ON CONVOLUTIONS IN HILBERT SPACES

B. KANGUZHIN, M. RUZHANSKY, AND N. TOKMAGAMBETOV

0.1. **Introduction.** Let $\mathcal{H}$ be a separable Hilbert space. Denote by $\mathcal{U} := \{u_\xi |\ u_\xi \in \mathcal{H}\}_{\xi\in\mathcal{I}}$ and $\mathcal{V} := \{v_\xi |\ v_\xi \in \mathcal{H}\}_{\xi\in\mathcal{I}}$ collections of elements of $\mathcal{H}$ parametrised by a discrete set $\mathcal{I}$. We assume that the system $\mathcal{U}$ is a Riesz basis of the space $\mathcal{H}$ and the system $\mathcal{V}$ is biorthogonal to $\mathcal{U}$ in $\mathcal{H}$. Then $(u_\xi, v_\eta)_{\mathcal{H}} = \delta_{\xi\eta}$, where $\delta_{\xi\eta}$ is the Kronecker delta, equal to 1 for $\xi = \eta$, and to 0 otherwise. In this case from the classical Bari's work [3] (see also Gelfand's paper [7]) it follows that the system $\mathcal{V}$ is also a Riesz basis in $\mathcal{H}$.

We define $\mathcal{U}$– and $\mathcal{V}$–convolutions in the following form:

$$f \star_{\mathcal{U}} g := \sum_{\xi\in\mathcal{I}} (f, v_\xi)(g, v_\xi)u_\xi \tag{0.1}$$

and

$$h \star_{\mathcal{V}} j := \sum_{\xi\in\mathcal{I}} (h, u_\xi)(j, u_\xi)v_\xi \tag{0.2}$$

for appropriate elements $f, g, h, j \in \mathcal{H}$.

0.2. **Example.** We give an example, which can be considered as an example of an extension of the periodic calculus studied in [12].

Consider an operator $\mathrm{O}_h^{(1)}$ defined by the action $\mathrm{O}_h^{(1)} := -i\frac{d}{dx}$ for $h > 0$, on the interval $(0, 1)$ with the boundary condition $hy(0) = y(1)$. When $h = 1$, $\mathrm{O}_1^{(1)}$ is defined by the periodic condition, and elements of systems $\mathcal{U}$ and $\mathcal{V}$ are eigenfunctions of the operator $\mathrm{O}_1^{(1)}$ and its conjugate $\mathrm{O}_1^{(1)^*}$, so they coincide, and are given by $\mathcal{U} = \mathcal{V} = \{u_j(x) = e^{2\pi ixj},\ j \in \mathbb{Z}\}$. That is, we get the basis elements for the classical Fourier analysis on a closed circle. Corresponding pseudo–differential calculus was developed in [12], developing the previous investigations of Agranovich [1, 2] and others.

For $h \neq 1$ the operator $\mathrm{O}_h^{(1)}$ is not self–adjoint. The spectral properties of $\mathrm{O}_h^{(1)}$ are well-known (see Titchmarsh [16] and Cartwright [4]). The corresponding biorthogonal families of eigenfunctions of $\mathrm{O}_h^{(1)}$ and its adjoint are given by $\mathcal{U} = \{u_j(x) = h^x e^{2\pi ixj},\ j \in \mathbb{Z}\}$ and $\mathcal{V} = \{v_j(x) = h^{-x}e^{2\pi ixj},\ j \in \mathbb{Z}\}$, respectively. They form Riesz bases in $\mathcal{H} = L^2(0, 1)$.

In this case, the convolution (0.1) has the form

$$(f \star_{\mathcal{U}} g)(x) = \int_0^x f(x - t)g(t)dt + \frac{1}{h}\int_x^1 f(1 + x - t)g(t)dt,$$

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which coincides with the standard convolution when $h = 1$. This convolution was investigated in papers [9, 11].

0.3. **Formulation of results.** We are aiming at a discussion of convolutions from a more abstract point of view, that is, when there is only given a Riesz basis in the Hilbert space without any additional assumptions on the operator whose eigenfunctions form this basis.

It is easy to check that the expressions (0.1) and (0.2) are correctly defined:

Theorem 0.1. *Let $f \star_{\mathcal{U}} g$ and $h \star_{\mathcal{V}} j$ be given by formulae* (0.1) *and* (0.2)*, respectively. Suppose that the families of functions $\mathcal{U}$ and $\mathcal{V}$ are uniformly bounded in $\mathcal{H}$. Then there exists a constant $M > 0$ such that we have*

$$\|f \star_{\mathcal{U}} g\|_{\mathcal{H}} \leq M \sup_{\xi\in\mathcal{I}} \|u_\xi\|_{\mathcal{H}} \|f\|_{\mathcal{H}} \|g\|_{\mathcal{H}}, \quad \|h \star_{\mathcal{V}} j\|_{\mathcal{H}} \leq M \sup_{\xi\in\mathcal{I}} \|v_\xi\|_{\mathcal{H}} \|h\|_{\mathcal{H}} \|j\|_{\mathcal{H}}, \tag{0.3}$$

for all $f, g, h, j \in \mathcal{H}$.

Let us introduce $\mathcal{U}$– and $\mathcal{V}$–Fourier transforms by formulae

$$\mathcal{F}_{\mathcal{U}}(f)(\xi) := (f, v_\xi) =: \widehat{f}(\xi) \tag{0.4}$$

and

$$\mathcal{F}_{\mathcal{V}}(g)(\xi) := (g, u_\xi) =: \widehat{g}_*(\xi), \tag{0.5}$$

respectively, for any $f, g \in \mathcal{H}$ and for arbitrary $\xi \in \mathcal{I}$. The inverse transforms have the following forms

$$(\mathcal{F}_{\mathcal{U}}^{-1} a)(x) := \sum_{\xi\in\mathcal{I}} a(\xi) u_\xi \tag{0.6}$$

and

$$(\mathcal{F}_{\mathcal{V}}^{-1} a)(x) := \sum_{\xi\in\mathcal{I}} a(\xi) v_\xi. \tag{0.7}$$

Now we formulate the statement about the relationship between $\mathcal{U}$- and $\mathcal{V}$–convolutions, and Fourier transforms:

Theorem 0.2. *For all $f, g, h, j \in \mathcal{H}$ we have*

$$\widehat{f \star_{\mathcal{U}} g} = \widehat{f}\,\widehat{g}, \quad \widehat{h \star_{\mathcal{V}} j}_* = \widehat{h}_*\,\widehat{j}_*.$$

Thus, the convolutions are commutative and associative.

Let $K : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be a bilinear operator. If for all $f, g \in \mathcal{H}$ the form $K(f, g)$ satisfies

$$\widehat{K(f,g)} = \widehat{f}\,\widehat{g}, \tag{0.8}$$

then K is a $\mathcal{U}$–convolution, that is $K(f, g) = f *_{\mathcal{U}} g$.

Analogously, if $K(f, g)$ satisfies the property

$$\widehat{K(f,g)}_* = \widehat{f}_*\,\widehat{g}_* \tag{0.9}$$

then K is a $\mathcal{V}$–convolution, that is $K(f, g) = f *_{\mathcal{V}} g$.

In particular, the last part of Theorem 0.2 means that the convolution (0.1) is uniquely determined by the condition that the Fourier transform of the convolution of elements is the product of Fourier transforms of elements.

0.4. **Discussion.** Let us discuss some additional aspects of the defined convolutions.

1. By identifying $\mathcal{H} = L^2(X)$ for some set X, we can write

$$(f \star_{\mathcal{U}} g)(x) = \int_X \int_X F(x, y, z) f(y) g(z) dy dz, \tag{0.10}$$

where

$$F(x, y, z) = \sum_{\xi \in \mathcal{I}} u_\xi(x) \, \overline{v_\xi(y)} \, \overline{v_\xi(z)}.$$

Here, we understand the integral (0.10) and the series for F in the sense of distributions. In the example from Section 0.2, when $h = 1$, we have $F(x, y, z) = \delta(x - y - z)$ (see [12]).

3. Obviously, convolutions (0.1) and (0.2) are commutative and associative. Moreover, they possess many properties of the standard convolution. The most important one is that by the Fourier transform the convolution is mapped to a product of Fourier transforms associated with $\mathcal{U}$ and $\mathcal{V}$.

4. Often biorthogonal families arise as systems of eigenfunctions of densely defined non-self-adjoint operators in $\mathcal{H}$, and the corresponding notion of the convolution leads to the development of the associated Fourier analysis. In this case, when eigenfunctions do not have zeros, then the corresponding global analysis of pseudo-differential operators is developed in the recent paper [13]. The no-zeros condition was later removed in [14], and an application of such analysis to the wave equation for the Landau Hamiltonian was given in [15]. Also, we note papers [5, 6], where properties of pseudo-differential operators on manifolds with and without boundary are studied.

5. Let us consider a convolution generated by the Ionkin operator from [8]. The Ionkin operator $\mathcal{Y}$ is defined in $\mathcal{H} := L^2(0, 1)$, and generated by the differential expression $-\frac{d^2}{dx^2}$, $x \in (0, 1)$, with boundary conditions $u(0) = 0$, $u'(0) = u'(1)$. The operator has the system of eigen– and associated functions

$$u_0(x) = x, \ \ u_{2\xi-1}(x) = \sin(2\pi\xi x), \ \ u_{2\xi}(x) = x\cos(2\pi\xi x), \ \xi \in \mathbb{N},$$

which forms a basis in $L^2(0, 1)$. Denote this basis by $\mathcal{U}$. The corresponding biorthogonal basis can be given by

$$v_0(x) = 2, \ \ v_{2\xi-1}(x) = 4(1 - x)\sin(2\pi\xi x), \ \ v_{2\xi}(x) = 4\cos(2\pi\xi x), \ \xi \in \mathbb{N},$$

for more information we refer to [8]. We define $\mathcal{Y}$–convolution by the formula

$$\begin{aligned} f \star_{\mathcal{Y}} g(x) := & \frac{1}{2}\int_x^1 f(1 + x - t) g(t) dt \\ & + \frac{1}{2}\int_{1-x}^1 f(x - 1 + t) g(t) dt + \int_0^x f(x - t) g(t) dt \\ & - \frac{1}{2}\int_0^{1-x} f(1 - x - t) g(t) dt + \frac{1}{2}\int_0^x f(1 + t - x) g(t) dt. \end{aligned} \tag{0.11}$$

Similar $\mathcal{Y}$–convolution is studied in the paper [10]. In particular, it satisfies the following properties:

$$\mathcal{Y}(f \star_{\mathcal{Y}} g) = (\mathcal{Y} f) \star_{\mathcal{Y}} g = f \star_{\mathcal{Y}} (\mathcal{Y} g).$$

For the basis

$$\mathcal{U} := \{u_\xi : \; u_0(x) = x, \, u_{2\xi-1}(x) = \sin(2\pi\xi x), \; u_{2\xi}(x) = x\cos(2\pi\xi x), \; \xi \in \mathbb{N}\},$$

we have $\widehat{f \star_{\mathcal{Y}} g}(0) = \widehat{f}(0)\widehat{g}(0)$, $\widehat{f \star_{\mathcal{Y}} g}(2\xi) = \widehat{f}(2\xi)\widehat{g}(2\xi)$, and

$$\widehat{f \star_{\mathcal{Y}} g}(2\xi - 1) = \widehat{f}(2\xi - 1)\widehat{g}(2\xi) + \widehat{f}(2\xi)\widehat{g}(2\xi) + \widehat{f}(2\xi)\widehat{g}(2\xi - 1), \; \xi \in \mathbb{N}.$$

Thus, by Theorem 0.2, it follows that the $\mathcal{Y}$–convolution not not arise as an $\mathcal{U}$–convolution.

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Baltabek Kanguzhin:
al–Farabi Kazakh National University
71 al–Farabi ave., Almaty, 050040
Kazakhstan
E-mail address kanbalta@mail.ru

Michael Ruzhansky:
Department of Mathematics
Imperial College London
180 Queen's Gate, London, SW7 2AZ
United Kingdom
E-mail address m.ruzhansky@imperial.ac.uk

Niyaz Tokmagambetov:
al–Farabi Kazakh National University
71 al–Farabi ave., Almaty, 050040
Kazakhstan,
and
Department of Mathematics
Imperial College London
180 Queen's Gate, London, SW7 2AZ
United Kingdom
E-mail address n.tokmagambetov@imperial.ac.uk