A Poincaré Inequality on Loop Spaces

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Abstract

We investigate properties of measures in infinite dimensional spaces in terms of Poincaré inequalities. A Poincaré inequality states that the L^2 variance of an admissible function is controlled by the homogeneous H^1 norm. In the case of Loop spaces, it was observed by L. Gross [17] that the homogeneous H^1 norm alone may not control the L^2 norm and a potential term involving the end value of the Brownian bridge is introduced. Aida, on the other hand, introduced a weight on the Dirichlet form. We show that Aida's modified Logarithmic Sobolev inequality implies weak Logarithmic Sobolev Inequalities and weak Poincaré inequalities with precise estimates on the order of convergence. The order of convergence in the weak Sobolev inequalities are related to weak L^1 estimates on the weight function. This and a relation between Logarithmic Sobolev inequalities and weak Poincaré inequalities lead to a Poincaré inequality on the loop space over certain manifolds.

1 Introduction

A Poincaré inequality is of the form

$$\int_N (f-\bar{f})^2 \mu(dx) \leqslant \frac{1}{C} \int_N |\nabla f|^2 \mu(dx),$$

where f ranges through an admissible set of real valued functions on a space N, ∇ is a gradient type operator, μ a finite measure on N and hence is often normalised to have total mass 1, and $\bar{f} = \int f d\mu$. For $N = [0, L]$, μ the normalised Lebesgue measure, the constant C is $\frac{4\pi^2}{L^2}$ for C^1 functions satisfying the Dirichlet boundary or the periodic boundary conditions. More generally if N is a compact closed Riemannian manifold, dx the volume measure and ∇ the Riemannian gradient operator, the best constant in the Poincaré inequality is given by taking infimum of the Raleigh quotient

$$\frac{\int_N |df|^2 dx}{\int_N f^2 dx}$$

over the set of non-constant smooth functions of zero mean. For this reason Poincaré inequality is associated with the study of the spectral properties of the Laplacian operator and hence the underlying Riemannian geometry. For quasi isometric Riemannian manifolds, if a Poincaré inequality holds for one manifold it holds for the other.

The Poincaré constant $C = \lambda_1$, that is the first non-trivial eigenvalue of the Laplacian on a compact manifold, is related to the isoperimetric constant in Cheeger's isoperimetric inequality. Standard isoperimetric inequalities say that for an open bounded set A in $\mathbf{R}^n$, the ratio between the area of its boundary ∂A and the volume of A to the power of $1 - \frac{1}{n}$ is minimised by the unit ball. In $\mathbf{R}^2$, it means that $L^2 \geqslant 4\pi A$ where A and L are respectively the area of an open set and L the length of its boundary. By the Federer-Fleming theorem the isoperimetric constant is the same as $\inf_{f \in C_K^\infty} \frac{\|\nabla f\|_{L_1}}{\|f\|_{\frac{n}{n-1}}}$.

In relation to Poincaré inequality, especially in infinite dimensions, the more useful form of isoperimetric inequality is that of Cheeger. Following Cheeger let

$$h = \inf_A \frac{\mu(\partial A)}{\min\{\mu(A), \mu(M/A)\}}.$$

where the infimum is taken over all open subsets of M. Then $h^2 \leqslant 4\lambda_1$ by Cheeger [7]. On the other hand let K be the lower bound of the Ricci curvature. Then it is shown by Buser [6] that $\lambda_1 \leqslant C(\sqrt{K}h + h^2)$ for which M. Ledoux [18] has a beautiful analytic proof. Versions of isoperimetric inequalities for Gaussian measures in infinite dimensional spaces are explained in Ledoux [19] and Ledoux-Talagrand [20].

We take the view that the Poincaré inequality describes properties of the measure μ for a given gradient operator. Poincaré inequality does not hold for $\mathbf{R}^n$ with Lebesgue measure. It does hold for the Gaussian measure. For the standard normalised Gaussian measure, the Poincaré constant is 1 and the corresponding eigenfunction of the Laplacian is the Hermitian polynomial $x/2$. If h is a smooth function μ a measure which is absolutely continuous with respect to the Lebesgue measure with density e^{-2h}, for any f in the domain of d,

$$\int_N |df|^2(x)\mu(dx) = -\int_N \langle f, \Delta f\rangle(x)\mu(dx) - 2\int_N \langle df, dh\rangle \mu(dx).$$

The corresponding Poincaré inequality is then related to the Raleigh quotient of the Bismut-Witten Laplacian $\Delta^h := \Delta + 2L_{\nabla h}$ on $L^2(M, e^{-2h}dx)$. The Bismut-Witten Laplacian

$$\Delta^h : L^2(M, e^{-2h}dx) \to L^2(M, e^{-2h}dx)$$

is unitarily equivalent to the following linear operator on $L^2(M, dx)$:

$$\Box^h = \Delta + (|dh|^2 + \Delta h).$$

The spectral property of Δ^h, hence the validity of the Poincaré inequality for μ is determined by the spectral property of the Schrödinger operator $\Box^h$ on $L^2(M; dx)$.

The state space. A number of infinite dimensional spaces have been the objects of study. They include the space of paths over a finite state space, in particular the space of loops, or more generally space of maps. Our interest in path spaces comes from the desire to understand regularity properties of measures which are distributions of important stochastic processes and to establish a related Sobolev calculus. By path space we mean the space of continuous paths which are not necessarily smooth, of which Wiener space Ω with Wiener measure $\mathbf{P}$ is a primary example. Other natural measures are those induced by stochastic processes such as the Brownian Bridge measure. The properties of Brownian Bridge measures are non-trivial. They are singular measures with respect to the Wiener measure. For the Wiener space the gradient operator would be that related to the Cameron-Martin space of the measure. Interesting functions on the Wiener space such as stochastic integrals are not in general differentiable as real valued functions on the Banach space Ω. They are on the other hand often differentiable in the sense of Malliavin calculus where the functions are differentiated in the directions of the Cameron-Martin space, also called H-differentiation. This will play the role of the standard differentiation on a differentiable manifold. The corresponding gradient operator will be used in the formulation of Poincaré inequality with respect to measures on the Wiener space and on more general spaces of continuous paths.

Main Results. Although a Logarithmic Sobolev inequality holds for the Brownian bridge measure on the Wiener space and for the Brownian motion measure on the path space over a compact manifolds, it may not hold on a general loop space. As noted by L. Gross, [17], Poincaré inequalities do not hold on the Lie group S^1 due to the lack of connectedness of the loop space. A. Eberle, [10], gave an example of a compact simply connected Riemannian manifold on which the Poincaré inequality does not hold for the Brownian bridge measure. Driver-Lohrenz [9] showed that Logarithmic Sobolev inequalities hold on loop groups for the heat kernel measure on loop spaces over a compact type Lie group. For the Brownian bridge measure a positive result was obtained by Aida for the Hyperbolic space H where he obtained a weak form of Logarithmic Sobolev inequality with a weight function. We show here that Aida's type weak logarithmic Sobolev inequality leads to a weak logarithmic Sobolev inequality using the non-homogeneous H_1 norm together with an L^∞ norm. We also show that there is a precise passage from weak Logarithmic Sobolev inequality to weak Poincaré inequality. As a corollary we obtain a Poincaré inequality for the Brownian bridge measure on loop spaces over the hyperbolic space where the Bismut tangent space is defined using the Levi-Civita connection.

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2 The Missing Arguments

On a compact manifold, Poincaré inequality for the Laplace-Beltrami operator is proved by showing that

$$\inf_{f\in H^1, |f|_{L^2}=1, \int f=0} \int_M |\nabla f|^2 dx$$

is attained, by a non constant function. The main ingredient for this method to work is the Rellich-Kondrachov compact embedding theorem of $H^{1,q}$ into L^p, which we do not have in the infinite dimensional situation. The other approach is the dynamic one which we will now explain. It is equivalent to consider the corresponding operator on differential 1-forms. By a Riemannian manifold we mean a connected Riemannian manifold.

We give the standard semi-group argument which in principle works for measures on infinite dimensional spaces. For better understanding assume that the measure concerned is on a finite dimensional Riemannian manifold. Let M be a smooth complete manifold and for $x_0 \in M$ let $(F_t(x_0, \omega), t \geqslant 0)$ be the solution flow to a stochastic differential equation

$$dx_t = \sum_{i=1}^{m} X_i(x_t) \circ dB_t^i + X_0(x_t)dt$$

with initial value x_0. Here X_i are smooth vector fields and ω the chance variable. Let μ_t be the law of F_t with initial distribution μ_0. It is given by

$$\mu_t(A) = \int_{x \in M} P(F_t(x) \in A)\mu_0(dx).$$

If the system is elliptic the $X_i's$ induces a Riemannian metric and the infinitesimal generator is of the form $\frac{1}{2}\Delta + A$ for Δ the Laplace-Beltrami operator for the corresponding Levi-Civita connection and A a vector field called the drift. Suppose that the drift is of gradient form given by a potential function h. Then the system has an invariant measure $\mu(dx) = e^{2h}dx$ which is finite for example if $Ric_x - 2Hess_x(h) > \rho$ for a positive number ρ. Here Ric denotes the Ricci curvature for the intrinsic Riemannian metric. More generally the finiteness of the invariant measure holds even if the lower bound ρ depends on x provided that the quantity

$$\sup_{x \in K} \int_0^\infty \mathbf{E} e^{-\int_0^t \rho(F_s(x,\omega))ds} dt,$$

is finite for any given compact subset K, see [23] [22]. In the following we assume that the system has an finite invariant measure μ and we assume that $P_t f$ converges in $L^2(M;\mu)$ as t goes to infinity. Then

$$\begin{aligned}
\int_M (f - \bar{f})^2 d\mu &= \int_M \left(f^2 - \bar{f}^2\right) \, d\mu = \lim_{t \to \infty} \int_M (f^2 - (P_t f)^2)(x) d\mu(x) \\
&= -\lim_{t \to \infty} \int_M \int_0^t \frac{\partial}{\partial s}(P_s f)^2 ds \, d\mu \\
&= \lim_{t \to \infty} \int_0^t \int_M (dP_s f)^2 \, d\mu \, ds = \int_0^\infty \int_M (dP_s f)^2 \, d\mu \, ds.
\end{aligned}$$

Here d^* is the L^2 adjoint of the differential operator d with respect to the measure μ. For $v_0 \in T_{x_0}M$, let $TF_t(\omega)(v_0)$ be the spatial derivative of $F_t(x_0, \omega_0)$ in the direction of v_0 which in general only exists in the L^2 sense. Define

$$\delta P_t(df)(v_0) = \mathbf{E} df(TF_t(\omega)(v_0)).$$

This extends to a semi-group on bounded differential 1-forms and under suitable conditions solves a corresponding partial differential equation on differential 1-forms. Assume that $d(P_tf) = \delta P_t(df)$, see [22] for conditions for this to hold. The condition $Ric_x - 2Hess_x(h) > \rho$ for some constant $\rho > 0$ implies that the norm of the conditional expectation of the derivative flow is controlled by $e^{-\rho t}$, see [21] for more precise estimate, and hence we have control for $d(P_tf)$ and

$$\begin{aligned}\int_M (f-\bar f)^2 d\mu &\leq \int_0^\infty \int_M \mathbf{E}|df|^2(F_t((x,\omega))\, d\mu\, e^{-\rho s} ds \\ &= \frac{1}{\rho}\int_M |df|^2(x)\, d\mu.\end{aligned}$$

This proof using the equivalence of Poincaré inequality and the semi-group inequality $|P_t|^2_{L^2} \leqslant e^{-\rho t}$. The condition $Ric-2Hess(h)$ is bounded from below is called Bakry-Emery condition [4]. In the case of $M = \mathbf{R}^n$, the standard Gaussian measure corresponding to a system with $Ric \equiv 0$ and the Bakry-Emery condition is exactly the log-convexity condition on measures. In this case $h(x) = -\frac{x^2}{4}$ and the constant in the Poincaré inequality is 1. The Poincaré theorem above can be considered as a generalisation to the Lichnerowicz Theorem, a standard theorem in Riemannian geometry which gives a lower bound for the first eigenvalue of the Laplacian in terms of the lower bound on the Ricci curvature.

In fact under the assumptions given above the stronger Logarithmic Sobolev inequality holds:

$$\int f^2 \log \frac{f^2}{\mathbf{E}|f|^2}\mu(dx) \leqslant \frac{2}{\rho}\int |\nabla f|^2 \mu(dx).$$

For the standard Gaussian measure the logarithmic Sobolev constant is 2. The proof is virtually the same. We apply the same argument to the function $P_tf \log P_tf$, with limit $\bar f \log(\bar f)$, instead of to f on functions bounded below by a positive constant. A Fatou lemma allows the extension to positive functions. The final result is obtained by applying the same argument to $|f|$ and observe that $|\nabla |f|| = |\nabla f|$.

Instead of the equilibrium measure μ on the finite dimensional Riemannian manifold, we study the law of a stochastic process $(F_t(\omega), 0 \leqslant t \leqslant T)$ on the space of paths over M, of which the Wiener measure on the Wiener space is a special case. To apply the semi-group argument we would need to have a good understanding of the semi-group associated to d^*d and corresponding semi-groups on differential 1-forms which is itself an issue to be resolved, except in the case of the classical Wiener space. The semi-group argument is modified and the standard method is the Clark-Ocone formula approach, which combines the problem of defining the unbounded operator d with the investigation of the measure itself.

2.1 Poincaré Inequality for Gaussian Measures

First let μ be a a Gaussian measure whose support is a finite dimensional vector space, $\mathbf{R}^n$. It is not surprising that a function f differentiable in $\mathbf{R}^n$ with $df = 0$ is a constant on this subspace. Let B be a Banach space and μ a mean zero Gaussian measure with B its topological support and covariance operator Γ. The Cameron-Martin space H is the intersection of all vector subspaces of B of full measure and it is a dense set of B. Yet the Gaussian measure μ does not charge H, $\mu(H) = 0$. And μ is quasi translation invariant precisely in the directions of vectors of H. Let $f : B \to \mathbf{R}$ be an L^2 function differentiable in the directions of H and let $\nabla f \equiv \nabla_H f$, an element of H, be the gradient of f defined by $\langle \nabla f, h\rangle_H = df(h)$. The square of the H-norm of the gradient f is precisely $\sum_i |df(h_i)|^2$ where h_i is an orthonormal basis of H. There is a corresponding quadratic form: $\int_B |\nabla f|_H^2(x)\mu(dx)$.

When B is a Hilbert space the Cameron-Martin space is the range of $\Gamma^{\frac{1}{2}}$ and Γ can be considered as a trace class linear operator on B. If f is a BC^1 function, $\nabla_B f$ is defined and $\nabla_H f = \Gamma \nabla_B f$. The associated quadratic form is $\int_B |\Gamma^{-1/2}\nabla_B f|_B^2 d\mu(x)$ and the Poincaré inequality becomes, for f with zero mean,

$$\int f^2(x)\mu(dx) \leqslant \frac{1}{C}\int_B |\Gamma^{-1/2}\nabla_H f|_B^2 d\mu(x).$$

To the quadratic form $\int_B |\Gamma^{-\frac{1}{2}}\nabla_H f|_B^2 d\mu(x)$ there associates a linear operator $\mathcal{L}$ given by

$$\int f\mathcal{L}g d\mu = \int \langle \nabla_H f, \Gamma^{-1}\nabla_H g\rangle_B \, d\mu.$$

The dynamic of the corresponding semi-group is given by the solution of the the Langevin equation $du_t = dW_t - \frac{1}{2}u_t dt$, where W_t is a cylindrical Wiener process on H.

For T any given positive number, define

$$C_0(\mathbf{R}^m) \equiv \Omega = \{\sigma : [0,T] \to \mathbf{R}^m : \sigma(0) = 0 \text{ continuous}\}.$$

The standard Wiener measure $\mathbf{P}$ on Ω is a Gaussian measure with Covariance

$$\Gamma(l_1, l_2) = \int_0^T \int_0^T (s \wedge t) d\mu_{\ell_1}(s) d\mu_{\ell_2}(t)$$

where μ_{ℓ_i} are measures on $[0,T]$ associated to $\ell_i \in \Omega^*$. Its associated Cameron-Martin space is the Sobolev space on $\mathbf{R}^n$ consisting of paths in Ω with finite energy

$$H = \left\{ h : [0,T] \to \mathbf{R}^m \text{ such that } \int_0^T |\dot{h}_t|^2 dt < \infty \right\}.$$

Denote by C_K^∞ the space of real valued functions on N with compact support. Let

$$\mathrm{Cyl} = \{f(\omega_{t_1}, \dots, \omega_{t_k}), f \in C_K^\infty(\overbrace{\mathbf{R}^m \times \cdots \times \mathbf{R}^m}^{k}), 0 < t_1 \leqslant \cdots \leqslant t_k \leqslant T\}.$$

For the cylindrical function f,

$$df(\omega)(h) = \sum_{i=1}^k \partial_i f(\omega_{t_1}, \dots, \omega_{t_k})(h_{t_i}),$$

where $\partial_i f$ stands for differentiation with respect to i-th variable. Hence

$$\nabla f(\omega)(t) = \sum_{i=1}^k \partial_i f(\omega_{t_1}, \dots, \omega_{t_k}) t \wedge t_i$$

where $t \wedge t_i$ denotes $\min(t, t_i)$. The gradient operator, more precisely the associated quadratic form, is associated to the Laplace operator $\mathcal{L} = -\frac{1}{2} d^* d$, where $d^* : L^2(\mathbb{L}(\Omega \to H), \mathbf{P}) \to L^2(\Omega, \mathbf{P})$ is the adjoint of the differential operator d. Note that d^* depends on the measure μ and the norm on the Cameron-Martin space. It is also called the number operator as it acts as a multiplication operator on each chaos of the Wiener Chaos decomposition of the L^2 space: $L^2(\Omega, \mu) = \oplus_{k=0}^\infty H_k$. Then $d^* df = \sum_{n=0}^\infty k I_k(f)$, where $I_k(f)$ is the orthogonal projection of f to the k-th chaos H_k. The operator d whose initial domain the set of smooth cylindrical functions with compact support is known to be a closable operator. Let $\mathbb{D}^{1,2}$ be the closure of d under the graph norm with the graph norm $|f|_{L^2}^2 + \int |\nabla f|^2 \, d\mu$. These are referred as the Sobolev space (defined by H-differentiation). The Gaussian Sobolev space structure can be given to any mean zero Gaussian measures and a Poincaré inequality related to the gradient can be shown to be valid for all functions in $\mathbb{D}^{2,1}$ with Poincaré constant 1. The classical approach to this is to use the symmetric property, rotation invariance, of the Gaussian measure. It is Gross, [16], who obtained the Logarithmic Sobolev inequality and notices its validity in an infinite dimensional space and its relation with Nelson's hypercontractivity. A number of simple proofs have since been given.

The dynamic argument we outlined earlier also works as the Ornstein-Uhlenbeck semi-group P_t for $\mathcal{L}$ has the commutation property: $\nabla P_t f = e^{-t} P_t(\nabla f)$.

The Brownian Bridge measure $\nu_{0,0}$ is the law of the Brownian bridge starting and ending at 0, one of whose realisation is $B_t - \frac{t}{T} B_T$. It can also be realised as solution to the time-inhomogeneous SDE $dx_t = dB_t - \frac{x_t}{T-t} dt$. The Brownian bridge measure is a Radon Gaussian measure and Gaussian measure theory applies to give the required Logarithmic Sobolev inequality as well as the Poincaré inequality with Poincaré constant 1.

2.2 The Path Spaces

Let M be a smooth finite dimensional Riemannian manifold which is stochastically complete. A Brownian motion on M is the strong Markov process x_t with values in M such that probability density of x_t is the heat kernel $p_t(x,y)$. By stochastically complete we mean that $\int_M p_t(x,y)dy = 1$, which holds true if the lower bound of the Ricci curvature , $Ric_x = \inf_{|v|=1} Ric_x(v,v)$, goes to minus infinity slower than $-d^2(x)$, where $d(x)$ denotes the Riemannian distance of x from a fixed point $x_0 \in M$, or by a result of Grigor'yan [15] if the growth of the volume of geodesic balls of radius r has an upper bound of the type: $\int^\infty \frac{rdr}{\log vol(B_{x_0}(r))} = \infty$. Fix a number $T > 0$. We define the path space on M based at x_0 as

$$\mathcal{C}_{x_0} M = \{\sigma : [0,T] \to M, \sigma(0) = x_0 | \ \sigma \text{ is continuous}\}.$$

It is Banach manifold modelled on the Wiener space $\mathcal{C}_0 \mathbf{R}^n$ for n the dimension of the manifold. It is also a complete separable metric space with distance function ρ given by:

$$\rho(\sigma_1, \sigma_2) = \sup_t d(\sigma_1(t), \sigma_2(t)\}.$$

For $y_0 \in M$, define

$$\begin{aligned} \mathcal{C}_{x_0,y_0} M &= \{\sigma \in \mathcal{C}_{x_0} M \mid \sigma(T) = y_0\} \\ L_{x_0} M &= \{\sigma \in \mathcal{C}_{x_0} M \,|\, \sigma(T) = x_0\}. \end{aligned}$$

Both $\mathcal{C}_{x_0,y_0} M$ and $L_{x_0} M$ are closed subspaces of $\mathcal{C}_{x_0} M$ viewed as a metric space.

The Brownian motion measure μ_{x_0} on $\mathcal{C}_{x_0} M$ is the pushed forward measure of $\mathbf{P}$ by the Brownian motion. We view the Brownian motion measure dynamically.

Define the space of cylindrical functions:

$$Cyl = \{F | F(\sigma) = f(\sigma_{t_1}, \dots, \sigma_{t_k}), f \in C_K^\infty(M^k), t_0 < t_1 < \dots < t_k \leqslant T\}.$$

Then

$$\begin{aligned}&\int_{\mathcal{C}_{x_0}M} f(\sigma_{t_1}, \dots, \sigma_{t_k}) d\mu_{x_0}(\sigma)\\ &= \int_M \dots \int_M f(x_1, \dots, x_k) p_{t_1}(x_0, x_1) p_{t_2 - t_1}(x_1, x_2) \dots p_{t_k - t_{k-1}}(x_{k-1}, x_k) \Pi_i dx_i.\end{aligned}$$

Let $ev_t : \mathcal{C}_{x_0}M \to \mathbf{R}$ be the evaluation map at time t. The conditional law of the canonical process $(ev_t, t \in [0,T])$ on $\mathcal{C}_{x_0}M$ given $ev_T(\sigma) = y_0$ is denoted by μ_{x_0,y_0}, hence for a Borel set A of $\mathcal{C}_{x_0}M$,

$$\mu_{x_0,y_0}(A) = \mu_{x_0}(\sigma \in A | \sigma_T = y_0). \tag{1}$$

Restricted to $\mathcal{F}_t$ for $t < T$ the two measures are absolutely continuous with respect to each other with Radon Nikodym derivative given by $\frac{p_{T-t}(y_0, \sigma_t)}{p_T(x_0, y_0)}$. Define

$$Cyl_t = \{F | F(\sigma) = f(\sigma_{s_1}, \dots, \sigma_{s_k}), f \in C_K^\infty(M^k), 0 < s_1 < \dots < s_k \leqslant t < T\}.$$

For $F \in Cyl_t$,

$$\begin{aligned}&p_T(x_0, y_0) \int_{\mathcal{C}_{x_0}M} f(\sigma_{s_1}, \dots, \sigma_{s_n}) d\mu_{x_0,x_1}(\sigma)\\ = &\int_{M^n} f(x_1, \dots, x_n) p_{s_1}(x_0, x_1) \dots p_{s_n - s_{n-1}}(x_{n-1}, x_n) p_{T-s_n}(x_n, y_0) \Pi_{i=1}^n dx_i.\end{aligned}$$

That this defines a measure on $C_{x_0}M$ due to Kolmogorov's theorem and the assumption that for $\beta > 0, \delta > 0$,

$$\int\int d(y,z)^\beta \frac{p_s(x_0, y) p_{t-s}(y, z) p_{T-t}(z, y_0)}{P_T(x_0, y_0)} dydz \leqslant C|t-s|^{1+\delta}, \tag{2}$$

whose validity we discuss later. The Brownian bridge measure μ_{x_0,y_0} starting at x_0 and ending at y_0 charges only the subspace, $\mathcal{C}_{x_0,y_0}(M)$. If $x_0 = y_0$ the Brownian bridge measure only charges the loop space $L_{x_0}M$.

2.3 Where is the Problem?

To see where the problem lies we look at the stochastic differential equation representation for the Brownian bridge measure. The fundamental difference between the dynamic representation for Brownian bridge measure and that for the Brownian motion measure is that the SDE for the Brownian bridge is no longer homogeneous and a singularity develops as t approaches the terminal time. The conditioned Brownian motion realisation of the Brownian bridge on the other hand poses a more artificial problem: the conditioned process is not adapted to the original filtration $\mathcal{F}_t$ of the Brownian motion we started with. It is however adapted to the enlarged filtration $\mathcal{G}_t = \mathcal{F}_t \vee \sigma\{B_T\}$.

Let $X : M \times \mathbf{R}^n \to TM$ be a smooth map with $X(x) : \mathbf{R}^m \to T_xM$ linear for each $x \in M$ and an isometric surjection. We assume that for $v \in T_xM$ and $U \in \Gamma TM$,

$$\nabla_v U = L_{Z^v} U(x)$$

where $Z^v(y) = X(y)Y(x)v$. That such a map X exists and defines the given connection was discussed in [12]. Consider the following stochastic differential equation:

$$dy_t = X(y_t) \circ dB_t. \tag{3}$$

Its infinitesimal generator is given by $\frac{1}{2}\Delta$ for Δ the Laplacian and the solution is the Brownian motion on M. The SDE perturbation by the gradient of the logarithm of the heat kernel

$$dy_t = X(y_t) \circ dB_t + \nabla \log p_{T-t}(y_t, y_0)dt \tag{4}$$

defines a process $(y_t, t < T)$. Here ∇ denotes the Levi-Civita connection. If

$$\int_0^T |\nabla \log p_{T-t}(y_t, y_0)| dt < \infty, \tag{5}$$

$\lim_{t\to T} y_t$ is well defined.

On $\mathbf{R}^n$, the time dependent vector field is $-\frac{y_t - y_0}{T-t}$ and exert a strong pull on the Brownian particle toward y_0. As the Brownian motion measure and the Brownian bridge measure are equivalent on $\mathcal{F}_t$ for $t < T$, the Brownian Bridge cannot explode before the terminal time. That the solution gives rise to the measure $\nu_{0,0}$ on the path space restricted to $\mathcal{F}_t, t < T$ is the consequence of the Girsanov transform: For $t < T$, the law of $\{y_s : s < t\}$ is absolutely continuous with respect to that of the Brownian motion with Radon-Nikodym derivative N_t on $\mathcal{F}_t$ given by

$$e^{\int_0^t \langle \nabla \log P_{T-s}(x_s), X^*(x_s) dB_s\rangle - \frac{1}{2}\int_0^t |\nabla \log P_{T-s}(x_s)|^2 ds} = \frac{P_{T-t}(x_t, y_0)}{P_T(x_0, y)}.$$

Hence they agree on cylindrical functions. To show that they agree everywhere, we only need to show that y_t has continuous sample path, i.e. for some $p > 0$, $\delta > 0$,

$$\mathbf{E} d(y_t, y_s)^p \leqslant C|t-s|^{1+\delta}. \tag{6}$$

We summarise now all conditions that we need so far

Assumption 2.1 (A.) *1.* $\int_M p_t(x, y) dy = 1$.

2. For some constant $p > 0$ *and* $\delta > 0$*,*

$$\int d(y_t, y_s)^p d\mu_{x_0, y_0} \leqslant C|t-s|^{1+\delta}$$

3.

$$\int_0^T |\nabla \log p_{T-t}(y_t, y_0)| dt < \infty, \qquad a.s.$$

4.

$$\int \int d(y, z)^\beta p_s(x_0, y) p_{t-s}(y, z) p_{T-t}(z, y_0) dy dz \leqslant C|t-s|^{1+\delta}.$$

Further gradient estimates on the heat kernel are needed for the validity of integration by parts formulae and Clark-Ocone formulae. See e.g. Driver [8] and Aida [2]. See also Gong-Ma [14] for an alternative formulation of the Clark-Ocone formula. We summarize the known heat kernel estimates here.

- For x not in the cut locus of y, for small t

$$P_t(x, y) = (2\pi t)^{-n/2} e^{\frac{-d(x,y)^2}{2t}} \theta_y(x)^{\frac{-1}{2}} (1 + o(t))$$

where $\theta_y(x)$ is Ruse's invariant. For hyperbolic space,

$$\theta_1(x_0) = \left(\frac{\sinh r(x_0)}{r(x_0)}\right)^{n-1}.$$

- On a compact manifold M, known estimates on the time dependent vector fields are:

$$|\nabla \log p_{T-t}(x,y_0)| \leqslant C\frac{d(x,y_0)}{T-t} + \frac{C}{\sqrt{T-t}}, \qquad t \in [0,T) \tag{7}$$

For the Hyperbolic space, the above assumption holds. For example it is shown in Aida [2] that (7) holds on the hyperbolic spaces. He used the iteration formula for heat kernels for H^n, iterated on n.

3 A weak Logarithmic Sobolev Inequality

For any torsion symmetric metric connection ∇ on the path space, whose parallel translation along a path σ is denoted by $/\!\!/.$, there is the tangent sub-space to $T_\sigma \mathcal{C}_{x_0}M$

$$H_\sigma = \{/\!\!/_s k_s : k \in L_0^{2,1}(T_{x_0}M)\},$$

which we call the Bismut tangent space with Hilbert space norm induced from the Cameron Martin space. The tangent space $T_{x_0}M$ is identified with a copy of $\mathbf{R}^n$ through a chosen linear frame u_0. Let μ be a probability measure on $C_{x_0}M$ including measures which concentrates on a subspace e.g. the loop space. When there is no confusion of which measure is used, we denote by the integral of an function f with respect to μ by $\mathbf{E}f$, its variance $\mathbf{E}(f-\mathbf{E}f)^2$ by $\mathbf{Var}(f)$ and its entropy $\mathbf{E}f\log\frac{f}{\mathbf{E}f}$ by $\mathbf{Ent}(f)$.

The differential operator d is cloasable whenever Driver's integration by parts formula holds. We define $\mathbb{D}^{1,2} \equiv \mathbb{D}^{1,2}(C_{x_0}M)$ to be the closure of smooth cylindrical function Cyl_t, $t < T$ under this graph norm:

$$\sqrt{\int_{C_{x_0}M} |\nabla f|^2_{H_\sigma}(\sigma)\mu(d\sigma) + \int f^2(\sigma)d\mu(\sigma)}.$$

3.1 Aida's inequality and weak Poincaré inequalities

Consider the Laplace Beltrami operator on a complete Riemannian manifold. A Poincaré inequality may not hold. By restriction to an exhausting relatively compact open sets U_n, local Poincaré inequality always exist. The problem is that the Poincaré constant may blow up as n goes to infinity. In [11] Eberle showed that a local Poincaré inequality holds for loops spaces over a compact manifold.

However the computation was difficult and complicated and there wasn't an estimate on the blowing up rate, although it is promising to obtain a concrete estimate from Eberle's frameworks. Once a blowing up rate for local Poincaré inequalities are obtained, we have the so called weak Poincaré inequality and in the case of Entropy we have the weak Logarithmic Sobolev inequality.

$$\begin{aligned}\mathbf{Var}(f) &\leqslant \alpha(s)\int|\nabla f|^2 d\mu + s|f|_\infty^2,\\ \mathbf{Ent}(f^2) &\leqslant \beta(s)\int|\nabla f|^2 d\mu + s|f|_\infty^2.\end{aligned}$$

We assume that α and β to be non-decreasing functions from $(0,\infty)$ to $\mathbf{R}_+$. These inequalities were studied by Aida [1], Röckner-Wang [26], Barthe-Cattiaux-Roberto [5], Cattiaux-Gentil-Guillin [25]. The rate of convergence to equilibrium for the dynamics associated to the Dirichlet form $\int|\nabla f|^2 d\mu$ is strongly linked to Poincaré inequalities. See Aida-Masuda-Shigekawa [3], Aida [1], Mathieu [24], and Röckner-Wang [26]. In the case of weak Poincaré inequalities, exponential convergence is no longer guaranteed. Also the weak Poincaré inequality holds for any α is equivalent to Kusuoka-Aida's weak spectral gap inequality which states that any mean zero sequence of functions f_n in $\mathbb{D}^{1,2}$ with $\mathbf{Var}(f_n)\leqslant 1$ and $\mathbf{E}(|\nabla f|^2)\to 0$ is a sequence which converges to 0 in probability.

Proposition 3.1 *Let μ be any probability measure on $\mathcal{C}_{x_0}M$ with the property that there exists a positive function $u\in\mathbb{D}^{1,2}$ such that Aida's type inequality holds:*

$$\mathbf{Ent}(f^2)\leqslant\int u^2|\nabla f|^2 d\mu,\qquad \forall f\in\mathbb{D}^{1,2}\cap L_\infty \tag{8}$$

Assume furthermore that $|\nabla u|\leqslant a$ and $\int e^{Cu^2}d\mu<\infty$ for some $C,a>0$. Then for all functions f in $\mathbb{D}^{1,2}\cap L_\infty$

$$\mathbf{Ent}(f^2)\leqslant\beta(s)\int|\nabla f|^2 d\mu + s|f|_\infty^2, \tag{9}$$

where $\beta(s)=C|\log s|$ for $s<r_0$ where C and r_0 are constants.

Proof. Let $\alpha_n:\mathbf{R}\to[0,1]$ be a sequence of smooth functions approximating 1 such that

$$\alpha_n(t)=\begin{cases}1 & t\leqslant n-1\\ \in[0,1], & t\in(n-1,n)\\ 0 & t\geqslant n\end{cases} \tag{10}$$

We may assume that $|\alpha_n'| \leqslant 2$. Define

$$f_n = \alpha_n(u) f$$

for u as in the assumption. Then f_n belongs to $\mathbb{D}^{1,2} \cap L_\infty$ if f does. We may apply Aida's inequality (8) to f_n. The gradient of f_n splits into two parts of which one involves f and the other involves ∇f. The part involving the gradient vanishes outside of the region of $A_n := \{\omega : u(\omega) < n\}$ and on A_n it is controlled by g and therefore by n. The part involving f itself vanishes outside $\{\omega : n-1 < u(\omega) < n\}$ and the probability of $\{\omega : n-1 < u(\omega) < n\}$ is very small by the exponential integrability of u. We split the entropy into two terms: $\mathbf{Ent}(f^2) = \mathbf{Ent}(f_n^2) + [\mathbf{Ent}(f^2) - \mathbf{Ent}(f_n^2)]$, to the first we apply the Sobolev inequality (8).

$$\begin{aligned}
\int f_n^2 \log \frac{f_n^2}{\mathbf{E} f_n^2} d\mu &\leqslant \int u^2 |\nabla f_n|^2 d\mu \\
&\leqslant \int u^2 \left[|\nabla f| \alpha_n(u) + |\alpha_n'| |\nabla u| f \right]^2 d\mu \\
&\leqslant \int_{u<n} 2u^2 |\nabla f|^2 \alpha_n^2(u) d\mu + 4a^2 \int_{n-1<u<n} u^2 f^2 d\mu \\
&\leqslant 2n^2 \int |\nabla f|^2 d\mu + 4a^2 n^2 |f|_\infty^2 \, \mu(n > u > n-1).
\end{aligned}$$

Next we compute the difference between $\mathbf{Ent}(f^2)$ and $\mathbf{Ent}(f_n^2)$.

$$\begin{aligned}
&\mathbf{Ent}(f^2) - \mathbf{Ent}(f_n^2) = \int \left(f^2 \log \frac{f^2}{\mathbf{E} f^2} - f_n^2 \log \frac{f_n^2}{\mathbf{E} f_n^2} \right) d\mu \\
&= \int \left(1 - \alpha_n^2(u)\right) f^2 \log \frac{f^2}{\mathbf{E} f^2} d\mu + \int f^2 \alpha_n^2(u) \left(\log \frac{f^2}{\mathbf{E} f^2} - \log \frac{\alpha_n^2(u) f^2}{\mathbf{E} \alpha_n^2(u) f^2} \right) d\mu \\
= \; & I + II.
\end{aligned}$$

Observe that

$$\begin{aligned}
I = \quad & \int \left(1 - \alpha_n^2(u)\right) f^2 \log \frac{f^2}{\mathbf{E} f^2} \, d\mu \\
& \leqslant \int_{u>n-1} f^2 (1 - \alpha_n^2(u)) \log \frac{f^2}{\mathbf{E} f^2} d\mu \\
& \leqslant 2|f|_\infty^2 \int_{u>n-1} \log \frac{|f|}{\sqrt{\mathbf{E} f^2}} \, d\mu.
\end{aligned}$$

By the elementary inequality $\log x \leqslant x$ and Cauchy-Schwartz inequality

$$
\begin{aligned}
I &\leqslant 2|f|_\infty^2 \int_{u>n-1} \left(\frac{|f|}{\sqrt{\mathbf{E} f^2}} \right) \, d\mu \\
&\leqslant 2|f|_\infty^2 \sqrt{\mathbf{E} \left(\frac{|f|}{\sqrt{\mathbf{E} f^2}} \right)^2} \sqrt{\mu(\{u > n-1\})} \\
&\leqslant 2|f|_\infty^2 \sqrt{\mu(u > n-1)}.
\end{aligned}
$$

For the second term of the sum, with the convention that $0 \log 0 = 0$,

$$
II = -\int_{n-1<u<n} f^2 \alpha_n^2(u) \log \alpha_n^2(u) d\mu + \int_{u<n} f^2 \alpha_n^2(u) \log \frac{\mathbf{E} f^2 \alpha_n^2(u)}{\mathbf{E} f^2} \, d\mu
$$

Using the fact that $\log \frac{\mathbf{E} f^2 \alpha_n^2(u)}{\mathbf{E} f^2} \leqslant 0$ from $\alpha_n^2(u) \leqslant 1$ and $x \log x \geqslant -\frac{1}{e}$, we see that

$$
II \quad \leqslant \quad \frac{1}{e} \int_{n-1<u<n} f^2 d\mu \leq \frac{1}{e} \left(|f|_\infty\right)^2 \cdot \; \mu(n-1 < u < n).
$$

Finally adding the three terms together to obtain

$$
\begin{aligned}
\int f^2 \log \frac{f^2}{\mathbf{E} f^2} d\mu \qquad &\leqslant 2n^2 \int |\nabla f|^2 d\mu + \big(4a^2 n^2 + \frac{1}{e}\big) |f|_\infty^2 \, \mu(n-1 < u < n) \\
&\quad + |f|_\infty^2 \sqrt{\mu(u > n-1)}
\end{aligned}
$$

which can be further simplified to the following estimate:

$$
\int f^2 \log \frac{f^2}{\mathbf{E} f^2} \, d\mu \leqslant 2n^2 \int |\nabla f|^2 d\mu + \big(4a^2 n^2 + \frac{1}{e} + 1\big) |f|_\infty^2 \sqrt{\mu(u > n-1)}. \tag{11}
$$

The exponential integrability of u will supply the required estimate on the tail probability,

$$
\sqrt{\mu(u > n-1)} \leqslant e^{-\frac{C}{2}(n-1)^2} \sqrt{\mathbf{E} e^{Cu^2}}
$$

Define $b(r) = (4a^2 r^2 + \frac{1}{e} + 1) e^{-\frac{C}{2}(r-1)^2}$. Then

$$
\int f^2 \log \frac{f^2}{\mathbf{E} f^2} d\mu \leqslant 2n^2 \int |\nabla f|^2 d\mu + b(n) |f|_\infty^2.
$$

For r sufficiently large, $b(r)$ is a strictly monotone function whose inverse function is denoted by $b^{-1}(s)$ which decreases exponentially fast to 0. Define $\beta(s) = b^{-1}(2s^2)$. For any s small choose $n(s)$ to be the smallest integer such that $s \geqslant b(n)$. Then

$$\int f^2 \log \frac{f^2}{\mathbf{E} f^2} d\mu \leqslant \beta(s) \int |\nabla f|^2 d\mu + s|f|_\infty^2$$

Here $\beta(s)$ is of order $|\log s|$ as $s \to 0$. □

Note that in the above proof we only needed the weak integrability of the function u^2, or the estimate $\mu(u > n-1)$. This leads to the following :

Remark 3.2 *If (8) holds for $u \in \mathbb{D}^{2,1}$ with the property $|\bigtriangledown u| \leqslant a, u \geqslant 0$ and*

$$\mu(u^2 > s^2) < m^2(s),$$

for a non-increasing function m of the order $o(s^{-2})$, then by (11), the weak Poincaré inequality holds with $\beta(s)$ of the order of the inverse function of $(s^2 + 2)m(s)$.

3.2 Relation between various inequalities

The functional inequalities for a measure describes how the L^2 or other norms of a function is controlled by its derivatives with a universal constant. They describe the concentration of an admissible function around its mean. A well chosen gradient operator is used to give these control. On the other hand concentration inequalities are related intimately with isoperimetric inequalities. For finite dimensional spaces it was shown in the remarkable works of Cattiaux-Gentil-Guillin [25] and Barthe-Cattiaux-Roberto [5] for measures in finite dimensional spaces one can pass from capacity type of inequalities to weak Logarithmic Sobolev inequalities and vice versa with great precision. Similar results holds for weak Poincaré inequalities. This gives a great passage between the two inequalities. We give here a direct proof that this works wonderfully in infinite dimensional spaces. The proof is somewhat standard and is inspired by the two previous mentioned articles and that of Ledoux [18].

Proposition 3.3 *If for all f bounded measurable functions in $\mathbb{D}^{2,1}(C_{x_0}M)$, the weak logarithmic Sobolev inequality holds for $0 < s < r_0$, some given $r_0 > 0$,*

$$\mathbf{Ent}(f^2) \leqslant \beta(s) \int |\nabla f|^2 d\mu + s|f|_\infty^2$$

where $\beta(s) = C\log\frac{1}{s}$ *for some constant* $C > 0$*, Then Poincaré inequality*

$$\mathbf{Var}(f) \leqslant \alpha \int |\nabla f|^2 d\mu.$$

holds for some constant $\alpha > 0$*.*

Proof. By the minimizing property of the variance for any real number m,

$$\mathbf{Var}(f) \leqslant \int ((f-m)^+)^2 d\mu + \int ((f-m)^-)^2 d\mu. \tag{12}$$

We choose m to be the median of f such that $\mu(f - m > 0) \leqslant \frac{1}{2}$ and $\mu(f - m < 0) \leqslant \frac{1}{2}$.

Let g be a positive function in $\mathbb{D}^{2,1}$ such that $\int g^2 d\mu = 1$ and $\mu\{g \neq 0\} \leqslant \frac{1}{2}$. Here we take $g = g_1$ or $g = g_2$ for

$$g_1 = \frac{(f-m)^+}{\sqrt{\int ((f-m)^+)^2 d\mu}} \text{ or } \quad g_2 = \frac{(f-m)^-}{\sqrt{\int ((f-m)^-)^2 d\mu}}. \tag{13}$$

For $\delta_0 > 0$ and $\delta > 1$ and $0 < \delta_0 < \delta_1 < \delta_2 < \dots$ with $\delta_n = \delta_0 \delta^n$,

$$\begin{aligned}
\mathbf{E}g^2 &= \int_0^{+\infty} 2s\mu(|g| > s)ds \\
&= \int_0^{\delta_1} 2s\ \mu(|g| > s)ds + \sum_{n=1}^{\infty} \int_{\delta_n}^{\delta_{n+1}} 2s\mu(|g| > s)ds \\
&\leqslant \int_0^{\infty} 2s\ \mu(|g| \wedge \delta_1 > s)ds + \sum_{n=1}^{\infty} \int_{\delta_n}^{\delta_{n+1}} 2s\mu(|g| > s)ds
\end{aligned}$$

Consequently we have,

$$\mathbf{E}g^2 \leqslant \mathbf{E}(g \wedge \delta_1)^2 + \sum_{n=1}^{\infty} \int_{\delta_n}^{\delta_{n+1}} 2s\mu(|g| > s)ds \tag{14}$$

Define

$$I_1 := \mathbf{E}(g \wedge \delta_1)^2, \qquad I_2 := \sum_{n=1}^{\infty} \int_{\delta_n}^{\delta_{n+1}} 2s\mu(|g| > s)ds.$$

Recall the following entropy inequality. If $\varphi : \Omega \to [-\infty, \infty)$ is a function such that $\mathbf{E}e^{\varphi} \leqslant 1$ and G is a real valued random function such that φ is finite on the support of G, then

$$\int G^2 \varphi d\mu \leqslant \mathbf{Ent}(G^2).$$

Here we take the convention that $G^2\varphi = 0$ where $G^2 = 0$ and $\varphi = \infty$. Let

$$\varphi := \begin{cases} \log 2 & \text{if } g > 0, \\ -\infty & \text{otherwise.} \end{cases}$$

a then $\int e^{\varphi} d\mu = 2\mu(g \neq 0) \leqslant 1$. Hence

$$\mathbf{Ent}((g \wedge \delta_1)^2) \geqslant \int (g \wedge \delta_1)^2 \varphi d\mu$$

so that

$$\mathbf{E}((g \wedge \delta_1)^2) \leqslant \frac{1}{\log 2} \mathbf{Ent}((g \wedge \delta_1)^2).$$

We apply the weak logarithmic Sobolev inequality

$$\mathbf{Ent}(f^2) \leqslant \beta(r)\mathbf{E}|\nabla f|^2 + r|f|_\infty^2$$

to $g \wedge \delta_1$ to obtain, for some $r < r_0$,

$$\mathbf{E}(g \wedge \delta_1)^2 \leqslant \frac{\beta(r)}{\log 2} \cdot \int |\nabla g|^2 1_{g<\delta_1} \, d\mu + \frac{r \cdot \delta_1^2}{\log 2}. \tag{15}$$

Now we are going to estimate I_2. For $n = 0, 1, \ldots$, let

$$g_n = (g - \delta_n)^+ \wedge (\delta_{n+1} - \delta_n).$$

Then $g_n \in \mathbb{D}^{1,2}$, $\mathbf{E}g_n^2 \leqslant 1$ and

$$|\nabla g_n| \leqslant |\nabla g| 1_{\delta_n \leqslant g \leqslant \delta_{n+1}}.$$

From $g_n \geqslant (\delta_{n+1} - \delta_n) I_{\{g > \delta_{n+1}\}}$,

$$\mu(g > \delta_{n+1}) \leqslant \frac{\mathbf{E}g_n^2}{(\delta_{n+1} - \delta_n)^2}.$$

Next we observe that for $n \geqslant 1$,

$$\int_{\delta_n}^{\delta_{n+1}} 2s\ \mu(|g| > s)ds \leqslant \mu(g > \delta_n)\cdot(\delta_{n+1}^2 - \delta_n^2)$$
$$\leqslant \frac{\delta_{n+1}^2 - \delta_n^2}{(\delta_n - \delta_{n-1})^2}\mathbf{E}g_{n-1}^2 \tag{16}$$
$$= \delta^2\frac{\delta+1}{\delta-1}\mathbf{E}g_{n-1}^2.$$

Next we compute $\mathbf{E}g_n^2$. We'll chose a function φ_n which can be used to estimate the L^1 norm of g_n^2 by its entropy. Define

$$\varphi_n := \begin{cases} \log\delta_n^2 & \text{if } g > \delta_n, \\ -\infty & \text{otherwise.} \end{cases}$$

Then $\int e^{\varphi_n} d\mu = \delta_n^2 \mu(g > \delta_n) \leqslant 1$, hence

$$\mathbf{Ent}(g_n^2) \geqslant \int g_n^2 \varphi_n d\mu.$$

Thus,

$$\mathbf{E}g_n^2 \leqslant \frac{1}{\log\delta_n^2}\mathbf{Ent}(g_n^2) \leqslant \frac{1}{2\log\delta_0 + 2n\log\delta}\mathbf{Ent}(g_n^2). \tag{17}$$

By (16) and (3.2) the second term in $\mathbf{E}g^2$ is controlled by the entropy of the functions g_n^2 to which we may apply the weak logarithmic Sobolev inequality with constants $r_n < r_0$. The constant r_n are to be chosen later.

$$\begin{aligned} &\int_{\delta_{n+1}}^{\delta_{n+2}} 2s\ \mu(|g| > s)ds \\ &\leqslant \delta^2\frac{\delta+1}{\delta-1}\cdot\frac{1}{2\log\delta_0 + 2n\log\delta}\mathbf{Ent}(g_n^2) \\ &\leqslant \delta^2\frac{\delta+1}{\delta-1}\cdot\frac{1}{2\log\delta_0 + 2n\log\delta}\left(\beta(r_n)\int|\nabla g|^2 I_{\delta_n\leqslant g<\delta_{n+1}}d\mu + r_n\cdot|g_n|_\infty^2\right). \end{aligned} \tag{18}$$

Note that $|g_n|_\infty \leqslant \delta_{n+1} - \delta_n$ and summing up in n we have,

$$\begin{aligned} I_2 \leqslant &\frac{\delta^2(\delta+1)}{2(\delta-1)}\sum_{n=0}^{\infty}\frac{\beta(r_n)}{\log\delta_0 + n\log\delta}\int|\nabla g|^2 I_{\delta_n\leqslant g<\delta_{n+1}}d\mu \\ &+ \frac{\delta^2-1}{2}\sum_{n=0}^{\infty}\frac{\delta_0^2\cdot\delta^{2n+2}}{\log\delta_0 + n\log\delta}\cdot r_n \end{aligned} \tag{19}$$

Denote

$$b_{-1} = \frac{\beta(r)}{\log 2}, \quad b_n = \frac{\delta^2(\delta+1)}{2(\delta-1)} \frac{\beta(r_n)}{\log \delta_0 + n \log \delta}$$

and

$$c_{-1} = \frac{r \cdot \delta_1^2}{\log 2}, \quad c_n = \frac{\delta^2 - 1}{2} \sum_{n=0}^{\infty} \frac{\delta_0^2 \cdot \delta^{2n+2}}{\log \delta_0 + n \log \delta} \cdot r_n$$

Finally combining (15) with (19) we have

$$\mathbf{E} g^2 \leqslant \sum_{n=-1}^{\infty} b_n \int |\nabla g|^2 1_{\{\delta_{n-1} \leqslant g < \delta_n\}} d\mu + \sum_{n=-1}^{\infty} c_n$$

We'll next choose r_n so that $\sum c_n < 1/2$ and that the sequence b_n has an upper bound. This is fairly easy by choosing that r_n of the order $\frac{\delta^{-(2n+2)}}{n}$. Taking $g = g_1$, we see that

$$\begin{aligned} 1 = \mathbf{E} g_1^2 &\leqslant \sup_n (b_n) \int |\nabla g_1|^2 d\mu + \sum c_n \\ &\leqslant \sup_n (b_n) \frac{1}{\mathbf{E}[(f-m)^+]^2} \int |\nabla f|^2 1_{\{f>m\}} d\mu + \sum c_n. \end{aligned}$$

Hence

$$\mathbf{E}[(f-m)^+]^2 \leq 2 \sup_n (b_n) \int |\nabla f|^2 1_{\{f>m\}} d\mu,$$

$$\mathbf{E}[(f-m)^-]^2 \leq 2 \sup_n (b_n) \int |\nabla f|^2 1_{\{f>m\}} d\mu.$$

The Poincaré inequality follows. □

Remark 3.4 *We could optimize the constant in the Poincaré inequality. For example when $r_0 = 1/2$, we let $\epsilon = 1/8, \delta = \sqrt{2}, \delta_0 = 2^{\frac{9}{2}}$, the Poincaré constant is approximately $40.82C$, which is smaller than that given in Cattiaux-Gentil-Guillin [25]. However we do not expect to have a sharp estimate on the constant.*

Proof. We need to choose the r_n, δ, δ_0 carefully to optimise on the constant. Assume that $\mathbf{E} g^2 = 1$ for simplicity. We choose suitable constants $\delta_0,\ \delta,\ \epsilon$ satisfying $\frac{\epsilon}{\delta_0^2 \cdot \delta^2} < r_0$ and take $r = \frac{\epsilon}{\delta_0^2 \cdot \delta^2}$ in b_{-1} and recall that $\beta(s) = C \log \frac{1}{s}$ here. Then

$$I_1 = \mathbf{E}(g \wedge \delta_1)^2 \leqslant \frac{C \cdot \log\left(\frac{\delta_0^2 \cdot \delta^2}{\epsilon}\right)}{\log 2} \cdot \int |\nabla g|^2 1_{g<\delta_1}\, d\mu + \frac{\epsilon}{\log 2} \qquad (20)$$

Next we take

$$r_n := \frac{\log \delta^2}{\delta_0^2 \cdot \delta^{2n+2} (\log \delta_0 + n \log \delta)^2} = \frac{1}{\delta_0^2 \cdot \delta^{2n+2}} \cdot \frac{1}{(\frac{\log \delta_0}{\log \delta} + n)^2}.$$

Choose δ_0, δ so that $r_n < r_0$ for each $n \geqslant 0$, and $\log \log \delta_0 > 0$. For simplicity we denote $A := \frac{\log \delta_0}{\log \delta}$. Note that $\beta(s) = C \log \frac{1}{s}$ in (11) and

$$\log \frac{1}{r_n} = 2 \log \delta_0 + 2(n+1) \log \delta + 2 \log(A+n)$$

It follows that

$$\begin{aligned} I_2 \leqslant\ & C\delta^2 \cdot \frac{\delta+1}{\delta-1} \sum_{n=0}^{\infty} \big(1 + \frac{1}{n+A} + \frac{\log(n+A)}{n+A} \cdot \frac{1}{\log \delta}\big) \int |\nabla g|^2 I_{\{\delta_n \leqslant g < \delta_{n+1}\}} d\mu \\ & + \frac{\delta^2 - 1}{2 \log \delta} \sum_{n=0}^{\infty} \frac{1}{(n+A)^3} \\ \leqslant\ & C\delta^2 \cdot \frac{\delta+1}{\delta-1} \big(1 + \frac{1}{A} + \frac{\log A}{A} \cdot \frac{1}{\log \delta}\big) \int |\nabla g|^2 I_{\{\delta_0 \leqslant g\}} d\mu + \frac{\delta^2-1}{4 \log \delta} \cdot \frac{1}{(A-1)^2} \end{aligned} \tag{21}$$

Let

$$C_1(\delta, \delta_0, \epsilon) := C\delta^2 \cdot \frac{\delta+1}{\delta-1} \big(1 + \frac{1}{A} + \frac{\log A}{A} \cdot \frac{1}{\log \delta}\big)$$

$$C_2(\delta, \delta_0, \epsilon) := \frac{C \cdot \log\big(\frac{\delta_0^2 \cdot \delta^2}{\epsilon}\big)}{\log 2}$$

$$C_3(\delta, \delta_0, \epsilon) := \frac{\delta^2 - 1}{4 \log \delta} \cdot \frac{1}{(A-1)^2} + \frac{\epsilon}{\log 2}$$

So from (20) and (21) and the assumption $\mathbf{E} g^2 = 1$, we have:

$$\mathbf{E} g^2 \leqslant \frac{C_1(\delta, \delta_0, \epsilon) + C_2(\delta, \delta_0, \epsilon)}{1 - C_3(\delta, \delta_0, \epsilon)} \int |\nabla g|^2 d\mu \tag{22}$$

provided we choose suitable constants $\delta, \delta_0, \epsilon$ to make $C_3(\delta, \delta_0, \epsilon) < 1$. Apply the above estimate to g_1, g_2 and these together with (12) give the required inequality.

$\square$

When the function $\beta(s)$ in weak logarithmic Sobolev inequality is of order greater than $\log\frac{1}{s}$, we no longer have a Poincaré inequality, but a weak Poincaré inequality is expected. In fact there is the following relation. The finite dimensional version can be found in [5]). We give here a direct proof without going through any capacity type inequalities.

Remark 3.5 *If for all bounded measurable functions f in $\mathbb{D}^{1,2}(C_{x_0}M)$, the weak logarithmic Sobolev inequality holds for $s < r_0$, some given $r_0 > 0$ and a non-increasing function $\beta : (0, r_0) \longmapsto R^+$,*

$$\mathbf{Ent}(f^2) \leqslant \beta(s)\mathbf{E}|\nabla f|^2 + s|f|_\infty^2$$

Then there exist constants $r_1 > 0, C_1, C_2$ such that for all $s < r_1$, the weak Poincaré inequality

$$\mathbf{Var}(f) \leqslant \frac{\beta\left(C_2 s \log\frac{1}{s}\right)}{C_1 \log\frac{1}{s}}\mathbf{E}(|\nabla f|^2) + s|f|_\infty^2.$$

holds.

Proof. As a Poincaré inequality is not expected, we need to cut off the integrand at infinity. We keep the notation of the proof of Proposition 3.3. Let $\delta_n = \delta_0 \cdot \delta^n$ for some $\delta_0 > 1,\ \delta > 1$ and the function g as in (13). We have

$$\begin{aligned}\mathbf{E}g^2 = \mathbf{E}(g\wedge\delta_1)^2 + \sum_{n=1}^{N+1}\int_{\delta_n}^{\delta_{n+1}} 2s\mu(g>s)ds \\ + \sum_{n=N+1}^{2N+1}\int_{\delta_n}^{\delta_{n+1}} 2s\mu(g>s)ds + \int_{\delta_{2N+2}}^{\infty} 2s\mu(g>s)ds\end{aligned} \tag{23}$$

First from $\mathbf{E}g^2 = 1$, we have the following tail behaviour:

$$\int_{\delta_{2N+2}}^{\infty} 2s\mu(g>s)ds = \mathbf{E}(g^2-\delta_{2N+2})_+^2 \leqslant |g|_\infty^2\mu(g>\delta_{2N+2}) \leqslant \frac{1}{\delta_0^2\delta^{4N+4}}|g|_\infty^2 \tag{24}$$

We now consider δ^{4N+4} to be of order $1/s$. For the first two terms of (23), we use estimates from the previous proof. First recall (15),

$$\mathbf{E}(g\wedge\delta_1)^2 \leqslant \frac{\beta(r)}{\log 2}\cdot\int|\nabla g|^2 1_{g<\delta_1}\,d\mu + \frac{r\cdot\delta_1^2}{\log 2}$$

Next by (18), we have:

$$\sum_{n=1}^{N+1}\int_{\delta_n}^{\delta_{n+1}} 2s\mu(g>s)ds$$
$$\leqslant C_2\sum_{n=0}^{N}\frac{\beta(r_n)}{n+C_3}\int|\nabla g|^2 I_{\delta_n\leqslant g<\delta_{n+1}}d\mu + C_2\sum_{n=0}^{\infty} r_n\cdot\frac{\delta^{2n}}{n+C_3}$$

Here C_2, C_3 are some constants depending on δ_0 and δ and $C_3 = \frac{\log\delta_0}{\log\delta}$. For $n = 0, 1, \ldots, N$, take

$$r_n = \frac{1}{\delta^{2n}\cdot(n+C_3)}.$$

We may assume that $\beta(r)$ is an increasing function of order greater than $\log(\frac{1}{r})$ for r small, in which case

$$\frac{\beta(\frac{1}{\delta^{2n}\cdot(n+C_3)})}{n+C_3}$$

is an increasing function of n for n sufficiently large. Hence

$$\sum_{n=1}^{N+1}\int_{\delta_n}^{\delta_{n+1}} 2s\mu(g>s)ds \leqslant C_2\frac{\beta\left(\frac{1}{\delta^{2N}\cdot(N+C_3)}\right)}{N+C_3}\int|\nabla g|^2 I_{\delta_0\leqslant g<\delta_{N+1}}d\mu + \frac{C_2}{C_3-1}. \tag{25}$$

If we apply this estimate to the whole range $n \leq 2N$, $\frac{\beta(r_{2N})}{2N}$ would be the order of $\beta(\frac{s}{|\log s|})$. However to make the estimate more precise, we take a different rate function r_n for $N+1 \leq n \leq 2N$. Let $r_n = \frac{N}{\delta^{4N}}$ in (26) and we will give a more precise estimate on $|g_n|_\infty$. Apply (18) again to the sum from $N+1$ to $2N$ in (23)

$$\sum_{n=N+1}^{2N+1}\int_{\delta_n}^{\delta_{n+1}} 2s\mu(g>s)ds$$
$$\leqslant C_2\sum_{n=N}^{2N}\frac{\beta(r_n)}{n+C_3}\int|\nabla g|^2 I_{\delta_n\leqslant g<\delta_{n+1}}d\mu + C_2\sum_{n=N}^{2N}\frac{r_n}{n+C_3}\cdot|g_n|_\infty^2. \tag{26}$$

Since g is bounded, there is k such that $\delta_k < |g|_\infty \leqslant \delta_{k+1}$ for some integer k.

$$\begin{aligned}\sum_{n=0}^{\infty}|g_n|_\infty^2 &= \sum_{n=0}^{k-1}(\delta_{n+1}-\delta_n)^2 + (|g|_\infty - \delta_k)^2 \\ &\leqslant \left(\sum_{n=0}^{k-1}(\delta_{n+1}-\delta_n) + |g|_\infty - \delta_k\right)^2 \\ &= (|g|_\infty - \delta_0)^2\end{aligned}$$

Hence

$$\sum_{n=0}^{\infty}|g_n|_\infty^2 \leqslant |g|_\infty^2.$$

Recall that $r_n = \frac{N}{\delta^{4N}}$,

$$\begin{aligned}&\sum_{n=N+1}^{2N+1}\int_{\delta_n}^{\delta_{n+1}} 2s\mu(g>s)ds \\ &\leqslant C_2 \cdot \frac{\beta(\frac{N}{\delta^{4N}})}{N+C_3}\int |\nabla g|^2 I_{\delta_N \leqslant g < \delta_{2N+1}} d\mu + \frac{C_2}{\delta^{4N}}|g|_\infty^2\end{aligned} \tag{27}$$

Now adding estimates to all terms in (23) together, (24-27), and rearrange the constants. We also note that $\mathbf{E}g^2 = 1$ and obtain for N large enough

$$1 \leqslant C_1 \frac{\beta(\frac{N}{\delta^{4N}})}{N}\int |\nabla g|^2 d\mu + \frac{C_2}{\delta^{4N}}|g|_\infty^2 + \frac{r\delta_1^2}{\log 2} + \frac{C_2}{C_3 - 1} \tag{28}$$

Here we use the monotonicity of β: $\beta(\frac{N}{\delta^{4N}}) \geqslant \beta(\frac{1}{\delta^{2N}\cdot(N+C_3)})$. Take r small and δ_0 large so that $\frac{r\delta_1^2}{\log 2} + \frac{C_2}{C_3-1} < 1$. Let $s = \frac{1}{\delta^{4N}}$ in (28), the required result follows.

□

Corollary 3.6 *Let μ be a probability measure. Suppose that there is a positive function $\mu(u^2 > s^2) \sim m^2(s)$ some increasing function m of order $o(s^{-2})$ for s small and such that $|\nabla u| \leqslant a, a > 0$ and for all $f \in \mathbb{D}^{2,1}$,*

$$\mathbf{Ent}(f^2) \leqslant \int g = u^2 |\nabla f|^2 d\mu \tag{29}$$

Then for s small,

$$\mathbf{Var}(f) \leqslant (r^2|\log r| + \frac{2}{|\log r|})m(r|\log r|)\int |\nabla f|^2 a\mu + s|f|_\infty^2.$$

Remark 3.7 *The results in this section hold for any Hilbert norm on $\mathcal{H}_\sigma$ including that used in Elworthy-Li [13]. It also works for a measure on the free path space $CM = \cup_{x_0 \in M} C_{x_0}M$ in the following sense. If μ_{x_0} is a probability measure on $C_{x_0}M$ and ν a probability measure on M, we consider on CM the measure $\mu = \int_M \mu_{x_0} d\nu$.*

4 Poincaré Inequality on Hyperbolic Space

Aida, [2], showed that, for M the standard hyperbolic space, of constant negative curvature. We may assume that the curvature is -1. Take the gradient ∇ to be that related to the Levi-Civita connection.

$$\int_{C_{x_0}H^n} f^2 \log \frac{f^2}{\log |f|_{L^2}^2} d\mu_{x_0,y_0}(\gamma) \leqslant \int_{C_{x_0}H^n} C(\gamma)|\nabla f|^2 d\mu_{x_0,y_0}(\gamma) \tag{30}$$

for $C(\gamma) = C_1(n) + C_2(n)\sup_{0\leqslant t\leqslant 1} d^2(\gamma_t, y_0)$. His method of proof is the Clark-Ocone formula approach. From an integration by parts formula he obtained the following Clark-Ocone formula by the integration representation theorem:

$$\mathbf{E}^{\mu_{x_0,y_0}}\{F|\mathcal{G}_t\} = \mathbf{E}^{\mu_{x_0,y_0}} F + \int_0^t \langle H_s(\gamma), dW_s\rangle,$$

where W_t is the anti-development of the Brownian bridge and

$$H(s,\gamma) = \mathbf{E}^{\mu_{x_0,y_0}}\{L(\gamma)\frac{d}{ds}\nabla F(\gamma)(s)|\mathcal{G}_s\}$$

almost surely with respect to the product measure $dt \otimes \mu_{x_0,y_0}$. Here $\mathcal{G}_t$ is the filtration generated by $\mathcal{F}_t$ and the end point of the Brownian bridge. The main obstruction here is that L is random and careful estimates on L leads to (30).

Theorem 4.1 *Let $M = H^n$, the hyperbolic space of constant curvature -1. Then Poincaré inequality holds for the Brownian bridge measure μ_{x_0,x_0}.*

Proof. Just note that by the time reversal of the Brownian bridge and its symmetric property and the concentration property of the Brownian motion measure

$$\int_{\mathcal{C}_{x_0}M} e^{Cd^2(\sigma,y_0)} d\mu_{x_0,y_0}(\sigma) < \infty.$$

Hence by Proposition 8 and Proposition 3.3 and (30), we finish the proof. □

Remark 4.2 *Aida has shown that inequality (8) holds for the loop space and each homotopy class of the free loop space over a compact Riemannian manifold of constant negative curvature. Our discussion earlier shows that Poincaré inequality holds in this case.*

A compact Riemannian manifold of constant negative curvature is of the form $M = G/H^n$ where G is a discrete subgroup of the isometry group of the hyperbolic space. The free loop space is the collection of all loops. See Aida [2] for precise formulation.

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