A Game Theoretic Approach to Distributed Control of Homogeneous Multi-Agent Systems

T. Mylvaganam

Abstract— **A distributed multi-agent system consisting of homogeneous agents is considered in this paper. Distributed differential games and their solutions in terms of Nash equilibria are defined for such systems, both in a linear-quadratic setting and in a general, nonlinear setting. As with standard differential games, obtaining exact solutions for nonlinear distributed differential games requires solving coupled partial differential equations, closed-form solutions for which are not readily available in general. A systematic method for constructing approximate solutions for a nonlinear distributed differential game with two players is provided. The method requires solving algebraic equations only and is illustrated on a numerical example.**

I. Introduction

A system consisting of several individual ''subsystems", such as a team of autonomous robots or a elements of a smart grid, form what is known as a multi-agent system. Such systems have gained much interest in various fields of engineering over the last decade (see, for instance, [1] and references therein) and have a variety of applications, with examples including mobile sensor networks, monitoring, power systems and space exploration [1]–[5]. From a control theoretic perspective, designing control laws for multi-agent systems may be challenging - in particular when the communication between agents is limited [6], [7]. In many practical situations inter-agent communication *is* limited and call for *distributed* control laws.

Different approaches for designing distributed control laws are available in the literature, often in the context of multi-agent systems and interconnected systems (see, for example, [8], [9]). For a general overview on distributed control see [10] and references therein. Several methods available in the literature make use of the framework provided by game theory to design distributed control laws. See, for instance, [11]–[13] . In [14]–[16] distributed differential games are considered in the context of multi-agent system coordination. In [17], [18] the multi-agent collision avoidance problem is considered (in a centralised setting) and solved by posing the problem as a nonlinear differential game. Formation flying is considered in [19]. Therein it is shown that under certain circumstances the proposed control laws are distributed.

In this paper the problem of designing distributed controllers for a multi-agent system consisting of several homogeneous[1] agents is considered in a game teoretic framework. The contributions of this paper are twofold: considering homogeneous multi-agent systems N-player distributed differential games and their solutions are defined and constructive approximate solutions for a distributed two-player nonlinear differential game are provided.

The remainder of the paper is organised as follows. In Section II some preliminaries, mainly related to graph theory which is used to represent the information available to each agent, are provided. Distributed differential games for multi-agent systems and their solutions are then considered in Section III. A systematic method for constructing approximate solutions for the two-player, nonlinear distributed differential game is then provided in Section IV. The method presented therein draws its inspiration from [20] wherein approximate solutions for (centralised) nonlinear differential games are provided. Similar ideas have been developed for constructing approximate solutions for optimal control problems [21]–[23]. The results presented in this paper are then illustrated on a numerical example before some concluding remarks and directions for future research are given in Sections V and VI, respectively.

Notation: $\mathbb{R}$ denotes the set of real numbers. The norm of a vector v weighted by a matrix $M = M^\top > 0$ is denoted by $\|v\|_M$. The $n \times n$ identity matrix is denoted by I_n and the zero matrix is denoted by 0. The kronecker product between a matrix M_1 and a matrix M_2 is denoted by $M_1 \otimes M_2$. Given a set $\mathcal{S}$, its cardinality is denoted by $|\mathcal{S}|$ and $\sum_{j \in \mathcal{S}} y_j$ is used to denote the summation of all y_j such that $j \in \mathcal{S}$.

II. Preliminaries

A system consisting of N homogenous agents is considered herein. Each agent is associated with a state $x_i(t) \in \mathbb{R}^n$, where the subscript i, $i = 1, \dots, N$, indicates a particular agent. The communication between the agents is described by a directed graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$. The set of N nodes $\mathcal{V} = \{1, \dots, N\}$ is such that a given node corresponds to the agent with the same index i, $i = 1, \dots, N$. The so-called edge set $\mathcal{E} \subset \mathcal{V} \times V$ is such that an edge (i, j) indicates that there is (directed) communication from agent i to agent j and agent i is said to be a *neighbour* of agent j. The set $\mathcal{N}_i$ denotes

T. Mylvaganam is with the Department of Aeronautics, Imperial College London, London SW7 2AZ, UK (Email:thulasi.mylvaganam06@imperial.ac.uk).

[1]Agents are said to be homogeneous if they can be described by the same dynamic model.

all neighbours of agent i and $|\mathcal{N}_i|$ is the number of neighbours of agent i has. Note that since we consider *directed graphs* $(i,j) \in \mathcal{E}$ does not imply $(j,i) \in \mathcal{E}$. Moreover, although we assume that each agent has knowledge about its own state, to simplify the notation, we require that $(i,i) \notin \mathcal{E}$.

The global state $x(t) = (x_1^\top, \ldots, x_N^\top)^\top$ describes the multi-agent system *as a whole* and, based on the communication topology each agent builds its own *local state* which contains its own state and the states of all neighbouring agents. The local state is denoted by $\hat{x}_i = (x_i, x_{\mathcal{N}_i})$, where $x_{\mathcal{N}_i}$ denotes the vector containing the states of all agents $j \in \mathcal{N}_i$, such that the k-th element of $x_{\mathcal{N}_i}$ corresponds to the k-the element of $\mathcal{N}_i$, for $k = 1, \ldots, |\mathcal{N}_i|$. Each of the local states can be written in terms of the global state as

$$\hat{x}_i = N_i x \,, \tag{1}$$

where the matrix $N_i \in \mathbb{R}^{|\mathcal{N}_i|n \times Nn}$ reflects the communication topology dictated by the graph $\mathcal{G}$. In the remainder of the paper it is assumed that each agent has knowledge of its own state and if $(i,j) \in \mathcal{E}$ agent j has acess to the local state and the control input of agent i.

III. Distributed Differential Games and their Exact Solutions

Consider a multi-agent system consisting of N homogeneous agents with communication topology described by a directed graph $\mathcal{G}$. It is assumed that each agent i, for $i = 1, \ldots, N$, has access to the states x_j and the control strategies u_j for all $j \in \mathcal{N}i$. We consider the problem in which each agent seeks to minimise its own individual cost functional based solely on the information availabe from its neighbours. This leads to a *distributed differential game*.

Consider the case in which each agent is described by the dynamics

$$\dot{x}_i = f(x_i) + g(x_i)u_i \,, \tag{2}$$

where $u_i \in \mathcal{R}^m$ is a control input and $f(x_i)$ and $g(x_i)$ are smooth mappings, for $i = 1, \ldots, N$.

Assumption 1: The origin of the system (2) is an equilibrium, *i.e.* $f(0) = 0$.

It follows from Assumption 1 that there exists a matrix-valued mapping $F(x_i)$ such that $f(x_i) = F(x_i)x_i$.

The *global system* is then described by the dynamics

$$\dot{x} = f_{gl}(x) + \sum_{i=1}^{N} g_i(x_i)u_i \,, \tag{3}$$

where $f_{gl}(x) = \left[f(x_1)^\top, \ldots, f_N(x_N)^\top\right]^\top$ and $g_1(x) = \left[g(x_1)^\top, 0 \ldots, 0\right]^\top, \ldots, g_N(x) = \left[0, \ldots, 0, g(x_N)^\top\right]^\top$.

Let $u_{\mathcal{N}_i}$ denote the set of feedback strategies corresponding to each neighbour of agent i and let $u_{\bar{\mathcal{N}}_i}$ denote the set of feedback strategies of each agent which is not a neighbour of agent i. Each agent seeks to minimise individual cost functionals of the form

$$J_i(\hat{x}_i, u_i, u_{\mathcal{N}_i}) = \frac{1}{2}\int_0^\infty q_i(\hat{x}_i) + u_i^\top u_i \, \mathrm{d}t \,, \tag{4}$$

where $q_i(\hat{x}_i) \geq 0$, for all $\hat{x}_i \neq 0$, and $q_i(0) = 0$ is a running cost, which is a function of the local state of agent i, and the second term is a penalty on the control effort, for $i = 1, \ldots, N$.

The *local dynamics* observed by an agent i can be written in the form

$$\dot{\hat{x}}_i = f_i(\hat{x}_i) + g_{ii}(\hat{x}_i)u_i + \sum_{j \in \mathcal{N}_i} g_{ij}(\hat{x}_i)u_j \,, \tag{5}$$

where $f_i(\hat{x}_i) = N_i f_{gl}(x)$, which by Assumption 1 can be written as $f_i(\hat{x}_i) = F_i(\hat{x}_i)\hat{x}_i$,

$$g_{ii} = \begin{bmatrix} 0 & 0 & g(x_i) & 0 & \ldots & 0 \end{bmatrix}^\top \in \mathbb{R}^{n(|\mathcal{N}_i|+1)\times m} \,,$$

and the functions $g_{ij}(\hat{x}_i) \in \mathbb{R}^{n(|\mathcal{N}_i|+1)\times m}$, for $i = 1, \ldots, N$ and for all $j \in \mathcal{N}_i$, are of the form $g_{ij}(x) = \left[0, \gamma_{i1}^\top, \ldots, \gamma_{i|\mathcal{N}_i|}^\top\right]^\top$, where $\gamma_{ik} = g(x_j)$ when j corresponds to the k-th element of $\mathcal{N}_i$ and $\gamma_{ik} = 0$ otherwise.

We define the distributed differential game as follows.

Problem 1: Consider the multi-agent system with N agents, each satisfying the dynamics (2), and consider the case in which each agent seeks to minimise its cost functional (4), $i = 1, \ldots N$. The inter-agent communication is described by the graph $\mathcal{G}$. Determine a set of admissible[2] strategies $(u_1^*, \ldots, u_N^*)$ satisfying the inequalities

$$J_i(\hat{x}_i, u_i^*, u_{\mathcal{N}_i}^*) \leq J_i(\hat{x}_i, u_i, u_{\mathcal{N}_i}^*) \,, \tag{6}$$

for all admissible sets of strategies $(u_i, u_{\mathcal{N}_i}^*, u_{\bar{\mathcal{N}}_i}^*)$, for $i = 1, \ldots, N$.

Solutions for Problem 1 are considered separately for the linear quadratic case and the general, nonlinear case in the remainder of this section.

A. Linear Quadratic Problem

Consider the case in which each agent is described the linear dynamics, *i.e.*

$$\dot{x}_i = Ax_i + Bu_i \,, \tag{7}$$

$A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, for $i = 1, \ldots, N$. Moreover, suppose the running costs $q_i(\hat{x}_i)$ in (4) are quadratic, *i.e.*

$$q_i(\hat{x}_i) = \hat{x}_i^\top Q_i \hat{x}_i \,, \tag{8}$$

with $Q_i \geq 0$, for $i = 1, \ldots, N$. In this setting the global system is characterised by the linear dynamics

$$\dot{x} = A \otimes I_{nN}x + \sum_{i=1}^{N} B_i u_i \,, \tag{9}$$

[2]A set of strategies $(u_1, \ldots, u_N)$ is said to be admissible if it renders the closed loop system (3) (locally) asymptotically stable.

where $B_1 = \left[B^\top, \ldots, 0, \ldots, 0\right]^\top, \ldots, B_N = \left[0, \ldots, 0, \ldots, B^\top\right]^\top$. The local state observed by an agent i is the linear equivalent of (5), namely $\dot{\hat{x}}_i = A_i\hat{x}_i + B_{ii}u_i + \sum_{j\in\mathcal{N}_i} B_{ij}u_j$. where $A_i = A \otimes I_{n(|\mathcal{N}_i|+1)}$, $B_{ii} = \left[\begin{array}{cccc} B^\top & 0 & \ldots & 0 \end{array}\right]^\top$ and, similarly to in the nonlinear setting, $B_{ij}(x) = \left[0, \gamma_{i1}^\top, \ldots, \gamma_{i|\mathcal{N}_i|}^\top\right]^\top$, where $\gamma_{ik} = B$ when j corresponds to the k-th element of $\mathcal{N}_i$ and $\gamma_{ik} = 0$ otherwise.

Assumption 2: The graph $\mathcal{G}$ and the running costs (8) are such that $\sum_{i=1}^N N_i^\top Q_i N_i > 0$.

In the following statemet, as is commonly done in the context of linear quadratic differential games (see, for example [20], [24]), we consider linear feedback strategies as solutions for Problem 1 when the agent dynamics are linear and the running costs are quadratic in the local states.

Proposition 1: Consider the (homogeneous) multi-agent system described by the directed graph $\mathcal{G}$ with the agent dynamics given by (7), for $i = 1, \ldots, N$. Consider Problem 1 with the running costs (8), for $i = 1, \ldots, N$, and suppose Assumption 2 is satisfied. Suppose we can find matrices $P_i = P_i^\top \geq 0$, for $i = 1, \ldots, N$, such that $\sum_{i=1}^N N_i P_i N_i > 0$ satisfying

$$\begin{aligned} &\hat{x}_i^\top Q_i \hat{x}_i - \hat{x}_i P_i B_{ii} B_{ii}^\top P_i \hat{x}_i + \hat{x}_i^\top P_i A_i \hat{x}_i \\ &+ \hat{x}_i^\top A_i^\top P_i \hat{x}_i - \sum_{j\in\mathcal{N}_i} \hat{x}_i^\top P_i B_{ij} B_{jj}^\top P_j \hat{x}_j \\ &- \sum_{j\in\mathcal{N}_i} \hat{x}_j^\top P_j B_{jj} B_{ij}^\top P_i \hat{x}_i = 0\,, \end{aligned} \tag{10}$$

for $i = 1, \ldots, N$, and for all $\hat{x}_i$ and $\hat{x}_j$. Then the set of feedback strategies

$$u_i^* = -B_{ii}^\top P_i \hat{x}_i\,, \tag{11}$$

for $i = 1, \ldots, N$, is a solution of Problem 1.

Remark 1: If the agents have knowledge of the graph topology, *i.e.* the matrices N_i, $i = 1, \ldots, N$, are known to each agent, the equations (10), $i = 1, \ldots, N$, in Proposition 1 can be replaced by the coupled algebraic Riccati-like equations

$$\begin{aligned} &N_i^\top Q_i N_i - N_i^\top P_i B_{ii} B_{ii}^\top P_i N_i + N_i^\top P_i A_i N_i \\ &+ N_i^\top A_i^\top P_i N_i - \sum_{j\in\mathcal{N}_i} N_i^\top P_i B_{ij} Bjj^\top P_j N_j \\ &- \sum_{j\in\mathcal{N}_i} N_j^\top P_j B_{jj} B_{ij}^\top P_i N_i = 0\,. \end{aligned} \tag{12}$$

B. Nonlinear Problem

Consider now the general nonlinear case in which the dynamics of each agent is given by (2) and the running cost in (4) is a general, nonlinear function. As in the case of standard differential games [25]–[27], obtaining a solution to Problem 1 requires solving a system of PDEs.

Assumption 3: The running costs $q_i(\hat{x}_i)$ are such that $q_x(x) = \sum_{i=1}^N q_i(\hat{x}_i) > 0$ for all $x \neq 0$.

Proposition 2: Consider the multi-agent system described by the directed graph $\mathcal{G}$ with the agent dynamics given by (2), for $i = 1, \ldots, N$. Consider Problem 1 and suppose Assumption 3 is satisfied. Suppose we can find a solution to the coupled Hamilton-Jacobi-Isaacs (HJI) partial differential equations (PDE)s

$$\begin{aligned} &\frac{\partial V_i}{\partial \hat{x}_i} f_i(\hat{x}_i) - \frac{1}{2}\frac{\partial V_i}{\partial \hat{x}_i} g_{ii}(\hat{x}_i) g_{ii}(\hat{x}_i)^\top + \frac{1}{2} q_i(\hat{x}_i) \\ &- \sum_{j\in\mathcal{N}_i} \frac{\partial V_i}{\partial \hat{x}_i} g_{ij}(\hat{x}_i) g_{jj}(\hat{x}_j)^\top \frac{\partial V_j}{\partial \hat{x}_j} = 0\,, \end{aligned} \tag{13}$$

such that $V_i(\hat{x}_i) \geq 0$ and $V_i(0) = 0$, for $i = 1, \ldots, N$, and $\sum_{i=1}^N V_i(\hat{x}_i) > 0$ for all $x \neq 0$. Then the set of feedback strategies

$$u_i^* = -g_{ii}(\hat{x}_i)^\top \frac{\partial V_i}{\partial \hat{x}_i}^\top\,, \tag{14}$$

for $i = 1, \ldots, N$, is a solution of Problem 1.

Remark 2: The existence of solutions of (10) and (12), $i = 1, \ldots, N$, arising in linear quadratic distributed differential games, and of (13), $i = 1, \ldots, N$, arising in nonlinear distributed differential games, depends on the communication topology described by the graph $\mathcal{G}$.

IV. Approximate Solution of the 2-Player Nonlinear Distributed Differential Game

The solution of general nonlinear differential games *without* communication constraints, *i.e.* when all players share the same knowledge of the global system, is characterised by HJI PDEs (see, for instance [25]). Similarly, the solution of Problem 1 requires solving the system of the N coupled HJI PDEs (13), $i = 1, \ldots, N$, as established in Proposition 2. Closed-form solutions to the PDEs (13), $i = 1, \ldots, N$, are not readily available in general. Thus, it may be of interest to determine *approximate solutions* for Problem 1.

In [20] constructive methods for obtaining approximate solutions for nonlinear differential games without communication constraints are provided, *i.e.* the methods proposed therein are inherently *centralised*. In this section the results therein are further developed to solve a distributed differential game with two players.

A two-player distributed differential game characterised by the dynamics (2), the cost functionals (4), for $i = 1, 2$, and the graph $\mathcal{G}$, with nodes $\mathcal{V} = \{1, 2\}$ and edge set $\mathcal{E} = \{(1, 2)\}$ is considered. In this scenario agent 1 has knowledge of its own state x_1, whereas agent 2 has knowledge the global state $x = (x_1^\top, x_2^\top)^\top$. In particular, $\hat{x}_1 = x_1 = \left[\begin{array}{cc} I_n & 0 \end{array}\right] x$, $\hat{x}_2 = \left[\begin{array}{cc} 0 & I_n \\ I_n & 0 \end{array}\right] x$, $g_{11} = g(x_1)$, $g_{21} = \left[\begin{array}{cc} g(x_2) & 0 \end{array}\right]^\top$ and $g_{12} = \left[\begin{array}{cc} 0 & g(x_1) \end{array}\right]^\top$. The HJI PDEs (13) associated with this problem are

$$
\begin{aligned}
&\frac{\partial V_1}{\partial \hat{x}_1} f_1(\hat{x}_1) - \frac{1}{2}\frac{\partial V_1}{\partial \hat{x}_1} g_{11}(\hat{x}_1) g_{11}(\hat{x}_1)^\top \\
&\qquad\qquad\qquad + \frac{1}{2} q_1(\hat{x}_1) = 0\,, \\
&\frac{\partial V_2}{\partial \hat{x}_2} f_2(\hat{x}_2) - \frac{1}{2}\frac{\partial V_2}{\partial \hat{x}_2} g_{22}(\hat{x}_2) g_{22}(\hat{x}_2)^\top \\
&\quad + \frac{1}{2} q_2(\hat{x}_2) - \frac{\partial V_2}{\partial \hat{x}_2} g_{21}(\hat{x}_2) g_{11}(\hat{x}_1)^\top \frac{\partial V_1}{\partial \hat{x}_1} = 0\,,
\end{aligned} \tag{15}
$$

and provided a solution $V_1(\hat{x}_1)$ and $V_2(\hat{x}_2)$ can be found, the solution to Problem 1 are the feedback strategies (14), $i = 1, 2$.

Notions similar to those introduced in [20] are used to systematically construct *distributed* control strategies which solve the distributed differential game defined in Problem 1 *approximately* for the case in which there are two players with the communication topology described by $\mathcal{G}$. Similarly to what is seen in [20], [21] the method utilises the notion of a so-called *algebraic $\bar{P}$ solution*. The systematic method for constructing approximate solutions to Problem 1 merely requires solving a system of algebraic matrix equations (in place of the PDEs (15)) and requires that the following conditions are satisfied.

Assumption 4: The multi-agent system satisfies the following conditions. Each agent i, for $i = 1, 2$, has access to

i) its own local state $\hat{x}_i$;
ii) the control strategies $u_{j\in\mathcal{N}_i}$;
iii) the graph $\mathcal{G}$.

Assumption 5: The running costs in (4) are of the form $q_i(\hat{x}_i) = \hat{x}_i Q_i(\hat{x}_i)\hat{x}_i$, for $i = 1, 2$.

Let $\bar{Q}_i = Q_i(0)$, for $i = 1, \ldots, N$. For forward reference the following notation, which is related to the linearisation of the local state dynamics, is introduced at this stage $A_i = \dfrac{\partial f_i(\hat{x}_i)}{\partial \hat{x}_i}\Big|_{\hat{x}_i=0} = F_i(0)$, $B_{ii} = g_{ii}(0)$ and $B_{ij} = g_{ij}(0)$, for $i = 1, 2$ and for all $j \in \mathcal{N}_i$.

Remark 3: Assumption 5 is such that the running costs $q_i(\hat{x}_i)$, for $i = 1, \ldots, N$, are at least locally quadratic. Assumptions 3 and 5 imply that $\sum_{i=1}^N N_i^\top \bar{Q}_i N_i > 0$.

A. Algebraic $\bar{P}$ Solution

Consider the (homogeneous) system with 2 agents described by the agent dynamics (2) and the graph $\mathcal{G}$. Consider the cost functionals (4) and the distributed differential game in Problem 1. The *algebraic $\bar{P}$ solution* for (15) is defined as follows.

Definition 1: Let $\Sigma_1 : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and $\Sigma_2 : \mathbb{R}^{2n} \to \mathbb{R}^{2n\times 2n}$, with $\Sigma_1(0) \geq 0$ and $\Sigma_2(0) \geq 0$, denote two matrix-valued functions. Let $\Sigma_1(\hat{x}_1)$ and $\Sigma_2(\hat{x}_2)$ be such that $\Sigma_1(\hat{x}_1) = \Sigma_1(\hat{x}_1)^\top > 0$ for all $\hat{x}_1 \in \mathbb{R}^n \setminus \{0\}$ and $\Sigma_2(\hat{x}_2) = \Sigma_2(\hat{x}_2)^\top > 0$, for all $\hat{x}_2 \in \mathbb{R}^{2n} \setminus \{0\}$. The $\mathcal{C}^1$ matrix-valued functions $P_1 : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and $P_2 : \mathbb{R}^{2n} \to \mathbb{R}^{2n\times 2n}$, such that $P_i(x) = P_i(x)^\top$, $i = 1, 2$, are said to be *$\mathcal{X}$-algebraic $\bar{P}$ solutions*[3] of (15), provided the following conditions hold.

(i) For all[4] $x \in \mathcal{X} \subseteq \mathbb{R}^{Nn}$, and for $i = 1, \ldots, N$

$$
\begin{aligned}
&N_1^\top P_1(\hat{x}_1) F_1(\hat{x}_1) N_1 + N_1^\top F_1(\hat{x}_1)^\top P_1(\hat{x}_1) N_1 \\
&- N_1^\top P_1(\hat{x}_1) g_{11}(\hat{x}_1) g_{11}(\hat{x}_1)^\top P_1(\hat{x}_1) N_1 \\
&+ N_1^\top \Sigma_{11}(\hat{x}_1) N_1 + N_1^\top Q_1(\hat{x}_1) N_1 = 0\,,
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
&N_2^\top P_2(\hat{x}_2) F_2(\hat{x}_2) N_2 + N_2^\top F_2(\hat{x}_2)^\top P_2(\hat{x}_2) N_2 \\
&- N_2^\top P_2(\hat{x}_2) g_{22}(\hat{x}_2) g_{22}(\hat{x}_2)^\top P_2(\hat{x}_2) N_2 \\
&- N_2^\top P_2(\hat{x}_2) g_{21}(\hat{x}_2) g_{11}(\hat{x}_1)^\top P_1(\hat{x}_1) N_1 \\
&- N_1^\top P_1(\hat{x}_1) g_{11}(\hat{x}_1) g_{21}(\hat{x}_2)^\top P_2(\hat{x}_2) N_2 \\
&+ N_2^\top Q_2(\hat{x}_2) N_2 + N_2^\top \Sigma_2(\hat{x}_2) N_2 = 0\,.
\end{aligned} \tag{17}
$$

(ii) $P_i(0) = \bar{P}_i$, such that $\left(N_1^\top \bar{P}_i N_1 + P_2\right) > 0$, with $\bar{P}_1$ and $\bar{P}_2$ solutions of the *coupled Riccati-like equations*

$$
\begin{aligned}
&N_1^\top \bar{Q}_1 N_1 - N_1^\top \bar{P}_1 B_{11} B_{11}^\top \bar{P}_1 N_1 + N_1^\top \bar{\Sigma}_1 N_1 \\
&\quad + N_1^\top \bar{P}_1 A_1 N_1 + N_1^\top A_1^\top \bar{P}_1 N_1 = 0\,, \\
&N_2^\top \bar{Q}_2 N_2 - N_2^\top \bar{P}_2 B_{22} B_{22}^\top \bar{P}_2 N_2 \\
&\quad + N_2^\top A_2^\top \bar{P}_2 N_2^\top + N_2^\top \bar{P}_2 A_2 N_2 \\
&\quad + N_2^\top \bar{\Sigma}_{22} N_2^\top - N_2^\top \bar{P}_2 B_{21} B_{11}^\top \bar{P}_1 N_1 \\
&\quad - N_1^\top \bar{P}_1 B_{11} B_{21}^\top \bar{P}_2 N_2^\top = 0\,.
\end{aligned} \tag{18}
$$

If $x \in \mathbb{R}^{2n}$, *i.e.* $\mathcal{X} = \mathbb{R}^{2n}$, then P_i, $i = 1, 2$, are said to be an *algebraic $\bar{P}$ solution*.

In what follows we assume the existence[5] of *algebraic $\bar{P}$ matrix solutions*, *i.e.* we assume $\mathcal{X} = \mathbb{R}^n$.

B. Approximate Solution to Distributed, Nonlinear Differential Games

In this section the notion of algebraic $\bar{P}$ solutions is used to systematically construct approximate solutions for Problem 1.

In what follows, we introduce a dynamic extension with state $\xi(t) = (\xi_1^\top, \xi_2)^\top \in \mathbb{R}^{nN}$, with $\xi_i(t) \in \mathbb{R}^n$, for $i = 1, 2$. The dynamic extension is utilised to design dynamic feedback strategies of the form

$$
\begin{aligned}
u_1 &= \beta_1(\hat{x}_1, \xi_1, \xi_{\mathcal{N}_\infty})\,, \\
u_2 &= \beta_2(\hat{x}_2, \xi_2, , \xi_{\mathcal{N}_\in})\,, \\
\dot{\xi} &= \tau(x, \xi)\,,
\end{aligned} \tag{19}
$$

where $\tau(0, 0) = 0$, $\beta_i(0, 0) = 0$, τ, β_i are smooth mappings, for $i = 1, 2$, and $\xi_{\mathcal{N}_i}$ denotes the vector containing the components ξ_j such that $j \in \mathcal{N}_i$. In the above it is apparent that agent 1 has access to ξ_1

[3] Provided the set $\mathcal{X}$ contains the origin.

[4] Since $\hat{x}_i \subseteq \mathbb{R}^{2n}$, for $i = 1, 2$, the algebraic $\bar{P}$ solution is defined on the space in which the *global state* x evolves.

[5] The existence of such as solution depends partly on the graph $\mathcal{G}$.

only and agent 2 has access to ξ_1 and ξ_2, *i.e.* $\xi_{\mathcal{N}_1} = \emptyset$ and $\xi_{\mathcal{N}_2} = \xi_1$. Let $\hat{\xi}_1 = \xi_1$ and $\hat{\xi}_2 = (\xi_2^\top, \xi_1^\top)^\top$. Note that the dynamic extension ξ "mimics" the structure of the global state x in terms of which components are avaialble to each of the individual agents.

Problem 2: Consider a multi-agent system with 2 agents, each satisfying the dynamics (2), and consider the case in which each agent seeks to minimise its cost functional (4), $i = 1, 2$. The inter-agent communication is described by the graph $\mathcal{G}$ with $\mathcal{V} = \{1, 2\}$, $\mathcal{E} = \{(1, 2)\}$. Determine a set of admissible *dynamic feedback strategies* (S_1, S_2), where the strategy S_i, $i = 1, 2$, is a dynamical system described by (19) and non-negative functions $c_1(\hat{x}_1, \hat{\xi}_1)$ and $c_2(\hat{x}_2, \hat{\xi}_2)$ such that for any admissible set of strategies (u_1, u_2, τ), with $u_i \neq \beta_i$, $i = 1, 2$,

$$\hat{J}_1((\hat{x}_1(0), \hat{\xi}_1(0)), \beta_1) \leq \hat{J}_1((\hat{x}_1(0), \hat{\xi}_1(0)), u_1),$$
$$\hat{J}_2((\hat{x}_2(0), \hat{\xi}_2(0)), \beta_2, \beta_1) \leq \hat{J}_2((\hat{x}_2(0), \hat{\xi}_2(0)), u_2, \beta_1),$$

where the extended cost functionals $\hat{J}_i$, $i = 1, 2$, are defined as

$$\hat{J}_i((\hat{x}_i(0), \hat{\xi}_i(0), u_i, u_{\mathcal{N}_i} \triangleq \frac{1}{2}\int_0^\infty \Big(q_i(\hat{x}_i(t)) + \|u_i(t)\|^2 + c_i(\hat{x}_i(t), \hat{\xi}_i(t))\Big)\mathrm{d}t. \quad (20)$$

Remark 4: The solution of Problem 2 constitutes an *approximate* solution, in terms of an ϵ_α-Nash equilibrium solution, of Problem 1 (see [20] for details).

Let $P_i(\hat{x}_i)$, $i = 1, 2$, denote an algebraic $\bar{P}$ solution for (15) and consider the *extended value functions*

$$V_1(\hat{x}_1, \xi_1) = \tfrac{1}{2}\hat{x}_1^\top P_1(\xi_1)\hat{x}_1 + \tfrac{1}{2}\|\hat{x}_1 - \hat{\xi}_1\|_{R_1},$$
$$V_2(\hat{x}_2, \xi) = \tfrac{1}{2}\hat{x}_2^\top P_2(\xi_2)\hat{x}_2 + \tfrac{1}{2}\|\hat{x}_2 - \hat{\xi}_2\|_{R_2}, \quad (21)$$

where $R_i = R_i^\top > 0$, for $i = 1, 2$. Let $\Phi_i(\hat{x}_i, \hat{\xi}_i)$ denote a matrix-valued mapping such that $P_i(\hat{x}_i)\hat{x}_i - P_i(\hat{\xi}_i)\hat{x}_i = \Phi_i(\hat{x}_i, \hat{\xi}_i)(\hat{x}_i - \hat{\xi}_i)$ and let $\Psi_i(\hat{x}_i, \hat{\xi}_i)$ denote that Jacobian matrix of $\frac{1}{2}P_i(\hat{\xi}_i)\hat{x}_i$ with respect to $\hat{\xi}_i$, for $i = 1, 2$. Moreover, note that Ψ_2 can be written as the block matrix $\Psi_2 = \begin{bmatrix} \Psi_{22} & \Psi_{21} \\ \Psi_{12} & \Psi_{11} \end{bmatrix}$.

Theorem 1: Consider Problem 2 and suppose Assumptions 1, 3, 4 and 5 are satisfied. Let P_i, $i = 1, 2$, be an *algebraic* $\bar{P}$ *solution* of (15), with $\bar{\Sigma}_i > 0$, for $i = 1, 2$. Suppose Ψ_2 is such that $\Psi_{12} = 0$ and let $R_2 = \text{blockdiag}\{R_{22}, R_{12}\} = R_2^\top$ and $R_1 = R_1^\top$ be such that

$$R_1(R_1 + R_{12}) + (R_1 + R_{12})R_1 > 0,$$
$$R_2(R_2 + N_1^\top R_1 N_1 N_2 + (N_2 N_1^\top R_1 N_1)R_2 > 0. \quad (22)$$

Then there exists constants $\bar{k} \geq 0$ and a set $\Omega \subseteq \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ such that the functions (21), solve the system of inequalities

$$\mathcal{HJ}_1 \triangleq \frac{\partial V_1}{\partial \hat{x}_1} f_1(\hat{x}_1) - \frac{1}{2}\frac{\partial V_1}{\partial \hat{x}_1} g_{11}(\hat{x}_1) g_{11}(\hat{x}_1)^\top + \frac{1}{2}q_1(\hat{x}_1) + \frac{\partial V_1}{\partial \hat{\xi}_1}\dot{\hat{\xi}}_1 \leq 0, \quad (23)$$

$$\mathcal{HJ}_2 \triangleq \frac{\partial V_2}{\partial \hat{x}_2} f_2(\hat{x}_2) - \frac{1}{2}\frac{\partial V_2}{\partial \hat{x}_2} g_{22}(\hat{x}_2) g_{22}(\hat{x}_2)^\top + \frac{1}{2}q_2(\hat{x}_2) - \frac{\partial V_2}{\partial \hat{x}_2} g_{21}(\hat{x}_2) g_{11}(\hat{x}_1)^\top \frac{\partial V_1}{\partial \hat{x}_1} + \frac{\partial V_2}{\partial \hat{\xi}_2}\dot{\hat{\xi}}_2 \leq 0, \quad (24)$$

with

$$\dot{\xi} = -k\left(\frac{\partial V_2}{\partial \hat{\xi}_2} N_2 + \frac{\partial V_1}{\partial \hat{\xi}_1} N_1\right)^\top,$$

for all $k > \bar{k}$ and for all $(x, \xi) \in \Omega$. It follows that

$$\begin{aligned}
\dot{\xi} = &- kN_2^\top \Big(\big(\Psi_2(\hat{x}_2, \hat{\xi}_2)^\top \hat{x}_2 - R_2(\hat{x}_2 - \hat{\xi}_2)\big) \\
&+ N_1^\top \Big(\Psi_1(\hat{x}_1, \hat{\xi}_1)^\top \hat{x}_1 - R_1(\hat{x}_1 - \hat{\xi}_1)\Big)\Big), \\
u_1 = &- g_{11}(\hat{x}_1)^\top \Big(P_1(\hat{x}_1)\hat{x}_1 \\
&+ (R_1 - \Phi_1(\hat{x}_1, \hat{\xi}_1))(\hat{x}_1 - \hat{\xi}_1)\Big), \\
u_2 = &- g_{22}(\hat{x}_2)^\top \Big(P_2(\hat{x}_2)\hat{x}_2 \\
&+ (R_2 - \Phi_2(\hat{x}_2, \hat{\xi}_2))(\hat{x}_2 - \hat{\xi}_2)\Big),
\end{aligned} \quad (25)$$

are admissible dynamic strategies that solve Problem 2 with $c_i(\hat{x}_i, \hat{\xi}_i) = -2\mathcal{HJ}_i(\hat{x}_i, \hat{\xi}_i)$, for $i = 1, 2$. Moreover, there exists a neighbourhood of the origin in which strategies (25) constitute an approximate solution for Problem 1.

V. Numerical Example

A numerical example is provided in this section to illustrate the results presented in Section IV. The example is one of multi-agent collision avoidance. In particular, we revisit a scenario with two agents seen in [18] and study it in the distributed setting considered in this paper.

Consider a system consisting of two agents moving on a plane, *i.e.* $p_i \in \mathbb{R}^2$ where p_i is the position of agent i, for $i = 1, 2$, and the inter-agent communication described by a graph $\mathcal{G}$ with the nodes $\mathcal{V} = \{1, 2\}$ and the edge set $\mathcal{E} = \{(1, 2)\}$. We consider the scenario in which each agent seeks to reach a predefined target $t_i \in \mathbb{R}^2$, $i = 1, 2$, and, in addition, agent 2 seeks to avoid collisions with agent i. To this end we define the running costs

$$\begin{aligned}
q_1(\hat{x}_1) &= a_1 x_1^\top x_1, \\
q_2(\hat{x}_2) &= a_2 \frac{x_2^\top x_2}{(\|x_1 + t_1 - x_2 - t_2\|^2 - r^2)^2},
\end{aligned} \quad (26)$$

where $a_i > 0$, for $i = 1, 2$, $x_i = p_i - t_i$, for $i = 1, 2$, and $r > 0$ is a minimum distance to be maintained between the two agents at all times. The agents are assumed to satisfy single-integrator dynamics, namely $\dot{x}_i = u_i$.

In the following we consider the case in which $a_1 = 1$ and $a_2 = 2$. The pair of agents, their dynamics, the running costs (26) and the graph $\mathcal{G}$ constitute a distributed differential game described in Problem 1. To solve Problem 2, and thus solve Problem 1 approximately, note that

$$P_1 = \alpha_1 I_2$$

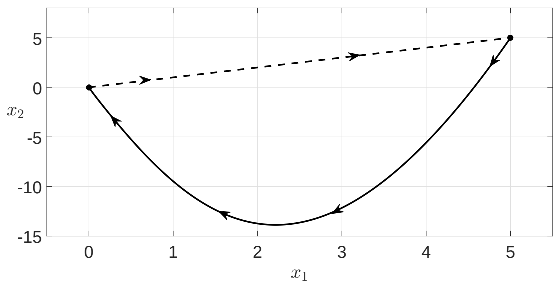

Fig. 1. Trajectories of p_1 (dashed line) and p_2 (solid line). The circular markers denote the initial and final positions of the agents and the arrows indicate the direction of travel.

and

$$P_2 = \text{blockdiag}\{\alpha_{22}\frac{1}{\|x_1+t_1-x_2-t_2\|}, \alpha_{21}I_2\}$$

with $\alpha_1 > \sqrt{a_1}$, $\alpha_{22} > \sqrt{a_2}$ and $\alpha_{21} > 0$, constitutes an algebraic $\bar{P}$ solution (satisfying $\Psi_{12} = 0$) for the PDEs (15) which characterise the solution of the distributed differential game. Applying the result of Theorem 1, the dynamic control strategies (25) are applied with the parameters selected as $k = 10$, $R_1 = 0.1I_2$, $R_2 = 0.1I_4$, $\alpha_1 = 10$, $\alpha_{22} = 2$, $\alpha_{21} = 2$ and $\xi(0) = \begin{bmatrix} 0 & 0 & 0 & -340 \end{bmatrix}^\top$. The initial and target positions of the two agents are such that the agents should switch positions with $p_1(0) = [0,0]^\top = t_2$ and $p_2(0) = [5,5]^\top = t_1$. The trajectories of agent 1 (dashed line) and agent 2 (solid line) are shown in Figure V. The simulation demonstrates that collision is avoided in the distributed setting considered herein by agent 2 maneuvering around agent 1.

VI. Conclusion

A system of agents, each seeking to minimise its own individual cost function subject to limited communication is considered in this paper. The available communication topology is described by a directed graph and the problem is defined as a distributed differential game. Exact solutions for the linear quadratic case and for the nonlinear case are classified. A method for constructing approximate solutions of the distributed differential game with two agents is then proposed and illustrated on a numerical example. Directions for further research include extending this method to distributed games with $N > 2$ players. It is also of interest to consider the influence of the communication topology on the existence of solutions for the game.

References

[1] T. Arai, E. Pagello, and L. E. Parker, "Editorial: Advances in multi-robot systems," *IEEE Transactions on robotics and automation*, vol. 18, no. 5, pp. 655–661, 2002.

[2] R. Olfati-Saber, "Flocking for multi-agent dynamic systems: algorithms and theory," *IEEE Transactions on Automatic Control*, vol. 51, no. 3, pp. 401–420, 2006.

[3] T. Mylvaganam and A. Astolfi, "Control of microgrids using a differential game theoretic framework," in *54th IEEE Conference on Decision and Control*, 2015, pp. 5839–5844.

[4] T. Mylvaganam and A. Astolfi, "Approximate optimal monitoring," in *14th European Control Conference*, 2014, pp. 1199–1204.

[5] T. Mylvaganam and A. Astolfi, "Approximate optimal monitoring: preliminary results," in *American Control Conference*, 2012, pp. 4745–4750.

[6] M. Ji and M. Egerstedt, "Distributed coordination control of multiagent systems while preserving connectedness," *IEEE Transactions on Robotics*, vol. 23, no. 4, pp. 693–703, 2007.

[7] M. Mesbahi and M. Egerstedt, *Graph Theoretic Methods in Multiagent Networks*, 1st ed. Princeton University Press, 2010.

[8] R. Olfati-Saber and R. M. Murray, "Distributed cooperative control of multiple vehicle formations using structural potential functions," *IFAC Proceedings Volumes*, vol. 35, no. 1, pp. 495–500, 2002.

[9] R. D'Andrea and G. E. Dullerud, "Distributed control design for spatially interconnected systems," *IEEE Transactions on Automatic Control*, vol. 48, no. 9, pp. 1478–1495, 2003.

[10] L. Bakule, "Decentralized control: An overview," *Annual Reviews in Control*, vol. 32, no. 1, pp. 87 – 98, 2008.

[11] A. Rantzer, "Using game theory for distributed control engineering," Department of Automatic Control, Lund Institute of Technology, Lund University, Tech. Rep., 2008.

[12] J. R. Marden, J. S. Shamma *et al.*, "Game theory and distributed control," *Handbook of game theory*, vol. 4, pp. 861–900, 2012.

[13] T. Mylvaganam and A. Astolfi, "Towards a systematic solution for differential games with limited communication," in *American Control Conference (ACC)*, 2016, pp. 3814–3819.

[14] W. Lin, Z. Qu, and M. A. Simaan, "A design of distributed nonzero-sum Nash strategies," in *49th IEEE Conference on Decision and Control (CDC)*, 2010, pp. 6305–6310.

[15] W. Lin, "Distributed UAV formation control using differential game approach," *Aerospace Science and Technology*, vol. 35, pp. 54 – 62, 2014.

[16] W. Lin, Z. Qu, and M. A. Simaan, "Distributed game strategy design with application to multi-agent formation control," in *53rd IEEE Conference on Decision and Control*, 2014, pp. 433–438.

[17] T. Mylvaganam, M. Sassano, and A. Astolfi, "A constructive differential game approach to collision avoidance in multi-agent systems," in *American Control Conference*, 2014, pp. 311–316.

[18] T. Mylvaganam, M. Sassano, and A. Astolfi, "A differential game approach to multi-agent collision avoidance," *IEEE Transactions on Automatic Control (In Press)*, 2017.

[19] T. Mylvaganam and A. Astolfi, "A differential game approach to formation control for a team of agents with one leader," in *American Control Conference (ACC)*, 2015, pp. 1469–1474.

[20] T. Mylvaganam, M. Sassano, and A. Astolfi, "Constructive ϵ-Nash equilibria for nonzero-sum differential games," *IEEE Transactions on Automatic Control*, vol. 60, no. 4, pp. 950–965, 2015.

[21] M. Sassano and A. Astolfi, "Dynamic approximate solutions of the HJ inequality and of the HJB equation for input-affine nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 57, pp. 2490–2503, 2012.

[22] G. Scarciotti and A. Astolfi, "Approximate finite-horizon optimal control for input-affine nonlinear systems with input constraints," *Journal of Control and Decision*, vol. 1, no. 2, pp. 149–165, 2014.

[23] M. Sassano and A. Astolfi, "Approximate finite-horizon optimal control without PDEs," *Systems and Control Letters*, vol. 62, no. 2, pp. 97–103, 2013.

[24] J. Engwerda, *LQ dynamic optimization and differential games*. Chichester: John Wiley & Sons, 2005.

[25] T. Basar and G. Olsder, *Dynamic Noncooperative Game Theory*. Academic Press, 1982.

[26] A. W. Starr and Y. C. Ho, "Nonzero-sum differential games," *Journal of Optimization Theory and Applications*, vol. 3, pp. 184–206, 1969.

[27] A. W. Starr and Y. C. Ho, "Further properties of nonzero-sum differential games," *Journal of Optimization Theory and Applications*, vol. 3, pp. 207–219, 1969.