

An approximation scheme for SDEs with non-smooth coefficients

Xin Chen and Xue-Mei Li*

Abstract

Elliptic stochastic differential equations (SDE) make sense when the coefficients are only continuous. We study the corresponding linearized SDE whose coefficients are not assumed to be locally bounded. This leads to existence of $W_{\text{loc}}^{1,p}$ solution flows for elliptic SDEs with Hölder continuous and $\cap_p W_{\text{loc}}^{1,p}$ coefficients. Furthermore an approximation scheme is studied from which we obtain a representation for the derivative of the Markov semigroup, and an integration by parts formula.

1 Introduction

Let $A_l, 1 \leq l \leq m$, be continuous vector fields on \mathbf{R}^n . We consider stochastic differential equations of Markovian type

$$d\xi_t = \sum_{l=1}^m A_l(\xi_t) dW_t^l + A_0(\xi_t) dt \quad (1.1)$$

where $(W_t^l, 1 \leq l \leq m)$ are independent Brownian motions. Denote by A_{il} the components of A_l hence $A_l = (A_{1l}, A_{2l}, \dots, A_{nl})^T$. Write $A = (A_1, \dots, A_m)$ for the $n \times m$ matrix with the induced $n \times n$ -matrix A^*A whose entries are $a_{ij}(x) = \sum_{l=1}^m A_{il}(x)A_{jl}(x)$. For $x \in \mathbf{R}^n$ let $\xi_t(x)$ be a solution to the SDE (1.1) with initial value x . If the function A_l are weakly differentiable there is formally the linearized SDE,

$$V_t(x) = \mathbf{I} + \sum_{l=1}^m \int_0^t DA_l(\xi_s(x))(V_s(x)) dW_s^l + \int_0^t DA_0(\xi_s(x))(V_s(x)) ds \quad (1.2)$$

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whose solution is a $n \times n$ matrix valued random function. Here $DA_l : \mathbf{R}^n \rightarrow L(\mathbf{R}^n, \mathbf{R}^n)$ denotes the function $DA_l(x) = (DA_{1l}(x), \dots, DA_{nl}(x))$. In the case of the vector fields A_l being smooth and when a global smooth solution flow to SDE (1.1) exists, the solution to (1.2) corresponds to the derivative of the solution (1.1) with respect to initial data. Here in Section 2 of the paper we do not assume local boundedness of DA_l . We state below our two basic sets of assumptions, which are used to show a key convergence theorem.

Assumption 1.1. (1) $A(x)$ is uniformly elliptic, for some $\theta > 0$,

$$\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2, \quad \forall x \in \mathbf{R}^n, \quad \xi = (\xi_1, \dots, \xi_n) \in \mathbf{R}^n. \quad (1.3)$$

(2) Each $A_l(x), 0 \leq l \leq m$ is uniformly Hölder continuous, i.e. for some positive K and $0 < \alpha < 1$,

$$|A_l(x) - A_l(y)| \leq K |x - y|^\alpha \quad \forall x, y \in \mathbf{R}^n$$

and $\sup_{x \in \mathbf{R}^n} |A_l(x)| \leq M$ for some $M > 0$.

(3) $A_{il} \in W_{\text{loc}}^{1,2n}(\mathbf{R}^n)$.

Assumption 1.1 is essential for existence and uniqueness of the strong solution of SDE (1.1), as well as the convergence of our approximating scheme. By Stroock-Varadhan theorem conditions (1) and (2), drawing from parabolic PDE theory on regularity of solutions, assure the existence of a weak solution to SDE (1.1). And the uniform Hölder continuity in (2) is crucial to derive some upper bound for the fundamental solution of parabolic PDE. Part (3) is the basic assumption, for the existence of a strong solution and pathwise uniqueness of the elliptic SDE, in Veretennikov [23]. In fact in Watanabe-Yamada's celebrated paper [24], it was shown that pathwise uniqueness holds for non-Lipschitz vector fields with regularity of the form $|A_l(x) - A_l(y)| \leq \rho_1(|x - y|), 1 \leq l \leq m, |A_0(x) - A_0(y)| \leq \rho_2(|x - y|)$, for e.g. $\rho_1(t) = t\sqrt{|\log t|}, \rho_2(t) = t|\log t|$ when t is small, essentially the same regularity required for the uniqueness of a deterministic differential equation. See also a recent work by Fang-Zhang [9] for latest progress. When the SDE is uniformly elliptic this condition weakens as in the work of Veretennikov [23]. About the existence and uniqueness of the strong solution of SDE (1.1), see also the work of Krylov-Röckner [18] who discussed SDEs with additive noise with drift in L^p and Flandoli-Gubinelli-Priola [11] for SDE with C_b^3

diffusion coefficients and a ‘locally uniformly α -Hölder continuous’ condition on the drift coefficient. See also Zvonkin and [27] and Zhang [25].

Here in this paper we take advantage of the elliptic system at the level of the derivative flow.

Condition $G(\sigma, T_0)$. Let $G(x) := \sum_{l=0}^m |DA_l(x)|^2$. There exist $\sigma > 0$ and $T_0 > 0$ such that for all bounded set S

$$\sup_{x \in S} \int_0^{T_0} \int_{\mathbf{R}^n} e^{\sigma G(y)} K_s(x, y) dy ds < \infty,$$

where $K_s(x, y)$ is the heat kernel on \mathbf{R}^n .

For example if $G \in L^1_{\text{loc}}(\mathbf{R}^n)$, $G(x) \leq C|x|^2$ for x outside of a compact set, this condition holds on the time interval $[0, T]$ if σT is sufficiently small. If the weak derivatives $DA_l(x)$ grows sub-linearly the integrability holds for all parameters.

Consider an one-dimensional example ($n = 1$), with $l = 1$, $A_1(x) = 1 + \int_0^x \left(\sqrt{\beta} |\log |y|| I_{|y| \leq 1} \right) dy$ where $\beta > 0$ is a positive constant, and $A_0(x) = 0$. The SDE (1.1) with above coefficients satisfy Assumption 1.1 and Condition $G(\sigma, T_0)$ for all $0 < \sigma < \frac{1}{\beta}$ and T_0 . In fact, it is obvious that Assumption 1.1 holds and note that for any $T_0 > 0$,

$$\int_0^{T_0} \int_{\mathbf{R}^1} e^{\sigma G(y)} K_s(x, y) dy ds \leq \int_0^{T_0} s^{-\frac{1}{2}} \left(\int_{\mathbf{R}^1} \left(\frac{1}{|y|^{\sigma\beta}} I_{|y| \leq 1} + 1 I_{|y| > 1} \right) e^{-\frac{|x-y|^2}{2T_0}} dy \right) ds$$

so Condition $G(\sigma, T_0)$ holds if $\sigma < \frac{1}{\beta}$.

The first result we state here is on the construction of a solution to the derivative SDE (1.2). We show that there is a regularising family of elliptic SDEs with parameter ε such that the derivative flows V_t^ε converge under a small time interval when ε tends to 0, and the limit process is the unique solution of SDE (1.2) on this time interval. And from that we construct a solution of SDE (1.2) in any time interval. In particular, the derivative in SDE (1.2) is the weak derivative. Under these conditions A_l are not regular enough for us to obtain the required bounds directly we employ the upper bound of the Markov kernel and the integrable condition of DA_l to estimate the moments of the derivative process.

Theorem 2.10. Under Assumption 1.1 and condition $G(\sigma, T_0)$, there exists a process $V_t(x), 0 \leq t < \infty$, such that for each $p > 0$, there is a $T_4 > 0$ as in Lemma 2.7,

$$\lim_{\varepsilon \rightarrow 0} \sup_{x \in S} \mathbf{E} \sup_{0 \leq s \leq T_4} |V_s^\varepsilon(x) - V_s(x)|^p = 0$$

holds for any bounded set S in \mathbf{R}^n . Furthermore, the process $V_t(x)$ is the unique strong solution of SDE (1.2).

In the case of locally Lipschitz coefficients the result to compare with is that of Blagovescenskii-Friedlin [1], which goes back to 1961, where it is stated that if the coefficients are globally Lipschitz continuous, there exists a version of the solution which is jointly continuous in time and space. This result has been strengthened in terms of the growth on the derivative of the vector fields if all the vector fields are differentiable. See e.g. Li [15], Fang [10] for the cases about SDEs with locally Lipschitz continuous coefficients. See also Zhang [26] for the case in which the coefficients are not Lipschitz continuous. In a recent work [11], Flandoli-Gubinelli-Priola study the case where diffusion coefficients are C_b^3 , drift coefficients are locally uniformly α -Hölder continuous and obtain the existence of a version of the solution which is C^1 with the space variable and a Bismut type formula.

As for the continuous flow property, let $\xi_t(x, \omega), 0 \leq t < \zeta(x, \omega)$ be its maximal solution starting from x and $\zeta(x)$ is the explosion time which we assume to be ∞ a.s. for each fixed $x \in \mathbf{R}^n$. It is indicated that if $V_t(x)$ is a version of $D_x \xi_t(x)$, the derivative of $\xi_t(-, \omega)$ at point x , moment bounds on $V_t(x)$ relates to both completeness and strong completeness [15, 16]. For example non-explosion from particular starting point and the condition that $\sup_{x \in S} \mathbf{E} \sup_{s \leq t} |V_s(x)|^p \chi_{t < \zeta(x)}$ for all bounded set S and some $p > n$ implies completeness from all initial points and the strong completeness, i.e. the existence of a version of the solution which is jointly continuous in time and space.

The essential analysis on SDE's whose coefficients are locally Lipschitz continuous are gathered in section 3. Suppose that Assumption 3.1 in Section 3 holds and A_l are elliptic there is a smooth approximation for the derivative process and the uniform convergence holds on any time interval. In this case a sequence of Lipschitz continuous cut-off functions are employed to approximate a Locally Lipschitz continuous system and we can remove the boundedness conditions on A_l and DA_l .

From this, for the SDE whose coefficients satisfy the conditions above, we obtain a representation for the derivative of the Markov semigroup associated with the SDE, the intertwining property of the differential d and the semigroup P_t . Another application is that it can be shown that under suitable conditions there is a continuous version of the solution, which is furthermore weakly differentiable and belongs to the Sobolev space $W_{loc}^{1,p}$ for some p in small time interval. We also prove an extrinsic integration by parts formula on path space.

Standard investigation with regularity of stochastic flows assumes a local Lipschitz condition. The following result compliments known results, see e.g. Kunita [19]. We say that an SDE has a global continuous solution flow is strong complete if for each starting point x there is a global solution $(\xi_t(x), t \geq 0)$ and that there is a modification with $(t, x) \in [0, \infty) \times \mathbf{R}^n \rightarrow \xi(\cdot, \omega) \in \mathbf{R}^n$ continuous almost surely.

Theorem 4.1. Under Assumption 1.1 and Condition $G(\sigma, T_0)$, SDE (1.1) has a global continuous solution flow. Furthermore for each $p > 0$ there is a constant T_5 , such that $\xi_t(\cdot, \omega) \in W_{\text{loc}}^{1,p}(\mathbf{R}^n)$ for each $t \in [0, T_5]$.

Theorem 5.1 Suppose the Assumption 1.1 and condition $G(\sigma, T_0)$ hold, then there is a constant T_6 , such that for each $t \in [0, T_6]$,

$$dP_t f(x)(v_0) = \frac{1}{t} \mathbf{E} \left[f(\xi_t(x)) \int_0^t \langle Y(\xi_s(x))(V_s(x, v_0)), dW_s \rangle_{\mathbf{R}^m} \right], \quad v_0 \in \mathbf{R}^n$$

holds for all f in $\mathcal{B}_b(\mathbf{R}^n)$. If moreover $f \in C_b^1(\mathbf{R}^n)$, for all $v_0 \in \mathbf{R}^n$ and $t \in [0, T_6]$,

$$d(P_t f)(x)(v_0) = \mathbf{E} df(V_t(x, v_0)).$$

Finally we have the following integration by parts formula . Let $C_x([0, t]; \mathbf{R}^n)$ be the space of continuous functions from $[0, t]$ to \mathbf{R}^n with initial value x .

Theorem 5.3. Assume Assumption 1.1 and Condition $G(\sigma, T_0)$. There is a positive constant T_8 , such that for any $t \in (0, T_8]$ the following integration by parts formula holds for every BC^1 function F on $C_x([0, t]; \mathbf{R}^n)$. Let $h : [0, t] \times \Omega \rightarrow \mathbf{R}^n$ be an adapted stochastic process with $h(\cdot, \omega) \in L_0^{2,1}([0, t]; \mathbf{R}^m)$ almost surely and $\mathbf{E}(\int_0^t |\dot{h}_s|^2 ds)^{\frac{1+\beta}{2}} < \infty$ for some $\beta > 0$. Then

$$\mathbf{E} dF(V^h(\xi)) = \mathbf{E} F(\xi(x)) \delta V_t^h(\xi)$$

for δV^h defined by (5.18).

Finally in the Appendix we analyse the non-smooth geometry induced by the SDE in terms of an approximation linear connection when \mathbf{R}^n is treated as a manifold. We overcome the difficulty that the limiting connection may not be torsion skew symmetric with respect to the relevant induced Riemannian metric and obtain an intrinsic integration by part formula.

2 An approximation scheme for the derivative flow

In this section we consider a family of SDEs whose coefficients are smooth,

$$d\xi_t^\varepsilon(x) = \sum_{l=1}^m A_l^\varepsilon(\xi_s^\varepsilon(x)) dW_s^l + A_0^\varepsilon(\xi_s^\varepsilon(x)) ds \quad (2.1)$$

with the property that if the solutions $\xi_t^\varepsilon(x)$ and $\xi_t(x)$ are the solution of SDE (2.1) and (1.1) respectively with the same starting point x , there is a convergence theorem.

Let $\eta : \mathbf{R}^n \rightarrow R$ be the smooth mollifier defined by $\eta(x) = Ce^{\frac{1}{|x|^2-1}} \mathbf{1}_{|x|<1}$ where C is a normalising constant such that $|\eta|_{L^1} = 1$. Define a sequence of smooth functions η_ε with support in the ball B_ε , of radius ε centred at 0 by $\eta_\varepsilon(x) = \varepsilon^{-n} \eta(\frac{x}{\varepsilon})$. For a locally integrable function f on \mathbf{R}^n let f_ε be its convolution with η_ε ,

$$f_\varepsilon(x) = \eta_\varepsilon * f(x) = \int_{\mathbf{R}^n} \eta_\varepsilon(x-y) f(y) dy = \int_{B_\varepsilon(0)} \eta_\varepsilon(y) f(x-y) dy \quad (2.2)$$

Then $f_\varepsilon \rightarrow f$ for almost surely all x and the approximation family are uniformly Lipschitz continuous if the original functions are and have uniform linear growth if the original function does. To summarise, we have the following lemma. (see [7])

Lemma 2.1. (1) *If $f : \mathbf{R}^n \rightarrow R$ such that $|f(x)| \leq \psi(|x|)$ for ψ a positive increasing function, then $|f_\varepsilon(x)| \leq \psi(|x| + \varepsilon)$. If f is uniformly Hölder continuous in \mathbf{R}^n i.e. $|f(x) - f(y)| \leq K|x - y|^\alpha$ for some $K > 0$, $0 < \alpha < 1$, we have*

$$\sup_\varepsilon |f_\varepsilon(x) - f_\varepsilon(y)| \leq K|x - y|^\alpha \quad \forall x, y \in \mathbf{R}^n$$

and

$$\lim_{\varepsilon \rightarrow 0} \sup_{x \in \mathbf{R}^n} |f_\varepsilon(x) - f(x)| = 0$$

If f is locally Lipschitz with rate function K then so is each f_ε with the same rate function, that is $|f_\varepsilon(x) - f_\varepsilon(y)| \leq K(n + \varepsilon)|x - y|$, for all $x, y \in B_n$.

(2) *If $f \in L^p_{\text{loc}}(\mathbf{R}^n)$, then for any $R > 0$, we have*

$$\int_{|x| \leq R} f_\varepsilon^p dx \leq \int_{|x| \leq R+\varepsilon} f^p dx$$

$$\lim_{\varepsilon \rightarrow 0} \int_{|x| \leq R} |f_\varepsilon - f|^p dx = 0$$

2.1 Basic Estimates

Unless otherwise stated we take $A_{li}^\varepsilon = \eta_\varepsilon * A_{li}$, $A_l^\varepsilon = (A_{1l}^\varepsilon, \dots, A_{nl}^\varepsilon)^T$. Let $A^\varepsilon = (A_1^\varepsilon, \dots, A_m^\varepsilon)$ and $a_{ij}^\varepsilon = \sum_{l=1}^m A_{il}^\varepsilon(x)A_{jl}^\varepsilon(x)$. We begin with a family of approximating SDEs, with the smooth coefficients A_l^ε defined as above,

$$d\xi_t^\varepsilon(x) = x + \sum_{l=1}^m \int_0^t A_l^\varepsilon(\xi_s^\varepsilon(x)) dW_s^l + \int_0^t A_0^\varepsilon(\xi_s^\varepsilon(x)) ds \quad (2.3)$$

We summarise below useful property of this approximation.

Lemma 2.2. *Suppose Assumption 2.1 holds, then for some $\varepsilon_0 > 0$, $\{a_{ij}^\varepsilon, \varepsilon < \varepsilon_0\}$ are elliptic with the same uniform elliptic constant.*

Proof. By Lemma 2.1, $A_{il}^\varepsilon(x)$ are uniformly bounded in all parameters and the family of functions $A_{il}^\varepsilon(x)$ converge uniformly in \mathbf{R}^n as ε tends to 0. Let ε_0 be such that $\varepsilon < \varepsilon_0$, $\sup_{x \in \mathbf{R}^n} |a_{ij}^\varepsilon(x) - a_{ij}(x)| \leq \frac{\theta}{2n}$. Hence for $x \in \mathbf{R}^n, \xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbf{R}^n$,

$$\begin{aligned} \sum_{i,j=1}^n a_{ij}^\varepsilon(x) \xi_i \xi_j &\geq \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j - \sum_{i,j=1}^n |a_{ij}^\varepsilon(x) - a_{ij}(x)| |\xi_i| |\xi_j| \\ &\geq \theta \sum_{i=1}^n |\xi_i|^2 - \frac{\theta}{2n} n \sum_{i=1}^n |\xi_i|^2 = \frac{\theta}{2} \sum_{i=1}^n |\xi_i|^2 \end{aligned}$$

□

From Theorem A in [17], we obtain the following results on the approximation of $\xi_t^\varepsilon(x)$ and $\xi_t(x)$.

Lemma 2.3. *If assumption 1.1 holds, for any $p > 0, R > 0, T > 0$,*

$$\lim_{\varepsilon \rightarrow 0} \sup_{|x| \leq R} \mathbf{E} \left[\sup_{0 \leq s \leq T} |\xi_s^\varepsilon(x) - \xi_s(x)|^p \right] = 0, \quad (2.4)$$

Proof. As indicated in the introduction, Assumption 1.1 implies that there is a unique strong solution for SDE (1.1), see Theorem 1 in [23]. Under the assumptions of the theorem, we may apply Lemma 2.1 and Theorem A in [17] to obtain (2.4) for $p = 2$. The case of $p < 2$ follows from the Hölder's inequality. If $p > 2$, we have for each $T > 0$,

$$\begin{aligned} \mathbf{E} \sup_{0 \leq s \leq T} |\xi_s^\varepsilon(x) - \xi_s(x)|^p &= \mathbf{E} \left[\left(\sup_{0 \leq s \leq T} |\xi_s^\varepsilon(x) - \xi_s(x)| \right) \left(\sup_{0 \leq s \leq T} |\xi_s^\varepsilon(x) - \xi_s(x)| \right)^{p-1} \right] \\ &\leq \sqrt{\mathbf{E} \left[\sup_{0 \leq s \leq T} |\xi_s^\varepsilon(x) - \xi_s(x)|^2 \right]} \sqrt{\mathbf{E} \left[\sup_{0 \leq s \leq T} |\xi_s^\varepsilon(x) - \xi_s(x)|^{2p-2} \right]} \end{aligned} \quad (2.5)$$

In fact, from the uniform boundedness of the coefficients of SDE (1.1) and that of (2.3), we can get the following for all bounded set S in \mathbf{R}^n , $T > 0, p > 0$

$$\sup_{0 < \varepsilon < \varepsilon_0} \sup_{x \in S} \mathbf{E} \sup_{0 \leq s \leq T} |\xi_s^\varepsilon(x)|^{2p-2} + \sup_{x \in S} \mathbf{E} \sup_{0 \leq s \leq T} |\xi_s(x)|^{2p-2} < \infty. \quad (2.6)$$

So the conclusion follows from (2.5) and (2.6). \square

Lemma 2.4. *Suppose $(\xi_s^\varepsilon(x), s \geq 0)$ are the solutions to the family of smooth SDEs (2.1). Assume that the coefficients of these SDEs are uniformly elliptic with a common uniform elliptic constant θ and uniformly bounded with a common bound M , and $\sup_\varepsilon |A_l^\varepsilon(x) - A_l^\varepsilon(y)| \leq K|x - y|^\alpha \forall x, y \in \mathbf{R}^n$ for some $K > 0, 0 < \alpha < 1$. For each (s, x) , we assume that $\xi_s^\varepsilon(x)$ converges to $\xi_s(x)$ almost surely as ε tends to zero. Then the distribution of $\xi_s(x)$ is absolutely continuous with respect to the Lebesgue measure in \mathbf{R}^n , and we have the following estimate for the transition kernel $p_s(x, \cdot)$,*

$$p_s(x, y) \leq C_1 s^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{2C_1 s}}, \quad \forall s \in (0, T], x, y \in \mathbf{R}^n \quad (2.7)$$

where the constant C_1 depends only on $K, \alpha, M, \theta, n, T$. In particular, under the Assumption 1.1, the estimate holds for the Markov kernel of SDE (1.1).

Proof. Denote by $\mathfrak{B}_b(\mathbf{R}^n)$ the set of bounded measurable functions in \mathbf{R}^n . Since the coefficients of SDE (2.1) are smooth and $A_l^\varepsilon, 1 \leq l \leq m$ are uniformly elliptic, the distribution of the solution $\xi_t^\varepsilon(x)$ is absolutely continuous with respect to the Lebesgue measure in \mathbf{R}^n for each $s > 0, \varepsilon > 0$. (for example, see [20]). Let $p_s^\varepsilon(x, y) : \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}$ be the Markov kernel, so for $f \in \mathfrak{B}_b(\mathbf{R}^n)$,

$$\mathbf{E}f(\xi_s^\varepsilon(x)) = \int_{\mathbf{R}^n} p_s^\varepsilon(x, y) f(y) dy.$$

By classical results in diffusion theory, $p_s^\varepsilon(x, y)$ is the fundamental solution of the following parabolic PDE,

$$\begin{cases} \frac{\partial u^\varepsilon}{\partial t} = \sum_{i,j} a_{ij}^\varepsilon \frac{\partial u^\varepsilon}{\partial x_i} \frac{\partial u^\varepsilon}{\partial x_j} + \sum_i A_{i0}^\varepsilon \frac{\partial u^\varepsilon}{\partial x_i} \\ u^\varepsilon(0, y) = \delta_x. \end{cases}$$

By the estimate for the fundamental solution of non-divergence form parabolic PDE, see [8] or [14], there are constants c_1, c_2 such that for $s \in (0, T]$,

$$p_s^\varepsilon(x, y) \leq c_1 s^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{2c_2 s}}, \quad s \in (0, T], \varepsilon \in (0, \varepsilon_0].$$

The constants depend only on the uniform elliptic constants of a_{ij}^ε , the bounds on a_{ij}^ε and A_{i0}^ε , the Hölder constants K, α of $a_{ij}^\varepsilon, A_{i0}^\varepsilon$, the dimension n and time interval T . In particular the constants are independent of ε when ε is sufficiently small by the condition of this lemma. Take $C_1 = \max(c_1, c_2)$ for simplicity.

As assumed in the condition, $\lim_{\varepsilon \rightarrow 0} \xi_s^\varepsilon(x) = \xi_s(x)$ almost surely. Taking $\varepsilon \rightarrow 0$, by the Lebesgue dominated convergence theorem, for each $f \in C_b(\mathbf{R}^n)$,

$$\mathbf{E}f(\xi_s(x)) \leq C_1 s^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{-\frac{|x-y|^2}{2C_1 s}} f(y) dy \quad (2.8)$$

For every bounded open sets O in \mathbf{R}^n , let f_k be a sequence of non-negative functions in $C_b(\mathbf{R}^n)$ such that $\lim_{k \rightarrow \infty} f_k = I_O$ where the convergence is pointwise. Fatou lemma leads to

$$\mathbf{E}I_O(\xi_s(x)) \leq C_1 s^{-\frac{n}{2}} \int_O e^{-\frac{|x-y|^2}{2C_1 s}} f(y) dy \quad (2.9)$$

The same inequality also holds for every open, not necessarily bounded set by Fatou lemma. It follows that if Γ in \mathbf{R}^n is a set with $\lambda(\Gamma) = 0$, where λ denotes the Lebesgue, then $\mathbf{E}I_\Gamma(\xi_s(x)) = 0$. In fact, by regularity of the measure, there are open subsets O_k with $\Gamma \subseteq O_k$, $\lambda(O_k) \downarrow 0$ and so

$$\mathbf{E}I_\Gamma(\xi_s(x)) \leq \mathbf{E}I_{O_k}(\xi_s(x)) \leq C_1 s^{-\frac{n}{2}} \int_{O_k} e^{-\frac{|x-y|^2}{2C_1 s}} f(y) dy \xrightarrow{k \rightarrow \infty} 0.$$

Now the distribution of $\xi_s(x)$ is absolutely continuous with respect to the Lebesgue measure in \mathbf{R}^n for each $x \in \mathbf{R}^n$ and by (2.8) for all $f \in C_b(\mathbf{R}^n)$, we have,

$$\int_{\mathbf{R}^n} p_s(x, y) f(y) dy \leq C_1 s^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{-\frac{|x-y|^2}{2C_1 s}} f(y) dy$$

By general approximation procedure the above inequality also holds for each $f \in \mathfrak{B}_b(\mathbf{R}^n)$, which implies the Markov kernel $p_s(x, y)$ has the same upper bound: $C_1 s^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{2C_1 s}}$. If Assumption 1.1 holds, by Lemma 2.1, 2.2 and Lemma 2.3, all the condition above are satisfied so the upper bounded for the Markov kernel of the solution of SDE (1.1) follows. \square

Let $K_s(x, y) := s^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{2s}}$, $s > 0, x, y \in \mathbf{R}^n$. Let $g : \mathbf{R}^n \rightarrow \mathbf{R}^+$ be a Borel measurable function. If there exists a constant $T_0 > 0$, such that $\int_0^{T_0} \int_{\mathbf{R}^n} g(y) K_s(x, y) dy ds < \infty$, then by Lemma 2.4, $\int_0^{\min(T_0, \frac{T_0}{C_1})} \mathbf{E}g(\xi_s(x)) ds < \infty$.

From now on we assume the estimate for the Markov kernel in Lemma 2.4 are considered in time interval $0 < s \leq \tilde{T}$ for some fixed \tilde{T}

Lemma 2.5. *Let $g : \mathbf{R}^n \rightarrow \mathbf{R}$ be a Borel measurable function. Assume Assumption 1.1 and that there exist $T_0 > 0$, and $p \geq 1$, such that*

$$\sup_{x \in S} \int_0^{T_0} \int_{\mathbf{R}^n} |g(y)|^p K_s(x, y) dy ds < \infty \quad (2.10)$$

for any bounded set S in \mathbf{R}^n . Set $T_1 = \min(\tilde{T}, \frac{T_0}{C_1})$ where C_1 is the constant in the transition kernel, c.f. (2.7), on the time interval $(0, \tilde{T}]$. Then for $g_\varepsilon = \eta_\varepsilon * g$ and any bounded set S in \mathbf{R}^n ,

$$\sup_{\varepsilon < \varepsilon_0} \sup_{x \in S} \int_0^{T_1} \mathbf{E} |g_\varepsilon(\xi_s^\varepsilon(x))|^p ds < \infty \quad (2.11)$$

where ε_0 is the constant in Lemma 2.2.

Proof. Recall the transition kernel estimates we use before,

$$p_s^\varepsilon(x, y) \leq C_1 s^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{2C_1 s}}, \quad \forall s \in (0, \tilde{T}), \varepsilon \in (0, \varepsilon_0).$$

In the remaining part of the proof, the constants C which appear in the computation may change from line to line and depend only on $K, \alpha, M, \theta, \delta, n, \tilde{T}, p$. Define $\tilde{K}_s(x) = K_s(x, 0)$, $s > 0$, $x \in \mathbf{R}^n$. For $T \in (0, \tilde{T}]$ and $\varepsilon \in (0, \varepsilon_0)$, we derive the following estimate:

$$\begin{aligned} \int_0^T \mathbf{E} |g_\varepsilon(\xi_s^\varepsilon(x))|^p ds &\leq C \int_0^{C_1 T} \int_{\mathbf{R}^n} |g_\varepsilon(y)|^p K_s(x, y) dy ds \\ &\leq C \int_0^{C_1 T} \int_{\mathbf{R}^n} \eta_\varepsilon * |g|^p(y) \tilde{K}_s(x - y) dy ds \\ &= C \int_0^{C_1 T} (\eta_\varepsilon * |g|^p) * \tilde{K}_s(x) ds \\ &= C \left(\int_0^{C_1 T} |g|^p * \tilde{K}_s ds \right) * \eta_\varepsilon(x) \end{aligned}$$

The last step is due to the property that $f * h = h * f$ for locally integrable functions and Fubini's Theorem. Since we assume that $\int_0^{T_0} |g|^p * \tilde{K}_s ds$ is locally bounded in \mathbf{R}^n for any bounded set S in \mathbf{R}^n , when $C_1 T \leq T_0$, i.e. $T \leq T_1 := \min(\tilde{T}, \frac{T_0}{C_1})$, the following holds:

$$\sup_{\varepsilon < \varepsilon_0} \sup_{x \in S} \int_0^T \mathbf{E} |g_\varepsilon(\xi_s^\varepsilon(x))|^p ds < \infty. \quad (2.12)$$

□

Lemma 2.6. *Let $g : \mathbf{R}^n \rightarrow \mathbf{R}$ be a Borel measurable function. Suppose Assumption 1.1 holds and that there exist $T_0 > 0$, $\delta \in (0, 1)$ and $p \geq 1$ such that*

$$\sup_{x \in S} \int_0^{T_0} \int_{\mathbf{R}^n} |g(y)|^{p(1+\delta/2)} K_s(x, y) dy ds < \infty \quad (2.13)$$

for any bounded set S in \mathbf{R}^n . If moreover,

$$g \in L_{\text{loc}}^{\bar{p}(n)}(\mathbf{R}^n) \quad (2.14)$$

where $\bar{p}(n) = \max\{p(1 + \delta), \frac{pn(1+\delta)}{2}\}$. Let $T_1 = \min(\tilde{T}, \frac{T_0}{C_1})$ (constant C_1 is the same as that in Lemma 2.5), then for $g_\varepsilon = \eta_\varepsilon * g$ and any bounded set S in \mathbf{R}^n

$$\limsup_{\varepsilon \rightarrow 0} \sup_{x \in S} \int_0^{T_1} \mathbf{E} |g_\varepsilon(\xi_s^\varepsilon(x)) - g(\xi_s(x))|^p ds = 0. \quad (2.15)$$

Proof. In the remaining part of the proof, the constants C which appear in the computation may change from line to line and depend only on $K, \alpha, M, \theta, \delta, n, \tilde{T}, p$. They do not depend on ε and R , which is essential for taking ε to 0 and R to infinity.

The required uniform convergence requires an uniform estimate. Since $W^{1,p}$ spaces are not included in $W^{1,\infty}$, we must sacrifice some integrability for this uniform estimate. Fixed $T \leq T_1$, let

$$\begin{aligned} & \int_0^T \mathbf{E} |g_\varepsilon(\xi_s^\varepsilon(x)) - g(\xi_s(x))|^p ds \\ & \leq C \int_0^T \mathbf{E} |g_\varepsilon(\xi_s^\varepsilon(x)) - g(\xi_s^\varepsilon(x))|^p ds + C \int_0^T \mathbf{E} |g(\xi_s^\varepsilon(x)) - g(\xi_s(x))|^p ds \\ & := I_1^\varepsilon(x, T) + I_2^\varepsilon(x, T) \end{aligned}$$

Note that

$$\begin{aligned} I_1^\varepsilon(x, T) & \leq C \int_0^T \mathbf{E} [|g^\varepsilon(\xi_s^\varepsilon(x)) - g(\xi_s^\varepsilon(x))|^p I_{\{|\xi_s^\varepsilon(x)| \leq R\}}] ds \\ & \quad + C \int_0^T \mathbf{E} [|g^\varepsilon(\xi_s^\varepsilon(x)) - g(\xi_s^\varepsilon(x))|^p I_{\{|\xi_s^\varepsilon(x)| > R\}}] ds \\ & := I_{11}^\varepsilon(x, T, R) + I_{12}^\varepsilon(x, T, R). \end{aligned}$$

For each $\frac{1}{p_1} + \frac{1}{q_1} = 1$, by Markov kernel estimate and Hölder inequality

$$\begin{aligned} & I_{11}^\varepsilon(x, T, R) \\ & \leq C \int_0^{C_1 T} C s^{-\frac{n}{2}} \left(\int_{\mathbf{R}^n} e^{-\frac{p_1|x-y|^2}{2s}} \right)^{\frac{1}{p_1}} \left(\int_{|y| \leq R} |g_\varepsilon - g|^{p q_1}(y) dy \right)^{\frac{1}{q_1}} ds \\ & \leq C \int_0^{C_1 T} s^{-\frac{n}{2}(1-\frac{1}{p_1})} \left(\int_{|y| \leq R} |g_\varepsilon - g|^{p q_1}(y) dy \right)^{\frac{1}{q_1}} ds \end{aligned}$$

When $n > 1$, we take $q_1 = \frac{n(1+\delta)}{2}$ in above inequality. Then

$$\begin{aligned} I_{11}^\varepsilon(x, T, R) & \leq C \int_0^{C_1 T} s^{-\frac{1}{1+\delta}} \left(\int_{|y| \leq R} |g_\varepsilon - g|^{\frac{pn(1+\delta)}{2}}(y) dy \right)^{\frac{2}{n(1+\delta)}} ds \\ & \leq C \left(\int_{|y| \leq R} |g_\varepsilon - g|^{\frac{np(1+\delta)}{2}}(y) dy \right)^{\frac{2}{n(1+\delta)}}. \end{aligned} \quad (2.16)$$

For $n = 1$ the corresponding estimate is

$$I_{11}^\varepsilon(x, T, R) \leq C \left(\int_{|y| \leq R} |g_\varepsilon - g|^{p(1+\delta)}(y) dy \right)^{\frac{1}{1+\delta}}$$

Under condition (2.14), by Lemma 2.1, for each fixed $R > 0$, we have

$$\limsup_{\varepsilon \rightarrow 0} \sup_{x \in S} I_{11}^\varepsilon(x, T, R) = 0. \quad (2.17)$$

For the term involving large $|\xi_s^\varepsilon(x)|$, Hölder inequality gives

$$I_{12}^\varepsilon(x, T, R) \leq C \left(\int_0^T \mathbf{E} |g^\varepsilon(\xi_s^\varepsilon(x)) - g(\xi_s^\varepsilon(x))|^{p(1+\delta/2)} ds \right)^{\frac{2}{2+\delta}} \left(\int_0^T \frac{\mathbf{E} |\xi_s^\varepsilon(x)|^2}{R^2} \right)^{\frac{\delta}{2+\delta}}.$$

The first factor on the right hand side is bounded uniformly in S , since by (2.13) and Lemma 2.5,

$$\sup_{x \in S} \int_0^T \mathbf{E} |g(\xi_s(x))|^{p(1+\delta/2)} ds + \sup_{\varepsilon < \varepsilon_0} \sup_{x \in S} \int_0^T \mathbf{E} |g(\xi_s^\varepsilon(x))|^{p(1+\delta/2)} ds < \infty. \quad (2.18)$$

To estimate the second factor note that the vector fields A_l , and A_l^ε are bounded uniformly in ε for sufficiently small ε and so by standard estimates for bounded set S in \mathbf{R}^n , and $T \in (0, T_1]$,

$$\sup_{\varepsilon < \varepsilon_0} \sup_{x \in S} \sup_{0 \leq s \leq T} |\xi_s^\varepsilon(x)|^2 + \sup_{x \in S} \sup_{0 \leq s \leq T} |\xi_s(x)|^2 < \infty. \quad (2.19)$$

Hence there exists a constant $C(K, \alpha, M, \theta, n, T_0, \tilde{T}, S)$, not depending on R or ε , such that,

$$\sup_{\varepsilon < \varepsilon_0} \sup_{x \in S} I_{12}^\varepsilon(x, T, R) \leq \frac{C}{R^{\frac{2\delta}{2+\delta}}}$$

Finally we obtain, for each $R > 0$, $\varepsilon < \varepsilon_0$,

$$\sup_{x \in S} I_1^\varepsilon(x, T_1) \leq \sup_{x \in S} I_{11}^\varepsilon(x, T_1, R) + \sup_{x \in S} I_{12}^\varepsilon(x, T_1, R) \leq \sup_{x \in S} I_{11}^\varepsilon(x, T_1, R) + \frac{C}{R^{\frac{2\delta}{2+\delta}}}.$$

First let ε tend to 0, taking into account of (2.17), then R tend to ∞ , we see

$$\lim_{\varepsilon \rightarrow 0} \sup_{x \in S} I_1^\varepsilon(x, T_1) = 0.$$

Next we observe that locally integrable functions are uniformly continuous on sufficiently large sets. By Egoroff theorem for finite measures if f_n converges almost surely, for any $\zeta > 0$ it converges uniformly outside of a set of measure ζ . So for any $g \in L_{loc}(\mathbf{R}^n)$, there is a function \tilde{g} , such that $g = \tilde{g}$ almost everywhere with Lebesgue measure in \mathbf{R}^n , and for positive numbers R and ζ , there exists a open set $U(R, \zeta)$ with the property that $\lambda(U(R, \zeta)) < \zeta$ and \tilde{g} is uniformly continuous on \overline{B}_R/U . So for any $r > 0$, there exists a $\vartheta \equiv \vartheta(R, \zeta, r)$ be such that

$$|\tilde{g}(x_1) - \tilde{g}(x_2)| < r, \quad \forall x_i \in \overline{B}_R \setminus U(R, \zeta), \quad |x_1 - x_2| < \vartheta(R, \zeta, r).$$

Note that by Lemma 2.4, given the function $g = \tilde{g}$ almost everywhere with Lebesgue measure, we have $g(\xi_s(x, \omega)) = \tilde{g}(\xi_s(x, \omega))$ and $g(\xi_s^\varepsilon(x, \omega)) = \tilde{g}(\xi_s^\varepsilon(x, \omega))$ almost surely in the probability space for each fixed $s > 0$ and $0 < \varepsilon < \varepsilon_0$. Then for each $0 \leq s \leq T_1$, let

$$\begin{aligned} O_1(s) &= \{\omega : |\xi_s^\varepsilon(x, \omega) - \xi_s(x, \omega)| < \vartheta(R, \zeta, r)\}, \\ O_2(s) &= \{\omega : \xi_s(x, \omega) \in \overline{B}_R \setminus U(R, \zeta)\} \cap \{\omega : \xi_s^\varepsilon(x, \omega) \in \overline{B}_R \setminus U(R, \zeta)\}. \end{aligned}$$

For each $0 < T \leq T_1$, we obtain,

$$\begin{aligned} I_2^\varepsilon(x, T) &= C \int_0^T \mathbf{E} |\tilde{g}(\xi_s^\varepsilon(x)) - \tilde{g}(\xi_s(x))|^p ds \\ &\leq C \int_0^T \mathbf{E} [|\tilde{g}(\xi_s^\varepsilon(x)) - g(\xi_s(x))|^p \mathbf{1}_{\{O_1(s) \cap O_2(s)\}}(\omega)] ds \\ &\quad + C \int_0^T \mathbf{E} [|\tilde{g}(\xi_s^\varepsilon(x)) - \tilde{g}(\xi_s(x))|^p \mathbf{1}_{\{O_1(s) \cap O_2(s)\}^c}(\omega)] ds \\ &\leq Cr^p + C \left(\int_0^T \mathbf{E} [|\tilde{g}(\xi_s^\varepsilon(x))|^{p(1+\delta/2)} + |\tilde{g}(\xi_s(x))|^{p(1+\delta/2)}] ds \right)^{\frac{2}{2+\delta}} \left(\int_0^T \mathbf{P}(O_1^c(s) \cup O_2^c(s)) ds \right)^{\frac{\delta}{2+\delta}} \end{aligned}$$

By the estimate (2.18),

$$\begin{aligned}
\sup_{x \in S} I_2^\varepsilon(x, T) &\leq Cr^p + C \sup_{x \in S} \left(\int_0^T (\mathbf{P}(|\xi_s^\varepsilon(x) - \xi_s(x)| > \vartheta(R, \zeta, r)) \right. \\
&\quad \left. + \mathbf{P}(|\xi_s^\varepsilon(x)| > R) + \mathbf{P}(|\xi_s(x)| > R) + \mathbf{P}(\xi_s^\varepsilon(x) \in U(R, \zeta)) + \mathbf{P}(\xi_s(x) \in U(R, \zeta))) ds \right)^{\frac{\delta}{2+\delta}} \\
&\leq Cr^p + C \sup_{x \in S} \left(\frac{\mathbf{E} \sup_{0 \leq s \leq T} |\xi_s^\varepsilon(x) - \xi_s(x)|^2}{\vartheta(R, \zeta, r)^2} + \frac{\mathbf{E}[\sup_{0 \leq s \leq T} (|\xi_s^\varepsilon(x)|^2 + |\xi_s(x)|^2)]}{R^2} \right) \\
&\quad + \int_0^{T_3} (\mathbf{P}(\xi_s^\varepsilon(x) \in U(R, \zeta)) + \mathbf{P}(\xi_s(x) \in U(R, \zeta))) ds \Big)^{\frac{\delta}{2+\delta}}
\end{aligned}$$

Then by Lemma 2.3 and estimate (2.19), let ε tend to 0, we have,

$$\begin{aligned}
\overline{\lim}_{\varepsilon \rightarrow 0} \sup_{x \in S} I_2^\varepsilon(x, T) &\leq Cr^p \\
&\quad + C \left(\frac{1}{R^2} + \int_0^T \sup_{x \in S} (\mathbf{P}(\xi_s^\varepsilon(x) \in U(R, \zeta)) + \mathbf{P}(\xi_s(x) \in U(R, \zeta))) ds \right)^{\frac{\delta}{2+\delta}}
\end{aligned} \tag{2.20}$$

The last two items can be estimated using the Markov kernel upper bounds and Hölder inequality as below,

$$\begin{cases} \sup_{x \in S} \mathbf{P}(\xi_s^\varepsilon(x) \in U(R, \zeta)) \leq Cs^{-\frac{1}{2}} (\lambda(U(R, \zeta)))^{\frac{1}{n}} \leq C\zeta^{\frac{1}{n}} s^{-\frac{1}{2}} \\ \sup_{x \in S} \mathbf{P}(\xi_s(x) \in U(R, \zeta)) \leq C\zeta^{\frac{1}{n}} s^{-\frac{1}{2}} \end{cases}$$

Put the above estimate into (2.20), we derive

$$\overline{\lim}_{\varepsilon \rightarrow 0} \sup_{x \in S} I_2^\varepsilon(x, T_1) \leq Cr^p + \left(\frac{1}{R^2} + \zeta^{\frac{1}{n}} \right)^{\frac{\delta}{2+\delta}}$$

Since R, ζ and r are arbitrary, then let ζ, r tend to 0 and R tend to ∞ , then we get $\lim_{\varepsilon \rightarrow 0} \sup_{x \in S} I_2^\varepsilon(x, T_1) = 0$ and by now we have completed the proof. \square

Remark 2.1. Let $n > 1$ and $g : \mathbf{R}^n \rightarrow \mathbf{R}$ be a function such that $g \in L_{\text{loc}}^{\overline{p}(n)}(\mathbf{R}^n)$, as (2.14) and the following, for some $R_0 > 0$ and $c > 0$,

$$|g(x)| \leq e^{c|x|^2}, \quad |x| \geq R_0. \tag{2.21}$$

Then condition (2.13) holds for $T < \frac{1}{4c}$. In fact

$$\begin{aligned}
&\sup_{x \in S} \int_0^T \int_{|y| \leq R_0} |g(y)|^{p(1+\delta/2)} K_s(x, y) dy ds \\
&\leq C \int_0^T s^{-\frac{2+\delta}{2+2\delta}} \left(\int_{|y| \leq R_0} |g|^{\frac{pn(1+\delta)}{2}}(y) dy \right)^{\frac{2+\delta}{n(1+\delta)}} ds < \infty
\end{aligned}$$

and, by a change of variable,

$$\begin{aligned}
& \sup_{x \in S} \int_0^T \int_{|y| \geq R_0} |g(y)|^{p(1+\delta/2)} K_s(x, y) dy ds \\
& \leq \sup_{x \in S} \int_0^T s^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^c |y|^2 e^{-\frac{|x-y|^2}{2s}} dy ds \\
& \leq C \sup_{x \in S} \int_0^T \int_{\mathbf{R}^n} e^{2c(s|y|^2+|x|^2)} e^{-\frac{|y|^2}{2}} dy ds \\
& \leq CT \sup_{x \in S} e^{2c|x|^2} \int_{\mathbf{R}^n} e^{2cT|y|^2} e^{-\frac{|y|^2}{2}} dy < \infty.
\end{aligned}$$

Lemma 2.7. Let $G(x) := \sum_{l=0}^m |DA_l(x)|^2$ and $G^\varepsilon(x) := \sum_{l=0}^m |DA_l^\varepsilon(x)|^2$. Assume Assumption 1.1 and that there exist positive constants σ and T_0 , such that for any bounded set S in \mathbf{R}^n ,

$$\sup_{x \in S} \int_0^{T_0} \int_{\mathbf{R}^n} e^{\sigma G(y)} K_s(x, y) dy ds < \infty. \quad (2.22)$$

Then for each $q > 0$,

$$\sup_{\varepsilon < \varepsilon_0} \sup_{x \in S} \mathbf{E} \left[e^{6q^2 \int_0^{T_2} G^\varepsilon(\xi_s^\varepsilon(x)) ds} \right] < \infty$$

for $T_2 := \min(T_1, \frac{\sigma}{6q^2})$ where $T_1 = \min(\tilde{T}, \frac{T_0}{C_1})$ for C_1 given by (2.7).

Proof. By Jensen's inequality,

$$\begin{aligned}
\sup_{x \in S} \mathbf{E} \left[e^{6q^2 \int_0^T G^\varepsilon(\xi_s^\varepsilon(x)) ds} \right] & \leq \frac{1}{T} \sup_{x \in S} \mathbf{E} \left[\int_0^T e^{6Tq^2 G^\varepsilon(\xi_s^\varepsilon(x))} ds \right] \\
& \leq \frac{1}{T} \sup_{x \in S} \mathbf{E} \left[\int_0^T (\eta_\varepsilon * e^{6Tq^2 G})(\xi_s^\varepsilon(x)) ds \right]
\end{aligned}$$

the function $e^{6Tq^2 G}$ satisfies (2.10) with power parameter $p = 1$, when $T \leq \frac{\sigma}{6q^2}$, then by Lemma 2.5 for $T_2 := \min(T_1, \frac{\sigma}{6q^2})$,

$$\sup_{\varepsilon < \varepsilon_0} \sup_{x \in S} \mathbf{E} \left[\int_0^{T_2} (\eta_\varepsilon * e^{6Tq^2 G})(\xi_s^\varepsilon(x)) ds \right] \leq \sup_{\varepsilon < \varepsilon_0} \sup_{x \in S} \mathbf{E} \left[\int_0^{T_1} (\eta_\varepsilon * e^{\sigma G})(\xi_s^\varepsilon(x)) ds \right] < \infty.$$

which implies the conclusion of the lemma. \square

Remark 2.2. In fact, in integrable condition (2.22), if we replace the function $e^{\sigma G(y)}$ by $F(G(y))$, where $F : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ is a convex non-negative function, then a corresponding uniform integrability conclusion holds for $F(G)$.

2.2 Convergence of derivative flows

Next we consider the convergence of the derivative flows. Since each A_j^ε is smooth and globally Lipschitz continuous, for each ε there is a smooth global solution flow $\xi_t^\varepsilon(x, \omega)$ to SDE (2.3). For $x \in \mathbf{R}^n$, let

$$V_t^\varepsilon(x) = D_x \xi_t^\varepsilon$$

be the space derivative of ξ_t^ε . Then V_t^ε satisfies the following SDE

$$V_t^\varepsilon(x) = \mathbf{I} + \sum_{l=1}^m \int_0^t DA_l^\varepsilon(\xi_s^\varepsilon(x))(V_s^\varepsilon(x)) dW_s^l + \int_0^t DA_0^\varepsilon(\xi_s^\varepsilon(x))(V_s^\varepsilon(x)) ds. \quad (2.23)$$

We prove the following uniform moment estimate for V_t^ε .

Lemma 2.8. *We assume the same condition as that in Lemma 2.7. Then for each $p > 0$, there is a constant $T_3 := \min(T_1, \frac{\sigma}{6p^2})$, such that for all $0 \leq T \leq T_3$ and bounded subset S of \mathbf{R}^n ,*

$$\sup_{\varepsilon < \varepsilon_0} \sup_{x \in S} \mathbf{E} \left[\sup_{0 \leq s \leq T} |V_s^\varepsilon(x)|^p \right] < \infty. \quad (2.24)$$

Proof. For simplicity we omit the starting point x in $V_t^\varepsilon(x)$. For SDE (2.3), with smooth coefficients, it holds that for all $p > 1$, see the analysis in [15],

$$|V_t^\varepsilon|^p = e^{M_t^{p,\varepsilon} - \frac{\langle M^{p,\varepsilon}, M^{p,\varepsilon} \rangle_t}{2} + a_t^{p,\varepsilon}} \quad (2.25)$$

where

$$M_t^{p,\varepsilon} = \sum_{l=1}^m p \int_0^t \frac{\langle DA_l^\varepsilon(V_s^\varepsilon), V_s^\varepsilon \rangle}{|V_s^\varepsilon|^2} dW_s^l, \quad a_t^{p,\varepsilon} = \frac{p}{2} \int_0^t \frac{H_p^\varepsilon(\xi_s^\varepsilon)(V_s^\varepsilon, V_s^\varepsilon)}{|V_s^\varepsilon|^2}$$

and

$$H_p^\varepsilon(v, v) = 2 \langle DA_0^\varepsilon(v), v \rangle + \sum_{l=1}^m |DA_l^\varepsilon(v)|^2 + (p-2) \sum_{l=1}^m \frac{\langle DA_l^\varepsilon(v), v \rangle^2}{|v|^2}. \quad (2.26)$$

This follows from an Itô formula applied to the function $|\cdot|^p$ and to the stochastic process V_t^ε . See Elworthy's book [2] for a nice Itô formula.

Let $G^\varepsilon(x) = \sum_{l=0}^m |DA_l^\varepsilon(x)|^2$. Note that

$$H_p^\varepsilon(x)(v, v) \leq (p+3)G^\varepsilon(x) + C.$$

By Lemma 2.7, for $T_3 := \min(T_1, \frac{\sigma}{6p^2})$,

$$\sup_{\varepsilon < \varepsilon_0} \sup_{x \in S} \mathbf{E} \left[e^{6p^2 \int_0^{T_3} G^\varepsilon(\xi_s^\varepsilon(x)) ds} \right] < \infty$$

According to the proof of Theorem 5.1 in [15], for any bounded set S in \mathbf{R}^n ,

$$\sup_{\varepsilon < \varepsilon_0} \sup_{x \in S} \mathbf{E} \left[\sup_{0 \leq s \leq T_3} |V_s^\varepsilon(x)|^p \right] \leq C \sup_{\varepsilon < \varepsilon_0} \sup_{x \in S} \mathbf{E} \left[e^{6p^2 \int_0^{T_3} G(\xi_s^\varepsilon(x)) ds} \right] < \infty$$

Here C is a constant only depending on n and p , not on ε . \square

Remark 2.3. The condition (2.22) used in the Lemma is a little stronger than it is needed for the uniformly moment estimate for $V_s^\varepsilon(x)$. In fact, let $F_1(x) := \sup_{|v|=1} \langle DA_0^\varepsilon(v), v \rangle(x)$ and $F_2(x) := \sum_{l=1}^m |DA_l(x)|^2$, we have for any $\varepsilon > 0$,

$$\begin{aligned} \sup_{|v|=1} \langle DA_0^\varepsilon(x)(v), v \rangle &= \sup_{|v|=1} \left\langle \int_{\mathbf{R}^n} \eta_\varepsilon(x-y) DA_0(y)(v) dy, v \right\rangle \\ &\leq \int_{\mathbf{R}^n} \eta_\varepsilon(x-y) \sup_{|v|=1} \langle DA_0(y)(v) dy, v \rangle \leq \eta_\varepsilon * F_1(x) \end{aligned}$$

So by Jensen's inequality, from (2.26) we see,

$$\sup_{|v|=1} H_p^\varepsilon(v, v) \leq \eta_\varepsilon * (F_1(x) + F_2(x))$$

So by (2.25), Hölder inequality and Jensen's inequality, if the following condition holds,

$$\sup_{x \in S} \int_0^{T_0} \int_{\mathbf{R}^n} e^{\sigma F_i(y)} K_s(x, y) dy ds < \infty, \quad i = 1, 2 \quad (2.27)$$

we obtain the uniformly moment estimate for $V_s^\varepsilon(x)$ at some small time interval.

Note that in condition (2.27), we only need the one-side bound of DA_0 , which is weaker than the two-side bound condition (2.22).

Theorem 2.9. *Suppose the Assumption 1.1 and condition (2.22) holds, Then for each $p > 0$, there is a constant T_4 , such that for any bounded set S in \mathbf{R}^n and $0 \leq T \leq T_4$*

$$\lim_{\varepsilon, \tilde{\varepsilon} \rightarrow 0} \sup_{x \in S} \mathbf{E} \sup_{0 \leq s \leq T} |V_s^\varepsilon(x) - V_s^{\tilde{\varepsilon}}(x)|^p = 0. \quad (2.28)$$

Proof. We only need to consider the case of $p \geq 2$. For simplicity, we use $\beta_l^\varepsilon(s)$, $\beta_l^{\tilde{\varepsilon}}(s)$ to denote $DA_l^\varepsilon(\xi_s^\varepsilon(x))$ and $DA_l^{\tilde{\varepsilon}}(\xi_s^{\tilde{\varepsilon}}(x))$, and the constants C may appear in the computation from line to line and depend only on $K, \alpha, M, \theta, \delta, n, \tilde{T}, p, \sigma$. Let \hat{T}_1 be the constant T_3 in Lemma 2.8 for the power parameter $2p$. By SDE (1.2), for any $T < \hat{T}_1$, we have,

$$\begin{aligned}
& \left(\mathbf{E} \sup_{0 \leq s \leq T} |V_s^\varepsilon(x) - V_s^{\tilde{\varepsilon}}(x)|^p \right)^{\frac{2}{p}} \\
& \leq C \sum_{l=1}^m \left(\mathbf{E} \sup_{0 \leq s \leq T} \left| \int_0^s \beta_l^\varepsilon(u)(V_u^\varepsilon(x)) - \beta_l^{\tilde{\varepsilon}}(u)(V_u^{\tilde{\varepsilon}}(x)) dB_u^l \right|^p \right)^{\frac{2}{p}} \\
& + C \left(\mathbf{E} \sup_{0 \leq s \leq T} \left| \int_0^s \beta_0^\varepsilon(u)(V_u^\varepsilon(x)) - \beta_0^{\tilde{\varepsilon}}(u)(V_u^{\tilde{\varepsilon}}(x)) ds \right|^p \right)^{\frac{2}{p}} \\
& \leq C \sum_{l=0}^m \left(\mathbf{E} \left[\int_0^T |\beta_l^\varepsilon(s)(V_s^\varepsilon(x)) - \beta_l^{\tilde{\varepsilon}}(s)(V_s^{\tilde{\varepsilon}}(x))|^2 ds \right]^{\frac{p}{2}} \right)^{\frac{2}{p}} \\
& \leq C \sum_{l=0}^m \int_0^T \left(\mathbf{E} |\beta_l^\varepsilon(s)(V_s^\varepsilon(x)) - \beta_l^{\tilde{\varepsilon}}(s)(V_s^{\tilde{\varepsilon}}(x))|^p \right)^{\frac{2}{p}} ds, .
\end{aligned}$$

where the second step of above inequality is due to BKG inequality and Hölder inequality, the third step is due to the inequality $\left(\mathbf{E} \left| \int_0^T f_s ds \right|^p \right)^{\frac{1}{p}} \leq \int_0^T \left(\mathbf{E} |f_s|^p \right)^{\frac{1}{p}} ds$ for measurable function $f(s, \omega)$ when $p \geq 1$. Now splitting up the terms,

$$\begin{aligned}
& \left(\mathbf{E} \sup_{0 \leq s \leq T} |V_s^\varepsilon(x) - V_s^{\tilde{\varepsilon}}(x)|^p \right)^{\frac{2}{p}} \\
& \leq C \sum_{l=0}^m \int_0^T \left(\mathbf{E} |\beta_l^\varepsilon(s)(V_s^\varepsilon(x)) - \beta_l^{\tilde{\varepsilon}}(s)(V_s^{\tilde{\varepsilon}}(x))|^p \right)^{\frac{2}{p}} ds \\
& + C \sum_{l=0}^m \int_0^T \left(\mathbf{E} |\beta_l^\varepsilon(s)(V_s^{\tilde{\varepsilon}}(x)) - \beta_l^{\tilde{\varepsilon}}(s)(V_s^{\tilde{\varepsilon}}(x))|^p \right)^{\frac{2}{p}} ds \\
& \leq CN^2 \int_0^T \left(\mathbf{E} \sup_{0 \leq u \leq s} |V_u^\varepsilon(x) - V_u^{\tilde{\varepsilon}}(x)|^p \right)^{\frac{2}{p}} ds \\
& + C \sup_{\varepsilon < \varepsilon_0} \left(\mathbf{E} \left[\sup_{0 \leq s \leq \hat{T}_1} |V_s^\varepsilon|^{2p} \right] \right)^{\frac{1}{p}} \left(\sum_{l=0}^m \left(\int_0^T \mathbf{E} |\beta_l^\varepsilon(s)|^{4p} ds \right)^{\frac{1}{2p}} \left(\int_0^T \mathbf{P}(|\beta_l^\varepsilon(s)| > N) ds \right)^{\frac{1}{2p}} \right) \\
& + C \sup_{\varepsilon < \varepsilon_0} \left(\mathbf{E} \left[\sup_{0 \leq s \leq \hat{T}_1} |V_s^\varepsilon|^{2p} \right] \right)^{\frac{1}{p}} \left(\sum_{l=0}^m \left(\int_0^T \mathbf{E} |\beta_l^\varepsilon(s) - \beta_l^{\tilde{\varepsilon}}(s)|^{2p} ds \right)^{\frac{1}{p}} \right).
\end{aligned} \tag{2.29}$$

The last step is due to Hölder's inequality. Note that by condition (2.22), we can find a constant $L > 0$ (depends on $M, \theta, n, \tilde{T}, \sigma$), such that the function $g = e^{L|DA_l|^2}$ satisfies the condition (2.10) with power parameter $p = 1$. And then $|DA_l|$ satisfies the conditions (2.14) and (2.13) for any power $p \geq 1$, so by Lemma 2.5 and Lemma 2.6, there are constants \hat{T}_2 depending on $K, \alpha, M, \theta, n, \tilde{T}, T_0, \sigma$, such that for any bounded set S ,

$$\sup_{\varepsilon < \varepsilon_0} \sup_{x \in S} \int_0^{\hat{T}_2} \mathbf{E} e^{L|\beta_l^\varepsilon(s)|^2} ds \leq \sup_{\varepsilon < \varepsilon_0} \sup_{x \in S} \int_0^{\hat{T}_2} \mathbf{E} \left[\eta_\varepsilon * e^{L|DA_l|^2}(\xi_s^\varepsilon(x)) \right] ds < \infty$$

$$\sup_{\varepsilon < \varepsilon_0} \sup_{x \in S} \int_0^{\hat{T}_2} \mathbf{E} |\beta_l^\varepsilon(s)|^{4p} ds < \infty \quad (2.30)$$

$$\lim_{\varepsilon, \tilde{\varepsilon} \rightarrow 0} \sup_{x \in S} \int_0^{\hat{T}_2} \mathbf{E} |\beta_l^\varepsilon(s) - \beta_l^{\tilde{\varepsilon}}(s)|^{2p} ds = 0 \quad (2.31)$$

Then by the Chebeshev inequality, for each $\varepsilon < \varepsilon_0$,

$$\int_0^{\hat{T}_2} \mathbf{P}(|\beta_l^\varepsilon(s)| > N) ds \leq \int_0^{\hat{T}_2} \frac{\mathbf{E} e^{L|\beta_l^\varepsilon(s)|^2}}{e^{LN^2}} ds \leq \frac{C}{e^{LN^2}} \quad (2.32)$$

So put (2.24), (2.32) and (2.30) into (2.29), when $T < \min(\hat{T}_1, \hat{T}_2)$ we derive,

$$\left(\mathbf{E} \sup_{0 \leq s \leq T} |V_s^\varepsilon(x) - V_s^{\tilde{\varepsilon}}(x)|^p \right)^{\frac{2}{p}} \leq CN^2 \int_0^T \left(\mathbf{E} \sup_{0 \leq u \leq s} |V_u^\varepsilon(x) - V_u^{\tilde{\varepsilon}}(x)|^p \right)^{\frac{2}{p}} ds$$

$$+ \frac{C}{e^{LN^2/2p}} + C \left(\sum_{l=0}^m \left(\int_0^T \mathbf{E} |\beta_l^\varepsilon(s) - \beta_l^{\tilde{\varepsilon}}(s)|^{2p} ds \right)^{\frac{1}{p}} \right)$$

By Gronwall lemma, let $\alpha^{\varepsilon, \tilde{\varepsilon}}(T, x) := \sum_{l=0}^m \left(\int_0^T \mathbf{E} |\beta_l^\varepsilon(s) - \beta_l^{\tilde{\varepsilon}}(s)|^{2p} ds \right)^{\frac{1}{p}}$, we have for any $T < \min(\hat{T}_1, \hat{T}_2)$,

$$\left(\mathbf{E} \sup_{0 \leq s \leq T} |V_s^\varepsilon(x) - V_s^{\tilde{\varepsilon}}(x)|^p \right)^{\frac{2}{p}}$$

$$\leq \frac{C}{e^{LN^2/2p}} + \alpha^{\varepsilon, \tilde{\varepsilon}}(T, x) + \int_0^T e^{CN^2(t-s)} \left(\frac{C}{e^{LN^2/2p}} + \alpha^{\varepsilon, \tilde{\varepsilon}}(s, x) \right) ds$$

$$\leq \alpha^{\varepsilon, \tilde{\varepsilon}}(T, x) + e^{CN^2T} \int_0^T \left(\frac{C}{e^{LN^2/2p}} + \alpha^{\varepsilon, \tilde{\varepsilon}}(s, x) \right) ds$$

Since by (2.31), $\lim_{\varepsilon, \tilde{\varepsilon} \rightarrow 0} \sup_{x \in S} \alpha^{\varepsilon, \tilde{\varepsilon}}(T, x) = 0$, first let ε tend to 0, then N tend to infinity in above inequality, so we can find a constant $T_4 > 0$ (take

$CT_4 < \frac{L}{2p}$ in above inequality), such that for any $T \leq T_4$,

$$\lim_{\varepsilon, \tilde{\varepsilon} \rightarrow 0} \sup_{x \in S} \mathbf{E} \sup_{0 \leq s \leq T} |V_s^\varepsilon(x) - V_s^{\tilde{\varepsilon}}(x)|^p = 0$$

□

By Theorem 2.9, we know V_t^ε is a Cauchy sequence in the space $L^p(\mathbf{P})$, so there is a limit process, in fact we have the following.

Theorem 2.10. *Suppose Assumption 1.1 and condition (2.22) hold, then there exists a process $V_t(x), 0 \leq t < \infty$, such that for each $p > 0$, there is a $T_4 > 0$ as in Lemma 2.7,*

$$\lim_{\varepsilon \rightarrow 0} \sup_{x \in S} \mathbf{E} \sup_{0 \leq s \leq T_4} |V_s^\varepsilon(x) - V_s(x)|^p = 0$$

holds for any bounded set S in \mathbf{R}^n . Furthermore, the process $V_t(x)$ is the unique strong solution of SDE (1.2) for all t .

Proof. We write $\xi_t^0 := \xi_t$ and $V_t^0 := V_t$. By Lemma 2.6 and Theorem 2.9, it is shown that the limit process $V_t(x)$ is the solution of SDE (1.2) in some time interval $0 \leq t \leq T_4$. This gives the moment estimate in Lemma 2.8 for the case that $\varepsilon = 0$ on $[0, T_4]$. In fact a direct computation as that in Lemma 2.8 gives the bound for any strong solution of SDE (1.2) on $[0, T_4]$. The key observation is that equation (2.25) holds for any strong solution of SDE (1.2), without further assumptions on the regularity on the vector fields. Hence if V_t and \tilde{V}_t are two solutions of SDE (1.2), the same method used for the proof of Theorem 2.9 gives

$$\sup_{x \in S} \mathbf{E} \sup_{0 \leq s \leq T_4} |V_s(x) - \tilde{V}_s(x)|^p = 0$$

and $V_t(x)$ is the unique strong solution of SDE (1.2) in the time interval $[0, T_4]$.

If we view SDE (1.1) and (1.2) together as a system with solution $(\xi_t(x), V_t(x))$ valued in $\mathbf{R}^n \times \mathbf{R}^{n \times n}$. Let $F_t(x, v_0, \omega) := (\xi_t(x), \langle V_t(x), v_0 \rangle v_0)$ which is the solution of that system with initial point (x, v_0) . When $T_4 < t \leq 2T_4$, let

$$F_t(x, v_0, \omega) := F_{t-T_4}(\xi_{\tilde{T}}(x), V_{T_4}(x), \theta_{T_4}(\omega)).$$

Here $\theta_{\tilde{T}_0}(\omega)_t = \omega_{t+T_4} - \omega_{T_4}$ is the shift operator. By the Markov property and the pathwise uniqueness at time interval $0 \leq t \leq T_4$ for any initial point $(x, v_0) \in \mathbf{R}^n \times \mathbf{R}^{n \times n}$, one may check that $V_t(x)$ is the solution for SDE (1.2) when $0 \leq t \leq 2T_4$. Taking this procedure repeatedly, we obtain a unique solution to SDE (1.2) for any time t . □

Remark 2.4. In particular, by Lemma 2.4, if we take different versions of weak derivative DA_l in SDE (1.2), the corresponding solutions V_s are indistinguishable.

Remark 2.5. In Theorem 2.10, $V_t(x)$ is shown to belong to $L^p(\mathbf{P})$ when $0 \leq t \leq T_4$. But this may fail when $t > T_4$.

3 The case of locally Lipschitz continuous coefficients

In a special case that $A_l, 0 \leq l \leq m$ are bounded, global Lipschitz continuous and uniformly elliptic in \mathbf{R}^n , the condition (2.22) are satisfied for every $T_0 > 0, \sigma > 0$. And for each $T > 0$, there exists a unique strong solution of SDE (1.2) since DA_l is bounded. Step by step checking the proof of Lemma 2.6 and Theorem 2.9 to determine the time interval, we can obtain,

Theorem 3.1. *Assume that the coefficients of the SDE (1.1) are bounded, Lipschitz continuous and uniformly elliptic. For each $T > 0, p > 0$ and bounded subset S in \mathbf{R}^n , we have*

$$\limsup_{\varepsilon \rightarrow 0} \mathbf{E} \sup_{x \in S} \sup_{0 \leq s \leq T} |V_s^\varepsilon(x) - V_s(x)|^p = 0.$$

We extend the approximation results to the elliptic SDE with locally Lipschitz continuous coefficients, in which case A_l is still weak differentiable and has a locally bounded version of the derivative. So SDE (1.2) is complete if SDE (1.1) is complete, i.e. non-explode for each fixed starting point.

Denote by (ρ, ϑ) , $\rho > 0, \vartheta \in \mathbb{S}^{n-1}$ the polar coordinate in \mathbf{R}^n . For any measurable function f on \mathbf{R}^n and integer $N > 0$, define a function f^N as,

$$f^N(\rho, \vartheta) := \begin{cases} f(\rho, \vartheta) & \text{if } |\rho| \leq N, \\ f(N, \vartheta) & \text{if } |\rho| > N. \end{cases} \quad (3.1)$$

and $f^N(0) = f(0)$. Suppose that SDE (1.1) is complete and coefficients A_l are locally Lipschitz continuous and elliptic. Set $A_l^N(x) = (A_{l1}^N(x), \dots, A_{ln}^N(x))$ then $A_l, 1 \leq l \leq m$ are bounded, Lipschitz continuous and uniformly elliptic. Let $\xi_t^N(x), V_t^N(x)$ be respectively the solutions to SDE (1.1) and (1.2) whose coefficients are A_l^N and DA_l^N . Let $A_l^{N,\varepsilon}$ be the smooth approximation of the vector fields A_l^N derived by convolution, as in Section 2. We denote the corresponding solution of approximation SDE (2.3) and (2.23) by $\xi_t^{N,\varepsilon}(x)$ and $V_t^{N,\varepsilon}(x)$.

Lemma 3.2. *Suppose the coefficients A_l of SDE (1.1) are locally Lipschitz continuous, and of linear growth, then for any $p > 0, T > 0$ and bounded set S in \mathbf{R}^n ,*

$$\lim_{\varepsilon \rightarrow 0} \sup_{x \in S} \mathbf{E} \sup_{0 \leq s \leq T} |\xi_s^\varepsilon(x) - \xi_s(x)|^p = 0, \quad (3.2)$$

Proof. Let $T_N^\varepsilon(x), T_N(x)$ be the first exit time of the ball B_N for the process $\xi_s^\varepsilon(x), \xi_s(x)$ respectively. Since $\xi_s^{N,\varepsilon}(x) = \xi_s^\varepsilon(x)$ a.s. for $s < T_N^\varepsilon(x)$, and $\xi_s^N(x) = \xi_s(x)$ a.s for $s < T_N(x)$, we have,

$$\begin{aligned} & \mathbf{E} \sup_{0 \leq s \leq T} \left[|\xi_s^{N,\varepsilon}(x) - \xi_s^\varepsilon(x)|^p + |\xi_s^N(x) - \xi_s(x)|^p \right] \\ & \leq C \sup_{0 < \varepsilon < \varepsilon_0, N > 0} \sup_{x \in S} \left(\sqrt{\mathbf{E} \sup_{0 \leq s \leq T} |\xi_s^{N,\varepsilon}(x)|^{2p}} \sqrt{\mathbf{P}(T > T_N^\varepsilon(x))} \right. \\ & \quad \left. + \sqrt{\mathbf{E} \sup_{0 \leq s \leq T} |\xi_s^N(x)|^{2p}} \sqrt{\mathbf{P}(T > T_N(x))} \right) \\ & \leq \sup_{0 < \varepsilon < \varepsilon_0, N > 0} \frac{\sup_{x \in S} \left(\mathbf{E} \sup_{0 \leq s \leq T} |\xi_s^{N,\varepsilon}(x)|^{2p} + |\xi_s^\varepsilon(x)|^{2p} \right)}{N^p}. \end{aligned} \quad (3.3)$$

This convergence to 0 as $N \rightarrow \infty$ from the uniform estimates below:

$$\sup_{0 < \varepsilon < \varepsilon_0, N > 0} \sup_{x \in S} \left(\mathbf{E} \sup_{0 \leq s \leq T} |\xi_s^{N,\varepsilon}(x)|^p + |\xi_s^\varepsilon(x)|^p \right) < \infty. \quad (3.4)$$

The uniform estimate holds for any $p \geq 1$ and follows from the common linear bounded on A_l^ε . Since A_l^N is bounded and global Lipschitz continuous for each $N > 0$, a Grownwall type argument shows that

$$\lim_{\varepsilon \rightarrow 0} \sup_{x \in S} \mathbf{E} \sup_{0 \leq s \leq T} |\xi_s^{N,\varepsilon}(x) - \xi_s^N(x)|^p = 0 \quad (3.5)$$

By (3.3), (3.5) and (3.4), we conclude the proof by taking $\varepsilon \rightarrow 0$ followed by $N \rightarrow \infty$ in the following inequality:

$$\begin{aligned} & \mathbf{E} \sup_{0 \leq s \leq T} |\xi_s^\varepsilon(x) - \xi_s(x)|^p \leq C \mathbf{E} \sup_{0 \leq s \leq T} \left(|\xi_s^{N,\varepsilon}(x) - \xi_s^\varepsilon(x)|^p + |\xi_s^N(x) - \xi_s(x)|^p \right) \\ & \quad + C \mathbf{E} \sup_{0 \leq s \leq T} |\xi_s^{N,\varepsilon}(x) - \xi_s^N(x)|^p. \end{aligned} \quad (3.6)$$

□

Remark 3.1. If we assume coefficients A_l of SDE (1.1) are locally Lipschitz continuous, elliptic and of linear growth. Since the coefficients A_l^N are bounded Lipschitz continuous and uniform elliptic for each $N > 0$, the distribution of $\xi_s^N(x)$ is absolutely continuous with respect to the Lebesgue measure in \mathbf{R}^n for each fixed $s > 0$ and $x \in \mathbf{R}^n$. Note that we have proved (3.2), by the same approximation methods we adopted in the proof of Lemma 2.4, we can prove the distribution of $\xi_s(x)$ is absolutely continuous with respect to the Lebesgue measure in \mathbf{R}^n for each fixed $s > 0$ and $x \in \mathbf{R}^n$. In particular that if we take different versions of DA_l in SDE (1.2), the solution V_s are indistinguishable.

As the same argument in the proof Lemma 3.2 above, especially triangle inequality (3.6) and the results of Theorem 3.1, we present below an approximation lemma for V_s in more general case and the remaining of the section devotes to the validity of the assumption there.

Lemma 3.3. *Let $S \subset \mathbf{R}^n$ be a bounded set and $T > 0$. If*

$$\lim_{N \rightarrow \infty} \sup_{0 < \varepsilon < \varepsilon_0} \sup_{x \in S} \mathbf{E} \sup_{0 \leq t \leq T} \left(|V_t^{N, \varepsilon}(x) - V_t^\varepsilon(x)|^p + |V_t^N(x) - V_t(x)|^p \right) = 0 \quad (3.7)$$

for all p , then

$$\lim_{\varepsilon \rightarrow 0} \sup_{x \in S} \mathbf{E} \sup_{0 \leq t \leq T} |V_t^\varepsilon(x) - V_t(x)|^p = 0$$

The following theorem, from Theorem 5.1 and observations from section 6 in [15], are valid for strong solutions ξ_t, V_t of SDE's (1.1) and (1.2), which are not necessarily elliptic or with smooth coefficients.

Theorem 3.4. (1) *Suppose that there is a point x_0 such that the solution $\xi_t(x_0)$ exists for all time and*

$$\sup_{|v|=1} \langle DA_0(v), v \rangle(x) \leq f(x)|v|^2 \quad \sum_{l=1}^m |DA_l|^2(x) \leq f(x)$$

for some function $f : \mathbf{R}^n \rightarrow \mathbf{R}$. Then

$$\mathbf{E} \sup_{s \leq t} |T\xi_t|^p < c\mathbf{E} \exp \left(6p^2 \int_0^t f(\xi_s(x)) ds \right)$$

and the SDE is strongly complete.

- If $g : \mathbf{R}^n \rightarrow \mathbf{R}$ is a function such that $g \in C^2(\mathbf{R}^n)$

$$\frac{1}{2} \sum_{l=1}^m |Dg(A_l)|^2 + \frac{1}{2} \sum_{l=1}^m D^2g(A_l, A_l) + Dg(A_0) \leq K \quad (3.8)$$

for some constant K , then for all σ and stopping times τ ,

$$\mathbf{E} \exp(\sigma g(\xi_{t \wedge \tau})(x)) \leq e^{\sigma g(x) + kt}$$

for some k depending on K and the SDE is complete if g has compact level sets.

The theorems and analysis we cited above from [15] are for the SDEs with smooth coefficients. Our key observation is that when the coefficients are not smooth, (2.25) still holds for a strong solution. The same argument and technicalities applies and we obtain the conclusion above.

The application of the theorem reduces to a well chosen function f, g for a particular SDE.

Assumption 3.1. Let $A_i : \mathbf{R}^n \rightarrow \mathbf{R}^n$, $i = 0, 1, \dots, m$, be locally Lipschitz continuous. Let $\psi_i : \mathbf{R} \rightarrow \mathbf{R}$, $i = 1, 2$ be positive non-decreasing functions and let $g_i(x) := \psi_i(|x|)$. Suppose that g_1 is C^2 and the following holds:

$$(1) \sum_{l=1}^m |DA_l|^2(x) \leq g_1(x), \sup_{|v|=1} \langle DA_0(x)(v), v \rangle \leq g_1(x)|v|^2$$

$$(2) \sum_{l=0}^m |A_l|(x) \leq g_2(x) \leq C_2(1 + |x|),$$

(3)

$$\frac{1}{2} \sum_{l=1}^m |Dg_1(A_l)|^2 + \frac{1}{2} \sum_{l=1}^m D^2g_1(A_l, A_l) + Dg_1(A_0) \leq C_3$$

Here C_2 and C_3 are some constants.

Condition (3) in Assumption 3.1 is satisfied if $\psi_1 \in C^2(\mathbf{R})$ and $\psi_1''(s)(\psi_2(s))^2$ and $\psi_1'(s)\psi_2(s)$ are bounded. It follows from the comments made earlier that $\mathbf{E}e^{\sigma g_1(\xi_t(x))}$ is finite for each $\sigma > 0$.

Remark 3.2. Assumption 3.1 holds under one of the following conditions:

$$(a) \sum_{l=1}^m |DA_l| \text{ is bounded.}$$

$$(b) \sum_{l=1}^m |DA_l(x)|^2 \leq C(1 + \ln(1 + |x|^2)), \quad \sum_{l=0}^m |A_l(x)| \leq C(1 + |x|), \\ \langle x, A_0(x) \rangle \leq C(1 + |x|^2), \quad \langle DA_0(x)(v), v \rangle \leq C(1 + \ln(1 + |x|^2))|v|^2.$$

(c) For some $\delta \geq 0$, the following holds,

$$\sum_{l=1}^m |A_l(x)| \leq C(1 + |x|^2)^{\frac{1}{2}-\delta} \text{ and } \sum_{l=1}^m |DA_l(x)|^2 \leq C(1 + |x|^2)^\delta \text{ and } \langle x, A_0(x) \rangle \leq C(1 + |x|^2)^{1-\delta}, \langle DA_0(x)(v), v \rangle \leq C(1 + |x|^2)^\delta |v|^2.$$

For part a) take g_1 to be a constant and g_2 a linear function. For (b) let $\psi_1(s) = \ln(1 + s^2)$ and $\psi_2(s) = 1 + s$. For (c) Let $\psi_2(s) = C(1 + s^2)^{\frac{1}{2}-\delta}$ and $\psi_1(s) = C(1 + s^2)^\delta$. See Corollary 6.2 and 6.3 in [15].

Proposition 3.5. *Suppose that $\text{span}\{A_1(x), \dots, A_m(x)\} = \mathbf{R}^n$ for each x so (1.1) is elliptic. If in addition that $\{A_0, A_1, \dots, A_m\}$ satisfies Assumption 3.1, condition (3.7) in Lemma 3.3 holds. In particular, for each $T > 0, p > 0$ and bounded set S in \mathbf{R}^n ,*

$$\limsup_{\varepsilon \rightarrow 0} \mathbf{E} \sup_{x \in S} \sup_{0 \leq s \leq T} |V_s^\varepsilon(x) - V_s(x)|^p = 0.$$

Proof. Here in the proof the constants C may change in different lines and only depend on p, S, T . By Lemma 2.1, for all $N > 0$ and $0 < \varepsilon < \varepsilon_0$,

$$|DA_l^\varepsilon(x)|^2 + |DA_l^{N,\varepsilon}(x)|^2 \leq 2\psi_1(|x|+1), \quad |A_l^\varepsilon(x)| + |A_l^{N,\varepsilon}(x)| \leq \psi_2(|x|+1).$$

So there is a global solution to approximation SDEs with smooth coefficients A_l^ε and $A_l^{N,\varepsilon}$ for any starting point and it follows from the assumption that $\tilde{g}(x) := \psi_1(|x|+1)$ is C^2 and the functions

$$\begin{aligned} & \frac{1}{2} \sum_{l=1}^m |(D\tilde{g})(A_l^\varepsilon)|^2 + \frac{1}{2} \sum_{l=1}^m (D^2\tilde{g})(A_l^\varepsilon, A_l^\varepsilon) + (D\tilde{g})(A_0^\varepsilon) \\ & \frac{1}{2} \sum_{l=1}^m |(D\tilde{g})(A_l^{N,\varepsilon})|^2 + \frac{1}{2} \sum_{l=1}^m (D^2\tilde{g})(A_l^{N,\varepsilon}, A_l^{N,\varepsilon}) + (D\tilde{g})(A_0^{N,\varepsilon}) \end{aligned}$$

are bounded above with the upper bound uniform in $\varepsilon \in (0, \varepsilon_0)$ and in $N > 0$. From the calculations in Lemma 6.1 and Theorem 5.1 of [15] or see Theorem 3.4 before, for every $p > 0$,

$$\sup_{0 < \varepsilon < \varepsilon_0, N > 0} \sup_{x \in S} \left(\mathbf{E} \sup_{0 \leq t \leq T} |V_t^{N,\varepsilon}(x)|^p + |V_t^\varepsilon(x)|^p \right) \leq C < \infty \quad (3.9)$$

As we remark before, the same theory can apply to SDE (1.1) and (2.3) with strong solution, we obtain,

$$\sup_{N > 0} \sup_{x \in S} \left(\mathbf{E} \sup_{0 \leq t \leq T} |V_t^N(x)|^p + |V_t(x)|^p \right) \leq C < \infty \quad (3.10)$$

As before, let $T_N(x)$, $T_N^\varepsilon(x)$ be the first exit times from the ball B_N of $\xi_t(x)$ and $\xi_t^\varepsilon(x)$. For $x \in S$, $0 < \varepsilon < \varepsilon_0$ and N large so that $S \subseteq B_N$, we have,

$$\begin{aligned}
& \mathbf{E} \sup_{0 \leq t \leq T} |V_t^{N,\varepsilon}(x) - V_t^\varepsilon(x)|^p \\
& \leq C \mathbf{E} \left[\left(\sup_{0 \leq t \leq T} |V_t^{N,\varepsilon}(x)|^p + \sup_{0 \leq t \leq T} |V_t^\varepsilon(x)|^p \right) I_{\{T > T_N^\varepsilon(x)\}} \right] \\
& \leq C \left(\sqrt{\mathbf{E} \sup_{0 \leq t \leq T} |V_t^{N,\varepsilon}(x)|^{2p}} \sqrt{\mathbf{P}(T > T_N^\varepsilon(x))} + \sqrt{\mathbf{E} \sup_{0 \leq t \leq T} |V_t^\varepsilon(x)|^{2p}} \sqrt{\mathbf{P}(T > T_N^\varepsilon(x))} \right) \\
& \leq C (\mathbf{P}(T > T_N^\varepsilon(x)))^{1/2} \\
& \leq \frac{C \sup_{0 < \varepsilon < \varepsilon_0, N > 0} \sup_{x \in S} \sqrt{\mathbf{E} \sup_{0 \leq s \leq T} |\xi_s^{N,\varepsilon}(x)|^{2p}}}{N^p} \leq \frac{C}{N^p}
\end{aligned} \tag{3.11}$$

Here in the last step we use the estimation (3.4) by linear growth condition of A_l . So by (3.11), we get,

$$\lim_{N \rightarrow \infty} \sup_{0 < \varepsilon < \varepsilon_0} \sup_{x \in S} \mathbf{E} \sup_{0 \leq t \leq T} |V_t^{N,\varepsilon}(x) - V_t^\varepsilon(x)|^p = 0.$$

Analogously, using (3.10), we have the results for the quantities without ε . \square

In the proof of Theorem 3.1, ellipticity condition is only needed for the estimate of the item $\mathbb{P}(\xi_s(x) \in U(R, \zeta))$ and Lemma 2.2 holds automatically if A_l are C^1 . The corresponding theorem for non-elliptic systems are given below.

Proposition 3.6. *Suppose the coefficients A_l of SDE (1.1) are C^1 and satisfies Assumption 3.1. Then*

$$\lim_{\varepsilon \rightarrow 0} \sup_{x \in S} \mathbf{E} \sup_{0 \leq s \leq T} |V_s^\varepsilon(x) - V_s(x)|^p = 0$$

for each $T > 0$, $p > 0$ and bounded subset S in \mathbf{R}^n .

4 Regularity of the solution flow

Theorem 4.5.1 in [19] states that for SDE (1.1), with A_l global Lipschitz continuous, there is a solution flow $\xi_t(x, \omega)$ such that for almost surely all

ω and every $t > 0$, $\xi_t(\cdot, \omega) \in C^{0,\delta}(\mathbf{R}^n)$ ($0 < \delta < 1$). See [15, 10, 26] for various generalisation. To our knowledge the following result on solution with Sobolev regularity is new.

Theorem 4.1. *Assume Assumption 1.1 and condition (2.22) hold. There is a global solution flow $\xi_t(x, \omega)$ for SDE (1.1), i.e. a version such that for almost surely all ω , $\xi(\cdot, \omega)$ is continuous in $[0, \infty) \times \mathbf{R}^n$. Furthermore for each $p > 0$, there is a constant $T_5(K, \alpha, M, \theta, n, p, T_0, \hat{T}, \sigma)$, such that $\xi_t(\cdot, \omega) \in W_{\text{loc}}^{1,p}(\mathbf{R}^n)$ for each $0 < t \leq T_5$.*

Proof. From the analysis in the proof of Theorem 4.1 in [15] for SDE (2.3) with smooth coefficients, given a bounded set S in \mathbf{R}^n , we have for each $x, y \in S$ and $T > 0, p \geq 1$,

$$\mathbf{E} \sup_{0 \leq s \leq T} |\xi_s^\varepsilon(x) - \xi_s^\varepsilon(y)|^p \leq |x - y|^p \sup_{z \in S} \mathbf{E} \sup_{0 \leq s \leq T} |V_s^\varepsilon(z)|^p$$

By Lemma 2.3 and Lemma 2.8, let ε tend to 0, there exists a $\hat{T} > 0$ which only depends on $K, \alpha, M, \theta, n, \hat{T}, T_0, p, \sigma$, such that,

$$\mathbf{E} \sup_{0 \leq s \leq \hat{T}} |\xi_s^\varepsilon(x) - \xi_s^\varepsilon(y)|^p \leq C|x - y|^p$$

and from that one note that

$$\mathbf{E} |\xi_t(x) - \xi_s(y)|^p \leq C(|x - y|^p + |t - s|^{\frac{p}{2}}) \quad 0 \leq t, s \leq \hat{T}, \quad x, y \in S$$

So in above estimate, we take $p > n$, then by the Kolmogorov's criterion, there is a version of the solution flow $\xi_t(x, \omega)$ for SDE (1.1), such that $\xi(\cdot, \omega)$ is continuous in $[0, \hat{T}] \times \mathbf{R}^n$. As for $t > \hat{T}$, note that by the uniqueness of the strong solution of SDE 1.1 under Assumption 1.1, it is satisfied that

$$\xi_t(x, \omega) = \xi_{t-\hat{T}}(\xi_{\hat{T}}(x, \omega), \theta_{\hat{T}}(\omega)) \quad a.s. \quad (4.1)$$

where $\theta_{\hat{T}}(\omega)_t = \omega_{t+\hat{T}} - \omega_{\hat{T}}$ is the shift operator and hence the solution flow $\xi(\cdot, \omega)$ is continuous in $[0, 2\hat{T}] \times \mathbf{R}^n$ and hence on $[0, \infty) \times \mathbf{R}^n$ by repeating the procedure.

By Lemma 2.3, Theorem 2.10 and the diagonal principle there exist a constant $T_5 > 0$, a subsequence ε_k with $\lim_{k \rightarrow \infty} \varepsilon_k = 0$ and a set $\tilde{\Lambda}_1$ with $\mathbb{P}(\tilde{\Lambda}_1) = 0$, such that if $\omega \in \tilde{\Lambda}_1^c$, for all $N > 0$,

$$\lim_{k \rightarrow \infty} \int_{|x| \leq N} \sup_{0 \leq t \leq T_5} |V_t^{\varepsilon_k}(x, \omega) - V_t(x, \omega)|^p dx = 0 \quad (4.2)$$

and

$$\lim_{k \rightarrow \infty} \int_{|x| \leq N} \sup_{0 \leq t \leq T_5} |\xi_t^{\varepsilon^k}(x, \omega) - \xi_t(x, \omega)|^p dx = 0. \quad (4.3)$$

Here $V_t(x)$, $t > 0$ is the solution of SDE (1.2) we get in the Theorem 2.10.

Let $\{e_r\}$ be an o.n.b. of \mathbf{R}^n . For simplicity write $V_t^{k,r}(x) = \langle V_t^{\varepsilon^k}(x), e_r \rangle_{\mathbf{R}^n}$ and $\xi_t^k(x) = \xi_t^{\varepsilon^k}(x)$. For the SDE (2.3) whose coefficients are smooth and with bounded derivatives, there is a smooth solution flow $\xi_t^k(\cdot, \omega)$ and $\frac{\partial \xi_t^k(\cdot, \omega)}{\partial r} = V_t^{k,r}(x, \omega)$. Therefore there exists a null set Λ_k , such that if $\omega \notin \Lambda_k$, the following integration by parts formula holds for $0 \leq t \leq T_5$ and any $\varphi \in C_0^\infty(\mathbf{R}^n)$,

$$\int_{\mathbf{R}^n} \frac{\partial \varphi}{\partial x_r}(x) \xi_t^k(x, \omega) dx = - \int_{\mathbf{R}^n} \varphi(x) V_t^{k,r}(x, \omega) dx. \quad (4.4)$$

Let $\tilde{\Lambda} := (\bigcup_{k=1}^\infty \Lambda_k) \cup \tilde{\Lambda}_1$, a null set. Taking the limit $k \rightarrow \infty$ in the above identity and using (4.2-4.3) to see when $\omega \notin \tilde{\Lambda}$ and $0 \leq t \leq T_5$,

$$\int_{\mathbf{R}^n} \frac{\partial \varphi}{\partial x_r}(x) \xi_t(x, \omega) dx = - \int_{\mathbf{R}^n} \varphi(x) V_t^r(x, \omega) dx \quad (4.5)$$

which means that $\xi_t(\cdot, \omega)$ is weak differentiable in distribution sense for almost surely all ω and $\frac{\partial \xi_t(\cdot, \omega)}{\partial x_r} = V_t^r(\cdot, \omega)$. Next we prove $\xi_t(\cdot, \omega) \in W_{\text{loc}}^{1,p}(\mathbf{R}^n)$ for each $p > 0$. For $N > 0$,

$$\mathbf{E} \int_{B_N} \sup_{0 \leq t \leq T_5} |V_t^r(x, \omega)|^p dx = \int_{B_N} \mathbf{E} \sup_{0 \leq t \leq T_5} |V_t^r(x, \omega)|^p dx \leq C\lambda(B_N)$$

Hence $\int_{B_N} \sup_{0 \leq t \leq T} |V_t^r(x, \omega)|^p dx$ is finite almost surely and so we can find a null set Γ_1 , such that $\int_{B_N} \sup_{0 \leq t \leq T} |V_t^r(x, \omega)|^p dx < \infty$ for every $N > 0$ when $\omega \notin \Gamma_1$. As the same way, we can prove the similar property for $\xi_t(x, \omega)$. Hence $\xi_t(x, \omega), V_t^r(x, \omega) \in L_{\text{loc}}^p(\mathbf{R}^n)$ for $\omega \notin \Gamma \cup \tilde{\Lambda}$ where Γ is a set with measure 0, which means almost surely $\xi_t(\cdot, \omega) \in W_{\text{loc}}^{1,p}(\mathbf{R}^n)$ for each $0 < t \leq T_5$. □

For SDE with locally Lipschitz continuous coefficients, we may get rid of the boundedness and the uniform ellipticity condition.

Theorem 4.2. *If Assumption 3.1 holds and the coefficients are elliptic, there is a solution flow $\xi_t(x, \omega)$ to SDE (1.1) with the property that $(t, x) \mapsto \xi_t(\cdot, \omega)$ is continuous for almost surely all ω and $\xi_t(\cdot, \omega) \in W_{\text{loc}}^{1,p}(\mathbf{R}^n)$ for each $t > 0, p > 0$.*

5 The differentiation formula

The uniqueness of a strong solution of SDE (1.1) leads to the Markov property of the solution $\xi_t(x)$, see a proof in [21] that can be easily followed under Assumption 1.1. For $f \in \mathcal{B}_b(\mathbf{R}^n)$ define $P_t f(x) := \mathbf{E}f(\xi_t(x))$ so P_t is the associated Markov semigroup to (1.1). In this section a representation for dP_t is given which leads to an integration by parts formula for the measure induced by the solution of the SDE (1.1). Let $\xi_t(x)$ be the solution flow of SDE (1.1) and $V_t(x) \in L(\mathbf{R}^n, \mathbf{R}^n)$ the solution of (2.23) constructed in Theorem 2.10. Let $V_t(x, v_0) = \langle V_t(x), v_0 \rangle_{\mathbf{R}^n}$ for $v_0 \in \mathbf{R}^n$ and $Y : \mathbf{R}^n \rightarrow L(\mathbf{R}^n, \mathbf{R}^m)$ the right inverse of map $A : \mathbf{R}^n \rightarrow L(\mathbf{R}^m, \mathbf{R}^n)$, where

$$A(x)(a) := \sum_{l=1}^m a_l A_l(x) \quad \text{for } a = (a_1, a_2, \dots, a_m) \in \mathbf{R}^m. \quad (5.1)$$

Theorem 5.1. *Suppose the Assumption 1.1 and condition (2.22) hold, then there is a constant $T_6(K, \alpha, M, \theta, n, T_0, \tilde{T}, \sigma)$, such that*

$$dP_t f(x)(v_0) = \frac{1}{t} \mathbf{E} \left[f(\xi_t(x)) \int_0^t \langle Y(\xi_s(x))(V_s(x, v_0)), dW_s \rangle_{\mathbf{R}^m} \right], \quad v_0 \in \mathbf{R}^n \quad (5.2)$$

for any f in $\mathcal{B}_b(\mathbf{R}^n)$ and $0 < t \leq T_6$. If moreover $f \in C_b^1(\mathbf{R}^n)$, then $d(P_t f)(x)(v_0) = \mathbf{E}df(V_t(x, v_0))$, for all $v_0 \in \mathbf{R}^n$ and such t .

Proof. Take $f \in C_b^1(\mathbf{R}^n)$. Since the coefficient of SDE (2.3) is smooth and with bounded derivatives, by the classical formula in [4], we have

$$d\mathbf{E}f(\xi_t^\varepsilon(x))(v_0) = \mathbf{E}df(V_t^\varepsilon(x, v_0)), \quad v_0 \in \mathbf{R}^n \quad (5.3)$$

and

$$d\mathbf{E}f(\xi_t^\varepsilon(x))(v_0) = \frac{1}{t} \mathbf{E} \left[f(\xi_t^\varepsilon(x)) \int_0^t \langle Y^\varepsilon(\xi_s^\varepsilon(x))(V_s^\varepsilon(x, v_0)), dW_s \rangle_{\mathbf{R}^m} \right] \quad (5.4)$$

where $Y^\varepsilon : \mathbf{R}^n \rightarrow L(\mathbf{R}^n, \mathbf{R}^m)$ is the right inverse of map $A^\varepsilon : \mathbf{R}^n \rightarrow L(\mathbf{R}^m, \mathbf{R}^n)$. Since $f \in C_b^1(\mathbf{R}^n)$, by Lemma 2.3 and Theorem 2.10, there is a constant $T_6 > 0$, such that for any bounded set S in \mathbf{R}^n ,

$$\limsup_{\varepsilon \rightarrow 0} \mathbf{E} \sup_{x \in S} \sup_{0 \leq s \leq T_6} |f(\xi_s^\varepsilon(x)) - f(\xi_s(x))|^4 = 0 \quad (5.5)$$

and

$$\limsup_{\varepsilon \rightarrow 0} \mathbf{E} \sup_{x \in S} \sup_{0 \leq s \leq T_6} |V_s^\varepsilon(x, v_0) - V_s(x, v_0)|^4 = 0 \quad (5.6)$$

So by (5.3) and (5.5-5.6) we obtain that $P_t f$ is differentiable if $f \in C_b^1(\mathbf{R}^n)$ and $d(P_t f)(x)(v_0) = \mathbf{E}df(V_t(x, v_0))$, $v_0 \in \mathbf{R}^n$ for each $0 \leq t \leq T_6$.

For the square matrix $(A^\varepsilon)^* A^\varepsilon$ (* here means the transpose of the matrix), related to A^ε , with entries (a_{ij}^ε) , we have

$$((A^\varepsilon)^* A^\varepsilon)^{-1} \rightarrow (A^* A)^{-1}$$

as the inverse map is smooth in the Lie group of $n \times n$ nonsingular matrices. Consequently $Y^\varepsilon = (A^\varepsilon)^* ((A^\varepsilon)^* A^\varepsilon)^{-1} \rightarrow Y$. In fact let $A_{il}^\varepsilon, a_{ij}^\varepsilon$ be defined as above, then for any $\eta = (\eta_1, \eta_2, \dots, \eta_n) \in \mathbf{R}^n$, $Y^\varepsilon(x)(\eta) = (\zeta_1^\varepsilon(x), \zeta_2^\varepsilon(x), \dots, \zeta_m^\varepsilon(x))$ where $\zeta_l^\varepsilon(x) = \sum_{i,j=1}^n A_{il}^\varepsilon(x) b_{ij}^\varepsilon(x) \eta_j$. Here

$$(b_{ij}^\varepsilon(x)) = (A^\varepsilon(x))^{-1}.$$

Since by Assumption 1.1, for ε small, a_{ij}^ε have the same uniform elliptic constants and A_l^ε are uniformly bounded with ε , we obtain that b_{ij}^ε converges to b_{ij} uniformly in \mathbf{R}^n , and b_{ij}^ε are uniformly bounded with ε . Also note that the bounded, uniform continuity of A_l and the uniformly positive lower bounded of the determination of matrix $A^* A$ (defined as (5.1)) in \mathbf{R}^n implies that b_{ij} is uniformly continuous in \mathbf{R}^n . So by Lemma 2.3 we can show that for any $T > 0$ and bounded set S in \mathbf{R}^n ,

$$\limsup_{\varepsilon \rightarrow 0} \mathbf{E} \sup_{x \in S} \sup_{0 \leq s \leq T} |b_{ij}^\varepsilon(\xi_s^\varepsilon(x)) - b_{ij}(\xi_s(x))|^4 = 0.$$

This together with the convergence (5.6) leads to

$$\limsup_{\varepsilon \rightarrow 0} \sup_{x \in S} \int_0^t \mathbf{E} |Y^\varepsilon(\xi_s^\varepsilon(x))(V_s^\varepsilon(x, v_0)) - Y(\xi_s(x))(V_s(x, v_0))|^2 ds = 0$$

for all $t \leq T_6$ and $S \subset \mathbf{R}^n$ bounded. Then by (5.5), we see that,

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \sup_{x \in S} \left| \mathbf{E} \left[f(\xi_t^\varepsilon(x)) \int_0^t \langle Y^\varepsilon(\xi_s^\varepsilon(x))(V_s^\varepsilon(x, v_0)), dW_s \rangle_{\mathbf{R}^m} \right] \right. \\ \left. - \mathbf{E} \left[f(\xi_t(x)) \int_0^t \langle Y(\xi_s(x))(V_s(x, v_0)), dW_s \rangle_{\mathbf{R}^m} \right] \right| = 0. \end{aligned} \quad (5.7)$$

which implies the differentiation formula (5.4) holds for each $f \in C_b^1(\mathbf{R}^n)$. For $f \in \mathcal{B}_b(\mathbf{R}^n)$. Let f_N be a sequence in $C_b^1(\mathbf{R}^n)$ with

$$\lim_{N \rightarrow \infty} \int_S |f_N(x) - f(x)|^4 dx = 0$$

for all bounded set S in \mathbf{R}^n . By the heat kernel upper bound $p_t(x, y) \leq c_1 t^{-n/2} e^{-\frac{|x-y|^2}{c_2 t}}$ from Lemma 2.4,

$$\lim_{N \rightarrow \infty} \sup_{x \in S} \mathbf{E} |f_N(\xi_t(x)) - f(\xi_t(x))|^4 = 0$$

also holds for each $0 < t \leq T_6$ and this completes the proof. \square

Theorem 5.2. *Suppose that Assumption 3.1 holds and*

$$\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq \frac{1}{e^{\sigma \psi_1(|x|)}} |\xi|^2. \quad \forall x \in \mathbf{R}^n, \xi = (\xi_1, \dots, \xi_n) \in \mathbf{R}^n. \quad (5.8)$$

Here σ is some constant and $\psi_1 : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ is the function in Assumption 3.1, then for any $f \in \mathfrak{B}_b(\mathbf{R}^n)$ and $t > 0$,

$$dP_t f(x)(v_0) = \frac{1}{t} \mathbf{E} \left[f(\xi_t(x)) \int_0^t \langle Y(\xi_s(x))(V_s(x, v_0)), dW_s \rangle_{\mathbf{R}^m} \right], \quad v_0 \in \mathbf{R}^n \quad (5.9)$$

Proof. In the proof the constants C may change in different lines and do not depend on N . As the same approximation methods in the above Theorem and by Theorem 3.1, we can prove that if the coefficients of SDE (1.1) are bounded, uniform elliptic and Lipschitz continuous, then the differentiation formula (5.9) holds for any $f \in \mathfrak{B}_b(\mathbf{R}^n)$ and $t > 0$. Let f^N, A_t^N be the cut-off functions defined by (3.1) and that $\xi_t^N(x), V_t^N(x)$ the solution of corresponding SDE. So by the analysis above, $P_t^N f(x) = \mathbf{E}(f(\xi_t^N(x)))$ is differentiable with x , and for any $f \in \mathfrak{B}_b(\mathbf{R}^n)$ and $t > 0$,

$$d\mathbf{E}f(\xi_t^N(x))(v_0) = \frac{1}{t} \mathbf{E} \left[f(\xi_t^N(x)) \int_0^t \langle Y^N(\xi_s^N(x))(V_s^N(x, v_0)), dW_s \rangle_{\mathbf{R}^m} \right] \quad (5.10)$$

where Y^N is a right inverse of A^N .

By the elliptic condition (5.8) and the expression of Y we use in the proof of Theorem 5.1, there are constants $C > 0, k \in \mathbb{N}^+$, such that $|Y(x)| \leq C e^{\sigma \psi_1(|x|)} (\psi_2(|x|))^k$, where $\psi_i, i = 1, 2$ are the functions as that in Assumption 3.1, and the same estimate also holds for Y^N . Let $T_N(x)$ be

the first exit time of $\xi_t(x)$ from the ball B_N , then for each $T > 0$, $p > 0$,

$$\begin{aligned} & \sup_{x \in S} \mathbf{E} \sup_{0 \leq s \leq T} |Y^N(\xi_s^N(x)) - Y(\xi_s(x))|^p \\ & \leq C \sup_{x \in S} \mathbf{E} \left[\sup_{0 \leq s \leq T} \left(\exp\{(\sigma p \psi_1(|\xi_s^N(x)|))\} \right) (\psi_2(|\xi_s^N(x)|))^{kp} I_{T > T_N(x)} \right] \\ & \leq C \sup_{x \in S} \mathbf{E} \left[\sup_{0 \leq s \leq T} \left(\exp\{(2\sigma p \psi_1(|\xi_s^N(x)|))\} + (\psi_2(|\xi_s^N(x)|))^{2kp} \right) I_{T > T_N(x)} \right] \end{aligned}$$

By the analysis in Section 3 (see also Theorem 5.1 in [15]), if Assumption 3.1 holds, we have,

$$\sup_N \sup_{x \in S} \mathbf{E} \left[\sup_{0 \leq s \leq T} \left(\exp\{(2\sigma p \psi_1(|\xi_s^N(x)|))\} + (\psi_2(|\xi_s^N(x)|))^{2kp} \right)^2 \right] < \infty \quad (5.11)$$

and

$$\sup_{x \in S} \mathbf{P}(T > T_N(x)) \leq \frac{\sup_{x \in S} \mathbf{E} \sup_{0 \leq s \leq T} |\xi_s^N(x)|^2}{N^2} \leq \frac{C}{N^2} \quad (5.12)$$

By (5.11) and (5.12), it follows that

$$\lim_{N \rightarrow \infty} \sup_{x \in S} \mathbf{E} \sup_{0 \leq s \leq T} |Y^N(\xi_s^N(x)) - Y(\xi_s(x))|^p = 0 \quad (5.13)$$

Also note that from the analysis of Section 3, if Assumption 3.1 holds, then,

$$\lim_{N \rightarrow \infty} \sup_{x \in S} \mathbf{E} \sup_{0 \leq s \leq T} (|\xi_s^N(x) - \xi_s(x)|^p + |V_s^N(x) - V_s(x)|^p) = 0 \quad (5.14)$$

By (5.13) and (5.14), (5.9) holds for each $f \in C_b^1(\mathbf{R}^n)$. For $f \in \mathfrak{B}_b(\mathbf{R}^n)$, take an approximating sequence $f_\varepsilon \in C_b^1(\mathbf{R}^n)$, such that for any bounded set S in \mathbf{R}^n ,

$$\lim_{\varepsilon \rightarrow 0} \int_S |f_\varepsilon(x) - f(x)|^p dx = 0$$

and $\|f_\varepsilon\|_{L^\infty} \leq \|f\|_{L^\infty}$. Note that $\xi_t^N(x)$ is the solution of a SDE with uniformly elliptic, global Lipschitz continuous and bounded coefficients, so by the Markov kernel estimate, we have for each fixed N and $t > 0$,

$$\lim_{\varepsilon \rightarrow 0} \sup_{x \in S} \mathbf{E} |f_\varepsilon(\xi_t^N(x)) - f(\xi_t^N(x))|^p = 0 \quad (5.15)$$

And then

$$\begin{aligned}
& \sup_{x \in S} \mathbf{E} |f_\varepsilon(\xi_t(x)) - f(\xi_t(x))|^p \leq C \left[\sup_{x \in S} \mathbf{E} |f_\varepsilon(\xi_t^N(x)) - f_\varepsilon(\xi_t(x))|^p \right. \\
& \quad \left. + \sup_{x \in S} \mathbf{E} |f(\xi_t^N(x)) - f(\xi_t(x))|^p + \sup_{x \in S} \mathbf{E} |f_\varepsilon(\xi_t^N(x)) - f(\xi_t^N(x))|^p \right] \\
& \leq C \left[\|f\|_{L^\infty} \sup_{x \in S} \mathbf{P}(T > T_N(x)) + \sup_{x \in S} \mathbf{E} |f_\varepsilon(\xi_t^N(x)) - f(\xi_t^N(x))|^p \right]
\end{aligned}$$

By (5.12) and (5.15), in above inequality first let $\varepsilon \rightarrow 0$, then $N \rightarrow \infty$, we obtain,

$$\lim_{\varepsilon \rightarrow 0} \sup_{x \in S} \mathbf{E} |f_\varepsilon(\xi_t(x)) - f(\xi_t(x))|^p = 0 \quad (5.16)$$

The proof is complete. \square

5.1 Integration by parts formula

Let $H = L_0^{2,1}(\mathbf{R}^m)$ be the space of real valued function from $[0, T]$ to \mathbf{R}^m , starting from 0 and with finite energy, which is also equipped with the usual Hilbert space structure. Let (Ω, \mathcal{F}, P) be the standard Wiener space and d the unbounded closed linear operator $L^p(\Omega, \mathbf{R}) \rightarrow L^p(\Omega, L(H, \mathbf{R}))$, $p > 1$ which agrees with the standard differentiation on BC^1 functions (see [6] and reference in that). Let $\mathcal{D}^{1,p}$ be the domain of d , which is the closure of smooth cylindrical functions under the graph norm, and by tradiiton we denote the extension by T .

Let $C_x([0, T]; \mathbf{R}^n) := \{\gamma : \gamma \in C([0, T], \mathbf{R}^n), \gamma(0) = x\}$ and

$$\mathcal{I} : \Omega \rightarrow C_x(\mathbf{R}^n) \quad \mathcal{I}(\omega)_t := \xi_t(x, \omega)$$

be the Itô map, where $\xi_t(x, \omega)$ is the solution of SDE (1.1). It is standard result that $\mathcal{I}_t : \Omega \rightarrow \mathbf{R}^n$ belong to the space of $D^{1,p}$ for all $p > 1$ and t if the coefficients of SDE (1.1) are Lipschitz continuous (see [20]). Furthermore, if the coefficients are smooth and with bounded derivatives, then by the results of Bismut,

$$T_\omega \mathcal{I} : H \rightarrow T_{\xi_t(x, \omega)} C_x([0, T]; \mathbf{R}^n)$$

the H derivative for the Itô map \mathcal{I} in the sense of Malliavin calculus exists in L^p for each $p > 1$, and for $v_t(\omega) := T_\omega \mathcal{I}_t(h)$,

$$v_t(\omega) = V_t(x) \int_0^t V_s^{-1}(x) (A(\xi_s(x))(\dot{h}_s)) ds. \quad 0 \leq t \leq T, \quad h \in H$$

where $V_t(x) \in \mathbf{R}^{n \times n}$ is the derivative process satisfying (1.2), V_t^{-1} is its inverse and A is defined as (5.1). And by [20], $v_t(\omega) := T_\omega \mathcal{I}_t(h)$ also satisfies the following SDE

$$v_t = 0 + \sum_{l=1}^m \left(\int_0^t DA_l(\xi_s(x)) v_s dW_s^l + \int_0^t A_l(\xi_s(x)) \dot{h}_s ds \right) + \int_0^t DA_0(\xi_s(x)) v_s ds \quad (5.17)$$

Define $V^h(\xi)_t := \langle V_t(x), h_t \rangle_{\mathbf{R}^n}$ and

$$\delta V_t^h(\xi) := \int_0^t \langle Y(\xi_s(x)) V^h(\xi(x))_s, dW_s \rangle_{\mathbf{R}^m} \quad (5.18)$$

where \dot{h} means the derivative of h with time and Y is the right inverse of A defined in (5.1). By the approximation theorem we derive the following result.

Theorem 5.3. *Suppose the Assumption 1.1 holds and there exist $\sigma > 0$ and $T_0 > 0$, the condition (2.22) is satisfied, then there is a constant $T_8 > 0$, such that for any $0 \leq T \leq T_8$, we consider the path space $C_x([0, T]; \mathbf{R}^n)$, given a $h : [0, T] \times \Omega \rightarrow \mathbf{R}^n$, an adapted stochastic process with $h(\omega) \in L_0^{2,1}([0, T_8]; \mathbf{R}^m)$ a.s. and $\mathbf{E}(\int_0^{T_8} |\dot{h}_s|^2 ds)^{\frac{1+\beta}{2}} < \infty$ for some $\beta > 0$,*

$$\mathbf{E}dF(V^h(\xi)) = \mathbf{E}F(\xi(x))\delta V_T^h(\xi)$$

where F is the BC^1 function on path space $C_x([0, T]; \mathbf{R}^n)$.

Proof. When the coefficients of SDE (1.1) are smooth, it was shown in [5] that the differentiation formula for $P_t f$ leads to an integration by parts formula. The theorem there was given for compact manifolds. However this results and its proof remain valid for non-compact manifold if the differentiation formula for $P_t f$ holds as the proof only involves the Markov property. Since the coefficients of the approximate SDE (2.3) are smooth, uniformly elliptic and with bounded derivatives,

$$\mathbf{E}dF(V^{h,\varepsilon}(\xi^\varepsilon(x))) = \mathbf{E}F(\xi^\varepsilon(x))\delta V_T^{h,\varepsilon}(\xi^\varepsilon),$$

for any $T > 0$ where $V^{h,\varepsilon}(\xi^\varepsilon(x)) = \langle V_t^\varepsilon(x), h_t \rangle_{\mathbf{R}^n}$ and

$$\delta V_T^{h,\varepsilon}(\xi^\varepsilon) = \int_0^T \langle Y^\varepsilon(\xi_s^\varepsilon(x)) V^{h,\varepsilon}(\xi^\varepsilon(x))_s, dW_s \rangle_{\mathbf{R}^m}.$$

Now by Theorem 2.10 and the analysis before for the convergence Y^ε , the proof is completed. \square

Remark 5.1. As the same cut-off methods in Section 3. suppose Assumption 3.1 holds, we can also prove the same results in Section 4 for any time interval $[0, T], T > 0$

6 Appendix: The Geometry of Regularization

Let \mathcal{L} be a smooth elliptic second order differential operator without zero order term with $a = (a_{ij})$ the matrix representation of its second order part. The non-singular symmetric matrix a has a square root which can be chosen to be locally Lipschitz continuous, see Stroock-Varadhan [22] and the book of Ikeda-watanabe[13]. Hence \mathcal{L} has a Hörmander form representation $\mathcal{L} = \frac{1}{2}L_{A_l}L_{A_l} + L_Z$ where A_l, Z are vector fields. This representation is far from being unique. Each representation produces a stochastic flow and corresponding geometry. We investigate the geometry and the properties of the stochastic flows for the decomposition involving non-global Lipschitz continuous vector fields.

Give $M = \mathbf{R}^n$ and the Riemannian metric $(a_{ij})^{-1}$ induced by the family of vector fields (A_1, \dots, A_m) . We consider \mathbf{R}^n as a trivial manifold with a non-trivial Riemannian structure. Uniform ellipticity condition and boundedness of the diffusion coefficients implies that the induced Riemannian metric is quasi-isometric to the Euclidean metric. The ellipticity condition (5.8) and Assumption 3.1 would mean the new Riemannian metric is ‘weakly’ quasi isometric with the Euclidean metric. For simplicity, from now on in this section, we assume the coefficients of SDE (1.1), $A_l \in C_b^1(\mathbf{R}^n), 1 \leq l \leq m, DA_l, 1 \leq l \leq m$ and A_0 is bounded and (global) Lipschitz continuous in \mathbf{R}^n . We can also obtain the results under more general condition by the cut-off methods used in Section 3.

For each $e \in \mathbf{R}^m$, define $A(x)(e) = \sum A_l(x)\langle e, e_l \rangle e_l$ where $\{e_l\}$ is an o.n.b. of \mathbf{R}^m . In the case when A_l are smooth and elliptic, this induces a smooth Riemannian metric on \mathbf{R}^n as well as an affine connection which is adapted to the metric such that $(\check{\nabla}.X)(e)(x) = 0$ for all $e \in [\ker X(x)]^\perp$. See the analysis in Elworthy-LeJan-Li [3]. This leads to a smooth decomposition of $(\check{\nabla}_t(x_t))^{-1}W_t$, where $\check{\nabla}_t$ is the stochastic parallel translation along the paths of ξ , defined using $\check{\nabla}$, into the sum of two independent Brownian motions in \mathbf{R}^m , one of which is intrinsic to ξ .

In the non-smooth case we discuss a smooth approximation which preserves much of the properties of the connection, which leads to a non-smooth Riemannian geometry. We use the approximation argument to prove a intrinsic integration by parts formula. Stronger regularity on A_l , than in

sections 2 and 3, are required.

For $(A_1^\varepsilon, \dots, A_m^\varepsilon)$, the smooth elliptic approximations of (A_1, \dots, A_m) there are the affine connection $\check{\nabla}^\varepsilon$ and its adjoint connection $\hat{\nabla}^\varepsilon$ given by

$$\check{\nabla}_v^\varepsilon U = A^\varepsilon(x)D(Y^\varepsilon(x)U)(v), \quad v \in T_x M, U \in \Gamma TM.$$

and

$$\hat{\nabla}_U^\varepsilon V = \check{\nabla}_V^\varepsilon U + [U, V], \quad U, V \in \Gamma TM.$$

In components this reads

$$(\check{\nabla}_v^\varepsilon U)_k(x_0) = (DU_k)_{x_0}(v) + \sum_{j=1}^n \langle A^\varepsilon(x_0)D(Y^\varepsilon(x_0))(v, e_j), e_k \rangle U_j e_k$$

where (e_j) is the standard basis of \mathbf{R}^n and $U = (U_1, \dots, U_n)$ and $DY^\varepsilon : \mathbf{R}^n \rightarrow L(\mathbf{R}^n \times \mathbf{R}^n; \mathbf{R}^m)$, see [3] and we follow the tradition there and call it the LW connection. The last term in the equation can be written as $\Gamma_{ij}^{\varepsilon,k} v_i U_j e_k$ where $\{\Gamma_{ij}^{\varepsilon,k}, 1 \leq i, j, k \leq n\}$ is a family of real valued smooth functions. In particular this is the unique connection such that $(\check{\nabla}_v^\varepsilon A)_{x_0} = 0$ for all $v \in [\ker A^\varepsilon(x_0)]^\perp$ and $x_0 \in \mathbf{R}^n$.

Given a vector field along a continuous curve there is the stochastic covariant differentiation with a fixed connection defined for almost surely all paths, given by $\frac{\hat{D}^\varepsilon}{dt} V_t = \hat{\jmath}_t^\varepsilon \frac{d}{dt} (\hat{\jmath}_t^\varepsilon)^{-1} V_t$ where $\hat{\jmath}_t^\varepsilon$ is the stochastic parallel translation using the connection $\hat{\nabla}^\varepsilon$.

Proposition 6.1. *Assume the SDE (1.1) is uniformly elliptic and $A_l \in C_b^1(\mathbf{R}^n)$ for $l = 1, \dots, m$. Suppose that DA_l , $l = 1, \dots, m$ and A_0 are bounded and (global) Lipschitz continuous in \mathbf{R}^n . Let $\check{\jmath}_s^\varepsilon : \mathbf{R}^n \rightarrow \mathbf{R}^n$ and $\hat{\jmath}_s^\varepsilon : \mathbf{R}^n \rightarrow \mathbf{R}^n$ be the stochastic parallel translations with the connection $\check{\nabla}^\varepsilon$ and $\hat{\nabla}^\varepsilon$ respectively. Then $\check{\jmath}_s^\varepsilon : \mathbf{R}^n \rightarrow \mathbf{R}^n$ and $\hat{\jmath}_s^\varepsilon : \mathbf{R}^n \rightarrow \mathbf{R}^n$ converge in L^p for any $p \geq 1$ and the martingale part of anti-stochastic development map also converges in L^p to a Brownian motion \check{B}_t . Furthermore the filtration of $\{\check{B}_s : 0 \leq s \leq t\}$ is the same as that of $\{\xi_s, 0 \leq s \leq t\}$.*

Proof. For any $v_0 \in \mathbf{R}^n$, let $\check{v}_t^\varepsilon := \check{\jmath}_t^\varepsilon(v_0)$ and $\hat{v}_t^\varepsilon := \hat{\jmath}_t^\varepsilon(v_0)$. Note that the k -th component of such process satisfy $d\check{v}_t^{\varepsilon,k} = -\Gamma_{i,j}^{\varepsilon,k}(\xi_t^\varepsilon) \check{v}_t^{\varepsilon,j} \circ d\xi_t^{\varepsilon,i}$ and $d\hat{v}_t^{\varepsilon,k} = -\Gamma_{j,i}^{\varepsilon,k}(\xi_t^\varepsilon) \hat{v}_t^{\varepsilon,j} \circ d\xi_t^{\varepsilon,i}$ respectively. For simplicity, we only prove the convergence of \check{v}_t^ε , and the same results can be proved for \hat{v}_t^ε as the same way. In fact, we have,

$$d\check{v}_t^\varepsilon = \sum_{l=1}^m G_l^\varepsilon(\xi_t^\varepsilon)(\check{v}_t^\varepsilon) dW_t^l + G_0^\varepsilon(\xi_t^\varepsilon)(\check{v}_t^\varepsilon) dt \quad (6.1)$$

where each $G_l^\varepsilon(x)$ for $l = 1, \dots, m$, is a $m \times n$ matrix with the (j, k) entry given by $\sum_{i=1}^n A_{il}^\varepsilon(x) \Gamma_{ij}^{\varepsilon, k}(x)$ and the drift term G_0 is the sum of some items only involving A_l , $\Gamma_{ij}^{\varepsilon, k}(x)$ and their first derivatives. Note that the Christoffel symbols are determined by $A^\varepsilon(DY^\varepsilon)$ (see the analysis in [3]) and

$$DY^\varepsilon(v) = D(A^\varepsilon)^*(v) + (A^\varepsilon)^*D((A^\varepsilon)^*A^\varepsilon)^{-1}(v).$$

From the assumption of the proposition we see that G_l^ε , $l = 1, \dots, m$, are bounded in \mathbf{R}^n , uniformly in ε for ε sufficiently small. Let \check{v}_t be the solution to the corresponding SDE (6.1) without the ε term and some items related to the second order derivatives of A_l are bounded modifications of the almost sure derivative of D^2A_l , so the first derivatives of Γ_{ij}^k makes sense. Then for the linear SDE (6.1), by a proof analogous to that of Lemma 2.6, we have for each $t > 0, p \geq 1$,

$$\lim_{\varepsilon \rightarrow 0} \mathbf{E} \sup_{0 \leq s \leq t} |\check{v}_s^\varepsilon - \check{v}_s|^p = 0$$

Let \check{B}_s^ε be the martingale part of the stochastic anti-development map $\int_0^t (\check{\jmath}_s^\varepsilon)^{-1} \circ d\xi_s^\varepsilon$. Note that the stochastic parallel translation $\check{\jmath}_s^\varepsilon$ is an isometry hence by the convergence results for $\check{\jmath}_s^\varepsilon$, it is straight forward to show that \check{B}_s^ε converges in L^p as ε tends to 0. Since for each ε , \check{B}_s^ε is a Brownian motion (see [3]), so the limit process \check{B}_s is also a Brownian motion. Moreover if $\check{\jmath}_s^{-1}$ is the inverse of $\check{\jmath}_s$, the limit process of $\check{\jmath}_s^\varepsilon$, \check{B}_s is the martingale part of $\int_0^t \check{\jmath}_s^{-1} d\xi_s$. The Brownian motion \check{B} is clearly adapted to the filtration of ξ . For the opposite inclusion of filtrations, let $h_u^\varepsilon(A_l^\varepsilon)$ be the horizontal lift of A_l^ε at frame u and respect to $\check{\nabla}^\varepsilon$ in the orthonormal frame bundle. Then the horizontal lift of the path ξ_t^ε starting from the initial frame u_0 satisfies:

$$d\tilde{\xi}_t^\varepsilon = \sum_{l=1}^m h_{\tilde{\xi}_t^\varepsilon}^\varepsilon A_l^\varepsilon(\xi_t^\varepsilon) \circ d\check{B}_t^{\varepsilon, l} + h_{\tilde{\xi}_t^\varepsilon}^\varepsilon A_0^\varepsilon(\xi_t^\varepsilon) dt.$$

Note that the horizontal lift h^ε only depends on the Christoffel symbols $\Gamma_{ij}^{\varepsilon, k}(x)$ and A_l^ε . So take $\varepsilon \rightarrow 0$ to see the above equation also holds without parameter ε (The second order derivatives of A_l are viewed as bounded version of the weak derivatives). And it implies that $\tilde{\xi}$, therefore ξ is adapted to the filtration of $\sigma\{\check{B}_s : \}$ from the stochastic differential equation which defines it. \square

The stochastic processes V^h defined in Section 5 are not intrinsic objects on the path space. And use the conclusion of Proposition 6.1, we show that

$\mathbf{E}\{T\mathcal{I}_t(h)|\sigma(\xi_s(x), 0 \leq s \leq T)\}$ is an intrinsic object where $T\mathcal{I}(h)$ is the Malliavin derivative of the Itô map \mathcal{I} .

Theorem 6.2. *Suppose the same assumption in Proposition 6.1 holds. Let $\hat{W}_t : \mathbf{R}^n \rightarrow \mathbf{R}^n$ be the damped parallel translation which satisfies the following equation*

$$\frac{\hat{D}}{dt}[\hat{W}_t(v_0)] = -\frac{1}{2}(\text{Ric})^\#(W_t(v_0))dt + \check{\nabla}_{W_t(v_0)}^\varepsilon A_0 dt \quad W_0(v_0) = v_0 \quad (6.2)$$

where $\frac{\hat{D}}{dt} = \hat{\jmath}_t \frac{d}{dt} ((\hat{\jmath}_t)^{-1})$ and $\text{Ric} : \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}$ is the Ricci tensor with the connection $\check{\nabla}$ and $\text{Ric}^\#$ is the corresponding linear map on defined by $\hat{\jmath} : \mathbf{R}^n \rightarrow \mathbf{R}^n$ (The $\hat{\jmath}$, $\hat{\jmath}_t$ are the limit process we get in Proposition 6.1). Then we have,

$$\mathbf{E}\{T\mathcal{I}_t(h)|\sigma(\xi_s, 0 \leq s \leq T)\}(\xi) = \hat{W}_t \int_0^t (\hat{W}_s)^{-1} A(\xi_s) \dot{h}_s ds \quad (6.3)$$

Proof. From equation (5.17), by the boundedness condition we have, it can be proved as before as that $T\mathcal{I}_t^\varepsilon(h)$ is a Cauchy sequence in $L^p(\mathbf{P})$ for any $p > 0$, where \mathcal{I}^ε is the Ito map defined by SDE (2.3). By the closibility of Malliavin derivative, we derive that \mathcal{I} is differentiable in the Malliavin calculus. By Theorem 3.3.7 in [3], for SDE (2.3) with smooth coefficients, $\mathbf{E}\{T\mathcal{I}_t^\varepsilon(h)|\sigma\{\xi_s^\varepsilon : 0 \leq s \leq T\}\}$ satisfies the equation (6.3) where process \hat{W}_t^ε is defined similarly by equation (6.2). Note that the parallel translation in the equation (6.2) is defined by the adjoint connection, which is in general not adapted with some metric, so $\hat{\jmath}$ is not a isometry in general. But we can tranform $\frac{\hat{D}}{dt}$ in the equation to $\frac{\check{D}}{dt}$ defined by original LW connection, and the pointwise ODE (6.2) will become a linear SDE with some torsion terms. Also note that we have the following formula for the curvature tensor (see [3]),

$$R^\varepsilon(u, v)(w) = \sum_i \check{\nabla}_u A_i^\varepsilon \langle \check{\nabla}_v A_i^\varepsilon, w \rangle + \sum_i \check{\nabla}_v A_i^\varepsilon \langle \check{\nabla}_u A_i^\varepsilon, w \rangle. \quad (6.4)$$

So by the conclusion of Proposition 6.1 and the methodology in Section 2, we obtain that

$$\lim_{\varepsilon \rightarrow 0} \mathbf{E} \sup_{0 \leq s \leq t} |\hat{W}_s^\varepsilon - \hat{W}_s|^p = 0$$

for any $p > 1$. Since W_t^{-1} also satisfies a linear SDE, but the driven Brownian motion is with backward filtration, we derive the L^p convergence of

$(\hat{W}_s^\varepsilon)^{-1}$. From that we know $\mathbf{E}\{T\mathcal{I}_t^\varepsilon(h)|\sigma\{\xi_s^\varepsilon : 0 \leq s \leq T\}\}$ converges to $\hat{W}_t \int_0^t (\hat{W}_s)^{-1} A(\xi_s) \dot{h}_s ds$ in L^p for any $p > 1$.

Note that for any BC^1 function F on path space, we have

$$\mathbf{E}\left[T\mathcal{I}_t^\varepsilon(h)F(\xi^\varepsilon)\right] = \mathbf{E}\left[\hat{W}_t^\varepsilon \left(\int_0^t (\hat{W}_s^\varepsilon)^{-1} A^\varepsilon(\xi_s^\varepsilon) \dot{h}_s ds\right) F(\xi^\varepsilon)\right]$$

Let ε tend to 0, by the convergence results for $T\mathcal{I}_t^\varepsilon(h)$ we have,

$$\mathbf{E}\left[T\mathcal{I}_t(h)F(\xi)\right] = \mathbf{E}\left[\hat{W}_t \int_0^t (\hat{W}_s)^{-1} A(\xi_s) \dot{h}_s ds F(\xi)\right]$$

which implies the conclusion (6.3). \square

As the same approximation argument, we can prove the following intrinsic integration by parts formula

Theorem 6.3. *We assume the assumptions of Proposition 6.1. Let $h : [0, 1] \times \Omega \rightarrow \mathbf{R}^n$ be an adapted stochastic process with $h(\omega) \in L_0^{2,1}([0, 1]; \mathbf{R}^n)$ for almost surely all ω and $\mathbf{E}(\int_0^1 |\dot{h}_s|^2 ds)^{\frac{1+\beta}{2}} < \infty$ for some $\beta > 0$. Then*

$$\mathbf{E}dF(\hat{W}.h.) = \mathbf{E}\left[F(\xi.) \int_0^1 \langle \hat{W}_s \dot{h}_s, \check{\int}_s d\check{B}_s \rangle\right]$$

for all BC^1 functions F on path space.

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