

Comments on (super)luminality

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Abstract

Recently, in an interesting work arXiv:1106.3972 a solution of the equations of motion of massive gravity was discussed, and it was shown that one of the fluctuations on that solution is superluminal. It was also stated that this rules out massive gravity. Here we find that the solution itself is rather unphysical. For this we show that there is another mode on the same background which grows and overcomes the background in an arbitrarily short period of time, that can be excited by a negligible cost in energy. This solution is triggered by the parameter governing the superluminality. Furthermore, we also show that the solution, if viewed as a perfect fluid, has no rest frame, or that the Lorentz transformation that is needed to boost to the rest frame is superluminal itself. The stress-tensor of this fluid has complex eigenvalues, and could not be obtained from any physically sensible matter. Moreover, for the same setup we find another background solution, fluctuations of which are all stable and subluminal. Based on these results, we conclude that the superluminality found in arXiv:1106.3972 is an artifact of using an inappropriate background, nevertheless, this solution represents an instructive example for understanding massive gravity. For instance, on this background the Boulware-Deser ghost is absent, even though this may naively appear not to be the case.

1 Introduction and summary

Massive gravity (classical theory) has had a turbulent past and present. To briefly account for works immediately relevant to the present paper: Fierz and Pauli (FP) constructed a ghost-less and tachyon-free linear theory [1]. Van Dam and Veltman, and Zakharov (vDVZ), have independently shown [2] that the FP theory has discontinuity in the zero mass limit, and argued that this excludes massive gravity. Soon after, Vainshtein showed that the vDVZ discontinuity is an artifact of the perturbative expansion that breaks precociously, and argued that upon inclusion of nonlinear terms there should be nonperturbative continuity to the massless theory, at least for the physical systems of observational relevance, thus evading the vDVZ conclusion [3]. However, subsequently Boulware and Deser (BD) [4] showed that in a broad class of nonlinear extensions of the FP theory one is not able to retain the needed five degrees of freedom of a massive graviton; instead, the sixth mode becomes propagating on certain backgrounds. This mode typically has negative energies, and is referred as the BD ghost.

More modern developments were triggered by the DGP model [5], for which it was argued by Deffayet et.al. [6] that the Vainshtein recovery does take place for sources of observational interest. This was followed by a covariant effective field theory formulation of massive gravity by Arkani-Hamed, Georgi, and Schwartz [7], who also proposed a program to construct a theory that would avoid the sixth mode (the BD ghost), starting from the analysis of the decoupling limit [7, 8, 9, 10], where things are easier to handle.

A positive progress toward this goal was made only recently: in Ref. [11] it was shown that in the decoupling limit the BD ghost can be avoided order-by-order to all orders. The absence of the BD ghost in the decoupling limit is a necessary consistency condition, but also turned out to be a powerful requirement leading to resummation of an infinite number of terms of the effective theory, resulting in a covariant Lagrangian with just a few terms [12]. The obtained Lagrangian reads:

$$\mathcal{L} = \frac{M_{\text{Pl}}^2}{2} \sqrt{-g} \left(R + m^2 (\mathcal{L}_{\text{der}}^{(2)}(\mathcal{K}) + \alpha_3 \mathcal{L}_{\text{der}}^{(3)}(\mathcal{K}) + \alpha_4 \mathcal{L}_{\text{der}}^{(4)}(\mathcal{K})) \right). \quad (1)$$

The tensor \mathcal{K} is defined as follows

$$\mathcal{K}_{\nu}^{\mu} = \delta_{\nu}^{\mu} - \sqrt{g^{\mu\alpha} \partial_{\alpha} \phi^a \partial_{\nu} \phi^b \eta_{ab}}, \quad (2)$$

where the square root above denotes a matrix element of the root of the matrix; $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$, and $\phi^a(x)$, $a = 0, 1, 2, 3$ are four spurious Stückelberg scalar fields introduced as a redundancy to provide for manifestly covariant description of massive gravity (for earlier works introducing these scalars, see, [13].) Finally, the mass and potential terms in (1) read as follows:

$$\mathcal{L}_{\text{der}}^{(2)}(\mathcal{K}) = [\mathcal{K}^2] - [\mathcal{K}]^2, \quad (3)$$

$$\mathcal{L}_{\text{der}}^{(3)}(\mathcal{K}) = [\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3], \quad (4)$$

$$\mathcal{L}_{\text{der}}^{(4)}(\mathcal{K}) = [\mathcal{K}]^4 - 6[\mathcal{K}^2][\mathcal{K}]^2 + 8[\mathcal{K}^3][\mathcal{K}] + 3[\mathcal{K}^2]^2 - 6[\mathcal{K}^4], \quad (5)$$

where we use the notations $[\mathcal{K}] \equiv (\text{Tr } \mathcal{K}_\nu^\mu)$, $[\mathcal{K}]^2 \equiv (\text{Tr } \mathcal{K}_\nu^\mu)^2$, while $[\mathcal{K}^2] \equiv \text{Tr } (\mathcal{K}_\nu^\mu \mathcal{K}_\alpha^\nu)$. The terms $\mathcal{L}_{\text{der}}^{(n)}$ give total derivatives upon substitution $\mathcal{K}_\nu^\mu \rightarrow \partial^\mu \partial_\nu \pi$, as indicated in their notation.

The above Lagrangian (1) has three free parameters (one of them being the graviton mass m), and for some values of these parameters the theory has been shown to be free of the BD ghost away from the decoupling limit up to (and including) the quartic order in nonlinearities [12]. Remarkably, Hassan and Rosen [14] have managed recently to show that it is free of the BD ghost away from the decoupling limit, to all orders¹. We note that the absence of the BD ghost guarantees the absence of the sixth mode. This however, does not prohibit one or more of the physical 5 polarizations to flip the sign of their kinetic terms on certain backgrounds and become ghosts. Such backgrounds should be considered unstable in the theory, but such cases should be distinguished from the ones with the sixth mode. For some work on cosmology and spherically symmetric solutions in massive gravity see, e.g., [17] - [24], and Ref. [25] for a theory review.

Recently, in a brief work Gruzinov [26] has found a certain solution of the theory (1), and showed that there is a fluctuation about this solution which is superluminal. Based on this observation, it was concluded that massive gravity is ruled out. Below we examine this conclusion more carefully. In Section 2 we show that there exist a growing solution which overcomes the background arbitrarily quickly, and can be excited with virtually no cost in energy. Moreover, this solution is not related to the BD ghost, as the latter is absent in this theory. In Section 3 we show that the solution of [26], if interpreted as a perfect fluid, has a stress-tensor with complex eigenvalues. Hence, the rest frame for this fluid can only be achieved via superluminal boosts. As such, this configuration could not, as an exact solution, be obtained from any known physically meaningful form of matter. In the appendix we discuss the decoupling limit of the linearized fluctuations.

The present work does not claim to exclude all possible superluminalities in massive gravity. Indeed, some of the terms obtained in the decoupling limit of massive gravity resemble the Galileon theories [27], which were shown to exhibit superluminalities. It is therefore reasonable to expect that when massive gravity reduces to a Galileon theory in the decoupling limit, the fluctuations of the helicity-0 mode around a spherically symmetric solution can also be superluminal. However, for more generic values of the parameters of the theory (1), in particular when $\alpha_3 + 4\alpha_4 \neq 0$, the decoupling limit of the theory cannot be written explicitly in a Galileon form, and it is possible that the decoupling limit does not capture the entire physics of the system. In that case a more careful treatment is required, and it is yet unclear whether the superluminalities around non-trivial backgrounds survive. We plan to report on this issue in a future work.

¹Our results [12], as well as the results of [14], are in conflict with the claim of Ref. [15] of the existence of the BD ghost in the quartic order. This controversy is addressed in Ref. [16], where it is shown that [15] missed a constraint. See also discussions in Section 2.

2 Superluminality and growing solutions

The system of equations of the theory (1) reads as follows:

$$G_{\mu\nu}(g) + m^2 X_{\mu\nu}(g, \phi) = 0, \quad (6)$$

$$m^2 \nabla^\mu X_{\mu\nu}(g, \phi) = 0. \quad (7)$$

Here, $X_{\mu\nu}$ is a tensor obtained by variation of the mass terms in (1), and is given explicitly in Section 3. Ref. [26] considers a classical scalar field configuration

$$\phi_{cl}^a = (\phi^0, \phi^1, \phi^2, \phi^3)_{cl} = (t, x + \epsilon t, y, z), \quad (8)$$

with an arbitrary constant ϵ , and studies a flat space fluctuation of ϕ^2 in the x direction, showing that this fluctuation is superluminal. Based on this, the work states that massive gravity is ruled out.

We start by presenting the results of Ref. [26] in more detail. We do this for a small value of $\epsilon \ll 1$, which is enough for our purposes. For this consider the field configuration (8). It produces some stress-tensor $X_{\mu\nu}$ which also depends on the metric $g_{\mu\nu}$; since $X_{\mu\nu}$ is multiplied by m^2 in eq. (6), one assumes that the back-reaction of $m^2 X_{\mu\nu}$ on the metric is negligible. In this approximation, the remaining equation is just an empty space Einstein equation, which certainly has a solution $g_{\mu\nu} = \eta_{\mu\nu}$. Hence, to summarize the solution of [26]:

$$g_{\mu\nu} = \eta_{\mu\nu} + \tilde{h}_{\mu\nu}, \quad \tilde{h}_{\mu\nu} \sim \mathcal{O}(\epsilon m^2 x x), \quad (9)$$

where for convenience we have included the correction \tilde{h} , where x denotes some components of x^μ (these corrections can straightforwardly be calculated for small ϵ , and take the form, $\tilde{h}_{01} \propto \epsilon m^2 t^2$, $\tilde{h}_{12} \propto \epsilon m^2 ty$). Since m can be arbitrarily small, one can neglect \tilde{h} in (9), at least in some region of space and time, and consider fluctuations on the approximate background $g_{\mu\nu} \simeq \eta_{\mu\nu}$.

To present the results of [26] more explicitly, we write down the quadratic Lagrangian for the fluctuations ζ^a of the four components of the ϕ^a field, while freezing all the other fields in the theory (the approximation in which this is justified will be discussed later, see below and the appendix):

$$\phi^a = x^a + \delta_1^a \epsilon t + \frac{\zeta^a(t, x, y, z)}{M_{\text{Pl}} m}. \quad (10)$$

The Lagrangian for ζ^a follows from an expansion of (1) on the background (8, 9), and in the quadratic approximation for the fluctuations ζ^a takes the form²:

$$\mathcal{L}_\zeta = -\frac{1}{4} F_{\mu\nu}^2(\zeta) - \frac{3\beta\epsilon}{2} F_{1\alpha} F_0^\alpha - \frac{\epsilon}{2} F_{01}(\partial_\alpha \zeta^\alpha) + \mathcal{O}(\epsilon^2), \quad (11)$$

²Note that ζ^a , in spite of its appearance, does not transform as a vector under diffeomorphisms, instead, it transforms as a four-coordinate.

where $\beta = -(\alpha_3 + 1/6)$, and the total derivative terms have been ignored. Furthermore, Ref. [26] focused on the ζ_2 component propagating only in the x direction (this does not excite other fluctuations in (11)). For $\zeta_2(t, x)$ the above Lagrangian reduces to

$$\mathcal{L}_{\zeta_2} = \frac{1}{2}\dot{\zeta}_2^2 - \frac{1}{2}\zeta_2'^2 - \frac{3\beta\epsilon}{2}\dot{\zeta}_2\zeta_2' + \mathcal{O}(\epsilon^2), \quad (12)$$

where an over-dot and prime denote t and x derivatives respectively. The dispersion relation that follows to leading order in ϵ , $\omega \simeq p(1 - 3\beta\epsilon/2)$, is superluminal, since for any nonzero β , the value of ϵ can always be chosen to give superluminality [26].

If the Lagrangian (12) is taken in isolation of all the other fields and interactions, as done in [26], then, the derived superluminality can be removed by a simple change of coordinates to \tilde{x} and \tilde{t} where, $\tilde{x} = x + t(3\beta\epsilon/2)$ and $\tilde{t} = t$ (which is just a galilean transformation with velocity equal to $-3\beta\epsilon/2$). Likewise, from an innocent field theory of a scalar φ coupled to a source J with the Lagrangian, $-(\partial_\mu\varphi)^2 + \varphi J$, one can get the Lagrangian of the type (12) by the above change of coordinates. Hence, if (12) were the entire Lagrangian one could quantize fluctuations in the $\{\tilde{x}, \tilde{t}\}$ coordinate system where no superluminality would appear.

The actual question, however, is what and how these fluctuations couple to other fluctuations and external sources, and what those other fluctuations do. In the full theory the field ζ does mix with the tensor and scalar modes at the linearized level, and has also nonlinear interactions. Also, there are $\mathcal{O}(\epsilon^2)$ terms neglected in (12)³. We study in turn all the fluctuations omitted in (12).

Let us first focus on other components of ζ^a which were not considered in [26]. Dropping the last term in (11), we write the equations of motion in the Lorentz gauge $\partial_\mu\zeta^\mu = 0$:

$$\begin{aligned} \square(\zeta_0 - q\zeta_1) + q\partial_0\partial_1\zeta_0 - q\partial_0^2\zeta_1 &= 0, \\ \square(\zeta_1 + q\zeta_0) + q\partial_0\partial_1\zeta_1 - q\partial_1^2\zeta_0 &= 0, \\ \square\zeta_b + q(-\partial_1\partial_b\zeta_0 + 2\partial_1\partial_0\zeta_b - \partial_0\partial_b\zeta_1) &= 0, \end{aligned} \quad (13)$$

where $b = 2, 3$, and $q \equiv 3\beta\epsilon/2$. These empty-space equations have many growing solutions. The one we focus on is

$$\zeta_1 \simeq \frac{1}{2}qm_0^3t^2 + m_0^3t(x - x_0) + \mathcal{O}(\epsilon^2), \quad \zeta_2 \simeq -m_0^3t(y - y_0) + \mathcal{O}(\epsilon^2), \quad (14)$$

where m_0, x_0, y_0 , are arbitrary integration constants, and other components of ζ are set to zero. In the leading order in ϵ , the ϵ^2 pieces in the above expressions should be ignored. Note that for (14), $F_{01} = qm_0^3t + m_0^3(x - x_0)$, and $F_{02} = -m_0^3(y - y_0)$.

³We thank Mehrdad Mirbabayi who pointed out to us that the Lagrangian (11), if considered in isolation, has a gauge symmetry, since the last term in it can simply be removed to the next, ϵ^2 order, by a field redefinition, $\zeta_0 \rightarrow \zeta_0 - \frac{\epsilon}{2}\zeta_1$, $\zeta_1 \rightarrow \zeta_1 - \frac{\epsilon}{2}\zeta_0$. This symmetry, is not present in the ϵ^2 order, and also in the full massive theory, due to the coupling of ζ_a to the tensor mode.

At each point in space one could choose corresponding x_0, y_0 , such that $F_{01} = qm_0^3t$ and $F_{02} = 0$ point-by-point in the whole space.

There are some important comments to be made about the solution.

(a) The solution (14) grows in time t and overcomes the background (8) for $t \gtrsim t_*$, where $t_* \equiv (mM_{\text{Pl}}/m_0^3)$. However, the mere existence of this solution cannot be interpreted as an instability of the background (8). The reason is that the solution (14) has a nonzero energy density proportional to $F_{01}^2 + F_{02}^2$, and in order to excite this field configuration within a finite volume in space one would need some energy; if such an energy is supplied, it is then not surprising that (14) does overcome the background after some period of time.

Nevertheless, there is an aspect of the solution (14) that suggests that the background (8) is unphysical. To see this let us introduce a length scale $L_* \equiv qt_*$. Let us now consider a small imaginary box of volume L^3 centered around the point $x = x_0, y = y_0, z = 0$. Most importantly, we take $L \lesssim L_*$. Furthermore, imagine that we supplied enough energy in the box, and the appropriate boundary conditions at its sides, so that inside the box the solution (14) is excited, while outside of the box the background is still given by (8). Let us now estimate how much energy density we need to supply for this to be the case. The average energy density in the box will consist of three terms, $D = D_1 + D_2 + D_3$, where $D_1 \sim q^2m_0^6t^2$, $D_2 \sim qtm_0^6L$, and $D_3 \sim m_0^6L^2$. Now, for any time moment $t > L/q$, the D_1 term dominates. What is important, however, is that $L/q \lesssim L_*/q = t_*$. Hence, at the time moment $t \sim t_*$, when the solution (14) in the box begins to dominate over the background (8), the energy density that is required to excite it is of the order $D_1 \sim q^2$; however, the latter happens to be zero in our approximation since we are ignoring terms of order $\epsilon^2 \sim q^2$ in the action. Therefore, we conclude that at the expense of the energy density that is zero in our approximation, we can excite the solution (14) in a volume of size $L^3 \lesssim L_*^3$, and this solution overcomes the background (8) after $t \sim t_*$ time. Since x_0 and y_0 in (14) are arbitrary, we can now consider the whole space populated by non-overlapping boxes in each of which an appropriate value of the parameters x_0, y_0 are chosen, and as a result, the growth described above develops in each of these boxes. Then, it is logical to interpret t_* as a characteristic time for the growing solution to dominate in the entire space. Most importantly, this time scale is independent of q , and can be made arbitrarily small by adjusting the integration constant m_0 . Moreover, the amount of the energy density needed to excite this solution, $D_1 \sim q^2m_0^4t^2$, although negligible in our approximation, in any event is much smaller than the characteristic scale of the stress-tensor for the background, $\epsilon m^2 M_{\text{Pl}}^2$, as long as $t < (mM_{\text{Pl}}/(\epsilon m_0^3))$; the latter is always the case, for $t \lesssim t_*$.

(b) Although, the above growth can be arbitrarily fast, it is triggered by the same parameter $q \sim \epsilon$ that sets the superluminality of the ζ_2 mode. This phenomenon disappears in the limit $q \rightarrow 0$, even though for $q = 0$ the solution (14) still grows and it may appear that the above-given argument would still hold in the $q \rightarrow 0$ limit. In this limit the solution (14) reduces to the terms $m_0^3t(x - x_0)$ and $m_0^3t(y - y_0)$, but

no t^2 term is remaining. Then, within a box of size L^3 (which is now necessarily larger than L_*^3 that tends to zero) one would be able to excite the solution at the cost of a non-negligible energy density $D \sim D_3 \sim m_0^6 L^2$. This energy density cannot be ignored as there are no neglected q^2 terms in the action any more. Clearly, such a solution does not indicate any problem of the background, it simply reflects the fact that the background has changed at the expense of the supplied finite energy density ⁴.

(c) The Lagrangian (11), as was pointed out above, is not the total Lagrangian of our theory, even at the linearized level and even in the leading order in ϵ . Then the question arises of whether the mixing with other fields plays any role. Since this is a bit technical, we address this question in the appendix, where we show that only in the $m \rightarrow 0$ limit the dynamics of the ζ_a modes in (11) can be decoupled from the tensor and scalar modes, assuming that one ignores nonlinear interactions as well.

In spite of the above described issue with the background (9) it is important to emphasize that this problem is not related in any way to the Boulware-Deser mode, i.e., to a potential sixth degree of freedom in a broad class of massive gravities, which is absent in the present model (1). To see that the sixth mode is not propagating on the background considered in [26], we look at the full action for the tensor fields. In unitary gauge $\phi^a = x^a$, the solution considered in [26] amounts to taking a background solution for the metric which is Minkowski, but not in Cartesian form. Specifically the background metric in unitary gauge is $ds^2 = -dt^2 + (dx - \epsilon dt)^2 + dy^2 + dz^2$. Expanding to quadratic order in perturbations around this solution and to first order in ϵ , the mass term which is now expressed entirely in terms of the tensor field since $\zeta^a = 0$ is

$$-\frac{m^2}{8} (\bar{h}_{\mu\nu}^2 - \bar{h}^2 + 6c_1 \epsilon \bar{h}_{1\alpha} \bar{h}_0^\alpha - (6c_1 + 1) \epsilon \bar{h}_{01} \bar{h}) , \quad (15)$$

with $c_1 = -\alpha_3 - 3/2$. We can use this form in order to count the physical degrees of freedom. In order for the BD ghost to be absent, one has to have the Hamiltonian constraint [4]. In the FP linearized theory the Hamiltonian constraint is enforced by h_{00} being a Lagrange multiplier, while h_{0j} is algebraically determined by an equation that is independent of h_{00} . In the Lagrangian (15), however, h_{00} mixes with h_{01} , and this may seem to forbid the presence of a constraint. However, this is not so, there still exists a linear combination of the fields that is a Lagrange multiplier in the approximation used. A convenient way to see this is to calculate the determinant of the 4×4 Hessian matrix for the Lagrangian $\mathcal{H}^{\mu\nu} \equiv \frac{\delta^2 \mathcal{L}}{\delta h_{0\mu} \delta h_{0\nu}}$. If the determinant is zero, then there are constraints. It is straightforward to calculate

⁴This is similar to the case of a free massless scalar field which has a solution, $\phi \sim t$; in any infinitesimal region of space one needs to supply a nonzero energy density to excite this solution. Likewise, to excite the solutions, $\phi \sim tx, ty$, which also exist in this theory, one would need finite energy density in any finite volume.

that the determinant of the Hessian that follows from (15) is of order ϵ^2 , i.e., it is zero in our approximation, while the rank of the Hessian is 3. Hence, there is one constraint in the system. Moreover, conservation of this constraint leads to a secondary constraint, as shown for these theories exactly in [14]; due to these one is able to eliminate the BD ghost.

One may also consider the counting of degrees of freedom in the non-unitary gauge considered in [26]. In such a gauge the tensor mode propagates two degrees of freedom, while there should be only three degrees of freedom for the four Stückelberg fields. The latter requirement at first sight seems unlikely since there is no gauge invariance for ζ in the full theory, and moreover, ζ_0 enters with a time derivative, even in the simplest case of $c_1 = 0$. Based on this one may be tempted to conclude that the Lagrangian for the ζ field (after gauge fixing $h_{\mu\nu}$) propagates 4 degrees of freedom. However, by more careful inspection one can show that there are constraints that render only 3 degrees of freedom in the ζ sector in general (detailed discussions of how this works in the full nonlinear theory are given in [16]).

In conclusion, the solution (9) seems problematic, despite being ghost-free. Is there another problem-free solution for the very same configuration of the Stückelberg fields (8)? The answer is positive. It is straightforward to find another solution to the system of eqs. (6) and (7), for given (8):

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon(\delta_\mu^0 \delta_\nu^1 + \delta_\mu^1 \delta_\nu^0) + \mathcal{O}(\epsilon^2). \quad (16)$$

or exact to all orders $ds^2 = -dt^2 + (dx + \epsilon dt)^2 + dy^2 + dz^2$. The above solution differs from (9), by ϵ , i.e., by the same parameter that sets superluminality found in [26]. Furthermore, it is easy to notice that the solution (8,16) is nothing but the Minkowski solution, $g_{\mu\nu} = \eta_{\mu\nu}$, $\phi^a = (t, x, y, x)$, transformed by the coordinate change $x^\mu \rightarrow x^\mu + \epsilon t \delta_1^\mu$. Therefore, the fluctuations above the solution (8,16) are just ordinary fluctuations of the Fierz-Pauli theory,

$$-\frac{m^2}{8} \eta^{\mu\alpha} \eta^{\nu\beta} (h_{\mu\nu} h_{\alpha\beta} - h_{\mu\alpha} h_{\nu\beta}), \quad (17)$$

which are known to be subluminal and stable.

3 Superluminality of the source

The solution considered in [26] is not an exact solution of massive gravity. As we have explained, it is at best a solution valid locally in a space-time region whose size/time scale is set by $L \sim 1/(\sqrt{\epsilon}m)$. Alternatively we can allow it to be an exact solution by adding an external source $T_{\mu\nu}^{\text{ext}}$ which is chosen so that

$$m^2 X_{\mu\nu} = T_{\mu\nu}^{\text{ext}}. \quad (18)$$

In principle we could imagine this external source being set up by a configuration of matter, a fluid, or a set of scalar or gauge fields. However it is easy to see that

the ‘fluid’ needed would itself be composed of superluminal matter. To see this, imagine $T_{\mu\nu}^{\text{ext}}$ were described by a perfect fluid. Let us assume that the fluid has a rest frame. If this is the case we can perform a Lorentz transformation so that in the vicinity of one point the fluid has zero velocity. At that point $T_{\text{ext}}^{0i} = 0$. Since the background metric is flat, $g_{0i} = 0$ and so $T_{\text{ext } i}^0 = T_{\text{ext } 0}^i = 0$. This in turn implies that the energy density $T^0_{\text{ext } 0}$ is one of the eigenvalues of the stress energy tensor $T^\mu_{\nu \text{ext}}$. The stress-energy tensor is expressed in terms of the tensor \mathcal{K}^μ_ν in the combination

$$X_{\mu\nu} = \mathcal{K}g_{\mu\nu} - \mathcal{K}_{\mu\nu} + (1 + 3\alpha_3) \left(\mathcal{K}_{\mu\nu}^2 - \mathcal{K}\mathcal{K}_{\mu\nu} + \frac{1}{2}([\mathcal{K}]^2 - [\mathcal{K}^2])g_{\mu\nu} \right) \quad (19)$$

$$+ \alpha \left(\mathcal{K}_{\mu\nu}^3 - \mathcal{K}\mathcal{K}_{\mu\nu}^2 + \frac{1}{2}\mathcal{K}_{\mu\nu}([\mathcal{K}]^2 - [\mathcal{K}^2]) - \frac{1}{6}([\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3])g_{\mu\nu} \right),$$

where $\alpha \equiv \alpha_3 + 4\alpha_4$. So the eigenvalues of $T^\mu_{\nu \text{ext}}$ are determined by the eigenvalues λ_K of \mathcal{K}^μ_ν , which in turn are expressed as

$$\lambda_{\mathcal{K}}^{(n)} = 1 - \sqrt{1 - \lambda_Y^{(n)}} \quad \text{for } n = 1, \dots, 4, \quad (20)$$

where λ_Y are the eigenvalues of Y^μ_ν :

$$Y^\mu_\nu = g^{\mu\alpha}\partial_\alpha\phi^a\partial_\nu\phi^b\eta_{ab}. \quad (21)$$

It is straightforward to show that the first two eigenvalues of this tensor are complex for the background solution, and so $T^0_{\text{ext } 0}$ is complex in this frame. Explicitly for the background considered the matrix (21) is

$$\begin{pmatrix} 1 - \epsilon^2 & -\epsilon & 0 & 0 \\ \epsilon & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (22)$$

and its eigenvalues are easily shown to be

$$\left(\lambda_Y^{(1)}, \lambda_Y^{(2)}, \lambda_Y^{(3)}, \lambda_Y^{(4)} \right) = \left(1 - \frac{1}{2}\epsilon^2 + \frac{i}{2}\epsilon\sqrt{4 - \epsilon^2}, 1 - \frac{1}{2}\epsilon^2 - \frac{i}{2}\epsilon\sqrt{4 - \epsilon^2}, 1, 1 \right). \quad (23)$$

We can therefore immediately infer that the eigenvalues of \mathcal{K} are also complex, and so are the eigenvalues of $T^\mu_{\nu \text{ext}}$.

This implies that there is no rest frame for the fluid, or that the Lorentz transformation needed to boost to the rest frame is superluminal (and hence a complex transformation), since it is not possible to perform a real Lorentz transformation to set $T_{0i}^{\text{ext}} = 0$. As such this configuration could not, as an exact solution, be obtained from any known physically sensible form of matter.

Even in the absence of a source, the same arguments hold. It is clear that for any solution of the equations $G_{\mu\nu} + m^2X_{\mu\nu} = 0$ which looks locally like flat space-time

with the field profile described in (8,9), it is not possible to boost to a frame in which $G^{0i} = 0$ in the local vicinity of a point. In this sense the solution already at the level of the background looks superluminal and rather unphysical.

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Appendix

Below we show that the Lagrangian (11) can be obtained from the full theory in the limit $m \rightarrow 0$. For this we start with the mass terms on the background (8, 9)

$$\mathcal{L}_m = -\frac{m^2 M_{\text{Pl}}^2}{8} (h_{\mu\nu}^2 - h^2 + \epsilon_1 h_{1\mu} h_0^\mu - \epsilon_2 h_{01} h + 4\epsilon h_{01}) , \quad (24)$$

where $\epsilon_1 \equiv \epsilon(6\beta - 4)$ and $\epsilon_2 \equiv \epsilon(6\beta - 3)$, and all the indices are contracted by $\eta_{\mu\nu}$. We express the Lagrangian in terms of the Stückelberg fields by using the substitution

$$h_{\mu\nu} \rightarrow \frac{h_{\mu\nu}}{M_{\text{Pl}}} - \frac{S_{\mu\nu}}{m M_{\text{Pl}}} - \frac{\partial_\mu \zeta^a \partial_\nu \zeta^b \eta_{ab}}{m^2 M_{\text{Pl}}^2} , \quad (25)$$

where we defined $S_{\mu\nu} \equiv \partial_\mu \zeta_\nu + \partial_\nu \zeta_\mu + \epsilon(\delta_\mu^0 \partial_\nu \zeta_1 + \delta_\nu^0 \partial_\mu \zeta_1)$, and introduced canonical normalizations for all fields.

Then, the total Lagrangian reads as follows:

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{EH}(h) + \mathcal{L}_\zeta(\zeta) + \frac{m}{4} S^{\mu\nu} (h_{\mu\nu} - \eta_{\mu\nu} h) \\ &+ \frac{m\epsilon_1}{8} (h_{1\mu} S_0^\mu + S_{1\mu} h_0^\mu) - \frac{m\epsilon_2}{8} (h_{01} S + S_{01} h) + \mathcal{O}(m^2) , \end{aligned} \quad (26)$$

where $\mathcal{L}_{EH}(h)$ denotes the linearized Einstein-Hilbert term, while $\mathcal{L}_\zeta(\zeta)$ is the Lagrangian given in (11). We also ignored the terms of order $\mathcal{O}(m^2)$ or smaller, $\mathcal{O}(\epsilon m^2)$, in (26). Note that the tadpole appearing in (24) gets canceled in (26) by the corresponding tadpole coming from the EH term taken on the background (9).

We see that the fields ζ^a mix to the tensor field. Our goal is to show that this mixing disappears in the $m \rightarrow 0$ limit. For this we note that we can remove the third term in $\mathcal{L}_\zeta(\zeta)$ by a linear field redefinition (see footnote 3), and then introduce the helicity-0 field by the change of variables $\zeta_\mu \rightarrow \zeta_\mu + \partial_\mu \pi / m$. As a result we get the following Lagrangian:

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{EH}(h) + \mathcal{L}_1(\zeta) + \frac{1}{4} \mathcal{P}^{\mu\nu}(\pi) (h_{\mu\nu} - \eta_{\mu\nu} h) + \\ &+ \frac{\epsilon_1}{4} (h_{1\mu} \partial^\mu \partial_0 \pi + \partial_1 \partial_\mu \pi h_0^\mu) - \frac{\epsilon_2}{4} (h_{01} \square \pi + \partial_0 \partial_1 \pi h) + \mathcal{O}(m, \epsilon m, m^2) , \end{aligned} \quad (27)$$

where $\mathcal{L}_1(\zeta)$ is the Lagrangian (11) less the last term, and $\mathcal{P}_{\mu\nu}(\pi) \equiv 2\partial_\mu\partial_\nu\pi + \frac{3\epsilon}{2}(\delta_\mu^0\partial_\nu\partial_1\pi + \delta_\nu^0\partial_\mu\partial_1\pi) + \frac{\epsilon}{2}(\delta_\mu^1\partial_\nu\partial_0\pi + \delta_\nu^1\partial_\mu\partial_0\pi)$. As we see, there is a mixing between the tensor mode and the helicity-0 mode π . Due to this mixing the helicity-0 gets a kinetic term via the shift $h_{\mu\nu} \rightarrow h_{\mu\nu} + \eta_{\mu\nu}\pi$; as a result, the helicity-0 would couple to an external source had we introduced it in the theory. However, the field ζ does not couple with anybody in this limit. This would be so even if we were to introduce a stress-tensor of an external matter. The coupling of ζ appears only at a nonlinear level.

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