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RESILIENCE TO CONTAGION IN FINANCIAL NETWORKS

HAMED AMINI

Swiss Finance Institute, École Polytechnique Fédérale de Lausanne

RAMA CONT

CNRS, Laboratoire de Probabilités et Modeles Aléatoires, Université Paris VI

ANDREEA MINCA

School of Operations Research and Information Engineering, Cornell University

We derive rigorous asymptotic results for the magnitude of contagion in a large counterparty network and give an analytical expression for the asymptotic fraction of defaults, in terms of network characteristics. Our results extend previous studies on contagion in random graphs to inhomogeneous-directed graphs with a given degree sequence and arbitrary distribution of weights. We introduce a criterion for the resilience of a large financial network to the insolvency of a small group of financial institutions and quantify how contagion amplifies small shocks to the network. Our results emphasize the role played by "contagious links" and show that institutions which contribute most to network instability have both large connectivity and a large fraction of contagious links. The asymptotic results show good agreement with simulations for networks with realistic sizes.

KEY WORDS: systemic risk, default contagion, random graphs, interbank network, financial stability, macroprudential regulation.

1. INTRODUCTION

Over the last decade, and especially since the financial crisis 2007–2009, systemic risk in financial markets has emerged as a major research topic. It is by now generally accepted that inadequate capital levels combined with a large degree of interconnectedness in the financial system may entail systemic instability but it remains a challenges to relate, in a precise manner, the systemic propagation of financial distress to measurable features of financial institutions and the structure of their linkages. Such understanding is crucial in order to correctly address systemic risk within the regulatory framework. Which features of financial institutions are relevant for monitoring systemic risk? Will the default of a small number of institutions propagate to a large fraction of the system, or will contagion be contained?

Network models have been considered as an interesting framework for answering some of these questions (Rochet and Tirole 1996; Allen and Gale 2000; Gai and Kapadia

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Address correspondence to Hamed Amini, Swiss Finance Institute @ EPFL, Quartier UNIL-Dorigny, Extranef 249, 1015 Lausanne, Switzerland; e-mail: hamed.amini@epfl.ch.

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2010). However, the existing theoretical literature on network models in finance has either focused on highly stylized networks, whose structure does not reflect the high degree of heterogeneity observed in interbank networks (Boss et al. 2004; Soramaki et al. 2007; Cont, Moussa, and Santos 2012) or have been focused on simulations (ElsingerLehar, Lehar, and Summer 2006; Nier et al. 2007; Cont et al. 2012) or heuristic analysis (Gai and Kapadia 2010). As a result, there is a lack of analytical results on contagion and stability for realistic network structures.

Our goal is to address these issues within an analytically tractable network model that is flexible enough to accommodate the empirical features of interbank networks. We consider a large counterparty network in which nodes represent financial institutions and each link represents an exposure, defined as the maximum loss incurred upon the default of the counterparty. In such a network, when exogenous shocks lead to the default of an institution, this leads to write-downs in balance sheets of all its counterparties. If the resulting loss exceeds the capital of an exposed counterparty, it will default, in turn, leading to contagion of defaults in the counterparty network. We investigate the magnitude of this contagion process on the network and its consequences for the resilience of the network with respect to such contagion phenomena, using asymptotic methods for large networks. Our contributions may be summarized as follows:

- *We derive asymptotic results on the size of a systemic outbreak in response to arbitrary shocks affecting banks' capital.*
  Our main contribution is to derive rigorous asymptotic results for the magnitude of contagion in financial networks and to give an analytical expression for the asymptotic fraction of defaults. Our analytic expression involves the following measurable quantities: for each node (institution) in the system, its in-degree (representing its number of creditors), its out-degree (representing its number of debtors), its sequence of exposure sizes, and its capital *after* the arbitrary shock. In order to ensure our convergence results, we make several assumptions on the empirical distributions of these quantities. These assumptions are mild enough to be verified by large sparse networks with sufficiently heterogenous degree distributions (the sparse nature of financial networks has been first pointed out by Furfine 1999, and subsequently by Bech and Atalay 2010).
  As a particular case of our model, we can consider a model in which degrees, exposure sequences, capital ratios, or shocks are random variables drawn from arbitrary distributions.
- *We identify features of a counterparty network, which determine its resilience to contagion.*
  A key insight of our work from the regulatory point of view is to show that the resilience of a counterparty network to contagion may be quantified by a quantity—which we call the *resilience measure*—whose computation only involves the distribution of connectivity across nodes (the degree distribution) and the distribution of "contagious links"—exposures that exceed a bank's capital. It may therefore be used as an assessment tool of the capital adequacy of each bank with respect to its exposures, under stress test scenarios (Amini, Cont, and Minca 2012a).
  In particular, our results show that unlike what is suggested by the analysis of simple, homogeneous networks, an increase in connectivity does not necessarily result in increased network stability, nor does it imply the opposite. Resilience to contagion and stability of realistic, heterogeneous networks are determined by the subnetwork of highly exposed counterparties, connected by "contagious links," and aggregate

measures of connectivity and size are not sufficient statistics for describing these properties.

- *We develop a probabilistic method for analyzing default cascades in financial networks.* Our method consists in constructing a weighted directed random graph with the same features as a given, observed, network. We show that with probability tending to one, as the size of the network grows to infinity, contagion on the random graph leads to the same number of defaults as the original contagion process on the financial network. The main idea of the proof consists in using a coupling argument to show that the contagion process depends on the sequence of weights only through some associated default thresholds. The threshold model of contagion on the random directed graph is easier to study because of its independence properties and, in particular, it may be described by a Markov chain. We show that as the network size increases, the rescaled Markov chain converges in probability to a fluid limit described by a system of ordinary differential equations, which can be solved in closed form. The fluid limit enables us to obtain analytical results on the final fraction of defaults in the network.

1.1. Related Literature

In the finance literature, network externalities are implicitly or explicitly present in various early discussions of systemic risk through the interlinkages between balance sheets, see, e.g., Hellwig (1995), Kiyotaki and Moore (2002) and Rochet and Tirole (1996). Allen and Gale (2000) pioneered the use of network models in the study of the stability of a system of interconnected financial institutions. The resulting insights are qualitative in nature and based on highly stylized networks.

More recently, Battiston et al. (2012) used random regular graphs models to study the impact of connectivity on financial stability. Conditions for stability of random directed financial networks with respect to contagion have been anticipated in the finance literature by Gai and Kapadia (2010). Our paper differs from these studies and improves on existing results in several ways:

First, our approach accounts for the strong heterogeneity of degrees and exposures, a feature that is underlined by empirical studies (Cont et al. 2012) but remarkably absent in the theoretical models analyzed so far (Allen and Gale 2000; Nier et al. 2007; Battiston et al. 2012; Gai and Kapadia 2010), where exposures are equally distributed across counterparties.

Second, our results generalize and provide a rigorous mathematical proof for the idea, suggested first in Watts (2002), that contagion in a large complex network can be approximated by a branching process. More importantly, if one aims to apply these results in a regulatory framework, the main limitation is the fact that the asymptotic results are given for the expected size of a cascade starting from a randomly chosen node, where expectation is taken over all random graphs with a given degree distribution. In contrast, our results represent much stronger statements: the final fraction of defaults is shown to converge in probability as the size of the network tends to infinity, to a limit that depends explicitly on the fundamentals of the model, i.e., degree sequence, interbank exposures, and capital ratios after shock.

Aside from the application to contagion in interbank networks, our results contribute to the literature on diffusion processes on random graphs. Related problems are the problem of existence of a giant component in random graphs (Cooper and Frieze 2004; Molloy and Reed 1998; Chung and Lu 2002) and the bootstrap percolation problem.

The bootstrap percolation process[1] is a diffusion model that has been studied on a variety of graphs (see, e.g., Holroyd 2003; Balogh and Bollobás 2006; Balogh and Pittel 2007; Amini 2010a, b). Our results generalize those of previous studies on contagion in random undirected graphs with prescribed degree sequences to the case of random directed graphs with prescribed degree and weight sequences. Note that while the directed version of the configuration model has been introduced in the literature (Cooper and Frieze 2004 who study the static property of existence of a giant component), our model is, to our best knowledge, the first to introduce an arbitrary sequence of weights and study the dynamic properties. Moreover, Cooper and Frieze (2004) require stronger conditions on the sequence of degrees (in particular a strong upper bound for the maximum degree), which we show that can be relaxed enough to apply in our setting. Also, while threshold contagion models have been studied in the literature by Amini (2010a) and Lelarge (2012) for undirected graphs, dealing with weights requires a rather involved coupling argument that allows us the aggregate the sequence of weights in the associated default thresholds.

1.2. Outline

The paper is structured as follows. Section 2 introduces a model for a network of financial institutions and describes a mechanism for default contagion in such a network. Section 3 gives our main result on the asymptotic magnitude of contagion. In Section 3.4, we use this result to define a measure of resilience for a financial network: We show that when this indicator of resilience crosses a threshold, small initial shocks to the network—in the form of the exogenous default of a small set of nodes—may generate a large-scale cascade of failures, a signature of *systemic risk*. Section 3.1 illustrates how the resilience measure allows us to quantify and predict the outcome of contagion on one sample network generated from a random network model that mimics the properties of a real interbank exposure network analyzed in Cont et al. (2012). We observe that networks with the same average connectivity may amplify initial shocks in very different manners and their resilience to contagion can vastly differ. In particular, the relation between "connectivity" and "contagion" is not monotonous. Technical proofs are given in Section 5.

Notations. We let $\mathbb{N}$ be the set of nonnegative integers. For nonnegative sequences x_n and y_n, we write $x_n = O(y_n)$ if there exist $N \in \mathbb{N}$ and $C > 0$ such that $x_n \leq Cy_n$ for all $n \geq N$, and $x_n = o(y_n)$, if $x_n/y_n \to 0$, as $n \to \infty$.

Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued random variables on a probability space $(\Omega, \mathbb{P})$. If $c \in \mathbb{R}$ is a constant, we write $X_n \xrightarrow{p} c$ to denote that X_n *converges in probability to* c. That is, for any $\epsilon > 0$, we have $\mathbb{P}(|X_n - c| > \epsilon) \to 0$ as $n \to \infty$.

Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers that tends to infinity as $n \to \infty$. We write $X_n = o_p(a_n)$, if $|X_n|/a_n$ *converges to 0 in probability*. Additionally, we write $X_n = O_p(a_n)$, to denote that for any positive sequence $\omega(n) \to \infty$, we have $\mathbb{P}(|X_n|/a_n \geq \omega(n)) = o(1)$. If $\mathcal{E}_n$ is a measurable subset of Ω, for any $n \in \mathbb{N}$, we say that the sequence $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ occurs *with high probability (w.h.p.)* if $\mathbb{P}(\mathcal{E}_n) = 1 - o(1)$, as $n \to \infty$.

$\text{Bin}(k, p)$ denotes a binomial distribution corresponding to the number of successes of a sequence of k independent Bernoulli trials each having probability of success p.

[1] A *bootstrap percolation process* on a graph G is an "infection" process that evolves in rounds. Initially, there is a subset of infected nodes and in each subsequent round, each uninfected node that has at least θ infected neighbors becomes infected and remains so forever. (The parameter $\theta \geq 2$ is fixed.)

TABLE 2.1
Stylized Balance Sheet of a Bank

Assets	Liabilities
Interbank assets $\sum_j e(i,j)$	Interbank liabilities $\sum_j e(j,i)$
	Deposits $D(i)$
Other assets $x(i)$	Net worth $c(i) = \gamma(i)A(i)$

2. A NETWORK MODEL OF DEFAULT CONTAGION

In this section, we introduce a model of a financial network, then describe the default cascade on this network, and finally the probabilistic setting we use throughout the paper.

2.1. Counterparty Networks

Interlinkages across balance sheets of financial institutions may be modeled by a weighted directed graph $G = (V, \mathbf{e})$ on the vertex set $V = \{1, \dots, n\} =: [n]$, whose elements represent financial institutions. The *exposure matrix* is given by $\mathbf{e} \in \mathbb{R}^{n \times n}$, where the entry $e(i, j)$ represents the exposure (in monetary units) of institution i to institution j. Table 2.1 displays a stylized balance sheet of a financial institution. The total interbank assets of an institution i are given by

$$A(i) := \sum_j e(i, j).$$

Note that $\sum_j e(j, i)$ represents the interbank liabilities of i. In addition to these interbank assets and liabilities, a bank may hold other assets and liabilities (such as deposits).

The net worth of the bank, given by its *capital*

$$c(i) = x(i) + \sum_{j \neq i} e(i, j) - \sum_{j \neq i} e(j, i) - D(i) \tag{2.1}$$

represents its capacity for absorbing losses while remaining solvent. We will refer to the ratio

$$\gamma(i) := \frac{c(i)}{A(i)}, \tag{2.2}$$

as the "capital ratio" of institution i, although technically, it is the ratio of capital to interbank assets and not total assets.

An institution is insolvent if its net worth is negative or zero, in which case we set $\gamma(i) = 0$.

DEFINITION 2.1. (Financial network). A financial network $(\mathbf{e}, \gamma)$ on the vertex set $V = [n]$ is defined by

- a matrix of exposures $\{e(i, j)\}_{1 \leq i, j \leq n}$,
- a set of capital ratios $\{\gamma(i)\}_{1 \leq i \leq n}$.

In this network, the *in-degree* of a node i is given by

$$d^{-}(i) := \#\{j \in V \mid e(j, i) > 0\},$$

which represents the number of nodes exposed to i, while its *out-degree*

$$d^{+}(i) := \#\{j \in V \mid e(i, j) > 0\}$$

represents the number of institutions i is exposed to.

The set of initially insolvent institutions is represented by

$$\mathbb{D}_0(\mathbf{e}, \gamma) = \{i \in V \mid \gamma(i) = 0\}.$$

The next section defines the default cascade triggered by nodes in $\mathbb{D}_0(\mathbf{e}, \gamma)$.

2.2. Default Contagion

In a network $(\mathbf{e}, \gamma)$ of counterparties, the default of one or several nodes may lead to the insolvency of other nodes, generating a *cascade* of defaults.

Starting from the set of initially insolvent institutions $\mathbb{D}_0(\mathbf{e}, \gamma)$ that represent *fundamental defaults*, we define a contagion process.

Denoting by $R(j)$ the recovery rate on the assets of j at default, the default of j induces a loss equal to $(1 - R(j))e(i, j)$ for its counterparty i. If this loss exceeds the capital of i, then i becomes, in turn, insolvent. Recall that $c(i) = \gamma(i)A(i)$. The set of nodes that become insolvent due to their exposures to initial defaults is

$$\mathbb{D}_1(\mathbf{e}, \gamma) = \{i \in V \mid \gamma(i)A(i) \leq \sum_{j \in \mathbb{D}_0} (1 - R(j))e(i, j)\}.$$

Iterating this procedure yields the *default cascade* initiated by a set of initial defaults:

DEFINITION 2.2. (Default cascade). Consider a financial network $(\mathbf{e}, \gamma)$ on the vertex set $V = [n]$. Set $\mathbb{D}_0(\mathbf{e}, \gamma) = \{i \in V \mid \gamma(i) = 0\}$ the set of initially insolvent institutions. The nondecreasing sequence $\mathbb{D}_0(\mathbf{e}, \gamma) \subseteq \mathbb{D}_1(\mathbf{e}, \gamma) \subseteq \ldots \subset \mathbb{D}_{n-1}(\mathbf{e}, \gamma) \subseteq V$ defined by

$$\mathbb{D}_k(\mathbf{e}, \gamma) = \{i \in V \mid \gamma(i)A(i) \leq \sum_{j \in \mathbb{D}_{k-1}(\mathbf{e}, \gamma)} (1 - R(j))e(i, j)\} \tag{2.3}$$

is called the *default cascade* initiated by $\mathbb{D}_0(\mathbf{e}, \gamma)$.

Thus, $\mathbb{D}_k(\mathbf{e}, \gamma)$ represents the set of institutions whose capital is insufficient to absorb losses due to defaults of institutions in $\mathbb{D}_{k-1}(\mathbf{e}, \gamma)$. Note that the losses caused by vertices in D_i for $i < k - 1$ are automatically included in the definition of D_k since by construction, the sequence D_i is a nondecreasing sequence of subsets of V.

It is easy to see that in a network of size n, the cascade ends after at most $n - 1$ iterations. Hence, $\mathbb{D}_{n-1}(e, \gamma)$ represents the set of all nodes that become insolvent starting from the initial set of defaults $\mathbb{D}_0(e, \gamma)$.

DEFINITION 2.3. Consider a financial network $(\mathbf{e}, \gamma)$ on the vertex set $V = [n]$. The *fraction of defaults* in the network $(\mathbf{e}, \gamma)$ (initiated by $\mathbb{D}_0(\mathbf{e}, \gamma)$) is given by

$$\alpha_n(\mathbf{e}, \gamma) := \frac{|\mathbb{D}_{n-1}(\mathbf{e}, \gamma)|}{n}. \tag{2.4}$$

The recovery rates $R(i)$ may be exogenous or, as in Eisenberg and Noe (2001), determined endogenously by redistributing assets of a defaulted entity among debtors, proportionally to their outstanding debt. As noted in Upper (2011) and Cont et al. (2012), the latter scenario is too optimistic since, in practice, liquidation takes time and assets may depreciate in value due to fire sales during liquidation. As argued in Cont et al. (2012) and Elsinger et al. (2006), when examining the short-term consequences of default, the most realistic assumption on recovery rates is zero: Assets held with a defaulted counterparty are frozen until liquidation takes place, a process that can, in practice, take months to terminate.

We assume in the sequel that recovery rates are constant for all institutions:
$R(i) = R$ *for all* $i \in [n]$.

2.3. A Random Network Model

Empirical studies on interbank exposures, see, e.g., Boss et al. (2004) and Cont et al. (2012), show such networks to have a complex and heterogeneous structure characterized by heavy-tailed (cross-sectional) distributions of degrees and exposures.

Given a description of the large-scale structure of the network in statistical terms, it is natural to model the network as a *random graph* whose statistical properties correspond to these observations.

Consider a sequence $(\mathbf{e}_n, \gamma_n)_{n\geq 1}$ of financial networks, indexed by the number of nodes n, where $\mathbf{d}_n^+ = \{d_n^+(i)\}_{i=1}^n$ (respectively, $\mathbf{d}_n^- = \{d_n^-(i)\}_{i=1}^n$) represents the sequence of out-degrees (respectively, in-degrees) of nodes in $\mathbf{e}_n$. We now construct a random network $\mathbf{E}_n$ such that $\mathbf{e}_n$ may be considered as a typical sample of $\mathbf{E}_n$.

DEFINITION 2.4. (Random network ensemble). Let $\mathcal{G}_n(\mathbf{e}_n)$ be the set of all weighted directed graphs with degree sequence $\mathbf{d}_n^+, \mathbf{d}_n^-$ such that for any node i, the set of exposures is given by the nonzero elements of line i in the exposure matrix $\mathbf{e}_n$. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. We define $\mathbf{E}_n : \Omega \to \mathcal{G}_n(\mathbf{e_n})$ as a random directed graph uniformly distributed on $\mathcal{G}_n(\mathbf{e}_n)$.

We endow the nodes in $\mathbf{E}_n$ with the capital ratios γ_n. Then, for all $i = 1, \dots, n$,

$$\{E_n(i,j), \quad E_n(i,j) > 0\} = \{e_n(i,j), \quad e_n(i,j) > 0\} \quad \mathbb{P} - a.s.,$$

$$\#\{j \in V,\ E_n(i,j) > 0\} = d_n^+(j) \quad \text{and} \quad \#\{j \in V,\ E_n(j,i) > 0\} = d_n^-(i).$$

Definition 2.4. is equivalent to the representation of the financial system by an unweighted graph chosen uniformly among all graphs with the degree sequence $(\mathbf{d}_n^+, \mathbf{d}_n^-)$, in which we assign to the links going out of node i the set of weights $\{e_n(i,j) > 0\}$.

3. ASYMPTOTIC RESULTS

We consider a sequence of random financial networks as introduced above. Our goal is to study the behavior of $\alpha_n(\mathbf{E}_n, \gamma_n)$ which represents the final fraction of defaults in the cascade generated by the set of initially insolvent institutions, i.e., $\mathbb{D}_0(\mathbf{E}_n, \gamma_n) = \{i \in [n] \mid \gamma_n(i) = 0\}$.

3.1 Assumptions

Consider a sequence $(\mathbf{e}_n, \gamma_n)_{n\geq 1}$ of financial networks, indexed by the number of nodes n. Let m_n denote the total number of links in the network $\mathbf{e}_n$:

$$m_n := \sum_{i=1}^{n} d_n^+(i) = \sum_{i=1}^{n} d_n^-(i).$$

The *empirical distribution of the degrees* is defined by

$$\mu_n(j,k) := \frac{1}{n}\#\{i \in [n] \mid \; d_n^+(i) = j, d_n^-(i) = k\}.$$

From now on, we assume that the degree sequences $\mathbf{d}_n^+$ and $\mathbf{d}_n^-$ satisfy the following conditions analogous to the ones introduced in Molloy and Reed (1998).

ASSUMPTION 3.1. *For each $n \in \mathbb{N}$, $\mathbf{d}_n^+ = \{(d_n^+(i))_{i=1}^n\}$ and $\mathbf{d}_n^- = \{(d_n^-(i))_{i=1}^n\}$ are sequences of nonnegative integers with $\sum_{i=1}^n d_n^+(i) = \sum_{i=1}^n d_n^-(i)$, and such that for some probability distribution μ on $\mathbb{N}^2$ independent of n and with finite mean $\lambda := \sum_{j,k} j\mu(j,k) = \sum_{j,k} k\mu(j,k) \in (0,\infty)$, the following holds:*

1. *$\mu_n(j,k) \to \mu(j,k)$ for every $j,k \geq 0$ as $n \to \infty$;*
2. *$\sum_{i=1}^n (d_n^+(i))^2 + (d_n^-(i))^2 = O(n)$.*

In particular, the second assumption implies (by uniform integrability) that $m_n/n \to \lambda$, as $n \to \infty$. This assumption is reasonable for empirical data sets of banking networks; for example, Cont et al. (2012) find that in the Brazilian banking system, $m_n/n \simeq 7$.

We now present our assumptions on the exposures. Let us denote by $\Sigma_n(i)$ the set of all permutations of the counterparties of i in the network $\mathbf{e}_n$, i.e., permutations of the set $\{j \in [n] \mid \; e_n(i,j) > 0\}$. For the purpose of studying contagion, the role of exposures and capital ratios may be expressed in terms of *default thresholds* for each node.

DEFINITION 3.2. (Default threshold). For a node i and permutation $\tau_n \in \Sigma_n(i)$ that specifies the order in which i's counterparties default, the default threshold

$$\Theta_n(i,\tau_n) := \min\{k \geq 0 \mid \; \gamma_n(i)\sum_{j=1}^{n} e_n(i,j) \leq (1-R)\sum_{j=1}^{k} e_n(i,\tau_n(j))\} \tag{3.1}$$

measures how many counterparty defaults i can tolerate before it becomes insolvent (in the financial network $(\mathbf{e}_n, \gamma_n)$), if its counterparties default in the order specified by τ_n.

We also define

(3.2)

$$p_n(j,k,\theta) := \frac{\#\{(i,\tau_n) \mid \; i \in [n], \; \tau_n \in \Sigma_n(i), \; d_n^+(i) = j, \; d_n^-(i) = k, \; \Theta_n(i,\tau_n) = \theta\}}{n\mu_n(j,k)j!}.$$

We will see in Section 5.2 that for large n, $p_n(j,k,\theta)$ gives the fraction of nodes with degree (j,k) that become insolvent after θ of their counterparties default, in the random financial network $\mathbf{E}_n$.

As we will see in Section 3.4, exposures whose magnitude exceeds the capital of the exposed counterparty play an important role in contagion: We call them *contagious links*.

DEFINITION 3.3. (Contagious link). We call a link $i \to j$ *contagious* if it represents an exposure larger than the capital of the exposed node:

$$(1-R)e_n(i,j) > c_n(i) = \gamma_n(i)\sum_{j=1}^{n} e_n(i,j).$$

In particular, $p_n(j,k,1)$ is the proportion of contagious exposures of nodes with degree (j,k).

In the following, we assume that $p_n(j,k,\theta)$ has a limit when $n \to \infty$:

ASSUMPTION 3.4. *There exists a function* $p : \mathbb{N}^3 \to [0,1]$ *such that for all*

$$\forall (j,k,\theta) \in \mathbb{N}^3 \ (\theta \le j): \quad p_n(j,k,\theta) \to p(j,k,\theta) \ \textit{ as } n \to \infty.$$

Some examples of exposures for which this assumption is fulfilled are given in Section 3.2. Under this assumption, we will see in Section 5 that $p(j,k,\theta)$ is also the limit in probability of the fraction of nodes with degree (j,k) that become insolvent after θ of their counterparties default. In particular,

- $p(j,k,0)$ represents the proportion of initially insolvent nodes with degree (j,k);
- $p(j,k,1)$ represents the proportion of nodes with degree (j,k) which are "vulnerable," i.e., may become insolvent due to the default of a single counterparty.

We now present some examples of models for counterparty networks that satisfy Assumption 3.4.

3.2. Examples of Networks That Satisfy Assumption 3.4

In this section, we give two important examples of exposures that satisfy Assumption 3.4.

EXAMPLE 3.5. (Independent exposures). Assume that for all n, the exposures of all nodes $i \in [n]$ with the same degree (j,k), i.e.,

$$\left\{e_n(i,l) > 0 \mid d_n^+(i) = j, d_n^-(i) = k\right\},$$

are independent and identically distributed (i.i.d.) random variables, with a law given by $F_X(j,k)$, depending on j and k but independent of n. We assume the same for the sequence of capital ratios, i.e., $\{\gamma_n(i) \mid d_n^+(i) = j, d_n^-(i) = k\}$ are i.i.d. random variables with a law given by $F_\gamma(j,k)$ which may depend on (j,k), but not on n. Then, it is easy to see that by the law of large numbers, Assumption 3.4 holds and the limit $p(j,k,\theta)$ is known (for all j, k, θ),

$$p(j,k,\theta) = \mathbb{P}\left((1-R)X_\theta > \gamma \sum_{l=1}^{j} X_l - (1-R)\sum_{l=1}^{\theta-1} X_l > 0\right),$$

where γ is a random variable with law $F_\gamma(j,k)$, and $(X_l)_{l=1}^{j}$ are i.i.d. random variables with law $F_X(j,k)$ and independent of γ.

EXAMPLE 3.6. (Exchangeable exposures). Empirical observations of banking networks, see, e.g., Cont et al. (2012), Boss et al. (2004), Soramaki et al. (2007) and May and Arinaminpathy (2010), show that they are hierarchical, "disassortative" networks, with a few large and highly interconnected dealer banks and many small banks, connected predominantly to dealer banks. This can be modeled in a stylized way by partitioning the set of nodes into two sets, a set $\mathcal{D}$ of n^D dealer banks, and a set $\mathcal{N}$ of n^N nondealer banks.

We assume that the exposures $\{e_n(i,l) > 0 \mid i \in \mathcal{D}\}$ and $\{e_n(i,l) > 0 \mid i \in \mathcal{N}\}$ are restrictions corresponding to the first m_n^D (respectively, m_n^N) elements of infinite sequences of exchangeable variables, where m_n^D and m_n^N denote the total number of exposures (links) belonging to dealer and, respectively, nondealer banks. Similarly, the capital ratios $\{\gamma_n(i) \mid i \in \mathcal{D}\}$ and $\{\gamma_n(i) \mid i \in \mathcal{N}\}$ are restrictions to the first n^D (respectively, n^N) elements of the sequence, independent of the sequence of exposures.

We can extend this example to a finite number of classes of nodes represented by their degrees, and also drop the assumption of independence between exposures and capital ratios. We assume that within each class, the sequence of a node's exposures and capital ratios are exchangeable random variables.

For each node i with $d_n^+(i) = j$, $d_n^-(i) = k$, we let

$$Y_n(i) := (\{e_n(i,\cdot) > 0\}, \gamma_n(i))$$

be a multivariate random variable with state space $\Im^{j,k} \subset \mathbb{R}_+^j \otimes \mathbb{R}$. We assume that the law of the finite sequence

$$\{Y_n(i) \mid i \in [n], d_n^+(i) = j, d_n^-(i) = k\}$$

is invariant under permutation.

Then, the family $\{Y_n(i) \mid i \in [n], d_n^+(i) = j, d_n^-(i) = k\}_{0 \le j,k \le M}$ represents a family of finite multiexchangeable systems, as defined in Graham (2008). It is proved in Graham (2008) that the conditional law of a finite multiclass system, given the value of the vector of the empirical measures of its classes, corresponds to independent uniform orderings of the samples within each class. A family of such systems converges in law if and only if the corresponding empirical measure vectors converge in law.

Let us consider the empirical measure sequence

$$\left\{\Lambda_n^{j,k} := \frac{\sum_i \mathbb{1}_{\{d_n^+(i)=j,\ d_n^-(i)=k\}} \delta_{Y_n(i)}}{n\mu_n(j,k)}\right\}_{0\le j,k\le M}.$$

We suppose that the family $\{Y_n(i) \mid i \in [n], d_n^+(i) = j, d_n^-(i) = k\}_{0\le j,k\le M}$ converges in law when $n \to \infty$ to an infinite multiexchangeable system

$$\lim_{n\to\infty} \{Y_n(i) \mid i \in [n], d_n^+(i) = j, d_n^-(i) = k\}_{0\le j,k\le M} \stackrel{\mathcal{L}}{=} \left\{Z_l^{j,k} \mid l \ge 1\right\}_{0\le j,k\le M}. \tag{3.3}$$

By theorem 2 of Graham (2008), the empirical measure converges in law to

$$\lim_{n\to\infty} \left\{\Lambda_n^{j,k}\right\}_{0\le j,k\le M} \overset{\mathcal{L}}{=} \left\{\Lambda^{j,k}\right\}_{0\le j,k\le M}. \tag{3.4}$$

For an arbitrary $Z \in \Im^{j,k}$, we define the function

$$h(Z,\theta) = \frac{\#\{\tau \mid \tau \in \Sigma(j),\, \Theta(Z,\tau) = \theta\}}{j!}.$$

Thus, by Equation (3.4) giving the convergence of empirical measures and the fact that the function h is bounded, we have

$$p_n(j,k,\theta) = \mathbb{E}^{\Lambda_n^{j,k}}(h(\mathbf{Z},\theta)) \overset{n\to\infty}{\longrightarrow} \mathbb{E}^{\Lambda^{j,k}}(h(\mathbf{Z},\theta)) = p(j,k,\theta),$$

with $\mathbf{Z}$ a random element of $\Im^{j,k}$ and $\mathbb{E}^{\Lambda_n^{j,k}}$ and $\mathbb{E}^{\Lambda^{j,k}}$ denoting expectation under the measures $\Lambda_n^{j,k}$ and $\Lambda^{j,k}$, respectively. A last observation is that Equation (3.3) is verified in our two tiered example since the sequences of exposures in the network of size n are restrictions of infinite exchangeable sequences.

3.3. The Asymptotic Magnitude of Contagion

Consider a sequence $(\mathbf{e}_n, \gamma_n)_{n\ge1}$ of financial networks satisfying Assumptions 3.1 and 3.4, and let $(\mathbf{E}_n, \gamma_n)_{n\ge1}$ be their corresponding sequence of random financial networks, see Definition 2.4. Let us denote by

$$\beta(j,\pi,\theta) := \mathbb{P}(\mathrm{Bin}(j,\pi) \ge \theta) = \sum_{l\ge\theta}^{j} \binom{j}{l} \pi^l (1-\pi)^{j-l},$$

the distribution function of a binomial random variable $\mathrm{Bin}(j,\pi)$ with parameters j and π.

We define the function $I : [0,1] \to [0,1]$ as

$$I(\pi) := \sum_{j,k} \frac{\mu(j,k)k}{\lambda} \sum_{\theta=0}^{j} p(j,k,\theta)\beta(j,\pi,\theta). \tag{3.5}$$

Indeed, $I(\pi)$ has the following interpretation (when the network size goes to infinity): If the end node of a randomly chosen edge defaults with probability π independent of everything else, $I(\pi)$ is the expected fraction of counterparty defaults after one iteration of the cascade. Clearly, in our model, obligors do default independently, but due to either the initial shock or to network effects. The proof that a law of large numbers holds for the contagion process is involved and deferred to Section 5.

Let π^* be the smallest fixed point of I:

$$\pi^* = \inf\{\pi \in [0,1] \mid I(\pi) = \pi\}.$$

The value π^* represents the probability that an edge taken at random ends in a defaulted node, at the end of the contagion process.

REMARK 3.7. I admits at least one fixed point. Indeed, I is a continuous increasing function and

$$I(1) = \sum_{j,k} \frac{\mu(j,k)k}{\lambda} \sum_{\theta=0}^{j} p(j,k,\theta) \le 1,$$

since $\sum_\theta p(j,k,\theta) \le 1$ by definition. Moreover,

$$I(0) = \sum_{j,k} \frac{\mu(j,k)k}{\lambda} p(j,k,0) \ge 0.$$

So, the function I has at least a fixed point in [0, 1].

We can now announce our main theorem.

THEOREM 3.8. *Consider a sequence* $(\mathbf{e}_n, \gamma_n)_{n\ge 1}$ *of financial networks satisfying Assumptions 3.1 and 3.4, and the corresponding sequence of random matrices* $(\mathbf{E}_n)_{n\ge 1}$ *defined on* $(\Omega, \mathcal{A}, \mathbb{P})$ *as in Definition 2.4. Let* π^* *be the smallest fixed point of* I *in* [0, 1].

1. *If* $\pi^* = 1$, *i.e., if* $I(\pi) > \pi$ *for all* $\pi \in [0, 1)$, *then asymptotically almost all nodes default during the cascades*

$$\alpha_n(\mathbf{E}_n, \gamma_n) \xrightarrow{p} 1.$$

2. *If* $\pi^* < 1$ *and furthermore* π^* *is a stable fixed point of* I, *i.e.,* $I'(\pi^*) < 1$, *then the asymptotic fraction of defaults is given by*

$$\alpha_n(\mathbf{E}_n, \gamma_n) \xrightarrow{p} \sum_{j,k} \mu(j,k) \sum_{\theta=0}^{j} p(j,k,\theta)\beta(j,\pi^*,\theta).$$

A proof of this theorem is given in Section 5.5.

3.4. Resilience to Contagion

The resilience of a network to small shocks is a global property of the network that depends on its detailed structure. However, the above results allow us to introduce a rather simple and easy to compute indicator for the resilience of a network to small shocks. Consider a sequence $(\mathbf{e}_n, \gamma_n)_{n\ge 1}$ of financial networks satisfying Assumptions 3.1 and 3.4.

DEFINITION 3.9. (Network resilience measure). We define the *network resilience* measure as

$$1 - \sum_{j,k} \frac{jk}{\lambda} \mu(j,k) p(j,k,1) \in (-\infty, 1].$$

The term $\sum_{j,k} \frac{jk}{\lambda} \mu(j,k) p(j,k,1)$ represents the *susceptibility* of the network (see Amini et al. 2012a). Note that for computing $p(k, j, 1)$, we look at each node with degree (j, k) and count the number of exposures that exceed its capital. For realistic networks, this is a small subset of nodes so that the computation is quite simple; we need to iterate over all exposures (links).

The following result, which is a consequence of Theorem 3.8, shows that this indicator measures the resilience of a network to the initial default of a small fraction of the nodes:

THEOREM 3.10. *Consider a sequence* $(\mathbf{e}_n, \gamma_n)_{n\geq 1}$ *of counterparty networks satisfying Assumptions 3.1 and 3.4. If*

$$1 - \sum_{j,k} \frac{jk}{\lambda} \mu(j,k) p(j,k,1) > 0, \tag{3.6}$$

then with high probability, as the fraction of initial defaults tends to zero, the final fraction of defaults is negligible: Formally, for every $\epsilon > 0$, *there exists* ρ_ϵ *such that if the initial fraction of defaults is smaller than* ρ_ϵ, *then the final fraction of defaults is smaller that* ϵ *with high probability:*

$$[\exists N_\epsilon > 0, \forall n > N_\epsilon, \frac{|\{i \in [n] \mid \gamma_n(i) = 0\}|}{n} < \rho_\epsilon] \Rightarrow [\forall n \geq N_\epsilon, \\ \mathbb{P}(\alpha_n(\mathbf{E}_n, \gamma_n) \leq \epsilon) > 1 - \epsilon].$$

Given the network topology, Condition (3.6) sets limits on the fraction of contagious links $p_n(j,k,1)$, i.e., on the magnitude of exposures relative to capital. The proof of the above theorem, given in Section 5.6, is based on the following observation: If contagion does not spread to nodes with contagion threshold one, then it will not spread at all. This provides an upper bound on the extent of contagion.

THEOREM 3.11. *Consider a sequence* $(\mathbf{e}_n, \gamma_n)_{n\geq 1}$ *of financial networks satisfying Assumptions 3.1 and 3.4. If*

$$1 - \sum_{j,k} \frac{\mu(j,k) jk}{\lambda} p(j,k,1) < 0, \tag{3.7}$$

then with high probability, there exists a set of nodes representing a positive fraction of the financial system, strongly interlinked by contagious links (i.e., there is a directed path of contagious links from any node to another in the component) such that any node belonging to this set can trigger the default of all nodes in the set.

A proof of this theorem is given in Section 5.7. However, this may be justified using the following heuristic argument.

REMARK 3.12. (Branching process approximation). We describe an approximation of the local structure of the graph by a branching process, the children being the in-coming neighbors: the root ϕ with probability $\mu^-(k_\phi) := \sum_j \mu(j,k_\phi)$ has an in-degree equal to k_ϕ. Each of these k_ϕ vertices with probability $\frac{\mu(j,k)j}{\lambda}$ has degree (j,k), and with probability equal to $p(j,k,1)$ default when their parent defaults. Let y be the extinction probability, given by the smallest solution of

$$y = \sum_{j,k} \frac{\mu(j,k) j}{\lambda} p(j,k,1) y^k. \tag{3.8}$$

If $\sum_{j,k} \frac{\mu(j,k)jk}{\lambda} p(j,k,1) < 1$, then the smallest solution of (3.8) is $y = 1$ (the population dies out with probability one), whereas if

$$\sum_{j,k} \frac{\mu(j,k)jk}{\lambda} p(j,k,1) > 1,$$

there is a unique solution with $y \in (0,1)$.

REMARK 3.13. (Amplification as a function of network susceptibility and connectivity of the initial default). We suppose that the resilience condition (3.6) is satisfied. Let π_ϵ^* be the smallest fixed point of I in $[0,1]$, when a fraction ϵ of all nodes represents fundamental defaults, i.e., $p(j,k,0) = \epsilon$ for all j,k.

We obtain then, by a first-order approximation of the function I, that

$$\pi_\epsilon^* = \frac{\epsilon}{1 - \sum_{j,k} \frac{\mu(j,k)jk}{\lambda} p(j,k,1)} + o(\epsilon).$$

By a first-order approximation of the function $\pi \to \sum_{j,k} \mu(j,k) \sum_{\theta=0}^{j} p(j,k,\theta)\beta(j,\pi,\theta)$ giving the asymptotic fraction of defaults in Theorem 3.8, we obtain that for any ρ, there exists ϵ_ρ and n_ρ such that for all $\epsilon < \epsilon_\rho$ and $n > n_\rho$

$$\mathbb{P}\left(|\alpha_n(\mathbf{E_n}, \gamma_\mathbf{n}) - \epsilon\left(1 + \frac{\sum_{j,k} j\mu(j,k)p(j,k,1)}{1 - \sum_{j,k} \frac{\mu(j,k)jk}{\lambda} p(j,k,1)}\right)| < \rho\right) > 1 - \rho. \tag{3.9}$$

Suppose now that initially, insolvent fraction involves only nodes with degree (d^+, d^-), and we denote $\pi_\epsilon^*(d^+, d^-)$ the smallest fixed point of I in $[0,1]$ in the case where $p(d^+, d^-, 0) = \epsilon$ and $p(j,k,0) = 0$ for all $(j,k) \neq (d^+, d^-)$. Then, we obtain that for any ρ, there exists ϵ_ρ and n_ρ such that for all $\epsilon < \epsilon_\rho$ and $n > n_\rho$,

$$\mathbb{P}\left(|\alpha_n(\mathbf{E_n}, \gamma_\mathbf{n}) - \epsilon\mu(d^+, d^-)\left(1 + \frac{d^-}{\lambda} \frac{\sum_{j,k} \frac{\mu(j,k)jk}{\lambda} p(j,k,1)}{1 - \sum_{j,k} \frac{\mu(j,k)jk}{\lambda} p(j,k,1)}\right)| < \rho\right) > 1 - \rho. \tag{3.10}$$

This simple expression shows that there are basically two factors that determine how small initial shocks are amplified by the financial network: the interconnectedness of the initial default (represented by its in-degree d^-) and the susceptibility of the network, i.e., $\sum_{j,k} \frac{\mu(j,k)jk}{\lambda} p(j,k,1)$.

4. NUMERICAL EXPERIMENTS WITH FINITE NETWORKS

The results of Section 3 hold in the limit of large network size. In order to assess whether these results still hold for networks whose size is large but finite, we now compare our theoretical results with numerical simulations for networks with realistic sizes. In particular, we investigate the effect of heterogeneity in network structure and the relation between resilience and connectivity.

For a given network of exposures $(\mathbf{e}, \gamma)$ of size n, one can define the following sample equivalent of the resilience measure (Definition 3.9), which we call the *empirical resilience* *measure* (Amini et al. 2012a):

$$1 - \frac{1}{m_n} \sum_i d^-(i) q(i), \tag{4.1}$$

where m_n is the total number of links in the network and $q(i)$ the number of contagious exposures of node i and $d^-(i)$ its in-degree.

4.1. Relevance of Asymptotics

Interbank networks in developed countries may contain several thousands of nodes. The Federal Deposit Insurance Corporation insured 7,969 institutions as of March 18, 2010, while the European Central Bank reports 8,350 monetary financial institutions in the Euro zone (80% credit institutions and 20% money market funds).

To assess the relevance of asymptotic formulae for studying contagion in networks with such sizes, we generate a scale-free network of 10,000 nodes with Pareto distributed exposures using the random graph model introduced by Blanchard, Chang, and Krüger (2003), which can be seen as a static version of the preferential attachment model. This model yields a network with statistical properties that are similar to the empirically observed properties of banking networks as described in Boss et al. (2004) for the Austrian network and Cont et al. (2012) for the Brazilian network. In this model, given the sequence of out-degrees, an arbitrary outgoing edge is assigned to an end-node i with probability proportional to the power $d_n^+(i)^\alpha$ where $\alpha > 0$. This leads to positive correlation between in-degrees and out-degrees.

The distribution of the out-degree in this model is a Pareto law with tail exponent γ^+:

$$\mu_n^+(j) := \#\{i \mid d_n^+(i) = j\} \stackrel{n\to\infty}{\to} \mu^+(j) \sim j^{\gamma^+ + 1},$$

and the conditional limit law of the in-degree is a Poisson distribution

$$P(d^- = k | d^+ = j) = e^{-\lambda(j)} \frac{\lambda(j)^k}{k!},$$

with $\lambda(j) = \frac{j^\alpha \mathbb{E}(D^+)}{\mathbb{E}((D^+)^\alpha)}$, where D^+ denotes a random variable with law μ^+. The main theorem in Blanchard et al. (2003) states that the marginal distribution of the in-degree has a Pareto tail with exponent $\gamma^- = \frac{\gamma^+}{\alpha}$, provided $1 \le \alpha < \gamma^+$.

Throughout this section, the capital ratio is assumed to be bounded from below by a minimal capital ratio and, we consider the worst case scenario where all nodes have a capital ratio equal to the minimal capital ratio: $\gamma(i) = \gamma_{min},\ \forall i \in V$. The distribution of this simulated network's degrees and exposures is given in Figure 4.1 and is based on the empirical analysis of the Brazilian network in June 2007; see Cont et al. (2012).

On the one hand, we make a simulation of the default contagion starting with a random set of defaults representing 0.1% of all nodes (chosen uniformly among all nodes). On the other hand, we plug the empirical distribution of the degrees and the fraction of contagious links into equation (3.9) for the amplification of a small number of initial defaults. Figure 4.2 plots the amplification for varying values of the minimal capital ratios. We obtain a good agreement between the theoretical and the simulated default amplification ratios. Moreover, we can clearly see that there exists a critical value

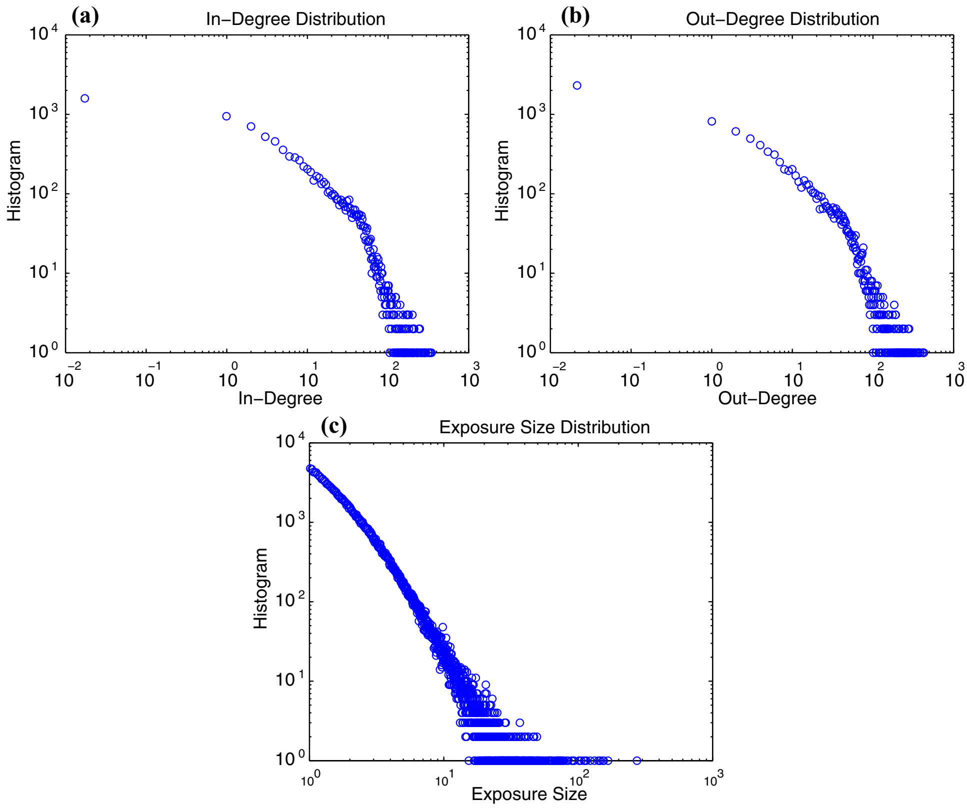

FIGURE 4.1. (a) The distribution of in-degree has a Pareto tail with exponent 2.19. (b) The distribution of the out-degree has a Pareto tail with exponent 1.98. (c) The distribution of the exposures (tail exponent 2.61).

$\gamma^*_{\min}$ for the minimal capital ratio such that if $\gamma_{\min} < \gamma^*_{\min}$, the simulated amplification explodes. This critical value corresponds to the value for which the resilience measure in Definition 3.9 becomes negative and consequently the theoretical amplification tends to ∞.

Figure 4.3 plots the simulated fraction of defaults in a scale-free network, starting from the initial default of a single node as a function of the in-degree of the defaulting node, versus the theoretical amplification given in equation (3.10). Recall that the theoretical amplification scales linearly with the in-degree and the slope is given by $\sum_{j,k} \frac{\mu(j,k)jk}{\lambda} p(j,k,1)$, which represents the susceptibility of the network to initial defaults.

The agreement is good for nodes with large in-degrees, demonstrating large amplification if such nodes were to default in a network with high susceptibility.

4.2. The Impact of Heterogeneity

In Figure 4.4, we compare the amplification of the number of defaults in three cases: a scale-free network with heterogeneous weights (exposures), a scale-free network with equal weights, and a "homogeneous" random network (the Erdős–Rényi random graph

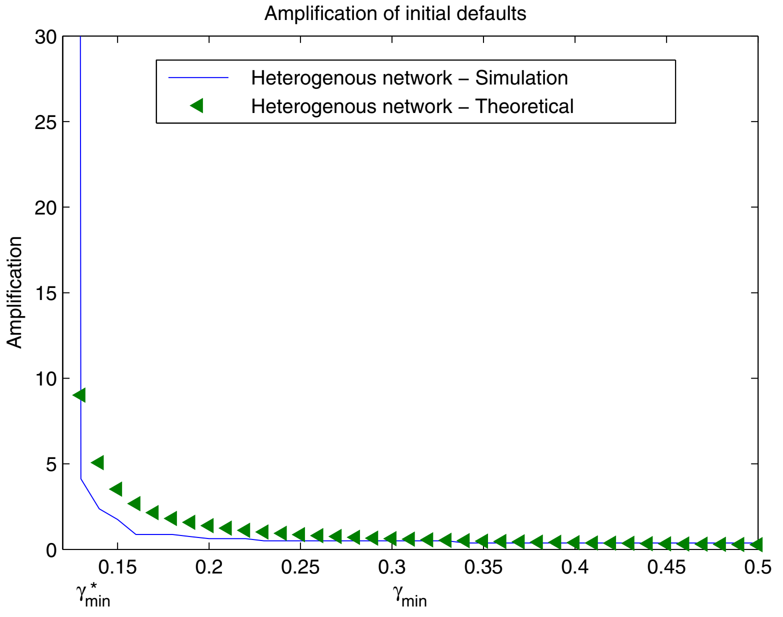

FIGURE 4.2. Amplification of the number of defaults in a scale-free network. The in- and out-degrees are Pareto distributed with tail coefficients 2.19 and 1.98, respectively, and the exposures are Pareto distributed with tail coefficient 2.61, $n = 10{,}000$.

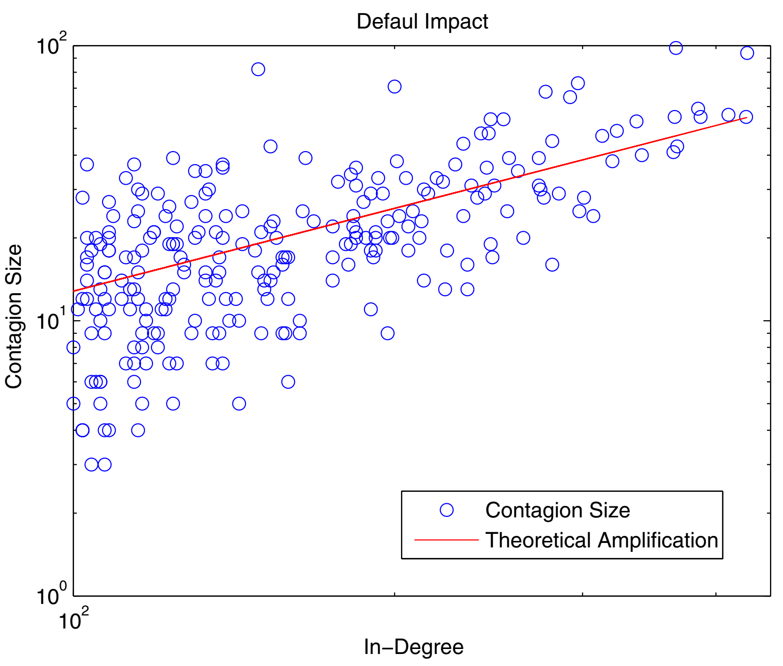

FIGURE 4.3. Number of defaulted nodes in a scale-free network resulting from one initial default. The in- and out-degree are Pareto distributed with tail coefficients 2.19 and 1.98, respectively, and the exposures are Pareto distributed with tail coefficient 2.61, $n = 10{,}000$.

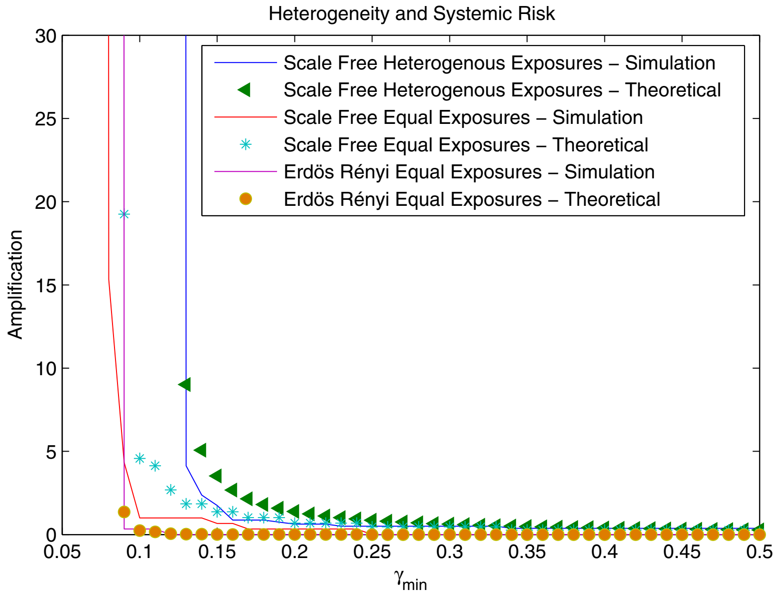

FIGURE 4.4. Amplification of the number of defaults in a scale-free network (in- and out-degrees of the scale-free network are Pareto distributed with tail coefficients 2.19 and 1.98, respectively, and the exposures are Pareto distributed with tail coefficient 2.61), the same network with equal weights and an Erdös Rényi Network with equal exposures $n = 10{,}000$.

where every directed edge is present with a fixed probability) with equal weights. All three networks have the same average degree, i.e., the same total number of links.

We vary $\gamma_{\min}$ from 0.08% to 0.5%. Recall that γ is not the actual capital ratio as defined by regulators, but, in fact, the ratio of capital to interbank assets. For most large US and European banks, interbank assets are approximately 20% of total assets, so the range $\gamma > 0.15$ corresponds, in fact, to a capital ratio greater than 3%. Basel II, by comparison, considers a ratio of 5% and Basel III 8%, so the range $\gamma_{\min} > 0.15$ is quite realistic.

It is interesting to note that in our example, the most heterogeneous network is also the least resilient, as opposed to the homogeneous Erdös–Rényi network with the same distribution of exposures. These simulations corroborate the crucial role played both by the network topology and the heterogeneity of weights.

4.3. Does Connectivity Increase or Decrease Resilience to Contagion?

One of the recurrent questions in the economics literature regards the impact of connectivity on resilience to contagion: Is increased global connectivity posing a threat to financial system or does it allow for more efficient risk sharing? While an influential paper by Allen and Gale (2000) finds that resilience increases with connectivity, Battiston et al. (2012) exhibit different model settings where this relation is nonmonotonous. Our results show nonetheless that in a model with heterogenous structure, the average connectivity is too simple a summary statistic of the network topology to explain resilience. To this end, we consider a simple example and use the resilience measure. We take networks in which nodes' exposures are equal and $1/3 \leq \gamma_{\min} < 1/2$ such that $p_n(j,k,\theta) = \mathbb{1}_{j\in\{1,2\}}$. The heterogeneity comes in our example only from the degree distribution. We consider three cases:

First, let $\tilde{\mu}(1,2) = 2/3$ and $\tilde{\mu}(4,2) = 1/3$. The average connectivity is 2 and the resilience measure is equal to $1/3$.

Second, let $\mu(1,3) = \mu(2,3) = \mu(4,3) = \mu(5,3) = 1/4$. The average connectivity in a network with this degree distribution is 3 and the resilience measure is equal to $1/4$.

Last, we take $\hat{\mu}(4,4) = 1$, i.e., a regular graph with degree 4. Here, the average connectivity is 4 and the resilience measure is 1.

In all three cases, a network constructed with the empirical degree distribution is resilient (w.h.p.). Nonetheless, we clearly observe that the resilience measure does not depend on the average connectivity in a monotonous way. While in the case of Battiston et al. (2012), this nonmonotonicity is due to a mechanism of "financial accelerators," which enacts on top of the network contagion effects, in our case, the trade-off between risk-sharing and contagion is inherent in the network topology. These examples show that the resilience of a real network cannot be assessed by examining a naive aggregate measure of connectivity such as the average degree or the number of links, as sometimes suggested in the literature, but requires a closer examination of features such as the distribution of degrees and the structure of the subgraph of contagious links.

5. PROOFS

In this section, we present the proofs of Theorems 3.8, 3.10, and 3.11. We begin by introducing a weighted configuration model—a multigraph related to the financial network—which has the same asymptotic behavior as the random financial network. We then show by a coupling argument that the default cluster in the weighted configuration model can be constructed sequentially. The contagion process in this sequential model may then be described by a Markov chain. We then use the generalized version of the differential equation method of Wormald (1995) to the case where the dimension of the Markov chain depends on size of the network and we show that as the network size increases, the rescaled Markov chain converges in probability to a limit described by a system of ordinary differential equations. We solve these equations and obtain an analytical result on the final fraction of defaults in the network. Finally, we show that if the resilience measure is negative, the skeleton of contagious links percolates.

In the sequel, we consider a sequence $(\mathbf{e}_n, \gamma_n)_{n\geq 1}$ of financial networks satisfying Assumptions 3.1 and 3.4.

5.1. Link with the Configuration Model

A standard method for studying random graphs with prescribed degree sequence is to consider (see, e.g., Molloy and Reed 1998; Bollobás 2001; Janson 2009a) a related random multigraph with the same degree sequence, known as the *configuration model*, then condition on this multigraph being simple. The configuration model in the case of random directed graphs has been studied by Cooper and Frieze (2004). Proceeding analogously, we introduce a multigraph with the same degrees and exposures as the network defined above, but which is easier to study because of the independence properties

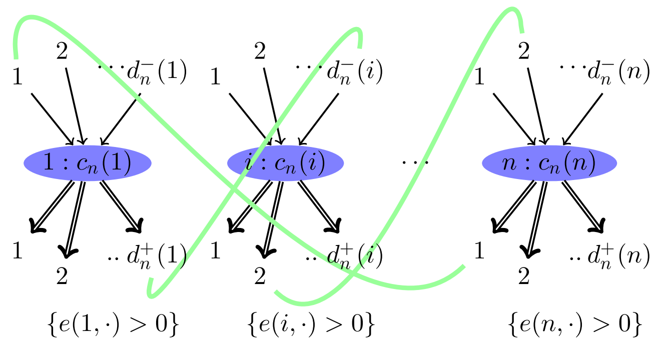

FIGURE 5.1. Configuration model.

of the variables involved. Conditioned on being a simple graph, it has the same law as the random financial network defined above.

DEFINITION 5.1. (Configuration model). Given a set of nodes $[n] = \{1, \dots, n\}$ and a degree sequence $(\mathbf{d}_n^+, \mathbf{d}_n^-)$, we associate to each node i two sets: $H_n^+(i)$ representing its outgoing half-edges and $H_n^-(i)$ representing its incoming half-edges, with $|H_n^+(i)| = d_n^+(i)$ and $|H_n^-(i)| = d_n^-(i)$. Let $H_n^+ = \bigcup_i H_n^+(i)$ and $H_n^- = \bigcup_i H_n^-(i)$. A *configuration* is a matching of H_n^+ with H_n^-. To each configuration, we assign a graph. When an outgoing half-edge of node i is matched with an incoming half-edge of node j, a directed edge from i to j appears in the graph. The *configuration model* is the random directed multigraph $G_n^*(\mathbf{e}_n)$ that is uniformly distributed across all configurations (Figure 5.1).

It is easy to see that conditioned on being a simple graph, $G_n^*(\mathbf{e}_n)$ is uniformly distributed on $\mathcal{G}_n(\mathbf{e}_n)$. Thus, the law of $G_n^*(\mathbf{e}_n)$ conditioned on being a simple graph is the same as the law of $\mathbf{E}_n$.

In particular, any property that holds with high probability for the random multigraph $G_n^*(\mathbf{e}_n)$ holds with high probability on the random network $\mathbf{E}_n$ provided

$$\liminf_{n\to\infty} \mathbb{P}(G_n^*(\mathbf{e}_n) \text{ is simple}) > 0. \tag{5.1}$$

Also, the condition $\sum_{i=1}^n (d_n^+(i))^2 + (d_n^-(i))^2 = O(n)$ implies (5.1) (Janson 2009b).

REMARK 5.2. Janson (2009b) has studied, in the case of undirected graphs, the probability of the random multigraph to be simple. One can adapt the proof to the directed case and show that the condition $\sum_{i=1}^n (d_n^+(i))^2 + (d_n^-(i))^2 = O(n)$ implies (5.1). Indeed, in the nondirected case, Janson (2009b) proves that when $m_n := \sum_{i=1}^n d_n(i) \to \infty$, ($d_n(i)$ is the degree of node i) one has

$$\mathbb{P}(G^*(n, (d_n(i))_1^n) \text{ is simple}) = \exp\left(-\frac{1}{2}\sum_i \lambda_{ii} - \sum_{i<j}(\lambda_{ij} - \log(1+\lambda_{ij}))\right) + o(1),$$

where for $1 \le i, j \le n$; $\lambda_{ij} := \frac{\sqrt{d_n(i)(d_n(i)-1)d_n(j)(d_n(j)-1)}}{m_n}$. The proof of these results is based on counting vertices with at least one loop and, pairs of vertices with at least two edges between them, disregarding the number of parallel loops or edges. The same argument applies to the directed case, and one can show that when

$m_n := \sum_{i=1}^n d_n^+(i) = \sum_{i=1}^n d_n^-(i) \to \infty$, then

$$\mathbb{P}(G_n^*(\mathbf{e}_n) \text{ is simple}) = \exp\left(-\frac{1}{2}\sum_i \lambda_{ii} - \sum_{i<j}(\lambda_{ij} - \log(1+\lambda_{ij}))\right) + o(1),$$

where for $1 \leq i, j \leq n$; $\lambda_{ij} = \frac{\sqrt{d_n^+(i)d_n^-(i)d_n^+(j)d_n^-(j)}}{m_n}$.

One can observe that a uniform matching of half-edges can be obtained sequentially: Choose an incoming half-edge according to any rule (random or deterministic), and then choose the corresponding outgoing half-edge uniformly over the unmatched outgoing half-edges. The configuration model is thus particularly appropriate for the study of contagion, as we will see in the proofs, since we can restrict the matching process to choosing only incoming half-edges entering defaulted nodes. In doing so, one constructs directly the contagion cluster in the random graph given by the configuration model and endowed with the sequence of capital ratios.

Due to this property, it is easier to study contagion on $G_n^*(\mathbf{e}_n)$ under conditions on the degree sequence for the assumption above (5.1) to hold, then translate all results holding with high probability to the initial network $\mathbf{E}_n$ defined in Definition 2.4.

5.2. Coupling

We are given the set of nodes $[n]$ and their sequence of degrees $(\mathbf{d}_n^+, \mathbf{d}_n^-)$. For each node i, we fix an indexing of its outgoing and incoming half-edges, ranging in $[d_n^+(i)] = \{1, \dots, d_n^+(i)\}$ and $[d_n^-(i)]$, respectively. Furthermore, all outgoing half-edges are given a global label in the range $[1, \dots, m_n]$, with m_n the total number of outgoing (in-coming) half-edges. Similarly, all incoming half-edges are given a global label in the range $[m_n]$.

Recall that $\Sigma_n(i)$ denotes the set of all permutations of the counterparties of i in the network $\mathbf{e}_n$. The set of weights on the links exiting node i is given by

$$W_n(i) := \{e_n(i, j) > 0\}. \tag{5.2}$$

For the sequence of edge weights and capital ratios, $(\mathbf{e}_n, \gamma_n)$, we generate the random graph $\tilde{G}_n(\mathbf{e}_n, \gamma_n)$, by the following algorithm:

1. For each node i, choose a permutation $\tau_n^i \in \Sigma_n(i)$ uniformly at random among all permutations of node i's outgoing half-edges.
2. Color all incoming and outgoing half-edges in black. Define the set of initially defaulted nodes

$$\mathcal{D}_0 := \bigcup_{i, \gamma_n(i)=0} \{i\}.$$

   Set for all nodes $i \in [n]\backslash\mathcal{D}_0$; $c(i) = \gamma_n(i)\sum_{j\in[n]} e_n(i, j)$.
3. At step $k \geq 1$, if the set of incoming black half-edges belonging to nodes in $\mathcal{D}_{k-1}$ is empty, denote $\mathcal{D}_f$ the set $\mathcal{D}_{k-1}$. Otherwise:
   (a) Choose among all incoming black half-edges of the nodes in $\mathcal{D}_{k-1}$ the incoming half-edge with the lowest global label and color it in red.
   (b) Choose a node i with probability proportional to its number of black outgoing half-edges and set $\pi_n(k) = i$. Let i have $l-1$ outgoing

half-edges colored in red. Choose its $\tau_n^i(l)$th outgoing half-edge and color it in red. Let its weight be w. If the node $i \notin \mathcal{D}_{k-1}$ and $(1-R)w$ is larger than i's remaining capital, then $\mathcal{D}_k = \mathcal{D}_{k-1} \bigcup \{i\}$. Otherwise, the capital of node i becomes $c(i) - (1-R)w$.

(c) Match node i's $\tau_n^i(l)$th out-going half-edge to the incoming half-edge selected at step (3a) to form an edge.

4. Choose a random uniform matching of the remaining outgoing half-edges and match them to the remaining incoming half-edges in increasing order and color them all in red.

LEMMA 5.3. *The random graph $\tilde{G}_n(\mathbf{e}_n, \gamma_n)$ has the same distribution as $G_n^*(\mathbf{e}_n)$. Furthermore, the set $\mathcal{D}_f$ at the end of the above algorithm is the final set of defaulted nodes in the graph $\tilde{G}_n(\mathbf{e}_n, \gamma_n)$ (endowed with capital ratios γ_n).*

Proof. The second claim is trivial. Let us prove the first claim. For a set A, we denote by Σ_A the set of permutations of A. Let σ_n^+ and σ_n^- be the random permutations in $\Sigma_{[m_n]}$, representing the order in which the above algorithm selects the incoming/outgoing edges. At step k of the above construction, incoming half-edge with global label $\sigma_n^-(k)$ is matched to outgoing half-edge with global label $\sigma_n^+(k)$ to form an edge. The permutation σ_n^+ is determined by the set of permutations $(\tau_n^i)_{i=1,\dots,n}$ and the sequence π_n of size m_n, representing the (ordered) sequence of nodes selected at step 3b (or step 4 when the set of incoming black half-edges belonging to nodes in $\mathcal{D}_{k-1}$ is empty—assume we choose sequentially uniformly at random) of the algorithm (each node i appears in sequence π_n exactly $d_n^+(i)$ times).

It is easy to see that σ_n^+ is a uniform permutation among all permutations in $\Sigma_{[m_n]}$, since $(\tau_n^i)_{i=1,\dots,n}$ are uniformly distributed and at each step of the algorithm, we choose a node with probability proportional to its black outgoing half-edges. On the other hand, the value of $\sigma_n^-(k)$ depends on a deterministic manner on

$$(\mathbf{e}_n, \gamma_n, \sigma_n^+(1), \dots, \sigma_n^+(k-1)).$$

The outgoing half-edge with global label j is matched with the incoming half-edge with global label $(\sigma_n^- \circ (\sigma_n^+)^{-1})(j)$. In order to prove our claim, it is enough to prove that the permutation $(\sigma_n^- \circ (\sigma_n^+)^{-1})$ is uniformly distributed among all permutations of m_n. We can easily show that for an arbitrary permutation ξ belonging to the set $\Sigma_{[m_n]}$, we have that

$$\mathbb{P}\left(\sigma_n^+(j) = \xi^{-1}(\sigma_n^-(j)) | \ \sigma_n^+(k) = \xi^{-1}(\sigma_n^-(k)) \quad \text{for all } \ k < j\right) = \frac{1}{m_n - j + 1}.$$

Indeed, conditional on the knowledge of $(\sigma_n^+(1), \dots, \sigma_n^+(j-1))$, $\sigma_n^-(j)$ is deterministic. Also, by conditioning on $\forall k < j$, $\sigma_n^+(k) = \xi^{-1}(\sigma_n^-(k))$, then $\xi^{-1}(\sigma_n^-(j)) \in \mathcal{T} := [m_n] \backslash \{\sigma_n^+(1), \dots, \sigma_n^+(j-1)\}$, of cardinal $m_n - j + 1$. In the above algorithm, $\sigma_n^+(j)$ has uniform law over $\mathcal{T}$. Then, the probability to choose $\xi^{-1}(\sigma_n^-(j))$ is $\frac{1}{m_n - j + 1}$.

Then, by the law of iterated expectations, we obtain that

$$\mathbb{P}\left(\sigma_n^- \circ \left(\sigma_n^+\right)^{-1} = \xi\right) = \mathbb{P}\left(\sigma_n^+ = \xi^{-1} \circ \sigma_n^-\right) = \frac{1}{m_n!}.$$

This and the fact that the last step of the algorithm is a conditionally uniform match conclude the proof. □

We can find the final set of defaulted nodes $\mathcal{D}_f$ of the above algorithm in the following manner: Once the permutation τ_n^i is chosen, assign to each node its corresponding threshold $\theta_n(i) = \Theta_n(i, \tau_n^i)$ as in Definition 3.2, and forget everything about $(\mathbf{e}_n, \gamma_n)$.

DEFINITION 5.4. Denote by $\tilde{G}_n(\mathbf{d}_n^+, \mathbf{d}_n^-, \theta_n)$ the random graph resulting from the above algorithm, in which we replace step 3b of the algorithm by the fact that node i defaults the first time it has $\theta_n(i)$ outgoing half-edges colored in red, i.e., at step

$$\inf\{k \geq 1, \text{ such that } \theta_n(i) = \#\{1 \leq l \leq k, \ \pi_n(l) = i\}\}.$$

COROLLARY 5.5. *The random graph $\tilde{G}_n(\mathbf{d}_n^+, \mathbf{d}_n^-, \theta_n)$ has the same law as the unweighted skeleton of $\tilde{G}_n(\mathbf{e}_n, \gamma_n)$.*

Let $N_n(j,k,\theta)$ denote the number of nodes with degree (j,k) and threshold θ after choosing uniformly the random permutations τ_n in the above construction.

LEMMA 5.6. *We have (as $n \to \infty$)*

$$\frac{N_n(j,k,\theta)}{n} \xrightarrow{p} \mu(j,k)p(j,k,\theta).$$

Proof. For any node i with with degree (j,k), the probability that its default threshold $\Theta_n(i, \tau_n^i)$ be equal to θ is

$$\nu_n(i,\theta) := \frac{\#\{\tau \in \Sigma_n(i) \mid \Theta_n(i,\tau) = \theta\}}{j!}.$$

Then,

$$N_{n(j,k,\theta)} = \sum_{i, d_n^+(i)=j, d_n^-(i)=k} \text{Be}(\nu_{n(i,\theta)}),$$

where Be$(\cdot)$ denotes a Bernoulli variable. By Assumption 3.4, we have

$$\mathbb{E}[N_n(j,k,\theta)/n] = \mu_n(j,k)p_n(j,k,\theta) \overset{n\to\infty}{\longrightarrow} \mu(j,k)p(j,k,\theta), \text{ and}$$

$$\text{Var}[N_n(j,k,\theta)/n] = \frac{\sum_{i,\ d_n^+(i)=j,\ d_n^-(i)=k} \nu_n(i,\theta)(1-\nu_n(i,\theta))}{n^2} \overset{n\to\infty}{\longrightarrow} 0.$$

Now, it is easy to conclude the proof by Chebysev's inequality. □

5.3. A Markov Chain Description of Contagion Dynamics

In the previous section, we have replaced the description based on default rounds by an equivalent one based on successive bilateral interactions. By *interaction*, we mean matching an incoming half-edge with an outgoing half-edge. At each step of the algorithm described in last section, we have one interaction only between two nodes (banks), yielding at most one default. This allows for a simpler Markov chain that leads to the same set of final defaults.

We describe now the contagion process on the unweighted graph $\tilde{G}_n(\mathbf{d}_n^+, \mathbf{d}_n^-, \theta_n)$ with thresholds $(\theta_n(i) = \Theta_n(i, \tau_n^i))_{1\le i\le n}$ in terms of the dynamics of a Markov chain.

At each iteration, we partition the nodes according to their state of solvency, degree, threshold, and number of defaulted neighbors. Let us define $S_n^{j,k,\theta,l}(t)$, the number of solvent banks with degree (j,k), default threshold θ, and l defaulted debtors before time t. We introduce the additional variables of interest:

- $D_n^{j,k,\theta}(t)$: the number of defaulted banks at time t with degree (j,k) and default threshold θ,
- $D_n(t)$: the number of defaulted banks at time t,
- $D_n^-(t)$: the number of black incoming edges belonging to defaulted banks,

for which it is easy to see that the following identities hold:

$$\begin{aligned} D_n^{j,k,\theta}(t) &= \mu_n(j,k)p_n(j,k,\theta) - \sum_{0\le l<\theta} S_n^{j,k,\theta,l}(t), \\ D_n^-(t) &= \sum_{j,k,0\le\theta\le j} kD_n^{j,k,\theta}(t) - t, \\ D_n(t) &= \sum_{j,k,0\le\theta\le j} D_n^{j,k,\theta}(t). \end{aligned}$$

Because at each step, we color in red one outgoing edge and the number of black outgoing edges at time 0 is m_n, the number of black outgoing edges at time t will be $m_n - t$.

By construction, $\mathbf{Y}_n(t) = (S_n^{j,k,\theta,l}(t))_{j,k,0\le l<\theta\le j}$ represents a Markov chain. Let $(\mathcal{F}_{n,t})_{t\ge0}$ be its natural filtration. We define the operator $\wedge$ as

$$x \wedge y = \min(x, y).$$

The length of the default cascade is given by

$$T_n = \inf\{0 \le t \le m_n,\ D_n^-(t) = 0\} \wedge m_n. \tag{5.3}$$

The total number of defaults is given by $D_n(T_n)$, which represents the cardinal of the final set of defaulted nodes.

Let us now describe the transition probabilities of the Markov chain. For $t < T_n$, there are three possibilities for the partner B of an incoming half-edge of a defaulted node A at time $t+1$:

1. B is in default, the next state is $\mathbf{Y}_n(t+1) = \mathbf{Y}_n(t)$.
2. B is solvent, has degree (j,k) and default threshold θ, and this is the $(l+1)$th deleted outgoing edge and $l+1<\theta$. The probability of this event is $\frac{(j-l)S_n^{j,k,\theta,l}(t)}{m_n-t}$. The changes for the next state will be
$$\begin{aligned} S_n^{j,k,\theta,l}(t+1) &= S_n^{j,k,\theta,l}(t) - 1, \\ S_n^{j,k,\theta,l+1}(t+1) &= S_n^{j,k,\theta,l+1}(t) + 1. \end{aligned}$$
3. B is solvent, has degree (j,k) and default threshold θ, and this is the θth deleted outgoing edge. Then, with probability $\frac{(j-\theta+1)S_n^{j,k,\theta,\theta-1}(t)}{m_n-t}$, we have
$$S_n^{j,k,\theta,\theta-1}(t+1) = S_n^{j,k,\theta,\theta-1}(t) - 1.$$

Let Δ_t be the difference operator: $\Delta_t Y := Y(t+1) - Y(t)$. We obtain the following equations for the expectation of $\mathbf{Y}_n(t+1)$, conditional on $\mathcal{F}_{n,t}$, by averaging over the possible transitions:

$$\mathbb{E}\left[\Delta_t S_n^{j,k,\theta,0} | \mathcal{F}_{n,t}\right] = -\frac{j S_n^{j,k,\theta,0}(t)}{m_n - t}, \tag{5.4}$$
$$\mathbb{E}\left[\Delta_t S_n^{j,k,\theta,l} | \mathcal{F}_{n,t}\right] = \frac{(j-l+1) S_n^{j,k,\theta,l-1}(t)}{m_n - t} - \frac{(j-l) S_n^{j,k,\theta,l}(t)}{m_n - t}.$$

The initial condition is

$$S_n^{j,k,\theta,l}(0) = N_n(j,k,\theta)\mathbb{1}(l=0)\mathbb{1}(0 < \theta \le j).$$

REMARK 5.7. We are interested in the value of $D_n(T_n)$, with T_n defined in (5.3). In case $T_n < m_n$, the Markov chain can still be well defined for $t \in [T_n, m_n)$ by the same transition probabilities. However, after T_n, it will no longer be related to the contagion process and the value $D_n^-(t)$, representing for $t \le T_n$ the number of incoming half-edges belonging to defaulted banks, becomes negative. We consider from now on that the above transition probabilities hold for $t < m_n$.

We will show in the next section that the path of these variables for $t \le T_n$ is, with high probability, close to the solution of the ordinary differential equations suggested by equations (5.4) .

5.4. A Law of Large Numbers for the Contagion Process

Define the following set of differential equations denoted by (DE):

$$\left(s^{j,k,\theta,0}\right)'(\tau) = -\frac{j s^{j,k,\theta,0}(\tau)}{\lambda - \tau},$$
$$\left(s^{j,k,\theta,l}\right)'(\tau) = \frac{(j-l+1) s^{j,k,\theta,l-1}(\tau)}{\lambda - \tau} - \frac{(j-l) s^{j,k,\theta,l}(\tau)}{\lambda - \tau}, \qquad \text{(DE)},$$

with initial conditions

$$s^{j,k,\theta,l}(0) = \mu(j,k)p(j,k,\theta)\mathbb{1}(l=0)\mathbb{1}(0 < \theta \le j).$$

LEMMA 5.8. *The system of ordinary differential equations (DEs) admits the unique solution*

$$y(\tau) := \left(s^{j,k,\theta,l}(\tau)\right)_{j,k,0 \le l < \theta \le j},$$

in the interval $0 \le \tau < \lambda$, *with*

$$s^{j,k,\theta,l}(\tau) := \mu(j,k)p(j,k,\theta)\binom{j}{l}\left(1 - \frac{\tau}{\lambda}\right)^{j-l}\left(\frac{\tau}{\lambda}\right)^{l} \mathbb{1}_{\{0<\theta \le j\}}. \tag{5.5}$$

Proof. We denote by DE^K the set of differential equations defined above, restricted to $j \wedge k < K$ and by $b(K)$ the dimension of the restricted system. Since the derivatives of the functions $\left(s^{j,k,\theta,l}(\tau)\right)_{j\wedge k<K,0\le l<\theta\le j}$ depend only on τ and the same functions, by a standard result in the theory of ordinary differential equations (Hurewicz 1958, ch. 2, theorem 11), there is an unique solution of DE^K in any domain of the type $(-\epsilon,\lambda)\times R$, with R a bounded subdomain of $\mathbb{R}^{b(K)}$ and $\epsilon>0$. The solution of DE is defined to be the set of functions solving all the finite systems $(\mathrm{DE}^K)_{K\ge1}$.

We solve now the system DE. Let $u=u(\tau)=-\ln(\lambda-\tau)$. Then, $u(0)=-\ln(\lambda)$, u is strictly monotone and so is the inverse function $\tau=\tau(u)$. We write the system of DEs with respect to u:

$$(s^{j,k,\theta,0})'(u)=-js^{j,k,\theta,0}(u),$$
$$(s^{j,k,\theta,l})'(u)=(j-l+1)s^{j,k,\theta,l-1}(u)-(j-l)s^{j,k,\theta,l}(u).$$

Then, we have

$$\frac{d}{du}(s^{j,k,\theta,l+1}e^{(j-l-1)(u-u(0))})=(j-l)s^{j,k,\theta,l}(u)e^{(j-l-1)(u-u(0))},$$

and by induction, we find

$$s^{j,k,\theta,l}(u)=e^{-(j-l)(u-u(0))}\sum_{r=0}^{l}\binom{j-r}{l-r}\left(1-e^{-(u-u(0))}\right)^{l-r}s^{j,k,\theta,r}(u(0)).$$

By going back to τ, we have

$$s^{j,k,\theta,l}(\tau)=\left(1-\frac{\tau}{\lambda}\right)^{j-l}\sum_{r=0}^{l}s^{j,k,\theta,r}(0)\binom{j-r}{l-r}\left(\frac{\tau}{\lambda}\right)^{l-r}.$$

Then, by using the initial conditions, we find

$$s^{j,k,\theta,l}(\tau)=\mu(j,k)p(j,k,\theta)\binom{j}{l}\left(1-\frac{\tau}{\lambda}\right)^{j-l}\left(\frac{\tau}{\lambda}\right)^{l}\mathbb{1}_{\{\theta>0\}}. \qquad \square$$

A key idea is to approximate, following Wormald (1995), the Markov chain by the solution of a system of differential equations in the large network limit. We summarize here the main result of Wormald (1995).

For a set of variables $Y^1,\dots,Y^b$ and for $U\subset\mathbb{R}^{b+1}$, define the stopping time $T_U=T_U(Y^1,\dots,Y^b)=\inf\{t\ge1,(t/n;Y^1(t)/n,\dots,Y^b(t)/n)\notin U\}$.

LEMMA 5.9. (Theorem 5.1 in Wormald 1995). *Let $b\ge2$ be an integer and consider a sequence of real valued random variables $(\{Y_n^l(t)\}_{1\le l\le b})_{t\ge0}$ and its natural filtration $\mathcal{F}_{n,t}$. Assume that there is a constant $C_0>0$ such that $|Y_n^l(t)|\le C_0 n$ for all $n,t\ge0$ and $1\le l\le b$. For all $l\ge1$ let $f_l:\mathbb{R}^{b+1}\to\mathbb{R}$ be functions and assume that for some bounded connected open set $U\subseteq\mathbb{R}^{b+1}$ containing the closure of*

$$\left\{(0,z_1,\dots,z_b):\exists\, n \text{ such that } \mathbb{P}\left(\forall\,1\le l\le b,\ Y_n^l(0)=z_l n\right)\ne0\right\},$$

the following three conditions are verified:

1. *(Boundedness). For some function $\beta(n) \geq 1$, we have for all $t < T_U$*

$$\max_{1 \leq l \leq b} |Y_n^l(t+1) - Y_n^l(t)| \leq \beta(n).$$

2. *(Trend). There exists $\lambda_1(n) = o(1)$ such that for $1 \leq l \leq b$ and $t < T_U$*

$$|\mathbb{E}\left[Y_n^l(t+1) - Y_n^l(t)|\mathcal{F}_{n,t}\right] - f_l\left(t/n, Y_n^1(t)/n, \ldots, Y_n^l(t)/n\right)| \leq \lambda_1(n).$$

3. *(Lipschitz). The functions $(f_l)_{1 \leq l \leq b}$ are Lipschitz-continuous on U.*

Then, the following conclusions hold:

- *For $(0, \hat{z}_1, \ldots, \hat{z}_b) \in U$, the system of differential equations*

$$\frac{dz_l}{ds} = f_l(s, z_1, \ldots, z_l), \quad l = 1, \ldots, b,$$

  has a unique solution in U, $z_l : \mathbb{R} \to \mathbb{R}$, which passes through $z_l(0) = \hat{z}_l$, for $l = 1, \ldots, b$, and which extends to points arbitrarily close to the boundary of U.
- *Let $\lambda > \lambda_1(n)$ with $\lambda = o(1)$. For a sufficiently large constant C, with probability $1 - O(\frac{b\beta(n)}{\lambda} \exp(-\frac{n\lambda^3}{\beta(n)^3}))$, we have*

$$\sup_{0 \leq t \leq \sigma(n)n} \left(Y_n^l(t) - nz_n^l(t/n)\right) = O(\lambda n),$$

  where $\mathbf{z}_n(t) = (z_n^1(t), \ldots, z_n^b(t))$ is the solution of

$$\frac{d\mathbf{z}_n}{dt} = f(t, \mathbf{z}_n(t)) \qquad z_n(0) = \mathbf{Y}_n(0)/n$$

$$\text{and} \qquad \sigma(n) = \sup\{t \geq 0, \qquad d_\infty(\mathbf{z}_n(t), \partial U) \geq C\lambda\}.$$

We apply this lemma to the contagion model described in Section 5.3. Let us define, for $0 \leq \tau \leq \lambda$

$$\delta^{j,k,\theta}(\tau) := \mu(j,k)p(j,k,\theta) - \sum_{0 \leq l < \theta} s^{j,k,\theta,l}(\tau),$$

$$\delta^{-}(\tau) := \sum_{j,k,\theta} k\delta^{j,k,\theta}(\tau) - \tau, \quad \text{and}$$

$$\delta(\tau) := \sum_{j,k,\theta} \delta^{j,k,\theta}(\tau),$$

with $s^{j,k,\theta,l}$ given in Lemma 5.8. With $\text{Bin}(j, \pi)$ denoting a binomial variable with parameters j and π, we have

$$\delta^{j,k,\theta}(\tau) = \mu(j,k)p(j,k,\theta)\mathbb{P}\left(\text{Bin}\left(j, \frac{\tau}{\lambda}\right) \geq \theta\right), \tag{5.6}$$

$$\delta^{-}(\tau) = \sum_{j,k,\theta} k\delta^{j,k,\theta}(\tau) - \tau$$

$$= \sum_{j,k,\theta\le j} k\mu(j,k)p(j,k,\theta)\mathbb{P}\left(\mathrm{Bin}\left(j,\frac{\tau}{\lambda}\right)\ge\theta\right)-\tau \tag{5.7}$$

$$= \lambda\left(I\left(\frac{\tau}{\lambda}\right)-\frac{\tau}{\lambda}\right),$$

and

$$\delta(\tau) := \sum_{j,k,0\le\theta\le j} \mu(j,k)p(j,k,\theta)\mathbb{P}\left(\mathrm{Bin}\left(j,\frac{\tau}{\lambda}\right)\ge\theta\right). \tag{5.8}$$

5.5. Proof of Theorem 3.8

We now proceed to the proof of Theorem 3.8 whose aim is to approximate the value $D_n(T_n)/n$ as $n\to\infty$. We base the proof on Lemma 5.9. However, several difficulties arise in our case since the number of variables depends on n. We first need to bound the contribution of higher order terms in the infinite sums (5.7) and (5.8). Fix $\epsilon>0$. By Condition 3.1, we know

$$\lambda=\sum_{j,k} k\mu(j,k)=\sum_{j,k} j\mu(j,k)\in(0,\infty).$$

Then, there exists an integer K_ϵ such that

$$\sum_{k\ge K_\epsilon}\sum_j k\mu(j,k)+\sum_{j\ge K_\epsilon}\sum_k j\mu(j,k)<\epsilon,$$

which implies that

$$\sum_{j\wedge k\ge K_\epsilon} k\mu(j,k)<\epsilon.$$

It follows that

$$\forall\, 0\le\tau\le\lambda,\qquad \sum_{j\wedge k\ge K_\epsilon,0\le\theta\le j} k\mu(j,k)p(j,k,\theta)\mathbb{P}\left(\mathrm{Bin}\left(j,\frac{\tau}{\lambda}\right)\ge\theta\right)<\epsilon. \tag{5.9}$$

The number of vertices with degree (j,k) is $n\mu_n(j,k)$. Again, by Condition 3.1,

$$\sum_{j,k} k\mu_n(j,k)=\sum_{j,k} j\mu_n(j,k)\to\lambda\in(0,\infty).$$

Therefore, for n large enough, $\sum_{j\wedge k\ge K_\epsilon} k\mu_n(j,k)<\epsilon$, and for all $0\le t\le m_n$,

$$\sum_{j\wedge k\ge K_\epsilon,0\le\theta\le j} kD_n^{j,k,\theta}(t)/n<\epsilon. \tag{5.10}$$

For $K\ge 1$, we denote

$$\mathbf{y}^K := \left(s^{j,k,\theta,l}(\tau)\right)_{j\wedge k<K,\ 0\le l<\theta\le j} \quad \text{and}$$

$$Y_n^K := \left(S_n^{j,k,\theta,l}(\tau)\right)_{j\wedge k<K,\ 0\le l<\theta\le j},$$

both of dimension $b(K)$, where $\delta^{j,k,\theta}(\tau)$, $s^{j,k,\theta,l}(\tau)$ are solutions to a system (DE) of ordinary differential equations. Let

$$\pi^* = \min\{\pi \in [0,1] | I(\pi) = \pi\}.$$

For the arbitrary constant $\epsilon > 0$ we fixed above, we define the domain U_ϵ as

$$U_\epsilon = \left\{ \left(\tau, y^{K_\epsilon}\right) \in \mathbb{R}^{b(K_\epsilon)+1} \ : \ -\epsilon < \tau < \lambda - \epsilon \ , \ -\epsilon < s^{j,k,\theta,l} < 1 \right\}. \tag{5.11}$$

The domain U_ϵ is a bounded open set that contains the support of all initial values of the variables. Each variable is bounded by a constant times n ($C_0 = 1$). By the definition of our process, the boundedness condition is satisfied with $\beta(n) = 1$. The second condition of the theorem is satisfied by some $\lambda_1(n) = O(1/n)$. Finally, the Lipschitz property is also satisfied since $\lambda - \tau$ is bounded away from zero. Then, by Lemma 5.9 and by using Lemma 5.6 for convergence of initial conditions, we have:

COROLLARY 5.10. *For a sufficiently large constant C, we have*

$$\mathbb{P}(\forall t \leq n\sigma_C(n), \mathbf{Y}_n^{K_\epsilon}(t) = n\mathbf{y}^{K_\epsilon}(t/n) + O(n^{3/4})) = 1 - O(b(K_\epsilon)n^{-1/4}exp(-n^{-1/4})) \tag{5.12}$$

uniformly for all $t \leq n\sigma_C(n)$, where

$$\sigma_C(n) = \sup\{\tau \geq 0, d(\mathbf{y}^{K_\epsilon}(\tau), \partial U_\epsilon) \geq Cn^{-1/4}\}.$$

When the solution reaches the boundary of U_ϵ, it violates the first constraint, determined by $\hat{\tau} = \lambda - \epsilon$. By convergence of $\frac{m_n}{n}$ to λ, there is a value n_0 such that $\forall n \geq n_0$, $\frac{m_n}{n} > \lambda - \epsilon$, which ensures that $\hat{\tau} n \leq m_n$. Using (5.9) and (5.10), we have for $0 \leq t \leq n\hat{\tau}$ and $n \geq n_0$:

$$\begin{aligned} \left|D_n^-(t)/n - \delta^-(t/n)\right| &= \left| \sum_{j,k} \sum_{\theta \leq j} k \left(D_n^{j,k,\theta}(t)/n - \delta^{j,k,\theta}(t/n)\right) \right| \\ &\leq \sum_{j,k} \sum_{\theta \leq j} k \left|D_n^{j,k,\theta}(t)/n - \delta^{j,k,\theta}(t/n)\right| \\ &\leq \sum_{j \wedge k \leq K_\epsilon} \sum_{\theta \leq j} k \left|D_n^{j,k,\theta}(t)/n - \delta^{j,k,\theta}(t/n)\right| + 2\epsilon, \end{aligned} \tag{5.13}$$

and

$$|D_n(t)/n - \delta(t/n)| \leq \sum_{j \wedge k \leq K_\epsilon} \sum_{\theta \leq j} \left|D_n^{j,k,\theta}(t)/n - \delta^{j,k,\theta}(t/n)\right| + 2\epsilon. \tag{5.14}$$

We obtain by Corollary 5.10 that

$$\sup_{t \leq \hat{\tau} n} \left|D_n^-(t)/n - \delta^-(t/n)\right| \leq 2\epsilon + o_p(1), \text{ and} \tag{5.15}$$

$$\sup_{t \leq \hat{\tau} n} |D_n(t)/n - \delta(t/n)| \leq 2\epsilon + o_p(1). \tag{5.16}$$

We now study the stopping time T_n defined in (5.3) and the size of the default cascade $D_n(T_n)$. First assume $I(\pi) > \pi$ for all $\pi \in [0, 1)$, i.e., $\pi^* = 1$. Then, we have

$$\forall \tau < \hat{\tau}, \delta^-(\tau) = \sum_{j,k,\theta} k\delta^{j,k,\theta}(\tau) - \tau > 0.$$

We have then that $T_n/n = \hat{\tau} + O(\epsilon) + o_p(1)$ and from convergence (5.16), since $\delta(\hat{\tau}) = 1 - O(\epsilon)$, we obtain by tending ϵ to 0 that $|D_n(T_n)| = n - o_p(n)$. This proves the first part of the theorem.

Now consider the case $\pi^* < 1$, and furthermore π^* is a stable fixed point of $I(\pi)$. Then, by definition of π^* and by using the fact that $I(1) \le 1$, we have $I(\pi) < \pi$ for some interval $(\pi^*, \pi^* + \tilde{\pi})$. Then, $\delta^-(\tau)$ is negative in an interval $(\tau^*, \tau^* + \tilde{\tau})$, with $\tau^* = \lambda\pi^*$.

Let ϵ such that $2\epsilon < -\inf_{\tau\in(\tau^*,\tau^*+\tilde{\tau})} \delta^-(\tau)$ and denote $\hat{\sigma}$ the first iteration at which it reaches the minimum. Since $\delta^-(\hat{\sigma}) < -2\epsilon$, it follows that with high probability $D^-(\hat{\sigma}n)/n < 0$, so $T_n/n = \tau^* + O(\epsilon) + o_p(1)$. The conclusion follows by taking the limit $\epsilon \to 0$.

5.6. Proof of Theorem 3.10

Denote by ρ the fraction of fundamental defaults

$$\rho := \sum_{j,k} \mu(j, k)p(j, k, 0).$$

We have

$$I(\alpha) = \sum_{j,k} \frac{\mu(j, k)k}{\lambda} \sum_{\theta=0}^{j} p(j, k, \theta)\beta(j, \alpha, \theta).$$

Using a first-order expansion of $\beta(j, \alpha, \theta)$ in α at 0 (when $\alpha \to 0$), we obtain

$$\beta(j, \alpha, \theta) = 1_{\{\theta=0\}} + \alpha j 1_{\{\theta=1\}} + o(\alpha).$$

Thus,

$$I(\alpha) = \sum_{j,k} \frac{\mu(j, k)k}{\lambda}(p(j, k, 0) + \alpha j p(j, k, 1)) + o(\alpha).$$

Let α^* be the smallest fixed point of $I(\alpha)$. Given Condition (3.6), for $\alpha > 0$ small enough,

$$\lim_{\rho\to 0} I(\alpha) = \alpha \sum_{j,k} \frac{\mu(j, k)jk}{\lambda} p(j, k, 1) + o(\alpha) < \alpha,$$

where we used the fact that if the fraction ρ of fundamental defaults tends to zero, so does the fraction of outgoing links belonging to fundamentally defaulted nodes, i.e., $\sum_{j,k} \frac{\mu(j,k)k}{\lambda} p(j, k, 0)$. On the other hand, we have seen that $I(0) \ge 0$. Thus, $\lim_{\rho\to 0} \alpha^* = 0$.

Let us now fix $\epsilon > 0$. By continuity of the function g defined by

$$g(\alpha) := \sum_{j,k} \mu(j, k) \sum_{\theta=0}^{j} p(j, k, \theta)\beta(j, \alpha, \theta),$$

appearing in Theorem 3.8, there exists ρ_ϵ such that $g(\alpha^*) < \epsilon/2$ as soon as $\rho < \rho_\epsilon$. In this case, by Theorem 3.8, we have that there exists an integer N_ϵ such that for $n \geq N_\epsilon$,

$$\mathbb{P}(|\alpha_n(\mathbf{E}_n, \gamma_n) - g(\alpha^*)| < \epsilon/2) > 1 - \epsilon.$$

We now complete the proof noting that since

$$\rho = \lim_{n\to\infty} \frac{|\{i \in [n] \mid \gamma_n(i) = 0\}|}{n} < \rho_\epsilon,$$

one can take N_ϵ sufficiently large such that for all $n > N_\epsilon$: $\frac{|\{i\in[n] \mid \gamma_n(i)=0\}|}{n} < \rho_\epsilon$.

5.7. Proof of Theorem 3.11

Let λ_n represent the average degree (then by Condition 3.1, $\lambda_n \to \lambda$ as $n \to \infty$), and $\mu_n(j,k)$ represent the empirical distribution of the degrees, assumed to be proper (as defined below), then Cooper and Frieze (2004, theorem 1.2) states that if

$$\sum_{j,k} jk\frac{\mu(j,k)}{\lambda} > 1, \tag{5.17}$$

then the graph contains w.h.p. a strongly connected giant component.

We remark that the above result is given in Cooper and Frieze (2004) under stronger assumptions on the degree sequence, adding to Assumption 3.1 the following three conditions, in which Δ_n denotes the maximum degree:

- Let $\rho_n = \max(\sum_{i,j} \frac{i^2 j\mu_n(i,j)}{\lambda_n}, \sum_{i,j} \frac{j^2 i\mu_n(i,j)}{\lambda_n})$. If $\Delta_n \to \infty$ with n, then $\rho_n = o(\Delta_n)$.
- $\Delta_n \leq \frac{n^{1/12}}{\log n}$.
- As $n \to \infty$, $\nu_n \to \nu \in (0,\infty)$.

Following Cooper and Frieze (2004), we call a degree sequence proper if it satisfies Assumption 3.1 together with the above conditions.

A first reason for adding these conditions in Cooper and Frieze (2004) is to ensure that Equation (5.1) holds. However, following Janson (2009b), the restricted set of conditions (3.1) is sufficient. The second reason is that Cooper and Frieze (2004) give more precise results on the structure of the giant component. For our purpose, to find the sufficient condition for the existence of strongly connected giant component, we show that these supplementary conditions may be dropped.

It is easy to see that a bounded degree sequence (i.e., $\Delta_n = O(1)$) that satisfies Assumption 3.1 is proper. We use this fact in the following.

LEMMA 5.11. *Consider the random directed graph $G_n^*(\mathbf{d}_n^-, \mathbf{d}_n^+)$ constructed by configuration model, where the degree sequence satisfies Assumption 3.1. If*

$$\sum_{j,k} jk\frac{\mu(j,k)}{\lambda} > 1, \tag{5.18}$$

then with high probability, the graph contains a strongly connected giant component.

Proof. By the second moment property and Fatou's lemma, there exists a constant C such that

$$\sum_{j,k} jk\mu(j,k) \le \sum_{j,k}(j^2+k^2)\mu(j,k)$$
$$\le \liminf_{n\to\infty} \sum_{j,k}(j^2+k^2)\mu_n(j,k) \le C.$$

Then, it follows that for arbitrary $\epsilon > 0$, there exists a constant Δ_ϵ such that

$$\sum_{j\wedge k>\Delta_\epsilon} jk\mu(j,k) \le \epsilon.$$

Thus, by choosing ϵ small enough, there exists a constant Δ_ϵ such that

$$\sum_{j\wedge k\le\Delta_\epsilon} jk\frac{\mu(j,k)}{\lambda} > 1.$$

We now modify the graph such that the maximum degree is equal to Δ_ϵ: for every node i such that $d_n^+(i) \wedge d_n^-(i) > \Delta_\epsilon$, all its incoming (respectively, outgoing) half-edges are transferred to new nodes with degree $(0, 1)$ (respectively, with degree $(1, 0)$). Since these newly created nodes cannot be part of any strongly connected component, it follows that if the modified graph contains such a component, then necessarily, the initial graph also does. It is then enough to evaluate equation (5.17) for this modified graph, which by construction verifies Assumption 3.1 for the new empirical distribution $\tilde{\mu}$ with the average degree $\tilde{\lambda}$. Also, since the degrees of the modified graph are bounded, the supplementary conditions above also hold, i.e., the degree sequence is proper, and we can apply Cooper & Frieze's result. It only remains to show that $\sum_{j,k} jk\frac{\tilde{\mu}(j,k)}{\tilde{\lambda}} > 1$. Indeed,

$$\sum_{j,k} jk\frac{\tilde{\mu}(j,k)}{\tilde{\lambda}} = \sum_{j\wedge k\le\Delta_\epsilon} jk\frac{\tilde{\mu}(j,k)}{\tilde{\lambda}}$$
$$= \sum_{0<j,k\le\Delta_\epsilon} jk\frac{\tilde{\mu}(j,k)}{\tilde{\lambda}}$$
$$= \sum_{0<j,k\le\Delta_\epsilon} jk\frac{\mu(j,k)}{\lambda} > 1.$$

The last equality follows from the fact that for $0 < j, k \le \Delta_\epsilon$, we have

$$\frac{\tilde{\mu}(j,k)}{\tilde{\lambda}} = \frac{\mu(j,k)}{\lambda}.$$

This is true since the total number of edges, and the number of nodes with degree j, k for $0 < j, k \le \Delta_\epsilon$, stays unmodified. □

We now proceed to the proof of Theorem 3.11. Our proof is based on ideas applied in Fountoulakis (2007) and Janson (2009a) for site and bond percolation in configuration model. Our aim is to show that the skeleton of contagious links in the random financial network is still described by configuration model, with a degree sequence verifying Assumption 3.1, and then apply Lemma 5.11.

For each node i, the set of contagious outgoing edges is given by

$$C_n(i) := \{l \mid (1-R)e_n(i,l) > \gamma_n(i)\}.$$

Let us denote their number by

$$c_n^+(i) := \# C_n(i).$$

We denote by G_n^c the unweighted skeleton of contagious links in the random network $G_n^*(\mathbf{e}_n)$, endowed with the capital ratios γ_n.

In order to characterize the law of G_n^c, we adapt Janson's method Janson (2009a) for the directed case.

LEMMA 5.12. *The unweighted skeleton of contagious links G_n^c has the same law as the random graph constructed as follows:*

1. *Replace the degree sequence $(\mathbf{d}_n^+, \mathbf{d}_n^-)$ of size n by the degree sequence $(\tilde{\mathbf{d}}_{n'}^+, \tilde{\mathbf{d}}_{n'}^-)$ of size n', with*

$$n' = n + m_n - \sum_{i=1}^{n} c_n^+(i),$$
$$\forall\, 1 \le i \le n, \quad \tilde{d}_{n'}^+(i) = c_n^+(i),\ \tilde{d}_{n'}^-(i) = d_n^-(i),$$
$$\forall\, n+1 \le i \le n', \quad \tilde{d}_{n'}^+(i) = 1,\ \tilde{d}_{n'}^+(i) = 0.$$

2. *Construct the random unweighted graph $G_{n'}^*(\tilde{\mathbf{d}}_{n'}^+, \tilde{\mathbf{d}}_{n'}^-)$ with n' nodes, and the degree sequence $(\tilde{\mathbf{d}}_{n'}^+, \tilde{\mathbf{d}}_{n'}^-)$ by configuration model.*
3. *Delete $n^+ = n' - n$ randomly chosen nodes with out-degree 1 and in-degree 0.*

Proof. The skeleton G_n^c can be obtained in a two-step procedure. First, disconnect all noncontagious links in $G_n^*(\mathbf{e}_n)$ from their end nodes and transfer them to newly created nodes of degree $(1,0)$. Then, delete all new nodes and their incident edges. The first step of this procedure may be dubbed as "rewiring." Looking at graphs as configurations, and since the first step changes the total number of nodes but not the number of half-edges, it is easy to see that there is a one to one correspondence between the configurations before and after the "rewiring." Thus, the graph after rewiring is still described by the configuration model, and has the same law as $G_{n'}^*(\tilde{\mathbf{d}}_{n'}^+, \tilde{\mathbf{d}}_{n'}^-)$. Finally, by symmetry, the nodes with out-degree 1 and in-degree 0 are equivalent, so one may remove randomly the appropriate number of them. □

Note that since the degree sequence before rewiring verifies Condition 3.1, so does the degree sequence after rewiring. Moreover, since we are interested in the strongly connected component and nodes of degrees $(1,0)$ will not be included, we can actually apply Lemma 5.11 to the random graph resulting by the above exploration process. Hence, we may study the strongly connected component in the intermediate graph $G_{n'}^*(\tilde{\mathbf{d}}_{n'}^+, \tilde{\mathbf{d}}_{n'}^-)$.

Let us denote by $l_{n'}(j,k)$, the number of nodes with out-degree j and in-degree k in the graph $G_{n'}^*(\tilde{\mathbf{d}}_{n'}^+, \tilde{\mathbf{d}}_{n'}^-)$, and by $\tilde{\lambda}_{n'}$, the average degree. Then, the average directed degree in this random graph is given by $\nu_n := \sum_{j,k} jk l_{n'}(j,k)/(\tilde{\lambda}_{n'} n')$.

We first observe that $\tilde{\lambda}_{n'} n' = \lambda n$, since the number of edges is unchanged after rewiring of the links. For every $k > 0$, the quantity $\sum_j j l_{n'}(j,k)$ represents the number of outgoing edges belonging to nodes with in-degree k in the graph after rewiring, which, in turn, represents the number of contagious outgoing edges belonging to nodes with in-degree k in the graph before rewiring. But so does $\sum_j p_n(j,k,1)n\mu_n(j,k)j$. So, for all k

$$\begin{aligned}\sum_j j\frac{l_{n'}(j,k)}{\lambda_{n'}n'} &= \frac{1}{\lambda_{n'}n'}\sum_j p_n(j,k,1)n\mu_n(j,k)j \\ &= \sum_j jp_n(j,k,1)\frac{\mu_n(j,k)}{\lambda_n} \\ &\overset{n\to\infty}{\longrightarrow} \sum_j jp(j,k,1)\frac{\mu(j,k)}{\lambda},\end{aligned}$$

where convergence holds by the second moment property in Assumption 3.1. Applying Lemma 5.11 to the sequence of degrees in the graph after rewiring shows that when

$$\sum_k k\lim_n \sum_j j\frac{l_{n'}(j,k)}{\lambda_{n'}n'} = \sum_k\sum_j jp(j,k,1)\frac{\mu(j,k)}{\lambda} > 1,$$

then with high probability, there exists a giant strongly connected component in the skeleton of contagious links.

6. CONCLUSIONS AND FURTHER DIRECTIONS

In this paper, we have analyzed distress propagation in a financial system represented as a large network. We obtained an asymptotic expression for the size of a default cascade in the network, in terms of the following characteristics: degree sequence, exposure sequences for all nodes, and capital sequences after an external shock.

Our analysis extends previous results on contagion processes in homogeneous, undirected random graphs to heterogeneous, weighted, and directed random graphs. Our asymptotic results were corroborated with a simulation study of contagion on a network with large but realistic size: On a *given network* (viewed as a sample from of our random network model), the spread of distress can be predicted by our measure of resilience. Our sample network has the same empirical properties as a real interbank network, e.g., the Brazilian one. As we vary the capital ratio, the point where the resilience condition fails indicates that the point where a large cascade would ensue should one bank default. This illustrates how one can use the resilience measure and the theoretical amplification ratios as a tool for monitoring the risk of interbank contagion *without resorting to large-scale simulation* of cascades. This point is further developed in the companion paper on stress testing, see Amini et al. (2012a).

The crucial question in the context of macroprudential regulation of banking systems is how to identify and mitigate those features that make nodes systemically important. We have identified institutions acting as potential hubs for default contagion as those highly connected and with a large fraction of *contagious links*. One natural way to mitigate the systemic impact of these nodes is to set minimal capital requirements with respect to contagious links (in various scenarios where shocks are applied to capital). This point of view is different from the current capital requirements as defined by the Basel II accords. Currently, minimal capital depends on the risk weighted sum of exposures, where the risk weights are given by the counterparty default probability. However, the default probability is computed by internal models and may not take into account knock-on effects. Our results suggest that for financial stability, minimal ratios of capital should be set with respect to those exposures that are contagious in scenarios where the network has been subjected to a shock.

While the insolvency contagion investigated in this paper has been mostly associated with balance sheet contagion, our results may be applied to the over-the-counter derivatives markets. Contagion is carried in these markets through intermediaries with a large fraction of critical receivables, defined similarly to contagious links, i.e., receivables on which the intermediary depends to meet its own payment obligations, see Minca (2011). We argue that financial stability would be significantly enhanced by setting minimal liquid reserve requirements with respect to critical receivables, for those nodes that are counterparties to a large number of contracts.

Whereas in this paper, we were concerned mostly with a network where the node's characteristics were observable, our results apply to the particular case where degrees and weights are sequences of (exchangeable) random variables with arbitrary correlation structure. This corresponds to a setting where the modeler cannot observe the sequence of exposures, but rather has some belief about their distribution. Such a perspective would have, for example, a market participant who models the default probability of its counterparties and takes into consideration network effects.

An important point related to the description of contagion in terms of a Markov chain, which in this paper allowed us to prove the convergence results is the fact that this approach can be extended to the study of intervention of a lender of last resort. As shown in Amini, Minca, and Sulem (2012b), the problem of optimal intervention by a lender of last resort becomes a stochastic control problem, which is tractable for stylized two-tiered financial networks.

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