

## A ROBUST SEISMIC PHASE UNWRAPPING METHOD

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### ABSTRACT

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One of the difficulties in seismic phase unwrapping occurs at the frequency components with energy close to zero, where the final unwrapped phase function should have  $\pm\pi$  discontinuities. At those frequency components, principal values of the wrapped phase spectrum and in turn the values of the final unwrapped phase function are contaminated severely by numerical errors. These errors also propagate along the frequency axis as the unwrapped phase function is calculated recursively. This paper presents a robust phase unwrapping method. Unwrapped phase function is defined as the sum of phase increments, where the phase increment of adjacent frequency components is evaluated in terms of the difference of principal values of the wrapped phase spectrum. In the vicinity of  $\pm\pi$  discontinuities, appropriate treatment of phase increments is required, preventing the numerical errors from propagating along the frequency axis.

KEY WORDS: phase spectrum, phase unwrapping, seismic phase, unwrapped phase.

### INTRODUCTION

Seismic phase unwrapping is a practically unsolved problem, although it is a conceptually simple processing (Oppenheim and Schaffer, 1989; Shatilo, 1992). One of the difficulties in seismic phase unwrapping occurs at frequency components with energy close to zero, where the amplitude spectrum appears as notches. Because of the low signal-to-noise ratio, the principal values of the modulo- $2\pi$  phase spectrum at those frequency components are contaminated severely by numerical errors, which are then retained in the final unwrapped phase function produced from the principal values. These numerical errors in the local phase values also propagate along the frequency axis as the unwrapped phase spectrum is calculated recursively.

At those frequency components with energy close to zero, the final unwrapped phase function should have  $\pm\pi$  discontinuities. To identify the phase discontinuities, McGowan and Kuc (1982) used the sign change of the product of the real and imaginary parts of the Fourier spectrum. But a rapid phase change can mask the true sign change of the product. Poggiagliolmi et al. (1982) suggested to distinguish the physical  $\pm\pi$  phase discontinuities from the  $2\pi$ -jumps of the phase wrapping by computing the finite difference of the phase principal values taken over progressively smaller frequency intervals. In the area where the unwrapped phase exhibits an intrinsic high variation, however, it is pointless to decrease the frequency interval because the phase function can change rapidly at a point (Stoffa et al., 1974).

This paper presents a robust seismic phase unwrapping method, in which the final unwrapped phase function is given by the sum of its phase increments. The phase increments of the *unwrapped* phase function (yet to be determined) are derived from the differences of principal values of the *wrapped*, modulo- $2\pi$  phase spectrum. The identification of the  $\pm\pi$  phase discontinuities is then based on the phase increments, in which any  $2\pi$ -jump of the phase spectrum has been removed unambiguously. Once the intrinsic  $\pm\pi$  discontinuity is identified, appropriate treatment to the phase increments in the area around the  $\pm\pi$  discontinuity is required to mitigate numerical errors. The phase unwrapping method and its implementation are presented in the next two sections, followed by verification examples of seismic traces.

#### PHASE UNWRAPPING METHOD

Considering a real-valued seismic trace  $\{x_n, 0 \leq n < 2N\}$ , and its frequency domain counterpart  $\{X_k, 0 \leq k \leq N\}$ ,

$$\{x_n\} \Leftrightarrow \{X_k\} \quad , \quad (1)$$

one has a wrapped phase spectrum,

$$\psi_k = \arctan[\Im(X_k), \Re(X_k)] \quad , \quad (2)$$

where  $\Im$  and  $\Re$  refer to the imaginary and real parts of the complex signal  $X_k$ , and  $\arctan$  is the four-quadrant arctangent function and referred to as the wrapping operator, which produces the principal value of the phase,

$$-\pi < \psi_k \leq \pi \quad . \quad (3)$$

Only one-half of the Fourier spectrum is considered here, as a real time-domain signal produces a symmetric amplitude spectrum and an anti-symmetric phase spectrum.

The modulo- $2\pi$  phase spectrum produced by equation (2) is unsuitable for further interpretation and processing in many cases, and needs to be unwrapped. The conventional unwrapping process involves detecting the  $2\pi$ -jumps in the wrapped phase and adding an appropriate multiple of  $2\pi$  (Oppenheim and Schaffer, 1989),

$$\phi_k = \psi_k + 2\pi r_k , \quad 0 \leq k \leq N , \quad (4)$$

where  $\{\phi_k\}$  are the unwrapped phase values, and  $\{r_k\}$  are integers. We will see in the next section that it is difficult to distinguish between the  $2\pi$ -jumps and the physical  $\pm\pi$  phase discontinuities.

If the unwrapped phase is assumed to be *continuous*, phase unwrapping may also be implemented by the integration (Stoffa et al., 1974; Wang, 1998),

$$\phi_{k+1} = \phi_k + \int_{\omega_k}^{\omega_{k+1}} \phi'[X(\theta)]d\theta , \quad 0 \leq k < N , \quad (5)$$

where  $\omega_k$  is the angular frequency of the  $k$ -th sample,  $\phi'[X(\omega)]$  is the derivative of (unwrapped) phase function with respect to  $\omega$  and given analytically by

$$\phi'[X(\omega)] = \Im[X'(\omega)/X(\omega)] , \quad (6)$$

and  $X'(\omega)$  is the derivative of  $X(\omega)$ . Note that the evaluation of phase derivatives is unstable, as it has a maximum when the spectrum magnitude is low.

The notion that the unwrapped phase is obtained by the integral of phase derivatives may lead to the implementation of the sum of phase increments,

$$\phi_{k+1} = \phi_k + \Delta\phi_k , \quad (7)$$

where  $\{\Delta\phi_k\}$  are the phase increments of unwrapped phase function between  $\omega_k$  and  $\omega_{k+1}$ . If the unwrapped phase function is not aliasing sampled, the phase increment should have the following two properties:

- Property I:  $-\pi < \Delta\phi_k \leq \pi$  ; (8)
- Property II:

If the unwrapped phase spectrum is a continuous function, then

$$|\Delta\phi_k| \rightarrow 0 , \quad \text{for } \Delta\omega \rightarrow 0 ; \quad (9)$$

and if the unwrapped phase spectrum has a  $\pm\pi$  discontinuity, then

$$|\Delta\phi_k| \rightarrow \pi, \quad \text{for } \Delta\omega \rightarrow 0. \quad (10)$$

The condition of unaliased frequency sampling is easily to be satisfied. Otherwise, one may always up-sample it in the frequency domain by zero-padding to the original time-domain seismic trace, prior to the Fourier transform. Property I, given originally by Itoh (1982), is exploited for estimating phase increments of the unwrapped phase function, while property II will be used in the next section to detect the phase discontinuities.

Suppose that the phase spectrum is unwrapped by equation (4), the phase increment,  $\Delta\phi_k \equiv \phi_{k+1} - \phi_k$ , of the unwrapped phase function is given by

$$\Delta\phi_k = (\psi_{k+1} + 2\pi r_{k+1}) - (\psi_k + 2\pi r_k).$$

That is,

$$\Delta\phi_k = \Delta\psi_k + 2\pi(r_{k+1} - r_k), \quad (11)$$

where  $\Delta\psi_k \equiv \psi_{k+1} - \psi_k$  is the difference of the phase principal values. We may now apply the wrapping operator, denoted as  $W$ , to equation (11). On the left-hand side we have

$$W[\Delta\phi_k] = \Delta\phi_k, \quad (12)$$

following property I [expression (8)]. On the right-hand side we have

$$W[\Delta\psi_k + 2\pi(r_{k+1} - r_k)] = W[\Delta\psi_k]; \quad (13)$$

since the difference  $(r_{k+1} - r_k)$  is an integer of  $-1, 0$  or  $+1$ . Finally, we have the phase increment of the unwrapped phase function,

$$\Delta\phi_k = \arctan[\sin(\Delta\psi_k), \cos(\Delta\psi_k)]. \quad (14)$$

Therefore, using equation (14) and equation (7), we obtain the unwrapped phase function  $\{\phi_k\}$  as the sum of the wrapped phase-differences of the principal phase values  $\{\psi_k\}$ .

#### ROBUST IMPLEMENTATION

The conclusion drawn above is the kernel of the seismic phase unwrapping method presented in this paper. Full implementation also includes

the following three pre-processing steps, prior to the final summation of the phase increments:

1. Removal of linear phase component by time shifting;
2. Removal of  $2\pi$ -jumps by wrapping phase differences;
3. Mitigation of numerical errors in the vicinity of the  $\pm\pi$  discontinuities.

### Removal of linear phase component

In practice, one frequently presents the unwrapped phase spectrum with a linear phase shift, which intuitively rotates the phase curve towards the frequency axis, and physically shifts the origin of the seismic series in the time domain to the centre of the seismic signal. The centre of signal is the point of equilibrium of the linearly weighted signal energy, and has a delay

$$\tau = \left[ \int_0^T tx^2(t)dt \right] / \left[ \int_0^T x^2(t)dt \right] , \quad (15)$$

where  $T$  is the duration of the signal  $x(t)$ .

This time delay in the seismic series is equivalent to adding a linear phase component into corresponding phase spectrum in the frequency domain. According to the phase-shift theorem (Karl, 1989), I express the linear phase component in a discrete form,

$$q_k = (\pi k/N) \left( \sum_{n=0}^{2N-1} nx_n^2 \right) / \left( \sum_{n=0}^{2N-1} x_n^2 \right) , \quad (16)$$

for  $0 \leq k \leq N$ . Any processing based on the unwrapped phase spectrum generated from a time-delayed trace should take this linear component into account.

Why do we shift the time origin of a seismic trace prior to the Fourier transform, instead of removing the linear component from the final unwrapped phase spectrum of the unshifted trace? Let us see an example shown in Fig. 1, where (a) is a seismic trace, (b) is its amplitude spectrum, (c) is the phase spectrum obtained from the original time series, and (d) is the phase spectrum obtained from the seismic trace with delay time  $\tau = 144$  ms. From Fig. 1(d) we see that the number of  $2\pi$  discontinuities is reduced to the minimum, and hence the phase curve is simplified significantly. The simplification of the wrapped phase will lead to a relatively easy unwrapping process. For example, I will in a later stage rely on the phase increment of adjacent samples to detect the  $\pm\pi$  discontinuity.

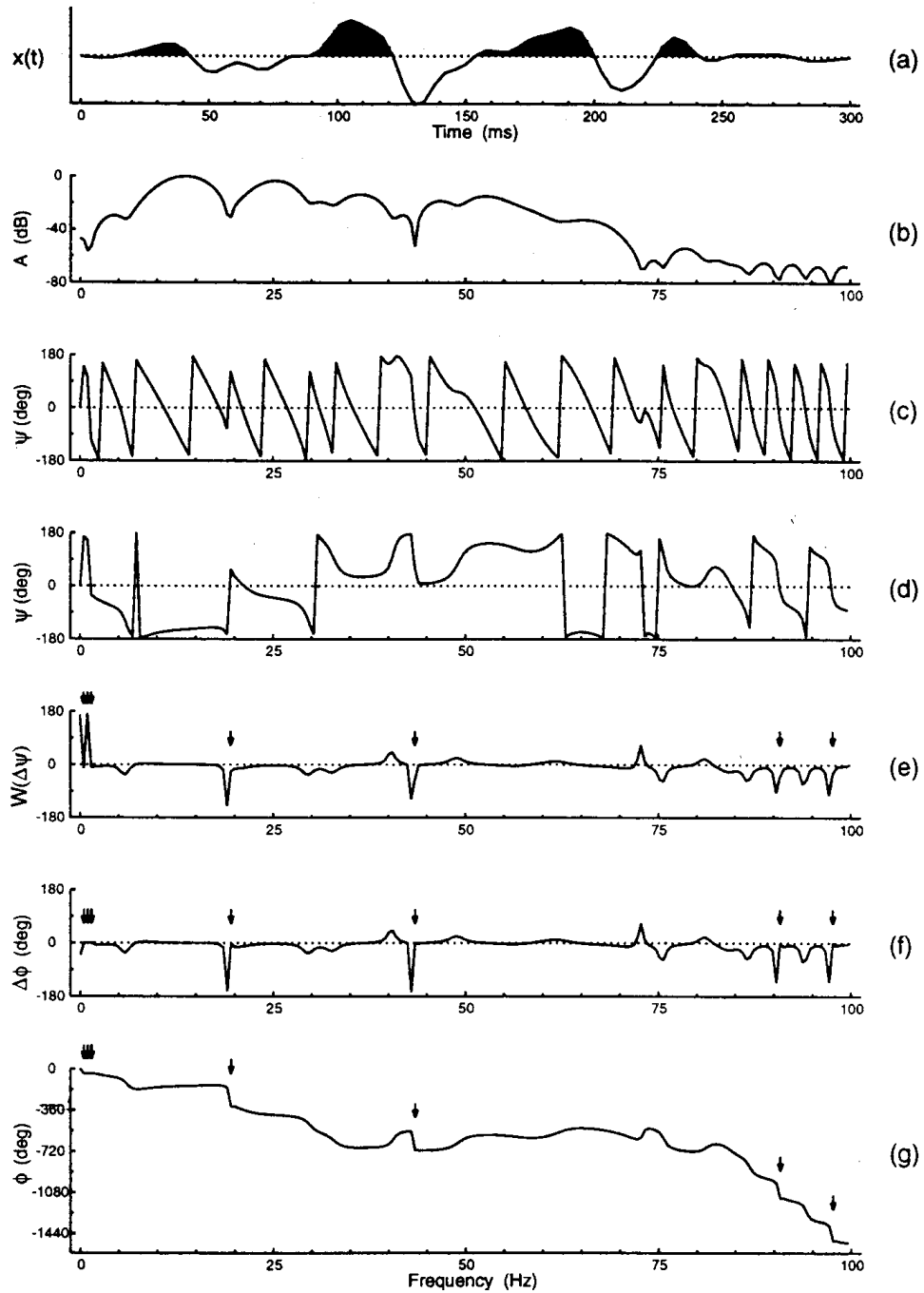


Fig. 1. (a) A time series; (b) The amplitude spectrum; (c) Wrapped phase spectrum; (d) Wrapped phase spectrum of the trace with time delay  $\tau = 144$  ms; (e) Wrapped phase differences; (f) Phase increments; (g) Resultant unwrapped phase function. Arrows indicate frequency components with  $\pm\pi$  phase discontinuity.

### Removal of $2\pi$ -jumps

The next step is to wrap the differences of wrapped phase values,

$$W[\Delta\psi_k] = W[\psi_{k+1} - \psi_k] , \quad (17)$$

following the condition (8). In this way, we effectively remove the  $2\pi$ -jumps in the wrapped phase curve. [Note here I did not consider the case of multiple poles or zeros between the  $k$ -th and  $(k+1)$ -th frequency samples, which can give rise to multiple  $2\pi$  shifts].

Let us now compare this approach with the conventional  $2\pi$ -jump detection method (equation 4), which may be presented as

$$\begin{aligned} \text{if} \quad & (\Delta\psi_k > 2\pi - \epsilon) \quad \text{then} \quad \{\text{there is } +2\pi \text{ jump}\}; \\ \text{elseif} \quad & (\Delta\psi_k < -2\pi + \epsilon) \quad \text{then} \quad \{\text{there is } -2\pi \text{ jump}\}; \\ \text{else} \quad & \quad \quad \quad \{\text{no } \pm 2\pi \text{ jump}\}. \end{aligned} \quad (18)$$

The parameter  $\epsilon$  above is a tolerance, recognising that the magnitude of the difference of the principal values between adjacent samples should always be less than  $2\pi$ . The choice of the  $\epsilon$  parameter however may cause problems. If  $\epsilon$  is too large, a  $2\pi$ -jump would be indicated where there is none or there is an intrinsic  $\pm\pi$  discontinuity. If  $\epsilon$  is too small, the algorithm would miss a  $2\pi$ -jump falling between two adjacent samples and treat it as a  $\pm\pi$  discontinuity of the unwrapped phase function. Solution of the conventional method depends on the choice of the  $\epsilon$  value.

This problem does not exist in the new approach of equation (17). For example, Fig. 1(e) shows the wrapped differences,  $W[\Delta\psi_k]$ , of wrapped phase values in Fig. 1(d). We see clearly from Fig. 1(e) that  $\pm 2\pi$ -jumps in Fig. 1(d) have been removed. Since this approach now gives an unique solution for removing the  $\pm 2\pi$ -jumps, any large value appearing in  $W[\Delta\psi_k]$  must relate to the physical  $\pm\pi$  discontinuity of the unwrapped phase function and must have numerical errors in it.

### Mitigation of numerical errors

To get rid of the numerical errors from the final unwrapped phase function, I first detect the frequency components with  $\pm\pi$  phase discontinuities by using property II of the phase increment. If the phase increment between adjacent samples is greater than a given threshold, say

$$W[\Delta\psi_k] > \pi/2 , \quad (19)$$

there is an intrinsic  $\pm\pi$  discontinuity in the phase function allocated between those adjacent frequency components, given that the  $k$ -th sample is not such a component. In the example of Fig. 1, detected components are indicated by arrows (Fig. 1(e)). Any single component between two detected components is also considered as a component within the notch. The identification here is based on the information in the phase spectrum and not based on the amplitude spectrum, but the notches appearing on the amplitude spectrum may verify the identification.

In the vicinity of the  $\pm\pi$  discontinuity, phase values inevitably have numerical errors. In other words, any large value  $W[\Delta\psi_k]$  may suspiciously have errors. To prevent the numerical errors effecting subsequent frequency components, I calculate the unwrapped phase value at the first point immediately after the discontinuity by

$$\phi_R = \phi_L + W[\psi_R - \psi_L] , \quad (20)$$

where subscripts R and L refer to the first samples on the right-hand and left-hand sides of the discontinuity range. Both  $\psi_L$  and  $\psi_R$  values used above are reliable and then  $\phi_R$  is not affected by the phase errors between L and R.

The asymptotical  $\pm\pi$  discontinuity exists somewhere between samples  $\phi_L$  and  $\phi_R$ . In practice, I set the unwrapped phase value of any component between samples L and R equal to  $\phi_R$ . This procedure calculates the phase increments of the unwrapped phase curve in the vicinity of a discontinuity by

$$\Delta\phi_k = \begin{cases} W[\psi_{k+1} - \psi_k] , & k < L , \\ W[\psi_R - \psi_L] , & k = L , \\ 0 , & L < k < R , \\ W[\psi_{k+1} - \psi_k] , & k \geq R . \end{cases} \quad (21)$$

Note that we are allowed to do so only because the linear phase shift has been removed in the first step. Otherwise, the quantity  $q_k$  given by equation (16) must be considered.

Once  $\{\Delta\phi_k\}$  are obtained, the final unwrapped phase function is produced simply by summing phase increments. In the example of Fig. 1, phase increments  $\{\Delta\phi_k\}$  and the resultant unwrapped phase are shown in parts (f) and (g). The unwrapping result has been checked by an inverse Fourier transform to reconstruct the original trace.



## VERIFICATION EXAMPLES

In all tests we have conducted, there is no visible discrepancy between the original seismic trace and the reconstructed trace from the resultant unwrapped phase spectrum. In the following two examples however, I verify the unwrapped phase spectrum by comparison with its analytical solution.

One of the examples is shown in Fig. 2, designed to test the effect of numerical errors which are introduced during the discrete Fourier transform. Given a real seismic trace  $\{x_n\}$  with amplitude spectrum  $\{A_k\}$  and unwrapped phase spectrum  $\{\phi_k\}$  drawn in solid lines, I generate a synthetic trace by

$$X_k = A_k[\cos(\phi_k) + i \sin(\phi_k)] \quad , \quad (22)$$

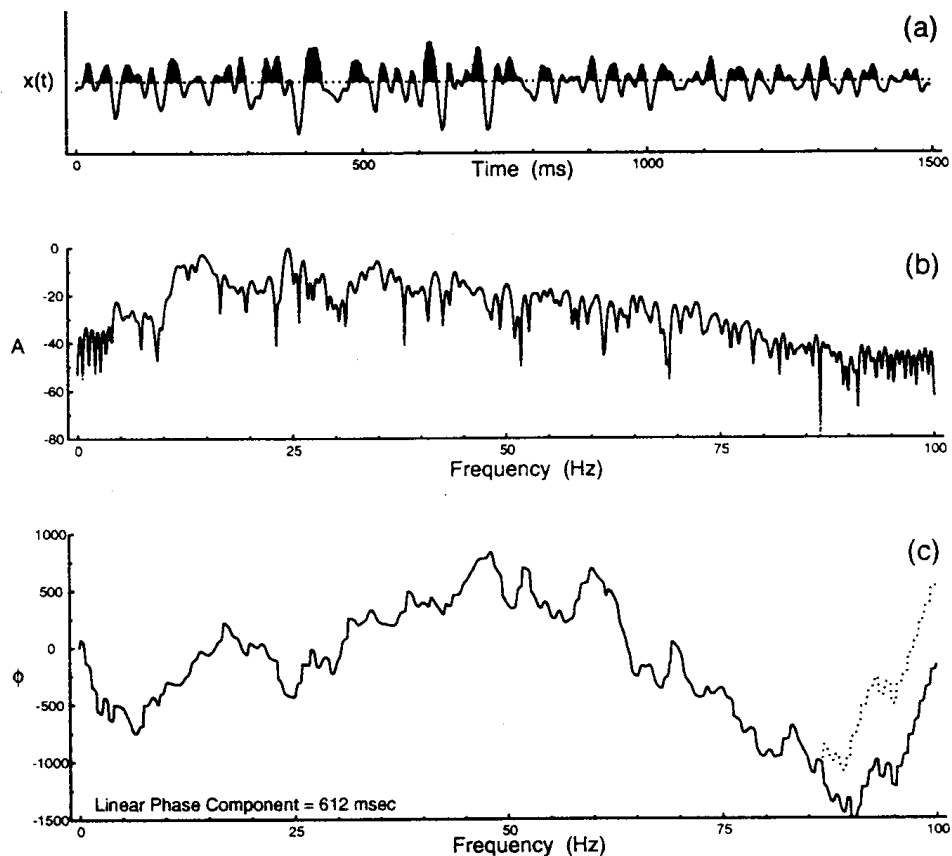


Fig. 2. Spectrum comparison of an original seismic trace and the synthetic trace: (a) the seismic traces; (b) the amplitude spectra; and (c) the phase spectra. The original trace and its spectra are plotted in solid lines, whereas the synthetic trace and its spectra are drawn in dotted lines.

followed by the inverse Fourier transform. By doing so, we introduce numerical errors into the seismic trace, and then see the effect of these errors on the phase unwrapping process, especially in the vicinity of the  $\pm\pi$  discontinuities. There is again no visible difference between the original trace and the synthetic trace. Note that the time shift of 612 ms has been removed from the synthetic trace.

Amplitude and phase spectra of the synthetic trace are drawn in dotted lines and superimposed on the spectra of the original trace. The amplitude spectrum shows that the synthetic trace contains numerical errors at notches, such as the frequency components of less than 1.5 Hz. But the phase spectrum matches the spectrum of the original trace very well within the range from 0 Hz to about 87 Hz. At about 87 Hz, the amplitude value is less than  $-60$  db, and the phase value changes from the  $-\pi$  discontinuity to  $+\pi$  discontinuity. This kind of change is also found at a frequency of about 90 Hz. As we well know, in the complex plane, the position of the signal zeros at the unit circle is very sensitive to noise. If the zero is just inside the unit circle, this is a  $-\pi$  phase change. If the zero is just outside the unit circle, there is a  $+\pi$  phase change. Hence, errors of  $2\pi$  are very likely to occur around notches as zeros might have been erroneously shifted across the unit circle (Poggiagliolmi et al., 1982).

Another example is shown in Fig. 3, designed to test the effect of numerical errors introduced in digital, finite convolution. Given two time series  $u(t)$  and  $v(t)$  with known spectra, I create a synthetic trace  $x(t)$  by convolution

$$x(t) = u(t) * v(t) . \quad (23)$$

The synthetic trace  $x(t)$  shall have analytical Fourier spectra

$$A(\omega) = A^{(u)}(\omega)A^{(v)}(\omega) , \quad (24)$$

and

$$\phi(\omega) = \phi^{(u)}(\omega) + \phi^{(v)}(\omega) , \quad (25)$$

where  $A^{(u)}$  and  $\phi^{(u)}$  are the frequency spectra of  $u(t)$ , and  $A^{(v)}$  and  $\phi^{(v)}$  are the frequency spectra of  $v(t)$ .

I now compare the analytical spectra with the spectra obtained from discrete Fourier transform. In Fig. 3, (a) is a time series  $u(t)$  and its spectra  $A^{(u)}$  and  $\phi^{(u)}$ , (b) is a wavelet  $v(t)$  with spectra  $A^{(v)}$  and  $\phi^{(v)}$ , and (c) is the synthetic  $x(t)$  and its spectra  $A$  and  $\phi$ , where numerically estimated spectra are plotted in solid lines, compared to dotted lines of the analytical solution. In phase  $\phi^{(u)}$ , a phase discontinuity occurs at a frequency of about 5.0 Hz, corresponding to a notch in the amplitude spectrum. To simplify the investigation,  $\phi^{(v)}$  is given as zero phase of a simple wavelet. As the convolution is implemented on the discrete finite series, the amplitude spectrum  $A$  differs from the analytical one. As a consequence, the phase spectrum has  $2\pi$  phase difference at and after the

notch point. Note that the phase spectrum alone does not provide any information about the sign of the  $\pm\pi$  discontinuities. To determine the sign requires a further assumption, such as the minimum phase in seismic.

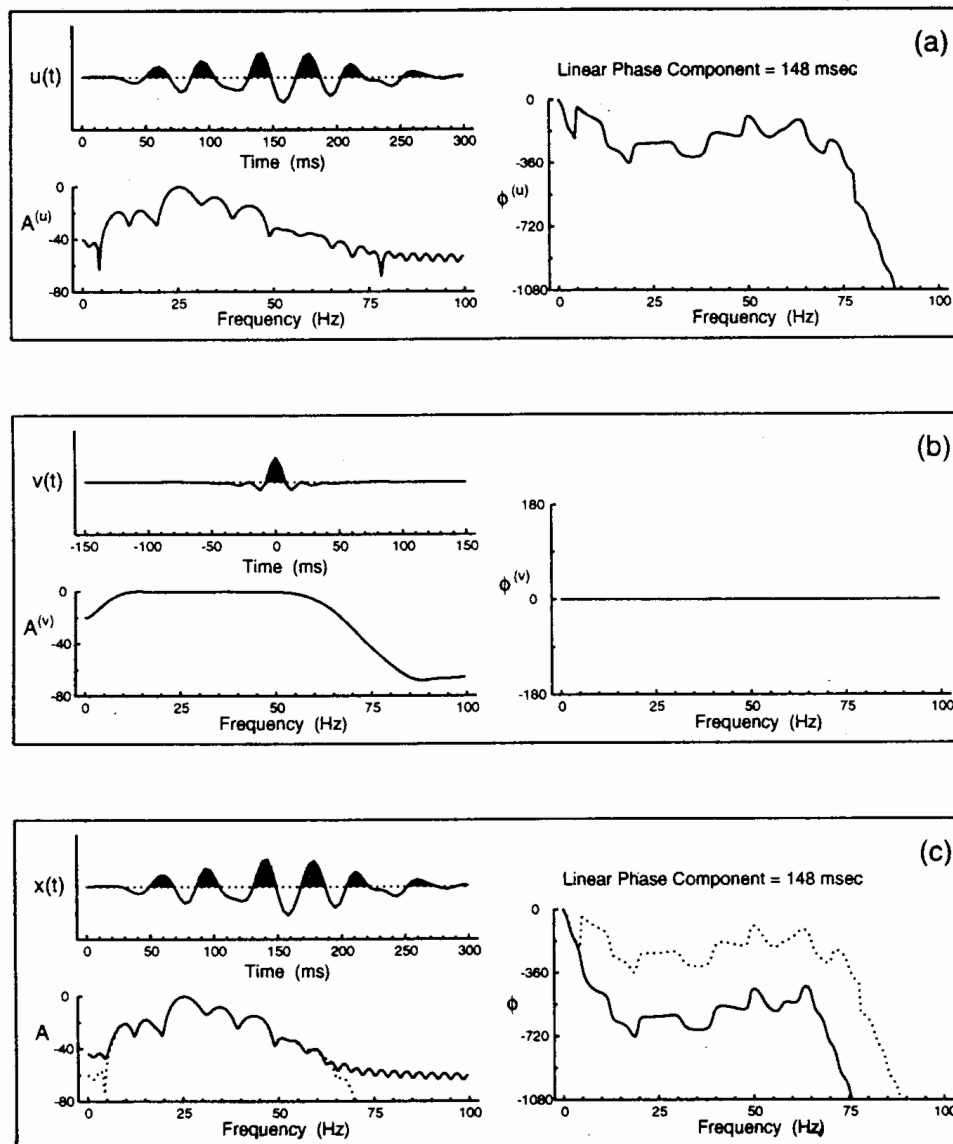


Fig. 3. (a) Time series  $u(t)$  and its spectra; (b) Wavelet  $v(t)$  and its spectra; (c) Synthetic trace  $x(t) = u(t) * v(t)$ , and its spectra obtained from the Fourier transform (in solid lines) and the analytical spectra (in dotted lines).

## CONCLUSIONS

The unwrapped phase function is defined as the sum of phase increments. These phase increments of the unwrapped phase function are obtained by wrapping the phase-differences of wrapped phase values.

This method is computationally efficient, compared to methods such as the adaptive integration. It is because this method does not require frequency re-sampling in an adaptive scheme, which reduces the frequency interval gradually to improve the accuracy of the estimation of the phase change (Wang, 1998).

This method is robust. It effectively prevents numerical errors in the vicinity of the  $\pm\pi$  discontinuities from propagating along the frequency axis, although the phase spectrum alone does not provide any information on the sign of phase discontinuities.

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