A Semi-Decentralized Scheme for Integration of Price-Responsive Appliances in the Electricity Market

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Abstract: A novel semi-decentralized control strategy is proposed for the integration in the power system of large populations of flexible loads, such as electric vehicles and "smart" appliances. To characterize the interactions between the single agents and their effects on the grid, a game theory framework is adopted. The price responsive appliances are modelled as competing players, characterizing a stable and efficient solution as a Nash equilibrium (no device has unilateral interest in changing its scheduled power consumption when the final electricity price is considered). We extend previous results on distributed control of flexible demand, proposing a partial centralization of the power scheduling at critical time instants. In this way, it is possible to ensure convergence to a Nash equilibrium for a wider range of scenarios, considering higher penetration levels of flexible demand and a wider range of parameters for the devices. The effectiveness of the proposed scheme is theoretically proved and its performance is evaluated in simulations, considering a future UK grid with high penetration of flexible demand.

Keywords: Smart grids; Price-responsive devices; Decentralized and distributed control; Game theories; Complex system management; Analysis and control in deregulated power systems; Optimization and control of large-scale network systems.

1. INTRODUCTION

This paper follows the significant transformations of the power system in its shifting towards the decentralized paradigm of the smart grid, as summarized by Ipakchi (2009). In particular, it deals with the increasing penetration of new devices, such as electric vehicles and "smart" appliances, that accommodate some flexibility in their power consumption when they are connected to the grid. The possible benefits deriving from the growing diffusion of these loads in the electric network have been assessed in detail by Albadi and El-Saadany (2008) and Rahimi and Ipakchi (2010), showing significant potential for an increased system reliability and a reduction of electricity bills for private customers. To obtain these results, it is necessary to devise suitable mechanisms that coordinate the power consumption of the appliances, taking into account their impact on global quantities such as power demand and energy prices. The main objective is to design a control strategy that guarantees energy cost minimization for the individual loads and ensures a reliable and efficient operation of the power system. Centralized approaches have been proposed by Samadi et al. (2012) and Papavasiliou and Oren (2014), determining the power consumption of the appliances so as to minimize some global quantity of the system, such as total generation costs. Given the computational complexity and communication requirements of these schemes for large populations of appliances, distributed mechanisms have also been evaluated, considering stochastic optimization (Chen et al. (2012)), congestion pricing (Fan (2012)) and iterative processes with Lagrange relaxation (Papadaskalopoulos and Strbac (2013)).

The present work proposes a game theory approach, modelling the individual appliances as rational players that compete for power consumption at the cheapest hours of the day. As a starting point, we consider the theoretical analysis presented in De Paola et al. (2015), where a suitable final solution has been characterized as a Nash equilibrium. Necessary and sufficient equilibrium conditions have been provided, determining the possibility of successfully coordinating, in a decentralized manner, large populations of price-responsive appliances. The possibility of taking additional control actions when such equilibrium conditions are not fulfilled (for example at high penetration levels of flexible demand) has also been investigated. We propose in De Paola et al. (2016a,b) a demand saturation technique that limits power consumption by the flexible devices at critical time instants, ensuring a Nash equilibrium while minimizing the total task time of the population. This paper presents a different approach and extends the initial equilibrium condition by removing some

⋆ This work was supported by the EPSRC grants "The Autonomic Power System" [EP/I031650/1] and "Energy Storage for Low Carbon Grids" [EP/K002252/1].

assumptions on the broadcast demand/price signal and introducing an increased level of centralization for a fraction of devices, on limited time intervals. After a detailed theoretical analysis, we provide a semi-decentralized scheme that allows to coordinate the appliances and induce a Nash equilibrium under very general conditions. Decentralized convergence of flexible appliances to a Nash equilibrium has already been obtained by introducing some quadratic term in the cost function of the agents, as proposed by Ma et al. (2013), or under some conditions on the price function and the number of devices, as presented for example in Chen et al. (2014). The main novelty of the proposed semi-decentralized scheme is the possibility to induce a stable solution for any penetration level of flexible demand, with no additional terms in the cost function of the agents nor precise knowledge of the electricity price function.

The rest of the paper is structured as follows: Section 2 presents the main modelling choices for the flexible loads and the energy market. The considered game-theoretical framework, with previous results and the current approach for convergence to equilibrium, is presented in Section 3. The proposed semi-decentralized scheme is described in Section 4 and tested in simulation in Section 5. Conclusive remarks are finally presented in Section 6.

2. MODELLING OF FLEXIBLE APPLIANCES AND INTERACTIONS WITH ELECTRICITY MARKET

Our analysis considers large populations of electrical loads that can flexibly allocate their power consumption within the considered time interval $[0, T]$. Each device can be characterized by two parameters: its total required energy E and its minimum task time τ. The latter corresponds to the time required for task completion if the device operates at maximum feasible power $P = E/\tau$. The sets of distinct energy and time parameters in the appliances populations are denoted by $\mathcal{E}$ and $\mathcal{T}$, respectively. The set $\mathcal{U}_{\tau,E}$ of feasible power profiles $u : [0, T] \to \mathbb{R}$, for a device with parameters τ and E, can be defined as follows:

$$\mathcal{U}_{\tau,E} := \left\{ u(t) : \int_0^T u(t)\, dt = E,\ 0 \le u(t) \le \frac{E}{\tau} \ \ \forall t \in [0, T] \right\}. \tag{1}$$

In a preliminary phase, each device communicates its parameters τ and E to the system operator which, on the basis of this data, can calculate their unnormalized distribution $m : \mathcal{T} \times \mathcal{E} \to \mathbb{R}^+$. For $\tau_1, \tau_2 \in \mathcal{T}$ and $E_1, E_2 \in \mathcal{E}$, the quantity $\int_{\tau_1}^{\tau_2} \int_{E_1}^{E_2} m(\tau, E)\, dE\, d\tau$ corresponds to the number of devices with $\tau_1 \le \tau \le \tau_2$ and $E_1 \le E \le E_2$. The properties of the appliances population can then be summarized by the energy density f, defined below:

$$f(\tau) := \int_{\mathcal{E}} E \cdot m(\tau, E)\, dE. \tag{2}$$

Specifically, $f(\tau)$ quantifies the total amount of energy required by the appliances with time parameter τ. If the number of appliances is sufficiently high and their parameters are adequately diversified, it is possible to describe the population as a continuum and assume that $f(\tau)$ is always a well-defined function. The results presented in this paper only require this hypothesis but, for a more compact presentation, the following assumption is introduced:

Assumption 1. The function $f(\tau)$ is bounded and with compact support:

$$\operatorname{supp}(f) = [q_{min}, q_{max}] \subset (0, T]. \tag{3}$$

The electricity market has been modelled trough the monotone increasing function $\Pi : [0, +\infty) \to [0, +\infty)$ that associates, to a certain value of aggregate power demand $D_a(t)$ at time t, the corresponding electricity price $\Pi(D_a(t))$. The demand profile $D_a(t)$, in turn, is the sum of two components: the power demand $D_i(t)$ of the inflexible loads (assumed to be known a priori) and the total power consumption $D_f(t)$ of the flexible loads. Assuming that the scheduled power of each device exclusively depends on its parameters τ and E and is denoted by $u(\cdot, \tau, E) \in \mathcal{U}_{\tau,E}$ (this is true in the subsequent analysis, when loads respond to a demand/price signal), we obtain:

$$D_f(t) = \int_{\mathcal{T}} \int_{\mathcal{E}} u(t, \tau, E) m(\tau, E)\, dE\, d\tau \tag{4}$$

$$D_a(t) = D_i(t) + D_f(t). \tag{5}$$

3. GAME-THEORETICAL FRAMEWORK AND NASH EQUILIBRIUM NOTION

The flexible appliances are modelled as rational agents that compete for power consumption at times with cheapest electricity. In absence of any coordination, they would consume power when prices are expected to be low, increasing total demand (and prices) at those times and making their initial strategy suboptimal. To avoid these rebound peaks phenomena, the appliances interaction are modelled through a differential game with the following elements:

- *Players:* Individual loads with parameters τ and E.
- *Strategies:* The set of feasible power profiles $\mathcal{U}_{\tau,E}$, defined in (1).
- *Objective function:* Minimization of the energy cost J, sustained when the final electricity price $\Pi(D_a(t))$ is considered:

$$J_{\tau,E} = \int_0^T \Pi(D_a(t)) \cdot u(t, \tau, E)\, dt.$$

The notion of Nash equilibrium for the considered game is the following:

Definition 1. The scheduled power consumption of the devices, characterized by $u^* : [0, T] \times \mathcal{T} \times \mathcal{E} \to \mathbb{R}^+$, corresponds to a Nash equilibrium in the electricity market if the following condition is fulfilled for all $\tau \in \mathcal{T}$ and $E \in \mathcal{E}$:

$$\int_0^T \Pi(D_a^*(t)) u^*(t, \tau, E)\, dt = \min_{u(\cdot) \in \mathcal{U}_{\tau,E}} \int_0^T \Pi(D_a^*(t)) u(t)\, dt \tag{6}$$

with the following expression for the total demand D_a^*:

$$D_a^*(t) = D_i(t) + \int_{\mathcal{T}} \int_{\mathcal{E}} u^*(t, \tau, E) m(\tau, E)\, dE\, d\tau. \tag{7}$$

When a Nash equilibrium is achieved, the individual devices are following a power scheduling u^* which is optimal for a certain aggregate demand D_a^* (and price $\Pi(D_a^*)$), as established in (6). At the same time, from (7), the whole set of power schedules induce that very same profile D_a^*. This means that each appliance has no interest in changing its initially scheduled power profile u^* when the resulting energy price $\Pi(D_a^*(t))$ is considered.

This notion of equilibrium is similar to the one proposed in mean field games, where the impact of the single agent on the global quantities of the system is negligible and only the behaviour of the whole population needs to be considered. The Nash equilibrium represents the main design objective of our study, as it corresponds to a stable allocation of power consumption which avoids rebound peaks and synchronicity phenomena, guaranteeing flat demand profiles and reduced generation costs.

3.1 Fully-Decentralized Scheduling

The initial equilibrium analysis, performed in De Paola et al. (2015), considers a completely decentralized setup. The appliances, on the basis of a broadcast demand profile D (or the corresponding price $\Pi(D)$), schedule their power consumption u_D through the following optimization:

$$u_D(\cdot,\tau,E) \in \underset{u(\cdot)\in\mathcal{U}_{\tau,E}}{\arg\min} \int_0^T \Pi(D(t))u(t)\,dt. \tag{8}$$

For a more compact presentation of the equilibrium results in this scenario, it is useful to consider the sublevel sets of the broadcast demand profile D, introducing its *cumulative distribution* Q_D:

$$Q_D(d) := \mu(\{t \in [0,T] : D(t) \leq d\}) \tag{9}$$

where μ denotes the Lebesgue measure. We also define $\mathcal{S}_D(q)$, corresponding to the set of time instants t for which $Q_D(D(t))$ is less or equal than q:

$$\mathcal{S}_D(q) := \{t \in [0,T] : Q_D(D(t)) \leq q\}\,. \tag{10}$$

The fully-decentralized analysis relies on the assumption of level sets of zero measure for D:

$$\mu\left(\{t \in [0,T] : D(t) = d\}\right) = 0 \quad \forall d \in \mathrm{Im}(D). \tag{11}$$

Under this hypothesis, we conclude in De Paola et al. (2015) that the Nash equilibrium presented in Definition 1 can be obtained (by broadcasting $D = D_i$ and considering $u^* = u_{D_i}$) if and only if:

$$\Lambda_{D_i}(q) \leq \Lambda_f(q) \qquad \forall q \in \mathrm{supp}(f) \tag{12}$$

where the *negotiable valley capacity* $\Lambda_D(q) = \frac{d}{dq}Q_D^{-1}(q)$ and the *power density of task durations* $\Lambda_f(q) = f(q)/q$ characterize the broadcast signal D and the appliances population, respectively.

3.2 Semi-Decentralized Scheduling of Power Consumption

In this work, we evaluate whether the removal of condition (11) allows to achieve a Nash equilibrium even when (12) is not fulfilled (and higher penetration of flexible demand is considered). In this respect note that, if (11) does not hold, the scheduled power consumption u_D in (8) is not unique. For example, if $D(t) = d\ \forall t \in [0,T]$, the right-hand side of (8) coincides with $\mathcal{U}_{\tau,E}$. Therefore, to characterize the aggregate power consumption of the flexible devices in the current scenario and induce a Nash equilibrium, central coordination of some appliances may be required at certain time intervals. As a preliminary step for the analysis presented in the next section, the scheduling of the appliances is characterized by the power density $h : [0,T] \times \mathcal{T} \to \mathbb{R}^+$ where $h(t,\tau)$ quantifies the total power consumed at time t by appliances with parameter $\tau \in \mathcal{T}$. For a certain power profile u, it holds:

$$h(t,\tau) = \int_{\mathcal{E}} u(t,\tau,E)m(\tau,E)\,dE. \tag{13}$$

Equivalent equilibrium conditions are now provided:

Lemma 1. Consider the power density h^* and the associated power consumption $\bar{u}^*$, defined as follows:

$$\bar{u}^*(t,\tau,E) := \frac{E}{f(\tau)}h^*(t,\tau). \tag{14}$$

A Nash equilibrium is achieved and (6)-(7) are satisfied for $u^* = \bar{u}^*$ if the following holds:

$$D_a^*(t) = D_i(t) + D_f^*(t) = D_i(t) + \int_{\mathcal{T}} h^*(t,\tau)\,d\tau \tag{15a}$$

$$0 \leq h^*(t,\tau) \leq \frac{f(\tau)}{\tau} \tag{15b}$$

$$\int_{\mathcal{S}_{D_a^*}(q)} h^*(t,\tau)\,dt = \min\left(f(\tau), \mu\left(\mathcal{S}_{D_a^*}(q)\right)\frac{f(\tau)}{\tau}\right) \tag{15c}$$

This result is not formally proved for length reason but one can verify that (15a) is equivalent to (7) and (15b) ensures feasibility of h^*. Finally, condition (15c) states that the integral of h^* over each sublevel set $\mathcal{S}_{D_a^*}(q)$ of $Q_{D_a^*}$ must be equal to its maximum feasible value (devices operate at maximum power when demand and prices are lowest).

4. COORDINATION SCHEME FOR EQUILIBRIUM

The main design objective is to induce, with minimum centralization, a power density h^* and an aggregate demand D_a^* which fulfil (15) and therefore correspond to a Nash equilibrium in the electricity market. In Section 4.1 we provide some preliminary assumptions and initial results for D_a^*, analysing their impact on the Nash equilibrium conditions in Section 4.2. Section 4.3 describes the centralized part of the control scheme. On the basis of the preliminary theoretical results, a general description of the final coordination strategy, is provided in Algorithm 1.

4.1 Assumptions and Properties of Demand Profiles

To simplify the analysis, a monotone increasing signal is considered for the inflexible demand D_i. This choice does not introduce any loss of generality: any arbitrary $\tilde{D}_i$ can always be reordered to obtain a demand function D_i taking increasing values. If (11) holds for $D = \tilde{D}_i$ (as it is usually the case), the reordered profile D_i can be calculated with the change of variable described in De Paola et al. (2015):

$$D_i(Q_{\tilde{D}_i}(\tilde{D}_i(t)) = \tilde{D}_i(t). \tag{16}$$

All the results obtained considering D_i can be extended to the original $\tilde{D}_i$ by applying the opposite transformation. Practical examples are shown in Fig. 2 and Fig. 3, in the simulation section. We also recall the expression of the aggregate demand D_a°, obtained in a fully decentralized framework by broadcasting $D = D_i$ to the appliances. Considering that, when D_i is increasing, each device with parameters τ and E operates at maximum power E/τ over the time interval $[0,\tau]$, we have:

$$D_a^\circ(t) = D_i(t) + \int_t^T \int_{\mathcal{E}} \frac{E}{\tau}m(\tau,E)dEd\tau = D_i(t) + \int_t^T \frac{f(\tau)}{\tau}\,d\tau. \tag{17}$$

The following properties of the sought power profile D_a^*, albeit not formally proved, can be easily inferred:

Lemma 2. For a monotone increasing D_i, an aggregate demand profile D_a^* fulfilling the equilibrium conditions (15) must be nondecreasing.

Lemma 3. Given the demand profiles D_a^* and D_a° in (17), let $D_I^*(t) = \int_0^t D_a^*(x)\,dx$ and $D_I^\circ(t) = \int_0^t D_a^\circ(x)\,dx$ denote their integrals. If D_a^* fulfils (15) for some h^*, it holds:

$$D_a^*(t) \le D_a^\circ(t) \quad \forall t \in [0,T] : D_I^*(t) = D_I^\circ(t). \tag{18}$$

The result of Lemma 2 follows from the supposed Nash equilibrium (all devices operate with higher power at lower prices). For Lemma 3, consider that D_a° is obtained by all appliances operating at maximum power until task completion. Therefore, when $D_I^*(t) = D_I^\circ(t)$, the same must have been occurred to induce D_a^*. It follows that, in both cases, only appliances with $\tau \ge t$ are available at time t. This corresponds to an upper bound on the aggregate demand D_a^*:

$$D_a^*(t) = D_i(t) + D_f^*(t) \le D_i(t) + \int_t^T \frac{f(s)}{s}\,ds = D_a^\circ(t).$$

The candidate solutions for D_a^* can now be limited to a specific class $\mathcal{D}_a$ of demand profiles, defined as follows:

Definition 2. Consider $D_a^* : [0,T] \to \mathbb{R}^+$ fulfilling Lemma 2 and 3. The signal D_a^* belongs to the class $\mathcal{D}_a$ if and only if there exists a scalar $k \in \{0,1\}$ and time instants $t_1, \dots, t_N$ (with $0 = t_1 < \dots < t_N = T$ and $N \ge 2$) that fulfil the following for $i = 1, \dots, N-1$:

$$\begin{array}{lll} \dot{D}_a^*(t) > 0 & \forall t \in (t_i, t_{i+1}) \text{ if } & (-1)^{k+i} < 0 \\ \dot{D}_a^*(t) = 0 & \forall t \in (t_i, t_{i+1}) \text{ if } & (-1)^{k+i} > 0. \end{array} \tag{19}$$

In other words, we are restricting our analysis to demand signals whose derivative has non-negative and piecewise-constant sign. Any trait with decreasing values is excluded a priori from Lemma 2. For any $D_a^*(\cdot) \in \mathcal{D}_a$ and corresponding $t_1, \dots, t_N$ and k, we can provide a simple expression of the sublevel sets $\mathcal{S}_{D_a^*}(q)$, introduced in (10). From the monotonicity of $D_a^*(t)$ and $Q_{D_a^*}(d)$, for $q \in [t_i, t_{i+1})$ it holds:

$$\mathcal{S}_{D_a^*}(q) = \begin{cases} [0, q] & \text{if } (-1)^{k+i} < 0 \\ [0, t_{i+1}] & \text{if } (-1)^{k+i} > 0 \end{cases} \tag{20}$$

4.2 Necessary Equilibrium Conditions

The equilibrium results presented in Section 3 give important indications for the design process if they are analysed on the basis of the new assumptions and notation.

Proposition 1. Consider a demand profile $D_a^*(\cdot) \in \mathcal{D}_a$ (as specified in Definition 2) and a time interval $[t_i, t_{i+1}) \subset [0,T]$ such that $(-1)^{k+i} < 0$. Equilibrium conditions (15) are satisfied only if:

$$h^*(t,\tau) = \begin{cases} 0 & \text{if } \tau < t \\ \dfrac{f(\tau)}{\tau} & \text{if } \tau \ge t \end{cases} \qquad \forall t \in [t_i, t_{i+1}) \tag{21a}$$

$$D_a^*(t) = D_i(t) + \int_t^T \frac{f(\tau)}{\tau}\,d\tau \qquad \forall t \in [t_i, t_{i+1}). \tag{21b}$$

Proof. We initially evaluate condition (15c) for $q = t \in [t_i, t_{i+1})$. Taking into account expression (20) for the sublevel set $\mathcal{S}_{D_a^*}$ under the current assumptions, we obtain:

$$\int_0^t h^*(x,\tau)\,dx = \begin{cases} f(\tau) & \text{if } \tau < t \\ t \cdot \dfrac{f(\tau)}{\tau} & \text{if } \tau \ge t \end{cases} \qquad \forall t \in [t_i, t_{i+1}). \tag{22}$$

Condition (21a) is necessary since it corresponds to the time derivative of (22). The proof is concluded by noticing that (15b) always holds in the present case and (15a) is equivalent to (21b) for the chosen h^*.

From Proposition 1, we can establish that a profile $D_a^* \in \mathcal{D}_a$ fulfilling (15) is equal, in its increasing traits, to D_a° in (17). Moreover, on the corresponding time intervals $[t_i, t_{i+1})$, the power density h^* and the repartitioned power $\bar{u}^*$, defined by (14), can be induced in a decentralized manner by simply broadcasting $D = D_i$ to the appliances.

Remark 1. Conditions (15) can only be satisfied by a function $D_a^*(\cdot) \in \mathcal{D}_a$ which is the composition of increasing profiles (equal to the corresponding values of D_a°), connected by traits at constant value. Given that the total integrals of D_a^* and D_a° must be the same (all devices must complete their task and therefore consume a specified total amount of energy), it is clear that in practical applications a limited number of candidate solutions D_a^* will need to be considered.

In the simulations of the present paper, the profile D_a^* has been calculated heuristically. In the journal version of this work it will be shown that at least one D_a^* with the aforementioned properties always exists and an algorithm for its analytical calculation will be provided.

A complementary result is now presented for time intervals with constant aggregate demand:

Proposition 2. Consider a demand signal $D_a^*(\cdot) \in \mathcal{D}_a$ (as specified in Definition 2) and the time interval $[t_i, t_{i+1}] \subseteq [0,T]$ such that $(-1)^{k+i} > 0$. Equilibrium conditions (15) are satisfied only if:

$$h^*(t,\tau) = \begin{cases} 0 & \text{if } \tau < t_i \\ \dfrac{f(\tau)}{\tau} & \text{if } \tau \ge t_{i+1} \end{cases} \qquad \forall t \in [t_i, t_{i+1}] \tag{23a}$$

$$\int_{t_i}^{t_{i+1}} h^*(t,\tau)\,dt = f(\tau)\left[1 - \frac{t_i}{\tau}\right] \qquad \forall \tau \in [t_i, t_{i+1}] \tag{23b}$$

Proof. Condition (15c) is evaluated for $q = t_i^-$ and $q \in [t_i, t_{i+1}]$, considering expression (20) for the sublevel set $\mathcal{S}_{D_a^*}$. Taking the difference of the resulting terms yields:

$$\int_{t_i}^{t_{i+1}} h^*(t,\tau)\,dt = \begin{cases} 0 & \text{if } \tau < t_i \\ f(\tau)\left[1 - \dfrac{t_i}{\tau}\right] & \text{if } t_i \le \tau \le t_{i+1} \\ (t_{i+1} - t_i)\dfrac{f(\tau)}{\tau} & \text{if } \tau > t_{i+1} \end{cases} \tag{24}$$

Note that (23b) can be inferred from the second case in (24). Moreover, since condition (15b) must hold, the first case in (24) is satisfied only if $h^*(t,\tau)$ is identically zero for $t \in [t_i, t_{i+1}]$ when $\tau < t_i$. For similar reasons, since the measure of the integration interval in (24) is equal to $t_{i+1} - t_i$, we have that $h^*(t,\tau)$ must be identically equal to $f(\tau)/\tau$ for $t \in [t_i, t_{i+1}]$ and $\tau \ge t_{i+1}$. Therefore, we can conclude that also (23a) must hold when the equilibrium conditions (15) are fulfilled.

From Proposition 2, three different behaviours of the appliances can be identified on intervals $[t_i, t_{i+1}]$ of constant demand, depending on their parameter τ. The devices with $\tau < t_i$ do not operate as they have already completed their task at lower times. The opposite result holds for the appliances with $\tau > t_{i+1}$ since they cannot complete their tasks within the interval $[0, t_{i+1}]$. Considering that they need to operate at time instants $t > t_{i+1}$ (characterized by higher values of demand $D_a^*(t)$ and price $\Pi(D_a^*(t))$), to achieve minimum cost they must consume maximum power on the interval $[t_i, t_{i+1}]$ of constant demand. Note that the appliances considered so far can still be coordinated in a decentralized manner by simply broadcasting D_i and inducing the desired behaviour. On the other hand, an increased level of coordination is required for appliances with $\tau \in [t_i, t_{i+1}]$. In this case Proposition 2 only provides an integral condition on h^* since there are multiple scheduled profiles of power consumption that ensure minimum cost. From the results of Proposition 1 for $i-1$, we can conclude that these devices need to operate at rated power on $[0, t_i]$ and then complete their task within the time interval $[t_i, t_{i+1}]$.

4.3 Centralized Scheduling

From previous considerations, in order to fulfil (15) and achieve a Nash equilibrium, it is necessary to centrally coordinate appliances with $\tau \in [t_i, t_{i+1}]$ on time intervals $[t_i, t_{i+1}]$ of constant aggregate demand D_a^*. In other words, we must determine a power density h^* which fulfils the following conditions:

$$\int_{t_i}^{t_{i+1}} h^*(t,\tau)\, d\tau = D_a^*(t) - D_i(t) - \int_{t_{i+1}}^{T} \frac{f(\tau)}{\tau}\, d\tau \quad \forall t \in [t_i, t_{i+1}] \tag{25a}$$

$$0 \le h^*(t,\tau) \le \frac{f(\tau)}{\tau} \qquad \forall t \in [t_i, t_{i+1}] \;\; \forall \tau \in [t_i, t_{i+1}] \tag{25b}$$

$$\int_{t_i}^{t_{i+1}} h^*(t,\tau)\, dt = f(\tau)\left[1 - \frac{t_i}{\tau}\right] \qquad \forall \tau \in [t_i, t_{i+1}]. \tag{25c}$$

Equation (25a) is obtained by substituting (23a) in (15a) and (25b) is the equivalent of (15b) while (25c), from Proposition 2, corresponds to (15c). In our analysis, instead of the power density h, we consider the remaining rated task time $\eta : [0, t_{i+1}-t_i] \times [t_i, t_{i+1}] \to \mathbb{R}^+$. The quantity $\eta(t,\tau)$ denotes the reduction in rated task time that a device with parameter τ needs to achieve (by operating at rated power $P = E/\tau$) on the interval $[t_i + t, t_{i+1}]$ of constant D_a^*. It can be defined as:

$$\eta(t,\tau) := \tau - t_i - \frac{\tau}{f(\tau)} \int_{t_i}^{t_i+t} h(x,\tau)\, dx. \tag{26}$$

The relationship between η and h in the opposite sense can be easily calculated by differentiating (26) with respect to t:

$$h(t_i + t, \tau) = -\frac{f(\tau)}{\tau}\eta_t(t,\tau) \qquad \forall t \in [0, t_{i+1} - t_i]. \tag{27}$$

If we substitute (27) in equations (25), the scheduling problem consists in determining a function η^* that fulfils the following set of equivalent conditions:

$$-\int_{t_i}^{t_{i+1}} \frac{f(\tau)}{\tau}\eta_t^*(t,\tau)\, d\tau = D_r(t) \quad \forall t \in [0, t_{i+1}-t_i] \tag{28a}$$

$$-1 \le \eta_t^*(t,\tau) \le 0 \quad \forall t \in [0, t_{i+1}-t_i] \;\; \forall \tau \in [t_i, t_{i+1}] \tag{28b}$$

$$\eta^*(t_{i+1} - t_i, \tau) = 0 \qquad \forall \tau \in [t_i, t_{i+1}] \tag{28c}$$

where $D_r(t)$ denotes the right-hand-side in equation (25a), evaluated at $t_i + t$:

$$D_r(t) := D_a^*(t_i + t) - D_i(t_i + t) - \int_{t_{i+1}}^{T} \frac{f(\tau)}{\tau}\, d\tau. \tag{29}$$

The proposed scheduling, which fulfils (28) and induces a Nash equilibrium, is characterized by the solution $\eta^\star$ of the following equations:

$$\eta(0,\tau) = \tau - t_i$$

$$\eta_t(t,\tau) = \begin{cases} -1 & \text{if} \quad \eta(t,\tau) > \eta(t, \theta_{D_r}(t)) \\ -1 & \text{if } \eta(t,\tau) = \eta(t,\theta_{D_r}(t)),\ W_-(t) = W_+(t) \\ -\dfrac{D_r(t) - W_+(t)}{W_-(t) - W_+(t)} & \text{if } \eta(t,\tau) = \eta(t,\theta_{D_r}(t)),\ W_-(t) > W_+(t) \\ 0 & \text{if} \quad \eta(t,\tau) < \eta(t,\theta_{D_r}(t)) \end{cases} \tag{30}$$

The quantity $\theta_{D_r}(t) \in [t_i, t_{i+1}]$ is such that $\int_{\theta_{D_r}(t)}^{t_{i+1}} \frac{f(\tau)}{\tau}\, d\tau = D_r(t)$. Its existence and uniqueness follow from the positivity of f and the following inequalities:

$$0 = \int_{t_{i+1}}^{t_{i+1}} \frac{f(\tau)}{\tau}\, d\tau \le D_r(t) \le \int_{t_i}^{t_{i+1}} \frac{f(\tau)}{\tau}\, d\tau.$$

The terms W_- and W_+ (for which the dependency from η and θ_{D_r} is not explicitly shown) correspond to:

$$W_-(t) = \int_{\mathcal{S}^-_{D_r(t)}} \frac{f(\tau)}{\tau}\, d\tau \qquad W_+(t) = \int_{\mathcal{S}^+_{D_r(t)}} \frac{f(\tau)}{\tau}\, d\tau \tag{31}$$

where the integration sets $\mathcal{S}^-_{D_r(t)}$ and $\mathcal{S}^+_{D_r(t)}$ are defined as:

$$\begin{aligned} \mathcal{S}^-_{D_r(t)} &:= \{\tau : \eta(t,\tau) \ge \eta(t, \theta_{D_r}(t))\} \\ \mathcal{S}^+_{D_r(t)} &:= \{\tau : \eta(t,\tau) > \eta(t, \theta_{D_r}(t))\}. \end{aligned} \tag{32}$$

By applying h in (27) with η_t as specified in (30), the total power $D_r(t)$ is consumed by the appliances with higher values of remaining task time η. Since $\eta(0,\tau)$ is monotone increasing with respect to τ, if such monotonicity is preserved over time the power will be allocated on the appliances with higher parameter τ. In this context, the quantity $\theta_{D_r}(t)$ represents the parameter threshold that determines which appliances are operating. When $\mathcal{T}_d(t) = (\mathcal{S}^-_{D_r(t)} \backslash \mathcal{S}^+_{D_r(t)}) = \{\tau : \eta(t,\tau) = \eta(t,\theta_{D_r}(t))\}$ has positive measure (and therefore $W_-(t) > W_+(t)$), the task time reduction is repartitioned equally among all devices with $\tau \in \mathcal{T}_d(t)$. Note that (30) is a non-canonical PDE (dependence of η_t from τ is implicit through the functions W_- and W_+) whose resolution is beyond the scope of this paper. Since a numerical integration of (30) is always possible in a simulative context, for the subsequent analysis the following hypothesis is introduced:

Assumption 2. Equations (30) have one and only one solution $\eta^\star : [0, t_{i+1} - t_i] \times [t_i, t_{i+1}] \to \mathbb{R}^+$ which is continuous and nondecreasing with respect to τ.

It is now possible to provide the main result of this section:

Theorem 1. For a signal $D_a^*(\cdot) \in \mathcal{D}_a$, the equilibrium conditions (28) on intervals $[t_i, t_{i+1}] \subset [0, T]$ of constant demand always hold for $\eta^* = \eta^\star$.

Proof. See Appendix.

Theorem 1 provides the remaining elements required to characterize the proposed semi-decentralized scheme. This can be implemented with the procedure presented below, inducing a Nash equilibrium in the electricity market.

Algorithm 1 Semi-decentralized power scheduling

(1) The desired profile of aggregate demand D_a^* is initially calculated. Using D_a° in (17) as a starting point, it is possible to determine a profile $D_a^* \in \mathcal{D}_a$ with the properties detailed in Remark 1.

(2) Proposition 1 and 2 return the power density h^* of the appliances that do not require a centralized control. Their power scheduling can be induced in a distributed manner by broadcasting the inflexible demand D_i.

(3) For each time interval $[t_i, t_{i+1}]$ characterized by constant aggregate demand D_a^*, we calculate the centralized scheduling for the subset of devices that require coordination.
  (a) The partial differential equation (30) is numerically integrated, obtaining the remaining rated task time $\eta^\star$.
  (b) The power density h^* and scheduled power $\bar{u}^*$ are given by (27) with $\eta = \eta^\star$ and (14), respectively.

5. SIMULATION RESULTS

The semi-decentralized scheduling presented in the previous section is now tested in simulations. The chosen inflexible demand $\tilde{D}_i$ (not reordered for increasing values) is equal to historical data of total power consumption in the UK grid as measured by National Grid (2016) (blue trace in Fig. 3) and the total energy required by the flexible appliances amounts to $55GWh$. In the considered case study the energy density f, which is defined in (2) and provides a general description of the appliances population, is given by the sum of two truncated gaussians centered at $4h$ and $8h$. This can correspond, for example, to two groups of devices that require about $4h$ and $8h$, respectively, to complete their tasks at rated power. It can be verified that, in the current scenario, a Nash equilibrium cannot be induced in a purely decentralized manner by broadcasting some demand/price signal to the appliances. This is clear from Fig. 1, where it is shown that the necessary and sufficient condition (12) for a decentralized equilibrium, introduced and proved in De Paola et al. (2015), is not satisfied.

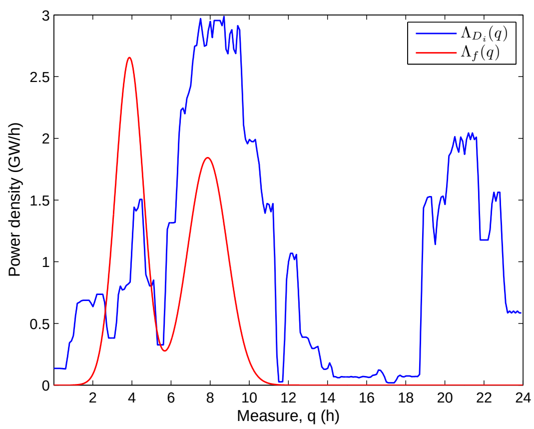

Fig. 1. Negotiable valley capacity Λ_{D_i} (blue trace) and power density of task durations Λ_f (red trace).

We now evaluate if the proposed semi-decentralized scheme can achieve better results. As a preliminary step in our analysis, we apply (16) and obtain the reordered (increasing) profile D_i of flexible demand (blue trace in Fig. 2). Algorithm 1 can now be used to determine the semi-decentralized coordination which induces a Nash equilibrium. In step 1 the desired profile of aggregate demand D_a^* (fulfilling the properties of Remark 1) is calculated. The result in the present case is shown in Fig. 2, where it is compared with the inflexible demand D_i and the profile D_a° in (17), induced by broadcasting D_i to the appliances. One can verify that D_a^* belongs to the class $\mathcal{D}_a$ of Definition 2, with $k = 1$, $N = 3$ and $t_2 = 6.63h$. In other words, D_a^* is constant on the interval $[0, t_2]$ and is equal to D_a° on the remaining time. The equivalent quantities in the original time coordinates (denoted with tilde accent) are presented in Fig. 3. The proposed semi-decentralized scheme will induce the aggregate demand $\tilde{D}_a^*$, guaranteeing a Nash equilibrium and avoiding the oscillation of the fully decentralized profile $\tilde{D}_a^\circ$.

The control actions required to induce D_a^* are now analysed. At step 2 of the design algorithm, applying Proposition 1 for $i = 2$ and Proposition 2 for $i = 1$, we can conclude that all appliances with parameter $\tau > t_2$ must operate at rated power (corresponding to power density $h^*(t, \tau) = f(\tau)/\tau$) until task completion. This behaviour can be induced in a decentralized fashion by broadcasting the inflexible demand D_i and letting them schedule their power consumption in order to minimize the total energy cost. The power consumption of devices with parameter $\tau \le t_2$ on the time interval $[0, t_2]$ is determined in step 3 of Algorithm 1. We numerically integrate equations (30) with $t_i = t_1 = 0h$. The resulting $\eta^\star : [0, t_2] \times [0, t_2] \to \mathbb{R}^+$ has been formally introduced in Assumption 2 and represents the remaining rated task time of the appliances during the centralized phase of the proposed scheduling. For a clearer exposition of the results, we will consider the remaining rated task time $\eta^* : [0, T] \times \mathcal{T} \to \mathbb{R}^+$, evaluated for all the appliances over the whole time interval $[0, T]$:

$$\eta^*(t, \tau) := \tau - \frac{\tau}{f(\tau)} \int_0^t h^*(x, \tau)\, dx. \tag{33}$$

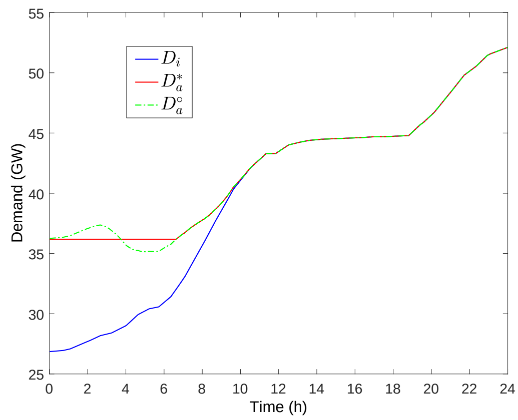

Fig. 2. Reordered profile of inflexible demand D_i (blue), candidate aggregate demand D_a^* for Nash equilibrium (red) and demand profile D_a° defined in (17) and obtained by broadcasting $D = D_i$ (green).

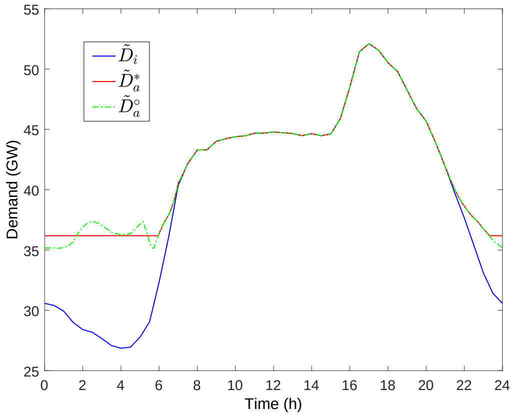

Fig. 3. Inflexible demand $\tilde{D}_i$, candidate aggregate demand $\tilde{D}_a^*$ for Nash equilibrium and aggregate demand $\tilde{D}_a^\circ$ with fully decentralized scheduling, in the original time variable.

The function $\eta^*(t,\tau)$ quantifies the rated task time reduction that a device with parameter τ needs to achieve in the interval $[t,T]$ when the power density h^* is being applied. The values of η^* across the variable τ, for different values of time t, are shown in Fig. 4. Note that an interval characterized by equal values of η^* appears almost immediately. This means that the appliances with higher parameters τ operate at maximum power rate. The remaining power required to track D_a^* is redistributed among the devices with τ in the interval of constant $\eta^*(t,\cdot)$, in order to achieve equal derivative η_t^*. As shown by the magenta trace in Fig. 4 at $t=t_2=6.63h$, we have $\eta^*(t_2,\tau)=0$ when $\tau \in [0,t_2]$ and $\eta^*(t_2,\tau)=\tau-t_2$ when $\tau \geq t_2$. This is coherent with previous results and shows that $\eta^*(t_2,\tau)$ can be brought to zero by imposing that devices with $\tau > t_2$ operate at rated power ($\eta_t(t,\tau)=-1$) until task completion. The last result shown in Fig. 5 is the normalized power profile $u_N^*(t,\tau)=\bar{u}^*(t,\tau,E)/P=\bar{u}^*(t,\tau,E)\frac{\tau}{E}$, where $\bar{u}^*(t,\tau,E)$ defined in (14) is the scheduled power consumption of devices with parameters τ and E associated to the power density at equilibrium h^*. Initially, all appliances are either operating at maximum or at zero power rate. Once the flat region in η^* appears, they gradually converge to some intermediate value. At $t=t_2=6.63h$ the flat region of η^* becomes zero, implying that all loads with $\tau \leq t_2$ have completed their tasks. The remaining ones operate at full power fot $t>t_2$ until they also achieve task completion.

6. CONCLUSIONS

This paper presents a novel semi-decentralized scheme for efficient integration of flexible demand in the power system. By relaxing some assumptions of previous studies and introducing partial coordination of the devices on certain time intervals, we can achieve a Nash equilibrium for large populations of price-responsive appliances, under very general assumptions. The theoretical analysis and the description of the proposed scheme are followed by simulation results that evaluate the achieved performance.

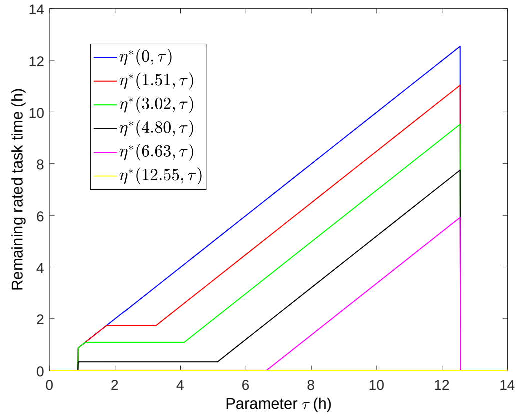

Fig. 4. Remaining rated task time $\eta^*(t,\tau)$ as function of the time parameter τ, evaluated at different time instants t.

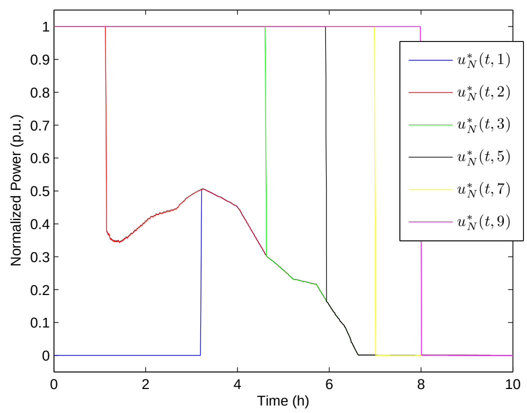

Fig. 5. Normalized power consumption u_N^* across time for devices with different parameters τ.

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Appendix A. PROOF OF THEOREM 1

It is initially shown that $\eta^\star$ in Assumption 2 always fulfils (28a) and (28b). For the latter, it is sufficient to verify that (28b) holds for the third nontrivial case in (30), showing that $W_-(t) \geq D_r(t) \geq W_+(t)$. This is always true given the positivity of f and the fact that, from (32), we have $\mathcal{S}^+_{D_r(t)} \subseteq [\theta_{D_r}(t), t_{i+1}] \subseteq \mathcal{S}^-_{D_r(t)}$. To check (28a), it is sufficient to evaluate its left-hand side for the two cases $W_-(t) = W_+(t)$ and $W_-(t) > W_+(t)$. Condition (28c) is now considered. We first introduce an alternative remaining task time $\eta^\diamond : [0, t_{i+1} - t_i] \times [t_i, t_{i+1}] \to \mathbb{R}^+$, obtained controlling the devices in a fully decentralized manner by broadcasting $D = D_i$:

$$\eta^\diamond(t,\tau) := \begin{cases} \tau - t_i - t & \text{if} \quad \tau > t_i + t \\ 0 & \text{if} \quad \tau \leq t_i + t. \end{cases} \tag{A.1}$$

We evaluate the total power $D_r^\star$ and $D_r^\diamond$ consumed by the considered subset of appliances (with $\tau \in [t_i, t_{i+1}]$) when $\eta^\star$ and $\eta^\diamond$ are applied, respectively:

$$D_r^\star(t) = -\int_{t_i}^{t_{i+1}} \frac{f(\tau)}{\tau} \eta_t^\star(t,\tau)\, d\tau = \int_{\theta_{D_r}(t)}^{t_{i+1}} \frac{f(\tau)}{\tau}\, d\tau \tag{A.2a}$$

$$D_r^\diamond(t) = -\int_{t_i}^{t_{i+1}} \frac{f(\tau)}{\tau} \eta_t^\diamond(t,\tau)\, d\tau = \int_{t+t_i}^{t_{i+1}} \frac{f(\tau)}{\tau}\, d\tau. \tag{A.2b}$$

We also introduce their integrals $D_I^\star$ and $D_I^\diamond$:

$$\begin{aligned} D_I^\star(t) &:= \int_0^t D_r^\star(x)\, dx = \int_{t_i}^{t_{i+1}} \frac{f(\tau)}{\tau} \left[\eta^\star(0,\tau) - \eta^\star(t,\tau)\right] d\tau \\ D_I^\diamond(t) &:= \int_0^t D_r^\diamond(x)\, dx = \int_{t_i}^{t_{i+1}} \frac{f(\tau)}{\tau} \left[\eta^\diamond(0,\tau) - \eta^\diamond(t,\tau)\right] d\tau. \end{aligned} \tag{A.3}$$

From the properties of the considered $D_a^*(\cdot) \in \mathcal{D}_a$, it holds:

$$D_I^\star(t_{i+1} - t_i) = \int_{t_i}^{t_{i+1}} \frac{f(\tau)}{\tau} [\tau - t_i]\, d\tau = \int_{t_i}^{t_{i+1}} \frac{f(\tau)}{\tau} \eta^\star(0,\tau). \tag{A.4}$$

Combining the results of (A.4) and (A.3) at $t = t_{i+1} - t_i$ yields:

$$\int_{t_i}^{t_{i+1}} \frac{f(\tau)}{\tau} \cdot \eta^\star(t_{i+1} - t_i, \tau)\, d\tau = 0.$$

Given the positivity of f, if (28c) is violated for $\eta^* = \eta^\star$, then $\eta^\star(t_{i+1} - t_i, \tau)$ takes negative values for some $\tau \in [t_i, t_{i+1}]$. Since we are assuming that $\eta^\star$ is continuous and its derivative $\eta_t^\star$ is always nonpositive, this only happens if there exists $\bar{t} \in [0, t_{i+1} - t_i)$ and $\bar{\tau} \in [t_i, t_{i+1}]$ such that:

$$\eta^\star(\bar{t}, \bar{\tau}) = 0 \qquad \eta_t^\star(\bar{t}, \bar{\tau}) < 0. \tag{A.5}$$

Having established the monotonicity of $\eta^\star$ in Assumption 2, its support at time t with respect to τ is equal to:

$$\operatorname{supp}(\eta^\star(t,\cdot)) = (\gamma(t), t_{i+1}] \qquad \forall t \in [0, t_{i+1} - t_i] \tag{A.6}$$

for some $\gamma(t) \in [t_i, t_{i+1}]$. It follows that, since $\eta^\star$ is the solution of (30), condition (A.5) is fulfilled (and (28c) is violated by $\eta^\star$) if and only if the following holds at some $\tilde{t} \in [t_i, t_{i+1}]$:

$$\theta_{D_r}(\tilde{t}) < \gamma(\tilde{t}). \tag{A.7}$$

If this is the case, we have:

$$\eta_t^\star(t,\tau) = -1 \qquad \forall \tau \geq \gamma(\tilde{t}) \quad \forall t \leq \tilde{t}. \tag{A.8}$$

To see this, consider that $\eta_t^\star$ fulfils (30) and therefore equality in the variable τ is preserved in $\eta^\star$ across time. Given $\bar{t}$, τ_1 and τ_2 such that $\eta^\star(\bar{t}, \tau_1) = \eta^\star(\bar{t}, \tau_2)$, it holds:

$$\eta^\star(t, \tau_1) = \eta^\star(t, \tau_2) \qquad \forall t \geq \bar{t} \tag{A.9}$$

from which it follows:

$$\eta^\star(t,\tau) > \eta^\star(\tilde{t}, \gamma(\tilde{t})) \qquad \forall \tau > \gamma(\tilde{t}) \quad \forall t < \tilde{t}. \tag{A.10}$$

Since $\theta_{D_r}(t) \leq \theta_{D_r}(\tilde{t}) < \gamma(\tilde{t})$ for $t \leq \tilde{t}$, we can exclude that $\eta_t^\star(t,\tau)$ with $\tau > \gamma(\tilde{t})$ is expressed by the third case in (30), proving (A.8). Similar results can be easily obtained for $\eta^\diamond$ in (A.1):

$$\eta_t^\star(t,\tau) = -1 \qquad \forall \tau \geq t_i + \tilde{t} \quad \forall t \leq \tilde{t} \tag{A.11a}$$

$$\operatorname{supp}(\eta^\star(\tilde{t}, \cdot) = (t_i + \tilde{t}, t_{i+1}]. \tag{A.11b}$$

It is straightforward to show that $\gamma(\tilde{t}) = t_i + \tilde{t}$ and the following equality is fulfilled:

$$\eta^\star(\tilde{t}, \tau) = \eta^\diamond(\tilde{t}, \tau) \qquad \forall \tau \in [t_i, t_{i+1}].$$

Considering (A.3), an equivalent relationship holds for the total power integrals:

$$D_I^\star(\tilde{t}) = D_I^\diamond(\tilde{t}). \tag{A.12}$$

Since we are considering demand profiles $D_a^*(\cdot) \in \mathcal{D}_a$, we can apply Lemma 3 at $\tilde{t} + t_i$ and derive the following inequality:

$$\begin{aligned} D_a^\star(\tilde{t} + t_i) &= D_r^\star(\tilde{t}) - D_i(\tilde{t} + t_i) - \int_{t_{i+1}}^T \frac{f(\tau)}{\tau}\, d\tau \\ &\leq D_r^\diamond(\tilde{t}) - D_i(\tilde{t} + t_i) - \int_{t_{i+1}}^T \frac{f(\tau)}{\tau}\, d\tau = D_a^\diamond(\tilde{t} + t_i). \end{aligned}$$

This can be rewritten as:

$$D_r^\star(\tilde{t}) = \int_{\theta_{D_r}(t)}^{t_{i+1}} \frac{f(\tau)}{\tau}\, d\tau \leq \int_{t_i + \tilde{t}}^{t_{i+1}} \frac{f(\tau)}{\tau}\, d\tau = D_r^\diamond(\tilde{t}) \tag{A.13}$$

Given the positivity of $f(\tau)/\tau$, inequality (A.13) implies that $\theta_{D_r}(\tilde{t}) \geq t_i + \tilde{t} = \gamma(\tilde{t})$. This contradicts (A.7), showing that $\eta^\star$ also fulfils (28c) and concluding the proof.