

On dynamic spectral risk measures, a limit theorem and optimal portfolio allocation

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Abstract. In this paper we propose the notion of continuous-time *dynamic spectral risk-measure* (DSR). Adopting a Poisson random measure setting, we define this class of dynamic coherent risk-measures in terms of certain backward stochastic differential equations. By establishing a functional limit theorem, we show that DSRs may be considered to be (strongly) time-consistent continuous-time extensions of iterated spectral risk-measures, which are obtained by iterating a given spectral risk-measure (such as Expected Shortfall) along a given time-grid. Specifically, we demonstrate that any DSR arises in the limit of a sequence of such iterated spectral risk-measures driven by lattice-random walks, under suitable scaling and vanishing time- and spatial-mesh sizes. To illustrate its use in financial optimisation problems, we analyse a dynamic portfolio optimisation problem under a DSR.

1 Introduction

Financial analysis and decision making rely on quantification and modelling of future risk exposures. A systematic approach for the latter was put forward in [3], laying the foundations of an axiomatic framework for coherent measurement of risk. A subsequent breakthrough was the development and application of the notion of backward stochastic differential equations (BSDEs) in the context of risk analysis, which gave rise to the (strongly) time-consistent extension of coherent risk-measures to continuous-time dynamic settings [39, 42]. Building on these advances, we consider in this article a new class of such continuous-time dynamic coherent risk measures, which we propose to call *dynamic spectral risk measures* (DSRs).

Quantile-based coherent risk measures, such as *Expected Shortfall*, belong to the most widely used risk-measures in risk analysis, and are also known as *spectral risk measures*, *Choquet expectations* (based on probability distortions) and *Weighted VaR*; see [1, 12, 34, 48]. In order to carry out, for instance, an analysis of portfolios involving dynamic rebalancing, one is lead to consider the (strongly) time-consistent extension of such coherent risk-measures to given time-grids, which are defined by iterative application of the spectral risk-measure along these particular grids. Due to its continuous-time domain of definition a DSR is, in contrast, independent of a grid structure. While the latter holds for any continuous-time risk measure we show that DSRs emerge as the limits of such iterated spectral risk measures when the time-step vanishes and under appropriate scaling of the parameters, by establishing a functional limit theorem.

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To explore its use in financial decision problems, we consider subsequently a dynamic portfolio optimisation problem under DSR, which we analyse in terms of its associated Hamilton-Jacobi-Bellman (HJB) equation. In the case of a long-only investor (who is allowed neither to borrow nor to short-sell stocks) we identify explicitly dynamic optimal allocation strategies.

DSR, like any dynamic risk-measure obtained from a BSDE, is (*strongly*) *time-consistent* in the sense that if the value of a random variable X is not larger than Y under DSR at time t almost surely, then the same relation holds at earlier times s , $s < t$. For dynamic risk-measures the property of strong time-consistency is well known to be equivalent to *recursiveness*, a tower-type property which is referred to as filtration-consistency in [16]. Such concepts have been investigated extensively in the literature; among others we mention [4, 11, 15, 17, 25, 30, 31, 40]. For studies on weaker forms of time-consistency we refer to [41, 47, 49].

The notion of strong time-consistency in *economics* goes back at least as far as [46] and has been standard in the economics literature ever since; see for instance [10, 20, 23, 24, 27, 32, 33].

Due to their recursive structure financial optimisation problems, such as utility optimisation under the entropic risk-measure and related robust portfolio optimisation problems satisfy the Dynamic Programming Principle and admit time-consistent dynamically optimal strategies (see for instance [6, 36] and references therein). In Section 6 we demonstrate that this also holds for the optimal portfolio allocation problem phrased in terms of the minimisation under a DSR, and phrase and solve this problem via the associated HJB equation.

For a given DSR, the functional limit theorem that we obtain (see Theorem 5.2) shows how to construct an approximating sequence of iterated spectral risk-measures driven by lattice random walks, suggesting an effective method to evaluate functionals under a given DSR and solutions to associated PIDEs, by recursively applying (distorted) Choquet expectations. The functional limit theorem involves a certain non-standard scaling of the parameters of the iterated spectral risk measures, which is given in Definition 5.1. The advantage of this approximation method is that it sidesteps the (typically non-trivial) task of computing the Malliavin derivatives. A numerical study is beyond the scope of the current paper, and is left for future research.

While one may prove the functional limit theorem directly through duality arguments, we present in the interest of brevity a proof that draws on the convergence results obtained in [37] for weak approximation of BSDEs. In the literature various related convergence results are available, of which we next mention a number (refer to [37] for additional references). The construction of continuous-time dynamic risk-measures arising as limits of discrete-time ones was studied in [45] in a Brownian setting. In a more general setting including in addition finitely many Poisson processes, [35] presents a limit theorem for recursive coherent quantile-based risk measures, which is proved via an associated non-linear partial differential equation. In [19] a Donsker-type theorem is established under a G -expectation.

Contents. The remainder of the paper is organised as follows. In Section 2 we collect preliminary results concerning dynamic coherent risk measures and related BSDEs, adopting a pure jump setting driven by a Poisson random measure. In Section 3 we are concerned with the Choquet-type integrals which appear in the definitions of dynamic and iterated spectral risk measures. With these results in hand, we phrase the definition of a DSR in Section 4 and identify its dual representation. In Section 5 we present the functional limit theorem for iterated spectral risk measures. Finally, in Section 6 we turn to the study of a dynamic portfolio allocation problems under a DSR.

2 Preliminaries

In this section we collect elements of the theory of time-consistent dynamic coherent risk measures and associated BSDEs, in both continuous-time and discrete-time settings. To avoid repetition we state some results and definitions in terms of the index set \mathcal{I} , which is taken to be either $\mathcal{I} = [0, T]$ or

$$\mathcal{I} = \pi_\Delta := \{t_i = i\Delta, i = 0, \dots, N\}, \text{ with } \Delta = T/N$$

for some $N \in \mathbb{N}$ and $T > 0$.

2.1 Time-consistent dynamic coherent risk measures

On some filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ with $\mathbf{F} = (\mathcal{F}_t)_{t \in I}$, we consider risks described by random variables $X \in \mathcal{L}^p = \mathcal{L}^p(\mathcal{F}_T)$, $p > 0$, the set of \mathcal{F}_T -measurable random variables X with $\mathbb{E}[|X|^p] = \int_\Omega |X|^p dP < \infty$. We denote by $\mathcal{L}_t^p = \mathcal{L}^p(\mathcal{F}_t)$ and $\mathcal{L}^p(\mathcal{G})$ the elements X in $\mathcal{L}^p(\mathcal{F})$ that are measurable with respect to the sigma-algebras \mathcal{F}_t and $\mathcal{G} \subset \mathcal{F}$, respectively, and by $\mathcal{L}^\infty, \mathcal{L}_t^\infty, \mathcal{L}^\infty(\mathcal{G})$ the collections of bounded elements in $\mathcal{L}^p, \mathcal{L}_t^p$ and $\mathcal{L}^p(\mathcal{G})$. Let $\mathcal{S}^2(\mathcal{I})$ denote the space of \mathbf{F} -adapted semi-martingales $Y = (Y_t)_{t \in \mathcal{I}}$ that are square-integrable in the sense that $\|Y\|_{\mathcal{S}^2(\mathcal{I})}^2 < \infty$, where

$$\|Y\|_{\mathcal{S}^2(\mathcal{I})}^2 := \mathbb{E} \left[\sup_{t \in \mathcal{I}} |Y_t|^2 \right].$$

For a given measure μ on a measurable space $(\mathbb{U}, \mathcal{U})$ we denote by $\mathcal{L}^p(\mu)$, $p > 0$, the set of Borel functions $v : \mathbb{U} \rightarrow \mathbb{R}$ with $|v|_{p, \mu} < \infty$, where

$$|v|_{p, \mu} := \left(\int_{\mathbb{U}} |v(x)|^p \mu(dx) \right)^{1/p},$$

and by $\mathcal{L}_+^p(\mu)$ the set of non-negative elements in $\mathcal{L}^p(\mu)$.

Dynamic coherent risk-measures and (strong) time-consistency, we recall, are defined as follows in an \mathcal{L}^2 -setting:

Definition 2.1 *A dynamic coherent risk measure $\rho = (\rho_t)_{t \in I}$ is a map $\rho : \mathcal{L}^2 \rightarrow \mathcal{S}^2(\mathcal{I})$ that satisfies the following properties:*

- (i) (cash invariance) for $m \in \mathcal{L}_+^2$, $\rho_t(X + m) = \rho_t(X) - m$;
- (ii) (monotonicity) for $X, Y \in \mathcal{L}^2$ with $X \geq Y$, $\rho_t(X) \leq \rho_t(Y)$;
- (iii) (positive homogeneity) for $X \in \mathcal{L}^2$ and $\lambda \in \mathcal{L}_+^\infty$, $\rho_t(|\lambda|X) = |\lambda|\rho_t(X)$;
- (iv) (subadditivity) for $X, Y \in \mathcal{L}^2$, $\rho_t(X + Y) \leq \rho_t(X) + \rho_t(Y)$.

Definition 2.2 *A dynamic coherent risk measure ρ is called (strongly) time-consistent if either of the following holds:*

- (v) (strong time-consistency) for $X, Y \in \mathcal{L}^2$ and s, t with $s \leq t$, $\rho_t(X) \leq \rho_t(Y) \Rightarrow \rho_s(X) \leq \rho_s(Y)$;
- (vi) (recursiveness) for $X \in \mathcal{L}^2$ and s, t with $s \leq t$, $\rho_s(\rho_t(X)) = \rho_s(X)$.

More generally, a map $\rho : \mathcal{L}^2 \rightarrow \mathcal{S}^2(\mathcal{I})$ is called a *time-consistent dynamic risk measure* if ρ satisfies conditions **(i)** and **(v)**. For a proof of the equivalence of items **(v)** and **(vi)** we refer to [26, Lemma 11.11]; for a discussion of (the unconditional version of) the properties **(i)**–**(iv)** see [3, 4]. One way to construct time-consistent dynamic risk measures is as solution to an associated backward stochastic differential equation (BSDE) or backward stochastic difference equation (BSΔE). To ensure that such dynamic risk measures are positively homogeneous and subadditive, the corresponding driver functions are to be positively homogeneous, subadditive and should not depend on the value of the risk-measure (see Proposition 11 in [42] and Lemma 2.1 in [16]). For background on the notion of strong time-consistency and its relation to g -expectations we refer to [7, 8, 39, 42]. Specifically, in our setting such driver functions are defined as follows:

Definition 2.3 *For a given Borel measure μ on $\mathbb{R}^k \setminus \{0\}$ we call a function $g : \mathcal{I} \times \mathcal{L}^2(\mu) \rightarrow \mathbb{R}$ a driver function if for any $z \in \mathcal{L}^2(\mu)$ $t \mapsto g(t, z)$ is continuous (in case $\mathcal{I} = [0, T]$) and the following holds:*

(i) (Lipschitz-continuity) *for some $K \in \mathbb{R}_+ \setminus \{0\}$ and any $t \in \mathcal{I}$ and $z_1, z_2 \in \mathcal{L}^2(\mu)$*

$$|g(t, z_1) - g(t, z_2)| \leq K |z_1 - z_2|_{2, \mu}.$$

A driver function g is called coherent if the following hold:

(ii) (positive homogeneity) *for any $r \in \mathbb{R}_+$, $t \in \mathcal{I}$ and $z \in \mathcal{L}^2(\mu)$, we have*

$$g(t, rz) = rg(t, z);$$

(iii) (subadditivity) *for any $t \in \mathcal{I}$ and $z_1, z_2 \in \mathcal{L}^2(\mu)$, we have*

$$g(t, z_1 + z_2) \leq g(t, z_1) + g(t, z_2).$$

We describe next the dynamic (coherent) risk-measure defined via the BSDEs (if $\mathcal{I} = [0, T]$) or BSΔEs (if \mathcal{I} is a finite partition of $[0, T]$).

2.2 Discrete-time lattice setting

We turn first to the discrete-time lattice setting, fixing a uniform partition $\pi = \pi_\Delta$ of $[0, T]$ with as before $\Delta = T/N$ for some $N \in \mathbb{N}$. Let $L^{(\pi)} = (L_t^{(\pi)})_{t \in \pi}$ denote a square-integrable zero-mean random walk starting at zero and taking values in $(\sqrt{\Delta}\mathbb{Z})^k$, and let $\mathbf{F}^{(\pi)} = (\mathcal{F}_t^{(\pi)})_{t \in \pi}$ denote the filtration generated by $L^{(\pi)}$. Furthermore, we let $g^{(\pi)}$ be a coherent driver function as in Definition 2.3 with $\mathcal{I} = \pi$ and $\mu(dx)$ equal to the scaled law $\nu^{(\pi)}(dx)$ of $\Delta L_t^{(\pi)} = L_{t+\Delta}^{(\pi)} - L_t^{(\pi)}$, $t \in \pi \setminus \{T\}$, given by

$$\nu^{(\pi)}(dx) := \frac{1}{\Delta} \mathbb{P}(\Delta L_t^{(\pi)} \in dx), \quad x \in (\sqrt{\Delta}\mathbb{Z})^k. \quad (2.1)$$

The BSΔE for $(Y^{(\pi)}, Z^{(\pi)})$ corresponding to final value $-X^{(\pi)} \in \mathcal{L}^2(\mathcal{F}_T^{(\pi)})$ and driver function $g^{(\pi)}$ takes the following form (in view of [37, Proposition 3.2]), which is analogous to the one in continuous-time case given in (2.6) below:

$$\begin{aligned} Y_t^{(\pi)} &= -X^{(\pi)} + \sum_{s=t}^{T-\Delta} g^{(\pi)}(s, Z_s^{(\pi)})\Delta \\ &\quad - \sum_{s=t}^{T-\Delta} \left(Z_s^{(\pi)}(\Delta L_s^{(\pi)}) I_{\{\Delta L_s^{(\pi)} \neq 0\}} - \mathbb{E} \left[Z_s^{(\pi)}(\Delta L_s^{(\pi)}) I_{\{\Delta L_s^{(\pi)} \neq 0\}} \middle| \mathcal{F}_s^{(\pi)} \right] \right) \end{aligned} \quad (2.2)$$

for $t \in \pi \setminus \{T\}$ and with $Y_T^{(\pi)} = -X^{(\pi)}$, where I_A denotes the indicator of a set A . In difference notation the BSΔE (2.2) is for $t \in \pi \setminus \{T\}$ given by

$$\begin{aligned} \Delta Y_t^{(\pi)} &= -g^{(\pi)}(t, Z_t^{(\pi)})\Delta \\ &+ Z_t^{(\pi)}(\Delta L_t^{(\pi)})I_{\{\Delta L_t^{(\pi)} \neq 0\}} - \mathbb{E} \left[Z_t^{(\pi)}(\Delta L_t^{(\pi)})I_{\{\Delta L_t^{(\pi)} \neq 0\}} \middle| \mathcal{F}_t^{(\pi)} \right] \end{aligned} \quad (2.3)$$

with $Y_T^{(\pi)} = -X^{(\pi)}$. A pair $(Y^{(\pi)}, Z^{(\pi)})$ is a solution of the BSΔE if, for any $t \in \pi$, it satisfies (2.2) with

$$Y_t^{(\pi)} \in \mathcal{L}^2(\mathcal{F}_t^{(\pi)}), \quad Z_t^{(\pi)} \in \mathbb{L}_t^2 := \mathcal{L}^2(\nu^{(\pi)}(dx) \times d\mathbb{P}, \mathcal{B}((\sqrt{\Delta}\mathbb{Z})^k) \otimes \mathcal{F}_t^{(\pi)}).$$

If the Lipschitz-constant $K = K^{(\pi)}$ of the driver function $g^{(\pi)}$ is strictly smaller than the reciprocal $1/\Delta$ of the mesh-size then it follows from [37, Propositions 3.1 and 3.2] (and using that, in the notation of [37], $F^{(p_i)}$ is independent of $W^{(p_i)}$) that there exists a unique solution $(Y^{(\pi)}, Z^{(\pi)})$ to the BSΔE which satisfies the following relations for $t \in \pi$:

$$Y_t^{(\pi)} = g^{(\pi)}(t, Z_t^{(\pi)})\Delta + \mathbb{E} \left[Y_{t+\Delta}^{(\pi)} \middle| \mathcal{F}_t^{(\pi)} \right], \quad (2.4)$$

$$\begin{aligned} Z_t^{(\pi)}(x) &= \mathbb{E} \left[Y_{t+\Delta}^{(\pi)} \middle| \mathcal{F}_t^{(\pi)} \vee \{\Delta L_t^{(\pi)} = x\} \right] \\ &- \mathbb{E} \left[Y_{t+\Delta}^{(\pi)} \middle| \mathcal{F}_t^{(\pi)} \vee \{\Delta L_t^{(\pi)} = 0\} \right] \end{aligned} \quad (2.5)$$

for $x \in (\sqrt{\Delta}\mathbb{Z})^k$, where $\mathcal{F}_t^{(\pi)} \vee \{\Delta L_t^{(\pi)} = x\} := \mathcal{F}_t^{(\pi)} \vee \sigma(\{\Delta L_t^{(\pi)} = x\})$ denotes the smallest sigma-algebra containing $\mathcal{F}_t^{(\pi)}$ as well as the sigma-algebra $\sigma(\{\Delta L_t^{(\pi)} = x\})$ generated by $\{\Delta L_t^{(\pi)} = x\}$. In analogy with the continuous-time case (reviewed below), the dynamic coherent risk-measure associated to the solution to the BSΔE is defined as follows:

Definition 2.4 For a driver function $g^{(\pi)}$ as in Definition 2.3 with $\mathcal{I} = \pi$ and $\mu(dx) = \nu^{(\pi)}$ and the solution $(Y^{(\pi)}, Z^{(\pi)})$ of the corresponding BSΔE (2.2), $\rho^{g^{(\pi)}, (\pi)} = (\rho_t^{g^{(\pi)}, (\pi)})_{t \in \pi}$ denotes the time-consistent dynamic risk measure given by $\rho_t^{g^{(\pi)}, (\pi)} : \mathcal{L}^2(\mathcal{F}_T^{(\pi)}) \rightarrow \mathcal{L}^2(\mathcal{F}_t^{(\pi)})$ with

$$\rho_t^{g^{(\pi)}, (\pi)}(X) = Y_t^{(\pi)}.$$

2.3 Continuous-time setting

In the continuous-time case ($\mathcal{I} = [0, T]$) we consider risky positions described by random variables X that are measurable with respect to \mathcal{F}_T , where $\mathbf{F} = \{\mathcal{F}_t\}_{t \in [0, T]}$ denotes the right-continuous and completed filtration generated by a Poisson random measure N on $[0, T] \times \mathbb{R}^k \setminus \{0\}$ for some $k \in \mathbb{N}$. We suppose throughout that the associated Lévy measure ν satisfies the following condition:

Assumption 2.5 The Lévy measure ν associated to the Poisson random measure N has no atoms and, for some $\varepsilon_0 > 0$, $\nu_{2+\varepsilon_0} \in \mathbb{R}_+ \setminus \{0\}$ where for $p \geq 0$

$$\nu_p := \int_{\mathbb{R}^k \setminus \{0\}} |x|^p \nu(dx).$$

We denote by $\tilde{N}(dt \times dx) = N(dt \times dx) - \nu(dx)dt$ the compensated Poisson random measure and by $L = (L_t)_{t \in [0, T]}$ the (column-vector) Lévy process given by

$$L_t = \int_{[0, t] \times \mathbb{R}^k \setminus \{0\}} x \tilde{N}(ds \times dx).$$

Under Assumption 2.5 we have $\mathbb{E}[|L_t|^{2+\varepsilon_0}] < \infty$ for any $t \in [0, T]$ (see [44, Theorem 25.3])

Let $\tilde{\mathcal{H}}^2$ denote the set of $\tilde{\mathcal{P}}$ -measurable square-integrable processes, where, with \mathcal{P} denoting the predictable sigma-algebra, $\tilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^k \setminus \{0\})$, and let \mathcal{U} denote the Borel sigma-algebra induced by the $L^2(\nu(dx))$ -norm. In particular, $U \in \tilde{\mathcal{H}}^2$ is such that $\|U\|_{\tilde{\mathcal{H}}^2} < \infty$, where

$$\|U\|_{\tilde{\mathcal{H}}^2} := \mathbb{E} \left[\int_0^T |U_t|_{2,\nu}^2 dt \right].$$

Moreover, let \mathcal{M}_2 denote the set of probability measures $\mathbb{Q} = \mathbb{Q}^\xi$ on (Ω, \mathcal{F}_T) that are absolutely continuous with respect to \mathbb{P} with square-integrable Radon-Nikodym derivatives $\xi \in \mathcal{L}_+^2(\mathcal{F}_T)$, and write $\mathcal{S}^2 := \mathcal{S}^2[0, T]$.

Let us next consider a coherent driver function g as in Definition 2.3 with $\mu = \nu$ and $\mathcal{I} = [0, T]$. and fix a final condition $X \in \mathcal{L}^2$. The associated BSDE for the pair $(Y, Z) \in \mathcal{S}^2 \times \tilde{\mathcal{H}}^2$ is given by

$$Y_t = -X + \int_t^T g(s, Z_s) ds - \int_{(t,T] \times \mathbb{R}^k \setminus \{0\}} Z_s(x) \tilde{N}(ds \times dx) \quad (2.6)$$

for $t \in [0, T]$. This BSDE, we recall from [5], admits a unique solution. By combining [38, 43, 42], we have that the BSDE (2.6) gives rise to a dynamic coherent risk-measure as follows:

Definition 2.6 For a given coherent driver function g , the corresponding dynamic coherent risk measure $\rho^g = (\rho_t^g)_{t \in [0, T]}$, $\mathcal{L}^2 \rightarrow \mathcal{S}^2$ is given by

$$\rho_t^g(X) = Y_t,$$

where $(Y, Z) \in \mathcal{S}^2 \times \tilde{\mathcal{H}}^2$ solves (2.6).

Remark 2.7 (i) Let $L^{\mathbf{d}} = (L_t^{\mathbf{d}})_{t \in [0, T]}$ be given by $L_t^{\mathbf{d}} = dt + L_t$ for some $\mathbf{d} \in \mathbb{R}^{k \times 1}$. For random variables $X \in \mathcal{L}^2$ of the form $X = f(L_T^{\mathbf{d}})$ for some function $f : \mathbb{R}^k \rightarrow \mathbb{R}$ the dynamic coherent risk-measure $\rho^g(X)$ is related to the following semi-linear PIDE (denoting $\dot{v} = \frac{\partial v}{\partial t}$):

$$\dot{v}(t, x) + \mathcal{G}v(t, x) + g(t, \mathcal{D}v_{t,x}) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^k, \quad (2.7)$$

$$v(T, x) = -f(x), \quad x \in \mathbb{R}^k, \quad (2.8)$$

where $\mathcal{D}v_{t,x} : \mathbb{R}^k \rightarrow \mathbb{R}$ and $\mathcal{G}v(t, x)$ are given by $\mathcal{D}v_{t,x}(y) = v(t, x + y) - v(t, x)$ and

$$\mathcal{G}v(t, x) = \mathbf{d}^\top \nabla v(t, x) + \int_{\mathbb{R}^k \setminus \{0\}} [\mathcal{D}v_{t,x}(y) - \nabla v(t, x)^\top y] \nu(dy),$$

where $\nabla v = (\frac{\partial v}{\partial x_1}, \dots, \frac{\partial v}{\partial x_k})^\top$. Specifically, if $v \in C^{1,1}([0, T] \times \mathbb{R}^k)$ solves (2.7)–(2.8) such that $\nabla v(t, x)$ is bounded (uniformly in $(t, x) \in [0, T] \times \mathbb{R}^k$) then we have the stochastic representation

$$\rho_t^g(X) = \mathbb{E} \left[-f(L_T^{\mathbf{d}}) + \int_t^T g(s, Z_s) ds \middle| \mathcal{F}_t \right] = v(t, L_t^{\mathbf{d}}), \quad (2.9)$$

$$Z_t(x) = v(t, L_{t-}^{\mathbf{d}} + x) - v(t, L_{t-}^{\mathbf{d}}), \quad x \in \mathbb{R}^k, \quad (2.10)$$

with $L_{0-}^{\mathbf{d}} = L_0^{\mathbf{d}}$. This non-linear Feynman-Kac result is shown by an application of Itô's lemma.

(ii) The risk measure ρ^g admits a dual representation

$$\rho_t^g(X) = \text{ess. sup}_{\mathbb{Q} \in \mathcal{S}^g} \mathbb{E}^{\mathbb{Q}}[-X | \mathcal{F}_t] \quad (2.11)$$

for a certain representing subset \mathcal{S}^g of the set \mathcal{M}_1 of probability measures that are absolutely continuous with respect to \mathbb{P} . The set \mathcal{S}^g is convex and closed (see [26, Theorem 11.22]).

We describe next a representation result for dynamic risk-measure ρ^g in terms of the representing processes (H^ξ) of the stochastic logarithms of the Radon-Nikodym derivatives $\xi \in L^2_+(\mathcal{F}_T)$ of the measure $\mathbb{Q}^\xi \in \mathcal{M}_2$, which are given by

$$\xi = \mathcal{E}\left(M^\xi\right)_T, \quad M^\xi = \int_{[0, \cdot] \times \mathbb{R}^k \setminus \{0\}} H^\xi_s(x) \tilde{N}(ds \times dx), \quad (2.12)$$

where $\mathcal{E}(\cdot)$ denotes the Doléans-Dade stochastic exponential. We call a $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{U}$ -measurable set $C = (C_t)_{t \in [0, T]}$ convex or closed if, for any $t \in [0, T]$, C_t is convex or closed.

Theorem 2.8 *Let g be a coherent driver function. Then for some $\mathcal{P} \otimes \mathcal{U}$ -measurable set C^g that is closed, convex, contains 0 and is bounded, we have for any $t \in [0, T]$ that $\rho_t^g(X)$ satisfies (2.11) with*

$$S^g = \left\{ \mathbb{Q}^\xi \in \mathcal{M}_1 : H^\xi_s \in C^g_s \text{ for all } s \in [0, T] \right\}. \quad (2.13)$$

Furthermore, the driver function g satisfies for $(t, z) \in [0, T] \times \mathcal{L}^2(\nu)$

$$g(t, z) = \sup_{h \in C^g_t} \int_{\mathbb{R}^k \setminus \{0\}} z(x) h(x) \nu(dx). \quad (2.14)$$

The proof of Theorem 2.8 follows by adaptation of the arguments given in [36, Theorem A.25], and is omitted.

Remark 2.9 (i) Note that two driver functions g_1 and g_2 are equal if and only if the corresponding sets C^{g_1} and C^{g_2} in the representation (2.14) are equal.

(ii) Let \bar{C} be a \mathcal{U} -measurable subset of $\mathcal{L}^2(\nu)$. If $C^g_t = \bar{C}$ for all $t \in [0, T]$ then the corresponding driver function is given by $g(t, z) = \bar{g}(z)$ where

$$\bar{g}(z) = \sup_{k \in \bar{C}} \int_{\mathbb{R}^k \setminus \{0\}} z(x) k(x) \nu(dx), \quad z \in \mathcal{L}^2(\nu). \quad (2.15)$$

2.4 Convergence

We next turn to the question of the convergence of a sequence $(\rho^{g^{(\pi)}, (\pi)})_\pi$ of dynamic coherent risk measures as in Definition 2.4 when the mesh size $\Delta = \Delta_\pi$ tends to zero. Let us suppose that $(\rho^{g^{(\pi)}, (\pi)})_\pi$ are driven by the random walks $(L^{(\pi)})_\pi$ that are defined as follows:

$$\Delta L_t^{(\pi)} = J_t \sqrt{\Delta}, \quad J_t \stackrel{\text{i.i.d.}}{\sim} (p_j^\Delta, j \in \mathbb{Z}^k), \quad t \in \pi \setminus \{T\}, \quad (2.16)$$

for some probability distribution $(p_j^\Delta, j \in \mathbb{Z}^k)$ on \mathbb{Z}^k that is given as follows in terms of a constant $c \geq 1$ (that will be specified shortly) and a partition $(B_j^\Delta, j = (j_1, \dots, j_k) \in \mathbb{Z}^k)$ of $(\sqrt{\Delta} \mathbb{Z})^k$ into block sets of the form

$$B_j^\Delta = \prod_{j_i} A_{j_i}^\Delta,$$

where $A_k^\Delta = [k\sqrt{\Delta}, (k+1)\sqrt{\Delta}]$ if $k > 0$, $A_k^\Delta = ((k-1)\sqrt{\Delta}, k\sqrt{\Delta}]$ if $k < 0$ and $A_0^\Delta = (-\sqrt{\Delta}, \sqrt{\Delta})$:

$$p_j^\Delta = \nu(B_j^\Delta) \Delta, \quad j \in \mathbb{Z}^k \setminus C_\Delta, \quad (2.17)$$

$$p_j^\Delta = 0, \quad j \in C_\Delta \setminus \{0\}, \quad (2.18)$$

$$p_0^\Delta = 1 - \sum_{j \neq 0} p_j^\Delta, \text{ where} \quad (2.19)$$

$$C_\Delta = \{j \in \mathbb{Z}^k : |j| \leq \sqrt{c_\Delta \nu_2}\}, \quad c_\Delta = c + (\log(\Delta))^{-}, \quad (2.20)$$

where, as before, $\nu_2 = \int_{\mathbb{R}^k \setminus \{0\}} |x|^2 \nu(dx)$.

When $\Delta \searrow 0$, we have by the dominated convergence theorem that

$$\mathbb{E} \left[\left(L_T^{(\pi),r} + L_T^{(\pi),s} \right)^2 \right] \longrightarrow T \int_{\mathbb{R}^k \setminus \{0\}} [(x_r + x_s)^2] \nu(dx), \quad r, s \in \{1, \dots, k\}, \quad (2.21)$$

where $L_T^{(\pi),m}$ and x_m , $m \in \{1, \dots, k\}$, denote the m th coordinates of $L_T^{(\pi)}$ and $x \in \mathbb{R}^k$.

Moreover, we have by functional weak convergence theory (see *e.g.* [29, Theorem VII.3.7])

$$L^{(\pi)} \xrightarrow{d} L, \quad \text{as } \Delta \searrow 0, \quad (2.22)$$

where \xrightarrow{d} denotes convergence in law in the Skorokhod J_1 -topology on the space $\mathbb{D}([0, T], \mathbb{R}^k)$ of \mathbb{R}^k -valued RCLL functions.

On a suitably chosen probability space $L_T^{(\pi)}$ converges to L_T in probability as $\Delta \searrow 0$. The latter convergence also holds in a stronger sense thanks to moment-conditions satisfied by $L_T^{(\pi)}$ that we show next. We define the value of c as follows in terms of $\varepsilon_0 > 0$ given in Assumption 2.5:

$$c = \sup_{x, y \in \mathbb{R}^k} \frac{|x + y|^{2+\varepsilon_0} \vee 1}{(|x|^{2+\varepsilon_0} \vee 1)(|y|^{2+\varepsilon_0} \vee 1)}, \quad (2.23)$$

where $x \vee y = \max\{x, y\}$ for $x, y \in \mathbb{R}$.

Lemma 2.10 *The collection $(L^{(\pi)})_\pi$ of random walks defined in (2.16) and (2.17)–(2.19) is such that we have, for any uniform partition π and $t \in \pi \setminus \{T\}$, $\mathbb{E} \left[\left| \Delta L_t^{(\pi)} \right| \right] / \sqrt{\Delta} \rightarrow 0$ as $\Delta \searrow 0$ and*

$$\mathbb{E} \left[\left| \Delta L_t^{(\pi)} \right|^{2+\varepsilon_0} \right] \leq \nu_{2+\varepsilon_0} \Delta, \quad \mathbb{P} \left(\left| \Delta L_t^{(\pi)} \right| = 0 \right) \geq 1 - \frac{1}{c_\Delta}, \quad (2.24)$$

where $\varepsilon_0 > 0$ and $\nu_{2+\varepsilon_0}$ are as in Assumption 2.5, and c_Δ is given in (2.20) and (2.23). Furthermore, we have

$$\sup_{\pi: \Delta_\pi \in \mathbb{R}_+ \setminus \{0\}} \mathbb{E} \left[\left| L_T^{(\pi)} \right|^{2+\varepsilon_0} \right] \in \mathbb{R}_+. \quad (2.25)$$

Remark 2.11 Under the bound in the right-hand side of (2.24) we have numerical stability of the solutions to sequence of BS Δ Es driven by $(L^{(\pi)})$ (see [37, Theorem 3.4]).

Proof. Letting $\pi = \pi_\Delta$ denote the partition with mesh $\Delta \in \mathbb{R}_+ \setminus \{0\}$ and $\varepsilon = \varepsilon_0$, a first observation is that, for any $t \in \pi \setminus \{T\}$, $a \in \mathbb{R}_+ \setminus \{0\}$ and $p \in [2, 2 + \varepsilon]$, we have by Chebyshev's inequality

$$\mathbb{P}(|\Delta L_t^{(\pi)}| > a) \leq \nu(z \in \mathbb{R}^k : |z| \geq a) \Delta \quad (2.26)$$

$$\leq \frac{\nu_p}{a^p} \Delta, \quad (2.27)$$

where, as before, $\nu_p = \int_{\mathbb{R}^k \setminus \{0\}} |x|^p \nu(dx)$. By multiplying (2.26) by $p a^{p-1}$ and integrating we have the estimate

$$\mathbb{E}[|\Delta L_t^{(\pi)}|^p] \leq \nu_p \Delta, \quad p \in [2, 2 + \varepsilon]. \quad (2.28)$$

Taking in (2.27) $p = 2$ and $a = b\sqrt{c_\Delta \nu_2 \Delta}$ and (a) setting $b = 1$ shows that

$$\mathbb{P}(|\Delta L_t^{(\pi)}| > 0) = \mathbb{P}(|\Delta L_t^{(\pi)}| > \sqrt{c_\Delta \nu_2 \Delta}) \leq c_\Delta^{-1} \quad (2.29)$$

which yields the bound in the right-hand side of (2.24), while (b) integrating over $b \geq 1$ shows that $\mathbb{E}[|\Delta L_t^{(\pi)}|]/\sqrt{\Delta} \leq \sqrt{\nu_2/c_\Delta}$, which tends to zero as $\Delta \searrow 0$ in view of the form of c_Δ .

To establish (2.25) the proof next proceeds analogously as that of the moment result for Lévy processes (see [44, Theorem 25.3]). The key step is to transfer the uniform estimate of moments of the increments to a uniform estimate of moments of the random walk at T is the following estimate for a sub-multiplicative functions g (a function $g : \mathbb{R}^k \rightarrow \mathbb{R}$ is called sub-multiplicative, we recall, if for some $b_g \in \mathbb{R}_+$ and any $x, y \in \mathbb{R}^k$ we have $g(x+y) \leq b_g g(x)g(y)$):

$$\begin{aligned} \mathbb{E} \left[g \left(L_T^{(\pi)} \right) \right] &= \mathbb{E} \left[g \left(\sum_{t \in \pi \setminus \{T\}} \Delta L_t^{(\pi)} \right) \right] \\ &\leq b_g^{N-1} \mathbb{E} \left[g \left(\Delta L_{t_1}^{(\pi)} \right) \right]^N, \end{aligned} \quad (2.30)$$

where we used that the increments $\Delta L_t^{(\pi)}$, $t \in \pi \setminus \{T\}$, are independent. For any $a \in \mathbb{R}_+$ the function g_a given by $g_a(x) := |x|^{2+\varepsilon} \vee a$, we recall from [44, Proposition 25.4] is sub-multiplicative. From (2.28) and (2.29) we have that $\mathbb{E} \left[g_1 \left(\Delta L_t^{(\pi)} \right) \right]$ is bounded above by

$$\mathbb{E} \left[g_0 \left(\Delta L_t^{(\pi)} \right) \right] + \mathbb{P} \left(\left| \Delta L_t^{(\pi)} \right| \in (0, 1) \right) \leq \nu_{2+\varepsilon} \Delta + c_\Delta^{-1}. \quad (2.31)$$

Combining the bounds (2.30) and (2.31) with the facts that c defined in 2.23 is such that $b_{g_1} = c$ and $c \leq c_\Delta$ we have for all $N \in \mathbb{N}$

$$\mathbb{E} \left[g_1 \left(L_T^{(\pi)} \right) \right] \leq c^{N-1} \left(\frac{1}{c} + \nu_{2+\varepsilon} \Delta \right)^N = \frac{1}{c} \left(1 + \frac{c \nu_{2+\varepsilon} T}{N} \right)^N. \quad (2.32)$$

As the right-hand side of (2.32) is bounded above by $c^{-1} \exp(c \nu_{2+\varepsilon} T)$ we have (2.25), and the proof is complete. \square

The moment-conditions in Lemma 2.10 carry over to those of path-functionals as follows:

Corollary 2.12 *Assume that $F : \mathbb{D}([0, T], \mathbb{R}^k) \rightarrow \mathbb{R}$ satisfies for some $c \in \mathbb{R}_+$*

$$|F(\omega)| \leq c \|\omega\|_\infty \text{ for all } \omega \in \mathbb{D}([0, T], \mathbb{R}^k), \quad (2.33)$$

where $\|\omega\|_\infty = \sup_{t \in [0, T]} |\omega(t)|$ for $\omega \in \mathbb{D}([0, T], \mathbb{R}^k)$. Then we have uniformly over partitions $\pi = \pi_\Delta$

$$\sup_{\Delta \in \mathbb{R}_+ \setminus \{0\}} \mathbb{E} \left[\left| F \left(L^{(\pi)} \right) \right|^{2+\varepsilon_0} \right] \in \mathbb{R}_+. \quad (2.34)$$

Proof. For any partition π , an application of Doob's inequality to the centered random walk $\bar{L}_t^{(\pi)} = L_t^{(\pi)} - t\mathbb{E}[L_1^{(\pi)}]$ shows that

$$\mathbb{E} \left[\sup_{t \in \pi} |\bar{L}_t^{(\pi)}|^{2+\varepsilon_0} \right] \leq \text{const} \mathbb{E}[|\bar{L}_T^{(\pi)}|^{2+\varepsilon_0}]. \quad (2.35)$$

The assertion now follows by combining the estimate (2.35) with (2.33), the triangle inequality, the convexity of $x \mapsto |x|^{2+\varepsilon_0}$ and (2.25) in Lemma 2.10. \square

To guarantee that the convergence of the random walks $(L^{(\pi)})_\pi$ carries over to the convergence of the corresponding BSDEs we impose the following condition on the sequence of coherent driver functions $(g^{(\pi)})_\pi$ and their piecewise-constant RCLL interpolations $(\tilde{g}^{(\pi)})_\pi$:

Condition 2.13 (i) *The collection of functions $(g^{(\pi)})_\pi$ is uniformly Lipschitz continuous with Lipschitz constants $K^{(\pi)}$ such that $\sup_\pi K^{(\pi)} \in \mathbb{R}_+$.*

(ii) *for any continuous function h for which $\sup_{x \in \mathbb{R}^k \setminus \{0\}} |h(x)|/|x|$ is bounded and any $t \in [0, T]$ we have*

$$\lim_{\Delta \rightarrow 0} \tilde{g}^{(\pi)}(t, h) = g(t, h).$$

The convergence result for BSDEs ([37, Theorem 4.1]) is phrased as follows in the current setting:

Theorem 2.14 *Let g be a coherent driver function and let $(L^{(\pi)})_\pi$ be as in (2.16) and (2.17)–(2.19) and suppose that the sequence of coherent driver functions $(g^{(\pi)})_\pi$ satisfies Condition 2.13. If $X^{(\pi)} \in \mathcal{L}^2(\mathcal{F}_T^{(\pi)})$ and $X \in \mathcal{L}^2$ are such that $X^{(\pi)} \rightarrow X$ in distribution and the collection $(\{X^{(\pi)}\}^2)_\pi$ is uniformly integrable, then we have (with $\tilde{\rho}^{g^{(\pi)}, (\pi)}$ the piecewise constant RCLL interpolation of $\rho^{g^{(\pi)}, (\pi)}$):*

$$\tilde{\rho}^{g^{(\pi)}, (\pi)}(X^{(\pi)}) \xrightarrow{d} \rho^g(X), \quad \text{as } \Delta \searrow 0. \quad (2.36)$$

3 Choquet-type integrals and iterated versions

3.1 Choquet-type integrals

We describe next the Choquet-type integrals that feature in the definition of dynamic spectral risk-measures given in the next section. We refer to [18] for a treatment of the theory of non-linear integration. The Choquet-type integrals that we consider are given in terms of measure distortions that we define next.

Definition 3.1 *Let $(\mathcal{U}, \mathbb{U}, \mu)$ be a measure space.*

(i) $\Gamma : [0, \mu(\mathbb{U})] \rightarrow [0, \infty]$ *is called a measure distortion if Γ is continuous and increasing with $\Gamma(0) = 0$. If $\Gamma(1) = 1$ then Γ is called a probability distortion.*

(ii) $\Gamma \circ \mu : \mathcal{U} \rightarrow [0, \infty]$ *denotes the set-function given by $(\Gamma \circ \mu)(A) := \Gamma(\mu(A))$ for $A \in \mathcal{U}$.*

On a given measure space $(\mathbb{U}, \mathcal{U})$ a set $A \in \mathbb{U}$ with $\mu(A) > 0$ is called an atom, we recall, if $C \subset A$ implies $\mu(C) \in \{0, \mu(A)\}$. We assume throughout that the measure distortions and associated measure spaces are of the following type:

Assumption 3.2 *The measure μ on $(\mathbb{U}, \mathcal{U})$ is sigma-finite and has no atoms, and the measure distortion $\Gamma : [0, \mu(\mathbb{U})] \rightarrow \mathbb{R}_+$ is bounded and such that*

$$K_\Gamma := \int_{(0, \mu(\mathbb{U}))} \frac{\Gamma(y)}{2y\sqrt{y}} dy \in \mathbb{R}_+. \quad (3.1)$$

The Choquet-type integrals that we consider are defined as follows:

Definition 3.3 *Let $(\mathcal{U}, \mathbb{U}, \mu)$ be a measure space and let Γ_+ and Γ_- be associated measure distortions which satisfy Assumption 3.2.*

(i) *The Choquet-type integral $\mathcal{C}_+^{\Gamma_+ \circ \mu} : \mathcal{L}_+^2(\mathbb{U}, \mathcal{U}, \mu) \rightarrow \mathbb{R}_+$ is given by*

$$\mathcal{C}_+^{\Gamma_+ \circ \mu}(f) := \int_{[0, \infty)} (\Gamma_+ \circ \mu)(f > x) dx, \quad f \in \mathcal{L}_+^2(\mathbb{U}, \mathcal{U}, \mu),$$

where $\{f > x\} = \{z \in \mathbb{U} : f(z) > x\}$.

(ii) The Choquet-type integral $\mathsf{C}^{\Gamma+\circ\mu, \Gamma-\circ\mu} : \mathcal{L}^2(\mathbb{U}, \mathcal{U}, \mu) \rightarrow \mathbb{R}$ is given by

$$\mathsf{C}^{\Gamma+\circ\mu, \Gamma-\circ\mu}(f) = \mathsf{C}_+^{\Gamma+\circ\mu}(f^+) - \mathsf{C}_+^{\Gamma-\circ\mu}(f^-), \quad (3.2)$$

where $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$ for $x \in \mathbb{R}$.

Remark 3.4 (i) To see that $\mathsf{C}_+^{\Gamma\circ\mu}(f) \in \mathbb{R}_+$ for $f \in \mathcal{L}_+^2(\mu)$ and μ and Γ satisfying Assumption 3.2 we note that by Chebyshev's inequality, monotonicity of Γ and a change of variables, we have

$$\mathsf{C}_+^{\Gamma\circ\mu}(f) = \int_0^\infty \Gamma(\mu(f > x))dx \leq \int_0^\infty \Gamma(|f|_{2,\mu}^2/x^2)dx = K_\Gamma |f|_{2,\mu}, \quad (3.3)$$

if $\mu(\mathbb{U}) = \infty$. If $\mu(\mathbb{U}) < \infty$ we find by a similar line of reasoning that $\mathsf{C}_+^{\Gamma\circ\mu}(f) \leq K'_\Gamma \|f\|_{2,\mu}$ with $K'_\Gamma = K_\Gamma + \Gamma(\mu(\mathbb{U}))/\sqrt{\mu(\mathbb{U})}$.

(ii) Taking in Definition 3.3, $(\mathbb{U}, \mathcal{U}, \mu) = (\Omega, \mathcal{F}_T, \mathbb{P})$, and taking the measure distortions Γ_+ and Γ_- equal to a continuous probability distortion Ψ and the function $\widehat{\Psi}$ given by $\widehat{\Psi}(x) = 1 - \Psi(1 - x)$ for $x \in [0, 1]$, it is straightforward to check that $\Psi \circ \mathbb{P}$ is a *capacity* and the Choquet-type integral of $X \in \mathcal{L}^2$ in (3.2) coincides with the classical Choquet expectation corresponding to $\Psi \circ \mathbb{P}$:

$$\mathsf{C}^{\Psi \circ \mathbb{P}, \widehat{\Psi} \circ \mathbb{P}}(X) = \int_0^\infty (\Psi \circ \mathbb{P})(X > x)dx - \int_{-\infty}^0 (1 - (\Psi \circ \mathbb{P})(X > x))dx. \quad (3.4)$$

Moreover, as we have $\widehat{\Psi}(x) \leq x \leq \Psi(x)$ for $x \in [0, 1]$ it follows

$$\mathsf{C}^{\Psi \circ \mathbb{P}, \widehat{\Psi} \circ \mathbb{P}}(X) \geq \mathbb{E}[X], \quad (3.5)$$

and we have equality in (3.5) for all $X \in \mathcal{L}^2$ if and only if $\Psi(x) = \widehat{\Psi}(x) = x$ for $x \in [0, 1]$.

We record next a robust representation result for Choquet-type integrals that plays an important role in the sequel. Let $\mathcal{M}_{p,\mu}$, $p \geq 1$, denote the set of measures m on $(\mathbb{U}, \mathcal{U})$ that are absolutely continuous with respect to a given measure μ on this space with Radon-Nikodym derivatives such that $\frac{dm}{d\mu} \in \mathcal{L}_+^p(\mu)$.

Proposition 3.5 For a given concave measure distortion Γ and measure μ on $(\mathbb{U}, \mathcal{U})$ satisfying Assumption 3.2 define

$$\mathcal{M}_{1,\mu}^\Gamma := \{m \in \mathcal{M}_{1,\mu}^{ac} : m(A) \leq \Gamma(\mu(A)) \text{ for all } A \in \mathcal{U} \text{ with } \mu(A) < \infty\}.$$

Then we have that $\mathsf{C}_+^{\Gamma\circ\mu} : \mathcal{L}_+^2(\mu) \rightarrow \mathbb{R}_+$ is K_Γ -Lipschitz-continuous and

$$\mathsf{C}_+^{\Gamma\circ\mu}(f) = \sup\{m(f) : m \in \mathcal{M}_{1,\mu}^\Gamma\} \quad \text{for } f \in \mathcal{L}_+^2(\mu). \quad (3.6)$$

In particular, $\mathsf{C}_+^{\Gamma\circ\mu}$ is positively homogeneous and subadditive, that is, for any $\lambda \in \mathbb{R}_+$ and $f, g \in \mathcal{L}_+^2$

$$\mathsf{C}_+^{\Gamma\circ\mu}(\lambda f) = \lambda \mathsf{C}_+^{\Gamma\circ\mu}(f), \quad \mathsf{C}_+^{\Gamma\circ\mu}(f + g) \leq \mathsf{C}_+^{\Gamma\circ\mu}(f) + \mathsf{C}_+^{\Gamma\circ\mu}(g). \quad (3.7)$$

Proof of Proposition 3.5. The representation in (3.6), we recall, is known to hold true when (a) $\Gamma(1) = 1$ and (b) μ has unit mass and (c) $\mathcal{M}_{1,\mu}^\Gamma$ is replaced by the set of $m \in \mathcal{M}_{1,\mu}^\Gamma$ with $m(\mathbb{U}) = 1$ (see [9] and [26, Corollary 4.80]). We note that, by positive homogeneity and (a) and (b), (c) is not needed for the representation in (3.6) to hold true. Let $\varepsilon > 0$, let μ be as given and let $m \in \mathcal{M}_{1,\mu}^\Gamma$,

and denote by O_ε , $\varepsilon > 0$, a collection of sets with finite non-zero μ -measure and such that $O_\varepsilon \nearrow \mathbb{U}$. Denoting

$$\begin{aligned} c_\varepsilon &:= \mu(O_\varepsilon), & \Gamma_\varepsilon(\cdot) &:= \Gamma(c_\varepsilon \cdot), \\ m_\varepsilon(dx) &:= I_{O_\varepsilon}(|x|)m(dx), & \mu_\varepsilon(dx) &:= c_\varepsilon^{-1}I_{O_\varepsilon}(|x|)\mu(dx), \end{aligned}$$

we thus have for any $f \in \mathcal{L}_+^2(\mu)$ that

$$\begin{aligned} \frac{1}{\Gamma_\varepsilon(1)} \mathbf{C}_+^{\Gamma_\varepsilon \circ \mu_\varepsilon}(f) &= \sup \left\{ m(f) : m \in \mathcal{M}_{1, \mu_\varepsilon}^{\Gamma_\varepsilon / \Gamma_\varepsilon(1)} \right\} \\ &= \sup \left\{ \frac{1}{\Gamma_\varepsilon(1)} m_\varepsilon(f) : m \in \mathcal{M}_{1, \mu}^\Gamma \right\}. \end{aligned} \quad (3.8)$$

Since, as is readily verified by an application of the monotone convergence theorem, $\mathbf{C}_+^{\Gamma_\varepsilon \circ \mu_\varepsilon}(f) \nearrow \mathbf{C}_+^{\Gamma \circ \mu}(f)$ and $m_\varepsilon(f) \nearrow m(f)$ as $\varepsilon \downarrow 0$, and $\Gamma_\varepsilon(1) \in \mathbb{R}_+ \setminus \{0\}$, we obtain (3.6) by taking $\varepsilon \searrow 0$ in (3.8).

The positive homogeneity and convexity of $\mathbf{C}_+^{\Gamma \circ \mu}(f)$ as stated in (3.7) follow as direct consequences of the robust representation in (3.6).

Next we turn to the proof of Lipschitz continuity. We observe that the robust representation (3.6) of $\mathbf{C}_+^{\Gamma \circ \mu}$ implies that for $u, v \in \mathcal{L}_+^2(\mu)$

$$|\mathbf{C}_+^{\Gamma \circ \mu}(u) - \mathbf{C}_+^{\Gamma \circ \mu}(v)| \leq |\mathbf{C}_+^{\Gamma \circ \mu}(v - u)| \vee |\mathbf{C}_+^{\Gamma \circ \mu}(u - v)|. \quad (3.9)$$

Using next a similar estimate as in Remark 3.4(i), we note that for $m \in \mathcal{M}_{1, \mu}^\Gamma$

$$\begin{aligned} \left| \frac{dm}{d\mu} \right|_{2, \mu}^2 &= \int \left(\frac{dm}{d\mu} \right)^2 d\mu = \int \frac{dm}{d\mu} dm = \int_0^\infty m \left(\frac{dm}{d\mu} > x \right) dx \\ &\leq \int_0^\infty \Gamma \left(\mu \left(\frac{dm}{d\mu} > x \right) \right) dx \leq c \left| \frac{dm}{d\mu} \right|_{2, \mu}, \end{aligned}$$

which implies $\sup_{m \in \mathcal{M}_{1, \mu}^\Gamma} \left| \frac{dm}{d\mu} \right|_{2, \mu} \leq c$ and hence for $u \in \mathcal{L}_+^2(\mu)$ we have $|\mathbf{C}_+^{\Gamma \circ \mu}(u)| \leq c|u|_{2, \mu}$ (by (3.6)).

The latter bound together with (3.9) yields the stated Lipschitz-continuity. \square

3.2 Conditional and iterated Choquet integrals

Analogously, we define \mathcal{F}_t -conditional Choquet-type integrals as follows:

Definition 3.6 For any $t \in [0, T]$ and probability distortions Ψ and $\bar{\Psi}$ satisfying Assumption 3.2 (relative to the measure \mathbb{P} restricted to (Ω, \mathcal{F}_t)), the conditional Choquet-type integral $\mathbf{C}^{\Psi \circ \mathbb{P}, \bar{\Psi} \circ \mathbb{P}}(\cdot | \mathcal{F}_t) : \mathcal{L}^2 \rightarrow \mathcal{L}_t^2$ is given by

$$\mathbf{C}^{\Psi \circ \mathbb{P}, \bar{\Psi} \circ \mathbb{P}}(X | \mathcal{F}_t) := \int_{\mathbb{R}_+} \Psi(\mathbb{P}(X^+ > x | \mathcal{F}_t)) dx - \int_{\mathbb{R}_+} \bar{\Psi}(\mathbb{P}(X^- > x | \mathcal{F}_t)) dx$$

for $X \in \mathcal{L}^2$, where $\{X^\pm > x\} = \{\omega \in \Omega : X^\pm(\omega) > x\}$.

Remark 3.7 (i) Reasoning similarly as in Remark 3.4(i) and as in the proof of Lemma 3.5, we have that (a) for any $X \in \mathcal{L}^2$, $\mathbf{C}^{\Psi \circ \mathbb{P}, \bar{\Psi} \circ \mathbb{P}}(X | \mathcal{F}_t)$ is square-integrable; and (b) the map $\mathbf{C}^{\Psi \circ \mathbb{P}, \bar{\Psi} \circ \mathbb{P}}(\cdot | \mathcal{F}_t)$ is Lipschitz-continuous on \mathcal{L}^2 with Lipschitz-constant $K_\Psi + K_{\bar{\Psi}}$ (which are given by the constant K_Γ in (3.1) with $\mu(\mathbb{U}) = 1$ and Γ equal to Ψ and $\bar{\Psi}$, respectively).

(ii) The conditional Choquet expectation in (3.2) of $X \in \mathcal{L}^2$ with $\bar{\Psi} = \widehat{\Psi}$ may equivalently be expressed as weighted integral of the conditional Expected Shortfall of X at different levels. Specifically, associated to any concave probability distortion Ψ is a unique Borel measure μ on $[0, 1]$ defined by $\mu(\{0\}) = 0$ and by $\mu(ds) = sF(ds)$ for $s \in (0, 1]$, where F is the locally finite positive measure given in terms of the right-derivative Ψ'_+ of Ψ by $F((s, 1]) = \Psi'_+(s)$ (see [26, Theorem 4.70]). It is straightforward to check that Ψ satisfies Assumption 3.2 if and only if

$$\int_{(0,1]} \frac{1}{\sqrt{s}} \mu(ds) \in \mathbb{R}_+ \setminus \{0\}. \quad (3.10)$$

The conditional Choquet expectation in Definition 3.6 can then be expressed in terms of the measure μ and the \mathcal{F}_t -conditional Expected Shortfall, as follows:

$$\mathbf{C}^{\Psi \circ \mathbb{P}, \widehat{\Psi} \circ \mathbb{P}}(X | \mathcal{F}_t) = \int_{(0,1]} \mathbf{ES}_\lambda(-X | \mathcal{F}_t) \mu(d\lambda), \quad X \in \mathcal{L}^2, \quad (3.11)$$

where the \mathcal{F}_t -conditional Expected Shortfall $\mathbf{ES}_\lambda(X | \mathcal{F}_t)$ of $X \in \mathcal{L}^2$ at level $\lambda \in (0, 1]$ is given in terms of the \mathcal{F}_t -conditional Value-at-Risk $\mathbf{VaR}_\lambda(X | \mathcal{F}_t) = \inf\{z \in \mathbb{R} : \mathbb{P}(X < -z | \mathcal{F}_t) < \lambda\}$ at level λ by

$$\mathbf{ES}_\lambda(X | \mathcal{F}_t) = \frac{1}{\lambda} \int_0^\lambda \mathbf{VaR}_u(X | \mathcal{F}_t) du, \quad \lambda \in (0, 1].$$

The proof of (3.11) follows by a straightforward adaptation to the conditional setting of the proof for the static setting given in Föllmer and Schied (2011).

(iii) It follows from the representation in (3.11) that the collection of the conditional Choquet expectations $X \mapsto \mathbf{C}^{\Psi \circ \mathbb{P}, \widehat{\Psi} \circ \mathbb{P}}(-X | \mathcal{F}_t)$, $t \in [0, T]$, $X \in \mathcal{L}^2$, is a dynamic coherent risk measure in the sense of Definition 2.1 (with $\mathcal{I} = [0, T]$).

One way to define a sequence of conditional spectral risk-measures that is adapted to the filtration $\mathbf{F}^{(\pi)} = (\mathcal{F}_t^{(\pi)})_{t \in \pi}$ is recursive in terms of conditional Choquet-integrals, as follows :

Definition 3.8 *Given a concave probability distortion Ψ satisfying Assumption 3.2 and a filtration $\mathbf{F}^{(\pi)} = (\mathcal{F}_t^{(\pi)})_{t \in \pi}$ the corresponding iterated spectral risk measure $\mathbf{S} = (\mathbf{S}_t)_{t \in \pi}$, $\mathbf{S}_t : \mathcal{L}^2(\mathcal{F}_T^{(\pi)}) \rightarrow \mathcal{L}^2(\mathcal{F}_t^{(\pi)})$ is defined recursively on the grid $\pi = \pi_\Delta$ by*

$$\mathbf{S}_t(X) = \begin{cases} \mathbf{C}^{\Psi \circ \mathbb{P}, \widehat{\Psi} \circ \mathbb{P}}(\mathbf{S}_{t+1}(X) | \mathcal{F}_t^{(\pi)}), & t \in \pi \setminus \{T\}; \\ -X, & t = T. \end{cases} \quad (3.12)$$

The class of iterated spectral risk-measures defined as such contains in particular the *Iterated Tail Conditional Expectation* proposed in [28] and is closely related to the *Dynamic Weighted V@R* that is defined in [13] for adapted processes via its robust representation. As already noted in the proof of Proposition 3.5, in the static case such a representation was derived in [9] for bounded random variables; see also [26, Theorems 4.79 and 4.94], and see [12] for the extension to the set of measurable random variables (we refer to [22] for families of dynamic risk measure defined via stochastic distortion probabilities in a binomial tree setting; see [14] for a general theory of finite state BSDEs).

We show next that iterated spectral risk measures are discrete-time time-consistent dynamic coherent risk measures and identify the driver function of the associated BSDE.

Proposition 3.9 *The iterated spectral risk measure $S = (S_t)_{t \in \pi}$ given in Definition 3.8 is a discrete-time coherent risk measure $\rho^{\bar{g}_\Delta, \pi}$ with driver function \bar{g}_Δ given by*

$$\bar{g}_\Delta(t, h) = \frac{1}{\Delta} \left(\mathbf{C}^{\Psi \circ (\nu^{(\pi)} \Delta), \widehat{\Psi} \circ (\nu^{(\pi)} \Delta)} (h(f)I_{\{f \neq 0\}}) - \Delta \int_{\mathbb{R}^k \setminus \{0\}} h(x) \nu^{(\pi)}(dx) \right), \quad (3.13)$$

where $\nu^{(\pi)}$ is defined in (2.1).

Proof. It follows from Proposition 3.5 that the function \bar{g}_Δ defined in (3.13) is a coherent driver function in the sense of Definition 2.3 with $\mathcal{I} = \pi$ and $\mu = \nu^{(\pi)}$. Let $X \in \mathcal{L}^2(\mathcal{F}^{(\pi)})$ be arbitrary and denote by $(Y^{(\pi)}, Z^{(\pi)})$ the solution of the BS Δ E with driver function \bar{g}_Δ . To show that the dynamic coherent risk measure corresponding to \bar{g}_Δ coincides with the spectral risk measure $S = (S_t)_{t \in \pi}$ it suffices to verify that

$$\bar{g}_\Delta(t, Z_t^{(\pi)}) \Delta = S_t(X) - \mathbb{E} \left[S_{t+1}(X) \mid \mathcal{F}_t^{(\pi)} \right]. \quad (3.14)$$

Letting $t \in \pi \setminus \{T\}$ and denoting $\Delta L = \Delta L_t^{(\pi)}$, we note from Definition 3.8 and (2.3) that $S_t(X) - \mathbb{E} \left[S_{t+1}(X) \mid \mathcal{F}_t^{(\pi)} \right]$ is equal to

$$\begin{aligned} & \mathbf{C}^{\Psi \circ \mathbb{P}, \widehat{\Psi} \circ \mathbb{P}} \left(S_{t+1}(X) \mid \mathcal{F}_t^{(\pi)} \right) - \mathbb{E} \left[S_{t+1}(X) \mid \mathcal{F}_t^{(\pi)} \right] \\ &= \mathbf{C}^{\Psi \circ \mathbb{P}, \widehat{\Psi} \circ \mathbb{P}} \left(Z_t^{(\pi)}(\Delta L) I_{\{\Delta L \neq 0\}} \mid \mathcal{F}_t^{(\pi)} \right) \\ & \quad - \mathbb{E} \left[Z_t^{(\pi)}(\Delta L) I_{\{\Delta L \neq 0\}} \mid \mathcal{F}_t^{(\pi)} \right] \\ &= \left(\mathbf{C}^{\Psi \circ (\nu^{(\pi)} \Delta), \widehat{\Psi} \circ (\nu^{(\pi)} \Delta)} (h(f)I_{\{f \neq 0\}}) - \Delta \int_{\mathbb{R}^k \setminus \{0\}} h(x) \nu^{(\pi)}(dx) \right) \Big|_{h=Z_t^{(\pi)}}, \end{aligned}$$

where we used that, due to stationarity of the increments of $L^{(\pi)}$, $\Delta L_t^{(\pi)}$ (which has law $\nu^{(\pi)} \Delta$) is independent of t . Thus we have (3.14) and the proof is complete. \square

4 Dynamic spectral risk measures

With the previous results in hand we move to the definition of dynamic spectral risk-measures in continuous time. Let us fix in the sequel a pair of concave measure distortions functions Γ_+ and Γ_- that satisfy Assumption 3.2 and are such that $\Gamma_-(x) \leq x$ for $x \in \mathbb{R}_+$. We define dynamic spectral risk measures to be those coherent spectral risk measures ρ^g for which the driver functions g are given in terms of Choquet integrals, as follows:

Definition 4.1 *The spectral driver function $\bar{g} : \mathcal{L}^2(\nu) \rightarrow \mathbb{R}_+$ is given by*

$$\bar{g}(u) := \mathbf{C}_+^{\Gamma_+ \circ \nu}(u^+) + \mathbf{C}_+^{\Gamma_- \circ \nu}(u^-)$$

for $u \in \mathcal{L}^2(\nu)$.

By Lemma 3.5 we have that \bar{g} is Lipschitz-continuous, positively homogeneous and convex, so that \bar{g} is a coherent driver function in the sense of Definition 2.3. The corresponding dynamic coherent risk-measure $\rho^{\bar{g}}$ is the object of study for the remainder of the paper, which we label as follows:

Definition 4.2 The dynamic coherent risk-measure $\rho^{\bar{g}}$ with spectral driver function \bar{g} given in Definition 4.1 is called the (continuous-time) dynamic spectral risk-measure corresponding to measure distortions Γ_+ and Γ_- .

We next show that dynamic spectral risk measure admit a dual representation of the form (2.11) and (2.13) with a representing set that is explicitly expressed in terms of the measure distortions Γ_+ and Γ_- , as follows:

Theorem 4.3 Let $X \in \mathcal{L}^2$, $t \in [0, T]$ and let \bar{g} be a spectral driver function. The dynamic spectral risk-measure $\rho^{\bar{g}}$ satisfies the dual representation in (2.11), (2.13) with representing set $C^{\bar{g}}$ given by

$$C^{\bar{g}} = \left\{ H \in \mathcal{L}^2(\nu) : \begin{array}{l} \text{for any } A \in \mathcal{B}(\mathbb{R}^k \setminus \{0\}) \text{ with } \nu(A) < \infty \\ -\Gamma_-(\nu(A)) \leq \int_A H d\nu \leq \Gamma_+(\nu(A)) \end{array} \right\}, \quad (4.1)$$

where $\int_A H d\nu = \int_A H(x)\nu(dx)$.

Example 4.4 The risk of a positive or negative jump arriving with a size larger than a , $a \in \mathbb{R}_+ \setminus \{0\}$, as quantified by the dynamic spectral risk measure $\rho^{\bar{g}}$ may be explicitly expressed in terms of ν , Γ_+ and Γ_- , as we show next. For any $a \in \mathbb{R}_+ \setminus \{0\}$, let $I(a) = I_{\{\sup_{t \in [0, T]} |\Delta L_t| \leq a\}} = \{N_T^a = 0\}$, $N_T^a = \#\{t \in [0, T] : |\Delta L_t| > a\}$ and $\bar{\nu}(a) = \nu(\{y : |y| > a\})$. While $\mathbb{E}[I(a)] = \exp(-\bar{\nu}(a)T)$ (since N_T follows a Poisson distribution with parameter $T\bar{\nu}(a)$), the values of $I(a)$ and $-I(a)$ under $\rho^{\bar{g}}$ are given as follows:

$$\begin{aligned} \rho_0^{\bar{g}}(I(a)) &= -\exp(-T\{\bar{\nu}(a) + \Gamma_+(\bar{\nu}(a))\}), \\ \rho_0^{\bar{g}}(-I(a)) &= \exp(-T\{\bar{\nu}(a) - \Gamma_-(\bar{\nu}(a))\}). \end{aligned}$$

These expressions follow by deploying the dual representation in Theorem 4.3 and Girsanov's theorem (e.g., Theorems III.3.24 and III.5.19 in Jacod and Shiryaev (1987)): we have that $\rho_0^{\bar{g}}(I(a))$ is equal to

$$\begin{aligned} \sup_{\mathbb{Q}^\xi \in \mathcal{S}^g} \mathbb{E}^{\mathbb{Q}^\xi}[-I(a)] &= \sup_{\mathbb{Q}^\xi \in \mathcal{S}^g} \mathbb{E} \left[-\exp \left(-\int_0^T \int_{(a, \infty)} (1 + H_t^\xi(y)) \nu(dy) dt \right) \right] \\ &= -\exp(-T\bar{\nu}(a)) \exp(-T\Gamma_+(\bar{\nu}(a))), \end{aligned}$$

while the expression for $\rho_0^{\bar{g}}(-I(a))$ follows in a similar manner.

Proof of Theorem 4.3. In view of Theorem 2.8 and Remark 2.9(i)–(ii) it suffices to verify that for any $h \in L^2(\nu)$ we have

$$\sup_{k \in C^{\bar{g}}} \int h k d\nu = C_+^{\Gamma_+ \circ \nu}(h^+) + C_+^{\Gamma_- \circ \nu}(h^-), \quad (4.2)$$

where $\int h k d\nu = \int_{\mathbb{R}^k \setminus \{0\}} h(x)k(x)\nu(dx)$.

Our next observation is that the set $C^{\bar{g}}$ in (4.1) admits the following equivalent representation:

$$C^{\bar{g}} = \left\{ U \in \mathcal{L}^2(\nu) : \begin{array}{l} \text{for any } A \in \mathcal{B}(\mathbb{R}^k \setminus \{0\}) \text{ with } \nu(A) < \infty \\ \int_A U^+ d\nu \leq \Gamma_+(\nu(A)), \int_A U^- d\nu \leq \Gamma_-(\nu(A)) \end{array} \right\}. \quad (4.3)$$

To see that this is the case, we note that, for any $U \in L^2(\nu)$, we have $-U^- \leq U \leq U^+$, while $U^+ = U_1$ and $-U^- = U_2$ for $U_1 = UI_{\{U \geq 0\}}$ and $U_2 = UI_{\{U < 0\}}$.

To see that (4.2) holds we note from (4.3), Proposition 3.5 and the identity

$$h k^* = h^+ k_1^+ + h^- k_2^-, \quad k^* = k_1^+ I_{\{h>0\}} - k_2^- I_{\{h<0\}},$$

for any $h, k_1, k_2 \in \mathcal{L}^2(\nu)$, that $\bar{g}(h) = \sup_{k \in C^{\bar{g}}} \int h k \, d\nu$ is bounded below by

$$\sup_{k_1, k_2 \in C^{\bar{g}}} \int h k^* \, d\nu = \sup_{k \in C^{\bar{g}}} \int h^+ k^+ \, d\nu + \sup_{k \in C^{\bar{g}}} \int h^- k^- \, d\nu,$$

which is by Proposition 3.5 equal to $C_+^{\Gamma_+ \circ \nu}(h^+) + C_+^{\Gamma_- \circ \nu}(h^-)$. Given this lower bound and the fact that $\bar{g}(h)$ is bounded above by

$$\sup_{k \in C^{\bar{g}}} \int h^+ k \, d\nu + \sup_{k \in C^{\bar{g}}} \int h^- (-k) \, d\nu \leq \sup_{k \in C^{\bar{g}}} \int h^+ k^+ \, d\nu + \sup_{k \in C^{\bar{g}}} \int h^- k^- \, d\nu,$$

we conclude that (4.2) holds true. \square

5 Limit theorem

We next turn to the functional limit theorem which shows that dynamic spectral risk measures arise as a limit of iterated spectral risk measures, under a suitable scaling of the corresponding probability distortions. We suppose that, uniformly in $p \in [0, 1]$, $\Psi_\Delta(p) - p$ scales in the mesh size Δ and the measure distortions Γ_+ and Γ_- as follows:

$$\Psi_\Delta(p) = p + \Delta \left\{ \Gamma_+(p/\Delta) I_{[0, \frac{1}{2}]}(p) + \Gamma_-((1-p)/\Delta) I_{(\frac{1}{2}, 1]}(p) \right\} + o(\Delta) \quad (\Delta \searrow 0).$$

Specifically, the condition that we require is phrased as follows:

Definition 5.1 *We denote by $(\Psi_\Delta)_{\Delta \in (0, 1]}$ a sequence of probability distortions that is such that Ψ_Δ and $\widehat{\Psi}_\Delta$ given by $\widehat{\Psi}_\Delta(p) = 1 - \Psi_\Delta(1-p)$ satisfy Assumption 3.2 with respect to the measure $\mu(dx) \equiv \mathbb{P}(\Delta L_{t_1}^{(\pi)} \in dx)$ and we have*

$$\lim_{\Delta \searrow 0} \Upsilon_\Delta = 0, \quad \Upsilon_\Delta = \sup_{x \in (0, 1)} \left| \frac{\Psi_\Delta(x) - x}{\Gamma_\Delta(x) \Delta} - 1 \right|, \quad (5.1)$$

where for $\Delta \in (0, 1]$ and $x \in [0, 1]$

$$\Gamma_\Delta(x) = \Gamma_+(x/\Delta) I_{[0, \frac{1}{2}]}(x) + \Gamma_-((1-x)/\Delta) I_{(\frac{1}{2}, 1]}(x).$$

Here, we recall, Γ_+ and Γ_- denote the given concave measure distortions which are such that $\Gamma_-(x) \leq x$ for $x \in \mathbb{R}_+$ and Assumption 3.2 holds with $\mu(dx) \equiv \nu(dx)$ and $\Gamma \equiv \Gamma_+$ or Γ_- .

The functional limit result is phrased as follows in terms of the sequence of piecewise-constant RCLL extensions $(\tilde{L}^{(\pi)})_\pi$ of the random walks $(L^{(\pi)})_\pi$ given by

$$\tilde{L}_t^{(\pi)} := L_{\Delta^{-1}[t\Delta]}^{(\pi)}, \quad t \in [0, T],$$

where $[r] = \sup\{n \in \mathbb{N} \cup \{0\} : n \leq r\}$ for $r \in \mathbb{R}_+$.

Theorem 5.2 Given a sequence of probability distortions $(\Psi_\Delta)_{\Delta \in (0,1]}$ as in Definition 5.1 and given filtrations $\mathbf{F}^{(\pi)} = (\mathcal{F}_t^{(\pi)})_{t \in \pi}$, let $\mathbf{S}^\Delta = (\mathbf{S}_t^\Delta)_{t \in \pi}$, $\Delta \in (0, 1]$, denote the corresponding iterated spectral risk-measures as given in Definition 3.8 and let \bar{g} denote the spectral driver function from Definition 4.1. Let the set of $\omega \in \mathbb{D}([0, T], \mathbb{R}^k)$ at which $F : \mathbb{D}([0, T], \mathbb{R}^k) \rightarrow \mathbb{R}$ is discontinuous in the Skorokhod J_1 -topology be a null-set under the law of L and assume that for some $k \in \mathbb{R}_+$

$$|F(\omega)| \leq k \|\omega\|_\infty \text{ for all } \omega \in \mathbb{D}([0, T], \mathbb{R}^k), \quad (5.2)$$

where $\|\omega\|_\infty = \sup_{t \in [0, T]} |\omega(t)|$ for $\omega \in \mathbb{D}([0, T], \mathbb{R}^k)$. Then we have

$$\tilde{\mathbf{S}}^\Delta \left(F \left(\tilde{L}^{(\pi)} \right) \right) \xrightarrow{d} \rho^{\bar{g}}(F(L)), \quad \Delta \searrow 0, \quad (5.3)$$

where $\tilde{\mathbf{S}}_t^\Delta = \mathbf{S}_{\Delta^{-1}[t\Delta]}^\Delta$, $t \in [0, T]$.

Remark 5.3 (i) Given two concave probability distortions Ψ_+ and Ψ_- satisfying the integrability condition (3.1) (with $\mu(\mathbb{U}) = 1$) one may explicitly construct a sequence $(\Psi_\Delta)_{\Delta \in (0,1]}$ satisfying Definition 5.1 as follows:

$$\Psi_\Delta(p) = p + (\Gamma_+(p/\Delta)I_{[0, \frac{1}{2}]}(p) + \Gamma_-((1-p)/\Delta)I_{(\frac{1}{2}, 1]}(p)) \Delta, \quad p \in [0, 1],$$

where, inspired by [21], we suppose that the functions $\Gamma_+, \Gamma_- : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are given by

$$\Gamma_+(x) = a \Psi_+(1 - e^{-cx}), \quad \Gamma_-(x) = \frac{b}{d} \Psi_-(1 - e^{-dx}), \quad x \in \mathbb{R}_+,$$

for some a, b, c and $d \in \mathbb{R}_+ \setminus \{0\}$ satisfying the restrictions

$$\Gamma_+(1/(2\Delta)) = \Gamma_-(1/(2\Delta)) < 1/(2\Delta), \quad b \Psi'_-(0^+) \in (0, 1), \quad (5.4)$$

where $f'(0^+)$ denote the right-derivative of a function f at $x = 0$. It is straightforward to check that, for any $\Delta \in (0, 1]$, Ψ_Δ is a concave probability distortion (the first condition in (5.4) guarantees continuity at $p = 1/2$ and $\Psi_\Delta(1/2) < 1$) and that $\Gamma_-(x) \leq x$ for any $x \in \mathbb{R}_+$ (as consequence of the second condition in (5.4)). Furthermore, we have that the limit in (5.1) holds.

(ii) Examples of functionals F that satisfy condition (5.2) include (a) a European call option payoff with strike $K \in \mathbb{R}_+$ ($F(\omega) = (\omega(T) - K)^+$); (b) the time-average ($F(\omega) = \frac{1}{T} \int_0^T \omega(s) ds$) and (c) the running maximum ($F(\omega) = \sup_{s \in [0, T]} \omega(s)$).

(iii) We note that Υ_Δ may be equivalently expressed in terms of Ψ_Δ and $\widehat{\Psi}_\Delta$ as follows:

$$\Upsilon_\Delta = \sup_{x \in (0, \frac{1}{2}]} \left| \frac{\Psi_\Delta(x) - x}{\Gamma_+(x/\Delta)\Delta} - 1 \right| \vee \sup_{x \in (0, \frac{1}{2})} \left| \frac{x - \widehat{\Psi}_\Delta(x)}{\Gamma_-(x/\Delta)\Delta} - 1 \right|.$$

(iv) We next provide an example to show the necessity of scaling the probability distortions. For a given uniform partition $\pi = \pi_\Delta$ of $[0, T]$ with mesh Δ , a probability distortion Ψ and $a_+, a_- \in \mathbb{R}_+ \setminus \{0\}$, let us consider the risk-charge under the iterated spectral risk measure \mathbf{S} corresponding to Ψ of the following statistic $X^{(\pi)}$ of the jump-sizes of $L^{(\pi)} = (L^{(\pi),1}, \dots, L^{(\pi),k})$:

$$X^{(\pi)} := N_\pi^+ - N_\pi^-, \quad N_\pi^\pm = \# \left\{ t \in \pi \setminus \{T\} : \sum_{i=1}^k (\Delta L_t^{(\pi),i})^\pm > a_\pm \right\}. \quad (5.5)$$

From the form (2.4)–(2.5) of the solution of the BSΔE associated to the iterated spectral risk measure \mathbb{S} we have that $Z^{(\pi)}$ is given by

$$Z_t^{(\pi)}(x) = z_+^{(\pi)}(x) - z_-^{(\pi)}(x), \quad z_{\pm}^{(\pi)}(x) = I_{A_{\pm}}(x), \quad (5.6)$$

$$A_{\pm} = \left\{ z \in \mathbb{R}^k \setminus \{0\} : \sum_{i=1}^k (z_i)^{\pm} > a_{\pm} \right\}. \quad (5.7)$$

As a consequence, we have from (3.13) in Proposition 3.9 that the driver function takes the form

$$\begin{aligned} \bar{g}_{\Delta}(t, Z_t^{(\pi)})\Delta &= \mathcal{C}^{\Psi \circ (\nu^{(\pi)}\Delta), \widehat{\Psi} \circ (\nu^{(\pi)}\Delta)} \left(z_+^{(\pi)}(f) - z_-^{(\pi)}(f) \right) \\ &\quad - \Delta \int_{\mathbb{R}^k \setminus \{0\}} (z_+^{(\pi)}(x) - z_-^{(\pi)}(x)) \nu^{(\pi)}(dx) \\ &= \Psi(\mathbb{P}(\Delta L_{t_1}^{(\pi)} \in A_+)) - \mathbb{P}(\Delta L_{t_1}^{(\pi)} \in A_+) \\ &\quad + \mathbb{P}(\Delta L_{t_1}^{(\pi)} \in A_-) - \widehat{\Psi}(\mathbb{P}(\Delta L_{t_1}^{(\pi)} \in A_-)). \end{aligned}$$

For given $t \in \pi \setminus \{T\}$ the iterated spectral risk-measure $\mathbb{S}_t(X^{(\pi)})$, may thus be expressed as follows in terms of the functions D_{Δ}^+ and $D_{\Delta}^- : [0, \Delta^{-1}] \rightarrow \mathbb{R}_+$ given by $D_{\Delta}^+(x) = \Psi(x\Delta) - x$ and $D_{\Delta}^-(x) = x - \widehat{\Psi}(x\Delta)$:

$$\begin{aligned} \mathbb{S}_t(X^{(\pi)}) - \mathbb{E}[X^{(\pi)} | \mathcal{F}_t^{(\pi)}] &= \mathbb{E} \left[\sum_{s \geq t, s \in \pi \setminus \{T\}} \bar{g}_{\Delta}(s, Z_s^{(\pi)})\Delta \middle| \mathcal{F}_t^{(\pi)} \right] \\ &= (T - t) \left(\frac{1}{\Delta} D_{\Delta}^+(\Delta^{-1} \mathbb{P}(\Delta L_{t_1}^{(\pi)} \in A_+)) + \frac{1}{\Delta} D_{\Delta}^-(\Delta^{-1} \mathbb{P}(\Delta L_{t_1}^{(\pi)} \in A_-)) \right). \end{aligned}$$

Note that, as $\Delta \searrow 0$, $\Delta^{-1} \mathbb{P}(\Delta L_{t_1}^{(\pi)} \in A_{\pm}) \rightarrow \nu(A_{\pm})$ and

$$\mathbb{E}[X^{(\pi)} | \mathcal{F}_t^{(\pi)}] \rightarrow (T - t)(\nu(A_+) - \nu(A_-)) + N_t^+ - N_t^-,$$

where $N_t^{\pm} = \#\{s \in (0, t] : L_s - L_{s-} \in A_{\pm}\}$. Hence, this suggests that for the sequence of iterated spectral risk-measures to converge, $\Delta^{-1} D_{\Delta}^+(x)$ and $\Delta^{-1} D_{\Delta}^-(x)$ are to admit limits as $\Delta \searrow 0$.

Proof of Theorem 5.2. We note first that, as $L^{(\pi)} \xrightarrow{d} L$ when $\Delta \searrow 0$, $F(L^{(\pi)})$ converges in distribution to $F(L)$, which is element of \mathcal{L}^2 . Furthermore, by Corollary 2.12, the collection $\{F(L^{(\pi)})^2\}_{\pi}$ is uniformly integrable. Thus, in view of Theorem 2.14 it suffices next to verify that the sequence of driver functions $(\bar{g}_{\Delta})_{\Delta \in (0, 1]}$ of the iterated spectral risk measures \mathbb{S}^{Δ} given in Proposition 3.9 satisfies Condition 2.13, which we proceed to do.

Let $t \in [0, T]$. Our first observation is by subadditivity and nonnegativity of \bar{g}_{Δ} we have for any $h, k \in L^2(\nu^{(\pi)})$

$$|\bar{g}_{\Delta}(t, h) - \bar{g}_{\Delta}(t, k)| \leq \bar{g}_{\Delta}(t, h - k) \vee \bar{g}_{\Delta}(t, k - h), \quad (5.8)$$

so that to verify Condition 2.13(i) it suffices to show that $\bar{g}_{\Delta}(t, h)/|h|_{2, \pi}$ is uniformly bounded. We

have for any $\Delta \in (0, 1]$ and $h \in L^2(\nu^{(\pi)})$ that

$$\begin{aligned}
\bar{g}_\Delta(t, h) &= \frac{1}{\Delta} \left(\mathbf{C}^{\Psi_\Delta \circ (\nu^{(\pi)} \Delta), \widehat{\Psi}_\Delta \circ (\nu^{(\pi)} \Delta)}(h) - \Delta \int h d\nu^{(\pi)} \right) \\
&= \frac{1}{\Delta} \left(\mathbf{C}_+^{\Psi_\Delta \circ (\nu^{(\pi)} \Delta)}(h^+) - \Delta \int h^+ d\nu^{(\pi)} \right) \\
&\quad + \frac{1}{\Delta} \left(\Delta \int h^- d\nu^{(\pi)} - \mathbf{C}_+^{\widehat{\Psi}_\Delta \circ (\nu^{(\pi)} \Delta)}(h^-) \right) \\
&= \mathbf{C}_+^{\Gamma_+ \circ \nu^{(\pi)}}(h^+) + R^\Delta(h^+) + \mathbf{C}_+^{\Gamma_- \circ \nu^{(\pi)}}(h^-) + \widehat{R}^\Delta(h^-), \tag{5.9}
\end{aligned}$$

where the remainder terms $R^\Delta(h^+)$ and $\widehat{R}^\Delta(h^-)$ are given as follows in terms of the identity function $I : [0, 1] \rightarrow [0, 1]$, $I(x) = x$:

$$\begin{aligned}
R^\Delta(h^+) &= \frac{1}{\Delta} \int_0^\infty \left[(\Psi_\Delta - I) \left(\nu^{(\pi)}(h^+ > x) \Delta \right) - \Gamma_+ \left(\nu^{(\pi)}(h^+ > x) \right) \Delta \right] dx \\
\widehat{R}^\Delta(h^-) &= \frac{1}{\Delta} \int_0^\infty \left[(I - \widehat{\Psi}_\Delta) \left(\nu^{(\pi)}(h^- > x) \Delta \right) - \Gamma_- \left(\nu^{(\pi)}(h^- > x) \right) \Delta \right] dx.
\end{aligned}$$

Since by Chebyshev's inequality $\nu^{(\pi)}(h^\pm > x) \leq |h^\pm|_{2, \nu^{(\pi)}}^2 / x^2$ for $x \in \mathbb{R}_+ \setminus \{0\}$, it follows that for

$$x \geq H^\pm := |h^\pm|_{2, \nu^{(\pi)}} \sqrt{2\Delta}$$

the mass of $\Delta \nu^{(\pi)}(h^\pm > x)$ is bounded above by $1/2$. Recalling the form of Υ_Δ (see Remark 5.3(iii)) and that $\Gamma_+ + \Gamma_-$ is bounded (by Γ_∞ say) we have

$$\begin{aligned}
|R^\Delta(h^+)| &\leq \Upsilon_\Delta \mathbf{C}_+^{\Gamma_+ \circ \nu^{(\pi)}}(h^+) \\
&\quad + \int_0^{H^+} (\Gamma_+(\nu^{(\pi)}(h^+ > x)) + \Gamma_-(\nu^{(\pi)}(h^- \leq x))) dx \\
&\leq \Upsilon_\Delta \mathbf{C}_+^{\Gamma_+ \circ \nu^{(\pi)}}(h^+) + H^+ \Gamma_\infty, \tag{5.10}
\end{aligned}$$

$$|\widehat{R}^\Delta(h^-)| \leq \Upsilon_\Delta \mathbf{C}_+^{\Gamma_- \circ \nu^{(\pi)}}(h^-) + H^- \Gamma_\infty. \tag{5.11}$$

Combining (5.8), (5.9), (5.10) and (5.11) and the K_{Γ_+} - and K_{Γ_-} -Lipschitz-continuity of $\mathbf{C}_+^{\Gamma_+ \circ \nu^{(\pi)}}$ and $\mathbf{C}_+^{\Gamma_- \circ \nu^{(\pi)}}$ (Proposition 3.5) and the fact that the values $\mathbf{C}_+^{\Gamma_+ \circ \nu^{(\pi)}}(0)$ and $\mathbf{C}_+^{\Gamma_- \circ \nu^{(\pi)}}(0)$ are equal to 0, we find

$$|\bar{g}_\Delta(h)| \leq \tilde{C} |h|_{2, \nu^{(\pi)}}, \tag{5.12}$$

where $\tilde{C} = (K_{\Gamma_+} + K_{\Gamma_-} + 2\sqrt{2}\Gamma_\infty)(1 + \sup_{\Delta \in (0, 1]} \Upsilon_\Delta)$ is finite by the limit (5.1) in Definition 5.1. This completes the proof of Condition 2.13(i).

We turn next to the proof of Condition 2.13(ii). Let h be a continuous function that is such that $c_h := \sup |h(x)/x| \in \mathbb{R}_+$. Since $\nu^{(\pi)}$ converges weakly to ν , we have that $\nu^{(\pi)}(h > x) \rightarrow \nu(h > x)$ at $x \in \mathbb{R}_+ \setminus \{0\}$ that are points of continuity. Hence, as Γ_\pm are continuous it follows that $\Gamma_\pm(\nu^{(\pi)}(h > x)) \rightarrow \Gamma_\pm(\nu(h > x))$ at such x . Next we show that the latter functions are dominated by an integrable function. By Chebyshev's inequality, $\Gamma_\pm(\nu^{(\pi)}(h > x)) \leq \Gamma_\pm(|h|_{2, \nu^{(\pi)}}^2 / x^2)$ while it follows from the inequality (2.28) that $\nu_2^{(\pi)} \leq \nu_2$, where $\nu_2^{(\pi)} = \int_{\mathbb{R}^k \setminus \{0\}} |x|^2 \nu^{(\pi)}(dx)$. Hence we have the bound

$$|h|_{2, \nu^{(\pi)}} \leq c_h \sqrt{\nu_2^{(\pi)}} \leq c_h \sqrt{\nu_2}.$$

Also, for any $d \in \mathbb{R}_+$, $\Gamma_{\pm}(d^2/x^2)$ is integrable:

$$\int_{d/\sqrt{\nu_0}}^{\infty} \Gamma_{\pm}(d^2/x^2) dx = K_{\Gamma_{\pm}} d,$$

where $\nu_0 = \nu(\mathbb{R}^k \setminus \{0\})$ and $K_{\Gamma_{\pm}}$ is given in (3.1). As a consequence, from the dominated convergence theorem we have that $C_{+}^{\Gamma_{\pm} \circ \nu^{(\pi)}}(h^{\pm}) \rightarrow C_{+}^{\Gamma_{\pm} \circ \nu}(h^{\pm})$ as $\Delta \searrow 0$. Further, in view of (5.1), $R^{\Delta}(h^+)$ and $\widehat{R}^{\Delta}(h^-)$ tend to zero as $\Delta \searrow 0$. This establishes Condition 2.13(ii), and the proof is complete. \square

6 Dynamically optimal portfolio allocation

We next consider dynamic portfolio problems concerning balancing gain and risk as quantified by the DSR. We suppose the investment horizon is equal to $T > 0$ and consider the DSR associated to the spectral driver function \bar{g} . In this section we impose the following restriction on the Lévy measure ν :

Assumption 6.1 The support of ν is included in the set $(-1, \infty)^k$.

We suppose that the financial market consists of a risk-free bond and n risky stocks with discounted prices $\widehat{S} = (\widehat{S}^1, \dots, \widehat{S}^n)$ evolving according to the following system of SDEs:

$$\begin{aligned} \frac{d\widehat{S}_t^i}{\widehat{S}_{t-}^i} &= \mathbf{d}^i dt + \int_{\mathbb{R}^k \setminus \{0\}} \mathbf{R}^i x \tilde{N}(dt \times dx), \quad i = 1, \dots, n, t \in (0, T], \\ \widehat{S}_0 &= s_0 \in (\mathbb{R}_+ \setminus \{0\})^k, \end{aligned}$$

where $\mathbf{d}^i \in \mathbb{R}$ is the excess log-return and $\mathbf{R}^i \in \mathbb{R}^k$ is the (row) vector of jump-coefficients with non-negative coordinates that are such that $(\mathbf{R}^i)^{\top} \mathbf{1} \leq 1$ (where $\mathbf{1} \in \mathbb{R}^k$ denotes the k -column vector of ones and where, for any vector v , v^{\top} denotes its transposition). Given the form of the model we have $\widehat{S}_t^i \in \mathcal{L}_t^2$ and $\widehat{S}_t^i > 0$ for any $i = 1, \dots, k$ and $t \in [0, T]$.

Let us consider the case of a small investor whose trades have a negligible impact on the price and let us adopt the classical frictionless and self-financing setting (no transaction cost, infinitely divisible assets, continuous-time trading, no funds are infused into or withdrawn from the portfolio at intermediate times, *etc.*). At any time $t \in [0, T]$ the investor decides to allocate part θ_t^i of the current wealth for investment into the stock \widehat{S}^i , $i = 1, \dots, n$, so that, if X_{t-}^{θ} denotes the discounted wealth just before time t , we have that $\theta_t^i X_{t-}^{\theta} / \widehat{S}_{t-}^i$ is the number of stocks i held in the portfolio at time t . We suppose that certain limits are placed on the leverage ratio of the portfolio and on the size of the short-holdings in the various stocks, and that this restriction is phrased in terms of a bounded and closed set $\mathcal{B} \subset \mathbb{R}^n$ as the requirement that

$$\theta_t(\omega) \in \mathcal{B} \quad \text{for any } (t, \omega) \in [0, T] \times \Omega. \quad (6.1)$$

Example 6.2 To impose constraints on the fractions of the current wealth invested in the bond account and the stock accounts we take

$$\mathcal{B} = \left\{ x \in (\mathbb{R}_+)^n : x_i \geq -L_i, \sum_{i=1}^n x_i \leq 1 + L_0 \right\}$$

for some $L_0, \dots, L_n \in \mathbb{R}_+$. In particular, by taking $L_i > 0$ we impose a limit on the borrowing ($i = 0$) or the number of stock i that may be shorted ($i \neq 0$). The case of a “long only” investor that has no short-sales and only invests own wealth (no borrowing) corresponds to taking in $L_0 = L_1 = \dots = L_n = 0$.

We call an allocation strategy $\theta = (\theta_t)_{t \in [0, T]}$ *admissible* if θ is predictable and (6.1) holds. We denote by \mathcal{A} the collection of admissible allocation strategies. Denoting by $\mathbf{R} = (\mathbf{R}^i)_{i=1, \dots, k}$ the $\mathbb{R}^{n \times k}$ -matrix with i th row equal to \mathbf{R}^i , we have that the discounted value $X^\theta = (X_t^\theta)_{t \in [0, T]}$ of a portfolio corresponding to $\theta \in \mathcal{A}$ evolves according to the SDE

$$\begin{aligned} \frac{dX_t^\theta}{X_{t-}^\theta} &= \theta_t^\top d\mathbf{d}t + \int_{\mathbb{R}^k \setminus \{0\}} \theta_t^\top \mathbf{R} x \tilde{N}(dt \times dx), \quad t \in (0, \tau^\theta \wedge T], \\ X_0^\theta &= x \in \mathbb{R}_+ \setminus \{0\}, \quad X_t^\theta = X_{\tau^\theta \wedge T}^\theta, \quad t \in (\tau^\theta \wedge T, T], \end{aligned}$$

where $\tau^\theta = \inf\{t \in [0, T] : X_t^\theta < 0\}$ (with $\inf \emptyset = +\infty$) is the first time that the value of the portfolio becomes negative, when the investor has to stop trading.

6.1 Portfolio optimisation under dynamic spectral risk measures

We consider next the stochastic optimisation problem given in terms of DSR by the following criterion that is to be minimised for $t \in [0, T]$:

$$\tilde{\mathcal{J}}_t^\theta = \rho_t^{\bar{g}}(X_{T \wedge \tau^\theta}^\theta), \quad (6.2)$$

The investor's problem is to identify a stochastic process $\tilde{\mathcal{J}}^* = (\tilde{\mathcal{J}}_t^*)_{t \in [0, T]}$ and an allocation strategy $\theta^* \in \mathcal{A}$ such that

$$\tilde{\mathcal{J}}_t^* = \text{ess. inf}_{\theta \in \mathcal{A}} \tilde{\mathcal{J}}_t^\theta = \tilde{\mathcal{J}}_t^{\theta^*}, \quad t \in [0, T]. \quad (6.3)$$

While the problem in (6.3) may be solved via a BSDE approach (as used in for instance [6, 36] to analyse utility optimisation and robust portfolio choice problems), due to its Markovian nature it may also be approached via classical methods based on an associated Hamilton-Jacobi-Bellman equation—this is the method that we present here. One class of allocation strategies are those of feedback-type that are defined as follows.

Definition 6.3 Denote by $\tilde{\Theta}$ the set of functions $\bar{\theta} : [0, T] \times \mathbb{R}_+ \rightarrow \mathcal{B}$ that are such that the following SDE admits a unique solution $X^{\bar{\theta}} = (X_t^{\bar{\theta}})_{t \in [0, T]}$:

$$\frac{dX_t^{\bar{\theta}}}{X_{t-}^{\bar{\theta}}} = \bar{\theta}(t, X_{t-}^{\bar{\theta}})^\top d\mathbf{d}t + \bar{\theta}(t, X_{t-}^{\bar{\theta}})^\top \mathbf{R} x \tilde{N}(dt \times dx), \quad t \in (0, \tau^{\bar{\theta}}], \quad (6.4)$$

$$X_0^{\bar{\theta}} = x, \quad X_t^{\bar{\theta}} = X_{\tau^{\bar{\theta}} \wedge T}^{\bar{\theta}}, \quad t \in (\tau^{\bar{\theta}} \wedge T, T], \quad (6.5)$$

where $\tau^{\bar{\theta}} = \inf\{t \in [0, T] : X_t^{\bar{\theta}} < 0\}$. A strategy $\theta \in \mathcal{A}$ is called a *feedback allocation strategy* if there exists a feedback function $\bar{\theta} \in \tilde{\Theta}$ such that

$$\theta_t = \bar{\theta}(\tau^{\bar{\theta}} \wedge t, X_{\tau^{\bar{\theta}} \wedge (t-)}^{\bar{\theta}}), \quad t \in [0, T],$$

where $X_{0-}^{\bar{\theta}} = X_0^{\bar{\theta}}$ and $X^{\bar{\theta}}$ solves the SDE in (6.4)–(6.5).

Associated to a given allocation strategy of feedback-type $\bar{\theta} \in \tilde{\Theta}$ there exists a value function $V^{\bar{\theta}}$ satisfying $J_t^* = V^{\bar{\theta}}(t, X_t^{\bar{\theta}})$ for $t \in [0, T]$ (as a consequence of the Markov property). If sufficiently regular, the function $V^{\bar{\theta}}$ satisfies a semi-linear PIDE that is given in terms of certain operators \mathcal{D}^θ and \mathcal{G}^θ indexed by $\theta \in \mathcal{B}$. For any function $f \in C^{1,1}([0, T] \times \mathbb{R})$, these operators are equal to the functions $\mathcal{D}_{t,x}^\theta f : \mathbb{R}^k \rightarrow \mathbb{R}$ and $\mathcal{G}^\theta f : [0, T] \times \mathbb{R}_+ \setminus \{0\} \rightarrow \mathbb{R}$ that are given in terms of

$$d_\theta = \theta^\top d, \quad \mathbf{R}_\theta = \theta^\top \mathbf{R}, \quad \theta \in \mathcal{B}, \quad (6.6)$$

by (denoting $f' = \frac{\partial f}{\partial x}$)

$$\begin{aligned} (\mathcal{D}_{t,x}^\theta f)(y) &= f(t, x + x \mathbf{R}_\theta y) - f(t, x) \\ \mathcal{G}^\theta f(t, x) &= \mathbf{d}_\theta f'(t, x) + \int_{\mathbb{R}^k \times \mathbb{1} \setminus \{0\}} \left\{ (\mathcal{D}_{t,x}^\theta f)(y) - f'(t, x) x \mathbf{R}_\theta y \right\} \nu(dy). \end{aligned}$$

The non-linear Feynman-Kac formula (see Remark 2.7) implies that if the following semi-linear PIDE has a sufficiently regular solution it is equal to $V^{\bar{\theta}}$:

$$\begin{aligned} \dot{v}(t, x) + \mathcal{G}^{\bar{\theta}(t,x)} v(t, x) + \bar{g} \left(\mathcal{D}_{t,x}^{\bar{\theta}(t,x)} v \right) &= 0, \quad (t, x) \in [0, T) \times \mathbb{R}_+ \setminus \{0\}, \\ v(t, x) = -x, \quad (t, x) &\in [0, T) \times (\mathbb{R} \setminus \mathbb{R}_+) \cup \{0\}, \\ v(T, x) = -x, \quad x &\in \mathbb{R}. \end{aligned}$$

Standard arguments suggest then that if the optimal allocation strategy θ^* is of feedback-type and the corresponding value-function V is sufficiently regular, then V satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\dot{V}(t, x) + \inf_{\theta \in \mathcal{B}} \left\{ \mathcal{G}^\theta V(t, x) + \bar{g}(\mathcal{D}_{t,x}^\theta V) \right\} = 0, \quad (t, x) \in [0, T) \times \mathbb{R}_+ \setminus \{0\}, \quad (6.7)$$

$$V(t, x) = -x, \quad t \in [0, T) \times (\mathbb{R} \setminus \mathbb{R}_+) \cup \{0\}, \quad (6.8)$$

$$V(T, x) = -x, \quad x \in \mathbb{R}. \quad (6.9)$$

Next we verify that a sufficiently smooth solution of the HJB equation gives rise to a solution of the optimisation problem in (6.3). Let $C_b^{1,1}([0, T] \times \mathbb{R})$ denote the set of $C^{1,1}$ -functions $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ with bounded first-order derivatives.

Theorem 6.4 *Let $w \in C_b^{1,1}([0, T] \times \mathbb{R})$ be a solution of the HJB-equation (6.7)–(6.9) and let the function $\tilde{\theta} : [0, T] \times \mathbb{R}_+ \rightarrow \mathcal{B}$, $(t, x) \mapsto \tilde{\theta}(t, x)$ given by*

$$\tilde{\theta}(t, x) \in \arg. \sup_{\theta \in \mathcal{B}} \left[\mathcal{G}^\theta w(t, x) + \bar{g}(\mathcal{D}_{t,x}^\theta w) \right]$$

be such that $\tilde{\theta} \in \bar{\Theta}$. Then the feedback strategy $\tilde{\theta}^ = (\tilde{\theta}_t^*)_{t \in [0, T]}$ with feedback function $\tilde{\theta}$ is optimal for (6.3) and we have $\tilde{\mathcal{J}}_t^* = \tilde{\mathcal{J}}_t^{\tilde{\theta}^*} = w(t, X_{t \wedge \tau^{\tilde{\theta}^*}}^{\tilde{\theta}})$, where $X^{\tilde{\theta}}$ solves the SDE in (6.4) and (6.5) with $\bar{\theta}$ replaced by $\tilde{\theta}$.*

Proof. Letting $\theta \in \mathcal{A}$ be an arbitrary admissible strategy, $t < \tau^\theta \wedge T$ and w as stated in the theorem, we find by an application of Itô's lemma that

$$\begin{aligned} w \left(T \wedge \tau^\theta, X_{T \wedge \tau^\theta}^\theta \right) - w(t, X_t^\theta) + \int_t^{T \wedge \tau^\theta} \bar{g}(\mathcal{D}^{\theta_s} w_{s, X_s^\theta}) ds \\ = \int_t^{T \wedge \tau^\theta} \left\{ \dot{w} + \mathcal{G}^{\theta_s} w \right\} \left(s, X_s^\theta \right) + \bar{g}(\mathcal{D}^{\theta_s} w_{s, X_s^\theta}) ds + M_{T \wedge \tau^\theta}^\theta - M_t^\theta, \end{aligned} \quad (6.10)$$

where M^θ is the square-integrable martingale given by

$$\begin{aligned} M_t^\theta &= \int_0^t w'(s, X_{s-}^\theta) \left(dX_s^\theta - \mathbf{d}_{\theta_s} X_s^\theta ds \right) \\ &\quad + \int_0^t \int_{\mathbb{R}^k \setminus \{0\}} \left(\mathcal{D}^{\theta_s} w_{s, X_s^\theta}(y) - w'(s, X_s^\theta) x \mathbf{R}_{\theta_s} y \right) \tilde{N}(ds \times dy). \end{aligned}$$

Note that by the HJB equation (6.7) the first term on the right-hand side of (6.10) is non-positive. Hence by taking conditional expectation in (6.10) and using (6.8)–(6.9) we have that

$$w(t, X_t^\theta) \leq \mathbb{E} \left[-X_{T \wedge \tau^\theta}^\theta + \int_t^{T \wedge \tau^\theta} \bar{g} \left(\mathcal{D}^{\theta_s} w_s, X_s^\theta \right) ds \middle| \mathcal{F}_t \right] = \mathcal{J}_t^\theta. \quad (6.11)$$

Since $\theta \in \mathcal{A}$ is arbitrary we have that

$$w(t, X_t^{\tilde{\theta}}) \leq \text{ess. inf}_{\theta \in \mathcal{A}} \mathcal{J}_t^\theta = \mathcal{J}_t^*. \quad (6.12)$$

If we choose $\theta = \tilde{\theta}^*$, we note that the first term on the right-hand side of (6.10) vanishes and the inequalities in (6.11)–(6.12) become equalities, so that $\mathcal{J}_t^* = w(t, X_t^{\tilde{\theta}^*})$. As the process $X^{\tilde{\theta}^*}$ coincides with the process $X^{\tilde{\theta}}$ solving the SDE in (6.4)–(6.5), the proof is complete. \square

6.1.1 Case of a “long-only” investor

We next restrict to the case of the “long-only” investor (see Example 6.2). In this case we note that for any admissible allocation strategy $\theta \in \mathcal{A}$ the solvency constraint $X_t^\theta \in \mathbb{R}_+$ is satisfied for all $t \in [0, T]$ so that $\tau^\theta = \infty$ a.s. We identify the optimal strategy as follows:

Theorem 6.5 *Let $\theta^* \in \mathcal{B}$ satisfy*

$$\theta^* \in \arg. \sup_{\theta \in \mathcal{B}} \{ \mathbf{d}_\theta - \bar{g}(-\mathbf{R}_\theta I) \}, \quad (6.13)$$

where \mathbf{d}_θ and \mathbf{R}_θ are given in (6.6) and $I : \mathbb{R}^k \rightarrow \mathbb{R}^k$ is given by the column vector $I(y) = y$. Then $\tilde{\theta}^* = (\tilde{\theta}_t^*)_{t \in [0, T]}$ given by $\tilde{\theta}_t^* \equiv \theta^*$ is an optimal strategy and

$$\mathcal{J}_t^* = -X_t^{\theta^*} \exp((T-t) \{ \mathbf{d}_{\theta^*} - \bar{g}(-\mathbf{R}_{\theta^*} I) \}). \quad (6.14)$$

Proof. The assertions follow by an application of the verification theorem (Theorem 6.4).

We note first that as the function $\theta \mapsto \mathbf{d}_\theta - \bar{g}(-\mathbf{R}_\theta I)$ is concave it attains its maximum on the compact set \mathcal{B} . Thus, the set in (6.13) is not empty and θ^* is well-defined. Moreover, given the positive homogeneity of g it is straightforward to verify that the function $C : [0, T] \rightarrow \mathbb{R}$ given by

$$C(t) = -\exp((T-t) \{ \mathbf{d}_{\theta^*} - \bar{g}(-\mathbf{R}_{\theta^*} I) \})$$

satisfies the ODE

$$\begin{aligned} \dot{C}(t) + \inf_{\theta \in \mathcal{B}} \{ \mathbf{d}_\theta C(t) + \bar{g}(C(t) \mathbf{R}_\theta I) \} &= 0, \quad t \in [0, T), \\ C(T) &= -1. \end{aligned}$$

As a consequence, we have that the candidate value function $V : [0, T] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ given by $V(t, x) = C(t)x$ satisfies the HJB equation (6.7)–(6.9) (here we used again the positive homogeneity of \bar{g}). The assertions follow now by an application of Theorem 6.4. \square

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