COINVARIANTS OF LIE ALGEBRAS OF VECTOR FIELDS ON ALGEBRAIC VARIETIES

PAVEL ETINGOF AND TRAVIS SCHEDLER

Abstract. We prove that the space of coinvariants of functions on an affine variety by a Lie algebra of vector fields whose flow generates finitely many leaves is finite-dimensional. Cases of the theorem include Poisson (or more generally Jacobi) varieties with finitely many symplectic leaves under Hamiltonian flow, complete intersections in Calabi-Yau varieties with isolated singularities under the flow of incompressible vector fields, quotients of Calabi-Yau varieties by finite volume-preserving groups under the incompressible vector fields, and arbitrary varieties with isolated singularities under the flow of all vector fields. We compute this quotient explicitly in many of these cases. The proofs involve constructing a natural $\mathcal{D}$-module representing the invariants under the flow of the vector fields, which we prove is holonomic if it has finitely many leaves (and whose holonomicity we study in more detail). We give many counterexamples to naive generalizations of our results. These examples have been a source of motivation for us.

Contents

1. Introduction

1.1. **Vector fields on affine schemes.** Let $\mathbf{k}$ be an algebraically closed field of characteristic zero, and let $X = \mathsf{Spec}\,\mathcal{O}_X$ be an affine scheme of finite type over $\mathbf{k}$ (we will generalize this to nonaffine schemes in §2.10 below). Our examples will be varieties, so the reader interested only in these (rather than the general theory, which profits from restriction to nonreduced subschemes) can freely make this assumption. We will be interested in the vector space $\mathrm{Vect}(X)$ of global vector fields on X, which is by definition the space of derivations $\mathrm{Der}(\mathcal{O}_X)$, a Lie algebra acting on $\mathcal{O}_X$.

We also remark that our results can be generalized to the analytic setting using the theory of analytic $\mathcal{D}$-modules, except that in these cases, the coinvariants need no longer be finite-dimensional, since analytic varieties can have infinite-dimensional cohomology in general (e.g., a surface with infinitely many punctures). But we will not discuss this here.

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When we say $x \in X$, we mean a closed point, which is the same as a point of the reduced subvariety X_{red}. Note that (since $\mathbf{k}$ has characteristic zero) it is well-known that all vector fields on X (which by definition means derivations of $\mathcal{O}_X$) are parallel to X_{red} (dating to at least [Sei67, Theorem 1]), i.e., they preserve the ideal of nilpotent elements. Hence, there is a restriction map $\text{Vect}(X) \to \text{Vect}(X_{\text{red}})$, although this is not an isomorphism unless $X = X_{\text{red}}$. In particular, for all global vector fields $\xi \in \text{Vect}(X)$ and all $x \in X$, $\xi|_x \in T_x X_{\text{red}}$.

Let $\mathfrak{v} \subseteq \text{Vect}(X)$ be a Lie subalgebra of the Lie algebra of vector fields (which is allowed to be all vector fields). We are interested in the coinvariant space,

$$(\mathcal{O}_X)_{\mathfrak{v}} := \mathcal{O}_X / \mathfrak{v}(\mathcal{O}_X).$$

This is called the coinvariant space because it is, by definition, the coinvariant space of the module $\mathcal{O}_X$ over the Lie algebra $\mathfrak{v} \subseteq \text{Vect}(X)$.

Remark 1.1. One could more generally replace $\mathfrak{v}$ above with an arbitrary set of vector fields that need not be a Lie algebra or even a vector space, but then the coinvariants coincide with those of the Lie algebra generated by that set. One could also allow $\mathfrak{v}$ to contain not merely vector fields (i.e., derivations of $\mathcal{O}_X$), but differential operators on $\mathcal{O}_X$ of order ≤ 1: see Remark 2.17.

Our main results show that, under nice geometric conditions, this coinvariant space is finite-dimensional, and in fact that the corresponding $\mathcal{D}$-module generated by $\mathfrak{v}$ is holonomic. This specializes to the finite-dimensionality theorems [BEG04, Theorem 4] and [ES10, Theorem 3.1] in the case of Poisson varieties. It also generalizes a standard result about coinvariants under the action of a reductive algebraic group (see Remarks 2.10 and 2.11 below).

Our first main result can be stated as follows.

Theorem 1.2. Suppose that, for all $i \geq 0$, the locus of $x \in X$ where the evaluation $\mathfrak{v}|_x$ has dimension $\leq i$ has dimension at most i. Then the coinvariant space $(\mathcal{O}_X)_{\mathfrak{v}}$ is finite-dimensional.

The theorem will be proved in a stronger form in Theorem 2.19 (after its reformulation in Theorem 2.9), hence we omit an explicit proof. Observe that the hypothesis of Theorem 1.2 implies that, on an open dense subvariety of X_{red}, $\mathfrak{v}$ generates the tangent bundle; as we will explain below, the hypothesis is equivalent to the statement that X_{red} is stratified by locally closed subvarieties with this property.

1.2. **Goals and outline of the paper.** First, in §2, we reformulate Theorem 1.2 geometrically and prove it, along with more general finite-dimensionality and holonomicity theorems. The main tool involves the definition of a right $\mathcal{D}$-module, $M(X, \mathfrak{v})$, generalizing [ES10], such that $\text{Hom}(M(X, \mathfrak{v}), N) \cong N^{\mathfrak{v}}$ for all $\mathcal{D}$-modules N, i.e., the $\mathcal{D}$-module which represents invariants under the flow of $\mathfrak{v}$. Then the theorem above is proved by studying when this $\mathcal{D}$-module is holonomic. This leads to Theorem 2.9 (a reformulation of Theorem 2.19), Theorem 2.19 (a $\mathcal{D}$-module generalization), 2.28 (a partial converse), and 2.57 (a generalization of all the preceding theorems, although the language is more technical).

The next goal, in §3, is to study examples related to Cartan's classification of simple infinite-dimensional transitive Lie algebras of vector fields on a formal polydisc which are complete with respect to the jet filtration. Namely, according to Cartan's classification [Car09, GQS70], there are four such Lie algebras, as follows. For $\xi \in \text{Vect}(X)$, let L_ξ denote the Lie derivative by ξ. Let $\hat{\mathbf{A}}^n$ be the formal neighborhood of the origin in $\mathbf{A}^n$, which is a formal polydisc of dimension n. Then, Cartan's classification consists of:

(a) The Lie algebra $\text{Vect}(\hat{\mathbf{A}}^n)$ of all vector fields on $\hat{\mathbf{A}}^n$;
(b) The Lie algebra $H(\hat{\mathbf{A}}^{2n}, \omega)$ of all Hamiltonian vector fields on $\hat{\mathbf{A}}^{2n}$, i.e., preserving the standard symplectic form $\omega = \sum_i dx_i \wedge dy_i$; explicitly, ξ such that $L_\xi \omega = 0$;

(c) The Lie algebra $H(\hat{\mathbf{A}}^{2n+1}, \alpha)$ of all contact vector fields on an odd-dimensional formal polydisc, with respect to the standard contact structure $\alpha = dt + \sum_i x_i dy_i$, i.e., those vector fields satisfying $L_\xi \alpha \in \mathcal{O}_X \cdot \alpha$;
(d) The Lie algebra $H(\hat{\mathbf{A}}^n, \mathsf{vol})$ of all volume-preserving vector fields on $\hat{\mathbf{A}}^n$ equipped with the standard volume form $\mathsf{vol} = dx_1 \wedge \cdots \wedge dx_n$, i.e., vector fields ξ such that $L_\xi \mathsf{vol} = 0$.

In §3, we define generalizations of each of these examples to the global (but still affine), singular, degenerate situation. For example, (a) becomes vector fields on arbitrary schemes of finite type. For (b)–(d), we define generalizations of the structure on the variety, which in case (b) yields Poisson varieties. Then, there are essentially two different choices of the Lie algebra of vector fields. In case (b), these are Hamiltonian vector fields or Poisson vector fields. We recall that Hamiltonian vector fields are of the form $\{f, -\}$ for $f \in \mathcal{O}_X$, and Poisson vector fields are all vector fields which preserve the Poisson bracket, i.e., such that $\xi\{f, g\} = \{\xi(f), g\} + \{f, \xi(g)\}$; this includes all Hamiltonian vector fields.

In each of the cases (a)–(d), we study the leaves under the flow of $\mathfrak{v}$ and the condition for the associated $\mathcal{D}$-module to be holonomic (and hence for $(\mathcal{O}_X)_{\mathfrak{v}}$ to be finite-dimensional). In this section, the examples, remarks, and propositions put together constitute the main content, although we mention in particular Theorem 3.34 and its corollaries as important results.

In §4 we discuss the globalization of these examples to the nonaffine setting, which turns out to be straightforward for Hamiltonian vector fields and all vector fields, but quite nontrivial for Poisson vector fields (and hence their generalizations). We do not need this material for the remainder of the paper. We mention Theorems 4.1 and 4.45 as important results here.

In the remainder of the paper we study in detail three specific examples for which the $\mathcal{D}$-module has an interesting and nontrivial structure which reflects the geometry. In these examples, we explicitly compute the $\mathcal{D}$-module and the coinvariants $(\mathcal{O}_X)_{\mathfrak{v}}$.

In §5, we consider the case of divergence-free vector fields on complete intersections in Calabi-Yau varieties. Holonomicity turns out to be equivalent to having isolated singularities, and we restrict to this case. Then, the structure of the $\mathcal{D}$-module and the coinvariant functions $(\mathcal{O}_X)_{\mathfrak{v}}$ is governed by the Milnor number and link of the isolated singularities. We mention Theorems 5.11 and 5.21 as important results.

In §6, we consider quotients of Calabi-Yau varieties by finite groups of volume-preserving automorphisms. In this case, it turns out that the $\mathcal{D}$-module associated to volume-preserving vector fields is governed by the most singular points, where the stabilizer is larger than that of any point in some neighborhood. More generally, rather than working on the quotient X/G where X is Calabi-Yau and G is a group of volume-preserving automorphisms, we study the Lie algebra of G-invariant volume-preserving vector fields on X itself (and we generalize this to the setting where G preserves volume up to scaling). This discussion culminates in Theorem 6.3.

Finally, in §7, we consider symmetric powers $(S^n X, \mathfrak{v})$ of smooth varieties $(X, \mathfrak{v})$ on which $\mathfrak{v}$ generates the tangent space everywhere (which we call *transitive*). This includes the symplectic, locally conformally symplectic, contact, and Calabi-Yau cases. In these situations, we explicitly compute the $\mathcal{D}$-module and the coinvariant functions. Dually, the main result says that the invariant functionals on $\mathcal{O}_{S^n X}$ form a polynomial algebra whose generators are the functionals on diagonal embeddings $X^i \to S^n X$ obtained by pulling back to X^i and taking a products of invariant functionals on each factor of X. For the $\mathcal{D}$-module, this expresses $M(S^n X, \mathfrak{v})$ as a direct sum of external tensor products of copies of $M(X, \mathfrak{v})$ along each diagonal embedding. The main result here is Theorem 7.9, which has a companion for symmetric powers of odd-dimensional contact varieties in Theorem 7.15. These results follow from the more general (but more abstract) Theorems 7.21, 7.24, and 7.29.

Although Theorem 1.2 and its reformulation and generalization in Theorems 2.9 and 2.19 can be viewed as main results of this paper (along with the example-driven discussion leading to the more general Theorem 2.57), more important than this is the study of examples in the subsequent sections and the main results on these, as mentioned above. We particularly highlight the examples and constructions of Section 3, as well as Theorems 3.34, 5.11, 6.3, and 7.9, as central to this work.

1.3. **Brief history of subject.** There are many works which deal with Lie algebras of vector fields on affine varieties; we survey just a few.

Many of these deal with the study, for each affine variety (or subscheme) $X \subseteq \mathbf{A}^n$, of the Lie algebra $\mathbf{D}_X$ of all vector fields on $\mathbf{A}^n$ which preserve the ideal I_X of X: this is called the *tangent algebra*. We will call vector fields in $\mathbf{D}_X$ those vector fields *parallel to* X below (and we will use the same terminology for any inclusion $X \subseteq Y$ of varieties, replacing $\mathbf{A}^n$ by Y). In fundamental work of Seidenberg [Sei67], it is shown that $\mathbf{D}_X \subseteq \mathbf{D}_{X_{\mathrm{red}}}$, where $X_{\mathrm{red}} \subseteq X$ is the reduced subscheme, i.e., its ideal $I_{X_{\mathrm{red}}} = \sqrt{I_X}$ is the radical of the ideal I_X of X. Moreover, he shows that $\mathbf{D}_{X_{\mathrm{red}}}$ is the intersection of $\mathbf{D}_Y$ over irreducible components Y of X_{red}. A generalization to nonreduced affine schemes is given in [HR99].

In work of Hauser and Müller [HM93], it is shown that, for $X \subseteq \mathbf{A}^n$, isomorphism classes of Lie subalgebras $\mathbf{D}_X$ correspond to isomorphism classes of embedded subvarieties $X \subseteq \mathbf{A}^n$, and that the same is true in the local analytic setting, i.e., when X is an analytic germ, provided $\dim X \geq 3$. This had been proved in the quasihomogeneous isolated singularity case in [Omo80]. In subsequent work by Hauser and Risler [HR99], these results were generalized to the real analytic setting.

In this paper, we are rather concerned with Lie subalgebras of the Lie algebra of vector fields on X, which we denote $\mathrm{Vect}(X)$, and which is sometimes denoted $\Theta(X)$ in the literature. This is the quotient of $\mathbf{D}_X$ by the Lie ideal $I_X \cdot \mathrm{Vect}(\mathbf{A}^n)$. The fact that this Lie algebra uniquely determines X up to isomorphsim is an old result: in the setting of C^∞ manifolds, it was proved in [SP54]; this was generalized to real analytic manifolds and (complex analytic) Stein spaces in [Gra79], and to normal algebraic varieties in [Sie96]. One of the main ideas in the analytic setting is that points of X correspond to maximal finite-codimensional subalgebras of $\mathrm{Vect}(X)$ (in the C^∞ setting, these are in fact ideals). As a consequence, X is smooth if and only if $\mathrm{Vect}(X)$ is simple [Jor86], [Sie96]. These results were generalized to the local complex analytic setting in [HM94].

We are particularly interested in sualgebras of $\mathrm{Vect}(X)$ such as, when X is a Poisson variety, the subalgebra of Hamiltonian vector fields. In the case of C^∞ and real analytic symplectic manifolds, this Lie algebra has been studied in many places, such as [ALDM74] and [Gra87].

The aim of this paper, in departure from the aforementioned and numerous other works on Lie algebras of vector fields, is to study the coinvariant space $(\mathcal{O}_X)_{\mathfrak{v}} := \mathcal{O}_X/(\mathfrak{v} \cdot \mathcal{O}_X)$ of functions under Lie algebras $\mathfrak{v} \subseteq \mathrm{Vect}(X)$ of vector fields, and to interpret this geometrically through a study of the $\mathcal{D}$-module (denoted $M(X, \mathfrak{v})$ below) which represents invariants under the flow of $\mathfrak{v}$. We believe that studying Lie algebras of vector fields via this $\mathcal{D}$-module (and more generally using the techniques of $\mathcal{D}$-modules) is profitable. Our work generalizes previous work of the authors in the case where X is a Poisson variety and $\mathfrak{v}$ is the Lie algebra of Hamiltonian vector fields, in e.g., [ES10, ES13].

Some of the most interesting examples include complete intersections: see §5), as well as the sequel to this work, [ES14]. This builds on [Gre75]; this case has been studied in many other places, notably [Yau82, MY82]. The Lie algebra of Hamiltonian vector fields in this case has been studied in many places, such as [MS96].

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2. General theory

Let $\Omega_X^\bullet := \wedge_{\mathcal{O}_X}^\bullet \Omega_X^1$ be the algebraic de Rham complex, where Ω_X^1 is the sheaf of Kähler differentials on X. We will frequently use the de Rham complex modulo torsion, $\tilde{\Omega}_X^\bullet := \Omega_X^\bullet/\text{torsion}$.

By *polyvector fields* of degree m on X, we mean skew-symmetric multiderivations $\wedge_{\mathbf{k}}^m \mathcal{O}_X \to \mathcal{O}_X$. Let T_X^m be the sheaf of such multiderivations. Equivalently, $T_X^m = \mathrm{Hom}_{\mathcal{O}_X}(\Omega_X^m, \mathcal{O}_X)$, where $\xi \in T_X^m$ is identified with the homomorphism sending $df_1 \wedge \cdots \wedge df_m$ to $\xi(f_1 \wedge \cdots \wedge f_m)$. This also coincides with $\mathrm{Hom}_{\mathcal{O}_X}(\tilde{\Omega}_X^m, \mathcal{O}_X)$.

When X is smooth, then $\tilde{\Omega}_X^\bullet = \Omega_X^\bullet$, and its hypercohomology (which, for X affine, is the same as the cohomology of its complex of global sections) is called the algebraic de Rham cohomology of X. Over $\mathbf{k} = \mathbf{C}$, this cohomology coincides with the topological cohomology of X under the complex topology, by a well-known theorem of Grothendieck. For arbitrary X, we will denote the cohomology of the space of global sections, $\Gamma(\tilde{\Omega}_X^\bullet)$, by $H_{DR}^\bullet(X)$, and the hypercohomology of the complex of sheaves $\tilde{\Omega}_X^\bullet$ by $\mathbf{H}_{DR}^\bullet(X)$ (very often we will use these when X is smooth and affine, where they both coincide with topological cohomology).

We caution that, when X is smooth, Ω_X (without a superscript) will denote the canonical right $\mathcal{D}_X$-module of volume forms, which as a $\mathcal{O}_X$-module coincides with $\Omega_X^{\dim X}$ under the above definition, when X has pure dimension.

By a *local system* on a variety, we mean an $\mathcal{O}$-coherent right $\mathcal{D}$-module on the variety. Moreover, from now on, when we say $\mathcal{D}$-module, we always will mean a right $\mathcal{D}$-module.

2.1. Reformulation of Theorem 1.2 in terms of leaves.

Recall that $(X, \mathfrak{v})$ is a pair of an affine scheme X of finite type and a Lie algebra $\mathfrak{v} \subseteq \mathrm{Vect}(X)$ of vector fields on X. We will give a more geometric formulation of Theorem 1.2 in terms of *leaves* of X under $\mathfrak{v}$, followed by a strengthened version in these terms.

Definition 2.1. An *invariant subscheme* is a locally closed subscheme $Z \subseteq X$ preserved by $\mathfrak{v}$; set-theoretically, this says that, at every point $z \in Z$, the evaluation $\mathfrak{v}|_z$ lies in the tangent space $T_z Z_{\mathrm{red}}$. A *leaf* is a connected invariant (reduced) subvariety Z such that, at every point z, in fact $\mathfrak{v}|_z = T_z Z$. A *degenerate invariant subscheme* is an invariant subscheme Z such that, at every point $z \in Z$, $\mathfrak{v}|_z \subsetneq T_z Z_{\mathrm{red}}$.

In the case of closed subschemes $Z \subseteq X$, the above can be rephrased in terms of the ideal I_Z of Z: Z is invariant if $\mathfrak{v}(I_Z) \subseteq I_Z$; it is a leaf if $\mathcal{O}_Z/I_Z$ has no nilpotents and the natural map $\mathcal{O}_Z \otimes \mathfrak{v} \to \mathrm{Der}(\mathcal{O}_Z)$ is surjective; and Z is a degenerate invariant subscheme if the cokernel of $\mathcal{O}_{Z_{\mathrm{red}}} \otimes \mathfrak{v} \to \mathrm{Der}(\mathcal{O}_{Z_{\mathrm{red}}})$ is fully supported on Z_{red} (i.e., for every $z \in Z_{\mathrm{red}}$, this cokernel does not vanish in any neighborhood of z).

The terminology "degenerate invariant subscheme" comes from the equivalent definition: the rank of $\mathfrak{v}$ on Z_{red} is everywhere less than the dimension of Z; equivalently, in a formal or analytic neighborhood of every point of Z, there are infinitely many leaves.

When an invariant subscheme is reduced, we call it an invariant subvariety. An invariant subscheme $Z \subseteq X$ is degenerate if and only if the invariant subvariety Z_{red} is degenerate. Note that the closure of any degenerate invariant subscheme is also such. Also, leaves are necessarily smooth. Although the same is clearly not true of degenerate invariant subschemes, we can restrict our attention to those with smooth reduction by first stratifying X_{red} by its (set-theoretic) singular loci, in view of the classical result:

Theorem 2.2. [Sei67, Corollary to Theorem 12] The set-theoretic singular locus of X_{red} is preserved by all vector fields on X.

We give a proof of a more general assertion in the proof of Proposition 2.6 below.

Remark 2.3. Note that, for the set-theoretic singular locus to be preserved by all vector fields, we need to use that the characteristic of $\mathbf{k}$ is zero; otherwise the singular locus is not preserved by all vector fields: e.g., in characteristic $p > 0$, one has the derivation ∂_x of $\mathbf{k}[x,y]/(y^2 - x^p)$, which does not vanish at the singular point at the origin.

On the other hand, in arbitrary characteristic, the scheme-theoretic singular locus of a variety of pure dimension $k \geq 0$ is preserved, where we define this by the Jacobian ideal: for a variety cut out by equations f_i in affine space, this is the ideal generated by determinants of $(k \times k)$-minors of the Jacobian matrix $(\frac{\partial f_i}{\partial x_j})$ (this is preserved by [Har74], where it is shown that it coincides with the smallest nonzero Fitting ideal of the module of Kähler differentials). In the above example it would be defined by the ideal (y) when $p > 2$. This is evidently preserved by all vector fields, which are all multiples of ∂_x. Note, however, that we will not make use of the scheme-theoretic singular locus in this paper (except in §5, where we will explicitly define it), nor will we consider the case of positive characteristic.

Definition 2.4. Say that $(X, \mathfrak{v})$ has *finitely many leaves* if X_{red} is a (disjoint) union of finitely many leaves.

For example, when X is a Poisson variety and $\mathfrak{v}$ is the Lie algebra of Hamiltonian vector fields, then this condition says that X has finitely many symplectic leaves.

We caution that, when $(X, \mathfrak{v})$ does not have finitely many leaves, it does *not* follow that there are infinitely many algebraic leaves, or any at all:

Example 2.5. Consider the two-dimensional torus $X = (\mathbf{A}^1 \setminus \{0\})^2$, and let $\mathfrak{v} = \langle \xi \rangle$ for some global vector field ξ which is not algebraically integrable, e.g., $x\partial_x - cy\partial_y$ where c is irrational. The analytic leaves of this are the level sets of $x^c y$, which are not algebraic. There are in fact *no* algebraic leaves at all.

However, it is always true that, in the formal neighborhood $\hat{X}_x$ of every point $x \in X$, there exists a formal leaf of X through x: this is the orbit of the formal group obtained by integrating $\mathfrak{v}$. In the above example, this says that the level sets of $x^c y$ do make sense in the formal neighborhood of every point $(x, y) \in X$.

The condition of having finitely many leaves is well-behaved:

Proposition 2.6. Let $X_i := \{x \in X \mid \dim \mathfrak{v}|_x = i\} \subseteq X_{\text{red}}$. Then X_i is an invariant locally closed subvariety. Moreover, X has finitely many leaves if and only if the connected components of the X_i are all leaves, and X does not have finitely many leaves if and only if some X_i contains a degenerate invariant subvariety.

Proof. First, to see that the X_i are locally closed, it suffices to show that $Y_j := \bigsqcup_{i \leq j} X_i$ is closed for all j. This statement would be clear if $\mathfrak{v}$ were finite-dimensional; for general $\mathfrak{v}$ we can write $\mathfrak{v}$ as a union of its finite-dimensional subspaces, and $Y_j(\mathfrak{v})$ is the intersection of $Y_j(\mathfrak{v}')$ over all finite-dimensional subspaces $\mathfrak{v}' \subseteq \mathfrak{v}$.

Next, we claim that, for all $i \leq k$, the subvariety $X_{i,k} \subseteq X$ of points $x \in X_i$ at which $\dim T_x X = k$ is preserved by all vector fields from $\mathfrak{v}$.

Let $S := \operatorname{Spec} \mathbf{k}[\![t]\!]$ and $X_S := \operatorname{Spec} \mathcal{O}_X[\![t]\!]$. For every $\xi \in \mathfrak{v}$, consider the automorphism $e^{t\xi}$ of $\mathcal{O}_{X_S}$. For any point $x \in X_{i,k}$, consider the corresponding S-point $x_S \in X_S$, i.e., $\mathcal{O}_S$-linear homomorphism $\mathcal{O}_{X_S} \to \mathcal{O}_S$. Let $\mathfrak{m} = \mathfrak{m}_{x_S}$ be its kernel, i.e., $\mathfrak{m}_x[\![t]\!]$. Then, let $\tilde{x}_S = e^{t\xi} x_S$, another S-point of X_S, and let $\tilde{\mathfrak{m}} = \mathfrak{m}_{x_S}$ be the kernel of its associated homomorphism $\mathcal{O}_{X_S} \to \mathcal{O}_S$.

Let the cotangent space to X_S at x_S be defined as $T^*_{x_S} X_S = \mathfrak{m}/\mathfrak{m}^2$, and similarly $T^*_{x'_S} X_S = \tilde{\mathfrak{m}}/\tilde{\mathfrak{m}}^2$. Since $T^*_{x_S} X_S$ is a free $\mathcal{O}_S$-module of rank k, the same holds for $T^*_{\tilde{x}_S} X_S$.

Moreover, we can view $\mathfrak{v}[\![t]\!]$ as a space of vector fields on X_S over S, i.e., as a subspace of $\mathcal{O}_S$-derivations $\mathcal{O}_{X_S} \to \mathcal{O}_{X_S}$. Since $e^{t\xi}$ is an automorphism preserving $\mathfrak{v}[\![t]\!]$, it follows as for $x_S \in X_S$

that the image of $\mathfrak{v}[\![t]\!] \to \operatorname{Hom}_{\mathcal{O}_S}(T^*_{\tilde{x}_S} X_S, \mathcal{O}_S)$ is a free $\mathcal{O}_S$-module of rank i. We conclude that $\tilde{x}_S \in (X_{i,k})_S = \mathsf{Spec}\, \mathcal{O}_{X_{i,k}}[\![t]\!] \subseteq X_S$.

We conclude from the preceding paragraphs that $\mathfrak{v}$ is parallel to $X_{i,k}$, i.e., that $\mathfrak{v}|_x \subseteq T_x X_{i,k}$ for all $x \in X_{i,k}$, as desired.

This can also be used to prove Theorem 2.2: setting $i = k = \dim X + 1$, we conclude that the intersection of the (set-theoretic) singular locus with the union of irreducible components of X of top dimension is preserved by all vector fields; one can then induct on dimension. Alternatively, one can apply the above argument, replacing $X_{i,k}$ by the set-theoretic singular locus of X.

For the final statement of the proposition, first note that, if one of the X_i contains a degenerate invariant subvariety, it cannot be a union of finitely many leaves, since this cannot hold for a degenerate invariant subvariety. Since X has finitely many leaves if and only if the same is true for all of the X_i, we deduce that this fails precisely when one of the X_i contains a degenerate invariant subvariety. It remains to show that, if X has finitely many leaves, then the connected components of the X_i are leaves. Since they cannot contain degenerate invariant subvarieties by the above, it follows that, for generic x in each irreducible component of X_i, we must have $\mathfrak{v}|_x = T_x X_i$. Thus the dimension of X_i is equal to i, and we have $\dim \mathfrak{v}|_x \geq \dim T_x X_i$ for all $x \in X_i$. The reverse inequality is automatic, so $\mathfrak{v}|_x = T_x X_i$ for all $x \in X_i$. This implies that the connected components of the X_i are leaves. □

Remark 2.7. We needed to use the formal power series ring $\mathbf{k}[\![t]\!]$ in the proof in order to integrate derivations to automorphisms for general $\mathbf{k}$ of characteristic zero. In the case that $\mathbf{k} = \mathbf{C}$, on the other hand, we could prove the proposition by embedding X into $\mathbf{C}^k$ and locally analytically integrating the flow of vector fields of $\mathfrak{v}$ (which individually noncanonically lift to $\mathbf{C}^k$), which must preserve the singular locus and the rank of $\mathfrak{v}$.

Corollary 2.8. There can be at most one decomposition of X_{red} into finitely many leaves. The following are equivalent:

(i) X has finitely many leaves;
(ii) X contains no degenerate invariant subvariety;
(iii) For all i, the dimension of X_i is at most i.

Proof. For the first statement, suppose that $X = \sqcup_i Z_i = \sqcup_i Z'_i$ are two decompositions into leaves. Then each nonempty pairwise intersection $Z_i \cap Z'_j$ is evidently a leaf. Now, for each i, $Z_i = \sqcup_j (Z_i \cap Z'_j)$ is a decomposition of Z_i as a disjoint union of locally closed subvarieties of the same dimension as Z_i. Since Z_i is connected, this implies that this decomposition is trivial, i.e., $Z'_j = Z_i$ for some j.

For the equivalence, first we show that (i) implies (ii). Indeed, if X were a union of finitely many leaves and also X contained a degenerate invariant subvariety Z, we could assume Z is irreducible. Then there would be some X_i such that $X_i \cap Z$ is open and dense in Z. But then the rank of $\mathfrak{v}$ along $X_i \cap Z$ would be less than the dimension of Z, and hence less than the dimension of X_i, a contradiction. To show (ii) implies (iii), note that, if $\dim X_i > i$, then any open subset of X_i of pure maximal dimension is degenerate. To show (iii) implies (i), note that the decomposition of Proposition 2.6 must be into leaves if $\dim X_i \leq i$ for all i (in fact, in this case, each X_i is a (possibly empty) finite union of leaves of dimension i). □

In view of Corollary 2.8, Theorem 1.2 above can be restated as:

Theorem 2.9. If $(X, \mathfrak{v})$ has finitely many leaves, then $(\mathcal{O}_X)_{\mathfrak{v}}$ is finite-dimensional.

In the aforementioned Poisson variety case, the theorem is a special case of [ES10, Theorem 1.1]. Note that the converse to the theorem does not hold: see Remark 2.22. As with Theorem 1.2, we will generalize Theorem 2.9 in Theorem 2.19 below, and hence we omit its proof here.

Remark 2.10. Suppose that X is irreducible and that $\mathfrak{v}$ acts locally finitely and semisimply on $\mathcal{O}_X$, e.g., if $\mathfrak{v}$ is the Lie algebra of a reductive algebraic group acting on X. In this situation, Theorem 2.9 is elementary. It is enough to assume that X is irreducible, so that if $\mathfrak{v}$ acts with finitely many leaves, then there is a unique open dense leaf. In this case, $\dim(\mathcal{O}_X)_{\mathfrak{v}} = 1$. This is because, by local finiteness and semisimplicity, the canonical map $(\mathcal{O}_X)^{\mathfrak{v}} \to (\mathcal{O}_X)_{\mathfrak{v}}$ is an isomorphism, and the former has dimension one since there is a unique open dense leaf, on which the invariant functions are all constant.

Remark 2.11. One can obtain examples where $\dim(\mathcal{O}_X)_{\mathfrak{v}} > 1$ when $\mathfrak{v}$ is semisimple and has only a single leaf (in particular, X is smooth), but does not act locally finitely. For example, let $X \subseteq \mathbf{A}^2$ be any nonempty open affine subvariety such that $0 \notin X$. Let $\mathfrak{sl}_2$ act on X by the restriction of its action on $\mathbf{A}^2$. This is the Lie algebra of linear Hamiltonian vector fields with respect to the usual symplectic structure on $\mathbf{A}^2$. Since X is affine symplectic, if $H(X)$ denotes the Hamiltonian vector fields, $(\mathcal{O}_X)_{H(X)} \cong H^{\dim X}(X) = H^2_{DR}(X)$, by the usual isomorphism $[f] \mapsto f \cdot \mathsf{vol}_X$.[1] On the other hand, $\mathfrak{sl}_2 \subseteq H(X)$, so $\dim(\mathcal{O}_X)_{\mathfrak{sl}_2} \geq \dim H^2(X)$ (in fact this is an equality since $\mathfrak{sl}_2 \cdot \mathcal{O}_X = H(X) \cdot \mathcal{O}_X$ inside the ring of differential operators $\mathcal{D}_X$ on X, since $\mathfrak{sl}_2$ generates the tangent space everywhere and is volume-preserving; see Proposition 2.53 below). There are many examples of such varieties X which have $\dim H^2(X) > 1$. For example, if X is the complement of $n+1$ lines through the origin, then $\dim H^2(X) = n$: the Betti numbers of X are $1, n+1$, and n, since the Euler characteristic is zero, each deleted line creates an independent class in first cohomology, and there can be no cohomology in degrees higher than two as X is a two-dimensional affine variety. This produces an example as desired for $n \geq 2$.

2.2. **The $\mathcal{D}$-module defined by $\mathfrak{v}$.** The proof of the theorems above is based on a stronger result concerning the $\mathcal{D}$-module whose solutions are invariants under the flow of $\mathfrak{v}$. This construction generalizes $M(X)$ from [ES10] in the case X is Poisson and $\mathfrak{v}$ is the Lie algebra of Hamiltonian vector fields. Namely, we prove that this $\mathcal{D}$-module is holonomic when X has finitely many leaves. We will explain a partial converse in §2.4, and discuss holonomicity in more detail in §2.9 below.

We will need to use right $\mathcal{D}$-modules on X as formulated by Kashiwara. Namely, first suppose V is a smooth affine variety. Then the category of right $\mathcal{D}$-modules on V is the category of right modules over the ring $\mathcal{D}_V$ of differential operators on $\mathcal{O}_V$ with polynomial coefficients (note that $\mathcal{D}_V = \mathsf{Sym}_{\mathcal{O}_V} \operatorname{Der}(\mathcal{O}_V)$). In particular, any right $\mathcal{D}$-module on V is a module over $\mathcal{O}_V$, and we can therefore define its support just as for $\mathcal{O}_V$-modules. Next, suppose $X \subseteq V$ is any closed affine subvariety. Then we define the category of right $\mathcal{D}$-modules on X to be the full subcategory of right $\mathcal{D}$-modules on V which are supported on X, i.e., whose support is contained in X. This all generalizes to define right $\mathcal{D}$-modules on any variety as follows: for smooth varieties the same definition applies where $\mathcal{O}_V$ and $\mathcal{D}_V$ are now sheaves of algebras and modules are quasicoherent sheaves of modules, and then $\mathcal{D}$-modules on $X \subseteq V$ is defined in the same way (where X and V need not be affine). As proved by Kashiwara, the category of right $\mathcal{D}$-modules on X does not depend, up to canonical equivalence, on the choice of embedding $X \subseteq V$. Therefore we can refer to $\mathcal{D}$-modules on X without a choice of embedding, and given any embedding $X \subseteq V$, we can call the resulting right $\mathcal{D}$-modules on V the *image under Kashiwara's equivalence* (of right $\mathcal{D}$-modules on X), i.e., under Kashiwara's equivalence between the category of right $\mathcal{D}$-modules on X and the category of right $\mathcal{D}$-modules on V supported on X. (We remark that there is another way to define the category of right $\mathcal{D}$-modules on X, under the name "crystals," which does not depend on a choice of embedding $X \subseteq V$ at all, and if one uses this definition, Kashiwara's equivalence is a theorem.)

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[1] Dually, in the complex case $\mathbf{k} = \mathbf{C}$, the second homology of X as a topological space produces the functionals on $\mathcal{O}_X$ invariant under $H(X)$ (and hence also those invariant under $\mathfrak{sl}_2$) by $C \in H_2(X) \mapsto \Phi_C$, $\Phi_C(f) = \int_C f \mathsf{vol}_X$.

For every variety X, there is a canonical right $\mathcal{D}$-module on X which we call $\mathcal{D}_X$. When $X = V$ is smooth, this is just the ring (or sheaf of rings) $\mathcal{D}_V$ of differential operators viewed as a right module over itself. When $X \subseteq V$ is an embedding, then $\mathcal{D}_X = I_X \cdot \mathcal{D}_V \setminus \mathcal{D}_V$, the quotient of $\mathcal{D}_V$ by the right ideal generated by the functions $I_X \subseteq \mathcal{O}_X$ vanishing on X. Note that $\mathcal{D}_X$ is canonically equipped with a left action by functions on X, as well as by vector fields on X, i.e., derivations on $\mathcal{O}_X$ (which are the same as derivations of $\mathcal{O}_V$ preserving I_X modulo derivations whose image is entirely in I_X).

From now on, since we will only deal with right $\mathcal{D}$-modules, we will often suppress the term "right." Our main object of study is the following $\mathcal{D}$-module on X:

$$M(X, \mathfrak{v}) := \mathfrak{v} \cdot \mathcal{D}_X \setminus \mathcal{D}_X, \tag{2.12}$$

where $\mathfrak{v} \cdot \mathcal{D}_X$ is the submodule generated by the action of $\mathfrak{v}$ on $\mathcal{D}_X$. (We will also use the same definition when X is replaced by its completion $\hat{X}_x$ at points $x \in X$, even though $\hat{X}_x$ does not have finite type.)

Explicitly, if $i : X \to V$ is an embedding into a smooth affine variety V, let $\widetilde{\mathfrak{v}} \subseteq \mathrm{Vect}(V)$ be the subspace of vector fields which are parallel to X and restrict on X to elements of $\mathfrak{v}$. Then, the image of $M(X, \mathfrak{v})$ under Kashiwara's equivalence is

$$M(X, \mathfrak{v}, i) = (I_X + \widetilde{\mathfrak{v}})\mathcal{D}_V \setminus \mathcal{D}_V.$$

Let $\pi : X \to \mathsf{Spec}\,\mathbf{k}$ be the projection to a point, and π_0 the functor of underived direct image from $\mathcal{D}$-modules on X to those on $\mathbf{k}$, i.e., $\mathbf{k}$-vector spaces. Explicitly, if M is a $\mathcal{D}$-module on a smooth affine variety V which is supported on X, then $\pi_0 M := M \otimes_{\mathcal{D}_V} \mathcal{O}_V$.

Proposition 2.13. $\pi_0 M(X, \mathfrak{v}) = (\mathcal{O}_X)_{\mathfrak{v}}$.

Proof. Fix an affine embedding $X \hookrightarrow V$. Then,

$$\pi_0 M(X, \mathfrak{v}) = (I_X + \widetilde{\mathfrak{v}})\mathcal{D}_V \setminus \mathcal{D}_V \otimes_{\mathcal{D}_V} \mathcal{O}_V = (\mathcal{O}_X)_{\mathfrak{v}}.$$ □

If $Z \subseteq X$ is an invariant closed subscheme, we will repeatedly use the following relationship between $M(X, \mathfrak{v})$ and $M(Z, \mathfrak{v}|_Z)$:

Proposition 2.14. If $i : Z \to X$ is the tautological embedding of an invariant closed subscheme, then there is a canonical surjection $M(X, \mathfrak{v}) \twoheadrightarrow i_* M(Z, \mathfrak{v}|_Z)$.

Proof. Since $i_* M(Z, \mathfrak{v}|_Z) = ((\mathfrak{v} + I_Z) \cdot \mathcal{D}_X) \setminus \mathcal{D}_X$, where I_Z is the ideal of Z, this is the quotient of $M(X, \mathfrak{v}) = \mathfrak{v} \cdot \mathcal{D}_X \setminus \mathcal{D}_X$ by its submodule $(\mathfrak{v} \cdot \mathcal{D}_X \cap I_Z \cdot \mathcal{D}_X) \setminus (I_Z \cdot \mathcal{D}_X)$. □

Remark 2.15. As pointed out in the previous subsection, one could more generally allow $\mathfrak{v}$ to be an arbitrary subset of $\mathrm{Vect}(X)$. However, it is easy to see that the $\mathcal{D}$-module is the same as for the Lie algebra generated by this subset. So, no generality is lost by assuming that $\mathfrak{v}$ be a Lie algebra.

Notation 2.16. By a Lie algebroid in $\mathrm{Vect}(X)$, we mean a Lie subalgebra which is also a coherent subsheaf.

Remark 2.17. One could more generally (although equivalently in a sense we will explain) allow $\mathfrak{v} \subseteq \mathcal{D}_X^{\leq 1}$ to be a space of differential operators of order ≤ 1. One then sets, as before, $M(X, \mathfrak{v}) = \mathfrak{v} \cdot \mathcal{D}_X \setminus \mathcal{D}_X$. In this case, one obtains the same $\mathcal{D}$-module not merely by passing to the Lie algebra generated by $\mathfrak{v}$, but in fact one can also replace $\mathfrak{v}$ by $\mathfrak{v} \cdot \mathcal{O}_X$. Let $\sigma : \mathcal{D}_X^{\leq 1} \to \mathrm{Vect}(X)$ denote the principal symbol. Then, we conclude that $\sigma(\mathfrak{v}) \subseteq \mathrm{Vect}(X)$ is actually a Lie algebroid (cf. Notation 2.16).

This is actually equivalent to using only vector fields, in the following sense: Given any pair $(X, \mathfrak{v})$ with $\mathfrak{v} \subseteq \mathcal{D}_X^{\leq 1}$, one can consider the pair $(\mathbf{A}^1 \times X, \hat{\mathfrak{v}})$ where, for x the coordinate on $\mathbf{A}^1$, $\hat{\mathfrak{v}}$ contains the vector field ∂_x together with, for every differential operator $\theta \in \mathfrak{v}$, $\sigma(\theta) - (\theta - \sigma(\theta))x\partial_x$.

Since $(x\partial_x + 1) = \partial_x \cdot x \in (\partial_x \cdot \mathcal{D}_{\mathbf{A}^1})$, one easily sees that $M(\mathbf{A}^1 \times X, \hat{\mathfrak{v}}) \cong \Omega_{\mathbf{A}^1} \boxtimes M(X, \mathfrak{v})$. So, in this sense, one can reduce the study of pairs $(X, \mathfrak{v})$ to the study of affine schemes of finite type with Lie algebras of vector fields. In particular, our general results extend easily to the setting of differential operators of order ≤ 1.

Remark 2.18. Similarly, one can reduce the study of pairs $(X, \mathfrak{v})$ to the case where X is affine space. Indeed, if $X \hookrightarrow \mathbf{A}^n$ is any embedding, and I_X is the ideal of X, we can consider the Lie algebroid

$$I_X \cdot \mathcal{D}^{\leq 1}_{\mathbf{A}^n} + \mathfrak{v} \subseteq \mathcal{D}^{\leq 1}_{\mathbf{A}^n}.$$

This makes sense by lifting elements of $\mathfrak{v}$ to vector fields on $\mathbf{A}^n$, and the result is independent of the choice. We can then apply the previous remark to reduce everything to Lie algebras of vector fields on affine space. (This is not really helpful, though: in our examples, $\mathfrak{v}$ is naturally associated with X (e.g., Hamiltonian vector fields on X), so it is not natural to replace X with an affine space.)

2.3. **Holonomicity and proof of Theorems 1.2 and 2.9.** Recall that a nonzero $\mathcal{D}$-module on X is *holonomic* if it is finitely generated and its singular support is a Lagrangian subvariety of T^*X (i.e., its dimension equals that of X). We always call the zero module holonomic. (Derived) pushforwards of holonomic $\mathcal{D}$-modules are well-known to have holonomic cohomology. Since a holonomic $\mathcal{D}$-module on a point is finite-dimensional, this implies that, if M is holonomic and $\pi : X \to \mathrm{pt}$ is the pushforward to a point, then $\pi_* M$ (by which we mean the cohomology of the complex of vector spaces), and in particular $\pi_0 M$, is finite-dimensional. Therefore, if we can show that $M(X, \mathfrak{v})$ is holonomic, this implies that $(\mathcal{O}_X)_{\mathfrak{v}} = \pi_0 M(X, \mathfrak{v})$ is finite-dimensional, along with the full pushforward $\pi_* M(X, \mathfrak{v})$. This reduces Theorem 2.9 and equivalently Theorem 1.2 to the statement:

Theorem 2.19. If $(X, \mathfrak{v})$ has finitely many leaves, then $M(X, \mathfrak{v})$ is holonomic. In this case, the composition factors are intermediate extensions of local systems along the leaves.

The converse does not hold: see, e.g., Example 2.32.

Proof of Theorem 2.19. The equations $\mathsf{gr}\,\mathfrak{v}$ are satisfied by the singular support of $M(X, \mathfrak{v})$. These equations say, at every point $x \in X$, that the restriction of the singular support of $M(X, \mathfrak{v})$ to x lies in $(\mathfrak{v}|_x)^\perp$. Thus, if $Z \subseteq X$ is a leaf, then the restriction of the singular support of $M(X, \mathfrak{v})$ to Z lies in the conormal bundle to Z, which is Lagrangian. If X is a finite union of leaves, it follows that the singular support of $M(X, \mathfrak{v})$ is contained in the union of the conormal bundles to the leaves, which is Lagrangian. The last statement immediately follows from this description of the singular support. $\square$

We will be interested in the condition on $\mathfrak{v}$ for $M(X, \mathfrak{v})$ to be holonomic, which turns out to be subtle.

Definition 2.20. Call $(X, \mathfrak{v})$, or $\mathfrak{v}$, holonomic if $M(X, \mathfrak{v})$ is.

We will often use the following immediate consequence, whose proof is omitted:

Proposition 2.21. If $\mathfrak{v}$ is holonomic, then $\mathcal{O}_{\mathfrak{v}}$ is finite-dimensional.

Remark 2.22. The converse to Proposition 2.21 does not hold in general (although we will have a couple of cases where it does: the Lie algebras of all vector fields (Proposition 3.3) and of Hamiltonian vector fields preserving a top polyvector field (Corollary 3.37)). A simple example where this converse does not hold is $(X, \mathfrak{v}) = (\mathbf{A}^2, \langle \partial_x \rangle)$ (where x is one of the coordinates on $\mathbf{A}^2$), where $M(X, \mathfrak{v}) = \Omega_{\mathbf{A}^1} \boxtimes \mathcal{D}_{\mathbf{A}^1}$ is not holonomic, but $\mathcal{O}_{\mathfrak{v}} = 0$. This example also has infinitely many leaves, namely all lines parallel to the x-axis.

2.4. **Incompressibility and a weak converse.** We say that a vector field ξ preserves a differential form ω if the Lie derivative L_ξ annihilates ω.

Definition 2.23. Say that $\mathfrak{v}$ flows *incompressibly* along an irreducible invariant subvariety Z if there exists a smooth point $z \in Z$ and a volume form on the formal neighborhood of Z at z which is preserved by $\mathfrak{v}$.

There is an alternative definition using divergence functions which does not require formal localization, which we discuss in §3.5; see also Proposition 2.24.(iii). When X is irreducible and $\mathfrak{v}$ flows incompressibly on X, we omit the X and merely say that $\mathfrak{v}$ flows incompressibly. Note that this is equivalent to flowing generically incompressibly.

In §2.6 we will prove

Proposition 2.24. Let X be an irreducible affine variety. The following conditions are equivalent:

(i) $\mathfrak{v}$ flows incompressibly;
(ii) $M(X, \mathfrak{v})$ is fully supported;
(iii) For all $\xi_i \in \mathfrak{v}$ and $f_i \in \mathcal{O}_X$ such that $\sum_i f_i \xi_i = 0$, one has $\sum_i \xi_i(f_i) = 0$.

Moreover, the equivalence (ii) $\Leftrightarrow$ (iii) holds when X is an arbitrary affine scheme of finite type, if one generalizes (ii) to the condition: (ii') The annihilator of $M(X, \mathfrak{v})$ in $\mathcal{O}_X$ is zero.

Remark 2.25. We can alternatively state (ii') and (iii) as follows, in terms of global sections of $\mathfrak{v} \cdot \mathcal{D}_Z \subseteq \mathcal{D}_Z$ (cf. §2.6 below): (ii') says that $(\mathfrak{v} \cdot \mathcal{D}_Z) \cap \mathcal{O}_Z = 0$, and (iii) says that $(\mathfrak{v} \cdot \mathcal{O}_Z) \cap \mathcal{O}_Z = 0$.

Motivated by this proposition, we will generalize the notion of incompressibility to the case of nonreduced subschemes in §2.8 below, to be defined by conditions (ii') or (iii) above.

Example 2.26. In the case that X is a Poisson variety, $\mathfrak{v}$ is the Lie algebra of Hamiltonian vector fields, and $Z \subseteq X$ is a symplectic leaf (i.e., a leaf of $\mathfrak{v}$), then $\mathfrak{v}$ flows incompressibly on Z, since it preserves the symplectic volume along Z, and hence also preserves the symplectic volume in a formal neighborhood of any point $z \in Z$.

Definition 2.27. Say that $\mathfrak{v}$ has *finitely many incompressible leaves* if it has no degenerate invariant subvariety on which $\mathfrak{v}$ flows incompressibly.

As before, if $\mathfrak{v}$ does not have finitely many incompressible leaves, one does *not* necessarily have infinitely many incompressible leaves, or any at all (see Example 2.5, which does not have finitely many incompressible leaves, but has no algebraic leaves).

In §2.7 below we will prove

Theorem 2.28. (i) For every incompressible leaf $Z \subseteq X$, letting $i : \bar{Z} \hookrightarrow X$ be the tautological embedding of its closure, the canonical quotient $M(X, \mathfrak{v}) \twoheadrightarrow i_* M(\bar{Z}, \mathfrak{v}|_{\bar{Z}})$ is an extension of a nonzero local system on Z to $\bar{Z}$.
(ii) If $(X, \mathfrak{v})$ is holonomic, then it has finitely many incompressible leaves.

Note that the converse to (i) does not hold: see Example 2.34. We will give a correct converse statement in §2.9 below. Also, the converse to (ii) does not hold, as we will demonstrate in Example 2.31.

We conclude from the Theorems 2.19 and 2.28 that

$$\text{finitely many leaves} \Rightarrow \text{holonomic} \Rightarrow \text{finitely many incompressible leaves}, \tag{2.29}$$

but neither converse direction holds, as mentioned (see Examples 2.32 and 2.31, respectively). However, we will see below that the second implication is generically a biconditional for irreducible varieties X, i.e., X generically has finitely many incompressible leaves if and only if X is generically holonomic.

Example 2.30. When X is Poisson and $\mathfrak{v}$ the Lie algebra of Hamiltonian vector fields, then Theorem 2.28 and Example 2.26 imply that $\mathfrak{v}$ is holonomic *if and only if* X has finitely many symplectic leaves. More precisely, if $Z \subseteq X$ is any invariant subvariety, then in the formal neighborhood of a generic point $z \in Z$, we can integrate the Hamiltonian flow and write $\hat{Z}_z = V \times V'$ for formal polydiscs V and V', where the Hamiltonian flow is along the V direction, and transitive along fibers of $(V \times V') \twoheadrightarrow V'$. Then Hamiltonian flow preserves the volume form $\omega_V \otimes \omega_{V'}$, where ω_V is the canonical symplectic volume, and $\omega_{V'}$ is an arbitrary volume form on V'. Therefore, all Z are incompressible. (In particular, this includes the case mentioned already in Example 2.26.) Then (2.29) shows that $H(X)$ is holonomic if and only if there are finitely many leaves.

Example 2.31. We demonstrate that $(\mathcal{O}_X)_{\mathfrak{v}}$ need not be finite-dimensional if we only assume that X has finitely many incompressible leaves. Therefore, $\mathfrak{v}$ is not holonomic (although non-holonomicity also follows directly in this example). Let $X = \mathbf{A}^2 \times (\mathbf{A} \setminus \{0\}) \subseteq \mathbf{A}^3$, with $\mathbf{A}^2 = \mathsf{Spec}\,\mathbf{k}[x,y]$ and $\mathbf{A} \setminus \{0\} = \mathsf{Spec}\,\mathbf{k}[z,z^{-1}]$. Let $\mathfrak{v} = \langle y^2\partial_x, y\partial_y + z\partial_z, \partial_z\rangle$. Then this has an incompressible open leaf, $\{y \neq 0\}$, preserving the volume form $\frac{1}{y^2} dx \wedge dy \wedge dz$. The complement consists of the leaves $\{x = c, y = 0\}$ for all $c \in \mathbf{k}$, which are not incompressible since the restriction of $\mathfrak{v}$ to each such leaf (or to their union, $\{y = 0\}$) includes both ∂_z and $z\partial_z$.

We claim that the coinvariants $(\mathcal{O}_X)_{\mathfrak{v}}$ are infinite-dimensional, and isomorphic to $\mathbf{k}[x] \cdot yz^{-1}$ via the quotient map $\mathcal{O}_X \twoheadrightarrow (\mathcal{O}_X)_{\mathfrak{v}}$. Indeed, $y^2\partial_x(\mathcal{O}_X) = y^2\mathcal{O}_X$, $(y\partial_y+z\partial_z)\mathcal{O}_X = \mathbf{k}[x]\cdot\langle y^i z^j \mid i+j \neq 0\rangle$, and $\partial_z(\mathcal{O}_X) = \mathbf{k}[x,y] \cdot \langle z^i \mid i \neq -1\rangle$. The sum of these vector subspaces is the space spanned by all monomials in x, y, z, and z^{-1} except for $x^i y z^{-1}$ for all $i \geq 0$.

Example 2.32. It is easy to give an example where $\mathfrak{v}$ is holonomic but has infinitely many leaves: for Y any positive-dimensional variety, consider $X = \mathbf{A}^1 \times Y$, $\mathfrak{v} := \langle \partial_x, x\partial_x\rangle$, where x is the coordinate on $\mathbf{A}^1$. Then the leaves of $(X, \mathfrak{v})$ are of the form $\mathbf{A}^1 \times \{y\}$ for $y \in Y$, but $M(X, \mathfrak{v}) = 0$, which is holonomic.

Example 2.33. For a less trivial example, which is a generically nonzero holonomic $\mathcal{D}$-module without finitely many leaves, let $X = \mathbf{A}^3$ with coordinates x, y, and z, and let $\mathfrak{v}$ be the Lie algebra of all incompressible vector fields (with respect to the standard volume) which along the plane $x = 0$ are parallel to the y-axis. Then we claim that the singular support of $M(X, \mathfrak{v})$ is the union of the zero section of T^*X and the conormal bundle of the plane $x = 0$, which is Lagrangian, even though there are not finitely many leaves. Actually, from the computation below, we see that $M(X, \mathfrak{v})$ is isomorphic to $j_!\Omega_{\mathbf{A}^1\setminus\{0\}} \boxtimes \Omega_{\mathbf{A}^2}$, where $j : \mathbf{A}^1 \setminus \{0\} \hookrightarrow \mathbf{A}^1$ is the inclusion (which is an affine open embedding, so $j_!$ is an exact functor on holonomic $\mathcal{D}$-modules). This is an extension of $\Omega_{\mathbf{A}^3}$ by $i_*\Omega_{\mathbf{A}^2}$, where $i : \mathbf{A}^2 = \{0\} \times \mathbf{A}^2 \hookrightarrow \mathbf{A}^3$ is the closed embedding, i.e., there is an exact sequence

$$0 \to i_*\Omega_{\mathbf{A}^2} \to M(X, \mathfrak{v}) \to \Omega_{\mathbf{A}^3} \to 0.$$

Thus, there is a single composition factor on the open leaf and a single composition factor on the degenerate (but not incompressible) invariant subvariety $\{x = 0\}$.

To see this, note first that $\partial_y \in \mathfrak{v}$. We claim that $1 + x\partial_x$ and ∂_z are in $\mathfrak{v} \cdot \mathcal{D}_X$:

$$\partial_y \cdot y - (y\partial_y - x\partial_x) = 1 + x\partial_x;$$
$$(1 + x\partial_x) \cdot \partial_z - (x\partial_z) \cdot \partial_x = \partial_z.$$

Thus, $\langle 1 + x\partial_x, \partial_y, \partial_z\rangle \subseteq \mathfrak{v} \cdot \mathcal{D}_X$. Conversely, we claim that $\mathfrak{v} \subseteq \langle 1 + x\partial_x, \partial_y, \partial_z\rangle \cdot \mathcal{D}_X$. Indeed, given an incompressible vector field of the form $\xi = xf\partial_x + g\partial_y + xh\partial_z \in \mathfrak{v}$ for $f, g, h \in \mathcal{O}_X$, we can write

$$\xi = (1 + x\partial_x) \cdot f + \partial_y \cdot g + \partial_z \cdot xh,$$

where the RHS is a vector field (and not merely a differential operator of order ≤ 1) because ξ is incompressible. Explicitly, the condition for this RHS to be a vector field, and the condition for ξ to be incompressible, are both that $\partial_x(xf) + \partial_y(g) + \partial_z(xh) = 0$.

We conclude that $\langle 1 + x\partial_x, \partial_y, \partial_z\rangle \cdot \mathcal{D}_X = \mathfrak{v} \cdot \mathcal{D}_X$. Therefore, $M(X, \mathfrak{v}) \cong j_! \Omega_{\mathbf{A}^1 \setminus \{0\}} \boxtimes \Omega_{\mathbf{A}^2}$, as claimed.

Example 2.34. We can slightly modify Example 2.33, so that (again for $X := \mathbf{A}^3$ and $i : \{0\} \times \mathbf{A}^2 \hookrightarrow \mathbf{A}^3$), $i_*\Omega_{\mathbf{A}^2}$ appears as a quotient of $M(X, \mathfrak{v})$ rather than as a submodule. More precisely, we will have $M(X, \mathfrak{v}) \cong j_* \Omega_{\mathbf{A}^1 \setminus \{0\}} \boxtimes \Omega_{\mathbf{A}^2}$. To do so, let $\mathfrak{v}$ be the Lie algebra of all incompressible vector fields preserving the volume form $\frac{1}{x^2} dx \wedge dy \wedge dz$ (cf. Example 2.31), which again along the plane $x = 0$ are parallel to the y-axis. Note also that, in this example, the subvariety $\{0\} \times \mathbf{A}^2$ is still not incompressible (since ∂_y and $y\partial_y$ are both in $\mathfrak{v}|_{0 \times \mathbf{A}^2}$, and these cannot both preserve the same volume form), even though this subvariety now supports a quotient $i_*\Omega_{\mathbf{A}^2}$ of $M(X)$.

To see this, we claim that $\mathfrak{v} \cdot \mathcal{D}_X = \langle 1 - x\partial_x, \partial_y, \partial_z \rangle \cdot \mathcal{D}_X$. For the containment $\supseteq$, we show that $1 - x\partial_x$ and ∂_z are in $\mathfrak{v} \cdot \mathcal{D}_X$. This follows from

$$\partial_y \cdot y - (y\partial_y + x\partial_x) = 1 - x\partial_x;$$
$$(1 - x\partial_x) \cdot \partial_z + (x\partial_z) \cdot \partial_x = \partial_z.$$

Then, as in Example 2.33, if $\xi = xf\partial_x + g\partial_y + xh\partial_z \in \mathfrak{v}$ preserves the volume form $\frac{1}{x^2} dx \wedge dy \wedge dz$, then

$$\xi = -(1 - x\partial_x) \cdot f + \partial_y \cdot g + \partial_z \cdot xh.$$

Therefore, we also have the opposite containment, $\mathfrak{v} \cdot \mathcal{D}_X \subseteq \langle 1 - x\partial_x, \partial_y, \partial_z \rangle \cdot \mathcal{D}_X$. As a consequence, $M(X, \mathfrak{v}) \cong j_* \Omega_{\mathbf{A}^1 \setminus \{0\}} \boxtimes \Omega_{\mathbf{A}^2}$. We therefore have a canonical exact sequence

$$0 \to \Omega_{\mathbf{A}^3} \to M(X, \mathfrak{v}) \to i_* \Omega_{\mathbf{A}^2} \to 0.$$

2.5. **The transitive case.** In this section we consider the simplest, but important, example of $\mathfrak{v}$ and the $\mathcal{D}$-module $M(X, \mathfrak{v})$, namely when $\mathfrak{v}$ has maximal rank everywhere:

Definition 2.35. A pair $(X, \mathfrak{v})$ is called *transitive at x* if $\mathfrak{v}|_x = T_x X$. We call the pair $(X, \mathfrak{v})$ transitive if it is so at all $x \in X$.

In other words, the transitive case is the one where every connected component of X is a leaf. Note that, in particular, X must be a smooth variety. Also, we remark that X is generically transitive if and only if it is generically not degenerate.

Proposition 2.36. If $(X, \mathfrak{v})$ is transitive and connected, then $M(X, \mathfrak{v})$ is a rank-one local system if $\mathfrak{v}$ flows incompressibly, and $M(X, \mathfrak{v}) = 0$ otherwise.

Proof. By taking associated graded of $M(X, \mathfrak{v})$, in the transitive connected case, one obtains either $\mathcal{O}_X$ (where $X \subseteq T^*X$ is the zero section) or zero. So $M(X, \mathfrak{v})$ is either a one-dimensional local system on X, or zero. In the incompressible case, in a formal neighborhood of some $x \in X$, a volume form is preserved, so there is a surjection $M(\hat{X}_x, \mathfrak{v}|_{\hat{X}_x}) \twoheadrightarrow \Omega_{\hat{X}_x}$, and hence in this case $M(X, \mathfrak{v})$ is a one-dimensional local system. Conversely, if $M(X, \mathfrak{v})$ is a one-dimensional local system, then in a formal neighborhood of any point $x \in X$, it is a trivial local system, and hence it preserves a volume form there. ∎

Example 2.37. In the case when X is connected and Calabi-Yau and $\mathfrak{v}$ preserves the global volume form (which includes the case where X is symplectic and $\mathfrak{v}$ is the Lie algebra of Hamiltonian vector fields), then we conclude that $M(X, \mathfrak{v}) \cong \Omega_X$. Thus, for $\pi : X \to \mathrm{pt}$ the projection to a point, $(\mathcal{O}_X)_{\mathfrak{v}} = \pi_0 \Omega_X = H_{DR}^{\dim X}(X)$, the top de Rham cohomology. Taking the derived pushforward, we conclude that $\pi_* M(X, \mathfrak{v}) = \pi_* \Omega_X = H_{DR}^{\dim X - *}(X)$. In the Poisson case, where $(\mathcal{O}_X)_{\mathfrak{v}}$ is the zeroth Poisson homology, in [ES10, Remark 2.27] this motivated the term *Poisson-de Rham homology*, $HP_*^{DR}(X) = \pi_* M(X, \mathfrak{v})$, for the derived pushforward. More generally, if $\mathfrak{v}$ preserves a *multivalued* volume form, then $M(X, \mathfrak{v})$ is a nontrivial rank-one local system and

$\pi_* M(X, \mathfrak{v}) = H_{DR}^{\dim X - *}(X, M(X, \mathfrak{v}))$ is the cohomology of X with coefficients in this local system (identifying $M(X, \mathfrak{v})$ with its corresponding local system under the de Rham functor). See the next example for more details on how to define such $\mathfrak{v}$.

Example 2.38. The rank-one local system need not be trivial when $\mathfrak{v}$ does not preserve a global volume form. For example, let $X = (\mathbf{A}^1 \setminus \{0\}) \times \mathbf{A}^1 = \mathsf{Spec}\, \mathbf{k}[x, x^{-1}, y]$. Then we can let $\mathfrak{v}$ be the Lie algebra of vector fields preserving the multivalued volume form $d(x^r) \wedge dy$ for $r \in \mathbf{k}$. It is easy to check that this makes sense and that the resulting Lie algebra $\mathfrak{v}$ is transitive. Then, $M(X, \mathfrak{v})$ is the rank-one local system whose homomorphisms to Ω_X correspond to this volume form, which is nontrivial (but with regular singularities) when r is not an integer. For $\mathbf{k} = \mathbf{C}$, the local system $M(X, \mathfrak{v})$ thus has monodromy $e^{-2\pi i r}$.

More generally, if X is an arbitrary smooth variety of pure dimension at least two, and ∇ is a flat connection on Ω_X, we can think of the flat sections of ∇ as giving multivalued volume forms, and define a corresponding Lie algebra $\mathfrak{v}$ so that $\mathrm{Hom}_{\mathcal{D}_X}(M(X, \mathfrak{v}), \Omega_X)$ returns these forms on formal neighborhoods. Precisely, we can let $\mathfrak{v}$ be the Lie algebra of vector fields preserving formal flat sections of ∇. We need to check that $\mathfrak{v}$ is transitive, which is where we use the hypothesis that X has pure dimension at least two: see §3.5.2 and in particular Proposition 3.62 (alternatively, we could simply impose the condition that $\mathfrak{v}$ be transitive, which is immediate to check in the example of the previous paragraph). Then $M(X, \mathfrak{v}) \cong (\Omega_X, \nabla)^* \otimes_{\mathcal{O}_X} \Omega_X$, via the map sending the canonical generator $1 \in M(X, \mathfrak{v})$ to the identity element of $\mathrm{End}_{\mathcal{O}_X}(\Omega_X)$. Conversely, if $(X, \mathfrak{v})$ is transitive and $M(X, \mathfrak{v})$ is nonzero (hence a rank-one local system), then $\mathrm{Hom}_{\mathcal{O}_X}(M(X, \mathfrak{v}), \Omega_X)$ canonically has the structure of a local system on Ω_X with formal flat sections given by $\mathrm{Hom}_{\mathcal{D}_X}(M(X, \mathfrak{v}), \Omega_X)$, and one has a canonical isomorphism

$$M(X, \mathfrak{v}) \cong \mathrm{Hom}_{\mathcal{O}_X}(M(X, \mathfrak{v}), \Omega_X)^* \otimes_{\mathcal{O}_X} \Omega_X.$$

On the other hand, if X is one-dimensional and $\mathfrak{v}$ is transitive, then $M(X, \mathfrak{v})$ cannot be a nontrivial local system, since there are no vector fields defined in any Zariski open set preserving a nontrivial local system. More precisely, assuming X is a connected smooth curve, in order to be incompressible, $\mathfrak{v}$ must be a one-dimensional vector space. Then, if $\xi \in \mathfrak{v}$ is nonzero, then the inverse ξ^{-1} defines the volume form preserved by $\mathfrak{v}$.

We can prove a converse generically: if $\mathfrak{v}$ is incompressible (which as we already stated in Proposition 2.24, but did not yet prove, is the same as $M(X, \mathfrak{v})$ being generically nonzero), then it is generically transitive if and only if $M(X, \mathfrak{v})$ is generically holonomic (hence, if and only if it is generically a local system of rank one). We actually prove a more general result, about the dimension of the singular support of $M(X, \mathfrak{v})$ generically:

Proposition 2.39. If $(X, \mathfrak{v})$ is a variety, then $M(X, \mathfrak{v})$ is fully supported on X if and only if $\mathfrak{v}$ flows incompressibly on every irreducible component of X. In this case, the dimension of the singular support of $M(X, \mathfrak{v})$ on each irreducible component $Y \subseteq X$ is generically $\dim Y + (\dim Y - r)$, where r is the generic rank of $\mathfrak{v}$ on Y.

Before we prove the proposition, we give the converse statement to Proposition 2.36:

Corollary 2.40. If $(X, \mathfrak{v})$ is an irreducible variety, then $\mathfrak{v}$ is generically holonomic if and only if it is either generically transitive or not incompressible.

Proof of Corollary 2.40. This follows because $M(X, \mathfrak{v})$ is generically holonomic if and only if the singular support generically has dimension equal to that of X, since X is generically transitive if and only if the generic rank of $\mathfrak{v}$ is equal to the dimension of X. ☐

Proof of Proposition 2.39. It suffices to assume X is irreducible, since the statements can be checked generically on each irreducible component. For generic $x \in X$, in the formal neighborhood $\hat{X}_x$, we can integrate the flow of $\mathfrak{v}$ and write $\hat{X}_x \cong (V \times V')$, where V and V' are two formal polydiscs about zero, mapping $x \in \hat{X}_x$ to $(0,0) \in (V \times V')$, and such that $\mathfrak{v}$ generates the tangent space in the V direction everywhere, i.e., $\mathfrak{v}|_{(v,v')} = T_v V \times \{0\}$ at all $(v,v') \in (V \times V')$.

Since $\hat{\mathcal{O}}_{X,x} \cdot \mathfrak{v} = T_V \boxtimes \mathcal{O}_{V'}$, inside $\mathfrak{v} \cdot \hat{\mathcal{O}}_{X,x}$ we have, for every $\xi \in T_V$, an element of the form $\xi + D(\xi)$, for some $D(\xi) \in \hat{\mathcal{O}}_{X,x}$. Namely, this is true because, when $\xi \in \mathfrak{v}$ and $f \in \hat{\mathcal{O}}_{X,x}$, $\xi \cdot f = f \cdot \xi + \xi(f) \in \mathfrak{v} \cdot \hat{\mathcal{O}}_{X,x}$, and T_V is contained in the span of such $f \cdot \xi$.

Now assume that $\mathfrak{v}$ preserves a volume form ω on $\hat{X}_x$. Recall that this means that, for all $\xi \in \mathfrak{v}$, one has $L_\xi \omega = 0$. Since the right $\mathcal{D}$-module action of vector fields $\xi \in \mathrm{Vect}(X)$ on Ω_X is by $\omega \cdot \xi := -L_\xi \omega$, we conclude that $D(\xi) = L_\xi \omega / \omega$. Write $\omega = f \cdot \omega_V \wedge \omega_{V'}$ where ω_V and $\omega_{V'}$ are volume forms on V and V' and $f \in \hat{\mathcal{O}}_{X,x}$ is a unit. Then we conclude that $M(\hat{X}_x, \mathfrak{v}|_{\hat{X}_x}) \cong \Omega_V \boxtimes \mathcal{D}_{V'}$, the quotient of $\mathcal{D}_{X,x}$ by the right ideal generated by ω_V-preserving vector fields on V.

Conversely, assume that $M(X, \mathfrak{v})$ is fully supported. Since x was generic, $M(\hat{X}_x, \mathfrak{v}|_{\hat{X}_x})$ is also fully supported. Thus, for every $\xi \in T_V$, there is a unique $D(\xi)$ such that $\xi + D(\xi) \in \mathfrak{v} \cdot \hat{\mathcal{D}}_{X,x}$ (and in fact this is in $\mathfrak{v} \cdot \hat{\mathcal{O}}_{X,x}$).

Let $\partial_1, \ldots, \partial_n$ be the constant vector fields on $V \times V'$. We conclude that $\mathfrak{v} \cdot \hat{\mathcal{D}}_{X,x} = \{\xi + D(\xi) : \xi \in T_V \boxtimes \mathcal{O}_{V'}\} \cdot \mathsf{Sym}\langle \partial_1, \ldots, \partial_n \rangle$. Since $M(\hat{X}_x, \mathfrak{v}|_{\hat{X}_x})$ is fully supported, this implies that $\mathsf{gr}(\mathfrak{v} \cdot \hat{\mathcal{D}}_{X,x}) = T_V \cdot \mathsf{Sym}_{\hat{\mathcal{O}}_{X,x}} T_{\hat{X}_x}$, and hence that $M(\hat{X}_x, \mathfrak{v}|_{\hat{X}_x}) \cong \Omega_V \boxtimes \mathcal{D}_{V'}$. Then, $\mathfrak{v}$ also preserves a formal volume form, since $\mathrm{Hom}(M(\hat{X}_x, \mathfrak{v}|_{\hat{X}_x}), \Omega_{V \times V'}) \neq 0$. (Explicitly, for the unique (up to scaling) volume form ω_V on V preserved by $\mathfrak{v}|_V$, these are of the form $\omega_V \boxtimes \omega_{V'}$ for arbitrary volume forms $\omega_{V'}$ on V'.)

For the final statement, the proof shows that, in the incompressible (irreducible) case, the dimension of the singular support is generically $\dim V + 2 \dim V'$, which is the same as the claimed formula when we note that $\dim V = r$ and $\dim V + \dim V' = \dim Y$. $\square$

2.6. **Proof of Proposition 2.24.** By Proposition 2.39, conditions (i) and (ii) are equivalent, when X is an irreducible affine variety. Now let X be an arbitrary affine scheme of finite type. We prove that (ii') and (iii) are equivalent.

In view of Remark 2.25, the implication (ii') $\Rightarrow$ (iii) is immediate. To make Remark 2.25 precise, we should define $\mathfrak{v} \cdot \mathcal{O}_X$ as a subspace of global sections of $\mathcal{D}_X$. One way to do this is to take an embedding $i : X \to V$ into a smooth affine variety V as in §2.2; in the notation there, the global sections of $i_*(\mathfrak{v} \cdot \mathcal{D}_X)$ then identify as

$$\Gamma(V, i_*(\mathfrak{v} \cdot \mathcal{D}_X)) = I_X \mathcal{D}_V \setminus \big((\widetilde{\mathfrak{v}} + I_X) \cdot \mathcal{D}_V\big). \tag{2.41}$$

Then, by $\mathfrak{v} \cdot \mathcal{O}_X$ we mean the subspace

$$\mathfrak{v} \cdot \mathcal{O}_X = (I_X \mathcal{D}_V \cap \widetilde{\mathfrak{v}} \cdot \mathcal{O}_V) \setminus \widetilde{\mathfrak{v}} \cdot \mathcal{O}_V. \tag{2.42}$$

Finally, by $\mathcal{O}_X$ itself, we mean the subspace

$$\mathcal{O}_X = (I_X \mathcal{D}_V \cap \mathcal{O}_V) \setminus \mathcal{O}_V. \tag{2.43}$$

Then, it follows that (ii') is equivalent to $(\mathfrak{v} \cdot \mathcal{D}_X) \cap \mathcal{O}_X = 0$ and that (iii) is equivalent to $(\mathfrak{v} \cdot \mathcal{O}_X) \cap \mathcal{O}_X = 0$, as desired. In other words, it is equivalent to ask that $(\widetilde{\mathfrak{v}} \cdot \mathcal{D}_V) \cap \mathcal{O}_V \subseteq I_X$ and $(\widetilde{\mathfrak{v}} \cdot \mathcal{O}_V) \cap \mathcal{O}_V \subseteq I_X$.

We now prove that (iii) implies (ii'). Assume that $V = \mathbf{A}^n = \mathsf{Spec}\, \mathbf{k}[x_1, \ldots, x_n]$. Note that

$$(\widetilde{\mathfrak{v}} + I_X) \cdot \mathcal{D}_V = (\widetilde{\mathfrak{v}} + I_X) \cdot \mathcal{O}_V \cdot \mathsf{Sym}\langle \partial_1, \ldots, \partial_n \rangle.$$

Thus, the fact that $\big((\widetilde{\mathfrak{v}} + I_X) \cdot \mathcal{O}_V\big) \cap \mathcal{O}_V = I_X$, i.e., (iii), implies also that $\big((\widetilde{\mathfrak{v}} + I_X) \cdot \mathcal{D}_V\big) \cap \mathcal{O}_V = I_X$, i.e., (ii').

Remark 2.44. For irreducible affine varieties, we can also show that (i) and (iii) are equivalent directly without using Remark 2.25, and hence by Proposition 2.39, that (ii) and (iii) are equivalent. Suppose (i). By Proposition 2.39, $\mathfrak{v}$ flows incompressibly on Z. Let $z \in Z$ be a smooth point and $\omega \in \Omega_{\hat{Z}_z}$ be a formal volume preserved by $\mathfrak{v}$. Then, if $f_i \in \mathcal{O}_Z$ and $\xi_i \in \mathfrak{v}|_Z$ satisfy $\sum_i f_i \xi_i = 0$, we have $0 = L_{f_i \xi_i} \omega = \sum_i \xi_i(f_i)$, which proves (iii).

Conversely, suppose that (iii) is satisfied. Let $z \in Z$ be a smooth point where the rank of $\mathfrak{v}|_Z$ is maximal. Then, in a neighborhood $U \subseteq Z$ of z, $\mathcal{O}_U \cdot \mathfrak{v}$ is a free submodule of T_U, and hence has a basis $\xi_1, \ldots, \xi_j$. In the language of §3.5, one can define a divergence function $D : \mathcal{O}_U \cdot \mathfrak{v} \to T_U, D(\sum_i f_i \xi_i) = \sum_i \xi_i(f_i)$. Therefore, by Proposition 3.52, $\mathfrak{v}$ flows incompressibly on U, and hence on Z.

2.7. **Proof of Theorem 2.28.** Part (i) is an immediate consequence of Proposition 2.14 and Proposition 2.39.

For part (ii), suppose that X does not have finitely many incompressible leaves. Then, there is a degenerate invariant subvariety $i : Z \hookrightarrow X$ such that $\mathfrak{v}$ flows incompressibly on Z. By Proposition 2.14 and Proposition 2.39, there is a nonholonomic quotient of $M(X, \mathfrak{v})$ supported on the closure of Z. So $M(X, \mathfrak{v})$ is not holonomic.

2.8. **Support and saturation.** To proceed, note that in some cases, $M(X, \mathfrak{v})$ is actually supported on a proper subvariety, e.g., in Example 2.32, where it is zero; more generally, by Proposition 2.24, this happens if and only if $\mathfrak{v}$ does not flow incompressibly. In this case, it makes sense to replace X with the support of $M(X, \mathfrak{v})$, and define an equivalent system there. More precisely, we define a scheme-theoretic support of $M(X, \mathfrak{v})$:

Definition 2.45. The support of $(X, \mathfrak{v})$ is the closed subscheme $X_\mathfrak{v} \subseteq X$ defined by the ideal $(\mathfrak{v} \cdot \mathcal{D}_X) \cap \mathcal{O}_X$ of $\mathcal{O}_X$.

To make sense of this definition, we work in the space of global sections of $\mathfrak{v} \cdot \mathcal{D}_X$, using (2.42) and (2.43). Note that here it is *essential* that we allow $X_\mathfrak{v}$ to be nonreduced (this was our motivation for working in the nonreduced context).

We immediately conclude (and therefore omit the proof of):

Proposition 2.46. Let $i : X_\mathfrak{v} \to X$ be the natural closed embedding. Then, there is a canonical isomorphism $M(X, \mathfrak{v}) \cong i_* M(X_\mathfrak{v}, \mathfrak{v}|_{X_\mathfrak{v}})$.

The above remarks say that, when X is a variety, $X = X_\mathfrak{v}$ if and only if $\mathfrak{v}$ flows incompressibly. Moreover, $\mathfrak{v}$ flows incompressibly on an invariant subvariety $Z \subseteq X$ if and only if $Z = Z_{\mathfrak{v}|_Z}$. With this in mind, we extend the definition of incompressibility to subschemes:

Definition 2.47. We say that $\mathfrak{v}$ flows incompressibly on an invariant subscheme Z if $Z = Z_{\mathfrak{v}|_Z}$.

With this definition, as promised, the conditions (i), (ii'), and (iii) of Proposition 2.24 are equivalent for arbitrary affine schemes of finite type.

Proposition 2.48. Let $Z \subseteq X$ be an irreducible closed subvariety. Then there exists a quotient of $M(X, \mathfrak{v})$ whose support is Z if and only if Z is invariant and $\mathfrak{v}$ flows incompressibly on some infinitesimal thickening of Z. In this case, this quotient factors through the quotient $M(X, \mathfrak{v}) \twoheadrightarrow i_* M(Z', \mathfrak{v}|_{Z'})$, for some infinitesimal thickening Z', with inclusion $i : Z' \hookrightarrow X$.

Here, an *infinitesimal thickening* of a subvariety $Z \subseteq X$ is a subscheme $Z' \subseteq X$ such that $Z'_{\text{red}} = Z$. Note that it can happen that $\mathfrak{v}$ flows incompressibly on Z' but not on Z, as in Example 2.34. We caution that, on the other hand, $M(X, \mathfrak{v})$ could have a *submodule* supported on Z even if $\mathfrak{v}$ does not flow incompressibly on any infinitesimal thickening of Z: see Example 2.33.

Proof of Proposition 2.48. $M(X, \mathfrak{v}) = \mathfrak{v} \cdot \mathcal{D}_X \setminus \mathcal{D}_X$ admits a quotient supported on Z if and only if, for some $N \geq 1$, $(\mathfrak{v} + I_Z^N) \cdot \mathcal{D}_X$ is not the unit ideal. This is equivalent to saying that $M(Z', \mathfrak{v}|_{Z'}) \neq 0$ for some infinitesimal thickening Z' of Z. This can only happen if Z is invariant. By definition, such a restriction is fully supported if and only if $\mathfrak{v}$ flows incompressibly on Z'. For the final statement, note that the quotient morphism must factor through a map $M(X, \mathfrak{v}) \twoheadrightarrow (\mathfrak{v} + I_Z^N)\mathcal{D}_X \setminus \mathcal{D}_X$, and the latter is $M(Z', \mathfrak{v}|_{Z'})$, where we define Z' by $I_{Z'} = I_Z^N$. □

Next, even if $X = X_{\mathfrak{v}}$, there can be many choices of $\mathfrak{v}$ that give rise to the same $\mathcal{D}$-module. This motivates

Definition 2.49. The *saturation* $\mathfrak{v}^s$ of $\mathfrak{v}$ is $\mathrm{Vect}(X) \cap (\mathfrak{v} \cdot \mathcal{D}_X)$. Precisely, in the language of §2.6 for an embedding $i : X \hookrightarrow V$,

$$\mathfrak{v}^s = \Big(\mathrm{Vect}(V) \cap \big((\widetilde{\mathfrak{v}} + I_X) \cdot \mathcal{D}_V\big)\Big)|_X.$$

It is easy to check that the definition of the saturation does not depend on the choice of embedding. We next define a smaller, but more computable, saturation:

Definition 2.50. The *$\mathcal{O}$-saturation* $\mathfrak{v}^{os}$ of $\mathfrak{v}$ is $\mathrm{Vect}(X) \cap (\mathfrak{v} \cdot \mathcal{O}_X)$, precisely,

$$\mathfrak{v}^{os} := \{\sum_i f_i \xi_i \mid f_i \in \mathcal{O}_X, \xi_i \in \mathfrak{v}, \text{ s.t. } \sum_i \xi_i(f_i) = 0\}.$$

Equivalently, for any embedding $X \subseteq V$ as above,

$$\mathfrak{v}^{os} = \Big(\mathrm{Vect}(V) \cap \big((\mathfrak{v}_V + I_X) \cdot \mathcal{O}_V\big)\Big)|_X.$$

Note that, by definition, $\mathfrak{v}^{os} \subseteq \mathcal{O}_X \cdot \mathfrak{v}$; however, the same does *not* necessarily hold for $\mathfrak{v}^s$, as in Examples 2.33 and 2.34. In particular, in those examples, $\mathfrak{v}^s$ has rank two on the locus $x = 0$, whereas $\mathfrak{v}^{os}$ has rank one.

However, generically on incompressible affine varieties, $\mathfrak{v}^{os} = \mathfrak{v}^s$. More precisely:

Definition 2.51. If $(X, \mathfrak{v})$ is incompressible, then call a vector field $\xi \in \mathcal{O}_X \cdot \mathfrak{v}$ *incompressible* if, writing $\xi = \sum_i f_i \xi_i$ for $f_i \in \mathcal{O}_X$ and $\xi_i \in \mathfrak{v}$, one has $\sum_i \xi_i(f_i) = 0$.

The meaning of this definition is explained in the following remark:

Remark 2.52. When X is a variety, $\xi \in \mathcal{O}_X \cdot \mathfrak{v}$ is incompressible if and only if, for every irreducible component of X, at a smooth point with a formal volume preserved by $\mathfrak{v}$, then ξ also preserves that volume. Indeed, $L_\xi = \sum_i f_i L_{\xi_i} + \sum_i \xi_i(f_i)$, so if $L_{\xi_i} = 0$ for all i, then the same is true for L_ξ if and only if $\sum_i \xi_i(f_i) = 0$.

Note that we used incompressibility for the definition to make sense; otherwise there could be multiple expressions $\sum_i f_i \xi_i$ for ξ which yield different values $\sum_i \xi_i(f_i)$.

Proposition 2.53. If $\mathfrak{v}$ flows incompressibly, then $\mathfrak{v}^{os}$ is the subspace of $\mathcal{O}_X \cdot \mathfrak{v}$ of incompressible vector fields. If X is additionally a variety, then for some open dense subset $U \subseteq X$, $(\mathfrak{v}|_U)^s = (\mathfrak{v}|_U)^{os}$ is the subspace of $\mathcal{O}_U \cdot \mathfrak{v}$ of incompressible vector fields.

Proof. For the first statement, if X is incompressible and $f_i \in \mathcal{O}_X, \xi_i \in \mathfrak{v}$ are such that $\sum_i \xi_i(f_i) = 0$, then it follows that $\sum_i f_i \cdot \xi_i = \sum_i \xi_i \cdot f_i$.

For the second statement, first note that, by Proposition 2.39, since $\mathfrak{v}$ is incompressible and X is a variety, on each irreducible component, $\mathfrak{v}^s$ must generically have the same rank as $\mathfrak{v}$. Now let $U \subseteq X$ be the locus of smooth points $x \in X$ such that, if $Y \subseteq X$ is the irreducible component containing x, the dimension $\mathfrak{v}|_x$ is maximal along Y. Then $\mathcal{O}_U \cdot \mathfrak{v}|_U$ is locally free. It follows that this also equals $\mathcal{O}_U \cdot (\mathfrak{v}|_U)^s$. Since $M(U, (\mathfrak{v}|_U)^s) = M(U, \mathfrak{v}|_U)$ is fully supported, $(\mathfrak{v}|_U)^s$ is incompressible. By the first part, we therefore have $(\mathfrak{v}|_U)^s \subseteq (\mathfrak{v}|_U)^{os}$; the opposite inclusion is true by definition. Finally, note that, by definition, U is open and dense. □

Example 2.54. When $X = X_{\mathfrak{v}}$ is reduced and irreducible, in the formal neighborhood of a generic point of $x \in X$, one has $\hat{X}_x \cong (V \times V')$ for formal polydiscs V and V', and $\mathfrak{v}^s = \mathfrak{v}^{os} = \mathcal{O}_{V'} \cdot H(V)$ where V is equipped with its standard volume form (this also gives an alternative proof of part of Proposition 2.53). So, up to isomorphism, this only depends on the dimension of X and the generic rank of $\mathfrak{v}$.

Remark 2.55. There is a close relationship between the saturation and the support ideal. In the language of Remark 2.17, if we generalize $\mathfrak{v}$ to the setting of differential operators of order ≤ 1, then the natural saturation becomes $(\mathfrak{v} \cdot \mathcal{D}_X) \cap \mathcal{D}_X^{\leq 1}$. In the case $\mathfrak{v} \subseteq \mathrm{Vect}(X)$, this saturation contains both $\mathfrak{v}^s$ and the ideal of $X_{\mathfrak{v}}$; by a computation similar to that of §2.6, in fact, this saturation is $\mathfrak{v}^s \cdot \mathcal{O}_X$.

Remark 2.56. By Remark 2.55, one obtains an alternative formula for the support ideal, call it $I_{X_{\mathfrak{v}}}$, of X: this is $I_{X_{\mathfrak{v}}} = (\mathfrak{v}^s \cdot \mathcal{O}_X) \cap \mathcal{O}_X$. This can be viewed as a generalization of the equivalence of Proposition 2.24, (ii') $\Leftrightarrow$ (iii), in the case that $\mathfrak{v} = \mathfrak{v}^s$ is saturated.

2.9. **Holonomicity criteria.**

Theorem 2.57. The following conditions are equivalent:
- (i) $(X, \mathfrak{v})$ is holonomic;
- (ii) For every (degenerate closed) invariant subscheme $Z' \subseteq X$ on which $\mathfrak{v}$ flows incompressibly, for $i : Z := Z'_{\mathrm{red}} \to Z'$ the inclusion, $i^! M(Z', \mathfrak{v}|_{Z'})$ is generically a local system;
- (iii) X has only finitely many invariant closed subvarieties Z on which $\mathfrak{v}$ flows incompressibly in some infinitesimal thickening $i : Z \hookrightarrow Z' \subseteq X$, and for all of them, in formal neighborhoods of generic $z \in Z$ there is a canonical isomorphism
$$i^! M(Z', \mathfrak{v}_{Z'}) \cong \Omega_{\hat{Z}_z} \otimes ((i_* \Omega_{\hat{Z}_z})^{\mathfrak{v}|_{Z'}})^*.$$

In this case, $M(X, \mathfrak{v})$ admits a filtration
$$0 \subseteq M_{\geq \dim X}(X, \mathfrak{v}) \subseteq M_{\geq \dim X - 1}(X, \mathfrak{v}) \subseteq \cdots \subseteq M_{\geq 0}(X, \mathfrak{v}) = M(X, \mathfrak{v}),$$
whose subquotients $M_{\geq j}(X, \mathfrak{v}) / M_{\geq (j+1)}(X, \mathfrak{v})$ are direct sums of indecomposable extensions of local systems on open subvarieties of the dimension j varieties appearing in (iii) by local systems on subvarieties of their boundaries.

Here $(i_* \Omega_{\hat{Z}_z})^{\mathfrak{v}|_{Z'}}$ is the (finite-dimensional) vector space of distributions along Z preserved by the flow of $\mathfrak{v}|_{Z'}$. For example, in the case that there exists a product decomposition $\widehat{Z'_z} \cong \hat{Z}_z \times S$ for some zero-dimensional scheme S, for which the inclusion of $\hat{Z}_z$ is the obvious one to $\hat{Z}_z \times \{0\}$, then $i_* \Omega_{\hat{Z}_z} \cong (\Omega_{\hat{Z}_z} \otimes_{\mathbf{k}} \mathcal{O}_S^*)$, where $\Omega_{\hat{Z}_z}$ is the space of formal volume forms on $\hat{Z}_z$ and $\mathcal{O}_S^*$ is the (finite-dimensional) space of algebraic distributions on S.

Proof of Theorem 2.57. Since holonomic $\mathcal{D}$-modules are always of finite length and their composition factors are intermediate extensions of local systems, and since in our case it is clear that any local systems must be on invariant subvarieties, it is immediate that (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii); we only need to explain the formula in (iii). First, note that, by Kashiwara's equivalence (i.e., via the restriction functor of $\mathcal{D}$-modules from Z' to Z), the categories of $\mathcal{D}$-modules on Z' and on Z are canonically equivalent. Then, the multiplicity space $((i_* \Omega_{\hat{Z}_z})^{\mathfrak{v}|_{Z'}})^*$ is explained by the canonical isomorphism
$$(i_* \Omega_{\hat{Z}_z})^{\mathfrak{v}|_{Z'}} \cong \mathrm{Hom}_{\widehat{\mathcal{D}}_{X,z}}(M(\widehat{Z'_z}, \mathfrak{v}|_{\widehat{Z'_z}}), i_* \Omega_{\hat{Z}_z}),$$
looking at the image of the canonical generator of $M(\widehat{Z'_z}, \mathfrak{v}|_{\widehat{Z'_z}})$, and viewing $\mathcal{D}$-modules on Z' as $\mathcal{D}$-modules on the ambient space X.

So, we prove that (ii) implies (i). Suppose (ii) holds. We prove holonomicity by induction on the dimension of X. There is an open dense subset $Y \subseteq X$ such that $M(X, \mathfrak{v})|_Y$ is a local system (viewed as a $\mathcal{D}$-module on Y_{red}). Take Y to be maximal for this property, i.e., the set-theoretic locus where $M(X, \mathfrak{v})$ is a local system in some neighborhood.

Let $j : Y \hookrightarrow X$ be the open embedding. Then by adjunction, since $j^*M(X, \mathfrak{v}) = M(X, \mathfrak{v})|_Y$ is holonomic, we obtain a canonical map $H^0 j_! M(X, \mathfrak{v})|_Y \to M(X, \mathfrak{v})$. The cokernel is supported on the closed invariant subvariety $Z := X \setminus Y$, which has strictly smaller dimension than that of X. By Proposition 2.48, the quotient factors through $M(Z', \mathfrak{v}|_{Z'})$ for some infinitesimal thickening Z' of Z. Then, by induction, $M(Z', \mathfrak{v}|_{Z'})$ is holonomic. This implies the result.

The final statement follows from the inductive construction of the previous paragraph, if we note that the image of $H^0 j_! M(X, \mathfrak{v})|_Y$ is an extension of the local system $M(X, \mathfrak{v})|_Y$ by local systems on boundary subvarieties, none of which split off the extension. □

Note that, by Example 2.33, in general the extensions appearing (iii) can contain composition factors supported on invariant subvarieties which do not themselves appear in (iii).

We remark that the theorem also gives another proof of Proposition 2.36 (which we don't use in the proof of the theorem), since a connected transitive variety $(X, \mathfrak{v})$ is a single leaf and therefore $\mathfrak{v}$ is holonomic.

Using part (iii) of the Theorem, we immediately conclude

Corollary 2.58. When $(X, \mathfrak{v})$ is holonomic, an invariant subscheme $Z' \subseteq X$ is incompressible if and only if, for generic $z \in Z := Z'_{\text{red}}$, with $i : Z \hookrightarrow Z'$ the inclusion, $(i_* \Omega_{\hat{Z}_z})^{\mathfrak{v}|_{Z'}} \neq 0$.

Note that, when Z' is a variety, the corollary is tantamount to the definition of incompressibility, and does not require holonomicity.

In particular, we can weaken the holonomicity criterion of Theorem 2.19, adding in the word "incompressible":

(2.59) no incompressible degenerate invariant closed subschemes $\Rightarrow$ holonomic.

For a counterexample to the converse implication, recall Example 2.34.

2.10. **Global generalization.** Although most of the phenomena discussed in this paper are already fully visible in the affine case, it is useful to generalize them to the non-affine setting. One reason for this is that we are interested in the leaves under the flow of $\mathfrak{v}$, and even when X is affine, the maximal leaves of $\mathfrak{v}$ in general need not be themselves affine (they are only locally closed, and hence quasi-affine). (In this case, by Example 2.63 below, this poses no problem if we consider as the Lie algebra of vector fields on each such locally closed $Y \subseteq X$ the restriction $\mathfrak{v}|_Y$). It is also interesting, however, to consider examples where X need not be affine, since for example, if X is a not-necessarily affine symplectic or Calabi-Yau variety and $\mathfrak{v}$ is the associated Lie algebra of Hamiltonian or volume-preserving vector fields, then $\pi_* M(X, \mathfrak{v}) \cong H^{\dim X - *}(X)$, by Example 2.37 (in fact, $M(X, \mathfrak{v}) \cong \Omega_X$), which shows that the $M(X, \mathfrak{v})$ has geometric meaning.

Suppose that X is not necessarily affine. Since X does not in general admit (enough) global vector fields, we need to generalize $\mathfrak{v}$ to a *presheaf* of vector fields, i.e., a sub-presheaf of $\mathbf{k}$-vector spaces of the tangent sheaf. As we will see, even for affine X, this is more natural and more flexible: for example, even in the case of Hamiltonian vector fields, we will see that Zariski-locally Hamiltonian vector fields need not coincide with Hamiltonian vector fields, so that the natural presheaf $\mathfrak{v}$ is not even a sheaf, let alone constant; see Remark 4.5 below.

Nonetheless, all of the main examples and results of this paper are already interesting for affine varieties and do not require this material, so the reader interested only in the affine case can feel free to skip this subsection.

Let X be a not necessarily affine variety and $\mathfrak{v}$ a presheaf of Lie algebras of vector fields on X. For any open affine subset $U \subseteq X$, we can define the $\mathcal{D}$-module on U, $M(U, \mathfrak{v}(U))$, as above. Recall that this is defined as a certain quotient of $\mathcal{D}_U$. Therefore, to show that the $M(U, \mathfrak{v}(U))$ glue together to a $\mathcal{D}$-module on X, it suffices to check that the restriction to $U \cap U'$ of the submodules of $\mathcal{D}_U$ and $\mathcal{D}_{U'}$ whose quotients are $M(U, \mathfrak{v}(U))$ and $M(U', \mathfrak{v}(U'))$, respectively, are the same. This does *not* hold in general, but it does hold if one has the following condition:

Definition 2.60. Say that $(X, \mathfrak{v})$ is (Zariski) *$\mathcal{D}$-localizable* if, for every open affine subset $U \subseteq X$ and every open affine $U' \subseteq U$,

$$\mathfrak{v}(U')\mathcal{D}_{U'} = \mathfrak{v}(U)|_{U'}\mathcal{D}_{U'}. \tag{2.61}$$

Remark 2.62. If X is already affine, the definition is still meaningful (and this is the case we will primarily be interested in here). In this case we can restrict to $U = X$ in (2.61).

Example 2.63. If X is irreducible and $\mathfrak{v}$ is a constant sheaf, then it is immediate that $\mathfrak{v}$ is $\mathcal{D}$-localizable. More generally, for reducible X and $\mathfrak{v} \subseteq \mathrm{Vect}(X)$, we can consider the associated presheaf $\mathfrak{v}(U) := \mathfrak{v}|_U$, and this is $\mathcal{D}$-localizable. If the irreducible components of X are X_j, then the sheafification of this $\mathfrak{v}$ is $\mathfrak{v}(U) = \bigoplus_{j | X_j \cap U \neq \emptyset} \mathfrak{v}(X_j \cap U)$.

We will use below the following basic

Lemma 2.64. Let X be an affine scheme of finite type and $\mathfrak{v} \subseteq \mathrm{Vect}(X)$ an arbitrary subset of vector fields. Then for every affine open $U \subseteq X$, one has the equality of sheaves on U,

$$(\mathfrak{v} \cdot \mathcal{D}_X)|_U = \mathfrak{v}|_U \cdot \mathcal{D}_U.$$

In particular, as a sheaf, the sections of $\mathfrak{v} \cdot \mathcal{D}_X$ on U coincide with the global sections of $\mathfrak{v}|_U \cdot \mathcal{D}_U$.

Similarly, for every $x \in X$, we have $(\mathfrak{v} \cdot \mathcal{D}_X)|_{\hat{X}_x} = \mathfrak{v}|_{\hat{X}_x} \cdot \hat{\mathcal{D}}_{X,x}$.

Proof. We use (2.41). In these terms, for $X \hookrightarrow V$ an embedding into a smooth affine variety V, let $U' \subseteq V$ be an affine open subset such that $U' \cap X = U$. Then $(\mathfrak{v} \cdot \mathcal{D}_X)|_U$ identifies with the $\mathcal{D}$-module restriction of (2.41) to U', which is then $\mathfrak{v}|_U \cdot \mathcal{D}_U$. We conclude the first assertion. The final assertion is similar. ☐

Given a presheaf $\mathcal{C}$, let $\mathrm{Sh}(\mathcal{C})$ be its sheafification.

Proposition 2.65. Suppose that $(X, \mathfrak{v})$ is $\mathcal{D}$-localizable. Then the following hold:

(i) The $M(U, \mathfrak{v}(U))$ glue together to a $\mathcal{D}$-module $M(X, \mathfrak{v})$ on X.
(ii) For every open affine U and every open affine $U' \subseteq U$, $M(X, \mathfrak{v})|_{U'} = M(U', \mathfrak{v}(U'))$.
(iii) $(X, \mathrm{Sh}(\mathfrak{v}))$ is also $\mathcal{D}$-localizable, and $M(X, \mathrm{Sh}(\mathfrak{v})) = M(X, \mathfrak{v})$.

Proof of Proposition 2.65. For (i), note that (2.61) applied to $U' := U \cap V$ implies that $M(U, \mathfrak{v}(U))$ and $M(V, \mathfrak{v}(V))$ glue. Then, (ii) is an immediate consequence of (2.61).

It remains to prove (iii). Suppose that U is an affine open, $U' \subseteq U$ is affine open, and $\xi \in \mathrm{Sh}(\mathfrak{v})(U')$. Let $u \in U'$. By definition, there exists a neighborhood $U'' \subseteq U'$ of u such that $\xi|_{U''} \in \mathfrak{v}(U'')$. By (2.61), $\xi|_{U''} \in \mathfrak{v}(U)|_{U''} \cdot \mathcal{D}_{U''}$. Thus, by Lemma 2.64, ξ is a section of the $\mathcal{D}$-module $\mathfrak{v}(U)|_{U'} \cdot \mathcal{D}_{U'} = (\mathfrak{v}(U) \cdot \mathcal{D}_U)|_{U'}$ on U'. This proves the first statement. This also proves the second statement, since $U' \subseteq U$ and $\xi \in \mathrm{Sh}(\mathfrak{v})(U')$ were arbitrary. ☐

Remark 2.66. As in Remark 2.15, we could have allowed $\mathfrak{v}$ to be an arbitrary presheaf of vector fields (rather than a sheaf of Lie algebras of vector fields). However, it is easy to see that it is then $\mathcal{D}$-localizable if and only if the presheaf of Lie algebras generated by it is, and that the resulting $\mathcal{D}$-module is the same. So, no generality is lost by requiring that $\mathfrak{v}$ be a presheaf of Lie algebras.

Using the above, in the $\mathcal{D}$-localizable setting, the results of this section extend to nonaffine schemes of finite type. We omit further details (but we will discuss $\mathcal{D}$-localizability more in §4 below).

3. Generalizations of Cartan's simple Lie algebras

In this section we state and prove general results on Lie algebras of vector fields on affine varieties which generalize the simple Lie algebras of vector fields on affine space as classified by Cartan. Namely, we will consider the Lie algebras of all vector fields; of Hamiltonian vector fields on Poisson varieties; of Hamiltonian vector fields on Jacobi varieties (this generalizes both the previous example and the setting of contact vector fields on contact varieties); and of Hamiltonian vector fields on varieties equipped with a top polyvector field, or more generally equipped with a divergence function. The last example, which seems to not have been studied before, generalizes the volume-preserving or divergence-free vector fields on $\mathbf{A}^n$ or on Calabi-Yau varieties. We also consider invariants of these Lie algebras under the actions of finite groups (we will continue this study in §§6 and 7).

Namely, in this section we compute the leaves under the flow of these vector fields and determine when they are holonomic, and hence their coinvariants are finite-dimensional.

We will state all examples in the affine setting; in §4 below we will explain how to generalize them to the nonaffine setting (which will at least work for the cases of all vector fields and Hamiltonian vector fields).

3.1. The case of all vector fields.

Consider the case where $\mathfrak{v}$ is the Lie algebra of all vector fields. In this case we have a basic result:

Proposition 3.1. The support, $Z = X_{\mathrm{Vect}(X)}$, of $\mathrm{Vect}(X)$ is the locus where all vector fields vanish, i.e., the scheme of the ideal $(\mathrm{Vect}(X)(\mathcal{O}_X))$. Moreover,

$$M(X, \mathrm{Vect}(X)) = \mathcal{D}_Z := \mathrm{Vect}(X)(\mathcal{O}_X) \cdot \mathcal{D}_X \setminus \mathcal{D}_X,$$

and $(\mathcal{O}_X)_{\mathrm{Vect}(X)} = \mathcal{O}_Z$.

The support is evidently incompressible, and is the union of zero-dimensional leaves at every point. Therefore, $\mathrm{Vect}(X)$ is holonomic if and only if this vanishing locus is finite.

Proof. Given $\xi \in \mathrm{Vect}(X)$, the submodule $\mathfrak{v} \cdot \mathcal{D}_X$ contains $[\xi, f] = \xi(f)$ for all $f \in \mathcal{O}_X$. These generate the ideal $(\mathrm{Vect}(X)(\mathcal{O}_X))$ over $\mathcal{O}_X$, which defines the vanishing scheme of $\mathrm{Vect}(X)$. Conversely, notice that the principal symbol of any product of vector fields lies in the submodule $(\mathrm{Vect}(X)(\mathcal{O}_X)) \cdot \mathcal{D}_X$. Thus, $\mathfrak{v} \cdot \mathcal{D}_X = (\mathrm{Vect}(X)(\mathcal{O}_X)) \cdot \mathcal{D}_X$. The last statement follows immediately. □

This motivates the

Definition 3.2. A point $x \in X$ is *exceptional* if all vector fields on X vanish at x.

Clearly, all exceptional points are singular, but not conversely: for example, if $X = Y \times Z$ where Z is smooth and of purely positive dimension, then X will have no exceptional points, regardless of how singular Y is.

Proposition 3.3. The following are equivalent:

(i) The quotient $(\mathcal{O}_X)_{\mathrm{Vect}(X)}$ is finite-dimensional;
(ii) X has finitely many exceptional points;
(iii) $\mathrm{Vect}(X)$ (i.e., $M(X, \mathrm{Vect}(X))$) is holonomic.

Proof. First, (ii) and (iii) are equivalent by Proposition 3.1, since $\mathcal{D}_Z$ is holonomic if and only if Z has dimension zero, i.e., set-theoretically Z is finite. By the proposition, with Z the support of $\mathrm{Vect}(X)$, then Z_{red} is the locus of exceptional points of X and $M(X, \mathrm{Vect}(X)) = \mathcal{D}_Z$, so the equivalence follows. Similarly, these are equivalent to (i), since $(\mathcal{O}_X)_{\mathrm{Vect}(X)} = \mathcal{O}_Z$. □

Remark 3.4. Note that the implication (i) $\Rightarrow$ (iii) above, a converse to Proposition 2.21, is special to the case $\mathfrak{v} = \mathrm{Vect}(X)$. See, e.g., Remarks 2.22 and 3.20.

Corollary 3.5. If X has a finite exceptional locus $Z \subseteq X$ (i.e., $\mathfrak{v}$ is holonomic), then

$$M(X, \mathrm{Vect}(X)) \cong \bigoplus_{z \in Z} \delta_z \otimes (\hat{\mathcal{O}}_{X,z})_{\mathrm{Vect}(\hat{\mathcal{O}}_{X,z})}.$$

Proof. This follows immediately by formally localizing at each exceptional point. $\square$

Corollary 3.6. Under the same assumptions as in the previous corollary, if $\pi : X \to \mathrm{pt}$ is the projection to a point,

$$\pi_* M(X, \mathrm{Vect}(X)) = \pi_0 M(X, \mathrm{Vect}(X)) \cong \bigoplus_{z \in Z} (\hat{\mathcal{O}}_{Z,z})_{\mathrm{Vect}(\hat{\mathcal{O}}_{Z,z})}.$$

Proof. This follows since $\pi_* \delta_x = \pi_0 \delta_x = \mathbf{k}$ for any point $x \in X$. $\square$

Example 3.7. Suppose that X has finitely many exceptional points. Then, the dual space $((\mathcal{O}_X)_{\mathrm{Vect}(X)})^* = (\mathcal{O}_X^*)^{\mathrm{Vect}(X)}$, of functionals invariant under all vector fields, includes the evaluation functionals at every exceptional point. These are linearly independent. However, they need not span all invariant functionals. In other words, the multiplicity spaces $(\hat{\mathcal{O}}_{X,x})_{\mathrm{Vect}(\hat{\mathcal{O}}_{X,x})}$ in the corollaries need not be one-dimensional.

For example, if one takes a curve $X \subset \mathbf{A}^2$ of the form $P(x,y) + Q(x,y) = 0$ in the plane with $P(x,y)$ and $Q(x,y)$ homogeneous of degrees n and $n+1$, then we claim that, if $n \geq 5$ and P and Q are generic, all vector fields on X vanish to degree at least two at the singularity at the origin. Therefore, the coinvariants $(\mathcal{O}_X)_{\mathrm{Vect}(X)}$ have dimension at least three, even though 0 is the only singularity of X.

Indeed, up to scaling, any vector field which sends P to a constant multiple of P up to higher degree terms is of the form $a\,\mathrm{Eu} + v$, where $a \in \mathbf{k}$ and v vanishes up to degree at least two at the origin. Suppose that such a vector field preserves the ideal $(P+Q)$, i.e., that it sends $P+Q$ to a multiple of $P+Q$. We claim that $a = 0$. Otherwise, we can assume up to scaling that $a = 1$. Then $(\mathrm{Eu} + v)(P+Q) = f(P+Q)$ for some polynomial f. By comparing the parts of degree n, we conclude that $f(0) = n$. Writing $f = n + bx + cy + g$, where g vanishes to degree at least two at the origin, we conclude that $Q = (-v + (bx+cy)\,\mathrm{Eu})P$. So there exists a quadratic vector field $w = -v + (bx+cy)\,\mathrm{Eu}$ which takes P to Q. The space of all quadratic vector fields is six-dimensional, whereas the space of all possible Q is of dimension $n+2$. So for $n \geq 5$, we obtain a contradiction, since P and Q are assumed to be generic.

Here is an explicit example for the smallest case, $n = 5$, of such a P and Q: Let $P = x^5 + y^5$ and $Q = x^3 y^3$. Then it is clear that the equation $Q = -v(P) + (bx+cy)P$ cannot be satisfied for any quadratic vector field v and any $b, c \in \mathbf{k}$.

Example 3.8. One example of a variety with infinitely many exceptional points, and hence infinite-dimensional $(\mathcal{O}_X)_{\mathrm{Vect}(X)}$ and non-holonomic $\mathrm{Vect}(X)$, is a nontrivial family of affine cones of elliptic curves: one can take $X = \mathsf{Spec}\,\mathbf{k}[x,y,z,t]/(x^3+y^3+z^3+txyz)$, which is a family over $\mathbf{A}^1 = \mathsf{Spec}\,\mathbf{k}[t]$ whose fibers are affine cones of elliptic curves in $\mathbf{P}^2$. Then, we claim that all singular points $x = y = z = 0$ are exceptional. This is true because, otherwise, there would be a vector field nonvanishing somewhere along the line $x = y = z = 0$, and then the family would, formally or analytically locally along this line, have to be isomorphic to a product of the line and some other analytic variety or formal scheme; this is impossible in this case since the affine cones at different values of t are nonisomorphic.

A direct algebraic proof is as follows: Take any vector field on X and lift it to a vector field ξ on $\mathbf{A}^4$ parallel to X. Then $\xi(x^3+y^3+z^3+txyz) = f(x^3+y^3+z^3+txyz)$ for some $f \in \mathcal{O}_{\mathbf{A}^4}$.

Replacing ξ by $\xi - (1/3)f \cdot (x\partial_x + y\partial_y + z\partial_z)$, we can assume that $f = 0$. Restricting to $t = t_0$, we obtain

$$\xi|_{t=t_0}(x^3 + y^3 + z^3 + t_0 xyz) = -\xi(t)|_{t=t_0} \cdot xyz. \tag{3.9}$$

Suppose that ξ did not vanish at $(0,0,0,t_0)$. We can assume ξ is homogeneous with respect to the grading $|x| = |y| = |z| = 1$ and $|t| = 0$. Then $\xi|_{t=t_0}$ is either constant or linear. By (3.9), $\xi|_{t=t_0}$ annihilates $x^3 + y^3 + z^3$, but no constant or linear vector field can do that, which is a contradiction.

Example 3.10. As pointed out by the referee, a simpler (although reducible) example where there are infinitely many singular points is a family of four lines in the plane, all passing through the origin, one of which rotates. Namely, take $X = \mathsf{Spec}\,\mathbf{k}[x,y,z]/(xy(x+y)(x+zy))$. Then, there cannot be a vector field nonvanishing along the singular line $x = y = 0$, since for different values of z (not including 0 and 1), the variety of four lines is nonisomorphic. Thus the singular line consists of exceptional points.

To see this algebraically, first note that any vector field ξ preserving the ideal $xy(x+y)(x+zy)$ must preserve the ideal generated by each linear factor (the vector field must be parallel to each of the four planes). Write $\xi = f\partial_x + g\partial_y + h\partial_z$. Then the conditions for ξ to preserve $(x), (y), (x+y)$, and $(x+zy)$ are equivalent to

$$x \mid f, \quad y \mid g, \quad (x+y) \mid (f+g), \quad \text{and} \quad (x+zy) \mid (f+gz+hy).$$

Let $f_0, g_0, h_0 \in \mathbf{k}[z]$ be such that

$$f \equiv xf_0 \pmod{(x,y)^2}, \quad g \equiv yg_0 \pmod{(x,y)^2}, \quad h \equiv h_0 \pmod{(x,y)}.$$

Then the conditions $(x+y) \mid (f+g)$ and $(x+zy) \mid (f+gz+hy)$, modulo $(x+y)^2$, become

$$f_0 + g_0 = 0, \quad f_0 + g_0 + h_0/z = 0.$$

This implies that $h_0 = 0$, and hence that ξ vanishes along the line $x = y = 0$, as claimed.

3.2. **The Poisson case.** Suppose that X is an affine Poisson scheme of finite type, i.e., $\mathcal{O}_X$ is a Poisson algebra. Let π be the Poisson bivector field on X. Then, we can let $\mathfrak{v}$ be the Lie algebra of Hamiltonian vector fields, $H(X) = H_\pi(X)$. In particular, these vector fields are $\xi_f := \pi(df)$ for $f \in \mathcal{O}_X$. In this case, $(\mathcal{O}_X)_\mathfrak{v} = \mathsf{HP}_0(\mathcal{O}_X)$, the zeroth Poisson homology of $\mathcal{O}_X$. As pointed out in Example 2.30, $H(X)$ is holonomic if and only if X has finitely many symplectic leaves.

There are several natural larger Lie algebras to consider than $H(X)$. Note that $H(X)$ is the space of vector fields obtained by contracting π with exact one-forms. So, one can consider instead $LH(X) = LH_\pi(X) = \pi(\tilde{\Omega}^1_X)$, the space of vector fields obtained by contracting π with *closed* one-forms modulo torsion (note that contracting π with torsion yields zero, since $\mathcal{O}_X$ is torsion-free). Here we will denote the resulting vector field by $\eta_\alpha := \pi(\alpha)$. Thus, when X is generically symplectic, $LH(U)/H(U) \cong H^1_{DR}(U)$ for all open affine $U \subseteq X$. (Recall from the beginning of §2 that, over $\mathbf{k} = \mathbf{C}$, if U is smooth, this coincides with the first topological cohomology of U).

Here, LH stands for "locally Hamiltonian;" in a smooth affine open subset, in the case that $\mathbf{k} = \mathbf{C}$, these are the vector fields which are locally Hamiltonian in the analytic topology. In general, in a smooth open subset, these are the vector fields which, restricted to a formal neighborhood of a point, are Hamiltonian. However, as explained in the next example, in formal neighborhoods of singular points not all locally Hamiltonian vector fields are Hamiltonian:

Example 3.11. In the formal neighborhood of singular points, locally Hamiltonian vector fields need not be Hamiltonian, since the first de Rham cohomology modulo torsion need not vanish in such a neighborhood, and as mentioned above, when X is generically symplectic, then $LH(X)/H(X) \cong H^1(\tilde{\Omega}^\bullet_X)$.

Here is an example where this cohomology does not vanish. Suppose $Z \subseteq \mathbf{A}^n$ is a complete intersection with an isolated singularity at $z \in Z$. By (5.5) below, in this case $\tilde{\Omega}^\bullet_{Z,z}$ is acyclic except in degree $k = \dim Z$, where $\dim H^k(\tilde{\Omega}^\bullet_{Z,z}) = \mu_z - \tau_z$, where μ_z and τ_z are the Milnor and Tjurina numbers of z (see §5 below; we will not use the general definition here). In the case when $Z \subseteq \mathbf{A}^2$ is a reduced curve cut out by $Q \in \mathbf{k}[x,y]$ with an isolated singularity at the origin, then all one-forms modulo torsion are closed, but they are not all exact in general. Explicitly, $H^1(\tilde{\Omega}^\bullet_{Z,0}) \cong (Q, \partial_x Q, \partial_y Q)_0/(\partial_x Q, \partial_y Q)_0$, where $(-)_0 \subseteq \hat{\mathcal{O}}_{\mathbf{A}^2,0}$ is the ideal in the completed local ring at the origin.

Specifically, take $Q = x^3 + x^2 y + y^4$, where

$$(Q, \partial_x Q, \partial_y Q) = (3x^2 + 2xy, x^2 + 4y^3, x^3 + x^2 y + y^4) = (3x^2 + 2xy, x^2 + 4y^3, y^4) \\ \neq (3x^2 + 2xy, x^2 + 4y^3) = (\partial_x Q, \partial_y Q).$$

One therefore obtains a nonexact (closed) one-form. Such a form is $\alpha := x \cdot dy$: one can compute that

$$\alpha \wedge dQ = (-3x^3 - 2x^2 y) \cdot dx \wedge dy \equiv 2y^4 \cdot dx \wedge dy \pmod{d\mathbf{k}[\![x,y]\!] \wedge dQ + (Q) dx \wedge dy + (x,y)^5 dx \wedge dy},$$

and this is not equivalent to zero modulo $d\mathbf{k}[\![x,y]\!] \wedge dQ + (Q)dx \wedge dy + (x,y)^5 dx \wedge dy$.

Then, consider the Poisson variety $X = Z \times \mathbf{A}^1$ with the Poisson structure $(\partial_x \wedge \partial_y)(dQ) \wedge \partial_t$, with t the coordinate on $\mathbf{A}^1$. This is generically symplectic, so provides an example where $LH(X) \neq H(X)$. Specifically, the vector field $\eta_\alpha = (-3x^3 - 2x^2 y)\partial_t$ is locally Hamiltonian on X, but in the formal neighborhood of the origin it is not Hamiltonian. By the above computation, this spans $LH(\hat{X}_0)/H(\hat{X}_0)$.

Note that the fact that $LH(X)$ and $H(X)$ are Lie algebras follow from the fact that $[LH(X), LH(X)] \subseteq H(X)$, since $\{\eta_\alpha, \eta_\beta\} = \xi_{i_{\eta_\alpha}\beta}$ for closed one-forms α and β.

Next, one can consider $P(X) = P_\pi(X)$, the space of all Poisson vector fields, i.e., those ξ such that $L_\xi(\pi) = 0$. Clearly, we have $H(X) \subseteq LH(X) \subseteq P(X)$. If X is symplectic (which for us in particular means X is smooth), then it is well-known that $LH(X) = P(X)$, but this may not be true in general (even if X has finitely many symplectic leaves: see Example 3.19). However, there is a certain generalization of this equality to the mildly singular case, as explained in the next remark.

Remark 3.12. In the case that X is normal and generically symplectic, then the following conditions are equivalent:

(i) X is symplectic on its smooth locus;
(ii) On each irreducible component, X is symplectic outside of a codimension two subset.

This is because the degeneracy locus of a Poisson structure is given by a single equation $\pi^{\wedge \lceil \dim Y/2 \rceil} = 0$, so on the smooth locus this consists of divisors (if it is generically nondegenerate).

If we assume that either of these conditions is satisfied, then letting $X^\circ \subseteq X$ be the smooth locus (which is not affine unless $X = X^\circ$) we claim that $P(X) = P(X^\circ) = LH(X^\circ)$, where here by $P(X^\circ)$ we mean global Poisson vector fields on the nonaffine X°, and by $LH(X^\circ)$ we mean the collection of vector fields η_α for $\alpha \in \Gamma(X^\circ, \Omega_{X^\circ})$ a closed one-form regular on X°.

Indeed, in this case, all vector fields which are regular on X° extend to all of X. Thus $P(X) = P(X^\circ)$. Moreover, if $\xi \in P(X)$ is a global Poisson vector field, then dividing by the Poisson bivector, we obtain a closed one-form regular on X°, and conversely.

The leaves of X under both $H(X)$ and $LH(X)$ are the symplectic leaves. For $H(X)$, this is the definition of symplectic leaves; for $LH(X)$, this is true because, since all one-forms (and in particular all closed one forms) are spanned over $\mathcal{O}_X$ by exact one-forms, the evaluations at each point of the contraction of π with either span the same subspace of the tangent space. That is, $H(X)|_x = LH(X)|_x$ for all $x \in X$, as subspaces of $T_x X$. In fact, $H(X)$ and $LH(X)$ define the same $\mathcal{D}$-module, since they have the same $\mathcal{O}$-saturation, as defined in §2.8:

Proposition 3.13. The $\mathcal{O}$-saturations are equal: $H(X)^{os} = LH(X)^{os}$. Hence, $M(X, H(X)) \cong M(X, LH(X))$.

Proof. Given any closed one-form $\alpha := \sum_i f_i dg_i \in T_X^*$, for $f_i, g_i \in \mathcal{O}_X$, we claim that $\eta_\alpha = \sum_i \xi_{g_i} \cdot f_i$. This follows because $\sum_i [\xi_{g_i}, f_i] = \sum_i \xi_{g_i}(f_i) = \pi(d\alpha) = 0$. Hence $LH(X) \cdot \mathcal{O}_X \subseteq H(X) \cdot \mathcal{O}_X$. For the opposite inclusion, note that $H(X) \subseteq LH(X)$. □

In the case that X has finitely many symplectic leaves, then $P(X)$ also has these as its leaves, since in this case every Poisson vector field must be parallel to the symplectic leaves. On the other hand, it can happen that $P(X)$ has finitely many leaves but not $LH(X)$:

Example 3.14. If $\pi = x\partial_x \wedge \partial_y$ on $\mathbf{A}^2$, then there are infinitely many symplectic leaves: the y-axis is a degenerate invariant subvariety with respect to $LH(X)$. On the other hand, the vector field ∂_y is Poisson, so the y-axis is a leaf with respect to $P(X)$.

For $LH(X)$, the same argument as for $H(X)$ shows that, in the notation of Proposition 2.6, all of the X_i are incompressible, and hence $LH(X)$ is holonomic if and only if it has finitely many leaves (the symplectic leaves); or one can use Proposition 3.13. So, again, Theorem 2.19 is the same as Theorem 2.9.

On the other hand, it can happen that $P(X)$ is holonomic even though it does not have finitely many leaves:

Example 3.15. If X is a variety equipped with the zero Poisson structure, then $P(X)$ is the Lie algebra of all vector fields, and as explained in §3.1, this is holonomic if and only if there are finitely many exceptional points. This can happen without having finitely many leaves, e.g., if one takes a product $X = \mathbf{A}^1 \times Y$ where Y has infinitely many exceptional points (cf. Example 3.8). Moreover, this is an example where the X_i are not incompressible (if x is the coordinate on $\mathbf{A}^1$, $P(X)$ contains both ∂_x and $x\partial_x$, so cannot be incompressible on any of the $X_i = \mathbf{A}^1 \times Y_{i-1}$).

Example 3.16. If Y is an n-dimensional Calabi-Yau variety (e.g., $Y = \mathbf{A}^n$) and $X = Z(f_1, \ldots, f_{n-2}) \subseteq Y$ is a surface which is a complete intersection $f_1 = \cdots = f_{n-2} = 0$, then there is a standard *Jacobian* Poisson structure on X, given by $i_\Xi df_1 \wedge \cdots \wedge df_{n-2}$, where $\Xi = \mathsf{vol}_Y^{-1}$ is the inverse to the volume form on Y, which we then contract with the exact $n-2$-form $df_1 \wedge \cdots \wedge df_{n-2}$. It is then standard that the result is a Poisson bivector field. Then $H(X)$ is holonomic if and only if X has only isolated singularities. Already in the case $Y = \mathbf{A}^3$ and $X = Z(f)$ for f a (quasi)homogeneous surface with an isolated singularity at zero, this is quite interesting; $\mathsf{HP}_0(\mathcal{O}_X) = (\mathcal{O}_X)_{H(X)}$ was computed in [AL98] (although, as we will explain in §5, it follows from older results of [Gre75]); we will compute $M(X, H(X))$ in [ES14]. See Example 3.39 and §5.

Example 3.17. If X and Y are Poisson schemes of finite type, then for any of the three Lie algebras defined above, the coinvariants are multiplicative in the sense that $(\mathcal{O}_{X\times Y})_{H(X\times Y)} = (\mathcal{O}_X)_{H(X)} \otimes (\mathcal{O}_Y)_{H(Y)}$ and similarly for LH and P. Similarly, the leaves of $X \times Y$ are the products of leaves from X and of leaves from Y. These facts follow from the following formula, which also holds for LH and P replacing H:

$$H(X) \oplus H(Y) \subseteq H(X \times Y) \subseteq (\mathcal{O}_X \boxtimes H(Y)) \oplus (H(X) \boxtimes \mathcal{O}_Y). \tag{3.18}$$

The first inclusion holds because, for $f \in \mathcal{O}_X$ and $g \in \mathcal{O}_Y$, $\xi_{(f\otimes 1)+(1\otimes g)} = \xi_f + \xi_g$. The second follows because, for $f \in \mathcal{O}_X$ and $g \in \mathcal{O}_Y$, $\xi_{f\otimes g}(h) = f\xi_g + g\xi_f$. To extend (3.18) to the case of $LH(X \times Y)$, it remains only to consider also the action of Hamiltonian vector fields of closed one-forms modulo torsion generating $H^1_{DR}(X \times Y) = H^1_{DR}(X) \oplus H^1_{DR}(Y)$ (assuming for simplicity that X and Y are connected). So it suffices to consider Hamiltonian vector fields of closed one-forms modulo torsion on X and Y separately. One concludes that (3.18) holds for LH replacing H. Finally, for $P(X \times Y)$, one also has (3.18) with P replacing H, since $\pi_{X\times Y} = \pi_X \oplus \pi_Y$ and $\mathrm{Vect}(X \times Y) = (\mathrm{Vect}(X) \boxtimes \mathcal{O}_Y) \oplus (\mathcal{O}_X \boxtimes \mathrm{Vect}(Y))$.

Example 3.19. Here we give an example of a variety X with finitely many symplectic leaves for which $LH(X) \subsetneq P(X)$. Namely, suppose X is a homogeneous cubic hypersurface, $Q = 0$, in $\mathbf{A}^3$ with an isolated singularity at the origin, i.e., the cone over a smooth curve of genus one. Then X is equipped with the Poisson bivector given by contracting the top polyvector field $\partial_x \wedge \partial_y \wedge \partial_z$ on $\mathbf{A}^3$ with dQ, where x, y, and z are the coordinate functions on $\mathbf{A}^3$. This has two symplectic leaves: the origin and its complement.

We claim that the Euler vector field is Poisson but not locally Hamiltonian. This is because the Poisson bracket preserves total degree, so the Euler vector field is Poisson, but it cannot be Hamiltonian since the Poisson bivector vanishes to degree two at the origin, i.e., $\pi(df \wedge dg) \subseteq \mathfrak{m}_0^2$ for all $f, g \in \mathcal{O}_X$, with $\mathfrak{m}_0$ the maximal ideal of functions vanishing at the origin. Hence all Hamiltonian vector fields vanish to degree two at the origin as well.

For example, X could be the hypersurface $x^3 + y^3 + z^3 = 0$, which is the cone over the Fermat curve. Then $\{x, y\} = 3z^2$, $\{y, z\} = 3x^2$, and $\{z, x\} = 3y^2$, and it is clear that the Euler vector field is Poisson but not (locally) Hamiltonian.

Remark 3.20. We note that, unlike for all vector fields, the converse to Proposition 2.21 does not hold in the Poisson case. Indeed, one can consider $\mathbf{A}^3$ with the Poisson structure $\partial_x \wedge \partial_y$, which has infinitely many leaves (hence is not holonomic) but vanishing HP_0.

Finally, if X is an affine Poisson scheme of finite type with finitely many symplectic leaves, and $f : X \to Y$ is a finite map, then the argument of [ES10] showed that the Lie algebra of Hamiltonian vector fields of Hamiltonian functions from $f^*\mathcal{O}_Y$ has finitely many leaves. We recover the result from *op. cit.* that $\mathcal{O}_X/\{\mathcal{O}_X, \mathcal{O}_Y\}$ is finite-dimensional. This includes the case, for example, where $X = V$ is a symplectic vector space, and $Y = V/G$ for $G < \mathsf{Sp}(V)$ a finite subgroup (or even any finite subgroup $G < \mathsf{GL}(V)$). If $G < \mathsf{Sp}(V)$ then we obtain the G-invariant Hamiltonian vector fields, $H(X)^G$. Note that, in this case, if $q : X \to X/G$ is the projection, then $q_* M(X, H(X)^G)^G \cong M(X/G, H(X/G))$.

3.3. **Jacobi schemes.** A Jacobi structure [Lic78] is a generalization of a Poisson structure, which includes both symplectic and contact manifolds (see the examples below), and can be thought of as a degenerate or singular version of both. By definition, it is a Lie bracket on $\mathcal{O}_X$ which need not satisfy the Leibniz rule, but instead satisfies that $\{f, -\}$ is a differential operator of order ≤ 1 for all $f \in \mathcal{O}_X$. Equivalently, the Lie bracket is given by a pair of a bivector field π and a vector field u via the formula

$$\{f, g\} = \pi(df \wedge dg) + u(fdg - gdf).$$

Here, by a degree k polyvector field, we mean a skew-symmetric multiderivation of $\mathcal{O}_X$ of degree k, i.e., a linear map $\wedge^k \mathcal{O}_X \to \mathcal{O}_X$ which is a derivation in each component.

The Jacobi identity is then equivalent to the identities

$$[u, \pi] = 0, \quad [\pi, \pi] = 2u \wedge \pi,$$

where $[-, -]$ is the Schouten-Nijenhuis bracket on polyvector fields.

To any affine Jacobi scheme X of finite type, one naturally associates the Lie algebra of Hamiltonian vector fields ξ_f for $f \in \mathcal{O}_X$, given by the principal symbol of the differential operator $\{f, -\}$, i.e.,

$$\xi_f = \pi(df) + fu, \quad \text{i.e.,} \quad \xi_f(g) = \{f, g\} + gu(f).$$

It is well-known and easy to verify that one has the identity

$$[\xi_f, \xi_g] = \xi_{\{f,g\}},$$

so this indeed forms a Lie algebra. Call it $H(X) := H_{\pi,u}(X)$.

We can also define a version $P(X) := P_{\pi,u}(X)$ of vector fields *preserving* the Jacobi structure, i.e., vector fields ξ such that $\xi(\{f, g\}) = \{\xi(f), g\} + \{f, \xi(g)\} = 0$ for all $f, g \in \mathcal{O}_X$. However, unlike before, it is no longer true that $H(X) \subseteq P(X)$. In particular, to have $\xi_f \in P(X)$, we require that $[u, \xi_f] = \xi_{u(f)} = 0$. So to have $H(X) \subseteq P(X)$, we would need to have $u = 0$, i.e., the structure has to be Poisson.

Remark 3.21. It seems that we cannot define an analogue of $LH(X)$ in this setting since there is no way to obtain Hamiltonian vector fields from closed one-forms. In a neighborhood of a smooth point, one could consider vector fields that restrict in a formal neighborhood of the point to a Hamiltonian vector field, but in general this will not coincide with the definition of $LH(X)$ in the Poisson case, in neighborhoods of singular points where the first de Rham cohomology does not vanish in the formal neighborhood; see Example 3.11.

Remark 3.22. Unlike the Poisson case, given Jacobi varieties X and Y, there is no natural way to define a Jacobi structure on the product $X \times Y$: if one set $\pi_{X\times Y} = \pi_X \oplus \pi_Y$ and $u_{X\times Y} = u_X \oplus u_Y$, then the identity $[\pi, \pi] = 2u \wedge \pi$ would no longer be satisfied: $\pi_X \wedge u_Y$ and $\pi_Y \wedge u_X$ would appear on the RHS but not the LHS. However, one can still equip $X \times Y$ with the Lie algebra of vector fields $\mathfrak{v}_X \oplus \mathfrak{v}_Y$; in this general situation (i.e., for any $\mathfrak{v}_X$ and $\mathfrak{v}_Y$), one always has $(\mathcal{O}_{X\times Y})_{\mathfrak{v}_X \oplus \mathfrak{v}_Y} \cong (\mathcal{O}_X)_{\mathfrak{v}_X} \otimes (\mathcal{O}_Y)_{\mathfrak{v}_Y}$.

Example 3.23. The analogue of symplectic varieties in this setting is a smooth Jacobi variety for which $H(X)$ has full rank everywhere, i.e., it has only one leaf (assuming X is connected). This is called a *transitive* Jacobi variety.

As pointed out in, e.g., [MS98] (this is in the smooth context, but the result is proved using a formal neighborhood and works in general), there are two types of connected transitive varieties. One is called *locally conformally symplectic*, and is the situation where π is nondegenerate (recall we assumed X was smooth). Therefore, X is even-dimensional. In this case, u is equivalent to the data of a closed one-form ϕ satisfying $d\omega = \phi \wedge \omega$, where ω is the inverse of π, and $\phi = u(\omega)$. Then, in the formal neighborhood of any point $x \in X$, we can write $\phi = df$ for some function f, and then $H(X)$ preserves the formal volume form $(e^{-f}\omega)^{\wedge \dim X}$ (cf. Example 3.25 below). This need not be a global volume form, so $M(X, H(X))$ is a rank-one local system which need not be trivial.

The other type of transitive Jacobi variety is an odd-dimensional contact variety. In this case, the Jacobi structure is equivalent to the structure of a *contact one-form* α, i.e., a one-form such that $\mathsf{vol}_X := \alpha \wedge (d\alpha)^{\wedge(\dim X - 1)/2}$ is a nonvanishing volume form. This determines u and π uniquely by the formulas

$$u(d\alpha) = 0, u(\alpha) = 1, \quad \pi(\alpha, \beta) = 0, \quad \pi(\beta \wedge d\alpha) = -\beta + u(\beta)\alpha, \forall \beta \in T_X^*.$$

By the next example, in this case $\mathfrak{v}$ does not flow incompressibly, so by Proposition 2.36, $M(X, H(X)) = 0$. On the other hand, we will see that $P(X)$ does flow incompressibly and transitively, preserving the volume form vol_X, so $M(X, P(X)) = \Omega_X$ and $\pi_* M(X, P(X)) \cong H_{DR}^{\dim X - *}(X)$. In particular, $(\mathcal{O}_X)_{P(X)} = H_{DR}^{\dim X}(X)$.

Example 3.24. The standard example of a contact variety is $\mathbf{A}^{2d+1}$ with the standard contact structure, $\alpha = dt + \sum_i x_i dy_i$. Also, note that an arbitrary contact variety restricts to one isomorphic to this in the formal neighborhood of any point. We claim that no volume form is preserved by $H(\mathbf{A}^{2d+1})$, and hence the flow of $H(X)$ on an arbitrary contact variety is not incompressible. Indeed, let Eu be the weighted Euler vector field on $\mathbf{A}^{2d+1}$ assigning weights $|x_i| = 1 = |y_i|$ and $|t| = 2$, i.e., $\mathrm{Eu} = 2t\frac{\partial}{\partial t} + \sum_i x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i}$. Then, we have $\pi = -\sum_i \frac{\partial}{\partial x_i} \wedge (\frac{\partial}{\partial y_i} - x_i \frac{\partial}{\partial t})$ and $u = \frac{\partial}{\partial t}$. In this case, $\xi_1 = \frac{\partial}{\partial t}$, $\xi_{x_i} = -\frac{\partial}{\partial y_i} + x_i \frac{\partial}{\partial t}$, $\xi_{y_i} = \frac{\partial}{\partial x_i}$, and $\xi_t = -\sum_i x_i \frac{\partial}{\partial x_i}$. In particular, the Lie algebra $H(X)$ does not preserve any volume form (if it did, for this form to be preserved by ξ_1, ξ_{x_i}, and ξ_{y_i}, it would have to preserve the constant vector fields, and hence the form would have to be the standard volume form, i.e., the one determined by the contact structure; however this form is not preserved by ξ_t.)

Finally, note that, in the above case, $P(\mathbf{A}^{2d+1})$, the Lie algebra of all vector fields that commute with both π and u, is the subspace of Hamiltonian vector fields ξ_f where f is independent of t. So $P(\mathbf{A}^{2d+1}) \subsetneq H(\mathbf{A}^{2d+1})$. This still flows transitively, since it includes the constant vector fields as above. As a result, for arbitrary odd-dimensional contact varieties, $P(X) \subsetneq H(X)$. In fact, $P(X)$ does flow incompressibly, since it preserves the standard volume form (it is clear that it preserves the inverse top polyvector field, $\pm\pi^{\wedge(\dim X-1)/2} \wedge u$).

Example 3.25. By the Darboux theorem, every locally conformally symplectic variety X of dimension $2d$ has the form, in a formal neighborhood of a point $x \in X$, $\omega = e^f \omega_0$ and $\phi = df$, where ω_0 is the standard symplectic form on $\hat{\mathbf{A}}^{2d} \cong \hat{X}_x$. In this case $\pi = e^{-f}\pi_0$ where π_0 is the standard Poisson bivector on $\mathbf{A}^{2d}$, and $u = \pi(df)$ is e^{-f} times the Hamiltonian vector field of f under the standard symplectic structure. Thus, $H(\hat{X}_x)$ is identical with the Lie algebra of Hamiltonian vector fields preserving the standard symplectic form ω_0 (in this formal neighborhood), so it flows incompressibly. However, as noted above, $H(\hat{X}_x) \not\subseteq P(\hat{X}_x)$. In fact, in this case, as in the case of odd-dimensional contact varieties, $P(\hat{X}_x) \subsetneq H(\hat{X}_x)$. Indeed, $P(\hat{X}_x)$ consists of ξ_g such that $u(g) = 0$, i.e., $\{f, g\} = 0$.

We see as a consequence of the above that, in general, the leaves of $H(X)$ consist of odd-dimensional contact varieties and locally conformally symplectic varieties. The former are not incompressible (without passing to an infinitesimal neighborhood), whereas the latter are. As a consequence, we conclude from Proposition 2.39 that

Proposition 3.26. Let X be a Jacobi variety. Then $X = X_{H(X)}$ if and only if the generic rank of $H(X)$ is even on each irreducible component.

(Recall from Definition 2.45 that $X_{H(X)}$ is the support of $M(X, H(X))$ on X.)

Example 3.27. Here is an example of a Jacobi variety where there is an odd-dimensional leaf having an infinitesimal neighborhood which is incompressible. Let $X = \mathbf{A}^2$ with $\pi = -x\partial_x \wedge \partial_t$ and $u = \partial_t$. Then $H(X)$ has rank two except along $x = 0$, where it has rank one. Moreover, the distribution $\phi := \partial_x(\delta_{x=0}) \boxtimes dt$ is preserved by $H(X)$: for $\xi_{x^i t^j}$ with $i \geq 2$ this clearly annihilates ϕ; then $\xi_{xt^j} = jx^2t^{j-1}\partial_x$ and $\xi_{t^j} = jxt^{j-1}\partial_x + t^j\partial_t$ also do (recall that the action of differential operators on distributions is a right action; the action of vector fields is given by $(\psi \cdot \xi)(f) = \psi(\xi(f))$ for ψ a distribution and f a function). The final vector field, ξ_{t^j}, can alternatively be rewritten in $H(X) \cdot \mathcal{O}_X$ as

$$\xi_{t^j} = j(x\partial_x - 1)t^{j-1} + \partial_t \cdot t^j,$$

and note that $x\partial_x - 1$ and ∂_t annihilate ϕ, which implied that ξ_{t^j} does.

Question 3.28. Let X_{even} be the closure of the locus where the rank of $\mathfrak{v}$ is even. Then, is the set-theoretic support, $(X_{H(X)})_{\text{red}}$, of $(X, H(X))$ equal to X_{even}? If the answer is negative, is there an example where $H(X)$ has everywhere odd rank, but $M(X, H(X)) \neq 0$?

3.4. **Varieties with a top polyvector field.** Motivated by the idea that a Poisson structure is a singular and/or degenerate generalization of a symplectic structure, we define a similar analogue of Calabi-Yau structures, and their associated Lie algebras of incompressible vector fields. These are also motivated by the relationship between incompressibility and holonomicity.

In the Poisson case, one replaces a nondegenerate two-form by a possibly degenerate two-bivector, which in the nondegenerate case is inverse to the symplectic form. Thus, by analogy, we replace a volume form by a top polyvector field, which is allowed to vanish on some locus. On the non-degenerate, smooth locus, one recovers a nonvanishing volume form by taking the inverse of the polyvector field.

Specifically, let X be an affine variety of dimension n equipped with a global top polyvector field, i.e., a multiderivation $\Xi : \wedge^n \mathcal{O}_X \to \mathcal{O}_X$. Then, as in the Poisson case, there are three natural Lie algebras to consider: the Lie algebra $H_\Xi(X)$ of vector fields obtained by contracting Ξ with exact $(n-1)$-forms; the Lie algebra $LH_\Xi(X)$ of vector fields obtained by contracting Ξ with closed $(n-1)$-forms; and the Lie algebra $P_\Xi(X)$ of all incompressible vector fields, i.e., vector fields ξ such that $L_\xi(\Xi) = 0$ (vector fields *preserving* Ξ). Note that, in this case, when X is irreducible and Ξ is nonzero, it is immediate that all three flow incompressibly on X.

As for generically symplectic varieties with their associated (locally) Hamiltonian vector fields, for arbitrary irreducible (X, Ξ) with $\Xi \neq 0$, one has $LH_\Xi(U)/H_\Xi(U) \cong H_{DR}^{\dim X - 1}(U)$ for all open affine $U \subseteq X$. Moreover, when U is additionally smooth, $LH_\Xi(X)$ coincides with those vector fields which, in formal neighborhoods of all $x \in U$, are Hamiltonian.

Remark 3.29. As in the Poisson case (see Example 3.11), in the formal neighborhood of a singular point $x \in X$, not all locally Hamiltonian vector fields need be Hamiltonian, since $H_{DR}^{\dim X-1}(\hat{X}_x)$ need not vanish. Indeed, as in Example 3.11, when $X = \mathbf{A}^1 \times Z$ where Z is a complete intersection with an isolated singularity at $z \in Z$, then $\dim H_{DR}^{\dim X-1}(\hat{X}_{(t,z)}) = \mu_z - \tau_z$, which need not be zero (already for the case of a hypersurface in $\mathbf{A}^n$). Then, equipped with the polyvector field $\Xi_{\mathbf{A}^1} \boxtimes \Xi_Z$ where Ξ_Z is as in Example 3.39 (which in the case $Z = \{Q = 0\} \subseteq \mathbf{A}^n$ is $\Xi_{\mathbf{A}^n}(dQ)$), one concludes that $LH(\hat{X}_{(t,z)})/H(\hat{X}_{(t,z)}) \cong H^{\dim X-1}(\hat{X}_{(t,z)}) \neq 0$.

As in the Poisson case, these are Lie algebras since $[LH(X), LH(X)] \subseteq H(X)$, as we explain. Given a $(n-2)$-form (modulo torsion) $\alpha \in \tilde{\Omega}_X^{n-2}$, let $\xi_\alpha := \Xi(d\alpha)$ be its associated Hamiltonian vector field. Similarly, given a closed $(n-1)$-form modulo torsion, $\gamma \in \tilde{\Omega}_X^{n-1}$, let $\eta_\gamma := \Xi(\gamma)$ be its associated locally Hamiltonian vector field. Then the fact that $[LH(X), LH(X)] \subseteq H(X)$ follows from the formula, where α and β are closed $(n-1)$-forms modulo torsion,

$$[\eta_\alpha, \eta_\beta] = \xi_{i_{\eta_\alpha}(\beta)}, \tag{3.30}$$

which can be verified in a formal neighborhood of a smooth point of X where Ξ is nonvanishing, and hence which holds globally.

As in the Poisson case (Proposition 3.13), $H(X)$ and $LH(X)$ define the same $\mathcal{D}$-modules on X:

Proposition 3.31. The $\mathcal{O}$-saturations are equal: $H(X)^{os} = LH(X)^{os}$. Thus, $M(X, H(X)) \cong M(X, LH(X))$.

Proof. Given a closed $n-1$ form $\alpha = \sum_i f_i d\beta_i$, we see that $\eta_\alpha = \sum_i \eta_\beta \cdot f_i$, since $\sum_i \eta_\beta(f_i) = \Xi(d\alpha) = 0$. Thus, $LH(X) \cdot \mathcal{O}_X \subseteq H(X) \cdot \mathcal{O}_X$, and the proposition follows since $H(X) \subseteq LH(X)$. ☐

Next, we compute the leaves of $H_\Xi(X)$ and of $LH_\Xi(X)$. All non-open leaves turn out to be points. We will use a general

Definition 3.32. Given a Lie algebra of vector fields $\mathfrak{v}$ on X, the *degenerate locus* of $\mathfrak{v}$ is the locus of $x \in X$ such that $\mathfrak{v}|_x \neq T_x X_{\text{red}}$.

Note that the degenerate locus includes the singular locus of X_{red} (which equals X in this subsection, although the preceding definition makes sense more generally).

Remark 3.33. If X is irreducible, then we claim that the degenerate locus is the same as the locus of x such that $\dim \mathfrak{v}|_x < \dim X$, i.e., such that $\mathfrak{v}$ does not have maximal rank. Thus, in terms of Proposition 2.6, the degenerate locus is the union of X_i for $i < \dim X$. To prove the claim, we only have to show that, along the singular locus, the rank of $\mathfrak{v}$ is strictly less than $\dim X$. This is true at generic singular points, where the singular locus is smooth, since $\mathfrak{v}$ must be parallel to the singular locus. Then, the result follows for the entire singular locus, by replacing X by its singular locus and inducting on the dimension of X.

Now return to our assumption that (X, Ξ) is a variety with a top polyvector field Ξ. For $\mathfrak{v} = H_\Xi(X), LH_\Xi(X)$, or $P_\Xi(X)$, it is clear that the degenerate locus is the union of the singular locus with the vanishing locus of Ξ. We will also call this *the degenerate locus of* Ξ.

Theorem 3.34. Let (X, Ξ) be a variety equipped with a top polyvector field. If $\mathfrak{v} := H_\Xi(X)$ or $LH_\Xi(X)$, then every degenerate point is a (zero-dimensional) leaf. That is, $\mathfrak{v}|_x \neq T_x X$ implies $\mathfrak{v}|_x = 0$.

We remark that the theorem is in stark contrast to the previous subsections, where in general there can exist leaves of positive dimension less than the dimension of X. For surfaces, where Ξ is the same as a Poisson structure, the theorem reduces to the statement that all symplectic leaves have dimension zero or two.

Proof of Theorem 3.34. It suffices to show that Ξ vanishes on the singular locus of X. Let Z be an irreducible component of the singular locus. Then $\dim Z < \dim X$, and $\mathfrak{v}$ is parallel to Z. Hence, $(\wedge^{\dim Z} \mathfrak{v})|_Z = 0$ (this holds at smooth points of Z, hence generically on Z, and hence on all of Z). □

Corollary 3.35. For $(X, \mathfrak{v})$ as in the theorem, assuming also that X is purely of positive dimension, the following are equal:

(i) The degenerate locus of $\mathfrak{v}$;
(ii) The set-theoretic support of the ideal generated by $\mathfrak{v}(\mathcal{O}_X)$;
(iii) The set of points x such that $(\hat{\mathcal{O}}_{X,x})_\mathfrak{v} \neq 0$.

Proof. It is easy to see that (ii) and (iii) coincide with the vanishing locus of $\mathfrak{v}$ since X is positive-dimensional. The theorem implies that this coincides with (i). □

Corollary 3.36. For $(X, \mathfrak{v})$ as in the theorem, X is the union of finitely many open leaves and the degenerate (set-theoretic) locus of Ξ. There are finitely many leaves if and only if the degenerate locus is finite.

Proof. The connected components of the open locus where $\mathfrak{v}|_x = T_x X$ are the open leaves (of which there are finitely many), and the vanishing locus of $\mathfrak{v}|_x$ is the union of all points which are leaves. By the theorem, the union of these is all of X. □

Corollary 3.37. Let $\mathfrak{v} := H_\Xi(X)$ or $LH_\Xi(X)$. Then, the following are equivalent:

(i) $(\mathcal{O}_X)_\mathfrak{v}$ is finite-dimensional;
(ii) The degenerate locus of Ξ is finite;
(iii) $\mathfrak{v}$ is holonomic.

Proof. By the corollary, X has finitely many leaves if and only if it has finitely many zero-dimensional leaves. Since zero-dimensional leaves are automatically incompressible, this shows that (ii) and (iii) are equivalent. Moreover, since zero-dimensional leaves always support linearly independent evaluation functionals in $((\mathcal{O}_X)_\mathfrak{v})^*$, (i) implies (ii) and (iii). The implication (iii) $\Rightarrow$ (i) is immediate. □

Note that, in contrast to $H_\Xi(X)$ and $LH_\Xi(X)$, $P_\Xi(X)$ can be holonomic even without having finitely many leaves (e.g., in the case when $\Xi = 0$, this happens if and only if X has finitely many exceptional points).

One example of a variety with a top polyvector field is an even-dimensional (affine) Poisson variety, with $\Xi = \pi^{\wedge \dim X/2}$, for π the Poisson bivector field. Note that $P_\Xi(X) \supseteq P_\pi(X)$. We claim that this is a proper containment if and only if $\dim X > 2$. For $\dim X = 2$ it is clear these are equal. Otherwise, since $\Xi \neq 0$ if and only if π is generically symplectic, passing to a formal neighborhood of a point, the statement reduces to the case $X = \mathbf{A}^{2n}$ with $n > 1$ and the usual symplectic structure, where it is well-known and easy to check.

Example 3.38. As noted in example 2.37, if X is a symplectic variety, then in particular it is Calabi-Yau and $M(X, H_\pi(X)) = M(X, H_\Xi(X)) = \Omega_X$, whether we use the Poisson bivector π or the top polyvector field $\Xi = \wedge^{\dim X/2}\pi$. However, for general Poisson varieties, again setting $\Xi = \wedge^{\dim X/2}$, this does not hold. For example, if the Poisson bivector field π has generic rank two and $\dim X \geq 4$, then the top exterior power, $\Xi = \pi^{\wedge(\dim X/2)}$, is zero, so $H_\Xi = LH_\Xi = 0$, and $P_\Xi = \mathrm{Vect}(X)$, but this is clearly not true of H_π, LH_π, and P_π, and the coinvariants will differ in general.

Example 3.39. Generalizing Example 3.16, we can let (Y, Ξ_Y) be any n-dimensional variety with a top polyvector field, and let $X = Z(f_1, \ldots, f_k) \subseteq Y$ be a complete intersection. Then we can set $\Xi_X = i_{\Xi_Y}(df_1 \wedge \cdots \wedge df_k)$, which is a top polyvector field on X. (Note that, when $Y = \mathbf{A}^n$, the Lie algebra $H(X)$ has been studied in many places, e.g., in [MS96]). Then, by Corollary 3.37, $H(X)$ is holonomic if and only if X has only isolated singularities, and the degenerate locus of Y meets X at only finitely many points. In this case, we explicitly compute $(\mathcal{O}_X)_{H(X)}$ in §5.

Remark 3.40. Unlike Example 3.17, a product formula does not hold for the above Lie algebras of vector fields on $X \times Y$, when X and Y are equipped with top polyvector fields Ξ_X and Ξ_Y and $X \times Y$ is equipped with the tensor product $\Xi_X \boxtimes \Xi_Y$. First of all, for the Lie algebras P, note that, in general,

$$P(X \times Y) \not\subseteq (P(X) \boxtimes \mathcal{O}_Y) \oplus (\mathcal{O}_X \boxtimes P(Y)).$$

For example, when X and Y admit vector fields $\mathrm{Eu}_X, \mathrm{Eu}_Y$ such that $L_{\mathrm{Eu}_X}(\Xi_X) = \Xi_X$ and $L_{\mathrm{Eu}_Y}(\Xi_Y) = \Xi_Y$, then $\mathrm{Eu}_X - \mathrm{Eu}_Y$ is in the LHS but not the RHS above. (This holds, for example, when X and Y are conical with top polyvector fields Ξ_X and Ξ_Y which are homogeneous of nonzero weight under the scaling action, replacing the standard Euler vector fields by suitable nonzero multiples).

Using this, one can see that a product formula does not hold for coinvariants: suppose $(\mathcal{O}_X)_{P(X)} \not\cong (\mathcal{O}_X)_{\mathrm{Vect}(X)}$. Suppose that $\xi \in \mathrm{Vect}(X)$ is a vector field such that $\xi(\mathcal{O}_X) \not\subseteq P(X)(\mathcal{O}_X)$ and $L_\xi(\Xi_X) = \Xi_X$. Then $P(X \times X)(\mathcal{O}_{X \times X})$ contains $(\xi \boxtimes 1 - 1 \boxtimes \xi)(\mathcal{O}_X \boxtimes 1) = \xi(\mathcal{O}_X)$, but this is not contained in $(P(X)(\mathcal{O}_X) \boxtimes \mathcal{O}_X) + (\mathcal{O}_X \boxtimes P(X)(\mathcal{O}_X))$. Since also $P(X \times X)$ contains horizontal and vertical vector fields, $P(X) \boxtimes 1$ and $1 \boxtimes P(X)$, we conclude that $(\mathcal{O}_{X \times X})_{P(X \times X)}$ is quotient of $(\mathcal{O}_X)^{\boxtimes 2}_{P(X)}$ by a nontrivial vector subspace.

For an explicit example, we could let X be the cuspidal curve $x^2 = y^3$ in the plane $\mathbf{A}^2$. Then, $P(X) = \langle 2x\partial_y + 3y^2\partial_x \rangle$ and hence $(\mathcal{O}_X)_{P(X)}$ surjects (in fact isomorphically by a special case of Corollary 5.23; cf. Remark 5.24) to $(\mathcal{O}_X)/(2x, 3y^2)$, which is two-dimensional; on the other hand, since $\mathrm{Vect}(X)$ contains the Euler vector field $3x\partial_x + 2y\partial_y$, $(\mathcal{O}_X)_{\mathrm{Vect}(X)} = (\mathcal{O}_X)/(x, y)$ is one dimensional. In particular, in this case, $(\mathcal{O}_{X^2})_{P(X^2)}$ is two-dimensional, whereas $(\mathcal{O}_X)^{\otimes 2}_{P(X)}$ is four-dimensional.

For the Lie algebras of Hamiltonian and locally Hamiltonian vector fields, let (Y, Ξ_Y) be any (affine) variety with $(\mathcal{O}_Y)_{H(Y)} = 0$ (by Corollary 3.35 and Example 2.37, this is equivalent to Y being Calabi-Yau with $H^{\dim Y}(Y) = 0$), and (X, Ξ_X) be a positive-dimensional (affine) variety. Then, we claim that $(\mathcal{O}_{X \times Y})_{H(X \times Y)} \cong (\mathcal{O}_X)/(H(X) \cdot \mathcal{O}_X) \boxtimes \mathcal{O}_Y$, where now $(H(X) \cdot \mathcal{O}_X)$ is the *ideal* generated by $H(X) \cdot \mathcal{O}_X$. That is, we claim that the vector space $H(X \times Y) \cdot \mathcal{O}_{X \times Y}$ is $(H(X) \cdot \mathcal{O}_X) \boxtimes \mathcal{O}_Y$.

To see this, note that the ideal $(H(X) \cdot \mathcal{O}_X)$ is identified with the image of the contraction of Ξ_X with top differential forms on X. Now, on the product variety $X \times Y$, top differential forms are spanned by exterior products of top differential forms on X with top differential forms on Y. The same is true for top polyvector fields: a derivation of $\mathcal{O}_X \otimes \mathcal{O}_Y$ is uniquely determined by its restriction to $\mathcal{O}_X \otimes 1$ and $1 \otimes \mathcal{O}_Y$, by the formula $D(f \otimes g) = D(f) \otimes g + f \otimes D(g)$. Thus, skew-symmetric multiderivations of degree $\dim X + \dim Y$ are of the form $\Xi_X \boxtimes \Xi_Y$ for Ξ_X and Ξ_Y top polyvector fields on X and Y, respectively.

Therefore, the contraction of top polyvector fields on $X \times Y$ with top differential forms lies in the ideal $(H(X) \cdot \mathcal{O}_X) \otimes \mathcal{O}_Y$ (in fact, they are equal, in view of the assumption that $(\mathcal{O}_Y)_{H(Y)} = 0$, or by the next argument). Thus we get the inclusion of $H(X \times Y) \cdot \mathcal{O}_{X \times Y}$ in $(H(X) \cdot \mathcal{O}_X) \otimes \mathcal{O}_Y$.

Conversely, for any element $f \in (H(X) \cdot \mathcal{O}_X) \subseteq \mathcal{O}_X$, suppose that $f = \Xi_X(\alpha)$ for some top differential form α. For any $g \in \mathcal{O}_Y$, write $g = \Xi_Y(d\beta)$ for some $(\dim Y - 2)$-form β. Then, $(f \otimes g) = (\Xi_X \wedge \Xi_Y)(\alpha \wedge d\beta)$. Therefore, $(f \otimes g) \in H(X \times Y) \cdot \mathcal{O}_{X \times Y}$. This gives the opposite inclusion.

Note that the ideal $(H(X) \cdot \mathcal{O}_X)$ is supported at the zero-dimensional leaves of X, which by Theorem 3.34 is the degenerate locus of Ξ_X. More generally, for arbitrary X and Y, the leaves of $H(X \times Y)$ and $LH(X \times Y)$ consist of the open leaves obtained as products of open leaves in X with open leaves in Y, and zero-dimensional leaves at every point of the degenerate locus.

Finally, as in the Poisson case, one can also consider, for every map $f : X \to Y$, the smaller Lie algebra of vector fields obtained by contracting Ξ with exact (or closed) $(n-1)$-forms pulled back from Y. The leaves of the resulting Lie algebra consist of open leaves, which are the restriction of the open leaves in X to the noncritical locus, together with zero-dimensional leaves at the critical points of f together with the degenerate locus of Ξ. This example includes, for every subgroup $G < \mathsf{SL}(n)$ (or even $\mathsf{GL}(n)$), the map $X = \mathbf{A}^n \to \mathbf{A}^n/G = Y$. The coinvariants of $\mathcal{O}_{\mathbf{A}^n}$ under the resulting Lie algebra is finite-dimensional if and only if the critical locus of f is finite, i.e., no nontrivial element of G has one as an eigenvalue; equivalently, this says that G acts freely on the $2n-1$-sphere of unit vectors in $\mathbf{C}^n$. More generally, we can take a quotient of an arbitrary pair (X, Ξ) by a finite group of automorphisms preserving Ξ, and the coinvariants of the resulting Lie algebra are finite-dimensional if and only if the degenerate locus of X is finite and all elements of the group have only isolated fixed points.

One can alternatively consider, for a finite group quotient $X \twoheadrightarrow X/G$, the Lie algebras $H(X)^G$, $LH(X)^G$, and $P(X)^G$. We can do this slightly more generally, where G only preserves Ξ up to scaling (then G still acts on $H(X), LH(X)$, and $P(X)$).

Proposition 3.41. Suppose $\dim X \geq 2$ and let G be a finite group of automorphisms of X which acts on Ξ by rescaling. Let $\mathfrak{v}$ be $H(X)$ or $LH(X)$. Then the leaves of $\mathfrak{v}^G$ consist of the points of the degenerate locus of X, together with the connected components of the subvarieties of the open leaf whose stabilizers are fixed subgroups of G.

If the degenerate locus of X is finite, then $\mathfrak{v}^G$ has finitely many leaves, and the same result holds for $\mathfrak{v} = P(X)$.

Call a subgroup $K < G$ *parabolic* if it occurs as the stabilizer of a point in X, i.e., it is the stabilizer of one of the leaves of $\mathfrak{v}^G$.

Proof. It is clear that $\mathfrak{v}^G$ must flow parallel to the given subvarieties. Therefore, since $H(X) \subseteq LH(X)$, we only have to show that $H(X)^G$ flows transitively along each of the given subvarieties. Also, the last statement is immediate from this, the fact that $P(X)$ preserves the given subvarieties (since the degenerate locus is finite, it cannot flow along it), and $H(X) \subseteq P(X)$.

Here we will make use of the fact that $H(X)$ is $\mathcal{D}$-localizable (which we will prove in Theorem 4.1 independently of the results in this section), and hence so is $\mathfrak{v} = H(X)^G$ as G is a finite group; therefore, by Proposition 2.65.(ii), for any open affine subset $U \subseteq X$, we have $M(U) = M(X)|_U$. It therefore suffices to prove the result for every open affine subset U of the nondegenerate locus of X, which is therefore Calabi-Yau.

Let $K < G$ be parabolic and let Z be a connected component of $\{x \in U \mid \operatorname{Stab}_G(x) = K\}$, as mentioned in the proposition. We have to show that, for $z \in Z$, $H(U)^G$ spans $T_z Z$.

Fix $z \in Z$ and $w \in T_z Z$. We will find $\xi \in H(U)^G$ such that $\xi|_z = w$. Since U is Calabi-Yau, there exists $\xi \in H(U)$ such that $\xi|_z = w$. Let $\phi \in \tilde{\Omega}_U^{\dim U - 2}$ be such that $\xi = \xi_\phi$. Let $f \in \mathcal{O}_U$ be such that $f(z) = 1$ and $f(y) = 0$ for all $y \in G \cdot z \setminus \{z\}$, and moreover such that $df|_{G \cdot z} = 0$.

Now, consider $\eta := |K|^{-1} \sum_{g \in G} g^* \xi_{f\phi} \in H(U)^G$. Then $(\xi_\psi)|_z = w$, as desired. □

Using Theorem 2.9, we immediately conclude:

Corollary 3.42. In the situation of Proposition 3.41, the coinvariants $(\mathcal{O}(X))_{H(X)^G}$ are finite-dimensional.

Note that, when X is normal and G acts by automorphisms on (X, Ξ) (preserving Ξ) with critical locus of codimension at least two, then $P(X)^G = P(X/G)$. This is because, by Hartogs' theorem, vector fields on X/G are the same as G-invariant vector fields on X, and such vector fields preserve Ξ_X if and only if they preserve $\Xi_{X/G}$. In particular, we conclude in this case that $(\mathcal{O}_{X/G})_{P(X/G)} = (\mathcal{O}(X))^G_{P(X)^G}$, and that this, as well as $(\mathcal{O}_X)_{P(X)^G}$ itself, are finite-dimensional if and only if the degenerate locus of X is finite. Moreover, $M(X/G, P(X/G)) \cong q_* M(X, P(X)^G)^G$, where $q : X \to X/G$ is the projection. We caution, however, that $H(X/G)$ and $LH(X/G)$ are in general much smaller than $P(X/G)$ (even for X Calabi-Yau), owing to the fact that G-invariant k-forms on X do not in general descend to k-forms on X/G when $k > 1$. In fact, by Theorem 3.34, $(\mathcal{O}_{X/G})_{H(X/G)}$ and $(\mathcal{O}_{X/G})_{LH(X/G)}$, as well as $(\mathcal{O}_X)_{H(X/G)}$ and $(\mathcal{O}_X)_{LH(X/G)}$, are finite-dimensional if and only if the critical locus of G is *finite* and X has a finite degenerate locus.

3.5. **Divergence functions and incompressibility.** The preceding example can be generalized to the setting of degenerate versions of *multivalued* volume forms (i.e., Calabi-Yau structures) rather than of ordinary volume forms. We formulate this in terms of *divergence functions*, which also yield an alternative definition of incompressibility (Proposition 3.52).

We assume throughout that X is irreducible and reduced. Recall the definitions of polyvector fields $T_X^\bullet$ and differential forms $\Omega_X^\bullet$ and $\tilde{\Omega}_X^\bullet$ from §2.

Definition 3.43. Let $N \subseteq T_X$ be an $\mathcal{O}_X$-submodule. A *divergence function* D on N is a morphism of sheaves of vector spaces $D : N \to \mathcal{O}_X$ satisfying $D(f\xi) = fD(\xi) + \xi(f)$ for all $\xi \in N$ and $f \in \mathcal{O}_X$. When $N = T_X$, we call this a divergence function on X.

As we will explain, divergence functions should be viewed as a degenerate, *multivalued* version of Calabi-Yau structures: they simultaneously generalize flat sections of flat connections on the canonical bundle (which includes volume forms), discussed in Example 2.38, and top polyvector fields on possibly singular schemes of finite type, discussed in §3.4.

For the latter, given (X, Ξ), we let $N \subseteq T_X$ be the submodule of $\xi \in T_X$ such that $L_\xi(\Xi)$ is a multiple of Ξ. This is a submodule in view of the identity $L_{f\xi}(\Xi) = fL_\xi(\Xi) - \xi(f) \cdot \Xi$, which can be checked in local formal coordinates where Ξ is nondegenerate (where we can take Ξ to be the inverse to the standard volume form on the formal neighborhood of the origin in affine space). Next, define D by the formula $D(\xi) \cdot \Xi = -L_\xi(\Xi)$. Note that, on the nondegenerate locus of Ξ, call it $X^\circ \subseteq X$, we have $N|_{X^\circ} = T_{X^\circ}$, since X° is Calabi-Yau.

Next, we explain how divergence functions generalize multivalued volume forms:

Proposition 3.44. If X is normal and of pure dimension n, then the following are in natural bijection:

(i) Divergence functions D on $N \subseteq T_X$;
(ii) Connections $N \times \tilde{\Omega}_X^n \to \tilde{\Omega}_X^n$ on $\tilde{\Omega}_X^n$ along N.
(iii) Connections $N \times T_X^n \to T_X^n$ on T_X^n along N.

The equivalence between (i) and (ii) is given by the correspondences, for $\xi \in N$ and $\omega \in \tilde{\Omega}^n_X$,

$$D \mapsto \nabla^D, \quad \nabla^D_\xi(\omega) = L_\xi(\omega) - D(\xi) \cdot \omega; \tag{3.45}$$

$$\nabla \mapsto D_\nabla, \quad D_\nabla(\xi) = L_\xi - \nabla_\xi \in \mathrm{End}_{\mathcal{O}_X}(\tilde{\Omega}^n_X) \cong \mathcal{O}_X. \tag{3.46}$$

The equivalence between (i) and (iii) is given by the formulas, for $\xi \in N$ and $\Xi \in T^n_X$,

$$D \mapsto \nabla^D, \quad \nabla^D_\xi(\Xi) = L_\xi(\Xi) + D(\xi) \cdot \Xi; \tag{3.47}$$

$$\nabla \mapsto D_\nabla, \quad D_\nabla(\xi) = \nabla_\xi - L_\xi \in \mathrm{End}_{\mathcal{O}_X}(T^n_X) \cong \mathcal{O}_X. \tag{3.48}$$

Finally, the constructions $D \mapsto \nabla^D$ are valid even when X is not normal.

We will need the elementary

Lemma 3.49. Suppose that X is normal and that F is a torsion-free coherent sheaf on X which is a line bundle outside of codimension two. Then $\mathrm{End}(F) = \mathcal{O}_X$.

Proof. For any $a \in \mathrm{End}(F)$, on some open subset $U \subseteq X$ where F is a line bundle and $X \setminus U$ has codimension two, $a|_U \in \mathrm{End}(F|_U) \cong \Gamma(\mathcal{O}_U)$ (the isomorphism holds because endomorphisms of any line bundle L are canonically identified with functions, via the map sending a function to the endomorphism of multiplication by that function). By normality, the resulting function extends (uniquely) to a function $f_a \in \mathcal{O}_X$ on all of X. Since $\mathcal{O}_X \subseteq \mathrm{End}(F)$, we conclude that $f_a - a \in \mathrm{End}(F)$ has zero restriction to U, and hence is zero since F is torsion-free. $\square$

Proof of Proposition 3.44. Suppose that D is a divergence function. Then $\nabla^D_\xi(f \cdot \omega) = f\nabla^D_\xi(\omega) + \xi(f) \cdot \omega$. Similarly, $\nabla^D_{f\xi}(\omega) = f\nabla^D(\omega) + \xi(f) - \xi(f) = f\nabla^D(\omega)$. We deduce that ∇^D_ξ is a connection. Similarly, if ∇ is a connection on $\tilde{\Omega}_X$, then first of all $L_\xi(f\omega) - \nabla_\xi(f\omega) = f\big(L_\xi(\omega) - \nabla_\xi(\omega)\big)$, so $D_\nabla(\xi)$ is indeed a well-defined $\mathcal{O}_X$-module endomorphism of $\tilde{\Omega}_X$. By Lemma 3.49, this is the same as an element of $\mathcal{O}_X$. Then, $D_\nabla(f\xi) = fD_\nabla(\xi) + \xi(f)$, so D_∇ is a divergence function. One immediately checks that $D_{\nabla^D} = D$ and $\nabla^{D_\nabla} = \nabla$.

The proof of the equivalence between (i) and (iii) is similar, so we omit the details. For the final statement, note that well-definition of ∇^D did not require normality. $\square$

Remark 3.50. In fact, in Proposition 3.44, we can replace T^n_X and $\tilde{\Omega}^n_X$ by any torsion-free coherent sheaves which coincide with T^n_X and $\tilde{\Omega}^n_X$, respectively, outside of codimension two; the proof then goes through unchanged.

Remark 3.51. For not necessarily normal X, but still of pure dimension n, Proposition 3.44 generalizes to give an equivalence between divergence functions of the form $D : N \to \mathrm{End}_{\mathcal{O}_X}(\tilde{\Omega}^n_X) \supseteq \mathcal{O}_X$ and connections $N \times \tilde{\Omega}^n_X \to \tilde{\Omega}^n_X$. Similarly, we obtain an equivalence between divergence functions valued in $\mathrm{End}_{\mathcal{O}_X}(T^n_X) \supseteq \mathcal{O}_X$ and connections on T^n_X along N.

Divergence functions yield the following alternative formulation of the incompressibility condition. Let $\mathcal{O}_X \cdot \mathfrak{v}$ denote the $\mathcal{O}_X$-linear span of $\mathfrak{v}$ and similarly for $\mathcal{O}_{X'}$ where X' is an open subvariety of X (we will also use this notation for formal neighborhoods, etc.).

Proposition 3.52. Let X be an arbitrary affine variety and $\mathfrak{v}$ a Lie algebra of vector fields $\mathfrak{v}$ on X. Then, the flow of $\mathfrak{v}$ along X is incompressible if and only if there exists an open dense subset $X^\circ \subseteq X$ and a divergence function on $\mathcal{O}_{X^\circ} \cdot \mathfrak{v}|_{X^\circ}$ annihilating $\mathfrak{v}|_{X^\circ}$. In this case, in the formal neighborhood of every point of X°, there exists a volume form preserved by $\mathfrak{v}$.

The proof is given in Section 3.5.3 below, after we develop some needed material. We can restate the proposition in terms of connections using the following basic result:

Proposition 3.53. In terms of Proposition 3.44, when X is normal and of pure dimension, a divergence function D on N annihilates $\mathfrak{v} \subseteq N$ if and only if $\nabla^D_\xi = L_\xi$ for all $\xi \in \mathfrak{v}$.

The proof of Proposition 3.53 is immediate from the definition of ∇^D and D_∇, and hence omitted. Using it, Proposition 3.52 becomes the following statement: when X is normal and of pure dimension, the flow of $\mathfrak{v}$ along X is incompressible if and only if there exists an open dense subset X° and a divergence function D on $\mathcal{O}_{X^\circ} \cdot \mathfrak{v}|_{X^\circ}$ such that $\nabla^D_\xi = L_\xi$ for all $\xi \in \mathfrak{v}|_{X^\circ}$.

3.5.1. *Flat divergence functions.* In terms of Proposition 3.44, we can describe what it means for a divergence function to be flat. As before, assume that X is a variety of pure dimension n. Assume that $N \subseteq T_X$ is a Lie subalgebroid.

Consider the extension of D to an operator $\tilde{D} : \wedge^\bullet_{\mathcal{O}_X} N \to \wedge^{\bullet-1}_{\mathcal{O}_X} N$ given by

$$\xi_1 \wedge \cdots \wedge \xi_k \mapsto \sum_{i=1}^{k} (-1)^{k-i} D(\xi_i) \xi_1 \wedge \cdots \wedge \hat{\xi}_i \wedge \cdots \wedge \xi_k + \sum_{i,j} (-1)^{i+j-1} [\xi_i, \xi_j] \wedge \xi_1 \wedge \cdots \wedge \hat{\xi}_i \wedge \cdots \wedge \hat{\xi}_j \wedge \cdots \wedge \xi_k. \tag{3.54}$$

Note that, since we take the exterior algebra over $\mathcal{O}_X$, one must check that the formula is well-defined, i.e., that one obtains the same result if we multiply ξ_i by f as if we multiply ξ_j by f, for all $i < j$ and all $f \in \mathcal{O}_X$. This is easy to check.

Definition 3.55. Call a divergence function D flat if the associated operator (3.54) has square zero: $\tilde{D}^2 = 0$.

Example 3.56. Suppose that $N = T_X$ and X is smooth. Then we can replace (3.54) with

$$\Omega_X^{\dim X - \bullet} \otimes_{\mathcal{O}_X} T_X^n, \tag{3.57}$$

equipped with the derivation $d_D = d \otimes \mathrm{Id} + \mathrm{Id} \otimes \bar{\nabla}^D$, where $\bar{\nabla}^D : T^n_X \to \Omega^1_X \otimes_{\mathcal{O}_X} T^n_X$ is the usual $\mathbf{k}$-linear operator associated to the connection ∇^D. This is isomorphic to (3.54) by contracting $\Omega^\bullet$ with T^n_X. Thus, $d_D^2 = 0$ if and only if D is flat.

Example 3.58. More generally than Example (3.56), suppose that X is smooth and N is locally free of rank $n - k$ and the vanishing locus of a collection of (linearly independent) one-forms $df_1, \ldots, df_k$. Then, we can consider the k-form $\alpha = df_1 \wedge \cdots \wedge df_k$, and replace (3.54) by

$$(\Omega_X^{\dim X - k - \bullet} \wedge \alpha) \otimes_{\mathcal{O}_X} T^n_X. \tag{3.59}$$

This is equipped with the derivation d_D defined as before, and with this derivation, the contraction map produces an isomorphism of (3.59) with (3.54). Thus, it remains true that $d_D^2 = 0$ if and only if D is flat. Moreover, by Frobenius's theorem, in a formal neighborhood of a smooth point $x \in X$, such $f_1, \ldots, f_k$ always exist since N is integrable.

Proposition 3.60. Let $D : N \to \mathcal{O}_X$ be a divergence function with $N \subseteq T_X$ a Lie subalgebroid, and let $\mathfrak{v} := \{\xi \in N \mid D(\xi) = 0\}$. Suppose moreover that $N = \mathcal{O}_X \cdot \mathfrak{v}$. Then D is generically flat if and only if $\mathfrak{v}$ is a Lie algebra.

By generically flat, we mean that, restricted to an open dense subset of X, D is flat. Note that the condition $N = \mathcal{O}_X \cdot \mathfrak{v}$ is automatic if we replace X with a formal neighborhood $\hat{X}_x$ for generic $x \in X$ and define $\mathfrak{v} \subseteq T_{\hat{X}_x}$ as above, since N is integrable, so we can write $\hat{X}_x \cong V \times V'$ for formal polydiscs V, V' such that N identifies with the subsheaf of $T_{\hat{X}_x}$ in the V direction.

Proof. First, if D is generically flat, then on some open dense subset of X, given any $\xi, \eta \in \mathfrak{v}$, we have $\tilde{D}^2(\xi \wedge \eta) = 0$ (since $D(\xi) = 0 = D(\eta)$), which implies that $[\xi, \eta] \in \mathfrak{v}$ as well.

Consider now the reverse implication. It suffices to restrict to a formal neighborhood of a smooth point $x \in X$ (on each connected component of X). Then, as noted in Example 3.58, we can assume N is the vanishing locus of k nonvanishing one-forms $df_1, \ldots, df_k$. Set $\alpha = df_1 \wedge \cdots \wedge df_k$ and replace (3.54) by (3.59). By Proposition 3.53, $\mathfrak{v}$ consists of those $\xi \in N$ such that $\nabla^D_\xi = L_\xi$ on $\Omega^n_{\hat{X}_x}$, or equivalently on $(\Omega^{n-k}_{\hat{X}_x} \wedge \alpha)$.

Assume that $\mathfrak{v}$ is a Lie algebra. Then, for $\xi, \eta \in \mathfrak{v}$,

$$[\nabla^D_\xi, \nabla^D_\eta] = [L_\xi, L_\eta] = L_{[\xi,\eta]} = \nabla^D_{[\xi,\eta]}.$$

Note that this also implies that $[\nabla^D_\xi, \nabla^D_\eta] = \nabla^D_{[\xi,\eta]}$ for all $\xi, \eta \in \hat{\mathcal{O}}_{X,x} \cdot N$, since this equality remains true when replacing ξ by $f \cdot \xi$ for $f \in \hat{\mathcal{O}}_{X,x}$, and it is biadditive in ξ and η. Since $N = \mathcal{O}_X \cdot \mathfrak{v}$, and hence $N|_{\hat{X}_x} = \hat{\mathcal{O}}_{X,x} \cdot \mathfrak{v}|_{\hat{X}_x}$, the equality holds for all $\xi, \eta \in \hat{\mathcal{O}}_{X,x}$. Now, the identity $[\nabla^D_\xi, \nabla^D_\eta] = \nabla^D_{[\xi,\eta]}$ on $\Omega^n_{\hat{X}_x}$ implies in the standard way that the derivation d_D on $(\Omega^{\dim X - k - \bullet}_{\hat{X}_x} \wedge \alpha) \otimes_{\mathcal{O}_X} T^n_{\hat{X}_x}$ has square zero. Namely, one can verify that d_D^2 is given by contraction with the two-form α given by

$$\alpha(\xi \wedge \eta) = [\nabla^D_\xi, \nabla^D_\eta] - \nabla^D_{[\xi,\eta]} \in \mathrm{End}(T^n_{\hat{X}_x}) = \hat{\mathcal{O}}_{X,x}. \qquad \square$$

3.5.2. *Hamiltonian vector fields on varieties with flat divergence functions.* Now we define, analogously to §3.4, Hamiltonian and incompressible vector fields preserving flat divergence functions (i.e., preserving the formal volumes associated to them).

Let X be a variety of pure dimension n and $N \subseteq T_X$ an $\mathcal{O}_X$-submodule, and $D : N \to \mathcal{O}_X$ be a flat divergence function. Then first we have the Lie algebra $P(X, D) \subseteq N$ of all incompressible vector fields in N. Note that the $\mathcal{O}_X$-linear span of $P(X, D)$ need not be all of N.

Next, given any element $\tau \in \wedge^2_{\mathcal{O}_X} N$, consider the image $\theta_\tau := \tilde{D}(\tau) \in N$. By construction, $D(\theta_\tau) = 0$. We call θ_τ the Hamiltonian vector field of τ. Since $[\theta_\tau, \theta_{\tau'}] = \theta_{L_{\theta_\tau}(\tau')}$, these form a Lie subalgebra of $P(X, D)$,

$$H(X, D) := \langle \theta_\tau \rangle \subseteq P(X, D).$$

Example 3.61. If X is Calabi-Yau and D is the associated divergence function, we again recover $H(X, D) = P(X, D) = H(X)$, the Lie algebra of volume-preserving vector fields.

As long as N has rank at least two, then $H(X, D)$ has enough vector fields, in the sense that $\mathcal{O}_X \cdot H(X, D) = N$; more precisely:

Proposition 3.62. Suppose that the image of N at the tangent fiber $T_x X$ has dimension at least two. Then $H(X, D)|_x = N|_x$, i.e., $H(X, D) \subseteq N$ spans the same tangent space at x as N. In particular, if $N = T_X$ and X has pure dimension at least two, then $H(X, D)$ is transitive.

As a consequence, the same result holds for $P(X, D) \supseteq H(X, D)$.

Proof. Let $x \in X$ be a point, and $\xi, \eta \in N$ two vector fields linearly independent at x. Let $f \in \mathcal{O}_X$ be a function such that $\xi(df)(x) = 1$ and $\eta(df)(x) = 0$. Then $\big(\tilde{D}(f\xi \wedge \eta) - f\tilde{D}(\xi \wedge \eta)\big)|_x = \eta|_x$. $\square$

On the other hand, if N has rank one, then $P(X, D)$ can be zero, e.g., when X is a smooth curve and D is a divergence function preserving a multivalued volume form which is not single valued (cf. Example 2.38).

Example 3.63. Consider the case of Example 3.58, i.e., where N is locally free of rank $n - k$ and the zero locus of (linearly independent) exact one-forms $df_1, \ldots, df_k$. Set $\alpha := df_1 \wedge \cdots \wedge df_k$ and replace (3.54) by (3.59). Given any element $\beta \in (\Omega_X^{n-k-2} \wedge \alpha) \otimes T_X^n$, we can define the Hamiltonian vector field

$$\xi_\beta = \operatorname{ctr}(\nabla^D(\beta)),$$

where ctr is the operator

$$\operatorname{ctr} : \tilde{\Omega}_X^\bullet \otimes_{\mathcal{O}_X} T_X^n \to T_X^{n-\bullet}, \operatorname{ctr}(\omega \otimes \tau) = i_\tau(\omega).$$

These vector fields coincide with $H(X, D)$ as defined above, since (3.59) is isomorphic to (3.54) via the contraction operation.

Next, call an element of $(\Omega_X^{n-k-\bullet} \wedge \alpha) \otimes_{\mathcal{O}_X} T_X^n$ ∇^D-closed if it is in the kernel of ∇^D. Then, if $\gamma \in (\Omega_X^{n-k-1} \wedge \alpha) \otimes T_X^n$ is ∇^D-closed, we can define the locally Hamiltonian vector field

$$\eta_\gamma := \operatorname{ctr}(\gamma).$$

These vector fields coincide with $P(X, D)$ as defined above, since via the contraction isomorphism of complexes (3.59) and (3.54), the vector fields η_γ are precisely those elements of N with zero divergence.

Example 3.64. Suppose (X, Ξ) is a variety of pure dimension n equipped with a generically nonvanishing top polyvector field Ξ as in §3.4, and define N and D as at the beginning of §3.5. Then we see immediately that $P(X) = P(X, D)$, consisting of the vector fields ξ such that $L_\xi \Xi = 0$.

3.5.3. *Proof of Proposition 3.52.* We can assume that $X = X^\circ$ is smooth and that $\mathfrak{v}$ has constant (i.e., maximal) rank. Therefore $\Omega_X = \tilde{\Omega}_X$, and we omit the tilde from now on. We show that $\mathfrak{v}$ flows incompressibly on X if and only if there exists a connection ∇ on Ω_X along $N := \mathcal{O}_X \cdot \mathfrak{v}$ such that $\nabla_\xi = L_\xi$ for all $\xi \in \mathfrak{v}$.

First, suppose that $\mathfrak{v}$ flows incompressibly on X. Let $x \in X$ be a point and $\omega \in \Omega_{\hat{X}_x}$ a formal volume form preserved by $\mathfrak{v}$. Let ∇ be the unique flat connection whose flat sections are multiples of ω. Then $\nabla_\xi \omega = 0 = L_\xi \omega$ for all $\xi \in \mathfrak{v}$. Therefore, the restriction of ∇ to N is as desired.

Conversely, suppose that ∇ is a connection on Ω_X along N such that $\nabla_\xi = L_\xi$ for all $\xi \in \mathfrak{v}$. Since $\mathfrak{v}$ is a Lie algebra, Proposition 3.60 implies that ∇ is generically flat. Thus, at a generic point $x \in X$, $\hat{N}_x$ is free over $\hat{\mathcal{O}}_{X,x}$, and we can write $T_{\hat{X}_x} = \hat{N}_x \oplus L$ for some complementary free $\hat{\mathcal{O}}_{X,x}$-submodule L. Then the connection ∇ can be extended to a flat connection on $T_{\hat{X}_x}$. Let $\omega \in \Omega_{\hat{X}_x}$ be a nonzero flat formal section of ∇. Then $\nabla_\xi(\omega) = 0$ for all $\xi \in T_X$. Hence $L_\xi(\omega) = 0$ for all $\xi \in \mathfrak{v}$. Therefore, ω is preserved by $\mathfrak{v}$.

3.6. **Smooth curves.** Let X be a smooth connected curve. In this section we explicitly compute $M(X, \mathfrak{v})$. We may assume that $\mathfrak{v}$ is nonzero. Let $Z \subseteq X$ be the vanishing locus of $\mathfrak{v}$, which is zero-dimensional. Let $X^\circ := (X \setminus Z) \subseteq X$ be the complement.

Lemma 3.65. If $\mathfrak{v}$ is one-dimensional, then $M(X, \mathfrak{v})|_{X^\circ} = \Omega_{X^\circ}$. Otherwise, $M(X, \mathfrak{v})|_{X^\circ} = 0$.

Proof. By our assumptions, $\mathfrak{v}|_{X^\circ}$ is transitive. Moreover, if $\mathfrak{v}$ is one-dimensional, then any nonzero element $\xi \in \mathfrak{v}$ is a top polyvector field on X vanishing on Z, so ξ^{-1} defines a nondegenerate volume form on X° preserved by $\mathfrak{v}$. Therefore we conclude that $M(X^\circ, \mathfrak{v}) \cong \Omega_{X^\circ}$ by Proposition 2.36. On the other hand, if $\mathfrak{v}$ is at least two-dimensional, then if $\xi_1, \xi_2 \in \mathfrak{v}$ are linearly independent, then on some open subset $U \subseteq X^\circ$, ξ_1^{-1} and ξ_2^{-1} both define nondegenerate volume forms which are not scalar multiples of each other. Then there can be no volume form on U preserved by both, even restricted to $\hat{U}_x$ for every $x \in U$. □

Proposition 3.66. If $\dim \mathfrak{v} \geq 2$, then $M(X, \mathfrak{v}) \cong \bigoplus_{z \in Z} \delta_z \otimes (\hat{\mathcal{O}}_{X,z})_{\mathfrak{v}}$. Moreover, $\dim(\hat{\mathcal{O}}_{X,z})_{\mathfrak{v}}$ is the minimum order of vanishing of vector fields of $\mathfrak{v}$ at z.

Note in particular that each $\dim(\hat{\mathcal{O}}_{X,z})_{\mathfrak{v}}$ is positive.

Proof. By the lemma, we immediately conclude that $M(X, \mathfrak{v})$ is a direct sum of copies of delta-function $\mathcal{D}$-modules at points of Z, which is finite. Then, the result follows from the fact that

$$\mathrm{Hom}(M(X, \mathfrak{v}), \delta_z) = \mathrm{Hom}(\mathcal{D}_X, \delta_z)^{\mathfrak{v}} = ((\hat{\mathcal{O}}_{X,z})^*)^{\mathfrak{v}}. \tag{3.67}$$

□

Now, assume that $\mathfrak{v} = \langle \xi \rangle$, so that $M(X, \mathfrak{v}) = \xi \cdot \mathcal{D}_X \setminus \mathcal{D}_X$ for $\xi \in \mathrm{Vect}(X)$. Then, by the lemma and the argument of the proposition, we have an exact sequence

$$0 \to j_! \Omega_{X^\circ} = \Omega_X \to M(X, \mathfrak{v}) \to \bigoplus_{z \in Z} \delta_z \otimes (\hat{\mathcal{O}}_{X,z})_{\mathfrak{v}} \to 0. \tag{3.68}$$

It turns out that this sequence is maximally nonsplit. Namely, at each $z \in Z$, $\mathrm{Ext}(\delta_z, \Omega_X) = \mathbf{k}$, since X is a smooth curve.

Proposition 3.69. When $\mathfrak{v}$ is one-dimensional, then $M(X, \mathfrak{v}) = N \oplus \bigoplus_{z \in Z} \delta_z \otimes (\hat{\mathcal{O}}_{X,z})_{\mathfrak{v}} / \mathbf{k}$, where N is an indecomposable $\mathcal{D}$-module fitting into an exact sequence

$$0 \to j_! \Omega_{X^\circ} = \Omega_X \to N \to \bigoplus_{z \in Z} \delta_z \to 0. \tag{3.70}$$

As before, $\dim(\hat{\mathcal{O}}_{X,z})_{\mathfrak{v}}$ is the minimum order of vanishing of vector fields of $\mathfrak{v}$ at z.

Proof. By formally localizing at $z \in Z$, it is enough to assume that $\mathfrak{v} = \langle x^k \partial_x \rangle$ for $\mathbf{A}^1 = \mathsf{Spec}\, \mathbf{k}[x]$ and $k \geq 1$. In this case, it suffices to prove that

$$\mathrm{Hom}(\mathcal{D}_{\mathbf{A}^1} / x^k \partial_x \cdot \mathcal{D}_{\mathbf{A}^1}, \Omega_{\mathbf{A}^1}) = 0.$$

But, no volume form on $\mathbf{A}^1$ is annihilated by $L_{x^k \partial_x}$ (even in a formal neighborhood of zero): the rational volume form annihilated by $L_{x^k \partial_x}$ is $x^{-k} dx$. The last statement follows as in the previous proof. □

3.7. **Finite maps.** Let $f : X \to Y$ be a finite surjective map of affine varieties. In this section we explain how to construct more examples using finite maps, which generalizes the aforementioned Lie algebras of Hamiltonian vector fields of Hamiltonians pulled back from Y. We will not need the material of this section for the remainder of the paper.

Definition 3.71. Let $\mathrm{Vect}_X(Y) \subseteq \mathrm{Vect}(Y)$ be the subspace of vector fields ξ on Y such that there exists a vector field $f^* \xi$ on X such that $f_*(f^* \xi|_x) = \xi|_{f(x)}$ for all $x \in X$.

Algebraically, $\mathrm{Vect}_X(Y)$ consists of the derivations of $\mathcal{O}_Y$ which extend to derivations of $\mathcal{O}_X$.

Since f is finite and X and Y are reduced, it is generically a covering map. Therefore, when $f^* \xi$ exists, it is unique.

Example 3.72. If X is a normal variety and the critical locus of f has codimension at least two, then by Hartogs' theorem, vector fields on X outside the singular and critical locus extend to all of X. Therefore, $\mathrm{Vect}_X(Y) = \mathrm{Vect}(Y)$, since f is a covering map when restricted to this latter locus.

Suppose that X and Y are varieties and $\mathfrak{v}_Y \subseteq \mathrm{Vect}_X(Y)$. Let $\mathfrak{v}_X := f^* \mathfrak{v}_Y$.

Proposition 3.73. (i) $(X, \mathfrak{v}_X)$ has finitely many leaves if and only if $(Y, \mathfrak{v}_Y)$ does.
(ii) $(X, \mathfrak{v}_X)$ has finitely many incompressible leaves of and only if $(Y, \mathfrak{v}_Y)$ does.
(iii) $(X, \mathfrak{v}_X)$ has finitely many zero-dimensional leaves if and only if $(Y, \mathfrak{v}_Y)$ does.

Proof. Restricted to any invariant subvariety $Z \subseteq X$, f is still finite and therefore generically a covering map. This reduced the statement to the case where f is a covering map of smooth varieties. Then, the statements (i)–(ii) follow from the basic facts that (i) X is generically transitive if and only if Y is; (ii) X is incompressible if and only if Y is. Statement (iii) follows from the fact that f restricts to a finite map from the vanishing locus of $\mathfrak{v}_X$ onto the vanishing locus of $\mathfrak{v}_Y$. □

Example 3.74. In the situation of Example 3.72, X has finitely many leaves under the flow of all vector fields if and only if the same is true of Y, and X has finitely many exceptional points if and only if Y does. Thus, $(\mathcal{O}_X)_{\mathrm{Vect}(X)}$ is finite-dimensional if and only if $(\mathcal{O}_Y)_{\mathrm{Vect}(Y)}$ is.

Example 3.75. If $f : X \to Y$ is a finite Poisson map of varieties with finitely many symplectic leaves and X is normal, one recovers the observation at the end of §3.2 in the setting of Poisson maps (note that the critical locus of f is automatically of codimension at least two, since f is nondegenerate over the open leaves of Y). Thus, one recovers [ES10, Theorem 3.1] in this setting, i.e., that $f^*H(Y)$ is holonomic (similarly one obtains that $f^*LH(Y)$ and $f^*P(Y)$ are holonomic). Here, we only used the conditions that X is normal and Y has finitely many symplectic leaves to assure that $H(Y) \subseteq \mathrm{Vect}_Y(X)$; to drop these assumptions, one can observe that $H(Y) \subseteq \mathrm{Vect}_Y(X)$, since $f^*\xi_h = \xi_{f^*h}$ (which also allows one to drop the condition that Y is Poisson altogether); similarly we can conclude in this setting that $LH(Y) \subseteq \mathrm{Vect}_Y(X)$.

Example 3.76. Suppose $f : X \to Y$ is a finite map of varieties equipped with top polyvector fields Ξ_X and Ξ_Y such that $f_*(\Xi_X|_x) = \Xi_Y|_{f(x)}$ for all $x \in X$ (an "incompressible" finite map). If X is normal, Ξ_Y has a finite degenerate locus, and the dimension of X is at least two, one concludes that $f^*H(Y)$ is holonomic (as well as $f^*LH(Y)$ and $f^*P(Y)$), and hence that $(\mathcal{O}_X)_{f^*H(Y)}$ is finite-dimensional; this recovers an observation at the end of §3.4 in a special case. As in the previous remark, we can drop the assumptions that X is normal and Ξ_Y has a finite degenerate locus, since those were only used to show that $H(Y) \subseteq \mathrm{Vect}_Y(X)$, but this is automatic since we can pull back closed $(n-1)$-forms from Y to X (this also applies to $LH(Y)$, but not necessarily to $P(Y)$).

In the case $Y = X/G$ where G is a finite group acting on (X, Ξ_X), one similarly recovers the observation from the end of §3.4, that $(\mathcal{O}_{X/G})_{P(X/G)} = (\mathcal{O}_X)^G_{f^*P(X/G)} = (\mathcal{O}_X)^G_{P(X)^G}$, as well as $(\mathcal{O}_X)_{P(X)^G}$, are finite-dimensional if and only if Ξ_X has a finite degenerate locus, i.e., if and only if $(\mathcal{O}_X)_{P(X)}$ is finite-dimensional.

4. Globalization and Poisson vector fields

4.1. **Hamiltonian vector fields are $\mathcal{D}$-localizable.** In order to prove that our main examples are $\mathcal{D}$-localizable (for all vector fields and Hamiltonian vector fields), we prove the following more general result, which roughly states that a Lie algebra of vector fields generated by a coherent sheaf E of "potentials" is $\mathcal{D}$-localizable (in the Poisson case with $\mathfrak{v} = H(X)$, or in the case $\mathfrak{v} = \mathrm{Vect}(X)$, $E = \mathcal{O}_X$, as we will explain):

Theorem 4.1. Let E be a coherent sheaf on an affine variety X equipped with a map $v : E \to T_X$ of $\mathbf{k}$-linear sheaves, such that, for all $e \in E$, the bilinear map

$$\pi_e(f, g) := v(f \cdot e)(g) - f \cdot v(e)(g)$$

defines a skew-symmetric biderivation $\mathcal{O}_X^{\otimes 2} \to \mathcal{O}_X$. Then (the Lie algebra generated by) $v(E)$ is $\mathcal{D}$-localizable.

The condition of the theorem can alternatively be stated as: $v : E \to T_X$ is a differential operator of order ≤ 1 whose principal symbol $\sigma(v) : E \to T_X \otimes T_X$ is skew-symmetric.

Proof. Let $X \subseteq \mathbf{A}^n$ be an embedding into affine space, and let $x_1, \ldots, x_n$ be the coordinate functions on $\mathbf{A}^n$. Let $U \subseteq X$ be an open affine subset. We need to show that, for every $g \in \mathcal{O}_U$ and $e \in E(X)$, then $v(g \cdot e) \in v(E(X)) \cdot \mathcal{D}_U$. Let $V \subseteq \mathbf{A}^n$ be an open affine subset such that $V \cap X = U$. We claim that, in $\mathcal{D}_U$, for all $f \in \mathcal{O}_V$,

$$v(f \cdot e) = v(e) \cdot f + \sum_{i=1}^{n} \big(v(x_i \cdot e) - v(e) \cdot x_i\big) \cdot \frac{\partial f}{\partial x_i}, \tag{4.2}$$

which immediately implies the statement. To prove (4.2), we first rewrite it (putting vector fields on the left-hand side and functions on the right hand side) as

$$(v(f\cdot e)-f\cdot v(e))-\sum_{i=1}^n \frac{\partial f}{\partial x_i}\big(v(x_i\cdot e)-x_i v(e)\big) = v(e)(f)+\sum_{i=1}^n\big(-\frac{\partial f}{\partial x_i}v(e)(x_i)+(v(x_i\cdot e)-x_i v(e))(\frac{\partial f}{\partial x_i})\big).$$

So the statement is equivalent to showing that both sides of the above desired equality are zero. For the LHS, this follows from the fact that, for fixed $e \in E$, the map $f \mapsto v(f\cdot e) - f\cdot v(e)$ is a derivation of f; in more detail, this implies that this is obtained from a linear map $\Omega^1 \to T_X, df \mapsto v(f\cdot e) - f\cdot v(e)$, and then we write $df = \sum_i \frac{\partial f}{\partial x_i} dx_i$. For the RHS, the fact that $v(e) \in T_X$ is a derivation implies that $v(e)(f) + \sum_{i=1}^n -\frac{\partial f}{\partial x_i} v(e)(x_i) = 0$, just as before. It remains to show that

$$\sum_{i=1}^n \big(v(x_i\cdot e) - x_i\cdot v(e)\big)\big(\frac{\partial f}{\partial x_i}\big) = 0.$$

Using the definition of π_e, we can rewrite the LHS of this expression as

$$\sum_i \pi_e\big(x_i, \frac{\partial f}{\partial x_i}\big).$$

Now, viewing π_e as a bivector field (i.e., a skew-symmetric biderivation), this can be rewritten as

$$\sum_i \pi_e\big(dx_i \wedge d(\frac{\partial f}{\partial x_i})\big) = \pi_e d(df) = 0. \qquad \square$$

Corollary 4.3. $(X, \mathrm{Vect}(X))$ is $\mathcal{D}$-localizable. More generally, if $E \subseteq \mathrm{Vect}(X)$ is a coherent subsheaf, then (the Lie algebra generated by) E is $\mathcal{D}$-localizable.

Proof. Take $v = \mathrm{Id}$ in the theorem. $\square$

Corollary 4.4. Let X be either Poisson, Jacobi, or equipped with a top polyvector field. Then the presheaf $\mathcal{H}(X)$ of Hamiltonian vector fields is $\mathcal{D}$-localizable. Moreover, in the Poisson and top polyvector field cases, the presheaf $\mathcal{LH}(X)$ of locally Hamiltonian vector fields is also $\mathcal{D}$-localizable, and defines the same $\mathcal{D}$-module.

Similarly, when X is equipped with a coherent subsheaf $N \subseteq \mathcal{T}_X$ and a divergence function $D : N \to \mathcal{O}_X$, then the presheaf $\mathcal{H}(X, D)$ is $\mathcal{D}$-localizable, setting $E := \wedge^2_{\mathcal{O}_X} N$.

Proof. In the Poisson and Jacobi cases, we can take $E = \mathcal{O}_X$ and $v(f) = \xi_f$. Then it is easy to check that π_e is a skew-symmetric biderivation for all $e \in E$, so the theorem implies that $\mathcal{H}_X$ is $\mathcal{D}$-localizable. In the case of a top polyvector field Ξ, we take $E = \tilde{\Omega}_X^{n-2}$ and again let $v(\alpha) = \xi_\alpha = \Xi(d\alpha)$.

For the second statement, it suffices to recall from Propositions 3.13 and 3.31 that, in the Poisson and top polyvector field cases, $H(X)\cdot\mathcal{O}_X = LH(X)\cdot\mathcal{O}_X$ for all affine X.

The final statement follows in the same manner. $\square$

On the other hand, $P(X)$ need not be $\mathcal{D}$-localizable: see §4.2 for a detailed discussion.

Remark 4.5. We note that, in general, $\mathcal{H}_X$ is *not* a sheaf, and neither is $\mathcal{LH}_X$.

For an example where $\mathcal{H}_X$ and $\mathcal{LH}_X$ are not sheaves, let X be the complement in $\mathbf{A}^3$ of the plane $x + y = 0$, equipped with the Poisson structure given by the potential $f(x, y, z) = \frac{xy}{x+y}$, i.e.,

$$\{x, y\} = 0, \{y, z\} = \frac{y^2}{(x+y)^2}, \{z, x\} = \frac{x^2}{(x+y)^2}.$$

Consider the vector field $\xi := (x + y)^{-2}\partial_z$. This is regular, and on the open set where $x \neq 0$, it is the Hamiltonian vector field of x^{-1}, and on the open set where $y \neq 0$, it is the Hamiltonian vector field of $-y^{-1}$. But it is not globally Hamiltonian, since if $\xi = \xi_f$ for some $f \in \mathcal{O}_X$, then we would have $f = x^{-1} + C$ for some Casimir function C regular on the locus $x \neq 0$ (recall that a Casimir function means a function that is Poisson central). Then, on the locus $y \neq 0$, we would obtain that $g := x^{-1} + C + y^{-1}$ is a Casimir function. But then, if h is any regular function such that $\{y, h\}$ does not vanish along $y = 0$, we would conclude that $\{g, h\}$ has a pole along $y = 0$, which is impossible since it must be zero.

The same argument shows that ξ is not given by a global one-form: in this case, writing the global one-form as $d(x^{-1}) + \beta$ for $\eta_\beta = 0$, we would again conclude that, for any regular h such that $\{y, h\}$ does not vanish along $y = 0$, the function $\eta_{d(x^{-1})+\beta}(h) = -\{y^{-1}, h\}$ has no pole at $y = 0$, a contradiction.

On the other hand, in the case that X is generically symplectic, it follows that $\mathcal{H}_X$ and $\mathcal{LH}_X$ are sheaves, since in this case any vector field which is Hamiltonian in some neighborhood must be given by a unique Hamiltonian function up to locally constant functions, and this is then defined and Hamiltonian on the regular locus of that function (and similarly in the locally Hamiltonian case).

Note similarly that, in the case of a variety with a top polyvector field Ξ, $\mathcal{H}_X$ and $\mathcal{LH}_X$ are sheaves, since if Ξ is nonzero, then on its nonvanishing locus a Hamiltonian vector field is once again given by a unique Hamiltonian.

Remark 4.6. In the examples above, the presheaves also are equipped naturally with spaces of sections on formal neighborhoods $\hat{X}_x$ of every point $x \in X$; the presheaf condition requires only that these contain the restrictions of sections on open subsets containing x. Thus it makes sense to define the notion of *formal $\mathcal{D}$-localizability*, i.e., that, for every open affine U and $x \in U$,

$$\mathfrak{v}(\hat{X}_x)\mathcal{D}_{\hat{X}_x} = \mathfrak{v}(U)|_{\hat{X}_x}\mathcal{D}_{\hat{X}_x}. \tag{4.7}$$

Formal localizability implies usual localizability: indeed, if $\mathfrak{v}$ is formally localizable, and $\xi \in \mathfrak{v}(U')$ for some $U' \subseteq U$, then at every $x \in U'$, it follows that $\xi|_{\hat{X}_x} \in \mathfrak{v}(U) \cdot \mathcal{D}_{\hat{X}_x}$, and hence $\xi \in \mathfrak{v}(U) \cdot \mathcal{D}_U$, by Lemma 2.64.

Theorem 4.1 extends to show that, under the assumptions there, $\mathfrak{v}$ is formally $\mathcal{D}$-localizable, by formally localizing the embedding $X \to \mathbf{A}^n$ to $\hat{X}_x \to \hat{\mathbf{A}}^n_0$. Then, the same proof applies. We conclude as before that the presheaves of (locally) Hamiltonian vector fields are formally $\mathcal{D}$-localizable, as well as $\mathrm{Vect}(X)$ and all coherent subsheaves thereof.

4.2. **$\mathcal{D}$-localizability of Poisson vector fields.** An interesting question raised in the previous subsection is whether $P(X)$ is $\mathcal{D}$-localizable. This turns out to have an interesting answer, which we discuss here. The material of this subsection will not be needed for the rest of the paper, and our motivation is partly to illustrate the nontriviality of $\mathcal{D}$-localizability. We will first give the statements and examples, and postpone the proofs of the propositions to the end of the subsection, for the purpose of emphasizing the statements and counterexamples to their generalization.

Proposition 4.8. Let X be an irreducible affine Poisson variety on which $P(X)$ flows incompressibly. If $P(X)$ is $\mathcal{D}$-localizable, then the generic rank of $P(X)$ must equal that of $P(U)$ for every open subset $U \subseteq X$.

Conversely, suppose that X is a smooth affine Poisson variety on which the rank of $P(U)$ equals that of $H(U)$ everywhere, for all affine open $U \subseteq X$, and that this rank is constant on X. Then, for all affine open $U \subseteq X$, one has an equality of $\mathcal{O}$-saturations $P(U)^{os} = H(U)^{os}$. Hence, $P(X)$ is $\mathcal{D}$-localizable.

The assertion of the second paragraph follows from the more general

Lemma 4.9. Suppose $\mathfrak{v} \subseteq \mathfrak{w}$ is an inclusion of Lie algebras of vector fields on a smooth affine variety X. Suppose that the rank of $\mathfrak{v}$ is constant and equals that of $\mathfrak{w}$ everywhere, and moreover that $\mathfrak{w}$ flows incompressibly. Then $\mathfrak{v}^{os} = \mathfrak{w}^{os}$. In particular, $M(X, \mathfrak{v}) = M(X, \mathfrak{w})$.

Proof. Since the ranks of $\mathfrak{v}$ and $\mathfrak{w}$ are constant and equal, we conclude that, for every $x \in X$, there exists an open subset $U \subseteq X$ such that $\mathcal{O}_U \cdot \mathfrak{v}|_U = \mathcal{O}_U \cdot \mathfrak{w}$, and hence in fact $\mathcal{O}_X \cdot \mathfrak{v} = \mathcal{O}_X \cdot \mathfrak{w}$. Now, if $\mathfrak{w}$ flows incompressibly, and hence also $\mathfrak{v}$, then $\mathfrak{v}^{os} = \mathfrak{w}^{os} =$ the subspace of $\mathcal{O}_X \cdot \mathfrak{v}$ of incompressible vector fields, by Proposition 2.53. □

Remark 4.10. Lemma 4.9 generalizes to affine schemes of finite type, if we replace the rank condition by the condition that $\mathcal{O}_X \cdot \mathfrak{v} = \mathcal{O}_X \cdot \mathfrak{w}$.

We also give a localizability result that does not require X to be smooth, in the situation of Remark 3.12, where $P(X) = LH(X^\circ)$ for X° the smooth locus of X.

Notation 4.11. Given any not necessarily affine scheme Y, we will let $H^\bullet_{DR}(Y) := H^\bullet(\Gamma(\tilde{\Omega}_Y))$ denote the cohomology of the complex of global sections of de Rham differential forms modulo torsion.

For all $x \in X$, Let $\mathcal{O}_{X,x}$ be the *uncompleted* local ring of X at x.

Proposition 4.12. Suppose X is Poisson, normal, and symplectic on its smooth locus. Let S be its singular locus.

(i) For every $s \in S$, let $E_s \subseteq S$ be the union of all irreducible components of S containing s. Suppose that, for all $s \in S$, the natural map

$$H^1_{DR}(X \setminus E_s) \oplus H^1_{DR}(\mathsf{Spec}\, \mathcal{O}_{X,s}) \to H^1_{DR}(\mathsf{Spec}\, \mathcal{O}_{X,s} \setminus E_s) \tag{4.13}$$

is surjective. Then, $P(X)$ is $\mathcal{D}$-localizable. Moreover, for all $s \in S$,

$$P(\mathsf{Spec}\, \mathcal{O}_{X,s}) = P(X)|_{\mathsf{Spec}\, \mathcal{O}_{X,s}} \cdot \mathcal{O}_{X,s} + LH(\mathsf{Spec}\, \mathcal{O}_{X,s}). \tag{4.14}$$

(ii) Now suppose that S is finite and $\mathbf{k} = \mathbf{C}$. Then the hypothesis of (i) is satisfied if, for all $s \in S$ and all affine Zariski open neighborhoods U of s, the natural map on *topological* cohomology,

$$H^1_{\text{top}}(X \setminus \{s\}) \oplus H^1_{\text{top}}(U) \to H^1_{\text{top}}(U \setminus \{s\}) \tag{4.15}$$

is surjective. In particular, in this case, $P(X)$ is $\mathcal{D}$-localizable, and (4.14) holds.

Example 4.16. When X has a contracting $\mathbf{G}_m$ action (where this is the multiplicative group), i.e., $\mathcal{O}_X$ is nonnegatively graded with $\mathbf{k}$ in degree zero, then $H^\bullet(X) = \mathbf{k}$, and in particular $H^2(X) = 0$. Therefore, in this case, when X also is normal, generically symplectic, and has an isolated singularity at the fixed point for the contracting action, the conditions of the proposition are satisfied, so $P(X)$ is $\mathcal{D}$-localizable. Also, in this case, $P(U) = P(X)|_U + LH(U)$ for all open sets (and for those U which don't contain the singularity we have $P(U) = H(U)$, since then U is symplectic).

For such an example where $P(U)/LH(U)$ is nonzero, let X be the locus $x^3 + y^3 + z^3 = 0$ (or a more general elliptic singularity); then $P(U)/LH(U)$ is generated by the Euler vector field in $P(X)$ for all open affine U.

Example 4.17. Here is a simple example of a non-normal X for which $P(X)$ is not $\mathcal{D}$-localizable: Suppose $X = \mathsf{Spec}\, \mathbf{k}[x^2, x^3, y, xy]$ and $\{x, y\} = y$. This is generically symplectic but not normal. Then we claim that every global Poisson vector field vanishes at $y = 0$. Indeed, $\xi = f\partial_x + g\partial_y$ is Poisson if and only if $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = \frac{g}{y}$. Writing $g = yh$, we obtain $\frac{\partial f}{\partial x} + y\frac{\partial h}{\partial y} = 0$. So $y \mid \frac{\partial f}{\partial x}$. Since ξ is a vector field on X, f vanishes at the origin, and hence $y \mid f$. This proves the claim. On the other hand, in the complement U of any hyperplane through the origin, ∂_x is a Poisson vector field; but this can only be in $P(X) \cdot \mathcal{D}_U$ when the hyperplane was $y = 0$. Thus $P(X)$ is not $\mathcal{D}$-localizable.

Remark 4.18. In the case X is smooth, if it has finitely many symplectic leaves, it is in fact symplectic. However, there are many cases where X is smooth and generically symplectic, and $P(X)$ has finitely many leaves even though X has infinitely many symplectic leaves; e.g., $\pi = x\partial_x \wedge \partial_y$ on $\mathbf{A}^2$, as mentioned in §3.2.

We give an example where X is smooth but $P(X)$ is not $\mathcal{D}$-localizable:

Example 4.19. Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{g} := \mathfrak{sl}_2$ and let $X = \mathfrak{g}^*$, equipped with the induced Poisson bracket on $\mathcal{O}_X = \mathsf{Sym}\,\mathfrak{g}$. Then, all global Poisson vector fields are Hamiltonian, since $H^1(\mathfrak{g}, \mathsf{Sym}\,\mathfrak{g}) = 0$ (this implies that all derivations $\mathfrak{g} \to \mathsf{Sym}\,\mathfrak{g}$ are inner, and hence all derivations of $\mathsf{Sym}\,\mathfrak{g}$, i.e., vector fields on $\mathfrak{g}^*$, are Hamiltonian). It is clear that the Poisson bivector has rank two, except at the origin, where the rank is zero; hence this is the rank of $P(X)$. However, we claim that the rank of the space of generic Poisson vector fields is three. Indeed, write $\mathfrak{g} = \langle e, h, f\rangle$ with the standard bracket $[e, f] = h, [h, e] = 2e, [h, f] = -2f$. So $e, h, f \in \mathcal{O}_X$ are linear coordinates. Let $C = 2ef + \frac{1}{2}h^2 \in \mathcal{O}_X$ be the Casimir function, so $\{C, g\} = 0$ for all $g \in \mathcal{O}_X$. Then, if we localize where $e \neq 0$, we can consider the coordinate system (e, h, C) and take the directional derivative in the C direction, which in the original coordinates (e, h, f) is $\xi := \frac{1}{2e}\partial_f$. Since the Poisson bivector field is tangent to the planes where C is constant, this vector field is Poisson, which is also immediate from explicit computation (it is enough to check that $\{\xi(x), y\} + \{x, \xi(y)\} = \xi\{x, y\}$ for $x, y \in \mathcal{O}_X$, which clearly reduces to the case $x = f, y = h$, where $\{\xi(f), h\} = \frac{1}{e} = \xi(2f) = \xi\{f, h\}$.)

Proof of Proposition 4.8. By incompressibility and Proposition 2.39, the generic rank of $P(X)$ equals $2\dim X$ minus the dimension of the singular support of $M(U', P(X)|_{U'})$ for small enough open subsets U' (viewing $P(X)|_{U'}$ as a vector space). Thus $\mathcal{D}$-localizability implies that this must also equal the generic rank of $P(U)$ for every open subset $U \subseteq X$.

The second statement follows from Lemma 4.9, provided we can show that $P(X)$ flows incompressibly. By assumption, $P(X)$ flows parallel to the symplectic leaves. But, to be Poisson, such vector fields must preserve the symplectic form along the leaves, and hence they are incompressible along the leaves. Thus, as for $H(X)$ (see Example 2.30), one concludes that $P(X)$ flows incompressibly on X. ☐

Proof of Proposition 4.12. (i) Suppose that $\xi \in P(\mathsf{Spec}\,\mathcal{O}_{X,s})$. As explained in Remark 3.12, this means that $\xi = \eta_\alpha$ where α is a closed one-form on $\mathsf{Spec}\,\mathcal{O}_{X,s} \setminus E_s$. By the hypothesis (4.13), we can write

$$\alpha = \alpha_{\mathsf{Spec}\,\mathcal{O}_{X,s}} + \alpha_{X \setminus E_s} + df, \tag{4.20}$$

where $\alpha_{\mathsf{Spec}\,\mathcal{O}_{X,s}}$ is a closed one-form modulo torsion on $Spec\mathcal{O}_{X,s}$, $\alpha_{X\setminus E_s}$ is a closed one-form modulo torsion on $X \setminus E_s$, and $f \in \Gamma(\mathsf{Spec}\,\mathcal{O}_{X,s} \setminus E_s)$. By normality, $f \in \mathcal{O}_{X,s}$. Thus $\xi_f \in H(\mathsf{Spec}\,\mathcal{O}_{X,s})$. Note that $\eta_{\alpha_{\mathsf{Spec}\,\mathcal{O}_{X,s}}} \in LH(\mathsf{Spec}\,\mathcal{O}_{X,s})$, by definition. As in Remark 3.12, we obtain that $\eta_{X\setminus E_s} \in P(X)$. Therefore, applying the operation $\beta \mapsto \eta_\beta$ to both sides of (4.20), we obtain (4.14), since ξ was arbitrary.

As a consequence, we deduce that $P(\mathsf{Spec}\,\mathcal{O}_{X,s}) \subseteq P(X)|_{\mathsf{Spec}\,\mathcal{O}_{X,s}} \cdot \mathcal{O}_{X,s}$. Now, $s \in X$ was an arbitrary singular point. At smooth points $x \in X$, we have $P(\mathsf{Spec}\,\mathcal{O}_{X,x}) = H(\mathsf{Spec}\,\mathcal{O}_{X,x}) \subseteq H(X)|_{Spec\mathcal{O}_{X,x}} \cdot \mathcal{O}_{X,x}$.

Now, for arbitrary open affine $U \subseteq X$, $P(X)|_U \cdot \mathcal{D}_U$ is a sheaf on U, by Lemma 2.64. By the above, $P(U)|_{\mathsf{Spec}\,\mathcal{O}_{X,x}} \subseteq P(X)|_{\mathsf{Spec}\,\mathcal{O}_{X,x}} \cdot \mathcal{D}_{X,x}$, where the latter is the Zariski localization of $\mathcal{D}_X$ at x. By Lemma 2.64, $P(X)|_{\mathsf{Spec}\,\mathcal{O}_{X,x}} \cdot \mathcal{D}_{X,x} = (P(X)|_U \cdot \mathcal{D}_U)|_{\mathsf{Spec}\,\mathcal{O}_{X,x}}$. We conclude that, for all $x \in U$,

$$P(U)|_{\mathsf{Spec}\,\mathcal{O}_{X,x}} \subseteq P(X)|_U \cdot \mathcal{D}_U)|_{\mathsf{Spec}\,\mathcal{O}_{X,x}},$$

and since $P(X) \cdot \mathcal{D}_U$ is a sheaf, this implies that $P(U) \subseteq P(X)|_U \cdot \mathcal{D}_U$. As U was arbitrary, we conclude the $\mathcal{D}$-localizability of $P(X)$.

(ii) In order to prove (4.13), it suffices to prove the statement when $\mathsf{Spec}\,\mathcal{O}_{X,s}$ is replaced by sufficiently small Zariski open neighborhoods U of s. This is because every closed one-form modulo torsion in $\mathsf{Spec}\,\mathcal{O}_{X,s} \setminus E_s$ is actually regular on $U \setminus E_s$ for some Zariski open neighborhood U of s, and we are free to shrink it.

Now, assuming that S is finite, $E_s = \{s\}$ for all $s \in S$. By the preceding paragraph, it suffices to show that (4.15) implies that the map

$$H^1_{DR}(X \setminus \{s\}) \oplus H^1_{DR}(U) \to H^1_{DR}(U \setminus \{s\}) \tag{4.21}$$

is surjective. To see this, we first note that, for Y smooth but not necessarily affine, we have an isomorphism by Grothendieck's theorem,

$$\mathbf{H}^\bullet_{DR}(Y) \cong H^\bullet_{\text{top}}(Y),$$

where $\mathbf{H}^\bullet_{DR}(Y)$ denotes the *hypercohomology* of the complex of sheaves $\Omega^\bullet_Y = \tilde{\Omega}^\bullet_Y$.

Next, there is a natural map $H^1_{DR}(U) \to H^1_{\text{top}}(U)$, obtained by integrating along cycles; one can slightly perturb a closed path in U to miss the isolated singularities of U, and integrating against a one-form on U which is closed mod torsion (hence closed when restricted to the smooth locus of U) produces a well-defined answer, which depends only on the homology class in U of the closed path.

Then, the restriction map $H^1_{DR}(U) = \mathbf{H}^1_{DR}(U) \to \mathbf{H}^1_{DR}(U \setminus \{s\}) = H^1_{\text{top}}(U \setminus \{s\})$ factors through the map $H^1_{DR}(U) \to H^1_{\text{top}}(U)$, which is surjective by the main result of [BH69].

Then, (4.15) implies that we have a surjection

$$\mathbf{H}^1_{DR}(X \setminus \{s\}) \oplus H^1_{DR}(U) \to \mathbf{H}^1_{DR}(U \setminus \{s\}), \tag{4.22}$$

where here we note that $H^1_{DR}(U) = \mathbf{H}^1_{DR}(U)$ since U is affine.

Since $X = (X \setminus \{s\}) \cup U$, we have an exact Mayer-Vietoris sequence on hypercohomology of the triple $(X, X \setminus \{s\}), U)$, which in part takes the form

$$\mathbf{H}^1_{DR}(X \setminus \{s\}) \oplus H^1_{DR}(U) \to \mathbf{H}^1_{DR}(U \setminus \{s\}) \to H^2_{DR}(X) \to \mathbf{H}^2_{DR}(X \setminus \{s\}) \oplus H^2_{DR}(U). \tag{4.23}$$

By (4.22), the first map is surjective, and hence the last map is injective.

We also have a Mayer-Vietoris sequence for ordinary $H^\bullet_{DR}$, associated to the exact sequence of complexes of global sections,

$$0 \to \Omega^\bullet_X \to \Gamma(\Omega^\bullet_{X \setminus \{s\}}) \oplus \Omega^\bullet_U \to \Gamma(\Omega^\bullet_{U \setminus \{s\}}) \to 0.$$

This has the form

$$H^1_{DR}(X \setminus \{s\}) \oplus H^1_{DR}(U) \to H^1_{DR}(U \setminus \{s\}) \to H^2_{DR}(X) \to H^2_{DR}(X \setminus \{s\}) \oplus H^2_{DR}(U). \tag{4.24}$$

Now, the final map in (4.23) factors through the final map in (4.24) (since X is affine). Therefore the last map in (4.24) must also be injective. We conclude that the first map of (4.24), which is the same as (4.21), is surjective. This completes the proof. $\square$

4.3. **Formal $\mathcal{D}$-localizability of Poisson vector fields.** It turns out that formal $\mathcal{D}$-localizability of Poisson vector fields is a stronger condition, which implies (in the incompressible case) that X is generically symplectic.

Proposition 4.25. If X is irreducible affine Poisson and $P(X)$ flows incompressibly, then if $P(X)$ is formally $\mathcal{D}$-localizable, then X must be generically symplectic.

Note that this in particular implies that the condition of Proposition 4.8 is satisfied: $P(U)$ has generic rank equal to $\dim X$ for all U.

Proof of Proposition 4.25. Suppose that X is not generically symplectic. Then, in the neighborhood of some sufficiently generic smooth point, $\hat{X}_x \cong V \times V'$ as a formal Poisson scheme, where V is a symplectic formal polydisc and V' is a positive-dimensional formal polydisc with the zero Poisson bracket. So $P(\hat{X}_x) = P(V) \otimes \mathcal{O}_{V'} \oplus \mathrm{Vect}(V')$. This is evidently not incompressible since V' is positive-dimensional. Thus $M(\hat{X}_x, P(\hat{X}_x)) = 0$. However, if we assume $P(X)$ flows incompressibly, then $M(X, P(X))|_{\hat{X}_x} \neq 0$ for sufficiently generic x (with $P(X)$ here the constant sheaf). Thus $P(X)$ is not formally $\mathcal{D}$-localizable. ☐

We can also give a positive result parallel to Proposition 4.12:

Proposition 4.26. Suppose X is affine Poisson, normal, and symplectic on its smooth locus. Let S be its singular locus.

(i) For every $s \in S$, let $E_s \subseteq S$ be the union of all irreducible components of S containing s. Suppose that, for all $s \in S$, the natural map

$$H^1_{DR}(X \setminus E_s) \oplus H^1_{DR}(\mathsf{Spec}\,\hat{\mathcal{O}}_{X,s}) \to H^1_{DR}(\mathsf{Spec}\,\hat{\mathcal{O}}_{X,s} \setminus E_s) \tag{4.27}$$

is surjective. Then, $P(X)$ is formally $\mathcal{D}$-localizable. Moreover, for all $s \in S$,

$$P(\mathsf{Spec}\,\hat{\mathcal{O}}_{X,s}) = P(X)|_{\mathsf{Spec}\,\hat{\mathcal{O}}_{X,s}} \cdot \hat{\mathcal{O}}_{X,s} + LH(\mathsf{Spec}\,\hat{\mathcal{O}}_{X,s}). \tag{4.28}$$

(ii) Suppose that S is finite and $\mathbf{k} = \mathbf{C}$. Then the hypothesis of (i) is satisfied if, for sufficiently small neighborhoods U of s in the complex topology, $H^1_{\mathrm{top}}(X \setminus \{s\}) \to H^1_{\mathrm{top}}(U \setminus \{s\})$, is surjective. In particular, in this case, $P(X)$ is formally $\mathcal{D}$-localizable, and (4.28) holds.

Remark 4.29. The condition of (ii) is equivalent to asking that $H^1_{\mathrm{top}}(X \setminus \{s\}) \to H^1_{\mathrm{top}}(U \setminus \{s\})$ be surjective for any fixed contractible neighborhood U of s (whose existence was proved in [Gil64]). Thus, the condition of (ii) is the same as that of (4.15), except replacing Zariski open subsets by analytic neighborhoods, and using holomorphic functions rather than algebraic functions.

Proof of Proposition 4.26. The proof of part (i) of the proposition is the same as in Proposition 4.12, except replacing U by $\hat{X}_x$. We omit the details. Note that, when $x \notin S$, one has $P(\hat{X}_x) = H(\hat{X}_x)$, since then $\hat{X}_x$ is symplectic.

For part (ii), we use holomorphic functions and analytic neighborhoods and various results about them, contained in §4.4 below. As in Proposition 4.12, for every analytic neighborhood U of s, the assumption of (ii) together with Grothendieck's theorem implies that the map on hypercohomology,

$$\mathbf{H}^1_{DR}(X \setminus \{s\}) \to \mathbf{H}^{1,\mathrm{an}}_{DR}(U \setminus \{s\}),$$

is surjective. Using the Mayer-Vietoris sequence for the exact sequence of complexes of sheaves ((4.38) below for $Y = X$, $Z = \{s\}$, and $V = U$), we conclude that the map

$$H^2_{DR}(X) \to \mathbf{H}^2_{DR}(X \setminus \{s\}) \oplus H^{2,\mathrm{an}}(U)$$

is injective. This map factors through the map from ordinary cohomology to hypercohomology, so we conclude that the map

$$H^2_{DR}(X) \to H^2_{DR}(X \setminus \{s\}) \oplus H^{2,\mathrm{an}}(U)$$

is also injective. Using the Mayer-Vietoris sequence for ordinary cohomology (using the global sections of (4.38), which is an exact sequence of complexes since X is affine), we conclude that

$$H^1_{DR}(X \setminus \{s\}) \oplus H^{1,\mathrm{an}}_{DR}(U) \to H^{1,\mathrm{an}}_{DR}(U \setminus \{s\}) \tag{4.30}$$

is surjective. Then, by Theorem 4.45 below, we conclude that

$$H^1_{DR}(X \setminus \{s\}) \oplus H^1_{DR}(\hat{X}_s) \to H^1_{DR}(\hat{X}_s \setminus \{s\}) \tag{4.31}$$

is also surjective, as desired. ☐

Remark 4.32. In fact, we did not need the full strength of Theorem 4.45 below, but only the fact that the maps $H^1_{DR}(U) \to H^1_{DR}(\hat{X}_s)$ and $H^1_{DR}(U \setminus \{s\}) \to H^1_{DR}(\hat{X}_s \setminus \{s\})$ are surjective. At least the first fact can be proved in an elementary way by lifting closed formal differential forms to closed analytic differential forms, and does not require resolution of singularities as used in the proof of Theorem 4.45.

Example 4.33. Here is an example of a surface with an isolated singularity, which is normal and symplectic away from the singularity, for which $P(X)$ $\mathcal{D}$-localizable (in fact satisfying (4.15)) but not formally $\mathcal{D}$-localizable (so in particular not satisfying (4.27)). This example was pointed out to us by J. McKernan.

Let $E \subseteq \mathbf{P}^2$ be a smooth cubic curve. Then, under the intersection pairing on $\mathbf{P}^2$, $E \cdot E = 9$. Now, blow up $\mathbf{P}^2$ at twelve generic points of E. Let Y be the resulting projective surface, and let $E' \subseteq Y$ be the proper transform of E. Then $E' \cdot E' = 9 - 12 = -3$, so we can blow down E' to obtain a new surface, call it Z, where the image of E' is a singular point, call it s, whose formal neighborhood $\hat{Z}_s$ is isomorphic to the cone over an elliptic curve.

Note that $H^1_{\text{top}}(Z \setminus \{s\}) \cong H^1_{\text{top}}(Y \setminus E') \cong H^1_{\text{top}}(\mathbf{P}^2 \setminus E) = 0$, since $E \subseteq \mathbf{P}^2$ has a nontrivial normal bundle.

Next, embed Z into projective space $\mathbf{P}^N$ of some dimension $N > 2$. Let $C \subseteq Z$ be the intersection of Z with a generic hyperplane, and let $X := Z \setminus C$ be the resulting affine surface. Since $\mathcal{O}(C)$ is (very) ample, C has a nontrivial normal bundle. Hence, the restriction map induces isomorphisms $H^1_{\text{top}}(Z) \xrightarrow{\sim} H^1_{\text{top}}(X)$ and $H^1_{\text{top}}(Z \setminus \{s\}) \xrightarrow{\sim} H^1_{\text{top}}(X \setminus \{s\})$. In particular, these are zero as well.

Thus, $\mathbf{H}^1_{DR}(X \setminus \{s\}) = 0$. We claim that $H^1_{DR}(X \setminus \{s\}) = 0$ as well. More generally, this follows from the following statement:

Lemma 4.34. Let V be a scheme or complex analytic space. Then the map $H^1_{DR}(V) \to \mathbf{H}^1_{DR}(V)$ is injective.

We remark that, in the case V is a smooth variety (as with $V = X \setminus \{s\}$ above), by Grothendieck's theorem we can replace $\mathbf{H}^1_{DR}(V)$ by the topological first cohomology of V, and then the statement follows because, if an algebraic or analytic one-form is the differential of a smooth (C^∞) function, then the function must actually be algebraic (or analytic).

Proof of Lemma 4.34. Consider the spectral sequence $H^i(R^j\Gamma(\Omega_V)) \Rightarrow \mathbf{H}^{i+j}_{DR}(V)$. In total degrees ≤ 2, the second page has the form

$$H^0_{DR}(V) \to H^1_{DR}(V) \oplus H^0(R^1\Gamma(\Omega_V)) \to H^2_{DR}(V) \oplus H^1(R^1\Gamma(\Omega_V)) \oplus H^0(R^2\Gamma(\Omega_V)).$$

The first map above is zero, and the restriction of the second map above to $H^1_{DR}(V)$ is zero. Therefore the summand of $H^1_{DR}(V)$ maps injectively to a summand of the third page of the spectral sequence. The same argument shows that, at every page, $H^1_{DR}(V)$ maps injectively to the next page, so the map $H^1_{DR}(V) \to \mathbf{H}^1_{DR}(V)$ is injective. □

Now, since $X \setminus \{s\}$ is symplectic, all global Poisson vector fields are locally Hamiltonian given by a global closed one-form. By the above, $H^1_{DR}(X \setminus \{s\}) = 0$, so that locally Hamiltonian vector fields are Hamiltonian. Therefore, all global Poisson vector fields are Hamiltonian.

On the other hand, not all Poisson vector fields on $\hat{X}_s$ are Hamiltonian, since $\hat{X}_s$ is isomorphic to the formal neighborhood of the vertex in the cone over an elliptic curve, and there one has the Euler vector field which is not Hamiltonian. Hence, $P(X)$ is not formally $\mathcal{D}$-localizable. (In fact, $P(X)$ is not étale-locally $\mathcal{D}$-localizable either, since the Euler vector field exists in an étale neighborhood of x, or equivalently in the strict Henselization of the local ring at x.)

On the other hand, we claim that $P(X)$ is $\mathcal{D}$-localizable, and in fact that (4.15) holds. Let $U \subseteq X$ be any affine open subset containing s. Since Z is rational (as Y, and hence Z, is birational to $\mathbf{P}^2$), so is U. Now, we claim that the map $H^1_{DR}(U) \to H^1_{DR}(U \setminus \{s\})$ is surjective. Consider the sequence (4.39) for the pair $(U, \{x\})$: this yields the exact sequence

$$H^1_{DR}(U) \to H^1_{DR}(U \setminus \{s\}) \oplus H^1_{DR}(\hat{U}_s) \to H^1_{DR}(\hat{U}_s \setminus \{s\}).$$

It suffices to show that the map $H^1_{DR}(U \setminus \{s\}) \to H^1_{DR}(\hat{U}_s \setminus \{s\})$ is zero. By Lemma 4.34 above, this is equivalent to showing that the map $H^1_{DR}(U \setminus \{s\}) \to \mathbf{H}^1_{DR}(\hat{U}_s \setminus \{s\})$ is zero. This map factors through the hypercohomology of any punctured neighborhood of s contained in $U \setminus \{s\}$, which by Grothendieck's theorem is the same as the topological cohomology of that punctured neighborhood. Such punctured neighborhoods, for sufficiently small contractible U, are homotopic to nontrivial S^1-bundles over an elliptic curve, and their fundamental group is isomorphic to that of the elliptic curve. If the map $H^1_{DR}(U \setminus \{s\}) \to \mathbf{H}^1_{DR}(\hat{U}_s \setminus \{s\})$ were nonzero, then a nontrivial period of the elliptic curve would be computable as integrals of closed algebraic one-forms on $U \setminus \{s\}$. However, as remarked, U is rational. Thus this would imply that a nontrivial period of the elliptic curve were computable as the integral of a rational closed one-form along a contour in $\mathbf{C}^2$. This is well-known to be impossible, since these periods are given by transcendental hypergeometric functions with infinite monodromy. Thus, $P(X)$ is $\mathcal{D}$-localizable. (Note that this paragraph also gives another proof that $P(X)$ is not formally $\mathcal{D}$-localizable, and in fact that $P(U)$ is not formally $\mathcal{D}$-localizable for every open affine neighborhood U of s: these periods are computable in a formal neighborhood of s, but by the above, they are not computable using global closed one-forms. Passing from closed one-forms to Poisson vector fields via the symplectic form on $U \setminus \{s\}$, this yields that $P(U)$ is not formally $\mathcal{D}$-localizable.)

Example 4.35. We give an example where X is smooth and $P(X)$ is $\mathcal{D}$-localizable but not formally $\mathcal{D}$-localizable. By Propositions 4.8 and 4.25, one way this happens is if the rank of $P(U)$ equals that of $H(U)$ and is constant but less than the dimension of X (which in particular is not generically symplectic).

Let $X = (\mathbf{A}^\times)^3 = \mathsf{Spec}\, \mathbf{k}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$ with the Poisson bracket $\{x, y\} = xyz$ and $\{x, z\} = \{y, z\} = 0$. Then $H(U)$ has rank two everywhere, for every open affine subset $U \subseteq X$. We claim that any rational Poisson vector field on X annihilates z. Therefore, the rank of every vector field in $P(U)$ is also everywhere two, for every open subset $U \subseteq X$, as desired.

To prove that every rational Poisson vector field annihilates z, it is enough to assume that $\mathbf{k} = \mathbf{C}$. Let ξ be a rational Poisson vector field and let $c \in \mathbf{C}$ be such that it does not have a pole at $z = c$. Then the irregular locus of ξ in $\{z = c\}$ is an algebraic curve in $\mathbf{A}^2$. One can show that such a curve must avoid a real two-torus $T = \{|x| = r, |y| = s\}$, and then $\int_{T \times \{c\}} \pi^{-1}$ is a nonzero constant multiple of $\frac{1}{z}$. Since ξ preserves π, one concludes that it must be parallel to the level sets of z, i.e., it annihilates z.

Remark 4.36. We can also give an elementary algebraic proof that, in the above example, every rational Poisson vector field annihilates z. Any rational Poisson vector field must send z to a rational function of z, since these are all the rational Casimirs. Moreover, any such vector field is still Poisson after multiplying by an arbitrary rational function of z. Hence, if such a vector field exists which does not annihilate z, then there must be one of the form $\partial_z + f\partial_x + g\partial_y$ for some rational functions f, g on X.

On the other hand, we can explicitly write one such non-rational vector field, $\partial_z + \frac{x \log x}{z}\partial_x$. This vector field is best understood by writing the Poisson bracket in coordinates $(u, v, z) = (\log x, \log y, z)$, as $\{u, v\} = z$, and the vector field as $\partial_z + \frac{u}{z}\partial_u$.

Thus, given a rational vector field $\partial_z + f\partial_x + g\partial_y$, taking the difference, we would obtain a non-rational Poisson vector field of the form $-\frac{x \log x}{z}\partial_x + f\partial_x + g\partial_y$. But no such vector field can be Poisson, since a vector field parallel to the symplectic leaf is Poisson if and only if its symplectic divergence vanishes, which here is $\frac{1}{z}\partial_x(f - x\log x) + \partial_y(g)$, and this cannot vanish for f and g rational.

4.4. **Analytic-to-formal comparison for de Rham cohomology.**

4.4.1. *Preliminaries on analytic forms and Mayer-Vietoris sequences.* We will need to use holomorphic differential forms, on an algebraic variety Y which need not be affine.

Definition 4.37. Let Y be an algebraic variety over $\mathbf{k} = \mathbf{C}$. Let $\Omega_Y^{\bullet,\mathrm{an}}$ denote the complex of sheaves of holomorphic Kähler differential forms, and $\tilde{\Omega}_Y^{\bullet,\mathrm{an}}$ its quotient modulo torsion. Let $\mathbf{H}_{DR}^{\bullet,\mathrm{an}}(Y)$ denote the hypercohomology of this complex, and $H_{DR}^{\bullet,\mathrm{an}}(Y)$ denote the cohomology of the complex of global sections $\Gamma(\Omega_Y^{\bullet,\mathrm{an}})$.

Grothendieck's theorem also extends to the holomorphic setting, where we obtain that $\mathbf{H}_{DR}^{\bullet,\mathrm{an}}(U) \cong H_{\mathrm{top}}^{\bullet}(U)$ if U is smooth.

For $Z \subseteq Y$ a subvariety and V an analytic neighborhood of Z, we will make use of the Mayer-Vietoris sequence associated to the exact sequence of complexes,

$$0 \to \tilde{\Omega}_Y^{\bullet} \to \tilde{\Omega}_{Y\setminus Z}^{\bullet} \oplus \tilde{\Omega}_V^{\bullet,\mathrm{an}} \to \tilde{\Omega}_{V\setminus Z}^{\bullet,\mathrm{an}} \to 0. \tag{4.38}$$

Similarly, we will need the corresponding exact sequence when V is replaced by a formal neighborhood of Z:

$$0 \to \tilde{\Omega}_Y^{\bullet} \to \tilde{\Omega}_{Y\setminus Z}^{\bullet} \oplus \tilde{\Omega}_{\hat{Y}_Z}^{\bullet} \to \tilde{\Omega}_{\hat{Y}_Z\setminus Z}^{\bullet} \to 0. \tag{4.39}$$

Note that there is a natural map by restriction from the sequence (4.38) to (4.39). This forms the commutative diagram with exact rows,

(4.40)
$$\begin{array}{ccccccccccc}
\cdots \to & \mathbf{H}_{DR}^{i-1}(Y) & \to & \mathbf{H}_{DR}^{i-1}(Y\setminus Z) \oplus \mathbf{H}_{DR}^{i-1,\mathrm{an}}(V) & \to & \mathbf{H}_{DR}^{i-1,\mathrm{an}}(V\setminus Z) & \to & \mathbf{H}_{DR}^{i}(Y) & \to \cdots \\
& \| & & \downarrow & & \downarrow & & \| & \\
\cdots \to & \mathbf{H}_{DR}^{i-1}(Y) & \to & \mathbf{H}_{DR}^{i-1}(Y\setminus Z) \oplus \mathbf{H}_{DR}^{i-1}(\hat{Y}_Z) & \to & \mathbf{H}_{DR}^{i-1}(\hat{Y}_Z\setminus Z) & \to & \mathbf{H}_{DR}^{i}(Y) & \to \cdots
\end{array}$$

Finally, note that, when Y is affine, we can also consider the same diagram for ordinary rather than hypercohomology, since the sequences (4.38) and (4.39) remain exact on the level of global sections.

4.4.2. *Comparison isomorphisms for smooth varieties.* Now consider the case that Y is smooth. Then, we will need the result that a small enough tubular neighborhood V of Z retracts onto Z. By Grothendieck's theorem, this implies

$$\mathbf{H}_{DR}^{\bullet,\mathrm{an}}(V) \cong H_{\mathrm{top}}^{\bullet}(Z), \tag{4.41}$$

where as before $\mathbf{H}$ denotes hypercohomology (which is necessary since we do not require Y to be affine).

Hartshorne's theorem [Har72, Har75] gives an algebraic analogue of the above statement:

$$\mathbf{H}_{DR}^{\bullet}(\hat{Y}_Z) \cong H_{\mathrm{top}}^{\bullet}(Z). \tag{4.42}$$

Moreover, the isomorphism (4.42) composed with the restriction $\mathbf{H}_{DR}^{\bullet,\mathrm{an}}(V) \to \mathbf{H}_{DR}^{\bullet}(\hat{Y}_Z)$ is the natural isomorphism (4.41). Put together, we deduce that the restriction map is an isomorphism,

$$\mathbf{H}_{DR}^{\bullet,\mathrm{an}}(V) \xrightarrow{\sim} \mathbf{H}_{DR}(\hat{Y}_Z). \tag{4.43}$$

Therefore, the five-lemma implies that the vertical arrows in (4.40) are all isomorphisms. In particular, this yields also

$$\mathbf{H}_{DR}^{\bullet,\mathrm{an}}(V\setminus Z) \xrightarrow{\sim} \mathbf{H}_{DR}^{\bullet}(\hat{Y}_Z\setminus Z). \tag{4.44}$$

Note that, when Y is affine, we can also replace hypercohomology with ordinary cohomology (in the second isomorphism), by using (4.40) for ordinary cohomology.

4.4.3. *Comparison theorem for isolated singularities.*

Theorem 4.45. Suppose that X is a complex algebraic variety with an isolated singularity at $x \in X$. Then, for sufficiently small contractible neighborhoods U of x, there are canonical isomorphisms

$$\mathbf{H}_{DR}^{\bullet,\mathrm{an}}(U) \xrightarrow{\sim} \mathbf{H}_{DR}^{\bullet}(\hat{X}_x), \quad \mathbf{H}_{DR}^{\bullet,\mathrm{an}}(U \setminus \{x\}) \xrightarrow{\sim} \mathbf{H}_{DR}^{\bullet}(\hat{X}_x \setminus \{x\}). \tag{4.46}$$

If in addition U is Stein, then we have canonical isomorphisms on cohomology of global sections,

$$H_{DR}^{\bullet,\mathrm{an}}(U) \xrightarrow{\sim} H_{DR}^{\bullet}(\hat{X}_x), \quad H_{DR}^{\bullet,\mathrm{an}}(U \setminus \{x\}) \xrightarrow{\sim} H_{DR}^{\bullet}(\hat{X}_x \setminus \{x\}). \tag{4.47}$$

Remark 4.48. The theorem also extends to the case where X is an analytic variety with an isolated singularity at x, with the same proof as below, since Hironaka's theorem on resolution of singularities also applies to analytic varieties. (This is a strict generalization of the theorem, since every algebraic variety is also analytic, and the objects above are the same.)

Proof. Let $Y \to X$ be a resolution of singularities, and let $Z \subseteq Y$ be the fiber over x. Let V be a tubular neighborhood of Z which retracts to Z and U its image under the resolution, which therefore retracts to x. Then the resolution maps restrict to isomorphisms $Y \setminus Z \xrightarrow{\sim} X \setminus \{x\}$, $V \setminus Z \xrightarrow{\sim} U \setminus \{x\}$, and $\hat{Y}_Z \setminus Z \xrightarrow{\sim} \hat{X}_x \setminus \{x\}$. By (4.44), we conclude the second isomorphism in (4.46).

Now, the above was for specific neighborhoods U, namely those obtainable from tubular neighborhoods V of $Z \subseteq Y$. For any smaller contractible neighborhood $U' \subseteq U$ of x, the restriction map $H_{\mathrm{top}}^{\bullet}(U \setminus \{x\}) \to H_{\mathrm{top}}^{\bullet}(U' \setminus \{x\})$ is an isomorphism by Grothendieck's theorem, and hence the second isomorphism of (4.46) holds for sufficiently small contractible neighborhoods of x.

Consider now (4.40) for the pair $(X, \{x\})$, with U such that the second isomorphism of (4.46) holds. The five-lemma then implies that the vertical arrows are all isomorphisms, which implies the first isomorphism of (4.46).

Next, the first isomorphism of (4.47) follows immediately, since U is Stein, so hypercohomology of U and $\hat{X}_x$ coincides with the cohomology of global sections. Finally, since X is affine, we can consider (4.40) for the pair $(X, \{x\})$ using ordinary rather than hypercohomology. The five-lemma now implies that the vertical arrows are once again isomorphisms, yielding the second isomorphism of (4.47). □

5. Complete intersections with isolated singularities

In this section, we explicitly compute $(\mathcal{O}_X)_{\mathfrak{v}}$, $M(X, \mathfrak{v})$, and $\pi_* M(X, \mathfrak{v})$, in the case that $X \subseteq Y$ is a locally complete intersection of positive dimension, Y is affine Calabi-Yau, and X has only isolated singularities; cf. Example 3.39. For $M(X, \mathfrak{v})$ itself, the assumption that Y (and hence X) is affine is not necessary, using §4.

We set $\mathfrak{v} = H(X)$ (one could equivalently use $LH(X)$, in view of Proposition 3.31.) Note that, in the case X is two-dimensional, then X is a Poisson variety and $H(X)$ is the Lie algebra of Hamiltonian vector fields.

5.1. **Complete intersections: Greuel's formulas.** Here we recall from [Gre75] an explicit formula for the de Rham cohomology of an analytic neighborhood of x.

Embed $\hat{X}_x \subseteq \mathbf{A}^n$ cut out by equations $f_1, \ldots, f_k$ such that $(f_1, \ldots, f_i)$ has only isolated singularities for all i. Then define the ideals

$$J_{X,x,i} = (f_1, \ldots, f_{i-1}, \frac{\partial(f_1, \ldots, f_i)}{\partial(x_{j_1}, \ldots, x_{j_i})}, 1 \leq j_1 \leq \cdots \leq j_i \leq n) \subseteq \mathcal{O}_{\hat{\mathbf{A}}^n, x}. \tag{5.1}$$

Here $\frac{\partial(f_1,\ldots,f_i)}{\partial(x_{j_1},\ldots,x_{j_i})}$ is the determinant of the matrix of partial derivatives $\partial_{x_{j_p}}(f_q), 1 \leq p, q \leq i$. Then, the Milnor number, μ_x, of the singularity of X at x is given by

$$\mu_x = \sum_{i=1}^{k} (-1)^{k-i} \operatorname{codim}_{\hat{\mathcal{O}}_{\mathbf{A}^n,x}} J_{X,x,i}. \tag{5.2}$$

Definition 5.3. Let X and x be as above. Define the singularity ring, $\mathcal{C}_{X,x}$, of X at x to be

$$\mathcal{C}_{X,x} := \hat{\mathcal{O}}_{\mathbf{A}^n,x}/(J_{X,x,k}, f_k),$$

and define the Tjurina number, τ_x, to be the dimension of $\mathcal{C}_{X,x}$.

Note that the ring $\mathcal{C}_{X,x}$ does not depend on the embedding $\hat{X}_x \subseteq \hat{\mathbf{A}}^n_x$ and is also definable intrinsically as the quotient of $\hat{\mathcal{O}}_{X,x}$ by the m-th Fitting ideal of $\Omega^1_{\hat{X}_x}$, cf. [Har74] and Remark 2.3).

Theorem 5.4. [Gre75, Proposition 5.7.(iii)] If x is an isolated singularity which is locally a complete intersection in the analytic topology, then

$$H^\bullet(\tilde{\Omega}^{\bullet,\mathrm{an}}_{X,x}) \cong \mathbf{k}^{\mu_x - \tau_x}[-\dim X]. \tag{5.5}$$

Here, $V[-\dim X]$ is the graded vector space concentrated in degree $\dim X$ with underlying vector space V.

5.2. **General structure.** Since $H(X)$ has finitely many leaves, $M(X, H(X))$ is holonomic. Let $i : Z \hookrightarrow X$ be the (finite) singular locus of X.

Note that $i_* H^0 i^* M(X, \mathfrak{v})$ is the maximal quotient of $M(X, \mathfrak{v})$ supported on Z. Let N be its kernel. Let $X^\circ := X \setminus Z$ and let $\mathrm{IC}(X) = j_{!*}\Omega_{X^\circ}$ be the intersection cohomology $\mathcal{D}$-module of X, i.e., the intermediate extension of Ω_{X°. Since $j^! M(X, \mathfrak{v}) \cong \Omega_{X^\circ}$, this is a composition factor of $M(X, \mathfrak{v})$, and all other composition factors are delta function $\mathcal{D}$-modules of points in Z. Since N has no quotient supported on Z, it must be an indecomposable extension given by an exact sequence of the form

$$0 \to K \to N \to \mathrm{IC}(X) \to 0, \tag{5.6}$$

where K is supported at Z. Then, the structure of $M(X, \mathfrak{v})$ reduces to computing $i_* H^0 i^* M(X, \mathfrak{v})$, the extension (5.6), and how these two are extended. The first question has a nice general answer:

Theorem 5.7. For every $z \in Z$, with $i_z : \{z\} \to X$ the embedding, there is a canonical exact sequence

$$0 \to H^{\dim X}_{DR}(\hat{X}_z) \to H^0 i_z^* M(X, \mathfrak{v}) \to \mathcal{C}_{X,z} \to 0. \tag{5.8}$$

By Theorem 4.45, there is a canonical isomorphism $H^{\dim X}(\Omega^{\bullet,\mathrm{an}}_{X,z}) \xrightarrow{\sim} H^{\dim X}_{DR}(\hat{X}_z)$. By Theorem 5.4, the former has dimension $\mu_z - \tau_z$. On the other hand, $\dim \mathcal{C}_{X,z} = \tau_z$. We conclude

Corollary 5.9. $i_* H^0 i^* M(X, \mathfrak{v}) \cong \bigoplus_{z \in Z} \delta_z^{\mu_z}$.

The following basic result will be useful in the theorem and later on. For an arbitrary scheme X and point $x \in X$, let $(\hat{\mathcal{O}}_{X,x})^*$ be the continuous dual of $\hat{\mathcal{O}}_{X,x}$ with respect to the adic topology.

Lemma 5.10. Let $(X, \mathfrak{v})$ and $x \in X$ be arbitrary. Then $\mathrm{Hom}(M(X, \mathfrak{v}), \delta_x) \cong ((\hat{\mathcal{O}}_{X,x})^*)^{\mathfrak{v}}$.

Proof. Note that $\mathrm{Hom}(\mathcal{D}_X, \delta_x) \cong (\hat{\mathcal{O}}_{X,x})^*$, since the latter are exactly the delta function distributions at x. By definition of $M(X, \mathfrak{v})$, each $\phi \in \mathrm{Hom}(M(X, \mathfrak{v}), \delta_x)$ is uniquely determined by $\phi(1)$, which can be any element of δ_x which is invariant under $\mathfrak{v}$. $\square$

The theorem can therefore be restated as

Theorem 5.11. For all $z \in Z$, there is a canonical exact sequence

$$0 \to H_{DR}^{\dim X}(\hat{X}_z) \to (\hat{\mathcal{O}}_{X,z})_{\mathfrak{v}} \to \mathcal{C}_{X,z} \to 0. \tag{5.12}$$

In particular, $\dim(\hat{\mathcal{O}}_{X,z})_{\mathfrak{v}} = \mu_z$.

In the case that $Y = \mathbf{A}^3$ and X is a quasihomogeneous hypersurface, the consequence that $(\hat{\mathcal{O}}_{X,z})_{\mathfrak{v}} = \mu_z = \tau_z$ was discovered in [AL98] without using the earlier results of [Gre75].

Proof. Let $n := \dim Y$, $m := \dim X$, and $k := n - m$. Let $I_X := (f_1, \ldots, f_k)$ be the ideal defining X. Consider the map

$$\Phi : \tilde{\Omega}_X^{\bullet} \to \Omega_Y^{\bullet+k}/I_X \cdot \Omega_Y^{\bullet+k}, \quad \alpha \mapsto \alpha \wedge df_1 \wedge \cdots \wedge df_k,$$

which induces also a map taking the completion at z, which we also denote by Φ. Note that, in this formula, we have to lift α to a form on Y, but the map is independent of the choice of lift. Furthermore, Φ is injective, since $X \setminus Z$ is locally transversely cut out by $f_1, \ldots, f_k$. Let $\widetilde{H(X)} \subseteq H(Y)$ be the Lie algebra of vector fields obtained from the $(n-2)$-forms $\Phi(\tilde{\Omega}_X^{m-2})$. Then we have an identification

$$(\hat{\mathcal{O}}_{X,z})_{\mathfrak{v}} \xrightarrow{\sim} \Omega^n_{\hat{Y}_z}/(\widetilde{H(X)}(\hat{\mathcal{O}}_{Y,z}) + I_X) \cdot \mathsf{vol}_{\hat{Y}_z}, \tag{5.13}$$

obtained by multiplying by $\mathsf{vol}_{\hat{Y}_z}$. In turn, $\widetilde{H(X)}(\hat{\mathcal{O}}_{Y,z}) \cdot \mathsf{vol}_{\hat{Y}_z}$ identifies with $d\Phi(\Omega^{m-1}_{\hat{Y}_z})$. Therefore,

$$(\hat{\mathcal{O}}_{X,z})_{\mathfrak{v}} \xrightarrow{\sim} \Omega^n_{\hat{Y}_z}/(d\Phi(\Omega^{m-1}_{\hat{X}_z}) + I_X\Omega^n_{\hat{Y}_z}). \tag{5.14}$$

We now compute the RHS. Recall that Φ is an injection of complexes. The image of $H^m(\tilde{\Omega}^{\bullet}_{\hat{X}_z})$ is a subspace of (5.14). Moreover, the quotient of $\Omega^n_{\hat{Y}_z}$ by this image is

$$\mathcal{C}_{X,z} = \Omega^n_{\hat{Y}_z}/(I_X\Omega^n_{\hat{Y}_z} + \Phi(\Omega^m_{\hat{X}_z})). \tag{5.15}$$

We obtain the desired canonical exact sequence (5.12). $\square$

We can be more specific about the meaning of K in (5.6) and use this to describe the derived pushforward $\pi_* M(X, \mathfrak{v})$, where $\pi : X \to \mathrm{pt}$ is the projection to a point. Let $\pi_i := H^i \pi_*$. If we apply π_* to (5.6), we obtain isomorphisms $\pi_i N \cong \pi_i \operatorname{IC}(X)$ for $i > 1$, and an exact sequence

$$0 \to \pi_1 N \to \mathrm{IH}^{\dim X - 1}(X) \to \pi_0 K \to \pi_0 N \to \mathrm{IH}^{\dim X}(X) \to 0. \tag{5.16}$$

Here $\mathrm{IH}^*(X)$ denotes the intersection cohomology of X, $\mathrm{IH}^*(X) := \pi_{\dim X - *} \operatorname{IC}(X)$.

Similarly, from the exact sequence $0 \to N \to M(X, \mathfrak{v}) \to i_* H^0 i^* M(X, \mathfrak{v}) \to 0$, we obtain isomorphisms $\pi_i(N) \cong \pi_i M(X, \mathfrak{v})$, $i \geq 1$, and a split exact sequence

$$0 \to \pi_0 N \to (\mathcal{O}_X)_{\mathfrak{v}} \to H^0 i^* M(X, \mathfrak{v}) \to 0.$$

Put together, we obtain

Corollary 5.17. For $i \geq 2$, $\pi_i M(X, \mathfrak{v}) \cong \mathrm{IH}^{\dim X - i}(X)$. For some decomposition $K = K' \oplus K''$, one has a split exact sequence

$$0 \to \pi_1 M(X, \mathfrak{v}) \to \mathrm{IH}^{\dim X - 1}(X) \to \pi_0 K' \to 0, \tag{5.18}$$

and an isomorphism

$$(\mathcal{O}_X)_{\mathfrak{v}} \cong \mathrm{IH}^{\dim X}(X) \oplus \bigoplus_{z \in Z} (\hat{\mathcal{O}}_{X,z})_{\mathfrak{v}} \oplus \pi_0 K''. \tag{5.19}$$

Remark 5.20. We will show in [ES14] that $N = H^0 j_! \Omega_{X \setminus \{0\}}$, so one obtains an exact sequence

$$0 \to K \to N \to IC(X) \to 0.$$

Moreover, $K = K' = \big(\mathrm{Ext}(\mathrm{IC}(X), \delta_0)^* \otimes \delta_0\big)$. Finally, will then conclude that $\pi_\bullet M(X, \mathfrak{v}) \cong H^{\dim X - \bullet}_{\mathrm{top}}(X) \oplus \mathbf{k}^{\mu_z}$.

5.3. **The quasihomogeneous case.** Now suppose that $X \subseteq \mathbf{A}^n$ where $\mathbf{A}^n = \mathsf{Spec}\, \mathbf{k}[x_1, \ldots, x_n]$, each of the x_i is assigned a weight $m_i \geq 1$, and X is cut out by $k := n - \dim X$ weighted-homogeneous polynomials in the x_i. In this case, $\mathsf{HP}_0(\mathcal{O}_X)$ is a nonnegatively graded vector space by weight. Moreover, $M(X, H(X))$ is a weakly $\mathbf{G}_m$-equivariant $\mathcal{D}$-module which decomposes into weight submodules. Hence, $H^0 i^* M(X, H(X))$ is weight-graded. Then, the proofs of the preceding results generalize to this context (considering also [Gre75] and references therein). Moreover, by [Fer70] (cf. [Gre75, Korollar 5.8]), in this case $H^\bullet_{DR}(X) = 0$ and (5.5) implies that $\mu_z = \tau_z$, which is the dimension of the singularity ring (see Definition 5.3). By using the weight-graded versions of the arguments of [Gre75] one deduces, for X_{sing} the scheme-theoretic singular locus of X, defined by the ideal $(J_{X,0,k}, f_k)$,

Theorem 5.21. The graded vector space $H^0 i^* M(X, H(X))$ has Poincaré polynomial
(5.22)

$$P(H^0 i^* M(X, H(X)); t) = P(\mathcal{O}_{X_{\mathrm{sing}}}; t) = P(\mathcal{O}_{\mathbf{A}^n}/(J_{X,0,k}, f_k); t) = \sum_{i=1}^{k} (-1)^{k-i} P(\mathcal{O}_{\mathbf{A}^n}/J_{X,0,i}; t).$$

Since $\mathcal{O}_X$ is nonnegatively graded and X is connected, $H(X)$ is spanned by homogeneous vector fields, and $(\mathcal{O}_X)_{H(X)}$ is finite-dimensional, we conclude that $(\hat{\mathcal{O}}_X)_{H(X)} \cong (\mathcal{O}_X)_{H(X)}$. Therefore, Lemma 5.10 implies

Corollary 5.23. $P((\mathcal{O}_X)_{H(X)}; t) = P(\mathcal{O}_{\mathbf{A}^n}/(J_{X,0,k}, f_k); t)$.

In particular, in this case, $\mathrm{IH}^{\dim X}(X) = 0$ and $K'' = 0$ (i.e., $K = K'$).

Remark 5.24. In the case that $k = 1$, i.e., X is a quasihomogeneous hypersurface $Z(f)$, the ideal of the singular locus of X is also known as the Jacobi ideal $J_X = (\partial_i f) = (\partial_i f, f)$. For the last equality, let m_i be the weight of x_i for all i as above, and set $m := \sum_i m_i$. Then $f = \frac{1}{m} \sum_i m_i x_i \partial_i f$.

In this case, one can prove the theorem in an elementary way. Namely, we need to show that

$$H(X)(\mathcal{O}_X) = J_X/(f).$$

Equivalently, we have to show that

$$\Omega^{n-1}_X \wedge df + I_X \cdot \Omega^n_{\mathbf{A}^n} = d\Omega^{n-2}_X \wedge df + I_X \cdot \Omega^n_{\mathbf{A}^n}. \tag{5.25}$$

For this, let $\mathrm{Eu} := \sum_i m_i x_i \partial_i$ be the Euler vector field on $\mathbf{A}^n$. Set $\mathrm{Eu}^\vee := i_{\mathrm{Eu}}(\mathsf{vol}_{\mathbf{A}^n}) \in \Omega^{n-1}_{\mathbf{A}^n}$. Then, for all $g \in \mathcal{O}_{\mathbf{A}^n}$, we have the identities

$$\mathrm{Eu}^\vee \wedge dg = \mathrm{Eu}(g) \cdot \mathsf{vol}_{\mathbf{A}^n}, \quad d(g\, \mathrm{Eu}^\vee) = (\mathrm{Eu}(g) + m \cdot g) \cdot \mathsf{vol}_{\mathbf{A}^n}.$$

Therefore, we conclude that, for all quasihomogeneous $\alpha \in \Omega^{n-1}_X$, letting $|\cdot|$ denote the weighted degree function,

$$\bar{\alpha} := \alpha - (|\alpha| + m)^{-1} (d\alpha/\mathsf{vol}_{\mathbf{A}^n})\, \mathrm{Eu}^\vee \in d\Omega^{n-2}_{\mathbf{A}^n}.$$

Moreover,

$$\alpha \wedge df \equiv \bar{\alpha} \wedge df \pmod{I_X \cdot \Omega^n_{\mathbf{A}^n}}.$$

We conclude (5.25), and hence the theorem in this case.

6. Finite quotients of Calabi-Yau varieties

Let X be an affine connected Calabi-Yau variety and Ξ the top polyvector field inverse to the volume form; for instance, we could have $X = \mathbf{A}^n$ with the inverse to the standard volume form. In this case, $H(X) = LH(X) = P(X)$. Let G be a finite group acting by automorphisms on X, such that the action on Ξ is by multiplication by a character $G \to \mathbf{k}^\times$. In this section we will compute the $\mathcal{D}$-module $M(X, H(X)^G)$. Everything generalizes without change to the case where X is not affine, using §4.

As noticed at the end of §3.4, in the case that G actually preserves Ξ, using the induced top polyvector field on X/G, $H(X)^G = P(X/G)$. So we also deduce $M(X/G, P(X/G)) = q_* M(X, H(X)^G)^G$ where $q : X \to X/G$ is the projection, and hence also its underived pushforward to a point, $(\mathcal{O}_{X/G})_{P(X/G)}$. We note that, by Proposition 3.41, when $\dim X \geq 2$, $H(X)^G$ has finitely many leaves and hence is holonomic, so $P(X/G)$ is as well; however, in general, $H(X/G)$ and $LH(X/G)$ are not holonomic (by Corollary 3.35, they are holonomic if and only if X/G has only finitely many singular points, i.e., only finitely many points of X have nontrivial stabilizers in G).

More generally, the statements of the preceding paragraph generalize to the setting that G acts by multiplication by a character on Ξ, if we consider X/G to be equipped with the multivalued volume form obtained from X. More precisely, the flat connection on the canonical bundle of X is G-invariant and therefore descends to X/G, so as in §3.5, X/G is equipped with a divergence function.

We will restrict our attention to the case where $\dim X \geq 2$. Note that, in the case $X = \mathbf{A}^1 = \mathsf{Spec}\, \mathbf{k}[x]$, then if $G < \mathsf{GL}(1)$ is nontrivial, then there are no G-invariant volume preserving vector fields on X. Thus more generally, if X is one-dimensional and G acts nonfreely, then there are no G-invariant volume-preserving vector fields on X. Therefore, there is nothing to compute for the case of dimension less than two.

Recall from §3.4 that we call a subgroup $K < G$ *parabolic* if there exists a point $x \in X$ such that $\mathrm{Stab}_G(x) = K$. Let $\mathrm{Par}(G)$ be the set of parabolic subgroups of G. For $K \in \mathrm{Par}(G)$, the connected components of X^K are called *parabolic subvarieties* of X. By Proposition 3.41, these are exactly the closures of the leaves of $\mathfrak{v}$, which are the connected components of $(X^K)^\circ = \{x \in X \mid \mathrm{Stab}_G(x) = K\}$.

Let $X^\circ \subseteq X$ be the inclusion of the open locus where G acts freely. Clearly, $M(X, H(X)^G)|_{X^\circ} \cong \Omega_{X^\circ}$. Therefore, by adjunction, we have a map $H^0 j_! \Omega_{X^\circ} = \Omega_X \to M(X, H(V)^G)$, and the cokernel of this map is supported on a union of proper parabolic subvarieties of V. Suppose that $U \subseteq V^K$ is a maximal such subvariety for $K \in \mathrm{Par}(G)$. We claim that U is zero-dimensional, i.e., a finite union of points. By formally localizing in the neighborhood of a generic point of U, it suffices to assume that $K = G$. This reduces the claim to:

Lemma 6.1. Suppose that U and W are positive-dimensional vector spaces and $G < \mathsf{GL}(W)$ is finite. Then $M(U \times W, H(U \times W)^G) \cong \Omega_{U \times W}$.

Proof. Let $W^\circ \subseteq W$ be the open subset where G acts freely. Then $H(U \times W^\circ)^G$ is transitive, so $M(U \times W^\circ, H(U \times W^\circ)^G) \cong \Omega_{U \times W^\circ}$. Let $j : (U \times W^\circ) \hookrightarrow U \times W$ be the inclusion. Therefore one obtains a map $H^0 j_! M(U \times W^\circ, H(U \times W^\circ)^G) = \Omega_{U \times W} \to M(U \times W, H(U \times W)^G)$, which is obviously injective. To prove the lemma, therefore, we have to show that $M(U \times W, H(U \times W)^G)$ has no quotients supported on a proper subvariety of $U \times W$, i.e., one of the form $U \times W^K$ for some parabolic subgroup $K < G$ (in terms of Proposition 2.48, we have to show that there is no infinitesimal thickening of such a subvariety which is incompressible; we will not use this interpretation).

Let $X := U \times W$ and $\mathfrak{v} := H(U \times W)^G$. Suppose there were a quotient of $M(X, \mathfrak{v})$ supported on $U \times W^K$ for some parabolic subgroup $K < G$. By formally localizing in a neighborhood of a generic point of $U \times W^K$, we can reduce to the case that $K = G$; let us assume this. So we have to show that there is no quotient supported at $U \times \{0\}$.

Since $\mathfrak{v}$ includes constant vector fields in the U direction, the defining quotient $\mathcal{D}_X \twoheadrightarrow M(X, \mathfrak{v})$ factors through $\mathcal{D}_X \twoheadrightarrow \Omega_U \boxtimes \mathcal{D}_W$. Moreover, given a vector field $\xi \in \mathfrak{v}$, write $\xi = \xi_1 + \xi_2$ where $\xi_1 \in \mathcal{O}_W \otimes \mathrm{Vect}(U)$ and $\xi_2 \in \mathcal{O}_U \otimes \mathrm{Vect}(W)^G$. Let $D : \mathrm{Vect}(X) \to \mathcal{O}_X$ be the standard divergence function, i.e., $D(\xi) = L_\xi \omega / \omega$, where ω is the standard volume form on X. Then, since $\mathfrak{v}$ includes constant vector fields in the U direction, $\xi_1 + D(\xi_1) \in \mathfrak{v} \cdot \mathcal{D}_X$. Thus, $\xi_2 - D(\xi_1) = \xi_2 + D(\xi_2) \in \mathfrak{v} \cdot \mathcal{D}_X$ as well. Conversely, the constant vector fields in the U direction together with elements $\xi_2 + D(\xi_2)$ span $\mathfrak{v} \cdot \mathcal{D}_X$. We conclude that $M(X, \mathfrak{v}) = \mathfrak{v} \cdot \mathcal{D}_X \setminus \mathcal{D}_X$ is of the form

$$M(X, \mathfrak{v}) = \Omega_U \boxtimes N, \quad N = \langle \xi + D(\xi) \mid \xi \in \mathrm{Vect}(W)^G \rangle \cdot \mathcal{D}_W \setminus \mathcal{D}_W.$$

Therefore, the lemma reduces to showing that N admits no quotient supported at $0 \in W$. First of all, let $\mathrm{Eu}_W \in \mathrm{Vect}(W)$ be the Euler vector field on U. Then $\mathrm{Eu}_W + D(\mathrm{Eu}_W) = (\mathrm{Eu}_W + \dim(W)) \in \mathfrak{v} \cdot \mathcal{D}_X$. On the other hand, since $\dim(W) > 0$, $\mathrm{Eu}_W + \dim(W)$ acts by an automorphism on every quotient supported at zero (note that sections of the delta function $\mathcal{D}$-module are in nonpositive polynomial degree, and homogeneous sections in degree $m \leq 0$ are annihilated by $\mathrm{Eu} + m$ (since we are using right $\mathcal{D}$-modules)). Thus, N admits no such quotient. □

We conclude that the cokernel of the inclusion $\Omega_X \hookrightarrow M(X, \mathfrak{v})$ is supported at finitely many points, i.e., it is a direct sum of delta-function $\mathcal{D}$-modules at these points. Since we assumed that $\dim X \geq 2$, $\mathrm{Ext}(\Omega_X, \delta) = 0$ when δ is such a delta-function $\mathcal{D}$-module (this follows because it is true in the case $X = \mathbf{A}^n$ and the point is the origin). Therefore, $M(X, \mathfrak{v})$ is semisimple, and we can explicitly conclude its structure, as follows.

Definition 6.2. Let $\mathrm{Parpt}(X, G)$ be the collection of points which are parabolic subvarieties; call them *parabolic points*.

Equivalently, the parabolic points $x \in X$ are those such that, for some open neighborhood U containing x, $\mathrm{Stab}_G(x)$ is strictly larger than the stabilizer of any point in $U \setminus \{x\}$.

Theorem 6.3. $M(X, \mathfrak{v}) \cong \Omega_X \oplus \bigoplus_{x \in \mathrm{Parpt}(X,G)} \delta_x \otimes (\hat{\mathcal{O}}_{X,x})_{\mathfrak{v}}$, and each $(\hat{\mathcal{O}}_{X,x})_{\mathfrak{v}}$ is finite-dimensional.

Proof. By the preceding material, it remains only to compute the multiplicity of δ_x. Note that this must be finite-dimensional since $M(X, \mathfrak{v})$ is holonomic. The result thus follows from Lemma 5.10. □

7. Symmetric powers of varieties

Given $(X, \mathfrak{v})$, note that $\mathfrak{v}$ also acts naturally on the symmetric powers $S^n X := X^n / S_n$. Then, the diagonal embedding of X into $S^n X$ is invariant, and more generally, arbitrary diagonal embeddings are invariant.

In this section, we compute the coinvariants $(\mathcal{O}_{S^n X})_{\mathfrak{v}}$ as well as the $\mathcal{D}$-module $M(S^n X, \mathfrak{v})$ for all $n \geq 1$ in the transitive (affine) cases of §3 (the "global" versions of the simple Lie algebras of vector fields). In the symplectic case this specializes to the main result of [ES13]. Our main result says that, in the Calabi-Yau and symplectic cases, this is a direct sum of the pushforwards under $X^n \twoheadrightarrow S^n X$ of the canonical $\mathcal{D}$-modules Ω_Δ as Δ ranges over the diagonal subvarieties $\Delta \subseteq X^n$ up to the action of S_n. In other words, these are the intersection cohomology $\mathcal{D}$-modules of the diagonal subvarieties of $S^n X$. In the locally conformally symplectic case, and in a more general transitive setting that includes all of these cases, we prove the same result, except replacing Ω_Δ by the diagonal embedding of $M(X, \mathfrak{v})$. Moreover, when X is a contact variety and $\mathfrak{v} = H(X)$, or X is smooth and $\mathfrak{v} = \mathrm{Vect}(X)$, we show that $M(S^n X, \mathfrak{v}) = 0$, and extend these cases to a more general transitive setting where $\mathfrak{v}$ does not flow incompressibly.

More generally, we will prove general structure theorems on $M(S^n X, \mathfrak{v})$ in the case that $\mathfrak{v}$ is transitive and satisfies a certain condition we call *quasi-locality*, which essentially says that its restriction to the m-th infinitesimal neighborhood of every finite set is equal to the sum of its restrictions to the m-th infinitesimal neighborhood of each point in the set. For convenience, we will also generally assume that X is connected; it is easy to remove this assumption.

7.1. **Relation to Lie algebras for $S^n X$.** The study of $S^n X$ under $\mathfrak{v}$ is closely related to the study of $S^n X$ under its own associated Lie algebras of vector fields. Note that $\mathcal{O}_{S^n X} = \mathsf{Sym}^n \mathcal{O}_X$ is spanned by elements $f^{\otimes n}$ for $f \in \mathcal{O}_X$. Let $\mathrm{symm} : \mathcal{O}_X^{\otimes n} \to \mathsf{Sym}^n \mathcal{O}_X$ be the symmetrization map,

$$\mathrm{symm}(f_1 \otimes \cdots \otimes f_n) = \frac{1}{n!} \sum_{\sigma \in S_n} f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)}.$$

Note that, if X is Poisson with bivector field π, then so is $S^n X$, using the unique Poisson bracket on $\mathsf{Sym}^n \mathcal{O}_X$ obtained from the Leibniz rule; in other words, one can consider the bivector field $\sum_{i=1}^n \pi^i$ on $X^n = \mathsf{Spec}\, \mathcal{O}_X^{\otimes n}$, where $\pi^i = \mathrm{Id}^{\otimes(i-1)} \otimes \pi \otimes \mathrm{Id}^{\otimes(n-i)} \in (\wedge_{\mathcal{O}_X} T_X)^{\otimes n}$ denotes π acting on the i-th component. This then restricts to symmetric functions $\mathcal{O}_{S^n X} = \mathsf{Sym}^n \mathcal{O}_X$.

If X is even-dimensional and equipped with a top polyvector field Ξ, then $S^n X$ is equipped with the top polyvector field $\wedge^n \Xi$.

As discussed in Remark 3.22, when X is Jacobi, there is no natural Jacobi structure induced on X^n and hence neither on $S^n X$.

We then have the following elementary proposition (the first part was essentially used in [ES13]):

Proposition 7.1. (i) If X is Poisson, then $M(S^n X, H(X)) \cong M(S^n X, H(S^n X))$;
(ii) For X even-dimensional and equipped with a top polyvector field, $P(X) \subseteq P(S^n X)$;
(iii) For X equipped with a divergence function D on a coherent subsheaf $N \subseteq T_X$, one has $P(X, D) \subseteq P(S^n X, D)$, where S^n is equipped with a divergence function on $\mathcal{O}_{S^n X} \cdot N$, using the natural embedding of vector spaces $N \subseteq T_X \hookrightarrow T_{S^n X}$ (via extending derivations from $\mathcal{O}_X$ to $\mathcal{O}_{S^n X} = \mathsf{Sym}^n_{\mathbf{k}} \mathcal{O}_X$);
(iv) For general X, $\mathrm{Vect}(X) \subseteq \mathrm{Vect}(S^n X)$.

Proof. (i) Given $f \in \mathcal{O}_X$, it is evident that (up to normalization) $n\xi_{\mathrm{symm}(f \otimes 1^{\otimes(n-1)})}$ identifies with $\xi_f \in H(X)$. Hence $H(X) \subseteq H(S^n X)$ (this is also a special case of part (ii)). Next, $H(S^n X)$ is spanned by the vector fields $\xi_{f^{\otimes n}} = \mathrm{symm}(\xi_f \otimes f^{\otimes(n-1)})$ for $f \in \mathcal{O}_X$. Note the identities $\xi_f(f) = 0$ and $\xi_{f^i} = i f^{i-1} \xi_f$. Thus, for all $i \geq 1$,

$$\begin{aligned}(7.2)\quad \mathrm{symm}(\xi_{f^i} \otimes 1^{\otimes(n-1)}) \cdot \mathrm{symm}(f^{\otimes(n-i-1)} \otimes 1^{\otimes(i+1)}) \\ = \frac{i}{n} \mathrm{symm}(\xi_{f^i} \otimes f^{\otimes(n-i-1)} \otimes 1^{\otimes i}) \\ + \frac{n-i}{n} \mathrm{symm}(\xi_{f^{i+1}} \otimes f^{\otimes(n-i-2)} \otimes 1^{\otimes(i+1)}).\end{aligned}$$

The LHS is in $H(X) \cdot \mathcal{D}_X$, and the RHS terms, taken over all $i \geq 1$, generate $\mathrm{symm}(\xi_f \otimes f^{\otimes(n-1)})$, as desired.

(ii) It is evident that, if a vector field preserves a top polyvector field Ξ on X, then it also preserves $\wedge^n \Xi$ on $S^n X$.

(iii) Similarly, if a vector field ξ preserves a divergence function D, i.e., $D(\xi) = 0$, then also it preserves the induced divergence function on $S^n X$, i.e., the induced divergence function on $S^n X$ by definition also kills ξ, viewed as a vector field on $S^n X$.

(iv) Similarly, given a vector field $\xi \in \mathrm{Vect}(X)$, we can take the sum $\sum_{i=1}^n \xi^i \in \mathrm{Vect}(X^n)$ which descends to $\mathrm{Vect}(S^n X)$. $\square$

Remark 7.3. Note that the isomorphism of (i) does not extend, in general, to the cases of top polyvector fields. For instance, when X is symplectic, then by part (i), viewed as a Poisson variety, $H(S^n X)$ and $H(X)$ determine the same $\mathcal{D}$-module, which is holonomic since $S^n X$ has finitely many symplectic leaves (the images of the diagonal embeddings). However, since the singular locus of $S^n X$ is infinite for $n \geq 2$, by Corollary 3.36, $H(S^n X, \wedge^n \mathsf{vol}_X^{-1})$ does not have finitely many leaves, and by Corollary 3.37 the associated $\mathcal{D}$-module is not holonomic.

7.2. **Diagonal embeddings.** Let $\Delta_i : X \to X^i$ be the standard diagonal embeddings for all $i \geq 1$. Let $\mathrm{pr}_n : X^n \to S^n X$ be the projection. Recall that a partition λ of n, which we denote by $\lambda \vdash n$, is a tuple $(\lambda_1, \ldots, \lambda_k)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$ and $\lambda_1 + \cdots + \lambda_k = n$. In this case the length, $|\lambda|$, of λ is defined by $|\lambda| := k$. Given a partition $\lambda \vdash n$, define the product of diagonal embeddings

$$\Delta_\lambda := \Delta_{\lambda_1} \times \cdots \times \Delta_{\lambda_{|\lambda|}} : X^{|\lambda|} \to X^n.$$

Now, composing with pr_n, we obtain a map $X^{|\lambda|} \to S^n X$. On the complement of diagonals in $X^{|\lambda|}$, this is a covering onto its image whose covering group is the subgroup $S_\lambda < S_{|\lambda|}$ preserving the partition λ. Explicitly, $S_\lambda = S_{r_1} \times \cdots \times S_{r_k}$ where, for all j,

$$\lambda_{r_1+\cdots+r_j} > \lambda_{r_1+\cdots+r_j+1} = \lambda_{r_1+\cdots+r_j+2} = \cdots = \lambda_{r_1+\cdots+r_j+r_{j+1}}.$$

7.3. **A morphism of graded algebras.** Consider the canonical morphism of graded algebras

$$\Phi : \mathsf{Sym}(t \cdot ((\mathcal{O}_X)^*)^{\mathfrak{v}}[t]) \to \bigoplus_{n \geq 0} ((\mathcal{O}_{S^n X})^*)^{\mathfrak{v}}, \tag{7.4}$$

given by the formula

$$\Phi(t^{r_1}\phi_1 \otimes \cdots \otimes t^{r_k}\phi_k)(f_1 \otimes \cdots \otimes f_{r_1+\cdots+r_k}) = \prod_{i=1}^{k} \phi_i(f_{r_1+\cdots+r_{i-1}+1} \cdots f_{r_1+\cdots+r_i}).$$

Let us explain the graded algebra structures in (7.4). First, the grading is by degree in t on the left-hand side and by degree in n on the right-hand side. The algebra structure on the left-hand side is as in a symmetric algebra. The algebra structure on the right-hand side is obtained from the natural inclusions

$$\mathcal{O}_{S^{n+m}(X)} \hookrightarrow \mathcal{O}_{S^n(X)} \otimes \mathcal{O}_{S^m(X)}.$$

In other words, the above maps are the symmetrization maps,

$$(f_1 \otimes \cdots \otimes f_{m+n}) \mapsto \frac{m!n!}{(m+n)!} \sum_{I \subseteq \{1,\ldots,m+n\}} f_I \otimes f_{I^c},$$

where $f_I := \prod_{i \in I} f_i$, and I^c is the complement of I.

This induces a coproduct on $\bigoplus_{n \geq 0} \mathcal{O}_{S^n X}$ and hence an algebra structure on $\bigoplus_{n \geq 0} \mathcal{O}^*_{S^n X}$. The $\mathfrak{v}$-invariants form a subalgebra.

Moreover, replacing $(\mathcal{O}_{S^n X})_{\mathfrak{v}}$ by the derived pushforward $\pi_\bullet M(S^n X, \mathfrak{v})$ for $\pi : S^n X \to \mathrm{pt}$ the projection to a point, we obtain a bigraded algebra $\bigoplus_{n \geq 0} \pi_\bullet M(S^n X, \mathfrak{v})^*$, in de Rham and homological degrees. Then (7.4) becomes

$$\Phi : \mathsf{Sym}(t \cdot \pi_\bullet M(X, \mathfrak{v})^*[t]) \to \bigoplus_{n \geq 0} \pi_\bullet M(S^n X, \mathfrak{v})^*. \tag{7.5}$$

Here, $\bullet$ is the homological degree, and the symmetric algebra is supersymmetric where the parity is given by the homological degree (note that this *differs* from the de Rham parity in the case that $\dim X$ is odd). By Proposition 2.36 and Example 2.37, in the case that X is symplectic or Calabi-Yau, (7.5) can be restated as

$$\mathsf{Sym}(t \cdot H^{\dim X - \bullet}(X)^*[t]) \to \bigoplus_{n \geq 0} \pi_\bullet M(S^n X, \mathfrak{v})^*. \tag{7.6}$$

7.4. **Quotients of $M(S^nX)$ supported on diagonals.** For arbitrary $(X, \mathfrak{v})$, since each Δ_λ is a closed embedding, one has a natural epimorphism

$$M(X^n, \mathfrak{v}) \twoheadrightarrow (\Delta_\lambda)_* M(X^{|\lambda|}, \mathfrak{v}),$$

Next, note that $M(X^n, \mathfrak{v})$ is an S_n-equivariant $\mathcal{D}$-module, and one has $(\mathrm{pr}_n)_* M(X^n, \mathfrak{v})^{S_n} \cong M(S^n X, \mathfrak{v})$. The morphism above descends to a natural map

$$M(S^n X, \mathfrak{v}) \twoheadrightarrow (\mathrm{pr}_n)_* (\Delta_\lambda)_* M(X^{|\lambda|}, \mathfrak{v})^{S_\lambda}.$$

Summing over λ, we obtain a natural map

$$M(S^n X, \mathfrak{v}) \to \bigoplus_{\lambda \vdash n} (\mathrm{pr}_n)_* \big((\Delta_\lambda)_* M(X^{|\lambda|}, \mathfrak{v})\big)^{S_\lambda} \tag{7.7}$$

In the case that X is symplectic or Calabi-Yau, by Proposition 2.36 and Example 2.37 (7.7) can be restated as

$$M(S^n X, \mathfrak{v}) \to \bigoplus_{\lambda \vdash n} (\mathrm{pr}_n)_* \big((\Delta_\lambda)_* \Omega_X^{\boxtimes |\lambda|}\big)^{S_\lambda}. \tag{7.8}$$

7.5. **Main result.**

Theorem 7.9. (i) If X has pure dimension at least two and is locally conformally symplectic or Calabi-Yau, then with $\mathfrak{v} = H(X)$, (7.7) and (7.5) are isomorphisms.

(ii) If $(X, \mathfrak{v})$ is an (odd-dimensional) contact variety with $\mathfrak{v} = H(X)$, or $(X, \mathfrak{v})$ is connected, smooth, and positive-dimensional with $\mathfrak{v} = \mathrm{Vect}(X)$, then $M(S^n X, \mathfrak{v}) = 0$.

For the case where X is a Calabi-Yau curve, $\mathfrak{v}$ is one-dimensional, and $M(S^n X, \mathfrak{v})$ is not holonomic for $n > 1$.

Remark 7.10. In the symplectic and Calabi-Yau cases, one can alternatively consider $H(S^n X)$, $LH(S^n X)$, and $P(S^n X)$, where now $S^n X$ is viewed as either a Poisson variety (when X is symplectic) or as a variety equipped with a top polyvector field (when X is even-dimensional Calabi-Yau) or more generally one can consider $H(S^n X, D)$ and $P(S^n X, D)$ when X is odd-dimensional and equipped with a divergence function on $T_{S^n X} = T_{X^n}^{S_n}$ obtained from the Calabi-Yau divergence function on X^n. It is easy to see that the image of the map in (7.6) is invariant under all of these, since on each leaf, i.e., the complement in a diagonal $\mathrm{pr}_n \circ \Delta_\lambda(X^{|\lambda|})$ of smaller diagonals, the image of the corresponding functionals on the left-hand side are supported on this diagonal and invariant under all vector fields that preserve the given structure. Moreover, in the symplectic case, using $H(S^n X)$ (and hence $LH(S^n X)$) must give the same result by Proposition 7.1.(i) (as already noticed in [ES13]). This recovers the main result of [ES13] (where this observation was also used in the proof).

In the Calabi-Yau case, one can replace $\mathfrak{v}$ on the RHS of (7.6) by $P(S^n X)$, since here one also has $P(X) \subseteq P(S^n X)$, so the isomorphism factors through the same expression with $P(S^n X)$-invariants.

However, in the Calabi-Yau case, one cannot replace the RHS with $H(S^n X)$ or $LH(S^n X)$-invariants, since $H(X)$ is not contained in these in general. In fact, for $n \geq 2$, these invariants are infinite-dimensional: already when $X = \mathbf{A}^2$ equipped with the standard volume form, $S^2 \mathbf{A}^2 \cong (\mathbf{A}^2/(\mathbf{Z}/2)) \times \mathbf{A}^2$, so the coinvariants $(\mathcal{O}_{S^2\mathbf{A}^2})_{H_{\wedge^2 \Xi}(S^2\mathbf{A}^2)} = (\mathcal{O}_{S^2\mathbf{A}^2})_{LH_{\wedge^2 \Xi}(S^2\mathbf{A}^2)}$ are infinite-dimensional by Remark 3.40.

Remark 7.11. Theorem 7.9 may generalize in some form to the case where X is not necessarily transitive, but has a finite degenerate locus. As a first step, in [ES12], the authors prove that, when $X \subseteq \mathbf{A}^3$ is a quasihomogeneous isolated surface singularity and $\mathfrak{v} = H(X)$, then abstractly one still has an isomorphism

$$\mathsf{Sym}(t \cdot ((\mathcal{O}_X)^*)^{\mathfrak{v}}[t]) \cong \bigoplus_{n \geq 0} ((\mathcal{O}_{S^n X})^*)^{\mathfrak{v}}, \tag{7.12}$$

but only as algebras graded by symmetric power degree, not by the weight degree in $\mathcal{O}_X$. (To correct this, one can assign t weight degree $-d$, where the hypersurface cutting out X has weight d (note that here $\mathcal{O}_X$ has nonnegative weight and $(\mathcal{O}_X)^*$ has nonpositive weight). Then one does obtain an isomorphism of graded algebras.)

Question 7.13. Does the abstract algebra isomorphism (7.12), graded only by symmetric power degree, extend to the case where $X \subseteq \mathbf{A}^n$ is an arbitrary quasihomogeneous complete intersection with an isolated singularity, equipped with its top polyvector field from Example 3.39? Can it be corrected to an abstract bigraded isomorphism by assigning t the appropriate weight?

Question 7.14. Does the abstract algebra isomorphism (7.12) extend to the case of arbitrary (not necessarily quasihomogeneous) complete intersections with isolated singularities? What about if the complete intersection condition is dropped?

Finally, we remark that, even as nonequivariant $\mathcal{D}$-modules, the two sides of (7.7) are *not* in general isomorphic, because $M(S^n X, \mathfrak{v})$ is not in general semisimple.

In the case of the du Val singularities, the two sides of (7.7) are only abstractly isomorphic as non-$\mathbf{G}_m$-equivariant $\mathcal{D}$-modules, by [ES13, §1.3]. One can introduce a correction analogous to the above one to the RHS which makes the two sides isomorphic as $\mathbf{G}_m$-equivariant $\mathcal{D}$-modules, but we do not know of any natural isomorphism between the two.

7.6. **Smooth and contact varieties.** By Theorem 7.9, in the case that $(X, \mathfrak{v})$ is either $(X, \mathrm{Vect}(X))$ for smooth X, or $(X, H(X))$ for X an odd-dimensional contact variety, then $M(S^n X, \mathfrak{v}) = 0$ for all $n \geq 0$. However, it turns out that $M(X^n, \mathfrak{v})$ itself is *nonzero* when $n > \dim X$. Moreover, this can be explicitly computed as an S_n-equivariant $\mathcal{D}$-module.

We first construct some canonical quotients $M(X^n, \mathrm{Vect}(X)) \twoheadrightarrow (\Delta_n)_* \Omega_X$. Let $d := \dim X$. We can identify global sections of $(\Delta_n)_* \Omega_X$ with $\mathcal{O}_{\Delta_n(X)}$-linear polydifferential operators $\hat{\mathcal{O}}_{X^n, \Delta_n(X)} \to \Omega_{\Delta_n(X)}$. Then, we consider the operator

$$(f_1 \otimes \cdots \otimes f_n) \mapsto f_{d+2} \cdots f_n \sum_{\sigma \in S_{d+1}} \frac{1}{(d+1)!} \mathrm{sign}(\sigma) f_{\sigma(1)} df_{\sigma(2)} \wedge \cdots \wedge df_{\sigma(d+1)}.$$

We can see that this is $\mathcal{O}_{\Delta_n(X)}$-linear (to ensure this, we had to skew-symmetrize over S_{d+1} rather than S_d). Moreover, the $\mathbf{k}[S_n]$-orbit is actually spanned by $\mathbf{k}[S_{n+1}]$, and as a representation of S_{n+1}, is

$$\mathrm{Ind}_{S_d \times S_{n-d-1}} (\mathrm{sign} \boxtimes \mathbf{k}).$$

Thus, it has dimension $\binom{n-1}{d}$. Let L_n be the S_n-equivariant local system supported on $\Delta_n(X)$ of rank $\binom{n-1}{d}$ corresponding to this quotient (as a nonequivariant local system, it is $((\Delta_n)_* \Omega_X)^{\oplus \binom{n-1}{d}}$). More generally, given a decomposition $\{1, \ldots, n\} = P_1 \sqcup \cdots \sqcup P_m$ into cells, let $L_{P_1} \boxtimes \cdots \boxtimes L_{P_m}$ denote the corresponding tensor product of local systems $L_{|P_i|}$ in the components P_i (i.e., these are all obtained by permutation of components from the local system $L_{|P_1|} \boxtimes \cdots \boxtimes L_{|P_m|}$). Note that this is nonzero if and only if $|P_i| > d$ for all i.

Theorem 7.15. Suppose that $(X, \mathfrak{v})$ is either $(X, \mathrm{Vect}(X))$ for smooth X, or $(X, H(X))$ for X an odd-dimensional contact variety. Then, we have an isomorphism

$$M(X^n, \mathfrak{v}) = \bigoplus_{m \geq 1, P_1 \sqcup \cdots \sqcup P_m = \{1,\ldots,n\}} L_{P_1} \boxtimes \cdots \boxtimes L_{P_m}. \tag{7.16}$$

7.7. **Quasi-locality and a generalization of Theorem 7.9.**

Definition 7.17. Say that $(X, \mathfrak{v})$ is *quasi-local* if, for every n-tuple of distinct points $x_1, \ldots, x_n \in X$, and every choice of positive integers $m_1, \ldots, m_{n-1} \geq 1$, the subspace of $\mathfrak{v}$ of vector fields vanishing to orders m_i at x_i for all $1 \leq i \leq n-1$ topologically span $\mathfrak{v}|_{\hat{X}_{x_n}}$.

Equivalently, as stated in the beginning of the section, the evaluation of $\mathfrak{v}$ at every subscheme supported at a finite subset $S \subseteq X$ is the direct sum of its evaluations at each connected component of S (i.e., at each subscheme of S supported on a point of S_{red}).

Proposition 7.18. If $(X, \mathfrak{v})$ is quasi-local, then the leaves of $(S^n X, \mathfrak{v})$ are the images of the products of leaves of X under pr_n. In particular, if $(X, \mathfrak{v})$ has finitely many leaves, so does $(S^n X, \mathfrak{v})$, and the latter is holonomic.

Proof. At each point $\mathrm{pr}_n \circ \Delta_\lambda(x_1, \ldots, x_{|\lambda|})$,

$$\mathfrak{v}|_{T_{\mathrm{pr}_n \circ \Delta_\lambda(x_1,\ldots,x_{|\lambda|})x_1} S^n X} \cong T^{S_\lambda}_{\Delta_\lambda(x_1,\ldots,x_{|\lambda|})x_1 X^n}. \tag{7.19}$$

Therefore, along each diagonal, the flow of $\mathfrak{v}$ is transitive along the images of the products of leaves of X. □

Proposition 7.20. If X is Jacobi or equipped with a top polyvector field, then $(X, H(X))$ is quasi-local. Similarly, $(X, \mathrm{Vect}(X))$ is quasi-local.

Proof. We first consider the Jacobi case. Given points $x_1, \ldots, x_n \in X$, and any orders $m_1, \ldots, m_{n-1} \geq 1$, we can consider functions which vanish up to order m_i at x_i for $1 \leq i \leq n-1$. Since the x_i are distinct, these functions topologically span $\hat{\mathcal{O}}_{X,x_n}$. Therefore, the Hamiltonian vector fields of such functions topologically span all Hamiltonian vector fields in the formal neighborhood $\hat{X}_{x_n}$.

Next consider the Calabi-Yau case. This is similar: we replace functions which vanish up to order m_i at x_i for $1 \leq i \leq n-1$ by $(\dim X - 2)$-forms with this vanishing property. Again, these topologically span $\Omega_{\hat{X}_{x_n}}$, and we conclude the result.

For the case of all vector fields, this is immediate. □

Theorem 7.21. Suppose that $(X, \mathfrak{v})$ is transitive and quasi-local and that X has pure dimension at least 2.

(i) If $\mathfrak{v}$ flows incompressibly, then (7.7) is an isomorphism if and only if:
  (*) For all n, and any (or every) $x \in X$, the space of $\mathfrak{v}$-invariant polydifferential operators $\mathsf{Sym}^n \hat{\mathcal{O}}_{X,x} \to \hat{\mathcal{O}}_{X,x}$ is spanned by the multiplication operator.

(ii) If $\mathfrak{v}$ does not flow incompressibly, then $M(S^n X, \mathfrak{v}) = 0$ for all $n \geq 1$ if and only if, for all $n \geq 1$, there are no $\mathfrak{v}$-invariant polydifferential operators $\mathsf{Sym}^n \hat{\mathcal{O}}_{X,x} \to \Omega_{\hat{X}_x}$.

We will prove this theorem as a consequence of a further generalization (Theorem 7.24) which relaxes condition (*) below (and this result will be further generalized from $\mathsf{Sym}^n X$ to X^n in Theorem 7.29). But, first, we explain why this theorem implies Theorem 7.9:

Proposition 7.22. (i) Let $(X, H(X))$ be locally conformally symplectic or Calabi-Yau of pure dimension at least two. Then (*) of Theorem 7.21 is satisfied.

(ii) In the case where $(X, \mathfrak{v})$ is either an odd-dimensional contact variety with $\mathfrak{v} = H(X)$, or smooth with $\mathfrak{v} = \mathrm{Vect}(X)$, then for all $x \in X$, all $\mathfrak{v}$-invariant polydifferential operators $\hat{\mathcal{O}}_{X,x}^{\otimes n} \to \Omega_{\hat{X},x}$ are spanned over $\mathbf{k}[S_n]$ by the operator

$$(f_1 \otimes \cdots \otimes f_n) \mapsto f_1 \cdots f_{n-\dim X} df_{n-\dim X+1} \wedge \cdots \wedge df_n.$$

In particular there are no symmetric such operators.

Proof. (i) This relies on the Darboux theorem, following [ES13, Lemma 2.1.8]. In a formal neighborhood $\hat{X}_x$, we can reduce to the case of the standard symplectic or Calabi-Yau structure, since in the locally conformally symplectic case, $H(\hat{X}_x)$ equals $H(\hat{X}_x, \omega_0)$, where ω_0 is a standard symplectic structure, as explained in Example 3.25.

Now, given a polydifferential operator $\phi : \mathsf{Sym}^n \hat{\mathcal{O}}_{X,x} \to \hat{\mathcal{O}}_{X,x}$, view it as a polynomial function $\bar{\phi} : \hat{\mathcal{O}}_{X,x} \to \hat{\mathcal{O}}_{X,x}$ on the pro-vector space $\hat{\mathcal{O}}_{X,x}$. Then $\bar{\phi}$ is uniquely determined by its restriction to functions with nonvanishing first derivative, since the complement has codimension at least two. Let $f \in \hat{\mathcal{O}}_{X,x}$ be such a function. Let $G_{X,x}$ be the formal group obtained by integrating $H(X)$, which acts on $\hat{\mathcal{O}}_{X,x}$. By the Darboux theorem, there is a coordinate change by $G_{X,x}$ that takes f to a coordinate function x_1 of X. Now, if a polydifferential operator is invariant under $H(X)$, it must take x_1 to a function invariant under the formal subgroup of $G_{X,x}$ preserving x_1, i.e., to a polynomial in x_1. Now, to be invariant under automorphisms in $G_{X,x}$ sending x_1 to λx_1, ϕ must have the form $x_1 \mapsto c \cdot x_1^n$ for some $c \in \mathbf{k}$. It remains to note that, if $f, g \in \hat{\mathcal{O}}_{X,x}$ are two functions with nonvanishing first derivative, again by the Darboux theorem there is an automorphism of $G_{X,x}$ sending f to g so the constant c must be independent of the choice of f. Therefore, $\bar{\phi}(g) = cg^n$ for all g. We can easily see that this is $\mathfrak{v}$-invariant.

(ii) Restricting to $\hat{X}_x$, suppose first that $\mathfrak{v}$ is arbitrary such that, in some coordinate system, it contains the constant vector fields and an Euler vector $\mathrm{Eu} = \sum_i m_i \partial_i$ for $m_i > 0$. Let $m := \sum_i m_i$. Let vol be the standard volume form in this coordinate system. The polydifferential operators $\hat{\mathcal{O}}_{X,x}^{\otimes n} \to \Omega_{\hat{X}_x}$ invariant under the aforementioned vector fields are spanned by

$$(F_1 \otimes \cdots \otimes F_n) \cdot \mathsf{vol}, \quad |F_1| + \cdots + |F_n| = -m,$$

where each F_i is a constant-coefficient monomial in the ∂_i, and here $|\cdot|$ denotes the weighted degree with respect to Eu. This is a finite-dimensional vector space.

Now, in the case where $\mathfrak{v} = \mathrm{Vect}(X)$, in order to be invariant under all possible Euler vector fields, the operator must be a linear combination of terms such that $F_1 \cdots F_n$ is linear in each coordinate. Moreover, to be invariant under volume-preserving linear changes of basis, i.e., under $\mathsf{SL}(T_x X)$, we conclude that the operator is spanned by images under S_n of $(\mathsf{vol}^{-1} \otimes 1^{\otimes(n-\dim X)}) \cdot \mathsf{vol}$, as desired.

In the case $\mathfrak{v} = H(X)$ and X is odd-dimensional contact variety, then we can take Eu as in Example 3.24, so that $F_1 \cdots F_n$ must have total degree $-(\dim X + 1)$ (since $|x_i| = |y_i| = 1$ and $|t| = 2$, and the partial derivatives have negative this degree). Also, the polydifferential operator must be preserved by all linear changes of basis preserving ∂_t. In particular, since it is preserved by $\mathsf{GL}(\langle \partial_{x_i}, \partial_{y_i} \rangle)$, the operator must be in the $\mathbf{k}[S_n]$-span of

$$\left(\wedge^{\dim X - 1} \langle \partial_{x_i}, \partial_{y_i} \rangle \otimes \partial_t \otimes 1^{\otimes(n-\dim X)}\right) \cdot \mathsf{vol}.$$

Since it is preserved by transformations $x_i \mapsto x_i + \lambda t$ for $\lambda \in \mathbf{k}$, we conclude in fact that it is in the $\mathbf{k}[S_n]$-span of $(\mathsf{vol}^{-1} \otimes 1^{\otimes(n-\dim X)}) \cdot \mathsf{vol}$, as desired. $\square$

7.8. **General decomposition statement.** First, we generalize Theorem 7.21 by replacing (*) by a general decomposition statement about $M(S^n X, \mathfrak{v})$. Then (*) becomes a multiplicity-one condition.

Definition 7.23. Given a smooth affine variety X and an integer $m \geq 1$, let $\mathrm{PDiff}(\mathcal{O}_X, \Omega_X, m+1)$ be the space of polydifferential operators $\mathcal{O}_X^{\otimes m} \to \Omega_X$ of degree m, i.e., linear maps which are differential operators in each component.

Note that there there is a natural action of S_{m+1} on $\mathrm{PDiff}(\mathcal{O}_X, \Omega_X, m+1)$ given by viewing these operators as distributions on the diagonal in X^{m+1}, i.e., as sections of the $\mathcal{D}_{X^{m+1}}$-module $(\Delta_{m+1})_*\Omega_X$, which has its natural S_{m+1}-action. The S_m action is just by permutation of components, and the extension to S_{m+1} is explicitly given by the integration by parts rule. For example, when $X = \mathbf{A}^1$ with the standard volume, this action restricted to the span of partial derivatives $\partial_1, \ldots, \partial_m$ is the reflection representation of S_{m+1} (viewed as a type A_m Weyl group); explicitly this can be viewed as the usual permutation action on $\partial_1, \ldots, \partial_{m+1}$ where we set $\partial_{m+1} = -\sum_{i=1}^n \partial_i$.

For all $m \geq 1$, let L_m be the maximal quotient of $M(S^m X, \mathfrak{v})$ supported on the diagonal, i.e., $L_m = (\mathrm{pr}_m \circ \Delta_m)_*(\mathrm{pr}_m \circ \Delta_m)^* M(S^m X, \mathfrak{v})$ (which at least makes sense when $M(S^m X, \mathfrak{v})$ is holonomic, as in the quasi-local transitive case).

Theorem 7.24. Suppose that $(X, \mathfrak{v})$ is quasi-local and transitive and has pure dimension at least two. Then, there is a canonical isomorphism

$$M(S^n X, \mathfrak{v}) \xrightarrow{\sim} \bigoplus_{\lambda \vdash n} (\mathrm{pr}_n)_*(\Delta_\lambda)_* L_\lambda^{S_\lambda}, \quad L_\lambda := L_{\lambda_1} \boxtimes \cdots \boxtimes L_{\lambda_{|\lambda|}}. \tag{7.25}$$

Moreover, the rank of L_m is equal to the dimension of $(\mathrm{PDiff}(\hat{\mathcal{O}}_{X,x}, \Omega_{\hat{X}_x}, m)^{\mathfrak{v}})^{S_m}$.

The canonical isomorphism is given by the direct sum of the morphisms

$$M(S^n X, \mathfrak{v}) \twoheadrightarrow (\mathrm{pr}_n)_*(\Delta_\lambda)_* L_\lambda^{S_\lambda}, \tag{7.26}$$

obtained by adjunction from the canonical quotients $(\mathrm{pr}_n \circ \Delta_\lambda)^* M(S^n X, \mathfrak{v}) \twoheadrightarrow L_\lambda$.

The theorem implies that composition factors from distinct leaves do not appear in nontrivial extensions:

Corollary 7.27. In the situation of the theorem, $M(S^n X, \mathfrak{v})$ is a direct sum of intermediate extensions of local systems on the leaves (locally closed diagonals).

Proof. For each diagonal $X_\lambda := \mathrm{pr}_n \circ \Delta_\lambda(X^{|\lambda|})$, let $j_\lambda : X_\lambda^\circ \hookrightarrow X_\lambda$ be the open embedding of the complement of smaller diagonals, i.e., such that X_λ° is a leaf of $S^n X$. Let $\tilde{j}_\lambda : \tilde{X}_\lambda^\circ \hookrightarrow \Delta_\lambda(X) \subseteq X^n$ be the preimage of X_λ°. Then, for each factor in (7.25),

$$j^*(\mathrm{pr}_n)_*(\Delta_\lambda)_* L_\lambda^{S_\lambda} \cong (\mathrm{pr}_n)_* \tilde{j}_\lambda^* (\Delta_\lambda)_* L_\lambda^{S_\lambda}.$$

Since pr_n is a covering of Y_λ° onto its image (with covering group S_λ), the above is a local system on X_λ°. It now suffices to prove that

$$(\mathrm{pr}_n)_*(\Delta_\lambda)_* L_\lambda^{S_\lambda} \cong j_{!*} j^* (\mathrm{pr}_n)_*(\Delta_\lambda)_* L_\lambda^{S_\lambda}. \tag{7.28}$$

This follows because, since pr_n is finite, the singular support of $(\mathrm{pr}_n)_*(\Delta_\lambda)_* L_\lambda^{S_\lambda}$ is the closure of the conormal bundle of the leaf $\mathrm{pr}_n \circ \Delta_\lambda((X^{|\lambda|})^\circ)$, where $(X^{|\lambda|})^\circ$ is the complement in $X^{|\lambda|}$ of the images of all diagonal embeddings of X^r for all $r < |\lambda|$. ☐

We can make a similar statement about $M(X^n, \mathfrak{v})$ itself: Let $\tilde{L}_m = (\Delta_m)_* \Delta_m^* M(X^m, \mathfrak{v})$ be the maximal quotient of $M(X^m, \mathfrak{v})$ supported on the diagonal. This is S_m-equivariant, and $(\tilde{L}_m)^{S_m} = L_m$.

Theorem 7.29. Let $(X, \mathfrak{v})$ be as in Theorem 7.24. Then, there is a canonical isomorphism

$$M(X^n, \mathfrak{v}) \xrightarrow{\sim} \bigoplus_{\lambda \vdash n} S_n(\widetilde{L_\lambda}), \quad \widetilde{L_\lambda} := \widetilde{L_{\lambda_1}} \boxtimes \cdots \boxtimes \widetilde{L_{\lambda_{|\lambda|}}}. \tag{7.30}$$

Here, $S_n(\widetilde{L_\lambda})$ is the S_n-equivariant local system on the S_n-orbit of $\Delta_\lambda(X)$ whose restriction to $\Delta_\lambda(X)$ is the $N_{S_n}(S_\lambda)/S_\lambda$-equivariant local system $\tilde{L}_m$.

Moreover, the rank of $\tilde{L}_m$ is the dimension of $\mathrm{PDiff}(\hat{\mathcal{O}}_{X,x}, \Omega_{\hat{X}_x}, m)^{\mathfrak{v}}$.

As in Corollary 7.27, it follows from this that the entire pushforward $(\mathrm{pr}_n)_* M(X^n, \mathfrak{v})$ on $S^n X$ is a direct sum of intermediate extensions of S_λ-equivariant local systems on the diagonals corresponding to partitions $\lambda \vdash n$.

7.9. **Proof of Theorems 7.24 and 7.29.** We will work with $M(X^n, \mathfrak{v})$. Since this is S_n-equivariant and $M(S^n X, \mathfrak{v}) = (\mathrm{pr}_n)_* M(X^n, \mathfrak{v})^{S_n}$, this will also compute the latter.

By transitivity and quasi-locality, the closures of the leaves of $M(X^n, \mathfrak{v})$ are the diagonals $\Delta_\lambda(X^{|\lambda|})$ together with the diagonals obtained from these by the action of S_n. Hence, $M(X^n, \mathfrak{v})$ is holonomic and its composition factors are intermediate extensions of local systems on these leaves. Similarly to (7.26), one has canonical surjections

$$M(X^n, \mathfrak{v}) \twoheadrightarrow (\Delta_\lambda)_* L_\lambda, \tag{7.31}$$

and similarly for the orbits of these under S_n (there is one of these for each coset in $S_n/(N_{S_n}(S_\lambda))$, and each is a local system on the image of $\Delta_\lambda(X^{|\lambda|})$ under the element of S_n which is equivariant under the corresponding conjugate of the subgroup $N_{S_n}(S_\lambda) < S_n$). It suffices to prove the following:

(i) The quotient (7.31) is the maximal quotient supported on $\Delta_\lambda(X^{|\lambda|})$, i.e., it is $(\Delta_\lambda)_*(\Delta_\lambda)^* M(X^n, \mathfrak{v})$;
(ii) For distinct λ or distinct orbits for a fixed λ, that the above factors have no nontrivial extensions (i.e., the Ext group of the two is zero).

For (i), by restricting to a formal neighborhood of a generic point $y = \Delta_\lambda(x)$ of $\Delta_\lambda(X^{|\lambda|})$, it suffices to find an isomorphism

$$\mathrm{Hom}_{\hat{\mathcal{D}}_{X^n,y}}(M(X^n, \mathfrak{v})|_{\widehat{X^n}_y}, (\Delta_\lambda)_* \Omega_{\widehat{X^{|\lambda|}}_x}) \cong \bigotimes_{i=1}^{|\lambda|} \mathrm{PDiff}(\hat{\mathcal{O}}_{X,x}, \Omega_{\hat{X}_x}, \lambda_i)^{\mathfrak{v}}.$$

By quasi-locality, it suffices to restrict to the case $|\lambda| = 1$ (for all n). For this, note that there is a canonical isomorphism

$$(\Delta_n)_* \Omega_{\widehat{X,x}} \cong \mathrm{PDiff}(\hat{\mathcal{O}}_{X,x}, n).$$

Moreover, for any $\mathcal{D}_{X^n}$-module N, we have a canonical isomorphism $\mathrm{Hom}(M(X^n, \mathfrak{v}), N) \cong N^{\mathfrak{v}}$, by considering the image of the canonical generator of $M(X^n, \mathfrak{v})$. Putting these together, we deduce part (i).

For (ii), note that the factors $(\Delta_\lambda)_* L_\lambda$, as well as their images under the action of S_n, are local systems on smooth closed subvarieties of X^n. Moreover, the intersection of two of these subvarieties has codimension a multiple of $\dim X$ in each, which in particular is codimension at least 2. Thus, the result follows from the following basic lemma:

Lemma 7.32. [ES13, Lemma 2.1.1] Suppose that Z is a smooth variety, and $Z_1, Z_2 \subseteq Z$ as well as $Z_1 \cap Z_2$ are smooth closed subvarieties, all of pure dimension. Let $\mathcal{L}_1, \mathcal{L}_2$ be local systems on Z_1 and Z_2, respectively, and let $i_1 : Z_1 \to Z$ and $i_2 : Z_2 \to Z$ be the inclusions. Then,

$$\mathrm{Ext}^j((i_1)_* \mathcal{L}_1, (i_2)_* \mathcal{L}_2) = 0, \text{ for } j < (\dim Z_1 - \dim Z_1 \cap Z_2) + (\dim Z_2 - \dim Z_1 \cap Z_2). \tag{7.33}$$

7.10. **Proof of Theorem 7.21.** (i) If $\mathfrak{v}$ flows incompressibly, then we have an isomorphism of modules over the Lie algebra $\mathfrak{v}$, $\hat{\mathcal{O}}_{X,x} \xrightarrow{\sim} \Omega_{\hat{X}_x}$, obtained from the formal volume at x preserved by $\mathfrak{v}$. Therefore, in the theorem, we can replace the polydifferential operators described by $(\mathrm{PDiff}(\hat{\mathcal{O}}_{X,x}, \Omega_{\hat{X}_x}, n+1)^{\mathfrak{v}})^{S_n}$. Then, the result is almost immediate from Theorem 7.24, except Theorem 7.21 deals with S_n-invariant polydifferential operators, whereas the multiplicity spaces of Theorem 7.24 are more symmetric: they are $(\mathrm{PDiff}(\hat{\mathcal{O}}_{X,x}, \Omega_{\hat{X}_x}, n+1)^{\mathfrak{v}})^{S_{n+1}}$.

Thus, it suffices to show that, if such $\mathfrak{v}$-invariant S_{n+1}-invariant polydifferential operators of degree n are spanned by the multiplication operator *for all* n, then the same is true requiring only S_n-invariance.

For this, note that, given a $\mathfrak{v}$-invariant polydifferential operator ϕ on $\hat{\mathcal{O}}_{X,x}$ of degree n, then the space of $\mathfrak{v}$-invariant polydifferential operators of degree $n+1$ includes the space $\mathrm{Ind}_{S_n \times S_1}^{S_{n+1}} \langle \phi \boxtimes \mathbf{k} \rangle$ spanned over S_{n+1} by the operator $f \boxtimes g \mapsto \phi(f) \cdot g$ for all $f \in \hat{\mathcal{O}}_{X,x}^{\otimes n}$ and $g \in \hat{\mathcal{O}}_{X,x}$.

To proceed, we will need the following technical combinatorial result, which we prove below:

Lemma 7.34. Suppose that ϕ generates a S_{n+1}-representation V, and that ϕ is S_n-invariant but not S_{n+1}-invariant. Then the S_{n+1}-representation $\mathrm{Ind}_{S_n \times S_1}^{S_{n+1}} (V|_{S_n} \boxtimes \mathbf{k})$ extends to a unique S_{n+2}-representation, up to isomorphism, and this has a nonzero S_{n+2}-invariant vector.

Let us use the lemma to finish the proof of the first statement. We conclude from the lemma that there exists a S_{n+2}-invariant, $\mathfrak{v}$-invariant polydifferential operator ϕ on $\hat{\mathcal{O}}_{X,x}$ of degree $n+1$. We claim that this is not the multiplication operator (up to scaling). Indeed, we could have assumed that ϕ were homogeneous of positive order (since $\mathfrak{v}$ preserves the grading by order of differential operators), so the latter S_{n+2}-operator can be assumed to have positive order. This contradicts our hypothesis. Hence, (*) of Theorem 7.21 is indeed satisfied.

(ii) If $\mathfrak{v}$ does not flow incompressibly, $M(X, \mathfrak{v}) = 0$, by Proposition 2.36. Next, suppose that there existed an S_n-invariant, $\mathfrak{v}$-invariant polydifferential operator $\phi : \hat{\mathcal{O}}_{X,x}^{\otimes n} \to \Omega_{\hat{X}_x}$ but not a S_{n+1}-invariant one. Again, we can form the polydifferential operator $(\phi \boxtimes 1)$, sending $f_1 \otimes \cdots \otimes f_{n+1}$ to $\phi(f_1 \otimes \cdots \otimes f_n) f_{n+1}$. So as before, we would obtain that, as S_{n+1}-representations, $\mathrm{PDiff}(\hat{\mathcal{O}}_{X,x}, \Omega_{\hat{X}_x}, n+2)^{\mathfrak{v}} \supseteq \mathrm{Ind}_{S_n \times S_1}^{S_{n+1}} (\mathbf{k}^n \boxtimes \mathbf{k})$. By the same argument as above, this would contain an S_{n+2}-invariant operator. Thus, $M(S^{n+2} X, \mathfrak{v}) \neq 0$. So, if $M(S^n X, \mathfrak{v}) = 0$ for all $n \geq 1$, then there are no S_n-invariant, $\mathfrak{v}$-invariant polydifferential operators $\phi : \hat{\mathcal{O}}_{X,x}^{\otimes n} \to \Omega_{\hat{X}_x}$, for all n. The converse is clear from Theorem 7.24.

Proof of Lemma 7.34. Under the assumption, V must include a summand isomorphic to the reflection representation $\mathbf{k}^n$ (V is either this or $\mathbf{k}^n$ with a trivial representation). As a representation of S_n, $\mathbf{k}^n$ is the standard representation.

Thus, we can assume that $V = \mathbf{k}^n$. As an S_n-representation, $V \cong \mathbf{k}^{n-1} \oplus \mathbf{k}$. Then, for $n \geq 3$, one computes the decomposition into irreducible S_{n+1}-representations:

$$\mathrm{Ind}_{S_n \times S_1}^{S_{n+1}} V|_{S_n} \boxtimes \mathbf{k} \cong \rho_{(1,1)[n+1]} \oplus \rho_{(2)[n+1]} \oplus \mathbf{k}^n \oplus \mathbf{k}^n \oplus \mathbf{k},$$

where, given a partition $\lambda \vdash n+1$, the representation ρ_λ is the irreducible representation with Young diagram λ. Moreover, given $\lambda' \vdash m$, we let $\lambda'[n+1]$ denotes the diagram obtained from λ' by adding a new row on top with $n+1-m$ boxes. Now, if the S_{n+1}-structure above extends to a S_{n+2}-structure, then the decomposition into irreducible S_{n+2}-representations (up to isomorphism) must be

$$\rho_{(1,1)[n+2]} \oplus \rho_{(2)[n+2]} \oplus \mathbf{k}.$$

We conclude that $\mathrm{Ind}_{S_n \times S_1}^{S_{n+1}} \langle \phi \boxtimes \mathbf{k} \rangle$ must contain a S_{n+2}-fixed vector.

In the case that $n = 2$, the second decomposition (as $S_{n+2} = S_4$-representations) above is still valid, so we still obtain the S_{n+2}-fixed vector. □

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MIT Dept of Math, Room 2-176, 77 Massachusetts Ave, Cambridge, MA 02139, USA
E-mail address: etingof@math.mit.edu

U. Texas at Austin, Math Dept, RLM 8.100, 2515 Speedway Stop C1200, Austin, TX 78712, USA
E-mail address: trasched@gmail.com