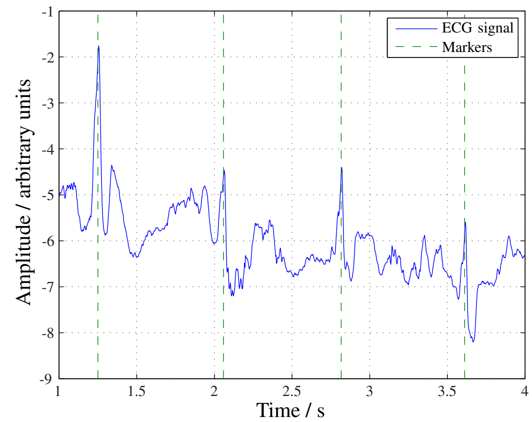

Fig. 3. ECG signal analysed with feature points marked.

The ordinary view of these filters, in the frequency domain, is given in Fig. 2 for three filter orders. The more interesting impulse response is given in Fig. 1. As for the LPCWT a resemblance to the Mexican hat is seen, especially in the higher order cases.

IV. PERFORMANCE TESTING AND DISCUSSION

To demonstrate the operation of the standard filter approximation wavelet filters an ECG signal from the MIT-BIT noise stress test database [7] is analysed. The ECG is illustrated in Fig. 3 and consists of a clean recording that has been corrupted by measured ambulatory ECG noise: baseline wander, muscle artefact and electrode motion artefact. The analysed record (number 118e06) has a 6 dB SNR. Four heart beat markers are present in the analysed period.

The aim here is to perform a time–frequency analysis on the ECG data to determine whether the marked heart beats can be identified using the standard filter approximation wavelet filters, as with the Mexican hat CWT. To keep the presentation of results tractable only three seconds are displayed here, starting at time 301 s in the record. One second on either side of this is also analysed and discarded to remove any edge effects.

To perform a detailed time–frequency analysis, for each filter approximation a bank of 98 bandpass filters is used. The filters have centre frequencies logarithmically spaced between 1.8 and 60 Hz. For the LPCWT approximation T also varies logarithmically from 400–12.5 ms such that higher centre frequency filters have smaller T values. For a fair comparison 98 Mexican hat CWT scales are also used.

The results of the time–frequency analysis using the five approximations and the idealised Mexican hat CWT are shown in Fig. 4. Here colour represents the absolute output amplitude from each filter, plotted against time and the filter centre frequency. In all six cases the ECG features are clearly highlighted. The form of the results is similar in all six cases, although for all of the analogue filter approximations the low frequency information is delayed (there is an apparent shift to the right compared to the Mexican hat case). This arises due to the delay T introduced explicitly in the LPCWT approximation procedure. Larger delays are introduced at lower centre frequency filters, giving the effect seen. This

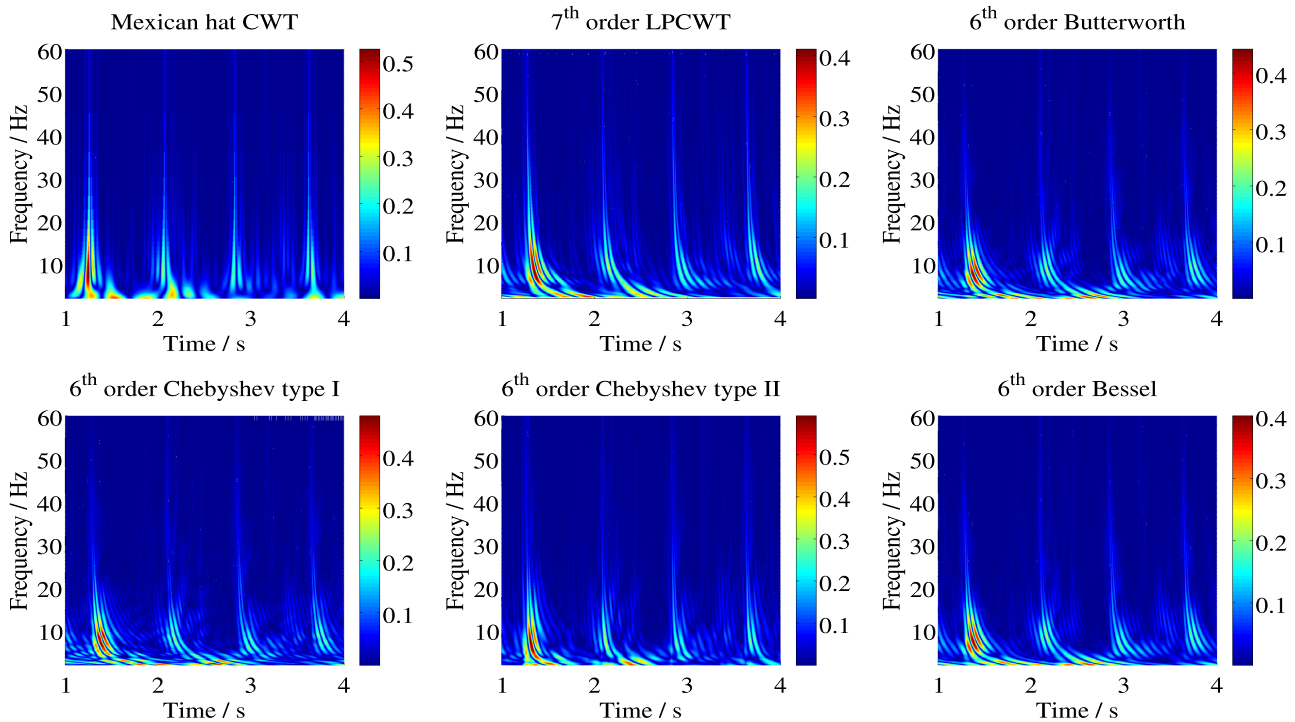

Fig. 4. Results of the wavelet analysis on 3 s of ECG data using the Mexican hat CWT and five approximations to it.

delay is implicitly present in the Butterworth, Chebyshev and Bessel filters as they too are not symmetrical around $t = 0$. It is important to note that the low frequency CWT information is preserved, only delayed compared to the high frequency information.

As would be expected minor differences are present between the different cases. However, the heart beat information can always be clearly seen, as desired. It is noted that in many applications the choice of mother wavelet to use is essentially arbitrary. The aim of the analogue CWT approximation is thus not necessarily to replicate the shape of the Mexican hat wavelet exactly, but to find a shape that achieves acceptable performance in the wanted application. Minor differences thus may not be significant.

Fig. 4 demonstrates that the standard filter approximations can work as *emphasis* stages: they highlight information which could then be passed to a classification stage or similar. Only a preliminary analysis of 3 s of data has been carried out here. Nevertheless it has been shown that standard filter approximation bandpass filters can potentially be used as alternatives to the idealised Mexican hat CWT.

By tuning the parameters available (Table I) other mother wavelets, for example the Morlet, may also be approximated. A numerical optimisation procedure, minimising the difference between the wanted wavelet shape and the impulse response of the standard filter approximation, could be used to help select suitable parameter values. Mathematical approximations of specific wavelet functions, such as the Padé, L_2 and LPCWT methods, will always have their place when the use of a specific wavelet is desired. However, this work has demonstrated that they are not the only approach available for analogue CWTs. Some situations requiring low power CWT implementations may benefit from the larger body of work available on the design of circuits implementing Butterworth, Chebyshev and Bessel transfer functions. For example, [8] presents a 70 nW, 6th order Butterworth bandpass filter.

V. CONCLUSIONS

This paper has presented the use of filter banks of Butterworth, Chebyshev and Bessel bandpass filters for implementing analogue domain Continuous Wavelet Transforms that can potentially be implemented with very low power consumption. By tuning the free parameters of each bandpass filter, approximations to common mother wavelet functions can be achieved. This provides an alternative technique to the mathematical approximation of mother wavelet functions that has been presented previously.

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