Control Space Reduction and Real-Time Accurate Modeling of Continuum Manipulators Using Ritz and Ritz-Galerkin Method

S.M. Hadi Sadati1, S. Elnaz Naghibi2, Ian D. Walker3, Kaspar Althoefer2, Thrishantha Nanayakkara4

Abstract—To address the challenges with real-time accurate modeling of multi-segment continuum manipulators in presence of significant external and body loads, we introduce a novel series solution for variable-curvature Cosserat rod static and Lagrangian dynamic method. By combining a modified Lagrange polynomial series solution, based on experimental observations, with Ritz and Ritz-Galerkin methods, the infinite modeling state space of a continuum manipulator is minimized to geometrical position of a handful of physical points (in our case two). As a result, a unified easy to implement vector formalism is proposed for nonlinear impedance and configuration control. We showed that by considering the mechanical effects of axial highly elastic deformation, the model accuracy is increased by up to 6%. The proposed model predicts experimental results with 6-8% (4-6 [mm]) mean error for the Ritz-Galerkin method in static cases and 16-20% (12-14 [mm]) mean error for the Ritz method in dynamic cases, in planar and general 3D motions. Comparing to five different models in the literature, our approximate solution shows to be more accurate with the smallest possible number of modeling states and suitable for real-time modeling, observation and control applications.

Index Terms—Flexible robots, Soft Material Robotics, Dynamics, Motion Control, Force Control.

I. INTRODUCTION

Taking inspiration from biological examples such as the octopus arms, chameleon tongues and elephant trunks, researchers are looking into the possibility of replicating similar maneuverability and grasping characteristics by harnessing the corresponding hyper-redundancy demonstrated in nature [1]. The class of continuum robots promises considerable performance improvements in different areas which currently witness the presence of traditional robots, such as surgical applications, underwater manipulation and inspection, etc. [2], [3], [4]. Real-time accurate modeling, observation and controller design suitable for this class of manipulators are shown to be complex and challenging tasks due to their high dexterity and deformability [5].

A Continuum manipulator model is described using a general kinematic map for the strain-configuration relation and a robot-specific mechanical map for the loads-strain relation. The Constant curvature (CC) model, which is the most common assumptions in continuum robotic research, simplifies the kinematics of a continuum manipulator by expressing the backbone deformation as a planar CC profile [5]. Although being commonly used as a simplifying assumption, the CC assumption suffers from local singularities and low accuracy in the presence of significant body and external loads. Variable curvature (VC) [6], [7] and identification based series-solutions [8] provide more accurate singularity free kinematic maps. The dynamic models introduced in the literature to provide a better modeling accuracy can be categorized into six groups, a comparative study of which in comparison with experiments is presented in [9].

1) Lumped system Lagrange dynamics with discrete VC kinematics, similar to a series rigid link mechanism, where a series of finite or infinite number of rigid disks are interconnected using compliant joints [10].
3) Continuum form of Lagrange dynamics [8], [13] or the Principle of Virtual Work (PVW) [14], [15] using CC, continuous VC or series-solutions as kinematic maps, where the kinematic map parameters are the dynamic model states.
4) Cosserat rod model [6], [16] and beam theory method, as a simplified version of that [9], which result in a

The identification based models are more accurate and computationally efficient; however, their validity is limited to their experiment conditions, external loads, input values and training data set [19], [8]. Besides, they do not account for the structural characteristics necessary for dealing with unknown conditions and design optimization [9]. On the other hand, the lumped system and Cosserat rod models provide a general solution, but not suitable for control design due to high computational cost and large number of modeling states. Use of series-solution based models for identification of kinematic map by Godage [8] have shown to be real-time but with a complex hard to interpret shape function and limitations related to identification based methods. Series-solution based Galerkin method for static modeling of complex deformation in a discretized finite element domain by Tunay [21] is accurate and comprehensive, but with a complex shape function and limitations related to finite element and Cosserat rod method.

Accuracy of these methods is compared based on their forward control performance [9] and a feedback term, such as PID, should be considered in the controller design to deal with real working disturbances. The large modeling state space results in hard impedance observation and control which are essential for safe surgical, inspection and human-robot interaction applications.

In our previous research [14], [15], we introduced a new geometry deformation based approach for the inverse static model of braided continuum manipulators. We utilized the experimental observation of the deformed cross-section to model the deformation energy of the continuum media and to show the importance of considering the cross-section highly elastic deformation, using the PVW and the CC kinematics in [14] and Cosserat rod method with VC kinematics in [9]. For the first time, our model provides an accurate semi-analytical solution for the manipulator cross-section deformation to increase the modeling accuracy, especially in the presence of regional coaxial homogeneous stiffness variation in the cross-section. There, different modeling assumptions are investigated, showing the advantageous of an exact analytical solution for design optimization based on sensitivity analysis of structural parameters. However, a controller was hard to design due to the infinite number of the modeling states.

In this paper, 1) we introduce a model with finite number of states while preserving the modeling accuracy and generality based on which the design of a nonlinear impedance and configuration feedback controller is possible; 2) the solution is extended to the dynamic case based on a vector formalism for PVW, in general 3D motions of a multi-segment manipulator; 3) the modeling accuracy is increased by considering the mechanical effects of the manipulator axial highly elastic deformation as well as in the planar direction. We introduce a novel, accurate yet real-time approximate series-solution for the Cosserat beam model in the static case and a PVW based model in the dynamic case for the general 3D motion of a multi-segment continuum manipulator in the presence of significant external and body loads. Based on the experimental observations, a modified Lagrange polynomial, as a differentiable polynomial of finite order, is chosen to drive a weak-form series-solution for the mechanics of a continuum manipulator using continues Ritz and Ritz-Galerkin methods.

The Ritz and Ritz-Galerkin series solutions are detailed in section III for static and dynamic models followed by a vector form derivation for the nonlinear control and observation problems in section IV. Experiments and discussion are described in section V and conclusions are presented in section VI.

II. Modeling Framework

Slenderness and compliance of most continuum manipulators make the external and body forces important in the modeling of their behavior. We use VC kinematics to relate a series-solution geometry in 3D space ($\in \mathbb{R}^3$) to the strain rates that are found from the Cosserat rod static model and to calculate the PVW terms for the dynamic model. The Ritz and Ritz-Galerkin series solutions are investigated, showing the advantageous of an exact analytical solution for the design optimization based on sensitivity analysis of structural parameters. However, a controller was hard to design due to the infinite number of the modeling states.

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In the following sections, first, the modeling framework is discussed consisting of the VC kinematics, the Cosserat and beam theory static model and PVW dynamic model. The Ritz and Ritz-Galerkin series solutions are investigated, showing the advantageous of an exact analytical solution for the design optimization based on sensitivity analysis of structural parameters. However, a controller was hard to design due to the infinite number of the modeling states.

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To the best of our knowledge and compared to the most recent similar research in the field [17], this is the first time that the infinite modeling state space of such problem minimizes to the geometrical positions of two points at the tip and in the middle of the manipulator. As a result, different dynamic impedance and configuration control scenarios are formulated using traditional nonlinear control theories in a unified and easy to implement vector formalism. We neglect the close loop terms in the first part to highlight the performance and accuracy of our method. A PID term is incorporated it in inverse control model later. Modeling results are verified against experimental results in the static and dynamic cases. The proposed model is the most accurate mechanics of materials based model with the smallest possible number of modeling states compared to the other well-known approaches in the literature [9] and can be used for precise static and dynamic modeling, observation and control in real-time applications.

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A. Variable Curvature Kinematics

The VC kinematics use curvilinear frames, $[\hat{d}_2, \hat{d}_1, \hat{d}_3]$ for cross-section physical frame or $[\hat{n}, \hat{t}, \hat{b}]$ for Frenet – Serret
VC Kinematics present the relation between the physical frame (Fig. 2), to represent the continuum manipulator backbone as a series of infinitesimal CC curves [7], [16], [6]. The physical curvilinear coordinates ([d_2, d_1, d_3]) and the curve spatial configuration (\(\rho\)) expressed in inertial Cartesian coordinates ([\(\hat{n}\), \(\hat{j}\), \(\hat{k}\)]). The physical frame [d_2, d_1, d_3], where d_1 is tangent to the backbone and d_2 is alongside the first air pressure chamber, is used in the forward VC kinematics. A 3 \times 3 rotation matrix (R) relates the local linear strain rates in the physical curvilinear coordinates (v) to the spatial deformation rates of the manipulator geometry in the inertial Cartesian coordinates (\(\rho\)) as

\[
\rho_{s} = Rv + \hat{d}_{1} = R(v + [0, 1, 0]^T),
\]

where subscript \((.,x)\) means partial differentiation w.r.t. variable \(x\). The spatial variation of \(R\) is found based on the local rotational strain rates (u)

\[
R_{s} = [u]_{x} R,
\]

and the angular velocity of each element is \([\omega]_{x} = R_{s} R\), where \([.\) is an operator to create a skew symmetric matrix. The curvilinear and Cartesian coordinates are aligned at the manipulator base (s = 0) where \(\rho_{0} = 0\), \(R_{0} = I\) and \(I\) is a 3 \times 3 unit matrix.

For the inverse VC kinematics, where the manipulator geometry is known as \(\rho^*\), we use Frenet – Serret curvilinear frames with tangent (\(\hat{t}\)), normal (\(\hat{n}\)) and binormal (\(\hat{b}\)) unit vectors. Here, \(\hat{b}\) expresses the direction of the local bending vector. The rotation matrix to relate Frenet – Serret to Cartesian coordinates is \(R^* = [\hat{n}, \hat{t}, \hat{b}]\), where \(\hat{t} = \rho^*/|\rho^*|\), \(\hat{n} = \hat{t}_{s}/|\hat{t}_{s}|\) and \(\hat{b} = \hat{t} \times \hat{n}\). Unlike curvilinear frames in the forward VC kinematics (R); for the Frenet – Serret frame in the inverse VC kinematics (\(R^*\)), the frame local twist around \(\hat{t}\) is a combination of a physical geometrical twist due to torsional stresses and a purely mathematical twist due to the side bending of the modules based on the mathematical definition of \(\hat{b}\). To cancel the mathematical twist and to find the physical curvilinear frame ([\(\hat{d}_2\), \(\hat{d}_1\), \(\hat{d}_3\)]), we use

\[
R_s^* = R^* \Pi_{i=1}^{n_p} R_{i=1}^{T},
\]

where \(\Pi\) is the post-multiplication operator, \(n_p\) is segment number, \(R_o\) is a 3 \times 3 rotation matrix around \(\hat{j}\), \(\phi = \tan(\sqrt{3}(p_2 + p_3 - 2p_1)/(p_2 - p_3)) - \pi/2\) is the angle of the bending plane due to internal pressure in the local cross-section plane [5] and \(p\) is the pressures vector in a pneumatic actuator [5] or tendon forces vector in a tendon driven design [3], [4]. Unlike [6], there is no need for a transformation between the segments. Strain rates (\(\sigma\) and \(\nu\)) relationship with the loads is found from the system mechanical map.

B. Static Model: Cosserat Rod and Beam Theory

The Cosserat rod model derives the equilibrium between the forces in an infinitesimal element of a continuum media using a free body diagram as in Fig. 2 and expressed in the inertial frame as [7]

\[
\begin{align*}
\dot{n}_{s} + f &= \sigma a_{c,u} \rho_{s,t}\tau \\
m_{s} + \rho_{s} v + n + \tau &= J_{d} \omega_{s},
\end{align*}
\]

where \(f\) and \(\tau\) are the uniform distributed force and torque on the element, \(\sigma\) is the material density, \(a_{c,u}\) is the cross-section area, \(J_{d}\) is the the deformed cross-section second moment of inertia and \(\omega_{s}\) is the element angular velocity. The time-dependent acceleration terms (\(\rho_{s,t}\) and \(\omega_{s}\)) are zero in a static model. Hooke’s law for the linear stress-strain relation can be used to solve a discrete finite element approaches can be used to solve a
large algebraic equation for the states of all the elements at once. Duriez presented a real-time approach for this method using the sparse format of the coefficient matrices[20]. Both methods are not suitable for control and design purposes with high computational cost.

Beam theory method is a simpler approach based on the resultant stresses due to the loads at each element which results in the same equations as the Cosserat rod method, but for $v$ and $u$, unlike $v_s$ and $u_s$ in Eq. (4) and (5).

$$v = K_{v_d}^{-1} \left( R_v^T \left( f_b + f_L + f_p \right) \right)$$

$$u = K_{u_d}^{-1} \left( R_u^T \left( \tau_b + \tau_L + \tau_p \right) \right),$$

where $f_L$ is the external load exerting at $s = s_L$, $s_L > s$, $f_b = \sigma a_c (l-s) g j$ is the body weight load, $f_p = p a_p [0,1,0]^T$ is the pneumatic pressure force, $\tau_L = (\rho(s_L) - \rho(s)) x f_L$ is the moment of $f_L$, $\tau_b = \sigma a_c j_s^T \left( \rho(s) - \rho(c) \right) dz x g j$ is the body weight moment, $\tau_p = p a_p [r_{j_d2},0,r_{j_d1}]^T$ is the pneumatic pressure moment and $\lambda_{d_i,s} = \bar{v}_{d_i} + 1. \bar{v}_{d_i}$. From the previous integration step is used in $\lambda_{d_i,s}$ to prevent added nonlinearity to the equations. The downside of the beam theory approach is that we need to guess the equilibrium geometry to find the cross-sections’ load, compared to the Cosserat rod model methods are used to find weak-form solutions for a BVP, in particular single shooting, multiple shooting and concatenation methods are not suitable for control and design purposes with high computational cost.

The virtual work of gravitational body load is

$$\Sum{\rho} \vec{w}_v = \sigma a c g y \vec{q} ds.$$  

The virtual work due to an external force $(dw_L, q) = C_{f_L,q} f_L ds$ and torque $(dw_{\tau_L,q} = C_{\tau_L,q} \tau ds)$ expressed in the inertial frame and exerting at $s = s_L$ with $0 \leq s \leq s_L$

$$C_{f_L,q} = \rho, s, \ C_{\tau_L,q} = u, q.$$  

A vector form for the body inertia virtual work can be found using the TMT method [24]. Here, the position vectors in the Cartesian space $(\rho)$ are mapped to the state space $(q)$ by $T = [p, q, \dot{q}]^T$, where $\dot{q}$ is found by rearranging $\omega = \vec{\omega}_q$. The mass matrix for disk elements with an infinitesimal thickness is $dM = \text{diag} \left( \left[ 1 1 [a, J_{zz}] \right] \right) ds$. Following the TMT derivations in [24], the transformed mass matrix is

$$dM_q = T^T dM T ds,$$  

and the transformed inertial damping matrix is

$$dD_{m,q} = T^T dM (T q_{tt}) ds.$$  

Then the inertial damping term is $dD_{m,q} = dD_{m,q} q_{tt}$, and the total inertial virtual work is $dw_{in} = (dM_{qtt} + dD_{m,q}) ds$. The linear and angular viscous damping of the material is $dw_{\mu,q} = C_{\mu} q_{tt}, ds$, where

$$C_{\mu} = [v u]^T \text{diag} \left( \left[ \mu_v, \mu_u \right] \right),$$  

where $\mu_v$ and $\mu_u$ are the material linear and rotational viscous damping coefficients.

The vector form of the dynamic EOM in each element is

$$\text{Acceleration \ Viscous \ Damping \ Conservative \ Actions}$$

$$dM_{q_{tt}} \dot{q}_{tt} + dD_{m,q} + dw_{\mu,q} + dw_{c,q} = \dot{dw}_{p,q} + dw_{f,L,q} + dw_{\tau_L,q}.$$  

which is integrated along the backbone curve at each time step. We use a trapezoidal rule for the spatial and a 4th order Runge-Kutta method for the time numerical integration.

III. RITZ AND RITZ-GALERKIN METHODS

Direct single shooting, multiple shooting and concatenation methods are used to find weak-form solutions for a BVP, in which the separation of variables method is used to separates the time and space domains and makes the forward integration possible. Using the direct single shooting method, an approximate series-solution is assumed for the backbone kinematics $(\rho^*)$ with a finite number of terms that satisfies the initial and boundary conditions

$$\rho^* = \Sum{\sum} C(t) \psi(s).$$  

using the Euler-Bernoulli linear strain-stress assumption or

$$dw_{c,q} = (E/6) a_{c_q} (v_{d_i}^2 + 2/(v_{d_i}^2 + 3) q ds,$$  

based on the Neo-Hookean method assuming a symmetric uniaxial deformation rates [14], [22], and the bending virtual work

$$dw_{c,q} = K_{u_d} u_{v,q} ds,$$  

using the Euler-Bernoulli assumption.
where $c_{(t)i}$ is a time dependent coefficient which is constant in the static case and $\psi_{(s)i}$ is a spatial shape function.

Based on the Ritz method for BVPs, widely being used in the finite element analysis [25], Eq. 20 is substituted in the BVP equation, Eq. 7 for the static or Eq. 19 for dynamic model, and the coefficients ($c_{(t)i}$) that approximate the BVP with minimum error are found. Numerical nonlinear solvers, i.e. Matlab "fsolve" function, can be used to find the coefficients ($c_{(t)i}$) that minimize the approximation error in the static case. The coefficients’ time series ($c_{(t)i}$) are found from the time numerical integration of the EOM in the dynamic case.

A better approximation is possible with the Continuous Galerkin method of weighted residuals, known as the Ritz-Galerkin method, where the weighted residuals of the system are minimized instead of the system function itself. The shape functions ($\psi_{(s)i}$) are as- 

plained physically and unsuitable for inverse mechanics, path planning, control and stability analysis.

A unified vector form for the modeling, nonlinear control and observer design, based on feedback linearization method is derived. Defining the input pressure and external load vector as the system states and adding algebraic constraints for the desired controls to the system EOM, for Eq. (19) after the spatial integration, we have

$$
\begin{bmatrix}
M_q & -[C_p & C_L & C_{TL}] \\
A_3 & A_2 & A_4 \\
q_{t,0} & q_{t,1} & q_{t,2}
\end{bmatrix}
\begin{bmatrix}
q_{t,0} \\
q_{t,1} \\
q_{t,2}
\end{bmatrix}
= (23)
$$

IV. NONLINEAR CONTROL AND OBSERVATION

A unified vector form for the modeling, nonlinear control and observer design, based on feedback linearization method is derived. Defining the input pressure and external load vector as the system states and adding algebraic constraints for the desired controls to the system EOM, Eq. (23) is numerically integrated w.r.t time by having the initial values for $[q_{t,0}, q_{t,1}, q_{t,2}]$. The control or observation values for $[q_{t,0}, q_{t,1}, q_{t,2}]$ is found by two times numerical differentiation of the integration results w.r.t time, since they are integrated twice alongside the system EOM.

For simple forward simulation we have $A_1 = 0, A_2 = I$ and $B = [p f_L, \tau_L]^{T}$, where $I$ is the identity matrix. In the case of tip position control in the presence of external loads, we have $A_1(3n_q + 1, n_q) = 1, A_1(3n_q + 2, 2n_q) = 1, A_1(3n_q + 3, 3n_q) = 1, A_2(3n_q + 4 : end, 4 : 9) = I$ and $B = [\rho_{ip_{t,tt}} + \text{pid}] f_L \tau_L]^{T}$, where $\rho_{ip_{t,tt}}$ is the desired tip position and $\text{pid}$ is a PID or any other feedback control term. Configuration control can be achieved by changing $A_1$ to control the position vector of any other intermediate point on the backbone.

Force control at the tip using the pressure inputs and for a known tip torque ($\tau_L$) and position profile ($\rho_{ip_{t,tt}}$) is possible by setting $A_1(3n_q + 1, n_q) = 1, A_1(3n_q + 2, 2n_q) = 1, A_1(3n_q + 3, 3n_q) = 1, A_2(3n_q + 4 : end, 4 : 9) = I$ for the tip vector position control and $B = [\rho_{ip_{t,tt}} + \text{pid}] f_L \tau_L]^{T}$, where $f_{Lb}$ is the desired force vector. For the tip torque control ($\tau_{Lb}$), we have the same $A$ matrix with $B = [\rho_{ip_{t,tt}} + \text{pid}] \tau_L]^{T}$.

Simultaneous force and position control is possible by fast enough switching between different control scenarios or having a combination of non-overlapping force and position directions to derive a proper control matrix set for.

Shape-based force estimation is possible by proper adjustment of the matrices. For example, for the tip force estimation based on the tip position vector we have $A_1(3n_q + 4, n_q) =
TABLE I
STIFF-FLOP PARAMETERS.

<table>
<thead>
<tr>
<th>Sym.</th>
<th>Value</th>
<th>Sym.</th>
<th>Value</th>
<th>Sym.</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_{p1}$ [mm]</td>
<td>12.5 (12)</td>
<td>$r_{p2}$ [mm]</td>
<td>3.0 (3)</td>
<td>$r_{g}$ [mm]</td>
<td>8.5 (9)</td>
</tr>
<tr>
<td>$l_{c1}$ [mm]</td>
<td>10 (17)</td>
<td>$m$ [gram]</td>
<td>24.5</td>
<td>$n_{r}$ [gram]</td>
<td>11</td>
</tr>
<tr>
<td>$g$ [m/s²]</td>
<td>9.81</td>
<td>$E_{sta}$ [KPa]</td>
<td>205**</td>
<td>$\sigma$ [Kg/m³]</td>
<td>1300</td>
</tr>
<tr>
<td>$\mu_{s}$ [Ns/m]</td>
<td>1e-3**</td>
<td>$\mu_{a}$ [Ns/rad]</td>
<td>4e-5**</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Values in brackets are for the tests with 3D deformation (Fig. 5d.e).
**Parameters found from identification to minimize the simulation errors.

1. $A_1 (3n_q + 5, 2n_q) = 1, A_1 (3n_q + 6, 3n_q) = 1, A_2 (3n_q + 1 : 3n_q + 3, 1 : 3) = I, A_2 (3n_q + 7 : 3n_q + 9, 7 : 9) = I$ and $B = [p (\rho_{tip,tt} + \rho{id}) \tau_L]^{T}$, in which the known tip geometry is assumed as geometrical constraint and the PID term is for numerical analysis error compensation not as a term for physical feedback control.

V. EXPERIMENTAL RESULTS AND DISCUSSION

We used a STIFF-FLOP (STIFFness controllable Flexible and Learn-able manipulator for surgical OPerations) pneumatic actuator module [26] as a standard, simple yet accurate and durable design without inhomogeneous deformations, i.e. due to structure swallow. Similar design is used in recent research [5], [6], [17]. The measured and identified structural parameters are presented in Table I with an experimental setup similar to the one in [14], [15] (Fig. 1). We choose STIFF-FLOP modules because of their high repeatability and negligible performance change due to aging and fatigue during the experimental trials. However, accurate modeling of these modules are challenging due to inhomogeneities as a result of the fabrication process.

Experimental conditions and inputs are used in the simulations where Matlab "fsolve", "ode113" and "fmincon" are used for direct single shooting optimization, time integration and parameter identification. 33 static points and 100 [s] dynamic motion recorded data of a module in planar motion with extensive external load (Fig. 1 and 3) and 43 static points and 200 [s] dynamic motion recorded data of a multi-segment manipulator in general 3D motion with one active module (Fig. 4.a-d and 5) are used to verify the accuracy of our model. Seven points from the static test are used to identify $E$ and $\mu_{v/\mu}$. Error is defined as the distance between the tip position in the model and experiments divided by the initial length of the manipulator as a reference length [6]. High external load brings the manipulator to an "S" shape configuration in the planar motion case (Fig. 1). Using a polynomial of order three, an average 6% ($\approx 4$ [mm]) mean error is observed for the static model in 2D and 8% ($\approx 6$ [mm]) in 3D motion using both the Ritz and Ritz-Galerkin method. For the Cosserat beam method, Our method shows to be at least twist faster using both the Ritz and Ritz-Galerkin method. For the Cosserat rod model (6-12%) and the approximate solutions similar to [8] (11%) in the static case and compared to the lumped system model (22% error) in the dynamic case [9]. The series-solution with two points (four states in a planar motion and six states in a general 3D motion) shows to be efficient and accurate enough to predict the "S" shape configuration of the manipulator under significant external loads. However, simulations for general 3D motion of a multi-segment manipulator with two active modules (Fig. 4.e) shows the different results from polynomial of order four and five (Fig. 6). This shows the importance of a proper choice for the polynomial order, especially, where the manipulator needs to perform in the confined maze-like space of a minimally invasive surgery as the final goal of our research. Using Matlab software on a regular laptop computer, the direct single shooting algorithm usually takes about 2 [s] to find the equilibrium configuration for a single-curve formation and 3 [s] for a double-curve formation ("S" shape) in the static case, and almost the same time as the actual experiment duration for the dynamic case. Compared to the presented established methods in the literature, our method has the highest static and a high dynamic modeling accuracy with exceptional real-time computational performance which shows the feasibility of real-time implementation of this approach in real-world applications. Our weak-form solution provides a powerful analytical means for sensitivity analysis of the model performance w.r.t. the system parameters for structural design optimization. We observed a slight change in the module cross-section dimensions affect the accuracy by 2% similar to [15] while a change in the module stiffness module, i.e. not considering the elasticity reduction due to axial highly elastic deformation, affect the accuracy by up to 6%. A detailed sensitivity analysis, as in [15], will be carried out in the future of our research to predict how inaccuracy in the model parameters can affect the simulation results for a successful control system design. This model is derived from a framework
Fig. 3. Results for planar deformation of a module with extensive external load at the tip in static and dynamic case (Fig. 1): pressure inputs, external force and torque at the tip, tip position time series and in the task space, simulation frames, and single shooting iterations for steps 4 and 9 in the static case.

Fig. 4. Results for general 3D deformation of a multi-segment manipulator in static and dynamic case: pressure inputs (a) and tip position time series in static (b) and dynamic (c) case for the manipulator with one actuated module (d), manipulator with two actuated modules (e).

Fig. 5. Tip position in the task space and simulation frames for general 3D motion of multi-segment manipulator with one active module (Fig. 4.d) in static (left) and dynamic (right) case.

Fig. 6. Tip position time series and simulation frames for a multi-segment manipulator with two active modules in general 3D motion (Fig. 4.e) in static case with a modified Lagrange polynomial of order four (4-p: left) and five (5-p: right).

with coaxial homogeneity in the manipulator cross-section and any other anomaly due to fabrication, aging and fatigue with a stochastic nature will be compensated with a feedback control term. These affects can be addressed using statistical approach...
and by incorporating stochastic terms in the future control model. A precise forward model contributes in increasing the stability, reducing the controller gains, better estimation of the system states and compensation of measurement noises. After using the absolute error for the verification and comparison of our model in the first step, we are going to investigate the performance of our model (Eq. (23)) for force estimation and control with application in soft tissue palpation and stiffness imaging. A continuum manipulator with 5-15 [gram-force] and 6-43% estimation error, achievable with STIFF-FLOP modules [27], is showed to be successful in such tasks [28]. The controller repeatability and durability are important for such tasks which we plan to address in our future work.

VI. CONCLUSION

In this paper, a modified Lagrange polynomial series-solution, based on experimental observation, was presented to produce a new technique for deriving the Cosserat rod static and Lagrangian dynamic model of a continuum manipulator. Our approach featured a small number of states (six in our analysis) which are the Cartesian positions of two points on the manipulator (at the tip and in the middle). We used a direct single shooting method to solve the Cosserat rod static model and forward numerical integration for the Lagrange dynamic model using the Ritz and Ritz-Galerkin approach. A unified vector form for the manipulator Lagrange dynamics was derived using which the modeling, control and observation scenarios can be easily implemented by adjusting three control matrices and a vector for the desired and input values. Noting the importance of considering the manipulator axial highly elastic deformation, the Ritz method is found to be more accurate in dynamic cases while the Ritz-Galerkin method is slightly more accurate in the static cases. The model is compared with experimental results and five other well-known models and shown to be the most accurate and efficient approach with the smallest possible number of states, suitable for real-time static and dynamic modeling and controller design. We plan to use this method to control a continuum manipulator for biological tissue palpation and stiffness imaging in medical applications.

REFERENCES