his article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2016.2598650, IEEI
Transactions on Automatic Control

A criterion for exponential consensus of time-varying non-monotone nonlinear networks

S. Manfredi* and D. Angeli◇

Abstract— **In this paper we present new results on exponential consensus for continuous-time nonlinear time varying networks. A key feature in the following is that the monotonicity property is not required, unlike most of existing literature on the subject. Moreover, we give an estimate of the exponential rate of convergence towards the agreement manifold. Finally, representative example and counterexample are given.**

Index Terms— **nonlinear networks, Consensus, Multi agent systems.**

I. Introduction

In recent years the scientific community has devoted considerable attention to the consensus problem (see [1], [4] and references therein). In the literature different conditions have been proposed to assess consensus in discrete and continuous time ([8], [9], just to cite a few) and under different class of both nonlinear time invariant and switching/time varying networks as outlined in [12], [7], [17], [16]. Most of the above frameworks assumed (in implicit or explicit way) the property monotonicity [20]. Given a function $f(t,x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, piecewise continuous in t and locally Lipschitz continuous with respect to x, the associated system of differential equations $\dot{x}(t) = f(t, x(t))$, is called monotone if for any $i \in \{1, 2, \ldots, n\}$, $f_i(t, x)$ is non-decreasing with respect to x_j for all $j \neq i$. Notice that this condition implies monotonicity of the flow $\phi(t; t_0, x_0)$ with respect to initial conditions, namely, for all t_0 and all $t \geq t_0$, it holds $\phi(t; t_0, x_1) \geq \phi(t; t_0, x_2)$ if $x_1 \geq x_2$ (where "$\geq$" is meant componentwise), [20]. This property is usually guaranteed in linear and nonlinear networks by respectively requiring the sign definiteness of off diagonal entries of the adjacency matrix (e.g. [9]) and Jacobian matrix $F(x)$ (e.g. [17]). An extension to the case of signed graph yielding cluster (bipartite) consensus (rather than standard consensus) is reported in [18], where the Jacobian has to fulfil the sign definiteness condition after a diagonal change of coordinates, thus implying monotonicity ([20]) with respect to the partial order induced by some arbitrary orthant. An extension to the case where the network topology is time-varying signed graph is presented in [19]. The assumption of monotonicity is widely and implicitly assumed in the literature, both in linear and nonlinear networks scenarios, as it appears natural because it models coupling influence growing with distance, thus allowing reasonable convergence speed to the consensus equilibria. However, many networks of theoretical and practical interest (i.e. opinion dynamics, swarm of robots, sensor networks) are characterized by limited or vanishing influence as the state distance goes to infinity. In this respect, the seminal work [2] proposed a linear-like second order swarming model where the weighting coefficients a_{ij} defining directed influence between birds are modelled by the following non-monotone function $a_{ij}(\|x_i - x_j\|^2) = \dfrac{K}{(\sigma^2 + \|x_i - x_j\|^2)^\beta}$, for some fixed K, $\sigma > 0$ and $\beta \geq 0$. Conditions to ensure that the birds velocities converge to a common one and their distance remain bounded are given. The analysis of the importance on the equilibrium (cluster consensus rather standard consensus) of the limited agents' communication with coupling function going to zero at some finite value is carried out in [3] for the opinion dynamic Krause's model.

*S. Manfredi is with the Department of Electrical Engineering and Information Technology, University of Naples Federico II, Italy and with the Control and Power Group, Electrical and Electronic Engineering Department, Imperial College, London. Email: sabato.manfredi@unina.it
◇D. Angeli is with the Control and Power Group, Electrical and Electronic Engineering Department, Imperial College, London and with the Dip. di Sistemi e Informatica, University of Florence, Italy. Email: d.angeli@imperial.ac.uk

A. Paper Contribution

In recent papers [10], [11], we introduced a condition for asymptotic agreement (*state frozen integral connectivity*), suitable for nonlinear time varying monotone networks that extended to this scenario the notion of integral connectivity introduced by Moreau for linear networks [9], with the additional merit to be frozen in state variables and therefore of simpler verification. Herein, we undertake a non trivial further step by removing the monotonicity assumption, thus extending remarkably the class of considered systems. Specifically the paper contributions are: i) guaranteed exponential consensus under weak connectivity properties (just existence of a spanning tree for a suitable averaged graph is required) for a large class of non monotone nonlinear networks with time-varying and state dependent dynamic and coupling. This encompasses most of the agents models normally adopted in the literature in the linear and non linear time varying setting. Specifically, both the dynamic at the node (self-feedback) and the coupling can be time varying and state dependent with the notable feature that the strength of attraction between two agents may vanish as the distance between their state values becomes larger. This is representative of several network scenarios where it is meaningful to assume that agents far away from each other have a low mutual influence. Differently from [2], the proposed condition focuses on first order consensus for general non monotone interactions and nonlinear time-varying agent dynamics. With the respect to [3], herein we address convergence towards standard consensus rather clustering, with the possibility of decreasing coupling strengths as the distance goes to infinity for a larger class of nonlinear time varying networks; ii) we extend the use of a "State Frozen" concept [10], [11] and integral connectivity to this non-trivial scenario of non monotone networks by introducing a suitable agents connectivity property (later called "Weak integral connectivity"). This has the merit to avoid the circular argument by which solutions depend on the connectivity and the latter is in turn influenced by state evolutions. This type of circular argument normally makes up for conditions that can hardly be tested, in the case of time-varying nonlinear agent dynamics and coupling, without explicit apriori knowledge of solutions; iii) for the described class of systems, we provide an estimate of the exponential rate of convergence towards the agreement manifold.

II. Notation and problem statement

Throughout the paper all vectors are assumed to be column vectors. To denote vectors we write $x = [x_1, \ldots, x_n]$ for the column vector $x \in \mathbb{R}^n$. $|x|$ denotes the Euclidean norm of x. $\mathbf{1}$ is the vector of all ones and e_j is the j-th element of the canonical basis of $\mathbb{R}^n$, where n should normally be clear from the context. The integer interval $N = \{1, 2, \ldots, n\}$ will be identified with the set of interacting agents. Let

his article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2016.2598650, IEEE Transactions on Automatic Control

a compact set $\mathcal{K} \in \mathbb{R}^n$, herein we denote $diam(\mathcal{K}) = \sup\{|x-y| : x, y \in \mathcal{K}\}$. Let $G(N, E)$ be a weighted directed graph (digraph) with the set of nodes $N = \{1, ..., n\}$, the set of edges $E \subseteq N \times N$. A node j is reachable from node i if there exists a path in a directed graph connecting nodes i and j, namely there is a finite sequence $n_1, n_2, ..., n_k$ of distinct nodes such that $(n_i, n_{i+1}) \in E$ for $i = 1, \ldots, k-1$ with $n_1 = i$ and $n_k = j$. A digraph $G(N, E)$ is quasi-strongly connected (or weakly connected) if there exists a node (root or center) from which any other node is reachable. $G(N, E)$ has a spanning tree if there exists a spanning tree that is a subgraph of G. Notice that the condition that $G(N, E)$ has a spanning tree is equivalent to quasi-strongly connectedness. A directed graph is connected if any two nodes can be joined by a path.

Consider a network of agents as described by the following system of nonlinear differential equations:

$$\dot{x}(t) = f(t, x(t)) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $t \in \mathbb{R}_+$ denotes time and f is a vector field $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ describing the dynamics of the interaction between agents.

We assume: i) f is locally Lipschitz continuous with respect to x uniformly in time, viz. for all compacts $K \in \mathbb{R}^n$ there exists $L_K > 0$, such that, for all $x_a, x_b \in K$ and all $t \geq 0$ it holds $|f(t, x_a) - f(t, x_b)| \leq L_K |x_a - x_b|$;[1] ii) that f admits an agreement equilibrium set, that is:

$$\mathcal{E} = \text{span}\{\mathbf{1}\} \subseteq \{x \in \mathbb{R}^n : f(t, x) = 0 \; \forall t \in \mathbb{R}_+\}. \tag{2}$$

The assumptions on f, imply the local existence and the unicity of the system's solution on some maximally extended open interval of definition.

Let $x(t)$ denote a solution of (1). At any time instant t the following quantities are of interest:

$$x_{\max}(t) = \max_{k \in N}\{x_k(t)\}, \quad x_{\min}(t) = \min_{k \in N}\{x_k(t)\}$$

and $\delta_k(t) = |x_{\max}(t) - x_k(t)|$ for all $k \in N$ (or symmetrically $\tilde{\delta}_k(t) = |x_{\min}(t) - x_k(t)|$).

Fixed an arbitrary solution $x(\cdot)$ and an arbitrary time t we define a time-dependent permutation $p_j(t)$ of indeces $j \in N$ such that it fulfills

$$x_{p_1(t)}(t) \leq x_{p_2(t)}(t) \leq x_{p_3(t)}(t) \leq \ldots \leq x_{p_n(t)}(t).$$

Notice that, if two or more entries of x take some given value, then the permutation is not uniquely defined. Nevertheless the permutation always exists and the value $x_{p_i(t)}$ is independent of how it is selected. Therefore, for any solution $x(t)$ of (1) we can define the corresponding re-ordered solution as $x_{p_i(t)}$.

III. Main Results

Next we state our our main assumption, which will guarantee exponential convergence towards a consensus state.

Definition 1 (***Connectivity indicator function***) *Given $i \neq j \in N$, we say that $\Psi_{ij}(t) : \mathbb{R} \to \{0, 1\}$ is a connectivity indicator function if for all compact intervals $\mathcal{K} \subset \mathbb{R}$ there exits $\varepsilon_{\mathcal{K}} > 0$ yielding for all $x \in \mathcal{K}^n$ and any $t \geq 0$:*

$$\begin{aligned} &sign(x_j - x_i)[f_i(t, x) - f_i(t, x + (x_i - x_j)e_j)] \\ &\quad \geq \Psi_{ij}(t)\varepsilon_{\mathcal{K}}|x_j - x_i|. \end{aligned} \tag{3}$$

[1]This holds, for instance, when the Jacobian is uniformly bounded as a function of time.

Definition 2 (***Averaged interaction graph***) *We say that $G(N, E)$ is an averaged interaction graph for (1) if for some $T > 0$ and for all $(i, j) \in E$ there exists a connectivity indicator function $\Psi_{ij}(t)$ and $\bar{\varepsilon} > 0$ such that for all $t \geq 0$:*

$$\int_t^{t+T} \Psi_{ij}(\tau)\, d\tau \geq \bar{\varepsilon}. \tag{4}$$

Assumption 1 (***Weak Integral Connectivity***) *We say that network (1) fulfills Weak Integral Connectivity if it admits a weakly connected averaged interaction graph and every pair $(i, j) \in N^2$ has an associated connectivity indicator function. We denote by $\mathcal{T}_r \subseteq E$ and $r \in N$ the spanning tree and root node in $G(N, E)$.*

Remark 1 *Notice that if equation (4) holds for some T, it holds a fortiori for all $\tilde{T} > T$.*

Remark 2 *In the light of equation (4) this is an assumption of averaged weak connectedness across uniform time intervals, while by condition (3), the node interaction property is defined on frozen state variables across the same interval, making its verification straightforward. Notice that $x + (x_i - x_j)e_j$ is a state configuration in which the agent j-th have already reached consensus with the i-th agent. Therefore, the proposed state frozen condition is a measure of how much a single agent is able to pull agent i.*

The following fact is well-known for monotone networks and continues to hold for the considered non monotone scenario under condition (3).

Lemma 1 *The functions $x_{\max}(t)$ and $x_{\min}(t)$ are (respectively) monotonically non-increasing and non-decreasing.*

Proof: Equivalently we show that the set:

$$\mathcal{M}_c := \{x : \max_{i \in N} x_i \leq c\},$$

is forward invariant for all $c \in \mathbb{R}$. Let x in $\mathcal{M}_c$ be arbitrary. Since $\mathcal{M}_c$ is convex, its tangent cone at x is simply given by $TC_x\mathcal{M}_c = \{z : z_i \leq 0, \forall\, i : x_i = c\}$ (see Proposition 5.5, [5]). Moreover, being $f_i(t, x_i\mathbf{1}) = 0$ and taking into account condition (3), for all i such that $x_i = c$ and any t it holds:

$$\begin{aligned} &f_i(t, x) = [f_i(t, x) - f_i(t, x + (x_i - x_1)e_1)] + \\ &[f_i(t, x + (x_i - x_1)e_1) - f_i(t, x + (x_i - x_1)e_1 + (x_i - x_2)e_2)] \\ &+[f_i(t, x + (x_i - x_1)e_1 + (x_i - x_2)e_2) - \\ &f_i(t, x + (x_i - x_1)e_1 + (x_i - x_2)e_2 \\ &+ (x_i - x_3)e_3)] + \ldots + [f_i(t, x + (x_i - x_1)e_1 + (x_i - x_2)e_2 \\ &+ \ldots + (x_i - x_{n-1})e_{n-1}) \\ &- f_i(t, x + (x_i - x_1)e_1 + (x_i - x_2)e_2 \\ &+ \ldots + (x_i - x_n)e_n)] + f_i(t, x_i\mathbf{1}) \\ &\leq -\sum_{j \in N} \Psi_{ij}(t)\varepsilon_{\mathcal{K}}|x_j - x_i| \leq 0 \end{aligned}$$

Hence $f(t, x) \in TC_x\mathcal{M}_c$. As this holds for all $x \in \mathcal{M}_c$ it proves forward invariance of $\mathcal{M}_c$ (by Nagumo's Theorem - [6]) and monotonicity of $x_{\max}(t)$. A symmetric argument can be used to prove monotonicity of $x_{\min}(t)$ by showing forward invariance of $\mathcal{N}_c = \{x : \min_{i \in N} x_i \geq c\}$. ∎

In what follows we will present a key lemma which will allow us to later prove exponential asymptotic consensus.

Lemma 2 *Let $r \in N$ be the root of the spanning tree as from Assumption 1. For all initial conditions $x(0) \in \mathbb{R}^n$, there exists a*

his article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2016.2598650, IEEE Transactions on Automatic Control

finite positive integer $\bar{k}$ and $\mu > 0$ (uniform in time) such that, for all $t \geq 0$, the following holds along the solutions of (1):

$$x_{\max}(t + \bar{k}T) \leq x_{\max}(t) - \mu|x_{\max}(t) - x_r(t)| \tag{5}$$

and:

$$x_{\min}(t + \bar{k}T) \geq x_{\min}(t) + \mu|x_{\min}(t) - x_r(t)|. \tag{6}$$

Proof: We prove the Lemma for $x_{\max}(t)$, a similar argument holds for $x_{\min}(t)$. Let $\varepsilon = \bar{\varepsilon}\varepsilon_{\mathcal{K}}$, and $d(q) : N \to \mathbb{N}$ denote the distance in the spanning tree of node q from the root r of the tree $\mathcal{T}_r$ as in Assumption 1. Let us deal first with nodes q at distance $d(q) = 1$. We carry out an iterative proof where each **STEP** is composed of several cases.

STEP 1

Case a): $x_r(\tau) \leq x_q(\tau)$, for all $\tau \in [t, t + 2T]$.

To come up with a suitable estimate we further need to consider the following subcases.

Subcase a_1) $\delta_q(t) \leq \frac{1}{2}\delta_r(t)$. Define $\bar{q}(\tau) \in \{1, \ldots, n\}$ so as to fulfill $p_{\bar{q}(\tau)}(\tau) = q$. In the following expressions, the time dependence of $\bar{q}$ will be omitted for the sake of simplicity of notation. Then, for any node q at distance 1 from the root it holds for all $\tau \in [t, t + 2T]$:

$$\begin{aligned}
&x_q(\tau) - x_q(t) = \int_t^\tau f_q(\theta, x(\theta))\, d\theta = \\
&\int_t^\tau \Big([f_q(\theta, x(\theta)) - f_q(\theta, x(\theta) + (x_q(\theta) - \\
&x_{p_{\bar{q}-1}(\theta)}(\theta))e_{p_{\bar{q}-1}(\theta)})] + [f_q(\theta, x(\theta) + \\
&(x_q(\theta) - x_{p_{\bar{q}-1}(\theta)}(\theta))e_{p_{\bar{q}-1}(\theta)}) - f_q(\theta, x(\theta) + \\
&(x_q(\theta) - x_{p_{\bar{q}-1}(\theta)}(\theta))e_{p_{\bar{q}-1}(\theta)} + (x_q(\theta) - \\
&x_{p_{\bar{q}-2}(\theta)}(\theta))e_{p_{\bar{q}-2}(\theta)})] + [f_q(\theta, x(\theta) + (x_q(\theta) \\
&- x_{p_{\bar{q}-1}(\theta))}(\theta))e_{p_{\bar{q}-1}(\theta)} \\
&+ (x_q(\theta) - x_{p_{\bar{q}-2}(\theta)}(\theta))e_{p_{\bar{q}-2}(\theta)}) - \\
&f_q(\theta, x(\theta) + (x_q(\theta) - x_{p_{\bar{q}-1}(\theta)}(\theta))e_{p_{\bar{q}-1}(\theta)} \\
&+ (x_q(\theta) - x_{p_{\bar{q}-2}(\theta)}(\theta))e_{p_{\bar{q}-2}(\theta)} \\
&+ (x_q(\theta) - x_{p_{\bar{q}-3}(\theta)}(\theta))e_{p_{\bar{q}-3}(\theta)})] + \\
&\ldots + [f_q(\theta, x(\theta) + (x_q(\theta) - x_{p_{\bar{q}-1}(\theta)}(\theta))e_{p_{\bar{q}-1}(\theta)} + \ldots \\
&+ (x_q(\theta) - x_{p_2(\theta)}(\theta))e_{p_2(\theta)}) \\
&- f_q(\theta, x(\theta) + (x_q(\theta) - x_{p_{\bar{q}-1}(\theta)}(\theta))e_{p_{\bar{q}-1}(\theta)} \\
&+ \ldots + (x_q(\theta) - x_{p_1(\theta)}(\theta))e_{p_1(\theta)})] \\
&+ f_q(\theta, x(\theta) + (x_q(\theta) - x_{p_{\bar{q}-1}(\theta)}(\theta))e_{p_{\bar{q}-1}(\theta)} \\
&+ \ldots + (x_q(\theta) - x_{p_1(\theta)}(\theta))e_{p_1(\theta)}) \Big) d\theta
\end{aligned}$$

The application of Assumption 1 to each of the terms in the integrand of the previous expression (except for the last one) leads to:

$$\begin{aligned}
&x_q(\tau) - x_q(t) \leq \\
&- \int_t^\tau \sum_{j: x_j(\theta) < x_q(\theta)} \Psi_{qj}(\theta)\varepsilon_{\mathcal{K}}|x_q(\theta) - x_j(\theta)| d\theta \\
&+ \int_t^\tau f_q(\theta, x(\theta) + (x_q(\theta) - x_{p_{\bar{q}-1}(\theta)}(\theta))e_{p_{\bar{q}-1}(\theta)} \\
&+ \ldots + (x_q(\theta) - x_{p_1(\theta)}(\theta))e_{p_1(\theta)})\ d\theta
\end{aligned}$$

The former calculations are instrumental for the subsequent exploitation of uniform Lipshitz continuity of f as detailed below:

$$\begin{aligned}
&x_q(\tau) - x_q(t) \leq \\
&- \int_t^\tau \sum_{j: x_j(\theta) < x_q(\theta)} \Psi_{qj}(\theta)\varepsilon_{\mathcal{K}}|x_q(\theta) - x_j(\theta)| d\theta \\
&+ \int_t^\tau \Big([f_q(\theta, x(\theta) + (x_q(\theta) - x_{p_{\bar{q}-1}})e_{p_{\bar{q}-1}} + \\
&\ldots + (x_q(\theta) - x_{p_1(\theta)}(\theta))e_{p_1(\theta)}) - f_q(\theta, x_q(\theta)\mathbf{1})] \\
&+ f_q(\theta, x_q(\theta)\mathbf{1}) \Big)\ d\theta.
\end{aligned}$$

Being $f_q(\theta, x_q(\theta)\mathbf{1}) = 0$, it results:

$$\begin{aligned}
&x_q(\tau) - x_q(t) \leq \\
&- \int_t^\tau \sum_{j: x_j(\theta) < x_q(\theta)} \Psi_{qj}(\theta)\varepsilon_{\mathcal{K}}|x_q(\theta) - x_j(\theta)| d\theta \\
&- L \int_t^\tau \sum_{j: x_j(\theta) \geq x_q(\theta)} [x_q(\theta) - x_j(\theta)]\ d\theta \\
&\leq - \int_t^\tau \sum_{j: x_j(\theta) < x_q(\theta)} \Psi_{qj}(\theta)\varepsilon_{\mathcal{K}}|x_q(\theta) - x_j(\theta)| d\theta \\
&- L \int_t^\tau \sum_{j: x_j(\theta) \geq x_q(\theta)} [x_q(\theta) - x_{\max}(t)]\ d\theta \\
&\leq - \int_t^\tau \sum_{j: x_j(\theta) < x_q(\theta)} \Psi_{qj}(\theta)\varepsilon_{\mathcal{K}}|x_q(\theta) - x_j(\theta)| d\theta \\
&- (n-1)L \int_t^\tau [x_q(\theta) - x_{\max}(t)]\ d\theta,
\end{aligned}$$

with L denotes the (time-independent) Lipschitz constant of f_q. In particular for all $\tau \in [t + T, t + 2T]$ we see that:

$$\begin{aligned}
&x_q(\tau) - x_{\max}(t) \leq x_q(\tau) - x_q(t) \\
&\leq - \int_t^\tau \sum_{j: x_j(\theta) < x_q(\theta)} \Psi_{qj}(\theta)\varepsilon_{\mathcal{K}}|x_q(\theta) - x_j(\theta)| d\theta \\
&- (n-1)L \int_t^\tau [x_q(\theta) - x_{\max}(t)]\ d\theta \\
&\leq - \int_t^\tau \Psi_{qr}(\theta)\, \varepsilon_{\mathcal{K}}|x_q(\theta) - x_r(\theta)| d\theta \\
&- (n-1)L \int_t^\tau [x_q(\theta) - x_{\max}(t)]\ d\theta.
\end{aligned} \tag{7}$$

By the triangular inequality it holds:

$$-|x_q(\theta) - x_r(\theta)| \leq -|x_{\max}(t) - x_r(\theta)| + |x_{\max}(t) - x_q(\theta)|,$$

moreover, by Lipschitz continuity of f_r, we may infer:

$$|x_{\max}(t) - x_r(\theta)| \geq e^{-L(\theta - t)}|x_{\max}(t) - x_r(t)|.$$

Combining the above inequalities, we may restate the bound for $x_q(\tau) - x_{\max}(t)$ expressed in (7) as detailed below:

$$\begin{aligned}
&x_q(\tau) - x_{\max}(t) \leq \\
&- \int_t^\tau \Psi_{qr}(\theta)\varepsilon_{\mathcal{K}} e^{-L(\theta - t)}|x_{\max}(t) - x_r(t)| d\theta \\
&- \int_t^\tau \Psi_{qr}(\theta)\varepsilon_{\mathcal{K}}[x_q(\theta) - x_{\max}(t)]\ d\theta \\
&- (n-1)L \int_t^\tau [x_q(\theta) - x_{\max}(\theta)]\ d\theta
\end{aligned} \tag{8}$$

$$\begin{aligned}&\leq -\varepsilon_{\mathcal{K}} e^{-2LT} |x_{\max}(t) - x_r(t)| \int_t^\tau \Psi_{qr}(\theta)\, d\theta \\ &- \varepsilon_{\mathcal{K}} \int_t^\tau [x_q(\theta) - x_{\max}(t)]\ d\theta \\ &- (n-1)L \int_t^\tau [x_q(\theta) - x_{\max}(t)]\ d\theta \qquad (9)\\ &\leq -\varepsilon e^{-2LT}|x_{\max}(t) - x_r(t)| - \varepsilon_{\mathcal{K}} \int_t^\tau [x_q(\theta) - x_{\max}(t)]\ d\theta \\ &- (n-1)L \int_t^\tau [x_q(\theta) - x_{\max}(t)]\ d\theta,\end{aligned}$$

with $\varepsilon = \varepsilon_{\mathcal{K}}\bar{\varepsilon}$. By defining $\Delta(\tau) = \int_t^\tau [x_q(\theta) - x_{\max}(t)]\, d\theta$ we can recast equation (8) as:

$$\begin{aligned}&\frac{d}{d\tau}\Delta(\tau) \leq -\varepsilon e^{-2LT}|x_{\max}(t) - x_r(t)| \\ &- ((n-1)L + \varepsilon_{\mathcal{K}})\Delta(\tau),\end{aligned}$$

which holds for all $\tau \in [t+T, t+2T]$. Since $\Delta(t+T) \leq 0$, by a standard comparison principle we see that:

$$\Delta(\tau) \leq -\mu_\Delta(\tau)|x_{\max}(t) - x_r(t)|, \qquad (10)$$

with

$$\mu_\Delta(\tau) = e^{-2LT}\frac{\varepsilon[1 - e^{-((n-1)L+\varepsilon_{\mathcal{K}})(\tau - T - t)}]}{((n-1)L + \varepsilon_{\mathcal{K}})},$$

which holds for all $\tau \in [t+T, t+2T]$. In particular, for $\tau = t+2T$ equation (10) yields:

$$\Delta(t+2T) \leq -\mu_\Delta|x_{\max}(t) - x_r(t)|, \qquad (11)$$

with

$$\mu_\Delta = \frac{\varepsilon}{((n-1)L + \varepsilon_{\mathcal{K}})}e^{-2LT}[1 - e^{-((n-1)L+\varepsilon_{\mathcal{K}})T}].$$

From the mean value theorem it results:

$$\exists t^* \in [t, t+2T] : x_q(t^*) - x_{\max}(t) = \frac{\Delta(t+2T)}{2T}. \qquad (12)$$

By Lipschitz continuity of f, convergence of $x_q(t)$ towards $x_{\max}(t)$ is at most exponential in time and therefore we may infer:

$$x_q(t+2T) - x_{\max}(t) \leq (x_q(t^*) - x_{\max}(t))e^{-2LT}. \qquad (13)$$

From (12) and (13) it results:

$$x_q(t+2T) - x_{\max}(t) \leq \frac{\Delta(t+2T)}{2T}e^{-2LT} \qquad (14)$$

Finally, in order to derive an estimate of how decreasing is $x_q(t)$ which is uniform in time we combine (14) and (11) and obtain:

$$x_q(t+2T) - x_{\max}(t) \leq -\mu_{a_1}\delta_r(t),$$

with $\mu_{a_1} = e^{-4LT}\frac{\varepsilon[1-e^{-((n-1)L+\varepsilon_{\mathcal{K}})T}]}{2((n-1)L+\varepsilon_{\mathcal{K}})T}$ and $\delta_r(t) = |x_{\max}(t) - x_r(t)|$.

Subcase a_2) $\delta_q(t) > \frac{1}{2}\delta_r(t)$

In this scenario, by Lipschitz continuity of f, convergence of $x_q(\tau)$ towards the value $x_{\max}(t)$ is at most exponential, and therefore we may infer:

$$|x_{\max}(t) - x_q(t+2T)| \geq e^{-2LT}|x_{\max}(t) - x_q(t)|.$$

that yields:

$$\begin{aligned}&x_q(t+2T) \leq x_{\max}(t) - e^{-2LT}|x_{\max}(t) - x_q(t)| \\ &= x_{\max}(t) - e^{-2LT}\delta_q(t) \leq x_{\max}(t) - \mu_{a_2}\delta_r(t),\end{aligned}$$

with $\mu_{a_2} = \frac{1}{2}e^{-2LT}$ and $\delta_r(t) = |x_{\max}(t) - x_r(t)|$.

Case b): $x_r(\tau) \geq x_q(\tau)$ for all $\tau \in [t, t+2T]$

In this case considering that $\delta_q(t) \geq \delta_r(t)$ and exploiting Lipschitz continuity of f, we may infer:

$x_q(t+2T) \leq x_{\max}(t) - e^{-2LT}|x_{\max}(t) - x_q(t)| = x_{\max}(t) - e^{-2LT}\delta_q(t) \leq x_{\max}(t) - \mu_b\delta_r(t)$, with $\mu_b = e^{-2LT}$.

Case c): $\exists\bar{\tau} \in (0, 2T]$ such that $x_q(t+\bar{\tau}) = x_r(t+\bar{\tau})$.

By Lipschitz continuity of f, convergence of x_r and x_q towards the value $x_{\max}(t)$ is at most exponential. This, along with assumption $x_q(t+\bar{\tau}) = x_r(t+\bar{\tau})$, yields:

$$\begin{aligned}&|x_{\max}(t) - x_q(t+2T)| \geq e^{-L(2T-\bar{\tau})}|x_{\max}(t) - x_q(t+\bar{\tau})| \\ &= e^{-L(2T-\bar{\tau})}|x_{\max}(t) - x_r(t+\bar{\tau})| \geq e^{-2LT}|x_{\max}(t) - x_r(t)|,\end{aligned}$$

and therefore $x_q(t+2T) \leq x_{\max}(t) - \mu_c\delta_r(t)$ with $\mu_c = e^{-2LT}$. Therefore, in any of cases a, b and c it results

$$x_q(t+2T) \leq x_{\max}(t) - \mu_1\delta_r(t)$$

or in other terms:

$$|x_{\max}(t) - x_q(t+2T)| \geq \mu_1\delta_r(t) \qquad (15)$$

with $\mu_1 = \min\{\mu_{a_1}, \mu_{a_2}, \mu_b, \mu_c\}$ and $\delta_r(t) = |x_{\max}(t) - x_r(t)|$.

STEP 2

Next we deal with nodes $k \in N$ with $d(k) = 2$. Let q be such that $d(q) = 1$ and $(q,k) \in \mathcal{T}_r$. We consider different cases.

Case a): $x_k(t+\tau) \geq x_q(t+\tau)$, for all $\tau \in [2T, 4T]$

Subcase a_1): $\delta_k(t+2T) \leq \frac{1}{2}\delta_q(t+2T)$. The analytical derivation is similar to that of the **STEP 1**-*Subcase* a_1) and here omitted for sake of brevity. It yields to the following estimates:

$$\begin{aligned}&x_k(t+4T) - x_{\max}(t) \leq \frac{\Delta(t+4T)}{2T}e^{-2LT} \qquad (16)\\ &\leq -\mu_{a_1}|x_{\max}(t) - x_q(t+2T)|\end{aligned}$$

with $\mu_{a_1} = e^{-4LT}\frac{\varepsilon[1-e^{-((n-1)L+\varepsilon_{\mathcal{K}})T}]}{2((n-1)L+\varepsilon_{\mathcal{K}})T}$.

Subcase a_2): $\delta_k(t+2T) \geq \frac{1}{2}\delta_q(t+2T)$. In this scenario taking into account that by Lipschitz continuity of f it results $|x_{\max}(t+4T) - x_k(t+4T)| \geq e^{-2LT}|x_{\max}(t+2T) - x_k(t+2T)|$, we may infer:

$$\begin{aligned}&x_k(t+4T) - x_{\max}(t) = x_k(t+4T) - x_{\max}(t+2T) \\ &- [x_{\max}(t) - x_{\max}(t+2T)] \leq x_k(t+4T) - x_{\max}(t+4T) \\ &- [x_{\max}(t) - x_{\max}(t+2T)] \\ &\leq -e^{-2LT}|x_{\max}(t+2T) - x_k(t+2T)| \\ &- [x_{\max}(t) - x_{\max}(t+2T)] \\ &= -e^{-2LT}\delta_k(t+2T) - [x_{\max}(t) - x_{\max}(t+2T)] \\ &\leq -e^{-2LT}\frac{1}{2}\delta_q(t+2T) - [x_{\max}(t) - x_{\max}(t+2T)] \\ &= -e^{-2LT}\frac{1}{2}[x_{\max}(t+2T) - x_q(t+2T)] \\ &- [x_{\max}(t) - x_{\max}(t+2T)] \\ &\leq -e^{-2LT}\frac{1}{2}[x_{\max}(t) - x_q(t+2T)] \\ &+ e^{-2LT}\frac{1}{2}[x_{\max}(t) - x_{\max}(t+2T)] - [x_{\max}(t) - x_{\max}(t+2T)] \\ &= -\mu_{a_2}[x_{\max}(t) - x_q(t+2T)] \\ &- (1-\mu_{a_2})[x_{\max}(t) - x_{\max}(t+2T)] \\ &\leq -\mu_{a_2}[x_{\max}(t) - x_q(t+2T)]\end{aligned}$$

being $\mu_{a_2} = e^{-2LT}\frac{1}{2} < 1$.

Case b): $x_q(t+\tau) \geq x_k(t+\tau)$ $\tau \in [2T, 4T]$

In this case, we may infer:

$$\begin{aligned}&x_k(t+4T) \leq x_{\max}(t) - e^{-2LT}|x_{\max}(t) - x_k(t+2T)| \\ &\leq x_{\max}(t) - \mu_b|x_{\max}(t) - x_q(t+2T)|,\end{aligned}$$

with $\mu_b = e^{-2LT}$.

Case c): $x_q(t+\bar{\tau}) = x_k(t+\bar{\tau})$ for some $\bar{\tau} \in (2T, 4T]$.
By Lipschitz continuity of f, it results:

$$\begin{aligned} |x_{\max}(t) - x_k(t+4T)| \geq\ & e^{-L(4T-\bar{\tau})}|x_{\max}(t) - x_k(t+\bar{\tau})| \\ = e^{-L(4T-\bar{\tau})}&|x_{\max}(t) - x_q(t+\bar{\tau})| \\ \geq e^{-2LT}&|x_{\max}(t) - x_q(t+2T)| = \mu_c|x_{\max}(t) - x_q(t+2T)|, \end{aligned}$$

with $\mu_c = e^{-2LT}$.

Therefore, in any of cases a, b and c it results:

$$x_k(t+4T) \leq x_{\max}(t) - \mu_2|x_{\max}(t) - x_q(t+2T)| \qquad (17)$$

with $\mu_2 = \min\{\mu_{a_1}, \mu_{a_2}, \mu_b, \mu_c\}$.

Consequently, in order to derive an estimate of how decreasing is $x_k(t)$ which is uniform in time by combining (17) and (15) we obtain:

$$x_k(t+4T) - x_{\max}(t) \leq -\mu_1\mu_2|x_{\max}(t) - x_r(t)|. \qquad (18)$$

A similar procedure can be used to construct an estimate of the convergence rate for an arbitrary node at distance $d(k)+1$ based on the estimate for nodes at distance $d(k)$. By induction, for any node k at distance $d(k)$ from the root, the following inequality holds:

$$\begin{aligned} x_k(t+2d(k)T) - x_{\max}(t) &\leq -\left(\prod_{i=1}^{d(k)}\mu_i\right)|x_{\max}(t) - x_r(t)| \\ &= -\mu(d(k))|x_{\max}(t) - x_r(t)|, \end{aligned}$$

with $\mu(d(k)) = \prod_{i=1}^{d(k)}\mu_i$ being a positive constant for any fixed $d(k)$. Given the fact that only a finite number of agents are present and by Assumption 1 every agent k has a finite distance from the root, a uniform estimate of the convergence rate can be provided. Estimate (19) is still not of the form needed to prove our claim as the estimated rate of contraction is $d(k)$-dependent and the number of T intervals needed in order to guarantee such decrease in $x_{\max}$ is proportional to $d(k)$. Nevertheless, $\mu(d(k)) \in (0,1)$ for any $d(k)$ and by monotonicity of $x_{\max}(t)$ (see Lemma 1) and finiteness of the number of agents, one can take without loss of generality $\bar{k} := 2(n-1)$ and $\mu = \mu(\bar{k}/2) = \mu(n-1)$. This concludes the proof of the Lemma. ■

Theorem 1 *Consider the network modeled by equations* (1), *if Assumptions 1 hold, then the equilibrium set is uniformly exponentially stable and, for any initial condition* $x(0)$, $x(t)$ *converges to an agreement equilibrium state with the following contraction rate:*

$$V(x(t+s\bar{k}T)) \leq (1-\hat{\mu})^s V(x(t)). \qquad (19)$$

with $\hat{\mu} = \mu(n-1)$.

The result follows by a Lyapunov argument, considering the function: $V(x) = \max_{k\in N} x_k - \min_{k\in N} x_k$, and exploiting Lemmas 1 and 2. It is omitted for sake of brevity.

IV. Example and Counterexample

In this Section we will discuss two illustrative examples. We consider the following nonlinear non-monotone network composed of agents N connected according to the topology of a connected graph $G(N,E)$ as detailed in the following equations:

$$\begin{aligned} \dot{x}_i = \sigma_i(t) &\sum_{j\in N_i} \frac{\max\{0, x_j - x_i\}}{1+[\max\{0, x_j - x_i\}]^2} \qquad (20) \\ + \sigma_i(t) &\sum_{j\in N_i} \frac{\min\{0, x_j - x_i\}}{1+[\min\{0, x_j - x_i\}]^2}, \end{aligned}$$

N_i being the set of neighbours of node i, $\sigma_i(t) = \sigma(t-(i-1)\bar{\tau})$, $i = 1\ldots4$, $\bar{\tau} = 0.25$ and $\sigma(t)$ is a periodic function represented in

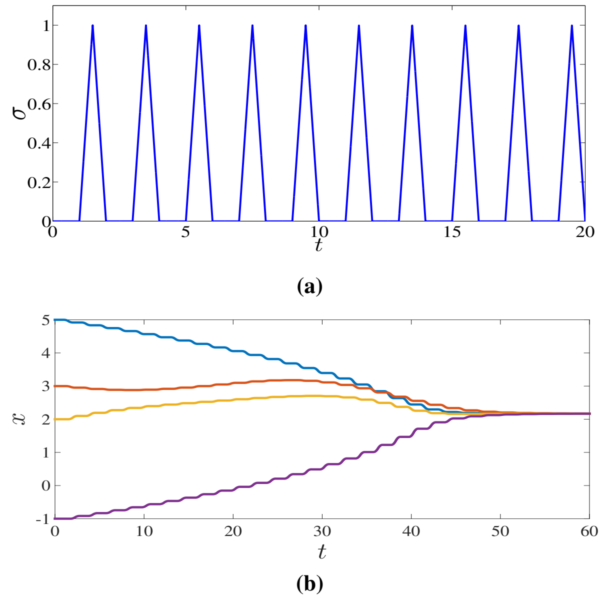

Fig. 1. Convergence to the consensus state: (a) $\sigma(t)$ (b) dynamic evolution of x.

Fig. 1-(a) with period $T_\sigma = 2$. Let the connectivity indicator function be:

$$\Psi_{ij}(t) = \begin{cases} 1, & \text{if } \sigma(t) \geq 0.2, \quad (i,j) \in \mathcal{T}_r; \\ 0, & otherwise, \end{cases}$$

we may take $T = 2$, so that $\int_t^{t+T}\Psi_{ij}(\tau)\,d\tau = \frac{2}{5}T_\sigma = 0.8 := \bar{\varepsilon} > 0$ holds for every $(i,j) \in E$. Moreover, by letting $\varepsilon_{\mathcal{K}} = 0.2/[1 + diam^2(\mathcal{K})]$ we see that

$$\begin{aligned} \mathrm{sign}(x_j - x_i)\sigma_i(t)&\left\{\frac{\max\{0, x_j - x_i\}}{1+[\max\{0, x_j - x_i\}]^2} + \right. \\ &\left.\frac{\min\{0, x_j - x_i\}}{1+[\min\{0, x_j - x_i\}]^2}\right\} \geq \Psi_{ij}(t)\varepsilon_{\mathcal{K}}|x_j - x_i|, \end{aligned}$$

for all t and all $(i,j) \in E$. Therefore, $G(N,E)$ is an Averaged interaction graph, and, being connected, it admits a spanning tree $\mathcal{T}_r$ as requested in Assumption 1. It is worth pointing out that the assumptions are stated in terms of "frozen" state variables, greatly simplifying the a priori verification of the conditions guaranteeing exponential consensus for non-monotone nonlinear time-varying networks (see Fig. 1-(b) for a simulation).

Remark 3 *Verification of connectivity conditions according to integral type ones (i.e. Moreau's definition in [9]) is not straightforward. Similar difficulties are encountered with all available criteria for consensus of time-varying linear networks ([8], [12], [13]), including the approach for nonlinear networks proposed in [16]. The fully time-varying set-up, with time-varying weights as in equation (20), is also not covered by asymptotic consensus conditions for state dependent nonlinear networks [14].*

Notice that the proposed analytical derivation does not cope with the scenario where the interaction can vanish at finite distance (in this respect a preliminary investigation is reported in [15]), rather with interactions vanishing asymptotically. Next we consider an example showing how Assumption 1 cannot be relaxed by allowing state and time dependent connectivity indicator functions even in the case of a pseudo linear embedding. Specifically, we show that if

$\Psi_{ij} = \Psi_{ij}(t, x)$, Assumption 1 is not sufficient to claim consensus. Let consider a network of 3 agents:

$$\begin{aligned}
\dot{x}_1 &= \max\left\{0, |x_2 - x_1 + 2 - \sin(t)| - \frac{1}{2}\right\} (x_2 - x_1) \\
\dot{x}_2 &= \frac{1}{2\cos\frac{\pi}{6}}\left(\sin^2\left(\frac{t}{2} - \frac{\pi}{12}\right)(x_3 - x_2)\right. \\
&\quad \left. + \cos^2\left(\frac{t}{2} - \frac{\pi}{12}\right)(x_1 - x_2)\right) \\
\dot{x}_3 &= \max\left\{0, |x_2 - x_3 - 2 - \sin(t)| - \frac{1}{2}\right\} (x_2 - x_3).
\end{aligned} \tag{21}$$

By direct computation it is possible to show that $\exists \bar{\varepsilon}_{\mathcal{K}} > 0 : \forall x_1, x_2, x_3 \in \mathcal{K}, \forall t \geq 0$,

$$\int_t^{t+2\pi} \max\left\{0, |x_2 - x_1 + 2 - \sin(\tau)| - \frac{1}{2}\right\} d\tau \geq \bar{\varepsilon}_{\mathcal{K}},$$

and

$$\int_t^{t+2\pi} \max\left\{0, |x_2 - x_3 - 2 - \sin(\tau)| - \frac{1}{2}\right\} d\tau \geq \bar{\varepsilon}_{\mathcal{K}},$$

which can be seen as a form of state dependent connectivity from x_2 respectively to x_1 and x_3. Moreover,

$$\forall t \geq 0, \quad \frac{1}{2\cos\left(\frac{\pi}{6}\right)} \int_t^{t+2\pi} \sin^2\left(\frac{\tau}{2} - \frac{\pi}{12}\right) d\tau = \frac{\pi}{\sqrt{3}},$$

which ensures (purely time dependent) connectivity from x_3 to x_2, and, similarly:

$$\forall t \geq 0, \quad \frac{1}{2\cos\left(\frac{\pi}{6}\right)} \int_t^{t+2\pi} \cos^2\left(\frac{\tau}{2} - \frac{\pi}{12}\right) d\tau = \frac{\pi}{\sqrt{3}}$$

for the connectivity from x_1 to x_2. It is possible to show that network (21) does not always converge to a consensus equilibrium, as the vector $[2, \sin(t), -2]$ is the solution corresponding to initial condition $[2, 0, -2]$. Indeed let the initial state condition $[2, 0, -2]$, the vector $\bar{x}(t) \doteq [2, \sin(t), -2]$ is the solution of the second equation in (21) being:

$$\begin{aligned}
\dot{\bar{x}}_2 &= \frac{1}{2\cos\frac{\pi}{6}}\left(\sin^2\left(\frac{t}{2} - \frac{\pi}{12}\right)(-2 - \sin(t)) + \right. \\
&\left.\cos^2\left(\frac{t}{2} - \frac{\pi}{12}\right)(2 - \sin(t))\right) = \frac{1}{2\cos\frac{\pi}{6}}\left(-2\sin^2\left(\frac{t}{2} - \frac{\pi}{12}\right)\right. \\
&\quad \left. + 2\cos^2\left(\frac{t}{2} - \frac{\pi}{12}\right) - \sin(t)\right) \\
&= \frac{1}{2\cos\frac{\pi}{6}}\left(2\cos\left(t - \frac{\pi}{6}\right) - \sin(t)\right) \\
&= \frac{1}{2\cos\frac{\pi}{6}}\left(2\cos(t)\cos\left(\frac{\pi}{6}\right) + 2\sin(t)\sin\left(\frac{\pi}{6}\right) - \sin(t)\right) \\
&= \frac{1}{2\cos\frac{\pi}{6}}\left(2\cos(t)\cos\left(\frac{\pi}{6}\right) + 2\sin(t)\frac{1}{2} - \sin(t)\right) \\
&= \frac{1}{2\cos\frac{\pi}{6}}\left(2\cos(t)\cos\left(\frac{\pi}{6}\right)\right) = \cos(t) = \frac{d}{dt}\sin(t),
\end{aligned}$$

Remark 4 *The example shows how lack of monotonicity allows the creation of state and time dependent "holes" where influence of neighbouring agents is not felt. While all state locations experience enough interaction strength over time, individual agents (which are generally not frozen at a particular location) may fail to perform their pulling task as they keep moving within such holes. To the best of our knowledge no criteria for consensus exist in the literature to allow existence of connectivity holes and yet ruling out the possibility of individual agents getting trapped within them as in the example. While the example may appear as pathological, this is meant to illustrate how our conditions are in some sense necessary, and where the main obstruction stands in order to possibly generalize our criterion.*

V. Conclusions

In this paper, we introduced a criterion for asymptotic exponential consensus avoiding the assumption of monotonicity that is widely adopted, explicitly or implicitly, in the literature. The criterion extends to this class of nonlinear scenario with state dependent dynamics and coupling the possibility of guaranteeing consensus through an averaged notion of connectivity allowing the coupling strength interactions to vanish as the state agent distance goes to infinity.

References

[1] R.M. Murray, *Recent Research in Cooperative Control of Multivehicle Systems*, Journal of Dynamic Systems, Measurement, and Control, 129, 5, 2007

[2] Cucker, F., Smale, S., *Emergent Behavior in Flocks*, IEEE Transactions on Automatic Control, 52, 5. 2007.

[3] J. M. Hendrickx, *Graphs and Networks for the Analysis of Autonomous Agent Systems*, Dpartement Dingnierie Mathmatique, 2008

[4] A. Olshevsky, J.N. Tsitsiklis, *Degree Fluctuations and the Convergence Time of Consensus Algorithms*, IEEE Transactions on Automatic Control, 58, 10, 2013

[5] F.H. Clarke, Y.S. Ledyaev, R.J. Stern and P.R. Wolenski, Nonsmooth Analysis and Control Theory, *Springer*, 1998.

[6] F. Blanchini and S. Miani, *Set Theoretic Methods in Control*, Systems and Control Foundations and Applications, Birkhauser, 2008.

[7] Chopra, N., Spong, M. *Passivity Based Control of Multi-Agent Systems*. Advances in Robot Control From Everyday Physics in Human-Like Movements, 107-134, 2007.

[8] V. D. Blondel, J. M. Hendrickx, A. Olshevsky, and J. N. Tsitsiklis. *Convergence in multiagent coordination, consensus, and flocking*. IEEE CDC and ECC, 2005.

[9] L. Moreau. *Stability of continuous-time distributed consensus algorithms*, Proc. 43rd IEEE Conf. Decision and Control, 2004.

[10] S. Manfredi and D. Angeli. *Frozen State Conditions for Asymptotic Consensus of Time-Varying Cooperative Nonlinear Networks*. 52nd IEEE Conference on Decision and Control, DOI:10.1109/CDC.2013.6760066, December 10-13, Firenze, Italy 2013.

[11] S. Manfredi and D. Angeli. *Frozen state conditions for exponential consensus of time-varying cooperative nonlinear networks*. Automatica, 64, 182 - 189, . DOI: 10.1016/$j.automatica$.2015.11.011, 2016

[12] J. M. Hendrickx and J. N. Tsitsiklis. *Convergence of type-symmetric and cut-balanced consensus seeking systems*. IEEE Transactions on Automatic Control, 58, 1, 2013

[13] G. Shi and H. Johansson, *The Role of Persistent Graphs in the Agreement Seeking of Social Networks*, IEEE Journal of Selected areas in communications, 31, 9, 595-606. 2012

[14] Z. Lin, B. Francis, and M. Maggiore. *State agreement for continuous time coupled nonlinear systems*. SIAM J. Contr., 46, 1, 2007.

[15] S. Manfredi, D. Angeli. *Consensus of time-varying nonlinear non-autonomous networks with application to field sampling by mobile robots*. IEEE 54th Annual Conference on Decision and Control (CDC), 4848 - 4853, DOI: 10.1109/CDC.2015.7402976, Osaka, Japan, 2015

[16] Z. Qu. *Control of Dynamical Systems*, Springer Verlag, London 2009.

[17] J.-J. E. Slotine and W. Wang, *A study of synchronization and group cooperation using partial contraction theory*, in Cooperative Control, vol. 309, Springer-Verlag, 2004.

[18] C. Altafini, *Consensus problems on networks with antagonistic interactions*, IEEE Transactions on Automatic Control, Vol. 58, No. 4, 2013.

[19] A. V. Proskurnikov, A. Matveev and M. Cao, *Opinion Dynamics in Social Networks with Hostile Camps: Consensus vs. Polarization*, IEEE Transactions on Automatic Control, DOI: 10.1109/TAC.2015.2471655,in press, 2016.

[20] H. L. Smith. *Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems*, Volume 41 of Mathematical Surveys and Monographs. AMS, Providence, RI, 1995.