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Rogue traders vs Value at Risk and Expected Shortfall

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Abstract

We show that, in a Black and Scholes market, value at risk and expected shortfall are irrelevant in limiting traders excessive tail-risk seeking behaviour as modelled via Kahneman and Tversky’s S-shaped utility. To have effective constraints one can introduce a risk limit based on a second but concave utility function.

1 Introduction

In this paper we aim to analyze how classic risk measures such as value at risk (VaR) and expected shortfall (ES) fare in limiting excessive tail-risk seeking behaviour in market players. We will model excessive tail risk seeking behaviour using Kahneman and Tversky [11] S-shaped utility. For example, S-shaped utility is naturally suited to model the behaviour of a trader who cares only about his pay packet and not about the overall loss of the bank, once the latter defaults. A traditional concave utility function would fail to reflect the limited liability of traders.

We point out, via a payoff optimization approach, that a trader may optimize expected S-shaped utility in presence of budget constraints regardless of VaR or ES constraints, in the sense that the optimum expected S-shaped utility without VaR or ES constraints will be exactly the same as the optimum with VaR or ES. VaR and ES are thus ineffective in limiting rogue traders. This
is particularly important in the light of the fact that ES has been officially endorsed and suggested as a risk measure by the Basel committee in 2012-2013 partly for its “coherent risk measure” properties [3, 1].

We will then hint at a solution to the problem: it will be enough to introduce a risk limit based on a second utility function, a traditional concave utility, for the expected concave utility constraint to be effective in curbing excessive tail risk seeking behaviour. Indeed, we will see that with this concave utility based risk constraint the optimum won’t be the same as in the case without constraint. We will not prove the result in this paper, referring the reader instead to [2]. The proof given there relies on the ideas used by Hardy and Littlewood to prove their inequality on symmetric decreasing rearrangements.

In this paper we limit the analysis mostly to the Black Scholes and Merton case [5, 12], since this is a benchmark model for derivatives valuation and it allows us to state our case without excessive mathematical infrastructure and in a familiar setting. However, our result is much more general and the general theory relies on a result that is similar to the theory of rearrangements behind Hardy and Littlewood inequality, see Armstrong and Brigo (2017) [2]. This is not needed or used here, however, as in this paper we use more direct techniques that allow us to avoid that complicated machinery and makes the result more immediate. The general result is in the same line of research of earlier contributions to behavioural finance, prospect theory and portfolio choice that we review now.

Expected utility maximization under risk measure and budget constraints was considered earlier in [4], but only under standard utility assumptions and no S-shaped utility in particular. In that paper it is shown that a market player who is forced by the VaR constraint to reduce portfolio losses in some states would finance these reduced losses by increasing portfolio losses in the costly states where the terminal state price density is large. As such states already have the lowest terminal portfolio value for the unconstrained problem, the VaR constraint ends up fattening the left tail of the terminal portfolio distribution. This leads to increased probability of extreme losses. In [7] it is shown that VaR constraints play a better role when, as is done in practice, the portfolio VaR is re-evaluated dynamically by incorporating available conditioning information. Again, this is done under standard utility and S-shaped utility is not considered.

Prospect theory has been studied in relation to risk measures and portfolio choice in a series of papers by Xunyu Zhou and co-authors [10, 13, 9, 8]. These papers tackle problems similar but not equivalent to the problem we consider here: papers [10, 9] do not study risk constraint in optimizing S-shaped utilities (with distortions), while in paper [8] the problem that is closest to our own is a problem where one optimizes expected returns but not expected utility. Still, these papers found the connections with rearrangements earlier, use law invariant portfolio optimization and use techniques and proofs that deal with and solve a wide spectrum of behavioural finance problems that are similar to the ones of this paper and of our general paper [2].
2 The Black Scholes market

We introduce briefly the Black-Scholes model [5, 12] for a market with a single risky asset and a bank account. We consider a probability space with a right continuous filtration $(\Omega, \mathcal{F}, (\mathcal{F}_t : 0 \leq t \leq T), \mathbb{P})$. In the given economy, two securities are traded continuously from time 0 until time $T$. The first security, the cash account or bank account, is locally risk free and its price $B_t$ evolves according to

$$
 dB_t = r B_t \, dt, \quad B_0 = 1, \quad \text{with solution} \quad B_t = e^{rt},
$$

where $r$ is a nonnegative number. Usually the risk-free rate $r$ is an $(\mathcal{F}_t)_{t \geq 0}$ adapted process but in this context we assume it to be a positive deterministic constant for simplicity.

As for the second security, given the $(\mathcal{F}_t, \mathbb{P})$-Wiener process $W_t$, consider the following stochastic differential equation for the price of such security, typically a stock,

$$
 dS_t = \mu S_t \, dt + \sigma S_t \, dW_t, \quad 0 \leq t \leq T,
$$

with initial condition $S_0 > 0$, and where $\mu$ and $\sigma$ are positive constants. Equation (2) has a unique (strong) solution which is given by

$$
 S_t = S_0 \exp \left\{ \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right\}, \quad 0 \leq t \leq T.
$$

We can write the probability law of $S_T$ easily by recalling that $W_t$ is normally distributed with mean zero and variance $t$. Let us call $F_t$ the cumulative distribution function (CDF) of $S_t$ under the measure $\mathbb{P}$, and $p_t$ the related density. We have a lognormal distribution for $S_t$.

$$
 F_t(y) = \mathbb{P}(S_t \leq y) = \mathbb{P} \left( W_t \leq \frac{\ln(y/S_0) - (\mu - \frac{1}{2} \sigma^2) t}{\sigma} \right) = \Phi \left( \frac{\ln(y/S_0) - (\mu - \frac{1}{2} \sigma^2) t}{\sigma \sqrt{t}} \right),
$$

where $\Phi$ is the CDF of the standard normal distribution.

Furthermore, in the basic Black and Scholes model there are no transaction costs; short selling is allowed without penalty or restrictions, and borrowing and lending occur at the risk free rate $r$, with no credit or default risk or funding costs.

The full development of valuation in presence of counterparty credit risk, liquidity risk, funding and capital costs has been addressed in the literature in recent years, see for example [6] for a work that tries to stay as close as possible to Black-Scholes while including credit, repo and funding effects.

Hence, even if the Black Scholes model neglects important aspects of valuation, here we will work under its assumptions since we are interested in s-shaped utility and ES limits in benchmark models. We expect our results to hold in extensions of the basic model, in particular in cases where the resulting valuation approach is very similar to the basic Black-Scholes setting, as in [6].
3 Simple claims and market price of risk

We consider a simple contingent claim. This is a contract guaranteeing a payoff of the form \( \phi(S_T) \) payable at maturity \( T \). If we assume that the market is arbitrage-free and complete, the unique no-arbitrage price of the simple claim above at time 0 is the expected value

\[
V_0 = \mathbb{E}^Q[e^{-rT}\phi(S_T)]
\]

where the expected value is taken under a probability measure \( Q \), the risk neutral measure, that is equivalent to \( P \) and under which

\[
dS_t = r S_t dt + \sigma S_t dW^Q_t
\]

where \( W^Q \) is a standard Brownian motion under \( Q \). It is immediate to check that the ratio \( S_t/B_t \) is a \( Q \)-martingale, and this is why sometimes \( Q \) is refereed to as the martingale measure. We can write the Radon-Nykodym derivative connecting the two equivalent measures as

\[
Z_t = \left. \frac{dQ}{dP} \right|_{F_t} = \exp \left( -\lambda W_t - \frac{1}{2} \lambda^2 t \right), \quad \lambda = \frac{\mu - r}{\sigma}.
\]

In the particular version of the Black Scholes setting we are working with, this formula can also be written as

\[
Z_t = \frac{\exp \left( -\frac{1}{2} \lambda^2 t + \lambda (\mu/\sigma - \sigma/2) t \right) (S_t)^{-\lambda/\sigma}}{S_0^{-\lambda/\sigma}} = g(S_t).
\]

The Radon Nykodym derivative \( Z_t \) becomes an explicit function of the underlying \( S_t \). This makes both the Radon-Nykodym derivative and the payoff functions of the same variable \( S_T \), and makes the analysis of their interaction explicit.

The constant \( \lambda \) is called market price of risk or a particular version of the Sharpe ratio. This has allowed us also to write down the Radon-Nykodym derivative as an explicit function of \( S \) for this market. We have

\[
\mathbb{E}^Q[e^{-rT}\phi(S_T)] = \mathbb{E}^P \left[ e^{-rT}\phi(S_T) \frac{dQ}{dp} \left. \right|_{F_T} \right] = \mathbb{E}^P \left[ e^{-rT}\phi(S_T)g(S_T) \right]. \tag{4}
\]

We will assume \( \lambda > 0 \), i.e. \( \mu > r \). This means that a trader would have interest in investing in the stock \( S \), since its expected return will exceed the risk free rate in the market.

4 Standardization to uniform risky asset

In our subsequent utility analysis, it will be helpful to re-scale the law of the risky asset to uniform. In the Black-Scholes context, this means that instead
of expressing simple claims as functions of $S_T$, we express them as functions of $X := F_T(S_T)$. Indeed, we know that $X$ has a standard uniform distribution. This allows us to write the price of the claim as

$$V_0 = \mathbb{E}^p \left[ e^{-rT} \phi(S_T)g(S_T) \right] = \mathbb{E}^p \left[ e^{-rT} \phi(F_T^{-1}(X))g(F_T^{-1}(X)) \right] = e^{-rT} \int_0^1 f(x)q(x)dx$$

where

$$f(x) = \phi(F_T^{-1}(x)), \quad q(x) = g(F_T^{-1}(x)).$$

Given our expressions above for $F_T$, $g$ and our assumption on $\phi$ and $\lambda$, it is immediate to check that $q$ is decreasing. Furthermore, a simple limit calculation based on the fact that $g(S)$ is essentially a power to the $-\lambda$ shows that

$$\lim_{x \to 0^+} q(x) = +\infty \text{ if } \lambda > 0.$$ 

5 S-shaped utility and tail-risk seeking behaviour

In [11], Kahneman & Tversky observed that individuals appear to have preferences governed by an S shaped utility function, $u$, that is increasing, strictly convex on the left, strictly concave on the right, non-differentiable at the origin, and asymmetrical: negative events are considered worse than positive events are considered good.

A typical S-shaped utility function is shown in Figure 1.

Whether the cause of S-shaped utility functions is irrationality or limited liability of market player, there is certainly good evidence that they are a useful
tool for modelling real world behaviour. A regulator or risk-manager could certainly consider them as a possibility.

Not all of the characteristics of S-shaped utility functions are important to us, so we adopt a slightly more general definition that we introduce now. We now suppose we have a candidate utility function as an increasing function $u(w)$ that represents the utility of holding the wealth $w$. Inspired by the definition of S-shaped utility in Kahneman and Tversky [11], we assume the following estimates holds:

$$u(w) \geq -c|w|^\eta \text{ for } w < N_R < 0, \quad \eta \in (0,1), \quad u(w) \leq k|w|^{\beta} \text{ for } w > N_I > 0, \quad \beta > 1$$

for some positive constants $c,k$. We call such a utility functions risk-seeking in the left tail and risk averse in the right tail.

In this work we finally define formally a S-shaped utility curve $u(w)$ as an increasing function that is negative for $w < 0$, positive for $w > 0$, concave for $w > 0$ and is risk-seeking in the left tail and risk-averse in the right tail.

6 Utility maximization under budget and ES constraints

We are now interested in a utility maximization problem in the Black-Scholes market. The trader or investor wishes to optimize over all simple claims $\phi$, to find the claim that gives her the maximum utility.

$$\sup_{\phi} \mathbb{E}^p[u(\phi(S_T))] \text{ under constraints}$$

$$\mathbb{E}^p[e^{-rT}\phi(S_T)g(S_T)] \leq C \text{ (budget),}$$

$$\text{ES}(p,T,\phi) \geq L_0 \text{ (ES constraint)}$$

where $\text{ES}(p,T,\phi)$ denotes the expected shortfall of $\phi(S_T)$ over the horizon $T$ at confidence level $p$. In this formulation of the problem we implicitly assume the additional constraint that the expectation and expected shortfall are both well defined and finite. We can reformulate the above problem after uniform re-scaling $X = F_T(S_T)$:

$$\sup_f \int_0^1 u(f(x))dx \text{ under constraints}$$

$$e^{-rT} \int_0^1 f(x)q(x)dx \leq C \text{ (budget),}$$

$$\frac{1}{p} \int_0^p f(x)dx \geq L_0 \text{ (ES constraint).}$$

Note that the expected shortfall representation in this last formulation comes from (3.3) in [1]. Again, we implicitly assume the constraints that the latter two integrals exist and are finite.
We will now show that, under the assumption $\lambda > 0$ (recall that $q$ depends on $\lambda$) the ES constraint is not relevant, in that the maximum attained under the constraint is the unconstrained $\sup_x u(f(x))$. Thus, for S-shaped utility functions that are risk seeking in the tail, as we expect from traders, the ES constraint is ineffective in curbing excessive risk taking.

First we could consider all constant functions $f$ and optimize on those. If we call $k$ the function constantly equal to $k$, the optimization problem is

$$\sup_k u(k) \text{ under constraints}$$

$$k \leq e^{rT} C \text{ (budget)},$$

$$k \geq L_0 \text{ (ES constraint)}.$$  (13)

We know that $u$ is increasing. Notice hence that if the limit of $u$ at $+\infty$ is infinite, then the supremum will be infinite as well.

So the first point we know is that, if $u^*$ is the optimal utility in the full problem, we have as lower bound the result for the “constant f” problem:

$$u^* \geq \sup_{y \in [L_0, e^{rT}C]} u(y).$$

It turns out it is enough to move to the next simplest possible function $f$, namely a two-step piecewise constant function, to obtain a much sharper result.

**Theorem 1** (Irrelevance of ES constraint in s-shaped utility maximization in a Black Scholes market). Consider a Black Scholes market where the bank account price $B$ and the risky asset (stock) price $S$ follow the differential equations

$$dB_t = r B_t \, dt, \quad B_0 = 1,$$

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t, \quad S_0$$

where $W$ is a standard Brownian motion under the measure $\mathbb{P}$, in a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, and $\sigma$ is a positive deterministic constant modeling the volatility. Assume $\lambda = (\mu - r)/\sigma > 0$, and assume we are given a s-shaped utility function $u$ that is risk seeking in the tail, namely there are constants $N$ and $\eta$ such that

$$u(w) \geq -c|w|^\eta \text{ for } w < N, \eta \in (0, 1).$$  (15)

Then one can show that for any claim $f$,

$$\left[ \sup_f \int_0^1 u(f(x)) \, dx \right] \text{ under constraints}$$

$$e^{-rT} \int_0^1 f(x)q(x) \, dx \leq C$$

$$\frac{1}{\rho} \int_0^p f(x) \, dx \geq L_0$$

$$= \sup_y u(y)$$
Proof. We will use non-decreasing piecewise constant functions taking only two values. Given three constants $k_1, k_2, \alpha$, with $0 < \alpha < p$ and $k_1 > k_2$, define
\[ f(x; k_1, k_2, \alpha) := k_2 1_{x < \alpha} + k_1 1_{x \geq \alpha}. \]
We will omit the arguments $k$ and $\alpha$ for brevity in this proof. Consider the expected utility for this function:
\[
\mathbb{E}^p[u(f(X))] = \int_0^1 u(f(x))dx = \alpha u(k_2) + (1 - \alpha)u(k_1). \tag{16}
\]
Let us now write the two constraints in the optimization problem: the budget constraint and the ES constraint. The budget constraint reads
\[
\int_0^\alpha f(x)q(x)dx + \int_\alpha^1 f(x)q(x)dx \leq e^{rT}C, \text{ or } k_2 \int_0^\alpha q(x)dx + k_1 \int_\alpha^1 q(x)dx \leq e^{rT}C.
\]
The ES constraint reads
\[
\frac{1}{p} \left[ \int_0^\alpha f(x)dx + \int_\alpha^p f(x)dx \right] \geq L_0 \text{ or } \frac{1}{p}[ak_2 + (p-\alpha)k_1] \geq L_0.
\]
Putting both constraints together we obtain
\[
\frac{pL_0 - (p-\alpha)k_1}{\alpha} \leq k_2 \leq \frac{Ce^{rT} - k_1 \int_\alpha^1 q(x)dx}{\int_0^\alpha q(x)dx}.
\]
In this constrained interval we now pick a special point, the initial point, thus assuming that the ES constraint is met as an equality:
\[ k_2 := \frac{pL_0 - (p-\alpha)k_1}{\alpha}. \]
We need to check that this constraint is consistent with our assumption that $k_1 > k_2$. This holds as long as $k_1$ is chosen sufficiently large and positive, namely $k_1 > (L_0)^+$. We denote the related function $f$ as
\[ \tilde{f}(x; k_1, \alpha) := f(x; k_1, (pL_0 - (p-\alpha)k_1)/\alpha, \alpha) \]
and the related expected utility (16) specializes to
\[
\mathbb{E}^p[u(\tilde{f}(X))] = \alpha u \left( \frac{pL_0 - (p-\alpha)k_1}{\alpha} \right) + (1 - \alpha)u(k_1). \tag{17}
\]
The budget constraint for $\tilde{f}$ becomes
\[
pL_0 - (p-\alpha)k_1 \leq \frac{\alpha}{\int_0^\alpha q(x)dx} \left( Ce^{rT} - k_1 \int_\alpha^1 q(x)dx \right). \tag{18}
\]
We can estimate the right hand side of the budget constraint (18) from below by noticing that

\[ \frac{\alpha}{\int_{\alpha}^{0} q(x)dx} \left( Ce^rT - k_1 \int_{\alpha}^{1} q(x)dx \right) > \frac{\alpha}{\int_{0}^{1} q(x)dx} \left( Ce^rT - k_1 \right) \]

If we now impose that \( Ce^rT < k_1 < M_1 \) for a sufficiently large positive constant \( M_1 \), then the right hand side becomes negative and we can estimate it further with

\[ \frac{\alpha}{\int_{0}^{1} q(x)dx} \left( Ce^rT - k_1 \right) > \frac{1}{q(\alpha)} \left( Ce^rT - k_1 \right) > \frac{1}{q(\alpha)} \left( Ce^rT - M_1 \right) \]

where the first estimate follows from \( q \) being decreasing, so that if we take \( \alpha \leq \varepsilon_\alpha \) for a positive \( \varepsilon_\alpha > 0 \), we conclude

\[ \frac{\alpha}{\int_{0}^{\alpha} q(x)dx} \left( Ce^rT - k_1 \right) > \frac{1}{q(\varepsilon_\alpha)} \left( Ce^rT - M_1 \right) . \]

The range for \( \varepsilon_\alpha \) will depend on \( M_1 \). On the other hand, we can estimate the left hand side of the budget constraint (18) from above by noticing that

\[ pL_0 - (p - \alpha)k_1 \leq pL_0 - (p - \varepsilon_\alpha)k_1 \]

and this can be made sufficiently negative, say \( < -1 \), by choosing \( k_1 \) large enough,

\[ k_1 \geq \frac{pL_0 + 1}{p - \varepsilon_\alpha} \]

where we are further assuming \( \varepsilon_\alpha < p \). We thus may write

\[ pL_0 - (p - \alpha)k_1 \leq pL_0 - (p - \varepsilon_\alpha)k_1 \leq -1 \quad \text{(19)} \]

\[ -1 \leq \frac{1}{q(\varepsilon_\alpha)} \left( Ce^rT - M_1 \right) \leq \frac{\alpha}{\int_{0}^{1} q(x)dx} \left( Ce^rT - k_1 \int_{\alpha}^{1} q(x)dx \right). \]

as long as we make sure that the central inequality

\[ -1 \leq \frac{1}{q(\varepsilon_\alpha)} \left( Ce^rT - M_1 \right) \]

holds. Since the denominator tends to plus infinity as \( \varepsilon \) tends to zero, it suffices, again, to choose \( \varepsilon_\alpha \) small enough.

We conclude that both the budget and the ES constraints are met if we require that \( k_1 \) and \( \alpha \) satisfy

\[ m_1 < k_1 < M_1, \quad \alpha < \varepsilon_\alpha \quad \text{(20)} \]

with the lower bound satisfying

\[ m_1 \geq \max \left( Ce^rT, L_0, \frac{pL_0 + 1}{p - \varepsilon_\alpha} \right). \]
with constants $M_1 > 0$ large enough and $\varepsilon_\alpha > 0$ small enough, with ranges determined by the other constants $C, r, T, L_0, \lambda$. We assume these to hold from now on.

We now go back to our expected utility (17). Given the estimate we obtained in (19), we can write

$$\frac{pL_0 - (p - \alpha)k_1}{\alpha} \leq \frac{-1}{\alpha} < \frac{-1}{\varepsilon_\alpha}. $$

We can now invoke inequality (15) for the utility function to deduce that

$$u \left( \frac{pL_0 - (p - \alpha)k_1}{\alpha} \right) \geq -c \left| \frac{pL_0 - (p - \alpha)k_1}{\alpha} \right|^{\eta}$$

as long as $\alpha \leq \varepsilon_\alpha$ with $\varepsilon_\alpha$ chosen sufficiently small. We can use this to estimate the expected utility in (17) as follows:

$$E^p[u(\bar{f}(X))] = \alpha u \left( \frac{pL_0 - (p - \alpha)k_1}{\alpha} \right) + (1 - \alpha) u(k_1) \geq \frac{1}{\alpha} \geq -\alpha c \left| \frac{pL_0 - (p - \alpha)k_1}{\alpha} \right|^{\eta} + (1 - \alpha) u(k_1)$$

Given that $0 < \eta < 1$, we can conclude that for $\alpha \to 0^+$ the first term on right hand side tends to zero and we are left with

$$\lim_{\alpha \to 0^+} E^p[u(\bar{f}(X; k_1, \alpha))] = u(k_1)$$

with the budget and ES constraints satisfied implicitly via (20) when taking the limit.

It follows that our general utility optimization problem has solution $u(k_1)$ when optimizing in the sub-class of admissible functions $\bar{f}$. It follows that the optimum on a larger class will yield an optimal expected utility larger than $u(k_1)$, for all possible $k_1$. On the other hand it is obvious that the optimal expected utility will be bounded from above by the supremum of the utility function. This then proves the claim that the optimal expected utility will be equal to the supremum of the utility function.

It may be interesting to check what the limiting function $x \mapsto \bar{f}(x; k_1, \alpha)$ looks like for small $\alpha$. Recall that

$$\bar{f}(x; k_1, \alpha) := \frac{pL_0 - (p - \alpha)k_1}{\alpha} \mathbb{1}_{\{x < \alpha\}} + k_1 \mathbb{1}_{\{x \geq \alpha\}} =: k_2(\alpha) \mathbb{1}_{\{x < \alpha\}} + k_1 \mathbb{1}_{\{x \geq \alpha\}}.$$

Fixing $k_1$, for very small $\alpha$ the first constant becomes negative and very large in absolute value, but on a very small interval $x \in [0, \alpha)$, whereas the second constant is equal to $k_1$ on a large interval $x \in [\alpha, 1]$. We thus have a digital option with an extremely negative constant payoff in a small range of the re-scaled uniform underlying, $[0, \alpha)$, and with a much smaller positive payoff in the remaining range $[\alpha, 1]$. This is illustrated in Figure 2.
7 An effective risk constraint based on a second concave utility

The focus of this short paper is the negative result above. However, we would like to hint at a possible solution for the ineffectiveness of the VaR and ES constraints, that is developed fully in Armstrong and Brigo (2017) [2].

Above we have seen a result related to the following fact: an investor with S-shaped utility function $u_I$ with $\lim_{x \to +\infty} u_I(x) = +\infty$ with a budget constraint and who is subject only to ES (or VaR) constraints for risk can find a sequence of portfolios satisfying these constraints whose expected $u_I$-utility tends to infinity. This implies that VaR or ES constraints cannot limit excessive tail risk seeking behaviour.

If the risk constraint is based instead on a second utility function of the type

$$u_R(x) = -(-x)^{\gamma_R}1_{\{x \leq 0\}}$$

with $\gamma_R > 1$, in that the constraint requires that the payoff $\phi$ is in the set

$$\{\phi : E[u_R(\phi(S_T))] \geq L\}$$

1We use the term utility function for $u_R$ somewhat informally here as what is important for our result is the functional form of the limit that is set and not whether or not it has been derived from any specific individual’s utility function. One might loosely think of $u_R$ as the regulator or risk manager’s utility function, although in practice the regulator or risk manager should choose any risk limits to reflect the risk preferences of whoever bears the risk. We have not attempted to consider how they should do this. We are simply assuming that one of the limits set has the given functional form.

Figure 2: Payoff used in the proof of the main theorem
for a negative loss level $L$, then any sequence of portfolios whose expected S-shaped utility $u_I$ tends to $+\infty$ will have expected $u_R$ tending to $-\infty$, and won’t thus be acceptable for the risk constraints. In this sense a classic concave utility $u_R$ adopted by the risk manager or regulator can be used more effectively than VaR or ES in limiting excessive tail risk seeking behaviour in presence of limited liability/s-shaped utility investors.

References


