A note on the stability of nonlinear differential-algebraic systems ⋆

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Abstract: The problem of the stability analysis for nonlinear differential-algebraic systems is addressed using tools from classical control theory. Exploiting Lyapunov Direct Method we provide linear matrix inequalities to establish stability properties of this class of systems. In addition, interpreting the differential-algebraic system as the feedback interconnection of a dynamical system and an algebraic system, a sufficient stability condition has been derived using the small-gain theorem. The proposed techniques are illustrated by means of simple examples.

Keywords: Differential-algebraic systems, descriptor systems, nonlinear systems, Lyapunov functions, small-gain theorem.

1. INTRODUCTION

Analysis and control of differential-algebraic (DA) systems (also known as descriptor systems, singular systems or semi-state systems) have been the subject of increasing interest from the research community in the last decades. Differential-algebraic systems arise, for example, in multi-body mechanical systems where they may represent environmental constraints or constraints related to kinematic joints, see Blajer (1992). For example, large mechanical systems involving thousands of bodies can be modeled as smaller purely differential subsystems interconnected via algebraic constraints, see Pogorelov (1998). A similar modeling approach applies to the interconnection of large-scale electrical networks, where differential-algebraic equations arise from the application of the Kirchoff's laws, see Riaza (2008). Other examples of differential-algebraic systems arise in social economic systems and chemical processes, see Dai (1989) and Kunkel and Mehrmann (2006). Despite significant advances in numerical analysis and simulation of DA systems, see Brenan et al. (1995), the problem of stability analysis and control for general DA systems remains open. Attempts to study DA systems have been undertaken by developing an equivalent ordinary differential equations (ODE) representation. ODE representations require multiple differentiation of the algebraic equations and further mathematical manipulations which poorly suit with the large-scale dimension of many engineering problems. Another approach consists in studying stability properties of DA systems in their original formulation by extending tools from classical and modern control theory. Successful attempts have been made for the case of linear time-invariant DA systems: for example in Müller (2006) an inertia theorem is presented, while the observer design problem is addressed in Müller and Hou (1991) and in Darouach and Boutayeb (1995). Contributions in extending optimal control theory to nonlinear DA systems are in Glad and Sjöberg (2006), where the Hamilton-Jacobi equation is directly formulated for DA systems, and in Sjöberg et al. (2007a), where a sampled-data nonlinear model predictive control scheme with guaranteed stability is presented. Stability analysis with Lyapunov methods is studied in Wang and Zhang (2012) for nonlinear DA systems with delays. In Wu and Mizukami (1995) the Lyapunov stability theory is extended to DA systems and a class of state feedback controllers that guarantee asymptotic stability of uncertain DA systems is proposed. In Coutinho et al. (2004) the stability analysis with guaranteed domain of attraction and control of DA nonlinear systems is studied by means of Lyapunov functions based on the linear matrix inequality framework.

For purely differential systems significant advances have been done in studying the nonlinear equivalent of the $\mathcal{H}_\infty$ control problem. One first contribution to this problem was given by van der Schaft (1992), which showed that the $\mathcal{L}_2$-induced norm can be calculated from the solution of a Hamilton-Jacobi-Isaacs equation or inequality. Other major contributions to nonlinear output feedback $\mathcal{H}_\infty$ control come from Isidori and Astolfi (1992) and Isidori and Kang (1995). However, few results have been developed on $\mathcal{H}_\infty$ schemes for DA systems, see for instance Wang et al. (2002), in which some necessary and sufficient conditions for the existence of a controller solving the $\mathcal{H}_\infty$ control problem for nonlinear DA systems are provided. More recently, $\mathcal{H}_\infty$ control and robust adaptive control for a class of nonlinear DA systems with external disturbances and parametric uncertainties have been studied in Sun and Wang (2013).

The objective of this paper is to present some stability conditions for nonlinear differential-algebraic systems ex-

⋆ This work was supported in part by Imperial College London under the Junior Research Fellowship Scheme.

tending classical tools from nonlinear control theory. In Section 2 two theorems based on Lyapunov Direct Method are introduced to study the local stability of DA systems. At the end of the section a simple example illustrates the results. In Section 3 we reformulate the differential-algebraic system as the feedback interconnection of a purely differential system and a purely algebraic system. With this approach we are able to exploit the small-gain theorem for the stability analysis of DA systems. From the main result we derive simple conditions to achieve global stability and we establish connections with the linear case. At the end of the section two examples are presented: one academic example and a nonlinear system describing an electrical circuit. Finally, Section 4 contains some concluding remark.

Notation. We use standard notation. The superscripts $\top$ and $-\top$ represent the transposition operator and the transposition of the inverse operator, respectively. I represents the identity matrix. The symbol $\mathcal{I}_r(0)$ denotes a ball of radius $r > 0$ and center $x = 0$. The symbols $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$ indicate, respectively, the set of strictly positive real numbers and the set of non-negative real numbers. Given a function $f : \mathbb{R}^n \to \mathbb{R}$ and a manifold $\mathcal{M}$, the symbol $f|_{\mathcal{M}}$ indicates the restriction of f to $\mathcal{M}$. Given a matrix A the symbols $\underline{\sigma}(A)$ and $\bar{\sigma}(A)$ represent the smallest and largest singular value, respectively, of the matrix A. In addition, $\sigma(A)$ indicates the spectrum of the matrix A. The symbol $\|\Sigma\|_\infty$ represents the $\mathcal{H}_\infty$ norm of the linear system Σ. The function $\mathrm{sinc}(x)$ is the cardinal sine function, *i.e.* $\mathrm{sinc}(x) = \frac{\sin(x)}{x}$.

2. LYAPUNOV DIRECT METHOD

In this section we provide some stability conditions for nonlinear DA systems using Lyapunov Direct Method. We conclude the section with an example.

2.1 Stability analysis

Consider a continuous-time, autonomous, differential-algebraic system in semi-explicit form, described by the equations

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2), \\ 0 &= g(x_1, x_2), \end{aligned} \tag{1}$$

where $x_1(t) \in \mathbb{R}^{n_1}$ and $x_2(t) \in \mathbb{R}^{n_2}$ denote the states of the system at time t and f and g are smooth mappings. Before undertaking the stability analysis, some clarifications are required on the nature of such a problem. Given a DA system, the differential index ν is the minimum number of differentiation steps required to transform the DA system into an ODE system. As outlined in Tarraf and Asada (2002) any solution of the differential-algebraic system with index ν must lay on the solution manifold

$$\mathcal{M} = \left\{ (x_1, x_2) : \frac{\partial^k g(x_1, x_2)}{\partial t^k} = 0,\ k = 1, ..., \nu - 1 \right\} \tag{2}$$

and satisfy the algebraic equation in (1) for all time. Note that the solution manifold is not attractive (invariant) in general. Hence, any perturbation of the state may cause the solution to diverge from the manifold. In conclusion, the stability property is addressed for perturbations of the solutions corresponding to consistent initial conditions, *i.e.* which remain on the manifold. Assuming that the origin is an equilibrium point, system (1) can be rewritten in the form

$$\begin{aligned} \dot{x}_1 &= A_{11}(x_1, x_2)x_1 + A_{12}(x_1, x_2)x_2, \\ 0 &= A_{21}(x_1, x_2)x_1 + A_{22}(x_1, x_2)x_2, \end{aligned} \tag{3}$$

where

$$\begin{aligned} &A_{11} : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_1 \times n_1}, \quad A_{12} : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_1 \times n_2}, \\ &A_{21} : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_2 \times n_1}, \quad A_{22} : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_2 \times n_2}. \end{aligned}$$

We also assume that A_{11}, A_{12}, A_{21} and A_{22} are smooth functions. For the sake of clarity the explicit dependence of the matrices $A_{11}, A_{12}, A_{21}, A_{22}$ from the state variables is omitted.

Theorem 1. *Consider system (3). Assume that the matrix A_{22} is square and has full rank for all $(x_1, x_2) \in \mathcal{I}_r(0,0) \subset \mathbb{R}^{n_1+n_2}$. Consider the Lyapunov function*

$$V = x_1^\top P x_1, \tag{4}$$

with $P \in \mathbb{R}^{n_1 \times n_1}$ a symmetric and positive definite matrix. If $V|_{\mathcal{M}} > 0$ and there exists $\alpha > 0$ such that

$$\begin{aligned} &A_{11}^\top P + PA_{11} + (\alpha A_{21}^\top A_{22}^{-\top} - PA_{12})A_{22}^{-1}A_{21} + \\ &+ A_{21}^\top A_{22}^{-\top}(\alpha A_{22}^{-1}A_{21} - A_{12}^\top P) < 0, \end{aligned} \tag{5}$$

for all $(x_1, x_2) \in \mathcal{I}_r(0,0) \subset \mathbb{R}^{n_1+n_2}$, then the origin is a locally asymptotically stable equilibrium point.

Consider now the case in which one wishes to avoid computing the inverse of the matrix A_{22}.

Theorem 2. *Consider system (3) and the Lyapunov function*

$$V = x_1^\top P x_1, \tag{6}$$

with $P \in \mathbb{R}^{n_1 \times n_1}$ a symmetric and positive definite matrix. If $V|_{\mathcal{M}} > 0$ and there exist constants $\alpha > 0$ and $\gamma > 0$ such that

$$\begin{aligned} &A_{11}^\top P + PA_{11} + 2\alpha A_{21}^\top A_{21} + \frac{1}{\gamma^2} PA_{12}A_{12}^\top P < 0, \\ &- 2\alpha A_{22}^\top A_{22} + \gamma^2 I < 0, \end{aligned} \tag{7}$$

for all $(x_1, x_2) \in \mathcal{I}_r(0,0) \subset \mathbb{R}^{n_1+n_2}$, then the origin is a locally asymptotically stable equilibrium point.

2.2 Example 1

Consider the DA system in Scarciotti (2018) described by the equations

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 2 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1+\tilde{\mu} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}, \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & \mathrm{sinc}(x_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}, \end{aligned} \tag{8}$$

where $\tilde{\mu} \in \mathbb{R}$ is a constant parameter. Note that the matrix $A_{22} = I$ is invertible. The linear matrix inequality in (5) becomes

$$\begin{bmatrix} 0 & -3 \\ 1-\tilde{\mu} & -\epsilon-2 \end{bmatrix} P + P \begin{bmatrix} 0 & 1-\tilde{\mu} \\ -3 & -\epsilon-2 \end{bmatrix} + 2\alpha \begin{bmatrix} 1 & -\epsilon \\ -\epsilon & \epsilon^2+1 \end{bmatrix} < 0, \tag{9}$$

which depends on the parameters $\tilde{\mu}$ and

$$\epsilon \in \left[\min_{x_2}\ \mathrm{sinc}(x_2),\ \max_{x_2}\ \mathrm{sinc}(x_2) \right].$$

Select, for instance,

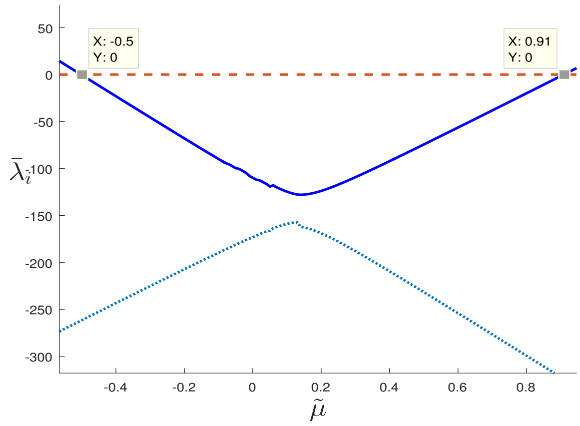

Fig. 1. The eigenvalues $\bar{\lambda}_1$ (dotted line) and $\bar{\lambda}_2$ (solid line) of the left hand side of (9) for different values of $\tilde{\mu}$.

$$P = \begin{bmatrix} 205 & 22 \\ 22 & 35 \end{bmatrix} \tag{10}$$

and $\alpha = 1$ in (9). Consider

$$\bar{\lambda}_i(\tilde{\mu}) = \max_{\epsilon \in [-1,1]} \lambda_i(\epsilon, \tilde{\mu}), \quad i = 1, 2, \tag{11}$$

where $\lambda_i(\epsilon, \tilde{\mu})$ are the eigenvalues of the matrix in the left hand side of (9). Fig. 1 shows that for $\tilde{\mu} \in (-0.5, 0.91)$ both eigenvalues have negative real part, thus the inequality (9) holds. By Theorem 1 we conclude that the origin is a locally asymptotically stable equilibrium of system (8).

3. A SMALL-GAIN THEOREM FOR DA SYSTEMS

In this section we first recall some useful definitions. We then provide a simple condition to have a finite $\mathcal{L}_2$-gain for an algebraic system and then we formulate a small-gain theorem for the stability analysis of nonlinear DA systems. We illustrate the results by mean of two examples.

3.1 Some definitions

We now give the definition of $\mathcal{L}_2$-gain in the spirit of Lemma 3.2.4 of van der Schaft (1996).

Definition 1. The nonlinear system

$$\begin{aligned} \dot{x} &= f(x, u), \\ y &= h(x), \end{aligned} \tag{12}$$

with state $x(t) \in \mathbb{R}^n$, input $u(t) \in \mathbb{R}$ has $\mathcal{L}_2$-gain less than γ if it is dissipative with respect to the supply rate $s(u, y) = \frac{1}{2}\gamma^2 \|u\|^2 - \frac{1}{2}\|h(x)\|^2$ and with a storage function that is differentiable and positive definite with a strict minimizer at $x = 0$.

Remark 1. In the linear case the $\mathcal{L}_2$-gain of a stable system is equal to the $\mathcal{H}_\infty$ norm of its transfer matrix, see Theorem 5.4 in Khalil (1996) or van der Schaft (1996).

3.2 Stability analysis

Consider again system (3) with $A_{22}(x_1, x_2)$ square and locally invertible around the origin. The system can be viewed as the feedback interconnection of the two systems

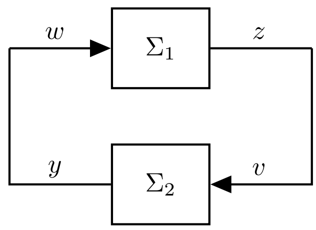

Fig. 2. Feedback interconnection representation of the DA system (3).

$$\Sigma_1 : \begin{cases} \dot{x}_1 = A_{11}(x_1, w)x_1 + A_{12}(x_1, w)w, \\ z = A_{21}(x_1, w)x_1, \end{cases} \tag{13}$$

with input $w(t) \in \mathbb{R}^{n_2}$ and output $z(t) \in \mathbb{R}^{n_2}$, and

$$\Sigma_2 : \begin{cases} 0 = v + A_{22}(x_1, x_2)x_2, \\ y = x_2, \end{cases} \tag{14}$$

with input $v = z$ and output $y = w$, see also Fig. 2. We now provide a preliminary result.

Lemma 1. *Consider system (14). Suppose*

$$\max_{(x_1,x_2) \in \mathbb{R}^{n_1+n_2}} \bar{\sigma}(A_{22}^{-1}(x_1, x_2)) \le k, \tag{15}$$

for some $k > 0$. Then system Σ_2 has finite $\mathcal{L}_2$-gain.

Note that while the differential equation in (3) is not affected by adding the algebraic equation multiplied by a function $\Gamma : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1 \times n_2}$ of the state, the system in (13) is affected by the addition of such a term. Performing this operation we obtain

$$\begin{aligned} \dot{x}_1 =& A_{11}(x_1, w)x_1 + A_{12}(x_1, w)w + \\ &+ \Gamma(x_1, w)(A_{21}(x_1, w)x_1 + A_{22}(x_1, w)w), \end{aligned} \tag{16}$$

or, after some rearrangements,

$$\begin{aligned} \dot{x}_1 =& (A_{11}(x_1, w) + \Gamma(x_1, w)A_{21}(x_1, w))x_1 + \\ &+ (A_{12}(x_1, w) + \Gamma(x_1, w)A_{22}(x_1, w))w. \end{aligned} \tag{17}$$

We now provide a small-gain theorem for the stability analysis of DA systems.

Theorem 3. *Consider system (3) and assume that condition (15) holds. Suppose that the systems Σ_1 is detectable in a closed and bounded set Ω and there exists Γ such that the $\mathcal{L}_2$-gain γ_1 of the modified system described by equation (17) is*

$$\gamma_1 < \frac{1}{\max_{(x_1,x_2) \in \Omega} \bar{\sigma}(A_{22}^{-1}(x_1, x_2))}. \tag{18}$$

Then the origin is a locally asymptotically stable equilibrium point.

Remark 2. The choice of Γ in equation (17) plays an important role in the calculation of the $\mathcal{L}_2$-gain of system Σ_1. As we show in Section 3.4, if a linear system Σ_1 is not asymptotically stable, then a proper choice of the matrix Γ transforms Σ_1 in an asymptotically stable system.

From the previous theorem the next result follows.

Corollary 1. *Consider system (3). Suppose the following conditions hold.*

(C_1) *There exists Γ such that*

$$
\begin{aligned}
&\frac{\partial}{\partial w}[(A_{11}(x_1,w)+\Gamma(x_1,w)A_{21}(x_1,w))x_1+ \\
&+(A_{12}(x_1,w)+\Gamma(x_1,w)A_{22}(x_1,w))w]=0.
\end{aligned} \tag{19}
$$

(C_2) *$x_1 = 0$ is a locally asymptotically stable equilibrium of the system*

$$\dot{x}_1 = A_{11}(x_1,0)+\Gamma(x_1,0)A_{21}(x_1,0))x_1. \tag{20}$$

(C_3) *$A_{22}(0,x_2)$ is invertible.*

Then the origin is an asymptotically stable equilibrium of system (3).

Remark 3. Consider the following linear differential-algebraic system

$$
\begin{aligned}
\dot{x}_1 &= A_{11}x_1 + A_{12}x_2, \\
0 &= A_{21}x_1 + A_{22}x_2,
\end{aligned} \tag{21}
$$

where $A_{11} \in \mathbb{R}^{n_1\times n_1}$, $A_{12} \in \mathbb{R}^{n_1\times n_2}$, $A_{21} \in \mathbb{R}^{n_2\times n_1}$ and $A_{22} \in \mathbb{R}^{n_2\times n_2}$ are constant matrices. Assume that the matrix A_{22} is invertible, then condition (C_1) of Corollary 1 holds for $\Gamma = -A_{12}A_{22}^{-1}$. In addition, condition (C_2) is equivalent to

$$Re(\lambda_i) < 0 \quad \forall \lambda_i \in \sigma(A_{11} - A_{12}A_{22}^{-1}A_{21}). \tag{22}$$

Note that (22) is a necessary and sufficient condition for asymptotic stability of the linear DA system (21) (provided A_{22} is invertible).

3.3 Example 2

Consider system (8). Such a system can be described as the interconnection of the two systems

$$
\Sigma_1 : \begin{cases}
\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 2\\ -4 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1+\tilde{\mu} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}w_1\\ w_2\end{bmatrix}, \\
\begin{bmatrix}z_1\\ z_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & \mathrm{sinc}(x_2)\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix},
\end{cases} \tag{23}
$$

and

$$
\Sigma_2 : \begin{cases}
\begin{bmatrix}0\\ 0\end{bmatrix} = \begin{bmatrix}v_1\\ v_2\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_3\\ x_4\end{bmatrix}, \\
\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}x_3\\ x_4\end{bmatrix}.
\end{cases} \tag{24}
$$

Since the matrix

$$A_{11} = \begin{bmatrix}0 & 2\\ -4 & -2\end{bmatrix} \tag{25}$$

has all eigenvalues with negative real part, system Σ_1 is asymptotically stable. Note that the function $\epsilon = \mathrm{sinc}(x_2)$ is bounded thus an upper bound on the $\mathcal{L}_2$-gain of system Σ_1 can be calculated as

$$\gamma_1 = \max_{\epsilon\in[-1,1]} \|\Sigma_1\|_\infty. \tag{26}$$

Fig. 3 shows the value of γ_1 as a function of $\tilde{\mu}$, from which it is clear that

$$\gamma_1 < 1, \quad \forall \tilde{\mu} \in (-1.44, -0.56). \tag{27}$$

Since $A_{22} = I$ it follows that $\gamma_2 = 1$. Hence, by Theorem 3 the origin is a (globally) asymptotically stable equilibrium point of system (8) for all $\tilde{\mu} \in (-1.44, -0.56)$. Note that the set of $\tilde{\mu}$ for which the equation is stable is different from that in Example 1.

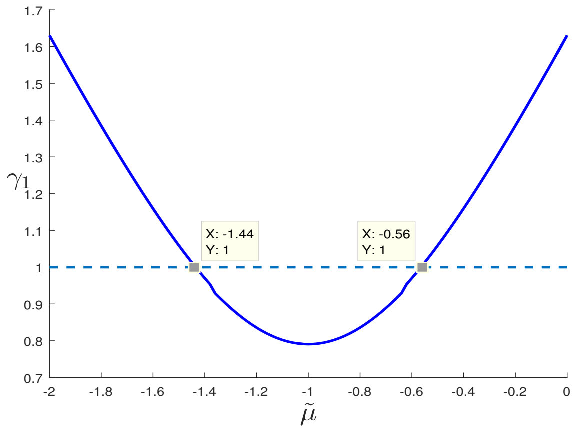

Fig. 3. Plot of γ_1 for different values of $\tilde{\mu}$.

3.4 Example 3

The result presented in Section 3 is validated on the model studied in Sjöberg et al. (2007b) and Scarciotti (2018), which describes the electrical circuit shown in Fig. 4. The model is described by the differential-algebraic equations

$$
\begin{aligned}
\dot{u}_{C_1} &= \frac{1}{2}\frac{i_1}{1+10^{-1}u_{C_1}} + \frac{1}{2}\frac{i_2}{1+10^{-2}u_{C_1}}, \\
\dot{\Phi} &= u_L, \\
0 &= u_R - 5i - 10i^3, \\
0 &= \frac{i_1}{1+10^{-1}u_{C_1}} - \frac{i_2}{1+10^{-2}u_{C_1}}, \\
0 &= u - u_R - u_{C_1} - u_L, \\
0 &= \Phi - \arctan(i), \\
0 &= i - i_1 - i_2,
\end{aligned} \tag{28}
$$

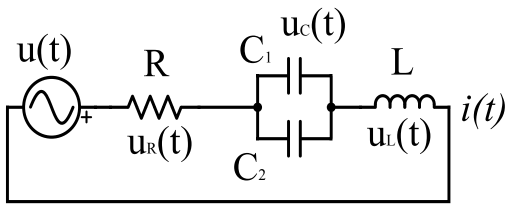

Fig. 4. The electrical circuit in Scarciotti (2018).

where u is an ideal voltage source, u_R is the voltage of the nonlinear resistor, i_1, i_2 and u_{C_1} represent, respectively, the currents and the voltage of the capacitor, and Φ and u_L represent, respectively, the saturated flux and the voltage of the inductor. In the following analysis we assume that the ideal voltage source u is set to zero. System (28) can be written according to the notation used in Section 3, where $(\tilde{x}_1, \tilde{x}_2) = (u_{C_1}, \Phi)$ are the dynamic variables, $(\tilde{x}_3, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6, \tilde{x}_7) = (i, i_1, i_2, u_L, u_R)$ are the algebraic variables and

$$A_{11} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix},\ A_{21} = \begin{bmatrix}0 & 1\\ 0 & 0\\ 0 & 0\\ 0 & 0\\ -1 & 0\end{bmatrix},$$

$$A_{12} = \begin{bmatrix} 0 & \frac{1}{2}\frac{1}{1+0.1\,\tilde{x}_1} & \frac{1}{2}\frac{1}{1+0.01\tilde{x}_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_{22} = \begin{bmatrix} -\frac{\arctan(\tilde{x}_3)}{\tilde{x}_3} & 0 & 0 & 0 & 0 \\ -10\,\tilde{x}_3^2 - 5 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 \\ 0 & \frac{1}{1+0.1\tilde{x}_1} & \frac{-1}{1+0.01\tilde{x}_1} & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}.$$

Since the matrix A_{11} is zero, *i.e.* system Σ_1 is not asymptotically stable, the alternative description (17) is required to apply Theorem 3. Consider, for instance, the choice

$$\Gamma = \begin{bmatrix} 0.5 & 0 & 0.5 & 0 & 0 \\ -5 & 1 & 0 & 0 & 1 \end{bmatrix}, \tag{29}$$

which is calculated according to Remark 3 from the linearized system. Then the alternative description of system Σ_1 is

$$\Sigma_1 : \begin{cases} \dot{x}_1 & = Ax_1 + B(x_1, w)w, \\ z & = Cx_1, \end{cases} \tag{30}$$

where $x_1 = [\tilde{x}_1\ \tilde{x}_2]^T$, $w = [\tilde{x}_3\ \tilde{x}_4\ \tilde{x}_5\ \tilde{x}_6\ \tilde{x}_7]$ and

$$A = A_{11} + \Gamma A_{21}, \tag{31}$$

$$B(x_1, w) = A_{12}(\tilde{x}_1) + \Gamma A_{22}(\tilde{x}_1, \tilde{x}_3), \tag{32}$$

$$C = A_{21}. \tag{33}$$

First note that the matrix A is constant and has all eigenvalues with negative real part, hence the modified system Σ_1 is asymptotically stable. The $\mathcal{L}_2$-gain of system Σ_1 is the smallest γ_1 which satisfies the linear matrix inequality, see Doyle et al. (1989),

$$A^T P + PA + \frac{P\hat{B}^T \hat{B} P}{\gamma_1^2} + CC^T \leq 0, \tag{34}$$

where

$$\hat{B} = \max_{\substack{|x_1|<0.1 \\ |w|<0.1}} B(x_1, w), \tag{35}$$

for some symmetric and positive definite matrix P. For instance, equation (34) is satisfied for

$$P = \begin{bmatrix} 53 & 5 \\ 5 & 1.5 \end{bmatrix} \tag{36}$$

and $\gamma_1 = 0.1225$. Moreover,

$$\max_{\substack{|\tilde{x}_1|<0.1 \\ |\tilde{x}_3|<0.1}} \bar{\sigma}(A_{22}^{-1}(\tilde{x}_1, \tilde{x}_3)) = 7.3677, \tag{37}$$

hence from Lemma 1 it follows that system Σ_2 has finite gain. Since

$$\gamma_1 = 0.1225 < \frac{1}{7.3677} = 0.1357, \tag{38}$$

all conditions of Theorem 3 are satisfied and the origin is a locally asymptotically stable equilibrium point of system (28).

4. CONCLUSION

In this paper we have extended the Lyapunov Direct Method to nonlinear DA systems by proposing two theorems. In addition, interpreting the DA system as the feedback interconnection of a dynamical system and an algebraic system, sufficient stability conditions have also been derived using the small-gain theorem. We have also shown that the proposed results yeld necessary and sufficient stability conditions when applied to linear DA systems. Finally, we have provided three examples to validate the theoretical results.

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