A generalized approach to Economic Model Predictive Control with terminal penalty functions

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Abstract: In this paper, we first introduce upper and a lower bounds of the best asymptotic average performance for nonlinear control systems based on the concepts of dissipativity and control storage functions. This allows to extend the formulation and analysis of Economic Model Predictive Control to more general optimal operation regimes, such as periodic solutions. A performance and stability analysis is carried out within this generalized framework. Finally two examples are proposed and discussed to show the merits of the proposed approach.

Keywords: nonlinear predictive control, optimal control, asymptotic stabilization, performance analysis, Lyapunov methods

1. INTRODUCTION

Because of its ability of handling nonlinearity and system constraints, model predictive control (MPC) is becoming increasingly popular in industrial applications and process control, see examples in Qin and Badgwell (2003). This control paradigm normally relegates economic and profitability issues to the design of optimal set-points and suitable pointwise in time constraints. Real-time control is, instead, only concerned with the resulting tracking problem which is translated as an optimization problem over a finite time horizon and with an objective function which is (for the sake of stability) chosen to be positive definite with respect to some equilibrium of interest. In recent years, however, an alternative approach, economic MPC (EMPC), has looked into the issue of directly addressing economic optimization in real time, and to this end, adopts cost functionals which are not required to be positive definite with respect to the equilibrium point.

In this respect, various tools in literature have been proposed and studied in the economic optimization setup. In analogy to Mayne et al. (2000), where three ingredients are elaborated in stabilizing MPC, consisting of terminal cost, terminal constraint and local controller, similar tools have been proposed for Economic MPC and have allowed feasibility, stability and performance analysis of the closed-loop system. In Rawlings et al. (2012), Angeli et al. (2012) and Amrit et al. (2011), asymptotic stability of EMPC with terminal constraints or terminal costs has been proved by using a rotated stage cost in an auxiliary optimization problem, provided that a condition called strict dissipativity is satisfied. Moreover, in these papers, concepts on EMPC are extended to periodic terminal constraint and average constraints. In order to obtain a larger feasibility set, a new generalized terminal state constraint where the terminal state-input pair can be a free variable in optimization process is studied by Fagiano and Teel (2013). Based on the generalized terminal equality constraint, several update rules for the self-tuning terminal weight are illustrated in Müller et al. (2013). Furthermore, in Müller et al. (2014), the closed-loop asymptotic average performance bounds can be improved if the generalized terminal equality is relaxed by regional constraint.

However, optimal regimes of operation may have complex nature, periodic operation can outperform steady-state and even more general regimes of operations could sometimes arise. To deal with such instances, this work will remove terminal equality constraints and employ a suitable notion of "control storage function" (CSF) as the terminal penalty function. The note is organized as follows. Notation and problem setup are described in Section 2. Section 3 provides an estimate to some upper and lower bound for system asymptotic average performance. The extension of EMPC formulation and the closed-loop stability are discussed in Section 4. Two examples indicating the convergence to the best periodic solution are included in Section 5. Section 6 concludes this paper.

2. PRELIMINARIES AND SETUP

2.1 Notation

The Euclidean norm of x is $|x|$. Let symbols $\mathbb{R}$ and $\mathbb{I}$ denote the sets of real numbers and integers, respectively. $\mathbb{I}_{[a,b]}$ denotes the integers $\{a,\ a+1,\ \cdots,\ b\}$ and $\mathbb{I}_{\geq 0}$ denotes the non-negative integers. A continuous function α: $[0,\ \infty)$ $\to$ $[0,\ \infty)$ is of *class* $\mathcal{K}$, if it is zero at zero and strictly increasing. A continuous function $\rho : \mathbb{R}^n \to \mathbb{R}$ is positive definite with respect to some point $x_e \in \mathbb{R}^n$ if $\rho(x_e) = 0$ and $\rho(x) > 0$ for all $x \neq x_e$. The distance of a point $x \in \mathbb{R}_n$ to a set Π is denoted as $|x|_\Pi := \min_{z\in\Pi} |x - z|$.

2.2 Problem setup

We consider finite dimensional discrete-time nonlinear control systems described by difference equations

$$x^+ = f(x,u) \tag{1}$$

with state $x \in \mathbb{X} \subset \mathbb{R}^n$, input $u \in \mathbb{U} \subset \mathbb{R}^m$, and a continuous state transition map $f : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$. Together with system (1), let us consider a time-invariant, nonlinear, nonconvex, but continuous stage cost given as

$$\ell(x,u) : \mathbb{Z} \to \mathbb{R} \tag{2}$$

where $\mathbb{Z}$ is a compact set capturing the pointwise-in-time state and input constraints which our system is subject to:

$$(x(k), u(k)) \in \mathbb{Z} \qquad \forall k \in \mathbb{I}_{\geq 0}. \tag{3}$$

Our goal is to enhance profitability by minimizing the economic costs incurred in the long term system operation:

$$V(x, \mathbf{u}) = \sum_k \ell(x(k), u(k)), \quad x^+ = f(x,u), \quad x(0) = x. \tag{4}$$

To this end, we need to identify a viable subset of state space and corresponding control actions. As well known, the notion of control invariant set is crucial in this respect.

Definition 1. A control invariant set is a non-empty set $\overline{\mathbb{X}} \subseteq \mathbb{X}$, such that

$$\forall x \in \overline{\mathbb{X}}, \exists u : \quad f(x,u) \in \overline{\mathbb{X}} \text{ and } (x,u) \in \mathbb{Z}. \tag{5}$$

The non-empty set of all admissible control input which keeps the system state inside $\overline{\mathbb{X}}$ is denoted as:

$$\overline{\mathbb{U}}(x) := \{u \,|\, (x,u) \in \mathbb{Z} \text{ and } f(x,u) \in \overline{\mathbb{X}}\}. \tag{6}$$

The set of state and corresponding admissible input pairs is given as:

$$\overline{\mathbb{Z}} := \bigcup_{x \in \overline{\mathbb{X}}} [\{x\} \times \overline{\mathbb{U}}(x)]. \tag{7}$$

Remark 2. We consider the largest control invariant set $\overline{\mathbb{X}} \subseteq \mathbb{X}$. This contains all control invariant sets in $\mathbb{X}$ and any given intial condition $x(0) \notin \overline{\mathbb{X}}$ generates trajectories which violate system constraints (3) at some point in time. Therefore, constraints (3) can be strengthened as follows:

$$(x(k), u(k)) \in \overline{\mathbb{Z}} \qquad \forall k \in \mathbb{I}_{\geq 0}, \tag{8}$$

and viability is still guaranteed whenever the system is initialized in $\overline{\mathbb{X}}$.

Notice that investigating state trajectories within a control invariant set is also a fundamental step in standard MPC Lyapunov stability analysis.

It will be convenient to also define an additional control invariant set for later use as in the assumption below

Assumption 1. There exists a control invariant set $\mathbb{X}_f \subseteq \overline{\mathbb{X}}$.

The set of all admissible control law which keeps the system state inside $\mathbb{X}_f$ is defined for all $x \in \mathbb{X}_f$ as:

$$\mathbb{U}_f(x) := \{u \in \overline{\mathbb{U}}(x) \mid f(x,u) \in \mathbb{X}_f\}. \tag{9}$$

The set of state and corresponding admissible input pairs is given as:

$$\mathbb{Z}_f := \bigcup_{x \in \mathbb{X}_f} [\{x\} \times \mathbb{U}_f(x)]. \tag{10}$$

3. DISSIPATIVITY AND CONTROL STORAGE FUNCTIONS

In order to have a grasp of the system long-run optimal average performance, three quantities ℓ^*_{av}, $\underline{\ell}$ and $\overline{\ell}$, which are explicitly defined below, will be discussed in this Section.

Definition 3. Let $x \in \overline{\mathbb{X}}$ be a given initial state, then the best average asymptotic cost is defined as:

$$\ell^*_{av}(x) := \inf_{\substack{u(\cdot) \\ x(0)=x \\ x^+ = f(x,u) \\ (x(t),u(t)) \in \overline{\mathbb{Z}} \\ \forall t \in \mathbb{I}_{\geq 0}}} \liminf_{T \to +\infty} \frac{\sum_{t=0}^{T-1} \ell(x(t), u(t))}{T}. \tag{11}$$

Moreover, we denote by $\ell^*_{av} = \inf_{x \in \overline{\mathbb{X}}} \ell^*_{av}(x)$.

Recall the notion of dissipativity as given in *Definition 4.1* in Angeli et al. (2012),

Definition 4. A discrete time system is dissipative with respect to a supply rate $s : \overline{\mathbb{Z}} \to \mathbb{R}$ if there is a continuous storage function $\lambda : \overline{\mathbb{X}} \to \mathbb{R}$ such that:

$$\lambda(f(x,u)) - \lambda(x) \leq s(x,u) \tag{12}$$

for all $(x,u) \in \overline{\mathbb{Z}}$. If in addition a positive definite function $\rho : \overline{\mathbb{X}} \to \mathbb{R}_{\geq 0}$ exists such that:

$$\lambda(f(x,u)) - \lambda(x) \leq -\rho(x) + s(x,u), \tag{13}$$

then the system is said to be strictly dissipative.

Alternatively, given the role of dissipativity in providing lower bounds to the best asymptotic performance, one may consider the following quantity.

Definition 5. The tightest lower bound of ℓ^*_{av} is defined as:

$$\begin{aligned} \underline{\ell} := \sup_c \{\, c \,|\, & \exists\, \lambda(\cdot) : \overline{\mathbb{X}} \to \mathbb{R}, \text{continuous, such that} \\ & \lambda(f(x,u)) \leq \lambda(x) + \ell(x,u) - c, \forall (x,u) \in \overline{\mathbb{Z}}\}. \end{aligned} \tag{14}$$

Next, along the lines of the well known tool of Control Lyapunov Function (CLF) (see definition in Rawlings and Mayne (2009)), we propose a similar concept referred to as Control Storage Function (CSF).

Definition 6. A control storage function is a function $V_f : \mathbb{X}_f \to \mathbb{R}$ that is continuous and such that for all $x \in \mathbb{X}_f$

$$\inf_{u \in \mathbb{U}_f(x)} V_f(f(x,u)) - s(x,u) \leq V_f(x), \tag{15}$$

where $s : \mathbb{Z}_f \to \mathbb{R}$ is the supply rate.

From this definition, we can see that CSF is a special case of CSF, in which $s(x,u) = 0$. Since the CLF is frequently used to approximate the tail of the infinite horizon cost in tracking MPC (see Jadbabaie (2000) for instance), our CSF is meant to be an appropriate choice of terminal cost in an economic setup. This will be discussed in a later Section.

In order to estimate upper bounds for the best asymptotic performance, the quantity below can be specified,

Definition 7. The tightest upper bound of ℓ^*_{av} is defined as:

$$\overline{\ell} := \inf_c \{ c \,|\, \exists V_f : \mathbb{X}_f \to \mathbb{R}, \text{ such that } \forall x \in \mathbb{X}_f, \inf_{u \in \mathbb{U}_f(x)} V_f(f(x,u)) + \ell(x,u) \leq V_f(x) + c\}. \tag{16}$$

Remark 8. Notice that the above CSF inequality in *Definition 6* follows the same form of the Hamilton-Jacobi-Bellman (HJB) inequality in Bardi and Capuzzo-Dolcetta (2008), so any CSF can also be regarded as a solution of the HJB inequality, which is a value function of an infinite horizon optimal control problem.

We are now ready to state the main result of this Section:

Theorem 9. Consider system (1) subject to constraints (8), then, the following inequality holds:

$$\underline{\ell} \leq \ell^*_{av}(x), \quad \forall x \in \overline{\mathbb{X}}. \tag{17}$$

In addition, if *Assumption 1* is fulfilled, we have the following upper bound for $\ell^*_{av}(x)$:

$$\ell^*_{av}(x) \leq \overline{\ell}, \quad \forall x \in \mathbb{X}_f. \tag{18}$$

Proof. i) We first prove inequality $\underline{\ell} \leq \ell^*_{av}(x), \forall x \in \overline{\mathbb{X}}$.
Suppose system (1) is, for all $\epsilon > 0$, dissipative with supply rate $s(x,u) = \ell(x,u) - \ell_a$ where $\ell_a = \underline{\ell} - \epsilon$. Then, there exists a continuous storage function λ^ϵ such that for all $(x,u) \in \overline{\mathbb{Z}}$

$$\lambda^\epsilon(f(x,u)) \leq \lambda^\epsilon(x) + \ell(x,u) - \ell_a.$$

Next, for any time K, and any given feasible solution, it holds

$$\sum_{t=0}^{K-1} \lambda^\epsilon(x(t+1)) - \lambda^\epsilon(x(t)) \leq \sum_{t=0}^{K-1} (\ell(x(t),u(t)) - \ell_a).$$

By applying liminf on both sides, we obtain

$$\liminf_{K\to+\infty} \frac{\lambda^\epsilon(x(K)) - \lambda^\epsilon(x(0))}{K} \leq \liminf_{K\to+\infty} \frac{\sum_{t=0}^{K-1} \ell(x(t),u(t))}{K} - \ell_a.$$

Moreover, exploiting boundedness of solutions, we see that

$$\liminf_{K\to+\infty} \frac{\lambda^\epsilon(x(K)) - \lambda^\epsilon(x(0))}{K} = 0,$$

and therefore,

$$\ell_a \leq \liminf_{K\to+\infty} \frac{\sum_{t=0}^{K-1} \ell(x(t),u(t))}{K}.$$

Then, taking infimums with respect to $u(\cdot)$ for any fixed $x(0) = x \in \overline{\mathbb{X}}$, we can see that

$$\ell_a \leq \ell^*_{av}(x).$$

Since $\epsilon > 0$ was taken arbitrary to start with,

$$\underline{\ell} \leq \ell^*_{av}(x).$$

ii) Next, we prove $\ell^*_{av}(x) \leq \overline{\ell}, \forall x \in \mathbb{X}_f$.
Suppose system (1) admits continuous CSFs with supply rate $s(x,u) = \ell_b - \ell(x,u)$ where $\ell_b = \overline{\ell} + \epsilon$ for all $\epsilon > 0$. The corresponding inequality is:

$$\inf_{u\in\mathbb{U}_f(x)} V^\epsilon_f(f(x,u)) + \ell(x,u) \leq V^\epsilon_f(x) + \ell_b, \quad \forall x \in \mathbb{X}_f.$$

Next, let us consider any state trajectory starting from arbitrary initial state $x(0) = x \in \mathbb{X}_f$ with corresponding control input sequence defined as:

$$u(t) \in \operatorname*{argmin}_{u\in\mathbb{U}_f(x(t))} V^\epsilon_f(f(x(t),u)) + \ell(x(t),u),$$
$$x(t+1) = f(x(t),u(t)), \; t \in \mathbb{I}_{\geq 0}.$$

The state-input pair at any time instant is denoted as $(x(t),u(t)) \in \mathbb{X}_f \times \mathbb{U}_f(x(t)), \forall t \in \mathbb{I}_{\geq 0}$; then, it holds

$$\sum_{t=0}^{K-1} V^\epsilon_f(x(t+1)) - V^\epsilon_f(x(t)) \leq \sum_{t=0}^{K-1} (\ell_b - \ell(x(t),u(t))).$$

Dividing by K and applying limsup on both sides, we see that

$$\limsup_{K\to+\infty} \frac{V^\epsilon_f(x(K)) - V^\epsilon_f(x(0))}{K} \leq \ell_b - \liminf_{K\to+\infty} \frac{\sum_{t=0}^{K-1} \ell(x(t),u(t))}{K},$$

and exploiting boundedness of solutions and continuity of V^ϵ_f,

$$\liminf_{K\to\infty} \frac{\sum_{t=0}^{K-1} \ell(x(t),u(t))}{K} \leq \ell_b.$$

Then,

$$\ell^*_{av}(x) \leq \liminf_{K\to\infty} \frac{\sum_{t=0}^{K-1} \ell(x(t),u(t))}{K} \leq \ell_b,$$

and, since $\epsilon > 0$ was taken arbitrary to start with,

$$\ell^*_{av}(x) \leq \overline{\ell}.$$

Remark 10. If a system is optimally operated at an equilibrium point x_e with corresponding input u_e, then $\ell^*_{av} = \ell(x_e,u_e)$ and one may consider $s(x,u) = \ell(x,u) - \ell(x_e,u_e)$. This supply rate is used in Angeli et al. (2012) as a sufficient condition to prove Lyapunov stability of the equilibrium point.

When a system is controllable within finite time to the best optimal operation, every initial condition gives the same best asymptotic average performance, that is $\ell^*_{av}(x) = \ell^*_{av}$, $\forall x \in \overline{\mathbb{X}}$. However, when this is not the case, the strict inequality $\ell^*_{av}(x) > \ell^*_{av}$ may hold. This gap arising from uncontrollable systems will be illustrated in the following *Example* 5.2.

4. STABILITY OF PERIODIC OPTIMAL OPERATION

4.1 Economic MPC formulation

Now, we consider the EMPC problem for a given finite prediction horizon $N \in \mathbb{I}_{[1,+\infty)}$. The open loop EMPC problem can be formulated as

$$\begin{aligned} V^0_N(x,\mathbf{u}) = \min_{\mathbf{u}} & \sum_{k=0}^{N-1} \ell(x(k),u(k)) + V_f(x(N)) \\ s.t. \quad & x(k+1) = f(x(k),u(k)), \quad x(0) = x \\ & (x(k),u(k)) \in \overline{\mathbb{Z}} \qquad \forall k \in \mathbb{I}_{[0,N-1]} \\ & x(N) \in \mathbb{X}_f \end{aligned} \tag{19}$$

where $\mathbf{u} := \{u(0), u(1), \cdots, u(N-1)\}$ is the control sequence, V_f is the terminal penalty function and a CSF according to *Definition 6*. The admissible set $\mathbb{Z}_N \subseteq \overline{\mathbb{Z}}$ for $(x,\mathbf{u})$ pairs is

$$\mathbb{Z}_N := \{(x,\mathbf{u}) \,|\, \exists x(1), \cdots, x(N) : x^+ = f(x,u), x(0) = x, x(N) \in \mathbb{X}_f, (x(k),u(k)) \in \overline{\mathbb{Z}}, \; \forall k \in \mathbb{I}_{[0,N-1]}\}, \tag{20}$$

and the projection of $\mathbb{Z}_N$ onto $\overline{\mathbb{X}}$ is

$$\mathcal{X}_N := \{x \in \overline{\mathbb{X}} \,|\, \exists\, \mathbf{u} \text{ such that } (x,\mathbf{u}) \in \mathbb{Z}_N\}. \tag{21}$$

A classic assumption from literature is

Assumption 2. The economic optimization problem (19) admits a feasible solution in the non-empty set $\mathbb{Z}_N$.

If the above assumption holds, the optimal feedback policy from EMPC controller is defined as:

$$u = u^0(0, x), \qquad x \in \mathcal{X}_N, \tag{22}$$

where $u^0(k, x)$ denotes the k^{th} element of the optimal control sequence $\mathbf{u}$ at the given initial state x.

In order to analyse the stability of the closed-loop trajectories induced from optimal solutions of EMPC problem (19), we propose an assumption as below:

Assumption 3. The optimal control policy as in (22) is a continuous function of x in neighborhood of the optimal states.

4.2 Performance analysis

Considering the optimal cost-to-go function in (19), at any time instant t, the following inequality holds,

$$\begin{aligned} V_N^0(x(t+1)) =& V_N^0(x(t)) - V_f(x(t+N)) + V_f(x(t+N+1)) \\ &+ \ell(x(t+N), u_f(x(t+N))) \\ &- \ell(x(t), u^0(0, x(t))) \\ \leq& V_N^0(x(t)) - \ell(x(t), u^0(0, x(t))) + \overline{\ell}, \end{aligned} \tag{23}$$

where $u_f(\cdot)$ is the terminal control policy and fulfills

$$u_f(x) \in \underset{u \in \mathbb{U}_f(x)}{\operatorname{argmin}} \; V_f(f(x, u)) - s(x, u). \tag{24}$$

Then, for any time K, we have,

$$\sum_{t=0}^{K-1} V_N^0(x(t+1)) - V_N^0(x(t)) \leq \sum_{t=0}^{K-1} (\overline{\ell} - \ell(x(t), u^0(0, x(t)))). \tag{25}$$

By applying liminf on both sides and exploiting boundness of solutions, we finally obtain

$$\limsup_{K \to \infty} \frac{\sum_{t=0}^{K-1} \ell(x(t), u(t))}{K} \leq \overline{\ell}. \tag{26}$$

Therefore, the closed-loop performance of EMPC problem is no worse than $\overline{\ell}$.

4.3 Stability analysis

This sub-section explores the asymptotic stability of the closed-loop system under EMPC control actions.

In this context, we only consider the no gap case, viz:

Assumption 4. There is no gap between the tightest upper and lower bounds of ℓ_{av}^*:

$$\underline{\ell} = \ell_{av}^* = \ell_{av}^*(x) = \overline{\ell}, \quad \forall x \in \overline{\mathbb{X}}. \tag{27}$$

Moreover, there exists a storage function $\lambda(\cdot)$ and a control storage function $V_f(\cdot)$ for which inequalities (13) and (15) are achieved with supply rates $\ell(x, u) - \ell_{av}^*$ and $\ell_{av}^* - \ell(x, u)$, respectively.

The rotated stage cost and terminal cost are defined as:

$$L(x, u) := \ell(x, u) + \lambda(x) - \lambda(f(x, u)) - \underline{\ell}, \tag{28}$$

$$\overline{V}_f(x) := V_f(x) + \lambda(x), \; \forall x \in \mathbb{X}_f, \tag{29}$$

then, the new objective function is

$$\overline{V}_N(x, \mathbf{u}) := \sum_{k=0}^{N-1} L(x(k), u(k)) + \overline{V}_f(x(N)). \tag{30}$$

It is noticed that the strict dissipation inequality (13) with supply rate $s(x, u) = \ell(x, u) - \underline{\ell}$ implies that the modified stage cost is positive definite with respect to some closed subset of state space $\overline{\mathbb{X}}$, denoted by $\Pi_{\overline{\mathbb{X}}}$. Thus it can be bounded from below by a *class* $\mathcal{K}$ function

$$L(x, u) \geq \alpha(|x|_{\Pi_{\overline{\mathbb{X}}}}) \geq 0. \tag{31}$$

We recall next a Lemma that was first introduced in Amrit et al. (2011),

Lemma 11. For every function $\lambda(\cdot)$ and constant $\underline{\ell} \in \mathbb{R}$, the solutions of optimization problem (19) and the one by using (30) as the objective function are identical.

Proof. The proof is similar to the proof of *Lemma 14* in Amrit et al. (2011).

Lemma 12. If *Assumption 4* holds, then for all $x \in \mathbb{X}_f$, the CSF inequality in (15) with $s(x, u) = \overline{\ell} - \ell(x, u)$ holds if and only if the following inequality is satisfied

$$\overline{V}_f(f(x, u_f(x))) - \overline{V}_f(x) \leq -L(x, u_f(x)). \tag{32}$$

Proof. The proof follows the same lines as that of *Lemma 9* in Amrit et al. (2011).

Then, the main result on asymptotic stability is given as:

Theorem 13. Let *Assumption 1, 2 and 4* hold, the set $\Pi_{\overline{\mathbb{X}}}$ by using control policy (22), is asymptotically stable within a region of attraction $\mathcal{X}_N$.

Proof. The proof is similar to the corresponding part of the proof of *Theorem 15* in Amrit et al. (2011).

4.4 Extension to periodic solutions

This sub-section introduces suitable terminal ingredients to specialize the stability analysis of EMPC to the case of systems with a periodic optimal regime of operation.

Denote the optimal period-T solution and associated control of the optimal control problem (4) on $\overline{\mathbb{Z}}$ as $\Pi := \{(x_i^*, u_i^*),\ i \in \mathbb{I}_{[0,T-1]}\}$ for some $T \in \mathbb{I}_{[1,+\infty)}$. For convenience, we denote the projection of Π on $\overline{\mathbb{X}}$ as $\Pi_{\overline{\mathbb{X}}}$, and we simply consider $i \in \mathbb{I}_{[0,T-1]}$ and use $i + j$ to respresent $(i + j) \bmod T$, $\forall j \in \mathbb{I}$.

The system cost over a time period T achieves its minimum at Π, viz.

$$\sum_{i=0}^{T-1} \ell(x_i^*, u_i^*) \leq \sum_{i=0}^{T-1} \ell(x_i, u_i), \forall (x_i, u_i) \in \overline{\mathbb{Z}}, x_{i+1} = f(x_i, u_i). \tag{33}$$

For every state $x_i^* \in \Pi_{\overline{\mathbb{X}}}$, three important terminal ingredients, as discussed on optimal steady state operation in Amrit et al. (2011), are adopted. A terminal region, containing x_i^*, is denoted by $\mathbb{X}_f^i$, with the corresponding terminal control policy and penalty functions denoted by $u_f^i(x)$ and $V_f^i(x)$, respectively.

Let us first make an assumption on the relationship between $\mathbb{X}_f^i$ and $u_f^i(x)$:

Assumption 5. There exists a family of compact sets $\mathbb{X}_f^i$, $i \in \mathbb{I}_{[0,T-1]}$, each containing x_i^*, such that for all $x \in \mathbb{X}_f^i$, the solutions of $x(t+1) = f(x(t), u_f^{i+t}(x(t)))$ converge exponentially to $\Pi_{\overline{\mathbb{X}}}$.

Accordingly, the feedback law $u_f^i(x)$ which drives any state $x \in \mathbb{X}_f^i$ to $\mathbb{X}_f^{i+1}$ takes values in:

$$\mathbb{U}_f^i(x) := \{u \in \overline{\mathbb{U}}(x) \,|\, f(x,u) \in \mathbb{X}_f^{i+1}\}. \tag{34}$$

Then, the i^{th} set of feasible state-input pairs is given as:

$$\mathbb{Z}_f^i := \bigcup_{x \in \mathbb{X}_f^i} [\{x\} \times \mathbb{U}_f^i(x)], \tag{35}$$

and we let

$$\mathbb{X}_f = \bigcup_i \mathbb{X}_f^i, \;\; \mathbb{U}_f = \bigcup_i \mathbb{U}_f^i(x). \tag{36}$$

With regards to the terminal penalty function $V_f^i(x)$ in period-T optimal operation, the concept of CSF in *Definition 6* is adapted to give us a T-CSF as follows:

Definition 14. A T-CSF is a family of continuous functions $V_f^i : \mathbb{X}_f^i \to \mathbb{R}, i \in \mathbb{I}_{[0,T-1]}$ and the following holds

$$\min_{u \in \mathbb{U}_f^i(x)} V_f^{i+1}(f(x,u)) - s(x,u) \leq V_f^i(x), \quad \forall x \in \mathbb{X}_f^i, \tag{37}$$

where $s : \bigcup_{i \in \mathbb{I}_{[0,T-1]}} \mathbb{Z}_f^i \to \mathbb{R}$ is the supply rate.

Notice that $V_f^i(x)$ is the i^{th} component of a T-CSF defined for all $x \in \mathbb{X}_f^i$. We can adopt the following convention

Assumption 6. $V_f^i(x)$ is a finite-valued function if $x \in \mathbb{X}_f^i$, whereas $V_f^i(x) := +\infty, \forall x \notin \mathbb{X}_f^i$.

Then, for any point $x \in \mathbb{X}_f$ a single terminal penalty function can be defined as:

$$V_f(x) := \min_i V_f^i(x). \tag{38}$$

Lemma 15. The minimum of a T-CSF as in (37) and (38) is a suitable CSF.

Proof. Denote the T-CSF by $\{V_f^0, V_f^1, \cdots, V_f^{T-1}\}$, and let $x \in \mathbb{X}_f$ be arbitrary. Then for any $i^* \in \underset{i}{\operatorname{argmin}}\, V_f^i(x)$, $\forall x \in \mathbb{X}_f^{i^*}$, the following inequality is satisfied,

$$V_f^{i^*+1}(f(x, u_f^{i^*}(x))) \leq V_f^{i^*}(x) + s(x, u_f^{i^*}(x))$$

hence, from (38) and the above inequality,

$$\begin{aligned} V_f(f(x, u_f^{i^*}(x))) &\leq V_f^{i^*+1}(f(x, u_f^{i^*}(x))) \\ &\leq V_f^{i^*}(x) + s(x, u_f^{i^*}(x)) \\ &= V_f(x) + s(x, u_f^{i^*}(x)). \end{aligned}$$

Therefore, there exists an admissible control policy $u_f^{i^*}(x) \in \mathbb{U}_f^{i^*}(x) \subseteq \mathbb{U}_f(x)$ such that

$$\inf_{u \in \mathbb{U}_f^{i^*}(x) \subseteq \mathbb{U}_f(x)} V_f(f(x,u)) - s(x,u) \leq V_f(x),$$

and $V_f(x)$ defined in (38) is a CSF.

Then, according to *Theorem 13*, the optimal periodic solution is orbitally asymptotically stable.

Remark 16. Notice that our notion of dissipativity, which is motivated by the continuous-time results proposed in L. Finlay and Lebedev (2008), is different from the three extended notions of dissipativity for periodic systems defined in Grüne and Zanon (2014). Exploring the link of our concept respect to those extensions and analysing the case in which *Assumption 4* does not hold is an interesting question for future investigations.

Lemma 17. Suppose a continuous automomous system $x^+ = f(x)$ has a periodic solution. Then this periodic solution is Lyapunov asymptotically stable if and only if it is orbitally asymptotically stable.

Proof. Let us denote $\overline{\Pi} := \{x_0^*, \cdots, x_{T-1}^*\}$ as the optimal T-period solution fulfilling $x_{i+1}^* = f(x_i^*, u_i^*), \forall i \in \mathbb{I}_{[0,T-2]}$ and $x_0^* = f(x_{T-1}^*, u_{T-1}^*)$. Suppose the state at any time $t \in \mathbb{I}_{\geq 0}$ with x as the initial condition is denoted as $\varphi(t,x)$. The optimal state at $t=0$ is assigned as x_0^* without loss of generality.
Since the periodic solution is orbitally stable, there exists a Lyapunov function $V(x)$ fulfilling
$\hat{\alpha}_1(|x|_{\overline{\Pi}}) \leq V(x) \leq \hat{\alpha}_2(|x|_{\overline{\Pi}})$, $V(f(x)) \leq V(x) - \hat{\alpha}_3(|x|_{\overline{\Pi}})$
where $\hat{\alpha}_1(\cdot), \hat{\alpha}_2(\cdot)$ and $\hat{\alpha}_3(\cdot)$ are *class* $\mathcal{K}$ functions.

We select $\varepsilon > 0$ and $\overline{\mathbb{X}}_f^i$ for every $i \in \mathbb{I}_{[0,T-1]}$ as the connected components of $\overline{\mathbb{X}}_f := \{x \,|\, V(x) \leq \varepsilon\}$ which contains x_i^*. Then, $\overline{\mathbb{X}}_f = \bigcup_{i=\{0,\cdots,T-1\}} \overline{\mathbb{X}}_f^i$ is forward invariant. Moreover, for $\varepsilon > 0$ sufficiently small, $\overline{\mathbb{X}}_f^i \cap \overline{\mathbb{X}}_f^j = \emptyset$ if $i \neq j$, and in addition we have

$$\underset{k}{\operatorname{argmin}}\, |x - x_k^*| = \{i\}, \; \forall x \in \overline{\mathbb{X}}_f^i.$$

Notice that, by *Proposition 4.6* in Bredon (2013), $f(\overline{\mathbb{X}}_f^i)$ is connected. Hence, there exist j such that $f(\overline{\mathbb{X}}_f^i) \subseteq \overline{\mathbb{X}}_f^j$. Since $f(x_i^*) = x_{i+1}^*$, we concluded that $f(\overline{\mathbb{X}}_f^i) \subseteq \overline{\mathbb{X}}_f^{i+1}$. Therefore, arguing by induction, we see that solutions initiated within $\overline{\mathbb{X}}_f^0$ will evolve in phase with the optimal periodic solution, that is $\varphi(t,x) \in \overline{\mathbb{X}}_f^{t \bmod T}, \forall x \in \overline{\mathbb{X}}_f^0$.

Thus, for any $x \in \overline{\mathbb{X}}_f^0$,

$$\underset{k}{\operatorname{argmin}}\, |\varphi(t,x) - x_k^*| = \{t \bmod T\},$$

or equivalently,

$$\min_k |\varphi(t,x) - x_k^*| = |\varphi(t,x) - \varphi(t, x_0^*)|,$$

since

$$\varphi(t, x_0^*) = x_{t \bmod T}^*.$$

By substituting the above equality into the following definition of orbital asymptotic stability:

$$\forall \epsilon_1 > 0, \; \exists \delta_1 > 0, \text{ such that}$$
$$\min_{k \in \mathbb{I}_{[0,T-1]}} |x - x_k^*| \leq \delta_1 \Rightarrow \min_{k \in \mathbb{I}_{[0,T-1]}} |\varphi(t,x) - x_k^*| \leq \epsilon_1 \quad \forall t \in \mathbb{I}_{\geq 0},$$

and

$$\exists \delta_2 > 0, \; \forall x \text{ such that } \min_{k \in \mathbb{I}_{[0,T-1]}} |x - x_k^*| \leq \delta_2$$
$$\text{it holds } \lim_{t \to \infty} \min_{k \in \mathbb{I}_{[0,T-1]}} |\varphi(t,x) - x_k^*| = 0,$$

we obtain the conditions for Lyapunov asymptotic stability as follows

$$\forall \epsilon_1 > 0, \exists \delta_1 > 0, \text{ such that}$$
$$|x - x_0^*| \leq \delta_1 \Rightarrow |\varphi(t,x) - \varphi(t, x_0^*)| \leq \epsilon_1 \; \forall t \in \mathbb{I}_{\geq 0}.$$

and

$$\exists\delta_2 > 0,\ \forall x \text{ such that } |x - x_0^*| \le \delta_2$$
$$\text{it holds } \lim_{t\to\infty} |\varphi(t,x) - \varphi(t,x_0^*)| = 0.$$

Hence, the conclusions of the lemma follow.

Now, it is sufficient to have a statement on Lyapunov stability of the optimal periodic solution.

Corollary 18. If *Assumption 3* holds, the optimal periodic solution is orbitally asymptotically stable and asymptotically stable in the sense of Lyapunov.

4.5 Construction of terminal penalty function

We propose next a constructive formula for computing T-CSF. In particular,

$$\overline{V}^i_f(x) = \sum_{k=0}^{\infty} L(x^i(k), u_f^{i+k}(x^i(k))). \tag{39}$$

where $x^i(k)$ is the solution of $x^i(k+1) = f(x^i(x), u_f^{i+k}(x^i(k)))$ initiated at $x^i(0) = x$ if $x \in \mathbb{X}_f^i$.

To analyze the convergence of (39), we consider the dissipation inequalities along the optimal periodic solution,

$$\lambda(x_1^*) - \lambda(x_0^*) \le \ell(x_0^*, u_0^*) - \underline{\ell}, \tag{40.1}$$
$$\lambda(x_2^*) - \lambda(x_1^*) \le \ell(x_1^*, u_1^*) - \underline{\ell}, \tag{40.2}$$
$$\vdots$$
$$\lambda(x_0^*) - \lambda(x_{T-1}^*) \le \ell(x_{T-1}^*, u_{T-1}^*) - \underline{\ell}. \tag{40.T}$$

Summing both sides of (40.2)-(40.T) and using the definition of $\ell^*_{av}(x)$ and (27), we have

$$\lambda(x_1^*) - \lambda(x_0^*) \ge \ell(x_0^*, u_0^*) - \underline{\ell}. \tag{41}$$

Comparing (40.1) and (41), we conclude

$$\lambda(x_1^*) - \lambda(x_0^*) = \ell(x_0^*, u_0^*) - \underline{\ell}. \tag{42}$$

Using the same technique, the following equalities hold

$$\lambda(x_{i+1}^*) - \lambda(x_i^*) = \ell(x_i^*, u_i^*) - \underline{\ell}. \tag{43}$$

Equivalently, the rotated stage cost at the periodic solution is

$$L(x_i^*, u_i^*) = 0 \quad \forall i \in \mathbb{I}_{[0,T-1]} \tag{44}$$

Assumption 7. There exists a $\delta_L > 0$ such that the functions $\ell(\cdot)$ and $\lambda(\cdot)$, and hence the rotated stage cost $L(\cdot)$ are Lipschitz continuous for all (x,u) fulfilling $|(x,u)|_\Pi < \delta_L$. Then, the corresponding Lipschitz constants are denoted by L_ℓ, L_λ and L_L. Moreover, for all $i \in \mathbb{I}_{[0,T-1]}$, $u_f^i(x)$ is Lipschitz continuous with respect to x.

Since *Assumption 5* implies that there exists constant real number A and $|a| < 1$ such that $|(x(k), u(k))|_\Pi \le A \cdot |a|^k$. Together with *Assumption 7* and (44), we conclude that $|L(x(k), u(k))| \le L_L \cdot A \cdot |a|^k$, and therefore the T-CSF in (39) is upper bounded by $\frac{L_L \cdot A}{1-a}$.

Notice that, provided all the series converge for arbitrary $i \in \mathbb{I}_{[0,T-1]}$, it is straightforward to have

$$\begin{aligned}\overline{V}^i_f(x) &= L(x, u_f^i(x)) + \sum_{k=1}^{\infty} L(x^i(k), u_f^{i+k}(x^i(k))) \\ &= L(x, u_f^i(x)) + \sum_{k=0}^{\infty} L(x^{i+1}(k), u_f^{i+k}(x^{i+1}(k))) \\ &= L(x, u_f^i(x)) + \overline{V}_f^{i+1}(f(x, u^i(x))).\end{aligned} \tag{45}$$

Thus, the proposed T-CSF follows *Lemma 12* and the terminal cost function can be selected as

$$\overline{V}_f(x) = \min_i \overline{V}^i_f(x). \tag{46}$$

Moreover, knowledge of the storage function $\lambda(\cdot)$ used in the definition of rotated stage and terminal costs is not needed in designing the EMPC controller. To see this, we consider the terminal cost

$$V_f(x) = \min_i V_f^i(x), \tag{47}$$

where

$$\begin{aligned}V_f^i(x) &= \overline{V}^i_f(x) - \lambda(x) \\ &= \sum_{k=0}^{\infty} L(x^i(k), u_f^{i+k}(x^i(k))) \ - \lambda(x) \\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{T-1} (\ell(x^i(mT+k), u_f^{i+k}(x^i(mT+k))) - \underline{\ell}).\end{aligned} \tag{48}$$

4.6 SOSTOOLs for storage function

The MATLAB toolbox SOSTOOLs which can be used to determine a candidate for $\lambda(\cdot)$ in polynomial form was proposed in Prajna et al. (2002). SOSTOOLs is a free MATLAB toolbox which can be used to solve two types of sum of squares programs: the feasibility and optimization problems. This toolbox has been applied to many control problems, such as construction of Lyapunov function, state feedback control synthesis, and nonlinear optimal control Parrilo (2000); Prajna et al. (2004).

In this paper, in order to find the construction of storage function, we consider the optimization problem as follows:

$$\min_{\underline{\ell}, c_0, c_1, \cdots, c_P} \quad \underline{\ell} \tag{49}$$

such that

$$\lambda(x) = \sum_{i=0}^{P} c_i x^i \tag{50.1}$$

$$\lambda(x) - \lambda(f(x,u)) + \ell(x,u) - \underline{\ell} \text{ is sum of squares} (\ge 0),$$
$$\forall (x,u) \in \overline{\mathbb{Z}}. \tag{50.2}$$

In this formulation, x and u are independent variables which are created as symbolic variables in MATLAB, whereas $\underline{\ell}$, c_i are decision variables.

The objective in (49) is to minimize the lower bound of asymptotic average. (50.1) shows the construction of the storage function that is formulated as a polynomial of order P. In addition, (50.2) is from the dissipation inequality or the rotated stage cost, this inequality must hold for all admissible state-input pairs.

By using the returned values of coefficients, the storage function can be formulated as a polynomial function of the system state.

5. EXAMPLES

5.1 No gap: $\overline{\ell} = \underline{\ell}$ - optimal period-2 operation

Consider the nonlinear system, known as logistic map or demographic model, described by the following difference euqation

$$x^+ = ux(1-x) \tag{51}$$

in which x is the ratio between current population and the maximum possible population with value in [0, 1] and a parameter u in interval (0, 4]. To avoid the trivial case $x = 0$, we only consider the robust control invariant set $\overline{\mathbb{X}} := [\epsilon, 1-\epsilon]$, $\bigcap_{x\in\overline{\mathbb{X}}} \overline{\mathbb{U}}(x) := [\frac{1}{1-\epsilon}, 4(1-\epsilon)]$ and $\overline{\mathbb{Z}} = \overline{\mathbb{X}} \times \bigcap_{x\in\overline{\mathbb{X}}} \overline{\mathbb{U}}(x)$, where ϵ is a small positive value.

The following stage cost

$$\ell(x, u) = -x^4 \tag{52}$$

results in an optimal periodic regime of operation which can be expressed as

$$x^*_{0,1} = \frac{(5-4\epsilon) \mp \sqrt{(1-4\epsilon)(5-4\epsilon)}}{8(1-\epsilon)}, \quad u^*_{0,1} = 4(1-\epsilon). \tag{53}$$

Particularly, when $\epsilon = 0.01$, the optimal periodic operation is $x^*_0 \approx 0.3507$ and $x^*_1 \approx 0.9018$ with optimal control $u^*_0 = u^*_1 = 3.96$, so that the best asymptotic average cost for all initial states $x \in \overline{\mathbb{X}}$ is

$$\underline{\ell} = \ell^*_{av} = \ell^*_{av}(x) = -\frac{(x^*_0)^4 + (x^*_1)^4}{2} = -0.3382. \tag{54}$$

To construct the terminal cost function, a terminal state feedback control law is needed

$$u^i_f(x) = u^*_i + K_i(x - x^*_i), \; i \in \mathbb{I}_{[0,1]}, \tag{55}$$

where K_i is determined from the linearized system of (51) along the optimal solution that is

$$\begin{aligned} \delta x^+_0 &= -3.1821\delta x_1 + 0.0886\delta u_1 \\ \delta x^+_1 &= 1.1821\delta x_0 + 0.2277\delta u_0 \end{aligned} \tag{56}$$

Correspondingly, constraints of this linearized system are

$$\begin{aligned} &-0.3407 \le \delta x_0 \le 0.6393, \quad -2.9499 \le \delta u_0 \le 0, \\ &-0.8918 \le \delta x_1 \le 0.0882, \quad -2.9499 \le \delta u_1 \le 0. \end{aligned} \tag{57}$$

Since there is no $[K_0\ K_1]^T$ being able to stabilize system (56) within the whole region in (57), we consider maximizing the region of attraction of the linearized system by solving a bilinear matrix inequality problem based on the method in Vassilaki et al. (2004). The resulting feedback gain is $[K_0\ K_1]^T = [-5.2\ 10.75]^T$ which is able to stabilize the linear system for $0 \le \delta x_0 \le 0.5681$ and $-0.2606 \le \delta x_1 \le 0$.

Next, we determine a potentially smaller set where the above control policy works for the nonlinear system (51). If the terminal state feedback control law is implemented into (51), the dynamic of state deviation becomes

$$\begin{aligned} \delta x^+_0 &= -10.75\,\delta x^3_1 - 12.6\,\delta x^2_1 - 2.23\,\delta x_1, \\ \delta x^+_1 &= 5.2\,\delta x^3_0 - 5.513\,\delta x^2_0 - 0.0016\,\delta x_0. \end{aligned} \tag{58}$$

Trajectories described by these two equations admit a small region $0 \le \delta x_0 \le 0.22$ and $-0.21 \le \delta x_1 \le 0$ such that any interior point can be attracted to the origin without constraints violation. Therefore, the terminal region is $\mathbb{X}_f = \mathbb{X}^0_f \cup \mathbb{X}^1_f = [0.3507, 0.5707] \cup [0.6918, 0.9018]$.

Then, according to the equations in (47) and (48), it is sufficient to construct the terminal cost function as follows

$$V_f(x) = \min_i V^i_f(x), \tag{59}$$

where

$$V^i_f(x) = \sum_{m=0}^{\infty} \sum_{k=0}^{T-1} (\ell(x^i(mT+k), u^{i+k}_f(x^i(mT+k))) - \underline{\ell}),$$

$$u^{i+k}_f(x^i(mT+k)) = u^*_{i+k} + K_{i+k}(x^i(mT+k) - x^*_{i+k}), \tag{60}$$

Moreover, by using SOSTOOLs, a candidate polynomial storage function of 3rd order which fulfills dissipativity approximately is

$$\lambda(x) = 0.30471x^3 - 0.81183x^2 + 1.2215x. \tag{61}$$

Fig.1 shows the closed loop state transition and input at initial condition $x = 0.5$ and prediction horizon $N = 6$. It can be seen that the state trajectory converges to the optimal periodic solution in several steps with the control input remaining at its upper bound. It also shows the convergence of asymptotic average performance to $\overline{\ell} = \underline{\ell}$ and the decreasing of optimal cost-to-go which is a candidate of Lyapunov function.

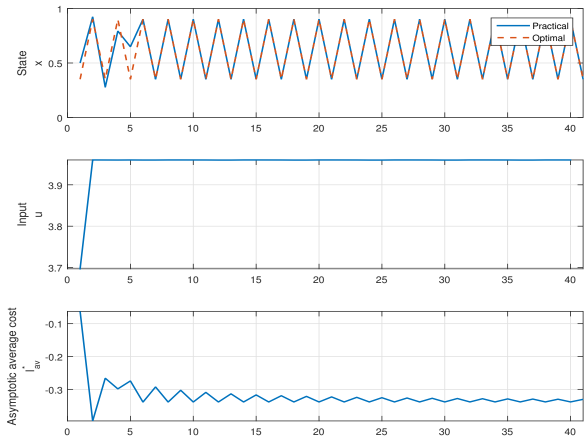

Fig. 1. Closed-loop behaviour of the logistic map system

5.2 *With gap: $\overline{\ell} > \underline{\ell}$ - optimal period-4 operation*

Next, we consider a bidimensional nonlinear system of equations:

$$x^+ = \begin{bmatrix} \frac{1-u^2}{1+u^2} & \frac{2u}{1+u^2} \\ \frac{-2u}{1+u^2} & \frac{1-u^2}{1+u^2} \end{bmatrix} x \tag{62}$$

where x is the state variable and u is the input. This nonlinear system gives a state trajectory which rotates on a circle with radius that is equal to the magnitude of the initial state. Thus, only states on this circle are reachable.

Consider the state space $\mathbb{X} := \{|x| \le 1\}$ and input constraint $u \in \mathbb{U} := (-\infty, 0]$, a stage cost

$$\ell(x, u) = -(x^T Q x)^2 + (u+1)^2, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{63}$$

results into the global optimal periodic solution $\overline{x}^*_0 = [1, 0]^T$, $\overline{x}^*_1 = [0, 1]^T$, $\overline{x}^*_2 = [-1, 0]^T$ and $\overline{x}^*_3 = [0, -1]^T$.

However, due to the structure of reachable sets, any given initial condition with distance to the origin $|x| = r$, $0 < r < 1$ moves within a control invariant set $\overline{\mathbb{X}} := \{x \,|\, |x| = r\}$. Then, a sub-optimal periodic operation is $x^*_0 = [r, 0]^T$, $x^*_1 = [0, r]^T$, $x^*_2 = [-r, 0]^T$ and $x^*_3 = [0, -r]^T$ with optimal control $u^*_0 = u^*_1 = u^*_2 = u^*_3 = -1$. Thus, there is a gap between ℓ^*_{av} and $\underline{\ell}$,

$$\underline{\ell} = -1 \le \ell^*_{av}(x) = -|x|^4. \tag{64}$$

In terms of terminal control policy, it is expected to be a continuous function of state x. Therefore, we consider $u_f(x)$, $x \in \mathbb{X}_f = \overline{\mathbb{X}} \setminus \{\frac{(2k+1)\pi}{4}\}, k \in \mathbb{R}$ which forces the system to follow the dynamic in polar coordinate,

$$\theta^+ = \theta + \frac{\pi}{2} - K\sin(4\theta), \tag{65}$$

where θ is the angle of the state vector with respect to the positive horizontal axis and K is the state feedback gain. Since the optimal solution evolves $\frac{\pi}{2}$ radius counter-clockwise every step, $K\sin(4\theta)$ is used to reduce the bias between the practical state to the optimal ones. To keep in phase with the rotation of periodic solution, K should be selected from $(0, \frac{1}{4}]$. In this example, $K = \frac{1}{4}$ for faster convergence.

Then, with $u_f(\cdot)$ defined above, the terminal cost function for $x \in \mathbb{X}_f$ is

$$V_f(x) = \sum_{m=0}^{\infty}\sum_{k=0}^{T-1}(\ell(x(mT+k), u_f(x(mT+k))) - \underline{\ell}). \tag{66}$$

Notice that, this terminal cost function will only be finite for x in $\{x|\ |x| = 1\}$ and Lyapunov asymptotic stability discussed in Section 4 applies for $\{x|\ |x| = 1\}$.

Furthermore, a candidate of storage function is

$$\lambda(x) = (x^T Q x)^2. \tag{67}$$

The state trajectory initialized at $x = [0.48\ 0.64]^T$ and with a prediction horizon $N = 6$ is shown in Fig.2 in which we see that closed-loop system catches the periodic solution after several control moves. The system average performance for this cost objectives convergences to $\ell_{av}^*(x)$ = -0.4096 instead of $\underline{\ell}$ = -1.

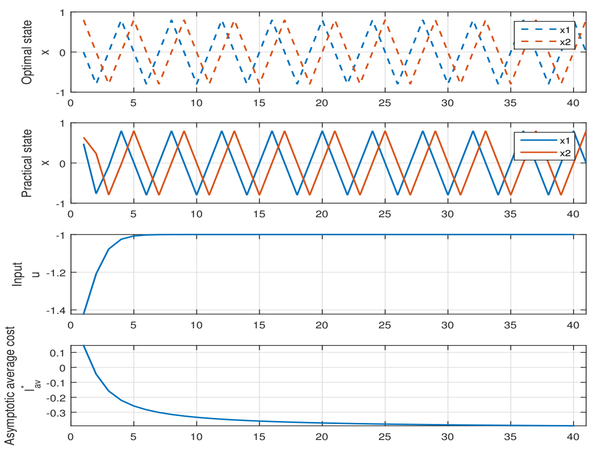

Fig. 2. Closed-loop behaviour of the circle system

6. CONCLUSION

In conclusion, this paper discusses a generalized approach for estimation of system asymptotic average performance from above and below by means of the CSF and dissipation inequalities. Such tools are adapted to formulate EMPC control schemes and eventually analyze their performance and stability. In the case of "no-gap", as previously defined, if the economic MPC controller generates a continuous input, the optimal periodic operation is Lyapunov asymptotically stable when the CSF is used as the terminal penalty function.

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