ALGEBRAIC PROPERTIES OF ANOMALIES

by

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To Riva

for her love and support

ABSTRACT

The work presented here is divided into three main sections. In the introduction some basic general properties of anomalies are reviewed, mainly through the example of the d=2 abelian anomaly. We also review the connection between this anomaly and the Kac-Moody algebra of free fermionic currents.

In chapter II we then investigate the algebra of free fermionic currents in d=4. It is proved for a specific case that the Jacobi identity is not compatible with a certain set of standard assumptions which are normally assumed to hold for field theories. The proof is supplemented with a full calculation of the relevant double commutators which suggests an interesting possibility of getting well-defined finite results. We then discuss the relationship between this result and the anomaly of the d=4 axial current.

In chapter III, the algebraic properties of abelian gauge theories are studied in the Hamiltonian formalism both in d=2 and d=4. In d=2 it is shown explicitly how the anomaly modifies the Poincaré algebra, besides the already known modifications of the Gauss-law constraint algebra. In d=4, employing a one loop BJL calculation, it is found that the anomaly leads to a breaking of the Jacobi identity in the algebra of the Hamiltonian with the Gauss-law. It is also shown that in order to reproduce the anomaly, a square diagram, with one fermionic energy-momentum tensor and three fermionic currents, should be considered.

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PREFACE

The work presented in this thesis was carried out in the Theoretical Particle Physics group at the Department of Physics, Imperial College, London, between April 1985 and April 1987, under the supervision of Doctor Ian G. Halliday. Unless otherwise stated, the work is original and has not been submitted before for a degree in this or any other university. The material in chapter II and appendix A was published by the author in Nuclear Physics B282 (1987) 367. The material in chapter III and appendices C,D,E has been submitted for publication (Imperial/TP/86-87/17).

THE THOUGHT-FOX / Ted Hughes

I imagine this midnight moment's forest
Something else is alive
Besides the clock's loneliness
And this blank page where my fingers move.

Through the window I see no star:
Something more near
Though deeper within darkness
Is entering the loneliness:

Cold, delicately as the dark snow,
A fox's nose touches twig, leaf;
Two eyes serve a movement, that now
And again now, and now, and now

Sets neat prints into the snow
Between trees, and warily a lame
Shadow lags by stump and in hollow
Of a body that is bold to come

Across clearings, an eye,
A widening deepening greenness
Brilliantly, concentratedly,
Coming about its own business

Till, with a sudden sharp hot stink of fox
It enters the dark hole of the head.
The window is starless still; the clock ticks,
The page is printed.

Notation and conventions

This work follows the conventions of ref. 3-10, which are basically those of ref. 3-15. Here is a short summary of some relevant ones (d=2 conventions are given in parenthesis):

a. A Greek letter index runs over 0,1,2,3 (0,1) while a Latin index runs over 1,2,3 (1) only. Repeated indices (Greek or Latin) are to be summed over.

b. $g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ $(\mathrm{diag}(1,-1))$; $\delta_{ij} = - g_{ij}$.

c. $\varepsilon_{0123} = -\varepsilon^{0123} = \varepsilon^{123} = 1$ $(\varepsilon_{01} = 1)$;
$\gamma_5 = \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ $(\gamma^5 = \gamma^0\gamma^1)$.

d. $B(x) \equiv B(t,\vec{x})$ and $B(\vec{x}) \equiv B(0,\vec{x})$, for any operator B.

e. $F_{\mu\nu} \equiv \partial_\nu A_\mu - \partial_\mu A_\nu$; $E^i \equiv F^{oi}$; $B^i \equiv (\nabla\times A)^i \equiv \varepsilon^{im\ell}\partial_m A^\ell$; In the Weyl gauge we have $E^i = - \partial^o A^i$, where A_μ is a U(1) gauge field.

f. The generators λ^a of a non-abelian group G are taken in the fundamental representation, and are normalized to give $\mathrm{tr}(\lambda^a\lambda^b) = \frac{1}{2}\delta^{ab}$. $d^{abc} \propto \mathrm{tr}(\lambda^a\{\lambda^b,\lambda^c\})$.

g. All commutators, unless otherwise specified, are equal-time commutators.

Other conventions or deviations from the above will be given at the introduction to each of the chapters or at the appropriate places when necessary.

We will also make use of the following abbreviations: J- Jacobi identity or the left-hand side of it (the sum of the three double commutators), VEV- vacuum expectation value, W.I.- Ward Identity, ST- Schwinger term, E.T.C- equal time commutator, C.C.R- canonical commutation relations, C.S.M. - Chiral Schwinger model.

CHAPTER I - Introduction to anomalies

1.1 Classical symmetries and conserved currents

By now, anomalies are recognized as a fundamental feature of quantum theories. Theoretically, their study led to the discovery of important topological, geometrical and algebraic structures and to a much better understanding of quantum theories themselves. In parallel to that, various experimental predictions based on the knowledge of anomalies were derived and verified. The principle of anomaly cancellation, for instance, is a clue to the understanding of the basic experimental fact of the equal magnitudes of the electron and proton charges.

The subject of anomalies and their implications has grown very rapidly, and it is even hard to give a good definition of it. In this brief introduction, I'll just outline a few properties and results, which are at the background of the present work.

We normally say that there is an anomaly when a symmetry of a classical theory is not respected by its quantum version. Noether's theorem tells us that when a classical theory is invariant under a continuous symmetry, there should exist corresponding conserved currents. In a theory containing fermions, one can define a transformation for the fermion field Ψ by :

$$\Psi(x) \rightarrow e^{i\alpha}\Psi(x) \;;\; \alpha \text{ real} \qquad (1.1)$$

These transformations form a U(1) group which will be denoted $U_V(1)$. If the theory in question is invariant under this group, we'll find a conserved current:

$$\partial^{\mu} J^{V}_{\mu}(x) = 0 \tag{1.2}$$

We can also define another type of U(1) transformations:

$$\Psi(x) \rightarrow e^{i\beta\gamma_5}\Psi(x) \ ; \ \beta \text{ real} \tag{1.3}$$

which we denote by $U_A(1)$. The classically conserved current will be denoted by $J^A_\mu(x)$. In the theories we'll deal with, these currents are given as bilinears in the fermion fields:

$$J^V_\mu(x)=\overline{\Psi}(x)\gamma_\mu\Psi(x) \ ; \ J^A_\mu(x)=\overline{\Psi}(x)\gamma_\mu\gamma_5\Psi(x) \tag{1.4}$$

In the quantum theory the conservation of the currents is translated into certain conditions on the Green's functions of the currents, known as the Ward-Takahashi identities (hereafter, W.I.). An anomaly then means that we cannot satisfy simultaneously all the W.I.'s and other physical conditions one would like to impose on a given Green's function. In order to demonstrate how this situation occurs we now specialize to d=2.

1.2 The axial anomaly in d=2

In many aspects, the gauge anomaly in two space-time dimensions, provides a good starting point to learn about anomalies in quantum field theory. The models are technically easy to handle, but still possess essential features.

Consider the following current-current Green's function:

$$\Pi_{\mu\nu}(x) = \langle 0|\, T(J_\mu(x)J_\nu(0)\, |0\rangle \tag{1.5}$$

where each of the J's can be either a vector or an axial current. The Fourier transform of $\Pi_{\mu\nu}(x)$ is defined by:

$$\Pi_{\mu\nu}(p) \equiv \int d^2x\; e^{-ip\cdot x}\Pi_{\mu\nu}(x) \tag{1.6}$$

We'll derive the W.I. for $\Pi_{\mu\nu}(p)$:

$$\partial^\mu\Pi_{\mu\nu}(x) = \partial^\mu \;\langle 0|\, \Theta(x^o)J_\mu(x)J_\nu(0)+\Theta(-x^o)J_\nu(0)J_\mu(x)\,|0\rangle \tag{1.7}$$

Since $\partial^\mu\Theta(x^o) = g^{\mu o}\delta(x^o)$, we get:

$$\partial^\mu\Pi_{\mu\nu}(x) = g^{\mu o}\delta(x^o)\;\langle 0|\,[J_\mu(x),J_\nu(0)]\,|0\rangle + \tag{1.8}$$

$$+\;\langle 0|\, T(\partial^\mu J_\mu(x)J_\nu(0))\,|0\rangle$$

Using current conservation, the second term is zero. The first term is the equal time commutator of the currents. It has a classical part which one can get by naive application of the canonical anti commutation relations of $\Psi(x)$ and its conjugate (see appendix A). For our 2-dim. case there is no contribution of this type. The canonical terms are normally (at least for internal symmetry currents) proportional to a Dirac delta function (whose argument is the space coordinates difference). There might also be other terms proportional to derivatives of delta. To get these terms one should define the equal time commutator more carefully, taking into account quantum divergences. Such calculations are at the heart of the present work. The non-canonical contributions to the commutator exposed by them are generally called Schwinger terms. It was Schwinger who proved that if $\langle 0| [J_o, J_i] |0\rangle$ were to vanish, one would be in conflict with very basic assumptions of quantum field theories. For this specific case, the Schwinger term is a c-number, but we'll call any non-canonical contribution to a commutator, a Schwinger term (abbreviated to ST). Another name in use is commutator anomalies.

We'll now proceed with the derivation of the W.I. ignoring the possible presence of a ST. In momentum space one gets:

$$ip^{\mu}\Pi_{\mu\nu}(p) \equiv \int d^2x\ e^{-ip\cdot x}\partial^{\mu}\Pi_{\mu\nu}(x) \qquad (1.9)$$

Therefore the naive W.I. is:

$$ip^{\mu}\Pi_{\mu\nu}(p) = 0 \qquad (1.10)$$

The reasoning behind neglecting the ST is the following: The T-product used to derive the W.I. is not a Lorentz covariant object, because of the presence of $\Theta(x^o)$. However, we normally define the Green's functions in a Lorentz covariant way. For example, in perturbation theory we use Lorentz covariant Feynman rules to evaluate $\Pi_{\mu\nu}(p)$. Therefore, we have neglected not only the ST, but also the derivative of the difference between the covariant and the non-covariant forms of the T-product, known as the seagull term. An implicit assumption in the derivation of the W.I. is that possible ST's cancel against derivatives of seagulls. This assumption is known as the Feynman conjecture. Thus we see that commutator anomalies do not necessarily correspond to W.I. anomalies. In ordinary QED, for instance, one encounters ST's but it is possible to define the T-product in such a way that the Feynman conjecture is satisfied.

We'll now check explicitly whether $\Pi_{\mu\nu}(p)$ can satisfy all the W.I.'s we want to impose on it. We'll consider a free massless fermionic theory, which is invariant under both $U_V(1)$ and $U_A(1)$. Then $\Pi_{\mu\nu}(p)$ is given exactly by the bubble diagram (Fig. 1a):

$$\Pi_{\mu\nu}(p) = -\int\frac{d^2r}{(2\pi)^2}\,\mathrm{tr}\left[\frac{i}{\not r+\not p}\,\Gamma_\mu\,\frac{i}{\not r}\,\Gamma_\nu\right] \qquad (1.11)$$

Where Γ_μ stands for γ_μ or $\gamma_\mu\gamma_5$.

We note that $\Pi_{\mu\nu}(p)$ is superficially log divergent. If all the expressions appearing in our derivation of the naive W.I. were well defined, the derivation would have been justified and the naive W.I. would hold. Therefore, having some sort of divergence is a necessary condition for having an anomaly. However, that doesn't mean that the starting Green's function must be divergent. In the cases where the canonical part of the commutator is non-trivial, the W.I. will connect two Green's functions. The second Green's function may have a divergence which will lead to an anomaly. Another important point is that it is the fact that we are forced to introduce some regularization in order to define $\Pi_{\mu\nu}(p)$ which is crucial for circumventing the naive W.I.. In fact we will see that the W.I. anomaly doesn't contain any infinite parts. Whatever regularization we choose for defining $\Pi_{\mu\nu}(p)$, one of our physical demands is Lorentz covariance. Assuming this, we can write a general expression for $\Pi_{\mu\nu}(p)$, using only the information that it has to be a two-index Lorentz tensor, depending on a single 2-vector p_μ:

$$\Pi_{\mu\nu}(p) = g_{\mu\nu}B_1(p^2) + (p^2 g_{\mu\nu}-2p_\mu p_\nu)B_2(p^2) \qquad (1.12)$$

$$+ \varepsilon_{\mu\nu}B_3(p^2) + \varepsilon_\mu^\alpha(p^2 g_{\alpha\nu}-2p_\alpha p_\nu)B_4(p^2)$$

The B_i's are as yet, undetermined functions. They will be called the invariant amplitudes. From (1.10) we get:

$$B_1 - p^2 B_2 = 0 \quad ; \quad B_3 + p^2 B_4 = 0 \qquad (1.13)$$

Similarly, we'll have a second W.I. from "dotting" p into the second index. Here we'll get:

$$B_1 - p^2 B_2 = 0 \quad ; \quad B_3 - p^2 B_4 = 0 \qquad (1.14)$$

Since $\Pi_{\mu\nu}(p)$ has zero mass dimensions, so have B_1 and B_3.

We can therefore expect them to correspond to the superficially log divergent part of the bubble. Hence their definition will depend on the chosen regularization. On the other hand, B_2 and B_4 have mass dimension -2. They correspond to the convergent parts and there is no ambiguity in their definition.

From parity considerations, B_3 and B_4 vanish for $\Pi_{\mu\nu}^{VV}$ and $\Pi_{\mu\nu}^{AA}$. Since B_1 is ambiguous, we might be able to choose it to satisfy (1.13-4). However, for the two mixed cases, namely, AV and VA, B_1 and B_2 are zero because of parity, and B_3 and B_4 must be zero to satisfy simultaneously (1.13-4) (assuming analyticity). But we can show by an explicit calculation that B_4 is non-zero. Since it is also unambiguous, there is no freedom left to redefine it to be zero. Thus, $\Pi_{\mu\nu}^{VA}(p)$ cannot satisfy both W.I.'s.

In fact, there is another constraint we would like to impose on $\Pi_{\mu\nu}(p)$. We have an algebraic identity (special to d=2), which says:

$$\gamma_\mu \gamma_5 = \varepsilon_\mu{}^\nu \gamma_\nu \tag{1.15}$$

It follows that:

$$\varepsilon_\mu{}^\nu J_\nu^V(x) = J_\mu^A(x) \tag{1.16}$$

If we want to impose it on the two-point function we get:

$$\Pi_{\mu\nu}^{AA}(p) = \varepsilon_\mu{}^\alpha \varepsilon_\nu{}^\beta \Pi_{\alpha\beta}^{VV}(p) \quad \therefore \quad B_1^V = -B_1^A \; ; \; B_2^V = B_2^A \tag{1.17}$$

The result of introducing this extra demand is that if we define B_1^V so that the W.I. for $\Pi^{VV}_{\mu\nu}(p)$ is satisfied, then $\Pi^{AA}_{\mu\nu}(p)$ won't satisfy its W.I. and vice versa. In d=4, extra constraints would typically come from demanding Bose symmetry.

We now return to (1.11) to prove some of the previous claims. After introducing a Feynman parameter, and doing a legal shift in the r-integration, we get:

$$\Pi_{\mu\nu}(p) = \int_0^1 dx \int \frac{d^2r}{(2\pi)^2} \left\{ \frac{\mathrm{tr}[(\not{r}+(1-x)\not{p})\Gamma_\mu(\not{r}-x\not{p})\Gamma_\nu]}{[r^2+x(1-x)p^2]^2} \right\} \tag{1.18}$$

In this form one can read off the Lorentz structure and make contact with our general decomposition (1.12). For example, the zero-dimensional amplitudes B_1 and B_3 are given by:

$$\text{log div part} = \int_0^1 dx \int \frac{d^2r}{(2\pi)^2} \left\{ \frac{\mathrm{tr}[\not{r}\Gamma_\mu\not{r}\Gamma_\nu]}{[r^2+x(1-x)p^2]^2} \right\} \tag{1.19}$$

We can define this integral through a symmetric integration formula (see appendix E). We then get:

$$\text{l.d. part} = \frac{1}{2}\,\mathrm{tr}[\gamma^\alpha\Gamma_\mu\gamma_\alpha\Gamma_\nu] \int_0^1 dx \int \frac{d^2r}{(2\pi)^2} \left\{ \frac{r^2}{[r^2+x(1-x)p^2]^2} \right\} \tag{1.20}$$

With this definition, the superficially log divergent amplitudes actually vanish. This follows from the algebraic identity (which is again special to d=2):

$$\gamma^{\alpha}\Gamma_{\mu_1}\Gamma_{\mu_2}\cdots\Gamma_{\mu_{2n+1}}\gamma_{\alpha} = 0 \quad \text{for any positive integer } n \tag{1.21}$$

Alternatively, we could have used the Pauli-Villars regularization method to define the bubble. This would have given us a non-zero but finite value for the zero dimensional amplitudes. The non-zero value would come from terms like:

$$\mathrm{tr}[M\Gamma_{\mu}M\Gamma_{\nu}] = M^2\,\mathrm{tr}[\Gamma_{\mu}\Gamma_{\nu}] \tag{1.22}$$

Where M is the mass of the regulating fermion.

We have seen that B_1 and B_3 turn out to be finite, but possess an ambiguity which reflects itself as a dependence on the regularization scheme. Next, we can identify B_2 and B_4 from:

$$\text{Finite part} = \mathrm{tr}[\not{p}\Gamma_{\mu}\not{p}\Gamma_{\nu}]\int_0^1 dx(1-x)x\int\frac{d^2r}{(2\pi)^2}\left\{\frac{1}{[r^2+x(1-x)p^2]^2}\right\} \tag{1.23}$$

The integral is convergent, and the result for it will be of the form $\frac{c}{p^2}$ with c a non-zero finite numerical constant.

Since all the invariant amplitudes are finite the anomaly (i.e. $p^{\mu}\Pi_{\mu\nu}(p)$) is also finite.

Let us summarize the main results of the analysis we have carried out. There is no choice of ambiguous quantities for which we can satisfy both W.I.'s in $\Pi^{VA}_{\mu\nu}(p)$. Moreover, imposing algebraic relations between the vector and the axial currents, we have to sacrifice W.I. for either $\Pi^{VV}_{\mu\nu}(p)$ or $\Pi^{AA}_{\mu\nu}(p)$ as well.

In another language we can say that there is a part in the ST which is not cancelled by the seagull term contribution when the divergence of $\Pi_{\mu\nu}(p)$ is taken and therefore the Feynman conjecture fails.

1.3 Implications of anomalies

The bubble diagram can appear in several theories of interest:

1. In the free theory. Here the anomaly doesn't have any direct physical consequence, since the bubble diagram doesn't enter in any physical process in this theory. The conservation equation of the current is true quantum mechanically since for any two physical states α and β we have:

$$\partial^{\mu}\langle\alpha|\, J_{\mu}^{V,A}(x)\, |\beta\rangle = 0 \tag{1.24}$$

(1.24) is true despite the fact that the set of naive W.I.'s cannot be maintained. For many purposes the free theory is a good place to study the anomaly, and the fact that the anomaly already appears there reflects its basic nature.

2. Suppose one of the currents is coupled to a gauge field while the symmetry associated with the other current remains global. QED with a global axial current is an example of this type. In our d=2 example, the bubble diagram with one external photon (FiG. 1b) then describes the following matrix element:

$$\langle 0|\, J_{\mu}^{A}(x)\, |\gamma\rangle \propto ie\varepsilon^{\mu}\Pi_{\mu\nu}^{VA}(p) \tag{1.25}$$

Where e is the gauge coupling constant and ε^{μ} is the photon polarization vector. One can choose to preserve the vector W.I.'s , but then the axial W.I.'s are broken. Because of (1.25), and in contrast to the free theory, this implies that the axial current is not conserved quantum mechanically:

$$\partial^{\mu} J^{A}_{\mu}(x) \propto e\epsilon^{\mu\nu} F_{\mu\nu}(x) \tag{1.26}$$

Thus, the global $U_A(1)$ is broken in the presence of vector photons. In a situation like this we talk about "The good anomaly", since the theory is consistent and we get a natural way to reproduce a desired physical effect of symmetry breaking. The first process to be understood in terms of this mechanism was the decay of π^0 into two photons. Since then, many other physical phenomena have been tied to this form of anomalies.

3. We can also think of a theory where both $U_V(1)$ and $U_A(1)$ are gauged (Fig. 1c). Because of the anomaly, at least one of the gauge symmetries must be broken when the theory is quantized. However, unlike the previous case, it is believed that the resulting theory is inconsistent. Chapter III is devoted to the problems arising in such theories. In this situation we are talking about "The bad anomaly". So far, the common strategy concerning bad anomalies has been to reject theories possessing them (The principle of anomaly cancellation). We'll remark on this later.

1.4 Generalizations and further results on anomalies

1. The type of anomaly arising in our d=2 example, i.e. an anomaly connected to a classically conserved fermionic current coming from the continuous axial symmetry, generalizes to space time dimensions d=2n. It occurs in one-loop diagrams containing at least $\frac{1}{2}n+1$ vertices where the number of axial vertices, i.e. vertices with γ_5 type of coupling, is odd. In d=4, for example, we encounter anomalies in the AVV and AAA triangle diagrams. Note that, because of the unavoidable anomaly in the AAA triangle, a theory with an axial gauge coupling in d=4 will always have a bad anomaly, in contrast to the d=2 case.

We can also discuss internal symmetries which are more general than the abelian U(1)'s appearing in our d=2 example. One can then show that if the fermionic current carries an index 'a' of a compact Lie group G, i.e., $J^a_\mu=\bar{\Psi}\Gamma_\mu\lambda^a\Psi$, where λ^a is a generator in the corresponding Lie algebra, the anomaly in the $\frac{1}{2}n+1$ diagram will be proportional to a symmetrized trace over the group generators. In d=4, this group-theoretic factor is d^{abc}. In the d=4 non-abelian case, we have anomalies also in the 4-point and 5-point Green's functions, which correspond to a square and a pentagon diagram. However, these anomalies are determined by the triangle anomaly. The practical meaning of rejecting bad anomalies of the type considered here is that we restrict the fermionic content of the theory to be consistent with the condition $d^{abc}=0$. This condition has been very useful in constructing the standard model and in analyzing grand unified theories.

2. Another interesting and important feature of axial anomalies is the fact that they are non-perturbative. In our example, one can see that there are no radiative corrections to the bubble diagram due to (1.21). The d=4 derivation is much more involved, but again the end result is that the numerical coefficient of the anomaly does not receive radiative corrections. Alternatively, this result is implied by the existence of manifestly non-perturbative methods to derive the anomaly.

3. Bosonic loops are known to be free of axial gauge anomalies. However, bosonic fields do play an important role in investigating them.

Consider the following action, which describes fermions coupled to an external gauge field:

$$S = \int d^4x \left(i\overline{\Psi}(x)\partial\!\!\!/\Psi(x) + \overline{\Psi}(x)\gamma^{\mu}B^a_{\mu}(x)\lambda^a\Psi(x)\right) \; ; \qquad (1.27)$$

$$B^a_{\mu}(x) = e_v V^a_{\mu}(x) + e_a\gamma_5 A^a_{\mu}(x)$$

where $V^a_{\mu}(x)$ and $A^a_{\mu}(x)$ are non-abelian vector and axial vector gauge fields. We now define an effective action related to S through functional integration over the fermion fields:

$$e^{-W(B)} \equiv \int D\Psi D\overline{\Psi}\; e^{iS(\overline{\Psi},\Psi,B)} \qquad (1.28)$$

Because of the anomaly, the effective action W(B) is not invariant under chiral gauge transformations and we have:

$$\left.\frac{\delta W(B^{\beta(x)})}{\delta\beta(x)}\right|_{\beta(x)=0} = G(B) \qquad (1.29)$$

Where B^{β} is the axially gauge transformed B. However, it is clear that W(B) is not local in B. If it had been local we could have subtracted it from S and the resulting theory would then be anomaly free. But the basic result about the anomaly is that it cannot be removed by subtracting a local counter term. Thus W(B) cannot be local in B.

Wess and Zumino[4] have shown that if a scalar field $\Theta(x)$ is introduced with an appropriate gauge transformation rule, an effective action $W(B,\Theta)$, local in both B and Θ, can be constructed, which will reproduce the anomaly, namely:

$$\left.\frac{\delta W(B^{\beta(x)},\Theta^{\beta(x)})}{\delta\beta(x)}\right|_{\beta(x)=0} = G(B) \qquad (1.30)$$

In the abelian case, the construction is very simple. For example:

$$W(A,\Theta) \propto e\Theta\varepsilon^{\mu\nu}F_{\mu\nu} \qquad (1.31)$$

reproduces (1.26) provided Θ transforms under $U_A(1)$:

$$\Theta \rightarrow \Theta' = \Theta + \beta \qquad (1.32)$$

In the non-abelian case the WZ action has a much more complicated and richer structure. Its main use so far has been in constructing effective actions in d=4 for describing QCD effects at low energies, which incorporate QCD good flavor anomalies. More recently, it has been proposed by Faddeev and Shatashvili (ref. 3-1) that adding a WZ action to a theory with bad anomalies may be an alternative to the normal strategy of anomaly cancellation, previously mentioned.

4. Other types of anomalies are known besides gauge anomalies. Another important anomaly connected with a continuous symmetry is the gravitational anomaly. It appears in one-loop diagrams involving energy momentum tensor vertices. An example in d=2 is again the bubble diagram, with the two current vertices replaced by $\Theta_{\mu\nu}$ vertices. In d=4 we have the Salam-Delbourgo anomaly (ref. 3-17) in the triangle diagram with two $\Theta_{\mu\nu}$'s and one $U_A(1)$ current. This is an example of a mixed gauge and gravitational anomaly. The symmetry connected with the conservation of $\Theta_{\mu\nu}$ is general coordinate invariance.

There are also anomalies associated with the topological nature of gauge or general coordinate transformations. They may arise in the case of "large" transformations which cannot be reached continuously from the identity. This type of anomalies will not be dealt with in the present work.

1.5 Connection of the d=2 anomaly with the K.M. algebra

It was said earlier that the anomaly already appears in the free quantum theory of fermions. It is also known that in d=2, the free quantum fermionic currents form a rich algebraic structure, called the Kac-Moody algebra. The defining relation for this algebra (using its continuum version) is:

$$[J^a(x),J^b(y)] = if^{abc}J^c(x)\delta(x-y) + \frac{i\hbar}{2\pi}k\delta^{ab}\delta'(x-y) \qquad (1.33)$$

Here f^{abc} are structure constants of a compact Lie algebra G, and the J's are linear combinations of vector and axial currents ($J^a \propto J_o^{Va} \pm J_o^{Aa}$). The K.M. algebra is an extension of G and reduces to G when the numerical constant k is set to zero. The central extension (the second term in (1.33)) is an example of a ST mentioned earlier. The $\hbar$ in front of it reflects its quantum nature. In order to derive (1.33) for the case that the J's are bilinears in elementary fermion fields, one should use a careful definition of a commutator. We'll use the BJL method which is defined and explained in appendix B. Since the BJL procedure calculates the commutator of two operators from their T-product, the connection with the W.I. will be more transparent.

The first term in (1.33) is a canonical term and therefore doesn't interest us here. If we take the vacuum expectation value of (1.33), on the r.h.s. only the central extension will contribute, since it's a c-number and the current is normal ordered. Hence the central extension is given by the VEV of the commutator. In order to evaluate the VEV of the commutator using the BJL definition, we have to know the VEV of the T-product of the currents in momentum space. This is nothing but $\Pi_{\mu\nu}(p)$. The fact that we deal with a general group G means that we have to replace Γ_μ by $\lambda^a\Gamma_\mu$. In the Feynman expression we'll encounter $tr(\lambda^a\lambda^b)$, which will give the δ^{ab} factor. Note that the T-product of the currents is given by an appropriate linear combination of $\Pi_{\mu\nu}(p)$'s, so it's enough to look at the BJL limit of a general $\Pi_{\mu\nu}(p)$.

Earlier on we found that B_1 and B_3 are ambiguous constants, while B_2 and B_4 are of the form $\frac{c}{p^2}$ and c is fixed. Going back to (1.12), and looking for the $\frac{1}{p_o}$ term we find:

$$\frac{1}{p_o}\,(-2cp_1)[g_{\mu o}g_\nu{}^1 + g_{\nu o}g_\mu{}^1 + \varepsilon_\mu{}^\alpha(g_{\alpha o}g_\nu{}^1 + g_{\nu o}g_\alpha{}^1)] \qquad (1.34)$$

p_1 will turn in x-space into an $i\delta'(x-y)$. We see two interesting things:

1. The non-vanishing value of the central extension k and the unambiguous and non-vanishing part of the W.I. anomaly arise from the same source - the non-zero value of c.

2. A ST appears in both the even and the odd parity parts, though a genuine W.I. anomaly is unavoidable only in the odd parity part. However, the two ST's can be distinguished by their different Lorentz structure.

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CHAPTER II - The failure of the Jacobi identity for free fermionic currents in d=4 and its relation to the axial anomaly

2.1 Introduction

The aim of this chapter is to examine the validity of Jacobi identity for current operators in a free massless fermionic theory, in four space-time dimensions. The main conclusion is that for certain cases, the Jacobi identity is not compatible with other "standard", well established assumptions which are made about quantum field theories. We begin by reviewing some previous results concerning the validity of the Jacobi identity. Next, it is shown that under a specified set of standard assumptions, the Jacobi identity for the time component of the axial current and two different space components of a vector current must fail at equal time. Then, all the components of the vacuum expectation value of the double equal time commutator of one axial current and two vector currents are calculated, using a double BJL limit and the previous result is verified and elaborated. The results of a similar calculation for three non-abelian vector currents, where another violation of the Jacobi identity occurs, is also given. It is shown that this result implies a third violation of the Jacobi identity, this time between the energy-momentum tensor and two currents. The properties of the various results are discussed and a connection is made with some previous work. A short discussion is given of what happens in two dimensions. We then suggest a possible connection with the W.I. anomaly of the axial current.

For convenience we change in this chapter to the following notation for the fermionic currents:

$V_\mu(x) \equiv \overline{\Psi}(x)\gamma_\mu\Psi(x)$ is the U(1) fermionic 4-vector current,

$V^a_\mu(x) \equiv \frac{1}{2}\overline{\Psi}(x)\gamma_\mu\lambda^a\Psi(x)$ is the non-abelian vector current, and

$A_\mu(x) \equiv \overline{\Psi}(x)\gamma_5\gamma_\mu\Psi(x)$ is the fermionic U(1) axial current.

2.2 A brief review of previous work on J breaking

The validity of J was previously questioned in several contexts:

1. The quantum mechanical non-relativistic problem of a charged point particle moving in an external magnetic field of a Dirac monopole. A possible failure of J for the velocity operators at the location of the monopole was first noted by the authors of ref 1 (1969), and much later aroused renewed interest[2-5,8] and debate[6,7]. The terminology used in these papers is that of cohomology theory, according to which a possible failure of J for a set of generators indicates the possible presence of a 3-cocycle. Moreover, if this third cocycle is non-trivial, i.e. if it is a non-integer multiple of 2π, then the finite transformations related to these generators are non-associative.

2. Johnson and Low[14] observed, back in 1966, a failure of J for spatial components of vector currents in the quark model. They sketched this possibility, using the BJL definition of a single commutator. Buccella et al.[10] proved, for the same model, that C-number ST's are not compatible with J. Jackiw[2] notes that there are experimental indications that the ST is indeed a C-number, consistent with the quark model calculations, but since the calculation produces a quadratically divergent term (which cannot be removed by usual renormalization), the mathematics of the problem seems to be ill-defined. It seems that no further investigations into this issue have been published. The results presented here are

closely related to those of ref's 10 and 14, and it is hoped that they will illuminate them further.

3. An argument by Brandt[16] seems to imply that whenever commutators are defined as limits of regularized expressions, there is no automatic guarantee that J will hold. His argument can be used to solve the following paradox: Suppose one chooses to define the E.T. double commutator as a limit of a regularized double commutator (for instance, by using point splitting or a double commutator at unequal time etc.). Then for the regularized double commutator J is expected to hold since the operators are now well defined. But if $J^{reg}=0$ for any value of the regulating parameters then in the limit it will remain zero and therefore we can always find a regularization procedure which respects J at equal times. However, we note that a definition of a double E.T.C is restricted by that of a single E.T.C. . It forces us to take the limits of the regulating parameters in a certain order. For a double commutator we'll need to take two successive limits, the first corresponds to the "inner" single commutator (See assumption 2 of section 2.3 . However, this restriction may leave some residual freedom, which is discussed later on). Therefore the limits of the regulators are taken in different orders in the three double commutators which form J. A failure of these limits to commute will result in a failure of J for the double E.T.C.'s even though it holds for the regulator-dependent ones.

4. A failure of J in a theory with a bad anomaly was noted by ref 9 (see also ref 11) for three spatial components of the

electric field. These results were also obtained from a BJL calculation. They are effectively reproduced in the next chapter and further implications are discussed there.

5. Possible associativity anomalies are also mentioned in the context of string theories[22].

2.3 The incompatibility of J(AVV) with standard assumptions

It was already mentioned in the introduction that Schwinger gave a proof[17] that the VEV of $[V_o, V_i]$ cannot be zero, independent of any definition (regularization) of this object, if one accepts a limited set of very general assumptions. In an analogous manner one would like to establish the possibility that J can fail in certain cases, without referring to any specific definition of the double commutators involved. Although we call the final result the failure of J, a more cautious statement to make is that we give a set of assumptions which are inconsistent with J.

Here are the assumptions:

(1) The existence of ETC, which for two local operators A and B will have the form:

$$[A(t,\vec{x}),B(t,\vec{y})] = \sum_n C_n D_n(\vec{x}-\vec{y}) O_n(t,\vec{x}) \qquad (2.1)$$

where:

$O_n(t,\vec{x})$ - are local operators constructed from basic fields of the theory and their derivatives including the unit operator. We'll assume that the O_n's belong to a set of independent operators (see ref 18).

$D_n(\vec{x}-\vec{y})$ - are delta functions and / or any of their derivatives.

C_n - are numerical quantities which are allowed to contain dimensional divergent parts of the form $(t)^{-1}$, $(t)^{-2}$ etc. where $t \to 0$.

(2) The free currents, together with a subset of $\{O_n\}$, close an algebra with the algebraic properties : $[A,B] = -[B,A]$; $[C,[A,B]] = [C,D]$ if $[A,B] = D$.

(3) In the free theory $[A,B]_{true} = [A,B]_{canonical}$ + possible different extra terms. The extra terms correspond to different values of the label n in (2.1) than the terms which can be derived from C.C.R. . Actually a weaker assumption is needed and we'll comment on this later.

(4) All properties used to prove the necessary existence of the ST hold, like Lorentz covariance, positivity of the Hamiltonian, etc. (see also assumption 6).

(5) The currents are hermitian operators of canonical dimension 3 (in mass units), and have normal transformation rules under P,C,T, symmetries (see appendix A). They are normal ordered to give zero VEV.

(6) We can use axial current conservation in the commutator of two axial currents.

Under these assumptions, it will now be shown that the VEV of:

$$[A_0(0,\vec{x}),[V_i(0,\vec{y}),V_j(0,\vec{0})]] + 2 \text{ Jacobi permutations} \qquad (2.2)$$

is non-zero for $i \neq j$ in a free massless fermionic theory since the first term is necessarily non-zero while the other two vanish.

We start by showing for the free theory that:

$$[A_0(0,\vec{x}),V_i(0,\vec{y})] = 0 \qquad (2.3)$$

Using (1) and (2) for the r.h.s. of (2.3) we have that dim O_n is less than or equal to 3 since dim D_n is at least 3. Since only fermion fields, their derivatives and the unit operator can be used to construct O_n, we get (using dimensions and Lorentz properties) that O_n can only be bilinears in the fermion fields or the unit operator. Writing down the most general sum of terms constructed from the five independent Lorentz fermion bilinears with complex coefficients we get, using hermiticity, discrete symmetries, anti-symmetry of the commutator, translation invariance, that all the coefficients vanish. In fact, c-numbers on the r.h.s of (2.3) do not matter, but it is easily seen, using charge conjugation, that they vanish. We also note that C.C.R also gives (2.3). It now remains to show that the first term of (2.2) cannot be zero. Repeating the same exercise for $[V_i(0,\vec{x}),V_j(0,\vec{y})]$, (This is done in detail in appendix A in order to demonstrate the general technique), we get that the only possible non c-number contribution, is a term of the form $\varepsilon_{ijk}A^k$. Again, this is exactly the result one gets from C.C.R. We now use assumption 3 to exclude any cancellation of the canonical term by a non-canonical contribution (here one can see that we are really using a weaker version of 3). The final step is to simply substitute the canonical result into the first term in (2.2) and use 4 and 6 to get that the VEV of what's left cannot vanish.

There are two comments to be made:

1. Assumption 3 already assumes that C.C.R. do not contain all the information about the true commutation relations. However,

rejecting the weaker version of it, which, as previously explained, is what is actually being used, is equivalent to saying that C.C.R. do not contain any information at all about the true commutation relation. Besides the fact that no acceptable definition of E.T.C is known to give such a result, it's hard to see how one can accept it and keep all other canonical results unaffected.

2. Assumption 6 relies on the fact that there is no W.I. anomaly in the AA two-point function, and therefore the proof that the $[A_o,A_i]$ ST cannot vanish goes parallel to that of the $[V_o,V_i]$ case.

Before concluding this part it is useful to give an explicit expression for the ST which appeared in our argument. It seems that the following structure is agreed upon by several[13,16,21] methods of defining the E.T.C. :

$$\langle 0|[V_o(0,\vec{x}),V_i(0,\vec{0})]|0\rangle = i(\Lambda\partial_i\delta^3(\vec{x})+\frac{1}{12\pi^2}S_{vv}\partial_i\vec{\nabla}^2\delta^3(\vec{x})) \quad (2.4)$$

where Λ is a quadratically divergent quantity and S_{vv} is a finite non-zero constant. For free currents the axial ST has the same structure as in (2.4) with $S_{aa}=S_{vv}$.

2.4 BJL calculation of the AVV double commutator

In this section we'll use the BJL definition of a double commutator (see appendix B) in order to calculate J for one axial and two vector currents of the U(1) free massless fermionic theory. Define:

$$J_{\sigma\rho\mu}(\vec{x},\vec{y}) \equiv \langle 0| [V_\sigma(0,\vec{x}),[V_\rho(0,\vec{y}),A_\mu(0,\vec{0})]]+ \tag{2.5}$$

$$[V_\rho(0,\vec{y}),[A_\mu(0,\vec{0}),V_\sigma(0,\vec{x})]]+[A_\mu(0,\vec{0}),[V_\sigma(0,\vec{x}),V_\rho(0,\vec{y})]]|0\rangle \equiv$$

$$\equiv B_1 + B_2 + B_3$$

The relevant three-point function needed to evaluate these double commutators is given in the free theory exactly by an appropriate triangle diagram (Fig. 2). The calculation of this diagram is well known and we will use the expression for it given in ref 12. We want to calculate:

$$B_1(\vec{k}_1,\vec{k}_2) = -\lim_{k_{10}\to i\infty} k_{10} \lim_{k_{20}\to i\infty} k_{20}\, T_{\sigma\rho\mu}(k_1,k_2) \tag{2.6}$$

$$B_2(\vec{k}_1,\vec{k}_2) = \lim_{k_{20}\to i\infty} k_{20} \lim_{k_{10}\to i\infty} k_{10}\, T_{\sigma\rho\mu}(k_1,k_2) \tag{2.7}$$

$$B_3(\vec{k}_1,\vec{k}_2) = \lim_{q_0\to i\infty} q_0 \lim_{k_{20}\to i\infty} k_{20}\, T_{\sigma\rho\mu}(k_1=-(q+k_2),k_2) \tag{2.8}$$

where:

$$T_{\sigma\rho\mu}(k_1,k_2) \equiv \int d^4x d^4y\, e^{ik_1x} e^{ik_2y} \langle 0|TV_\sigma(x)V_\rho(y)A_\mu(0)|0\rangle = \quad (2.9)$$

$$= \frac{-i}{(2\pi)^4} R_{\sigma\rho\mu}(k_1,k_2,m_0=0)$$

and $R_{\sigma\rho\mu}(k_1,k_2,m_0=0)$ is given[12] by:

$$\frac{1}{16\pi^2} R_{\sigma\rho\mu}(k_1,k_2) = \quad (2.10)$$

$$= k_1^\tau \varepsilon_{\tau\sigma\rho\mu}[-k_1\cdot k_2 I_{11}(k_1,k_2)+k_2^2(I_{20}(k_1,k_2)-I_{10}(k_1,k_2))]$$

$$+k_2^\tau \varepsilon_{\tau\sigma\rho\mu}[-k_1^2(I_{20}(k_2,k_1)-I_{10}(k_2,k_1))+k_1\cdot k_2 I_{11}(k_2,k_1)]$$

$$-k_{1\rho} k_1^\xi k_2^\tau \varepsilon_{\xi\tau\sigma\mu} I_{11}(k_1,k_2) + k_{2\rho} k_1^\xi k_2^\tau \varepsilon_{\xi\tau\sigma\mu}[I_{20}(k_1,k_2)-I_{10}(k_1,k_2)]$$

$$-k_{1\sigma} k_1^\xi k_2^\tau \varepsilon_{\xi\tau\rho\mu}[I_{20}(k_2,k_1)-I_{10}(k_2,k_1)] + k_{2\sigma} k_1^\xi k_2^\tau \varepsilon_{\xi\tau\rho\mu} I_{11}(k_2,k_1)$$

where $I_{st}(k_1,k_2)$ is defined by:

$$I_{st}(k_1,k_2) \equiv \int_0^1 dx \int_0^{1-x} dy \frac{x^s y^t}{y(1-y)k_1^2+x(1-x)k_2^2+2xyk_1\bullet k_2} \quad (2.11)$$

Let us calculate B_2. We first take the limit $k_{10}\to i\infty$. We can rewrite (2.11) as follows:

$$I_{st}(k_1,k_2) = \frac{f_{st}(k_1,k_2)}{k_{10}^2} \quad (2.12)$$

Examining (2.11) we find f to be at most logarithmically divergent as $k_{10} \to i\infty$. Hence, we write an expansion:

$$f_{st}(k_1,k_2) = c_1 + \frac{c_2}{k_{10}} + O\left(\frac{1}{k_{10}^2}\right) \tag{2.13}$$

Suppose now that c_1 and c_2 are finite. It is then legitimate to expand under the integral sign. We can see from dimensional reasoning and from (2.11) that c_1 is a numerical constant and c_2 is proportional to k_{20}. In this case there will be no contribution to the second limit since there are no negative powers of k_{20}. Checking the s,t values relevant to (2.10), we find that only the combination $(I_{20}(k_1,k_2) - I_{10}(k_1,k_2))$ has a logarithmically divergent c_1, while all other I_{st} have finite c_1 and c_2, and therefore won't contribute. We also see from (2.10) that $I_{20}(k_1,k_2) - I_{10}(k_1,k_2)$ is multiplied at most by one power of k_{10} so it's enough to consider only c_1.

For $I_{20}(k_1,k_2) - I_{10}(k_1,k_2)$ c_1 behaves at $k_{10} \to i\infty$ like $\ln\left(\frac{k_{10}^2}{a}\right)$. The quantity 'a' should have the dimension of mass^2. Considering (2.11) again, we note that the logarithmic behaviour of f at the above limit is caused by the pole at y=0 , which is the lower limit of the y integration. The behaviour near the pole is dominated by the coefficient of the x(1-x) term, which is $\frac{k_2^2}{k_{10}^2}$. For later purpose we'll write it:

$$\left(\frac{k_{20}^2}{k_{10}^2}\right)\left(1 - \frac{\vec{k}_2^2}{k_{20}^2}\right) \tag{2.14}$$

and by the coefficient of y, which in the leading order has no momentum dependence. Therefore 'a' should be proportional to k_2^2 and dependence on $\vec{k}_1^2$ or on $\vec{k}_1 \cdot \vec{k}_2$ can come only through higher orders in $\frac{1}{k_{10}}$ which don't contribute anyhow, as was previously explained. This is what we expect from the analysis of the previous section. Since each $\vec{k}_n$ factor corresponds to a derivative of a δ-function, and since for B_2 we expect only derivatives of $\delta^3(\vec{y})$, we get only $\vec{k}_2$ dependence.

Our conclusion that c_1 should be proportional to the logarithm of (2.14) can be verified explicitly and the constant of proportionality can be found since the y integration in (2.11) is elementary and indeed the expected logarithm is found. Now, the logarithm of (2.14) is a sum of two logarithms of which one is finite at $k_{10} \to i\infty$ and gives rise to negative powers of k_{20}. Performing the second limit $k_{20} \to i\infty$ we will get a contribution only from the fourth term in (2.10) since only one k_{20} factor is needed from the polynomial to get a final $\frac{1}{k_{20}}$ contribution. We finally get for B_2:

$$B_2(\vec{k}_1, \vec{k}_2) = \tag{2.15}$$

$$= \frac{-i}{6\pi^2} g_{\rho o} \varepsilon_{oi\sigma\mu} k_2^i [\vec{k}_2^2 - \lim_{k_{20} \to i\infty} k_{20} \lim_{k_{10} \to i\infty} k_{10} (2\frac{k_{20}}{k_{10}} \ln(\frac{k_{20}}{k_{10}}))]$$

Most of the features of the result are those expected from the previous section. Examining the Lorentz structure it is easily seen that to get a non-zero result we must have $\rho=0$ and $\sigma\neq\mu$ both spatial. Comparing with (2.4), (2.15) is also a sum of two terms : a finite third derivative of a δ-function $(\partial_i \vec{\nabla}^2 \delta^3(\vec{y}))\delta^3(\vec{x})$ and a first derivative of a δ-function with a coefficient which suggests a quadratic divergence, i.e. $k_{20}\ln(k_{20})$. However, unlike the calculation of a single commutator, we are left here with a possibility to get rid of the divergent part by fixing the final limit to be taken on a $k_{20}=\text{constant}\cdot k_{10}$ line, which will leave us with a constant that can be thought of as a part of a pure polynomial in k_{20} (taking constant=1 will set it immediately to zero and save us the trouble of dropping it!).

The interpretation of the second term in (2.15) deserves further discussion. To get (2.15) we had to take a successive double limit. The first step is to identify the leading contribution to the first limit and then to identify in what remains the leading contribution to the second limit. If no divergences occur, this procedure should agree with what was done in the previous section, i.e. forming the double commutator from two single ones. However, we find in this case that we need a further specification of the definition to get a full agreement and that there are other possible definitions which do not agree with the previous section about the presence of the quadratic divergent term. Now, one may argue that we should insist on consistency with the single BJL limit for the VEV of $[A_o, A_i]$, even if it means that we have to keep a divergent term. However, since associativity is lost whether we keep the divergent part or not, it doesn't seem at all necessary to force such a consistency condition. Rather, it is more natural to use the freedom created by non-associativity to get rid of the infinite term. Of course, we must keep in mind the possibility, that another, yet unknown physical or mathematical argument will fix the ambiguity in the second term of (2.15) in another way. Nonetheless, we'll encounter more hints as we continue that everything we need is in the finite third derivative term.

For completness we write down the full expression for the VEV of J(AVV), dropping the ambiguous term:

$$J_{\sigma\rho\mu}(\vec{k}_1,\vec{k}_2) = \frac{-i}{6\pi^2}\left[g_{\rho o}\varepsilon_{oi\mu\sigma}\, k_2^i \vec{k}_2^{\,2} + g_{\sigma o}\varepsilon_{oi\rho\mu} k_1^i \vec{k}_1^{\,2} + g_{\mu o}\varepsilon_{oi\rho\sigma} q^i \vec{q}^{\,2}\right] \qquad (2.16)$$

where $\vec{q} = -(\vec{k}_1 + \vec{k}_2)$.

2.5 The VVV double commutator

Using the double BJL limit we can repeat the previous calculation for three vector currents. Because of charge conjugation, only non-abelian currents will give a non-zero result for the VEV of the double commutator, which is therefore proportional to the totally anti-symmetric structure constant f^{abc}. We get for J(VVV):

$$J^{abc}_{\sigma\rho\mu}(\vec{k}_1,\vec{k}_2)=\frac{-i}{12\pi^2}f^{abc}\Big[(g_{\sigma i}g_{\mu o}g_{\rho o}+g_{\rho i}g_{\sigma o}g_{\mu o}+g_{\mu i}g_{\sigma o}g_{\rho o})\qquad(2.17)$$

$$(k_1^i\vec{k}_1^{\,2}+k_2^i\vec{k}_2^{\,2}+q^i\vec{q}^{\,2})-g_{\sigma i}g_{\rho k}g_{\mu k}k_1^i\vec{k}_1^{\,2}-g_{\rho i}g_{\sigma k}g_{\mu k}k_2^i\vec{k}_2^{\,2}$$

$$-g_{\mu i}g_{\sigma k}g_{\rho k}q^i\vec{q}^{\,2}\Big]$$

where the spatial index k is not under summation. We immediately note that the momentum structure is the same as that of the AVV double commutator, including an ambiguous first derivative term which has already been omitted from (2.17). This shouldn't be a surprise since the final result arises in both cases from the VEV of the ST between one time and one space components of the currents (which, as has already been mentioned, is the same for AA and VV cases). As for the Lorentz structure, we see that J is violated in two cases; when two of the indices pick up the time value and the third a spatial value, or when two of them take the same spatial value and the third a different spatial value. The second case is the one discussed by ref's 10 and 14, and it resembles J(AVV) in the sense that only one of the three VEV's which form J is different from zero. This property was the basis for the arguments previously constructed here and in ref 10 for the failure of J. It enables one to rely only on the non-vanishing of the ST and not on any knowledge about its form. It is conceivable that the analysis of section 2.3 can be repeated for the J(VVV) case as well, with the appropriate modification needed to take into account the presence of a non-abelian G. For two time components and one spatial such an argument cannot be constructed without using the explicit form (2.4) of the ST, since all three terms are different from zero. However, it is interesting to note that for this case the ambiguous first derivatives simply cancel when we form J since $k_1+k_2+q = 0$. Thus, for this choice of indices, J(VVV) comes out finite, well-defined and non-zero even if the double commutator is left ambiguous. This result clearly demonstrates the importance of the third derivative terms.

We can see the intimate connection between the presence of third derivatives and the failure of J yet from another direction, and as a byproduct to discover a third failure of J. Ref 13 (chap. II) gives a formal proof that:

$$[V_0^a(t,\vec{x}),V_i^b(t,\vec{y})]= if^{abc}V_i^c(\vec{x})\delta^3(\vec{x}-\vec{y}) + S_{ij}^{ab}\partial^j\delta^3(\vec{x}-\vec{y}) \quad (2.18)$$

i.e. there are no ST's with higher derivatives of a delta function. This, of course, contradicts the previous results (2.16) and (2.17), and the form (2.4) for the ST. Examining the proof given in ref 13 we find that J is assumed to hold for the VEV of the following double commutator:

$$[\theta^{00}(0,\vec{z}),[V_0^a(0,\vec{x}),V_0^b(0,\vec{y})]] \qquad (2.19)$$

where θ stands for the energy momentum tensor. It is reasonable to expect that the formal proof of (2.18) is incorrect since J fails also for (2.19). In fact, it will now be shown, using arguments similar to those of section 2.3 that (2.18) is in fact a minimal requirement for $J(\theta VV)=0$ to hold. Hence the formal proof is invalidated since it is actually a circular argument.

We begin by observing that the double commutator (2.19) is expected to have a vanishing VEV since:

$$\text{(a)}\ [V_0^a(0,\vec{x}),V_0^b(0,\vec{y})] = if^{abc}V_0^c(0,\vec{x})\delta^3(\vec{x}-\vec{y}) \qquad (2.20)$$

$$\text{(b)}\ \langle 0|[\theta^{00},V_0^c]|0\rangle = 0$$

The relation (2.20b) follows from the invariance of the vacuum under group transformations. To check the other two terms in $J(\theta VV)$ we'll use the canonical commutator[13] (assuming that the currents are conserved):

$$[\theta^{oo}(0,\vec{x}),V^{oa}(0,\vec{y})] = iV^{ja}(0,\vec{x})\partial_j\delta^3(\vec{x}-\vec{y}) \qquad (2.21)$$

After substituting (2.21) into the remaining terms in J(ΘVV) one can immediately see that the final result depends crucially on the nature of the ST. Only if we assume the form (2.18) may we hope that a cancellation between the two terms will occur, because in this case, both of them are proportional to $\vec{\nabla}\delta^3(\vec{x}-\vec{y})\cdot\vec{\nabla}\delta^3(\vec{y}-\vec{z})$. Higher derivatives of a δ-function, like those in (2.4) spoil this possibility. One may still worry whether a cancellation can occur from extra non-canonical terms on the r.h.s. of (2.21). However, it's easy to see that no such other terms which are proportional to a first derivative of a δ-function besides the canonical one can be present in the free theory. Using assumption 3 of section 2.3 which states that the canonical term must be present closes the argument. It should also be noted that (2.21) is insensitive to the question of which form of Θ is chosen for the l.h.s. of it (i.e. Θ_c or Θ_B. See ref. 13).

2.6 The Jacobi identity in two space-time dimensions

In d=2, the Jacobi identity is repeatedly used to derive important results concerning the K.M. and Virasoro algebra, and no inconsistencies are encountered. It is therefore natural to ask how our previous arguments are evaded in d=2. We can note immediately that the argument of section 2.3 and the argument of ref 10 do not carry through, since they both require two different space indices. Also, from dimensional arguments (in d=2 the currents have mass dimension 1), the ST for $[V_o, V_i]$ can contain no more then a finite first derivative of δ-function, as was shown explicitly in the previous chapter. Hence, one can easily verify the following:

1. Though VEV's of certain current double commutators are non-zero, J will be zero, again due to the fact that the ST contains only first derivatives of a δ-function and momentum conservation implies $k_1+k_2+q = 0$.
2. Since the ST is finite from the start the double BJL results completely agree with the single BJL ones and no ambiguities arise in the final limit.

2.7 A possible connection between J breaking and the axial anomaly

It is natural to ask whether the breaking of J is somehow connected with the anomaly in the divergence of the axial current. Recall the main feature of the analogous connection in d=2 discussed in the introduction. In the first place we have found that an anomaly in the commutator of certain components of a specific current doesn't necessarily imply a breakdown of the W.I. connected with the conservation of this current. However, the reverse is true, and a W.I. breaking does imply the appearance of a commutator anomaly. Moreover, we have found a numerical connection between the W.I. anomaly and the ST.

We will show here, that the d=4 free theory case reveals a remarkable similarity to the d=2 case, if we take the VEV of the double commutator or that of J to be the analog of the d=2 ST. In order to do this we look for a numerical connection between the non-conservation of the axial current and the double commutator. Actually, a connection between the axial anomaly and ST's in a four-dimensional fermionic theory was pointed out by the authors of ref 15. They define two numerical constants K_{vv} and S_{aa}, where the definition of S_{aa} is given by (2.4), and K_{vv} is defined through:

$$[V_i(0,\vec{y}),V_j(0,\vec{x})]= 2iK_{vv}\varepsilon_{ijk}A^k(0,\vec{x})\delta^3(\vec{x}-\vec{y}) + \dots \qquad (2.22)$$

The ... refers to extra terms which might appear in a theory with interactions. Now, the claim of ref. 15, following a short-distance analysis of the triangle graph by Wilson[16] and Crewther[17], is that the Crewther relations connect S_{aa} and K_{vv} to the divergence anomaly of the axial current via:

$$S = S_{aa} K_{vv} \tag{2.23}$$

where S is Adler's anomalous constant[12] defined for the fermionic part of the current by:

$$\partial_\mu A^\mu(x) = S \frac{\alpha_0}{4\pi} F^{\xi\sigma}(x) F^{\tau\rho}(x) \varepsilon_{\xi\sigma\tau\rho} \tag{2.24}$$

In (2.24), $F^{\tau\rho}$ is the field strength tensor of the E.M. field and α_0 is the "bare" fine structure constant.

It is trivial to see that the r.h.s. of (2.23) is uniquely related to the numerical coefficient of the VEV of the double commutator (2.2) by simply combining (2.4) with (2.22). We note that the three numerical coefficients in (2.23) were defined so to fulfil $S=S_{aa}=K_{vv}=1$ for the U(1) free currents. Therefore, the result we get from combining (2.4) with (2.22) agrees with our BJL calculation (2.15) as well.

To summarize, we find a similarity between the d=2 and d=4 free fermionic theories, concerning the relation of the W.I. anomaly to anomalous commutators. A breaking of the Jacobi identity for certain components of a given current does not necessarily imply a breakdown of the W.I. connected with this current (e.g. the non-abelian vector triangle is free from W.I. anomalies, but the Jacobi identity of the free non-abelian currents is broken); however, we do find a breaking of J associated with a W.I. breaking for the axial current. We also find a clear numerical connection between the two phenomena.

The similarity between d=2 and d=4 ceases when we look at the actual algebraic structure which emerges in d=4. Two important differences are immediately apparant:

1. There are no direct means of getting rid of the divergence in the single commutator ST, and it is therefore not clear how to incorporate it into the algebra.

2. The role of the space components of the currents is more fundamental in d=4. In d=2, with the help of the special identity (1.16), A_1 is actually V_0, and if we start in the classical theory with a gauge group G, the ST provides an extension of this algebra. In d=4, already for G=U(1) we have to consider a bigger algebra in order to incorporate the space components of the currents which take part in forming the anomalous contribution. However, it is interesting to note that although in d=4 the vector and axial currents are not tied together by an identity of the type (1.16), the algebra of the V_μ's doesn't close without the A_μ's (eq. (A.5) app. A).

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CHAPTER III - On algebraic properties of anomalous abelian theories in the Hamiltonian formalism

3.1 Introduction

The aim of the present chapter is to look at the problems caused by the bad anomaly and at the ways various commutators are affected by it when the theory is formulated using the Hamiltonian approach. In a set of papers by Faddeev and Faddeev and Shatashvili[1] it was suggested that when an anomalous theory is quantized in the Hamiltonian formalism, using the Weyl gauge ($A_0=0$), information about the anomaly appears as an extra term in the commutator of the Gauss-law constraints. These constraints are also the generators of the residual time independent gauge transformations which are left after fixing the Weyl gauge. In a non-anomalous theory they satisfy the normal algebra of the gauge group, namely:

$$[G^a(\vec{x}),G^b(\vec{y})] = if^{abc}G^c(\vec{x})\delta^3(\vec{x}-\vec{y}) \qquad (3.1)$$

In this case it is consistent to impose:

$$G^a(\vec{x})|\text{physical}\rangle = 0 \qquad (3.2)$$

which eliminates the remaining gauge freedom. However, for an anomalous theory, ref.1 proposes, that as a consequence of a proper regularization of the quantum operator $G^a(\vec{x})$, eq. (3.1) is modified to:

$$[G^a(\vec{x}),G^b(\vec{y})] = if^{abc}G^c(\vec{x})\delta^3(\vec{x}-\vec{y}) + \alpha_2^{ab}(\vec{x},\vec{y}) \qquad (3.3)$$

Now, eq. (3.2) and (3.3) are inconsistent when taken together. Since eq. (3.2) is an essential step in the normal quantization procedure, $\alpha_2^{ab}(\vec{x},\vec{y})$ represents an obstruction to this procedure due to the fact that it appears only after passing to the quantum theory. Yet, Faddeev has suggested that maybe after taking into account the presence of α_2, a new quantization scheme, which will yield a consistent theory, can be found. This hope provides us with the motivation to look more closely at anomalous gauge theories in the Hamiltonian formulation. In general, the potential problems one might expect, in an anomalous gauge theory, are a lack of unitarity when the theory is quantized in a manifestly Lorentz covariant gauge, or a lack of Lorentz invariance when the theory is quantized in a unitary gauge (like the Weyl gauge).

Most of the investigations around Faddeev's proposal have so far been carried on two dimensional theories. In particular, Halliday, Rabinovici, Schwimmer and Chanowitz[2] have shown, by an exact solution of the abelian chiral Schwinger model, that the resulting spectrum of states of the full Hilbert space is indeed non relativistic and that there is no subspace of it which possesses the desired Lorentz invariance. Before discussing this point, it's important to clarify a certain issue concerning the status of α_2, for the case of an abelian gauge group.

It is known that for the abelian case, α_2 is cohomologically trivial in the sense that one can change the definition of $G(\vec{x})$ by a term with local dependence on the gauge field in such a way that the new $G(\vec{x})$ will satisfy (3.1). However, this is not likely to remove the obstruction to quantization, since in order to impose (3.2) consistently, one should also demand:

$$[H,G^a(\vec{x})] \approx 0 \tag{3.4}$$

Where H is the Hamiltonian operator, and $\approx$ stands for "weak equality"[4,5]. Eq.(3.4) guarantees that the time evolution of physical states is consistent with the constraints imposed at some initial time. If we start with a situation where (3.4) is satisfied but $\alpha_2 \neq 0$, a redefinition of $G(\vec{x})$ by an $A_\mu(\vec{x})$ dependent term in order to remove α_2, may result in an appearance of a new term on the r.h.s. of (3.4). In fact, this situation is known to occur in the two dimensional abelian models, as will be shown explicitly later on.

This interplay between (3.3) and (3.4) in the abelian chiral Schwinger model has already caused some confusion, more so because different quantization procedures can differ in an implicit manner about the way the above mentioned ambiguity is fixed and to produce what look like contradictory results (As an example see some of the remarks of ref.3 concerning ref.2). The conclusion that should be drawn from this "abelian phenomenon" for the general case (abelian and non-abelian), is that we should look how the anomaly affects both equations (3.1) and (3.4).

Returning to the non-relativistic spectrum found in ref.2, it can be argued to be the result of the impossibility of imposing (3.2) due to problems with either (3.1) or (3.4). It is known that in a non-anomalous theory like QED, Poincaré invariance is guaranteed only when Gauss-law is imposed, since the equal-time Poincaré algebra closes up to G-dependent terms. It will be shown that this picture of the source of Poincaré non-invariance needs some modification. The conclusion will be that the anomaly can affect a third commutator, connected with the Poincaré algebra, in a way that doesn't follow directly from the two anomalous commutators which were already discussed. The rest of the chapter is divided into two parts as follows: In the first part (3.2) we study the Poincaré algebra of the abelian chiral Schwinger model (hereafter C.S.M) in its bosonized form and show that when G(x) is fixed to commute with the Hamiltonian, a "G-independent" anomaly appears in the Poincaré algebra, and discuss the implications. In the second part (3.3) we try to examine the same questions of how the anomaly affects key commutators in a d=4 abelian example. Again, because the model is abelian, α_2 is trivial and there should exist a local $G(\vec{x})$ that satisfies (3.1). However, it is not a priori clear that there is a choice of a local $G(\vec{x})$ which can satisfy (3.4). We study this question for the case of massless axial QED in perturbation theory, using the BJL definition of commutators. Two interesting features, compared with the d=2 case, emerge. The first one is that the algebra of G and H does not satisfy the Jacobi identity (As was mentioned in the previous chapter, the breaking of the Jacobi identity by commutators calculated from anomalous diagrams in an anomalous gauge theory was first noted by Jo[6], but its manifestation in the G,H algebra was not discussed). The second interesting feature is that the relevant commutators have contributions from a square diagram on top of the expected contributions from the AAA triangle diagram.

3.2 Poincaré non-invariance of C.S.M

In this section we will study the question of Poincaré invariance in the C.S.M. using its bosonized representation. For simplicity we take one left-handed and one right-handed fermion. As was shown in ref.2 through the bosonization of the fermion operators, the original theory is completely equivalent to a quadratic bosonic model. Hence, we'll work with the following expressions: a Hamiltonian H given by:

$$H = \int dx\ H(x) = \frac{1}{2}\int dx\ \{E^2(x) + H^L(x) + H^R(x)\} \tag{3.5}$$

where:

$$H^{\stackrel{L}{R}}(x) \equiv [\sqrt{\tfrac{1}{2}}\ (\Pi^{\stackrel{L}{R}}(x) \pm \partial_x \Phi^{\stackrel{L}{R}}(x))\ \pm\ e^{\stackrel{L}{R}} A(x)]^2 \tag{3.6}$$

and a constraint G(x) given by:

$$G(x) \equiv \partial_x E(x) - \sqrt{\tfrac{1}{2}}e^L(\Pi^L(x)+\partial_x\Phi^L(x)) - \sqrt{\tfrac{1}{2}}e^R(\Pi^R(x)-\partial_x\Phi^R(x)) \tag{3.7}$$

In these formulas $\Phi(x)$ is a dimensionless scalar field and $\Pi(x)$ is its canonical conjugate momentum. They satisfy canonical equal-time commutation relations:

$$[\Pi^L(x),\Phi^L(y)] = [\Pi^R(x),\Phi^R(y)] = -i\delta(x-y) \tag{3.8}$$

Similarly, the gauge field A(x) and the electric field E(x) satisfy:

$$[E(x),A(y)] = i\delta(x-y) \tag{3.9}$$

while all other equal time commutators between these fields are zero. In this formulation of the theory, all commutators we want to calculate are realized canonically, that is, we use (3.8) and (3.9) directly. The standard normal ordering of composite operators is to be understood in all of our expressions.

With the above definitions one gets:

$$[G(x),G(y)] = i(e_L^2-e_R^2)\delta'(x-y) \tag{3.10}$$

$$[H,G(x)] = 0 \tag{3.11}$$

As was mentioned in the introduction we can modify G(x) by a locally A(x)-dependent term which will set the r.h.s. of (3.10) to zero. Define:

$$G_M(x) \equiv G(x) - \frac{1}{2}(e_L^2-e_R^2)A(x) \tag{3.12}$$

$$[G_M(x),G_M(y)] = 0 \tag{3.13}$$

$$[H,G_M(x)] = -\frac{i}{2}(e_L^2-e_R^2)E(x) \tag{3.14}$$

The modification (3.12) can be thought of as a modification of the fermionic current, reflecting finite A(x)-dependent ambiguities in the definition of the normal-ordered current, and is therefore strongly related to the current conservation equation. We will discuss this point in more detail in the next section where we work directly with the fermionic variables. The possible presence of such an A(x)-dependent term in the definition of the fermionic current explains quite naturally how the so called seagull commutator can be realized in the present canonical framework[3].

We now want to understand whether we can construct a set of Poincaré generators satisfying the appropriate equal-time algebra. These generators have to satisfy two demands:

1. to generate the relevant Poincaré transformations (infinitesimal translations and Lorentz transformations) when acting on the basic degrees of freedom.
2. to satisfy the Poincaré algebra.

The two-dimensional Poincaré symmetry consists of three operations: Time translations that should be generated by the Hamiltonian H, space translations generated by the momentum P, and boosts generated by M. The algebra these generators should satisfy is:

$$\text{(a)}\quad [H,P] = 0 \tag{3.15}$$

$$\text{(b)}\quad [M,P] = -iH$$

$$\text{(c)}\quad [M,H] = -iP$$

Assuming H and P are given as space integrals over densities which are local functions of the basic degrees of freedom:

$$H = \int dx\ H(x)\ ;\ P = \int dx\ P(x) \tag{3.16}$$

the boost generator will be given by:

$$M = tP - \int dx\ xH(x) \tag{3.17}$$

Based on experience from non-anomalous QED, we allow terms of the form $\int dx\ U(x)G(x)$, to appear on the r.h.s. of (3.15).

We first note that our information on P is somewhat different in nature from that concerning H. Since we are working in the Hamiltonian formulation, the Hamiltonian generates the dynamics, and in the equation:

$$i[H,\phi(t,x)] = \partial_0\phi(t,x) \tag{3.18}$$

the r.h.s. is defined by the l.h.s., and we are looking for a $\phi(t,x)$ which solves it. In other words, (3.18) cannot in general be used to restrict the form of H. In contrast, the equation:

$$i[P,\phi(0,x)] = \partial_x\phi(0,x) \tag{3.19}$$

serves as a restriction on P. We are looking for a P which satisfies it. There is one crucial exception to this. Since we don't want the constraint to be a dynamical variable, the case ϕ = a constraint, does restrict H as a Poincaré generator, and H must be chosen to commute with the constraints (eq.(3.4)), as a part of the first condition we have placed on the Poincaré generators. Trying to treat possible terms appearing on the r.h.s. of (3.4) as new constraints, without introducing new degrees of freedom into the theory, is not likely to succeed because the theory will become over constrained and may be "pushed back" to the free theory.

Following the above, it is clear that with the choice (3.13-14) the Poincaré algebra is anomalous. What about the choice (3.10-11) ? Here we make use of an observation due to Schwinger[7], that once H(x) is given, P is fixed by (3.15). We substitute (3.17) into (3.15c) and use (3.15a) to get:

$$-iP = -\int dxdy\; x[H(x),H(y)] - i\int dx\; U_P(x)G(x) \qquad (3.20)$$

The commutator appearing in (3.20) must be anti-symmetric under $x \leftrightarrow y$ so only terms proportional to $\delta'(x-y)$, can appear. Substituting from (3.5) we get:

$$[H(x),\; H(y)] = \frac{1}{4}\{[H^L(x),H^L(y)]+[H^R(x),H^R(y)]\} \qquad (3.21)$$

The evaluation of the remaining commutators using (3.8) is elementary. We get:

$$P = -\frac{1}{2}\int dx\ \left(H^L(x)-H^R(x)\right) + \int dx\ U_P(x)G(x) \tag{3.22}$$

We now check to see whether this P satisfies (3.15a):

$$[P,H] = \int dx[U_P(x),H]G(x) - \frac{1}{4}\int dxdy[H^L(x)-H^R(x),E^2(y)] = \tag{3.23}$$

$$= \int dx\ [U_P(x),H]G(x) + \frac{i}{2}(e_L^2-e_R^2)\int dx\ \left(E(x)A(x)+A(x)E(x)\right) +$$

$$+ i\int dx\ E(x)\left[\sqrt{\frac{1}{2}}e_L\left(\Pi^L(x)+\partial_x\Phi^L(x)\right)+\sqrt{\frac{1}{2}}e_R\left(\Pi^R(x)-\partial_x\Phi^R(x)\right)\right]$$

We can replace the square brackets in the last line of (3.23) by $-(G(x)-\partial_x E(x))$. Doing this we get:

$$[P,H] = \int dx\left([U_P(x),H]-iE(x)\right)G(x) \tag{3.24}$$

$$+\frac{i}{2}(e_L^2-e_R^2)\int dx\left(E(x)A(x)+A(x)E(x)\right)$$

The first term in (3.24) is proportional to G(x), and in fact vanishes for the correct choice of $U_P(x)$. However, the second term which is "G-independent", remains and prevents the closure of the Poincaré algebra. Note that this term has the characteristic coefficient of the anomaly (i.e. it vanishes for $e_L = \pm e_R$). It comes from the following term in P (see (3.22)):

$$-\frac{1}{2}(e_L^2-e_R^2)\int dx\ A^2(x) \tag{3.25}$$

This in turn can be traced back to the commutator of the interaction part in the Hamiltonian (i.e. the eAJ^1 part of the Hamiltonian) with itself (see (3.21)). The reason that this last commutator is anomalous is of course the presence of the Schwinger term in the current-current commutator, precisely the same Schwinger term which is the cause of (3.10). In appendix C we check whether this "Poincaré anomaly" cannot be removed by local redefinitions of both the Gauss-law G(x) and the Hamiltonian. We show that under a suitable set of assumptions about these possible modifications, as long as one demands that the modified H will commute with the modified G(x), the Poincaré algebra based on the modified H is anomalous in a similar "G-independent" manner.

The "Poincaré anomaly" can place some restrictions on possible suggestions for a consistent quantization of an anomalous theory. Suppose that we choose an H and a G(x) which satisfy (3.10-11). In Dirac's terminology, G(x) is a second class constraint[4,5]. Dirac has suggested that a set of classical second class constraints can be set consistently to zero, if the normal Poisson brackets are replaced by what is termed Dirac brackets[4,5]. In a classical theory, these brackets are defined in such a way that the bracket of any function of the phase space coordinates q_i and p_i with a second class constraint, vanishes. Therefore, after introducing them, it is consistent to set the second class constraints to zero. In our case, we cannot apply Dirac's method directly, since G(x) turns into a second class constraint only after quantization. Still we may try to borrow the idea and define a new quantum commutator:

$$[O_1(x),O_2(y)]^* \equiv [O_1(x),O_2(y)] \qquad (3.26)$$
$$-\int dzdw[O_1(x),G(z)]([G(z),G(w)])^{-1}[G(w),O_2(y)]$$

where the inverse of the commutator of G(x) with itself will be defined by:

$$\int dz([G(x),G(z)])^{-1}[G(z),G(y)] = \qquad (3.27)$$

$$=\int dz[G(x),G(z)]([G(z),G(y)])^{-1}= \delta(x-y)$$

In the general case, the commutator of the constraints can be some quantum operator, so finding a solution to (3.27), and in particular assuring that the left inverse and the right inverse are equal (if they exist at all), can range from a complicated task to an impossible one. However, the r.h.s. of (3.10) is a c-number. By substituting (3.10) into (3.27) it's easy to see that:

$$([G(x),G(y)])^{-1}= \frac{i}{(e_L^2-e_R^2)} \varepsilon(x-y) \qquad (3.28)$$

Where (θ is the step function):

$$\varepsilon(x-y) \equiv \frac{1}{2}(\theta(y-x)-\theta(x-y))$$

With (3.28) substituted into (3.26) the Dirac "trick" will work, and whenever one of the O's in (3.26) is G(x) we'll get a zero. Note, that under the new commutation rule we have:

$$[G(x),G(y)]^{*}= 0; \ [H,G(x)]^{*}= 0 \qquad (3.29)$$

One may wonder how we overcame previous obstructions to satisfying both of these equations simultaneously. This can be understood by recalling that using Dirac brackets is equivalent to using normal brackets but modifying the operators by non-local terms[4,5]. Since previously we imposed locality on the possible modifications, we were not able to achieve (3.29). But does (3.29) solve the problem of Poincaré invariance ? The answer is no. Because of (3.11) (which is also true for the Hamiltonian density H(y)),for any operator O, the following holds:

$$[H,O]^{*}= [H,O] \qquad (3.30)$$

Hence the new commutator definition doesn't modify eqs.(3.20-24), and the Poincaré anomaly remains. We note that the treatment of the Chiral Schwinger model in ref.8 seems to match our last discussion. Through the introduction of certain non-local terms the analog of (3.29) is achieved (see ref.8 eqs. 11 and 22), but still Poincaré invariance is not recovered.

Compared with this, the main suggestion so far for a consistent quantization of an anomalous gauge theory[1] is to add to the original action a Wess-Zumino term, which includes a new field (θ), and therefore should be considered as a non-local modification. This proposal is again studied in the context of C.S.M. in ref.2. It is left to the reader to convince him/herself by using the formulas given there that this time, after correctly fixing the local modifications, not only G(x) does return to be a first class constraint, but also the "G-independent" anomaly in Poincaré vanishes as well.

3.3 Axial massless QED in d=4

For d=4 anomalous gauge theories we again expect problems with Poincaré invariance when the starting classical theory is formulated in a physical gauge. Again, for the abelian case, α_2 in eq. (3.3) is cohomologically trivial, and therefore one expects the anomaly to affect (3.4) as well. The question we want to address in this section is whether the features of the d=2 abelian case generalize to the d=4 abelian theory. Can we, for instance, still shift the anomaly between (3.1) and (3.4) ?

Since in d=4 one cannot obtain exact solutions to the relevant models, we will work out one-loop contributions to anomalous commutators, to the leading order in perturbation theory, using the BJL definition, and check the answers at this level, as was done in refs. 6 and 9. As a convenient model for studying these questions we choose massless axial electrodynamics, in which the U(1) gauge field is coupled to a single axial fermionic current. Contrary to d=2, where the "pure axial" theory is non-anomalous, the AAA triangle diagram (Fig.3a) has a fundamental anomaly[10]. Including a vector current, we would have to consider the AVV diagram as well.

The classical Lagrangian density for our model is given by:

$$L(x) = -\frac{1}{4} F_{\mu\nu}(x)F^{\mu\nu}(x)+i\overline{\Psi}(x)\not{\partial}\Psi(x)-eJ^5_\mu(x)A^\mu(x) \qquad (3.31)$$

where $J^5_\mu(x) \equiv \overline{\Psi}(x)\gamma_\mu\gamma_5\Psi(x)$.

In the Weyl gauge, the Hamiltonian density derived from this Lagrangian is:

$$H(x) = \frac{1}{2}(\vec{E}^2(x)+\vec{B}^2(x))-i\overline{\Psi}(x)\gamma_i\partial^i\Psi(x)+eJ_i^5(x)A^i(x) \qquad (3.32)$$

and the Gauss-law constraint is:

$$G(\vec{x}) = \vec{\nabla}\cdot\vec{E}(\vec{x})-eJ_0^5(\vec{x}) \approx 0 \qquad (3.33)$$

Before proceeding to the actual calculation of the relevant commutators, we'll try to use simple considerations in order to analyze the possibilities. We'll start by examining the possible form of $[G_M(\vec{x}),G_M(\vec{y})]$, where the label M indicates, as before, a possible modification of (3.33) by a local function of $A_i(\vec{x})$. Standard assumptions[11] (see (2.1)) imply that the terms on the r.h.s. must be proportional to $\delta^3(\vec{x}-\vec{y})$ or its derivatives. The required $\vec{x} \leftrightarrow \vec{y}$ anti-symmetry, and dimensions allow only for a first derivative. We also expect to have the Levi-Civita tensor on the r.h.s. since the anomaly is coming from the parity violating part. Moreover, the r.h.s. should depend only on A_i and must be a local function of it. All of this leads to:

$$[G_M(\vec{x}),G_M(\vec{y})] = c\,\frac{ie^3}{\pi^2}\,\vec{B}\cdot\vec{\nabla}\delta^3(\vec{x}-\vec{y}) \qquad (3.34)$$

where $\vec{B}$ is the magnetic field, and c is an undetermind numerical constant. e^3 anticipates the fact that non-zero contributions to the r.h.s. of (3.34) will be connected to the triangle diagram. In fact, (3.34), apart from a group theoretic factor, is precisely Faddeev's suggestion[1]. However, the triviality of α_2 in the abelian case means that there is a choice of $G_M(\vec{x})$ for which c=0.

Using similar considerations, we can assume the following form for the second commutator of interest:

$$[H,G_M(\vec{x})] = \frac{ie^3}{\pi^2}\left(c_1\vec{E}(\vec{x})\cdot\vec{B}(\vec{x})+c_2\vec{A}(\vec{x})\cdot\vec{\nabla}\times\vec{E}(\vec{x})\right) \qquad (3.35)$$

Contrary to (3.34), we must allow $\vec{E}(\vec{x})=-\partial_0\vec{A}(\vec{x})$ dependence on the r.h.s. because the l.h.s. gives us $\partial_0 G(\vec{x})$. A shortcut argument for no fermion field dependence on the r.h.s. is that anomalous diagrams from which possible contributions to the r.h.s. may arise, do not contain external fermion lines.

When we examine the possible terms which we can use to modify $G(\vec{x})$, we find only one term which is a local function of $A_i(\vec{x})$ alone and is invariant under space rotations. We can therefore write:

$$G_M(\vec{x}) = G(\vec{x})+c_3\frac{e^3}{\pi^2}\,\vec{A}(\vec{x})\cdot\vec{B}(\vec{x}) \qquad (3.36)$$

The term we added to $G(\vec{x})$ should start from order e^3 because it has a non-vanishing canonical commutator with $G(\vec{x})$, so only e^3 will be consistent with (3.34).

Now we'll examine how the various undetermined constants are constrained by the Jacobi identity and by the demand that there is a choice of c_3 for which $G_M(\vec{x})$ commutes with the Hamiltonian. Starting with the Jacobi identity we have:

$$J(G_M(\vec{x}),G_M(\vec{y}),H) \equiv [[G_M(\vec{x}),G_M(\vec{y})],H] + [[H,G_M(\vec{x})],G_M(\vec{y})]+[[G_M(\vec{y}),H],G_M(\vec{x})]= \quad (3.37)$$

$$= i\frac{e^3}{\pi^2}\{[c\vec{B}\cdot\vec{\nabla}\delta^3(\vec{x}-\vec{y}),H] + ([c_1\vec{E}(\vec{x})\cdot\vec{B}(\vec{x})+c_2\vec{A}(\vec{x})\cdot\vec{\nabla}\times\vec{E}(\vec{x}),G_M(\vec{y})] - \vec{x}\leftrightarrow\vec{y})\}$$

The commutators that are left to evaluate on the r.h.s. of (3.37) are $O(e^3)$; therefore, to this order, only their canonical contributions are important.

$$J(G_M(\vec{x}),G_M(\vec{y}),H) \equiv \frac{ie^3}{\pi^2}\{c\partial_j\delta^3(\vec{x}-\vec{y})\int d^3z[B^j(\vec{y}),\frac{1}{2}\vec{E}^2(\vec{z})] + \quad (3.38)$$

$$+(c_1E^j(\vec{x})[B^j(\vec{x}),\partial_iE^i(\vec{y})]+c_2[A^j(\vec{x}),\partial_iE^i(\vec{y})](\vec{\nabla}\times\vec{E}(\vec{x}))^j-(\vec{x}\leftrightarrow\vec{y}))\}=$$

$$= \frac{ie^3}{\pi^2}\, i(-c+2c_2)\vec{\nabla}\times\vec{E}(\vec{x})\cdot\vec{\nabla}_x\delta^3(\vec{x}-\vec{y})$$

So satisfying the Jacobi identity implies $c = 2c_2$. On the other hand, from eq. (3.35) we see that we have to have $c_1=c_2=0$ for the modified Gauss-law to commute with the Hamiltonian. Combining the two conditions together we get $c=0$. The net result of this analysis is, that under our assumptions and at least to order e^3 we are facing four distinct possibilities:

1. There is a choice of c_3 which will set the r.h.s. of (3.34)

and (3.35), simultaneously to zero.

2. There is no choice of c_3 which will set (3.35) to zero.

3. The Jacobi identity is not satisfied. As in the d=2 case, one can shift the anomaly between (3.34) and (3.35), by choosing an appropriate c_3 each time. It's important to note here that $J(G_M,G_M,H)$ does not really depend on c_3, since the term connected with c_3 is of order e^3 and therefore contributes through canonical commutators which automatically satisfy the Jacobi identity.

4. The Jacobi identity is not satisfied and still the r.h.s. of eq. (3.35) cannot be set to zero.

While the first possibility looks very unlikely, since it simply tells us that there is a local modification of $G(\vec{x})$, which allows to impose it consistently, a miracle that hasn't occured in d=2, the fourth possibility looks too ugly. In any case, it is clear that the d=4 abelian case possesses some new qualitative features compared with d=2.

In fact, we already know from published results[6,9] that BJL commutators give c=0 for c_3=0. Moreover, the calculation in ref.6 indicates that some of the anomalous commutators do break the Jacobi identity (note eq. 1.1 in ref.6b - It holds for the abelian case as well). So we can say that existing results hint that possibilities 3 or 4 may be realized. Assuming for the moment that there exists a value of c_3 for which $c_1=c_2=0$, we can speculate a bit more on what this value may be. First note, that in order to achieve $c_1=c_2=0$, for some value of c_3, we must have $c_1=c_2$ for all values of c_3. This can be seen directly by evaluating the following commutator canonically:

$$[H,\vec{A}(\vec{x})\cdot\vec{B}(\vec{x})] = \int d^3y[\frac{1}{2}\vec{E}^2(\vec{y}),\vec{A}(\vec{x})\cdot\vec{B}(\vec{x})] = \tag{3.39}$$
$$= i\left(\vec{E}(\vec{x})\cdot\vec{B}(\vec{x}) + \vec{A}(\vec{x})\cdot\vec{\nabla}\times\vec{E}(\vec{x})\right)$$

Next, recall[10] that the expression for the divergence of the axial current which reproduces the anomaly in the AAA triangle is:

$$\partial^\mu J^5_\mu(x) = \frac{e^2}{48\pi^2}\,\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}(x)F_{\alpha\beta}(x) = \frac{e^2}{6\pi^2}\,\vec{E}(x)\cdot\vec{B}(x) \tag{3.40}$$

where the ambiguities in the AAA triangle were fixed by the demand of full Bose symmetry between the three photonic legs, and the second equality holds in the Weyl gauge. As is shown in ref.10, one can define:

$$\bar{J}^5_\mu(x) \equiv J^5_\mu(x) - \frac{e^2}{12\pi^2}A^\xi(x)\partial^\rho A^\tau(x)\varepsilon_{\xi\mu\tau\rho} \tag{3.41}$$

which satisfies:

$$\partial^\mu\bar{J}^5_\mu(x) = 0 \tag{3.42}$$

We'll now couple $A^\mu(x)$ to $\bar{J}^5_\mu(x)$. In the Weyl gauge the Hamiltonian (3.32) remains unchanged, but the Gauss-law is modified since $J^5_0(\vec{x})$ is replaced by $\bar{J}^5_0(\vec{x})$. We'll have:

$$\overline{G}(\vec{x}) = G(\vec{x}) - \frac{e^3}{12\pi^2}\vec{A}(\vec{x})\cdot\vec{B}(\vec{x}) \qquad (3.43)$$

from which we can guess that $c_3 = -\frac{1}{12}$ is a serious candidate to set $c_1=c_2=0$. This guess is motivated by the classical (Poisson bracket) result:

$$[H,G(x)] = \partial^\mu J^5_\mu(x) \qquad (3.44)$$

A derivation of a related relation can be found in Jackiw[12] (eq. 2.18). A direct proof is presented in a paper by Hwang, who also studies the status of (3.44) for C.S.M. . From (3.40) we see that if (3.44) is not modified in the quantum theory, we'll have $c_2=0$ and $c_1 \neq 0$ when $c_3=0$, and therefore there will be no way to get the modified Gauss-law to commute with H. This is to be contrasted with the d=2 case, where (3.44) is true in the quantum case, and consistent with $[H,\overline{G}]=0$. For d=4 it's attractive to conjecture that (3.44) is modified in such a way as to allow $\overline{G}$ to commute with H.

We now turn to the actual calculation to see which of the possibilities mentioned is realized. The standard assumption made in calculations of anomalous commutators is that they get contributions only from anomalous diagrams (to be more precise, that other possible contributions from non-anomalous diagrams cancel when we calculate a classically gauge-invariant object like [H,G]). Following this, we consider contributions from the AAA diagram (Fig.3a). Here we'll just give the final results. Details of the method of calculation are given in app. D and E. Our application of the BJL definition follows closely that of Jo[6] and complementary details concerning the method can be found there. In fact, our triangle results should be deduceable from the results of ref.6 after taking care of differences in conventions. I've redone the part of the calculation which is relevant to the present work for the sake of completness and to check the method.

$$\text{(a)}\quad [J_0^5(\vec{x}),J_0^5(\vec{y})] = -\frac{ie}{2\pi^2}\varepsilon^{ijk}\partial_i A^j \partial_k \delta^3(\vec{x}-\vec{y}) \qquad (3.45)$$

$$\text{(b)}\quad [J_0^5(\vec{x}),J_i^5(\vec{y})] = -\frac{ie}{4\pi^2}\varepsilon^{ijk}E_j(\vec{y})\partial_k \delta^3(\vec{x}-\vec{y})$$

$$\text{(c)}\quad [E^i(\vec{x}),J_0^5(\vec{y})] = \frac{ie^2}{6\pi^2}\varepsilon^{ijk}(A_j(\vec{y})\partial_k+2\partial_j A_k)\delta^3(\vec{x}-\vec{y})$$

$$\text{(d)}\quad [E^i(\vec{x}),J_j^5(\vec{y})] = \frac{ie^2}{12\pi^2}\varepsilon^{ij\ell}E^\ell \delta^3(\vec{x}-\vec{y})$$

$$\text{(e)}\quad [E^i(\vec{x}),E^j(\vec{y})] = -\frac{ie^3}{6\pi^2}\varepsilon^{ij\ell}A^\ell \delta^3(\vec{x}-\vec{y})$$

$$\text{(f)}\quad [A^j(\vec{x}),J_0^5(\vec{y})] = 0$$

$$\text{(g)}\quad [A^j(\vec{x}),J_i^5(\vec{y})] = \frac{ie^2}{6\pi^2}\varepsilon^{ij\ell}A^\ell \delta^3(\vec{x}-\vec{y})$$

Using (3.45) (a),(c) and (e) one can verify that indeed:

$$[G(\vec{x}),G(\vec{y})] = 0 \qquad (3.46)$$

i.e. c=0 for $c_3=0$. The contribution of the triangle to the commutator of the Hamiltonian with the Gauss-law is given by:

$$[H,G(\vec{x})]_T = \frac{ie^3}{12\pi^2}\left(4\vec{E}(\vec{x})\cdot\vec{B}(\vec{x})-2\vec{A}(\vec{x})\cdot\vec{\nabla}\times\vec{E}(\vec{x})\right) \qquad (3.47)$$

Comparing with (3.35) we find that for $c_3=0$, we have $c_1=\frac{1}{3}$ and $c_2=-\frac{1}{6}$. These values correspond to our possibility 4.

We can get another indication that considering the triangle contributions is not enough, by trying to reproduce eq. (3.40) in the present formalism through:

$$\partial^\mu J^5_\mu(x) = i[H,J^5_0(\vec{x})] + i[P^i,J^5_i(\vec{x})] \qquad (3.48)$$

where the space translations operator is given by:

$$P^i = \int d^3x\{i\Psi^+(\vec{x})\partial^i\Psi(\vec{x})+(\partial^i\vec{E}(\vec{x}))\cdot\vec{A}(\vec{x})\} \qquad (3.49)$$

Using (3.45) we get:

$$i[H,J^5_0(\vec{x})]_T+i[P^i,J^5_i(\vec{x})]_T = \frac{e^2}{12\pi^2}\left(8\vec{E}(\vec{x})\cdot\vec{B}(\vec{x})-3\vec{A}(\vec{x})\cdot\vec{\nabla}\times\vec{E}(\vec{x})\right) \qquad (3.50)$$

which doesn't reproduce (3.40).

All of this suggests that we have missed contributions to the anomalous commutators and that (3.45) is not the full list. It's not difficult to see that the only other diagram which can potentially contribute to order e^3 is the square diagram in Fig.3b . It can contribute to commutators of the free fermionic energy momentum tensor (hereafter denoted by Θ_F), with various operators. Details of the calculations, which are much more cumbersome than the triangle calculations, are to be found in app. D and E. The relevant results are:

$$\text{(a)}\ \left[\int d^3y(-i\overline{\Psi}(\vec{y})\gamma_i\partial^i\Psi(\vec{y})), J_0^5(\vec{x})\right]_{\text{anomalous}} = \tag{3.51}$$

$$= \frac{6ie^2}{12\pi^2}(\vec{E}(\vec{x})\cdot\vec{B}(\vec{x})-\vec{A}(\vec{x})\cdot\vec{\nabla}\times\vec{E}(\vec{x}))$$

$$\text{(b)}\ \left[\int d^3y(-i\overline{\Psi}(\vec{y})\gamma_i\partial^i\Psi(\vec{y})), \vec{\nabla}\cdot\vec{E}(\vec{x})\right] =$$

$$= \frac{3ie^3}{12\pi^2}(\vec{E}(\vec{x})\cdot\vec{B}(\vec{x})-\vec{A}(\vec{x})\cdot\vec{\nabla}\times\vec{E}(\vec{x}))$$

$$\text{(c)}\ \left[\int d^3y(i\Psi^+(\vec{y})\partial^i\Psi(\vec{y})), J_i^5(\vec{x})\right]_{\text{anomalous}} =$$

$$= \frac{3ie^2}{12\pi^2}\vec{A}(\vec{x})\cdot\vec{\nabla}\times\vec{E}(\vec{x})$$

The contribution of (3.51) (a) and (b) to $[H,G(\vec{x})]$ is:

$$[H,G(\vec{x})]_S = -\frac{3ie^3}{12\pi^2}(\vec{E}(\vec{x})\cdot\vec{B}(\vec{x})-\vec{A}(\vec{x})\cdot\vec{\nabla}\times\vec{E}(\vec{x})) \tag{3.52}$$

Combining (3.47) with (3.52) we now get for $c_3=0$, $c_1=c_2=\frac{1}{12}$. The Jacobi identity is still not saved but for the predicted value of $c_3=-\frac{1}{12}$, the Gauss-law constraint commutes with H. It's also easy to see that (3.51) (a) and (c) combine with (3.50) to reproduce (3.40). This completes the proof that the square diagram contributions are essential to reproduce the anomaly in the Hamiltonian formalism. We'll end up by making a few remarks on the d=4 results and their implications.

1. The results of ref.13 indicate that in d=2, Θ_F does not contribute in a fundamental way to the anomalous commutators. The same result can be also easily seen from the formalism of ref.2.

2. We have mentioned in chapter I that another axial Ward identity anomaly is known to exist in our model, namely the Delbourgo-Salam anomaly[17], which appears in a triangle diagram with two Θ_F vertices and one J^5_μ. However, it can't contribute to the commutators discussed above.

3. After showing $[H,\overline{G}(\vec{x})]=0$, we can ask whether the d=2 result of a "G-independent" anomaly in Poincaré, also generalizes to d=4. The direct diagrammatic evaluation of the analog to (3.20), involves a lot of labour, since one should check for possible contributions from the triangle, the square, and possibly from a pentagon diagram, giving the matrix element of $[\Theta_F(\vec{x}),\Theta_F(\vec{y})]$ between vaccum and three photon state. The full calculation hasn't been done but there are some things that can be said on the expected result. Let us assume by analogy with the d=2 case that an extra anomalous term in P^i, will be of order e^3 and a local function of $\vec{A}(\vec{x})$ alone, containing a Levi-Civita tensor. There are three terms which answer this

requirement:

1) $\int d^3x\, A^i(\vec{x})\vec{A}(\vec{x})\cdot\vec{B}(\vec{x})$ 2) $\int d^3x\, B^i(\vec{x})\vec{A}^2(\vec{x})$ 3) $\int d^3x\, B^i(\vec{x})\vec{\nabla}\cdot\vec{A}(\vec{x})$

We can discard the last term since potential contributions are from terms containing three photon fields. Now, in d=2 we were able to construct a non-anomalous P^i when $G_M(\vec{x})$ commuted with itself. In d=4 using the BJL definition this happens for $c_3=0$. As we have seen, the canonical Poincaré algebra closes up to a G-dependent term, where this G is the unmodified one ($c_3=0$). So if d=2 results are to be paralleled, we expect the anomalous contributions to the analog of (3.20) to vanish (When we work with $c_3\neq 0$, P^i will be anomalous exactly because the canonical Poincaré algebra still closes up to $G(\vec{x})$ and not $G_M(\vec{x})$).

It is not hard to evaluate the contribution of the triangle diagram, since we have a complete expression for it[10]. One can immediately see that none of the commutators in (3.45) can contribute. The only remaining triangle commutator which is not in (3.45) is $[J^5_i(\vec{x}),J^5_j(\vec{y})]$. This commutator contains several pieces, including a divergent term. However, only one term, with a non-divergent coefficient, contributes to the final answer. One finds:

$$\int d^3x d^3y\, x^i[e\vec{A}(\vec{x})\cdot\vec{J}^5(\vec{x}),e\vec{A}(\vec{y})\cdot\vec{J}^5(\vec{y})] = \tag{3.53}$$
$$= c_4 \frac{ie^3}{\pi^2}\int d^3x A^i(\vec{x})\vec{A}(\vec{x})\cdot\vec{B}(\vec{x})$$

with $c_4\neq 0$ and finite. We see once more that if d=2 results are to be generalized, we'll need "non-triangle" contributions in order to cancel (3.53).

4. Earlier on it was mentioned that a basic assumption in calculating anomalous commutators is that contributions to them come from diagrams with true Ward identity anomalies. If we stick to this assumption, we should conclude that the square diagram in Fig.3b has a Ward identity anomaly. Seen from a slightly different angle, this diagram describes an effective coupling of a graviton to three axial photons. In order to completely clarify the nature of the square anomaly, we should therefore study the one-loop diagrams of fermions coupled to external gauge and gravitational fields, and try to impose all desired physical requirements on the square. This investigation is beyond the scope of the present work. However, one can think of three distinct possible results:

(a) There is no fundamental anomaly in the square - one can fix the ambiguities of the square in such a way that all Ward identities and symmetry requirements are satisfied.

(b) There is a fundamental anomaly in the square, but it is connected to the already known anomalies in the triangle and may be calculated from them via some consistency condition.

(c) The anomaly in the square is new, and cannot be deduced from the already known anomalies.

It seems that our commutator results offer some (not conclusive) evidence, against the first possibility. Recall chapter I, where it was explained that a Ward identity anomaly is often described as arising in a situation where the ST doesn't cancel against seagull contributions when the divergence of the relevant T-product is taken. In fact, the occurrence of this cancellation for the bubble diagram is precisely the mechanism which guarantees that $[H,G(x)]=0$ in ordinary QED. Since in our case, the square has a non-zero contribution to this commutator, it suggests that a failure of Feynman conjecture does occur. Another sign for a non-trivial nature of the square commutators is the fact that the contribution of the square to $\partial^{\mu} J_{\mu}^{5}$ via (3.51a,c) is not sensitive to various ambiguities in the relevant one-loop diagrams. This issue is explained in appendix D.

If we take the fermions to be in some representation of a gauge group, our square commutators will all be multiplied by d^{abc} (a,b,c are group indices of the currents - Θ_F does not carry any). Thus, a cancellation of the square anomaly doesn't place any new restrictions on the fermion representation and occurs automatically if the triangle anomaly is cancelled in the standard way. This is consistent with the second possibility we have mentioned, but by no means can be considered as a sufficient proof of it. In connection with the last point we note the following: Any loop diagram which one obtains from a basic anomalous diagram by inserting into it an arbitrary number, N, of Θ_F vertices will have the same degree of divergence and the same group and discrete symmetry properties of the original diagram. Moreover, the new diagram will be connected through the naive Ward identities to a similar diagram with N-1 Θ_F insertions. Therefore, the answer to the problem of the nature of the square anomaly should clarify the status of all of these other diagrams as well. Finally, we mention that Capper, Jones and Linden[18] discuss the process of one graviton → three photons in the context of an anomalous gauge theory, formulated in the light cone gauge. However, no specific claim about the square diagram or any other diagram with Θ_F insertions is made.

5. Another issue related to our d=4 results is their implications concerning the suggestion for a possible consistent quantization procedure of an anomalous gauge theory. First of all we remark that the main qualitative features of our results, namely the breaking of the Jacobi identity and the contributions from the square diagram, are expected to generalize to the non-abelian case as well. In fact, as was mentioned earlier, the failure of the Jacobi identity for the electric field components was first calculated[6] in a general non-abelian theory. It is interesting, however, to know whether one can still achieve eq. (3.4) through a local modification, when α_2 is non trivial.

The failure of the Jacobi identity for H and G implies that they cannot be both represented on the same Hilbert space, making the mathematical inconsistency of the theory evident (In d=2 the inconsistency of the theory is physical and not mathematical). If by adding a Wess-Zumino term we can remove this inconsistency, it means that we should find a breaking of the Jacobi identity in the commutators involving the Wess-Zumino part. This is an interesting problem, since the Wess-Zumino term is supposed to reproduce the effects of the anomaly on a "classical" level, i.e. through Poisson brackets relations.

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SUMMARY AND CONCLUSIONS

The main feature of our results in chapters II and III is the connection in d=4 between the W.I. anomaly and a Jacobi identity failure for certain commutators. We started by looking for a d=4 analogy to the clear connection that exists in d=2 between the Kac-Moody algebra and the d=2 axial anomaly. In d=2 one first notices this connection in the free fermionic theory. Then, if the fermions are coupled to gauge fields in such a way that a bad anomaly is present, one can see in a clear manner how the presence of the non vanishing K.M. central term affects the theory and destroys a desired physical property like Poincaré invariance. Moreover, it was also shown (ref. 3-2), at least for d=2 abelian models, how the suggestion for a consistent quantization of an anomalous theory via the addition of a WZ action can be understood from a "K.M. point of view", as a mechanism for cancelling the anomaly by adding another K.M. representation, based on a scalar field, in such a way that in the combined theory the K.M. algebra is trivialized. In fact, the question whether this picture is true also for the d=2 non-abelian case is open at the moment.

In d=4 we haven't seen any such direct connection between the algebraic structure of the free theory and the problems encountered when the anomalous current is gauged. The c-number terms calculated in chapter II do not seem to be related to the commutators of chapter III. Nevertheless, a Jacobi identity breaking does appear in crucial commutators and does create severe problems for the gauged theory. This is intriguing and deserves further attention.

Finally, and in connection with the last point, it is interesting to know whether some sort of a "K.M. point of view" exists also for Faddeev and Shatashvili's suggestion in d=4. Can we think about their procedure as adding a representation of the same algebra as that of the original fermionic theory, based on a scalar field, in such a way that in the combined theory the Jacobi identity is restored ?

A P P E N D I C E S

Appendix A

The purpose of this appendix is to show in more detail how symmetry considerations constrain the current E.T.C.'s in a free massless fermionic theory. As an example, consider the non c-number part of $[V_i(0,\vec{x}),V_j(0,\vec{y})]$. As explained in section 2.3, it's enough to analyze the contributions from the five independent fermion bilinears (the scalar, pseudo-scalar, vector, axial vector and anti-symmetric tensor), which have the form $\bar{\Psi}(0,\vec{x})\Gamma_i\Psi(0,\vec{x})$, where Γ_i is the appropriate combination of Dirac matrices. For the definitions of the fermion bilinears and other notational conventions see ref 2-20 as well as for the P,C,T transformation properties (table (3-199) p.157 there). We now write a general expression for the above commutator:

$$[V_i(0,\vec{x}),V_j(0,\vec{y})] = \sum_n C_n O_{ij}(\vec{x})\delta^3(\vec{x}-\vec{y}) \qquad (A.1)$$

where C_n are complex numerical coefficients and $O_{ij}(\vec{x})$ are the fermion bilinears (or any combination of their components we can form using the tensors g_{ij} and ε_{ijk}). All our O_{ij}'s are defined to be hermitian; therefore, by taking the hermitian conjugate of both sides of (A.1), we get that all C_n's should be pure imaginary. Furthermore, since the l.h.s. of (A.1) should be anti-symmetric under the simultaneous interchange $i\leftrightarrow j$, $\vec{x}\leftrightarrow\vec{y}$ (see assumption 2 section 2.3), and since the r.h.s. of (A.1) is symmetric under $\vec{x}\leftrightarrow\vec{y}$ it follows that there are no g_{ij} terms on the r.h.s. Therefore:

$$[V_i(0,\vec{x}),V_j(0,\vec{y})] = (C_1\varepsilon_{ijk}A^k(0,\vec{x}) + C_2\varepsilon_{ijk}V^k(0,\vec{x}) + \qquad (A.2)$$

$$+ C_3T_{ij}(0,\vec{x}) + C_4\varepsilon_{ijk}T^{0k}(0,\vec{x}))\delta^3(\vec{x}-\vec{y})$$

By operating with charge conjugation on both sides of (A.2) and using the known transformation properties of all the bilinears appearing in this equation we get $C_2=C_3=C_4=0$. One can furthermore verify that an imaginary C_1 is consistent with P and T transformations. Finally, naive use of the canonical anti-commutation relations:

$$\text{(a)}\ \{\Psi_\alpha(t,\vec{x}),\Psi_\beta(t,\vec{y})\} = \{\Psi^+_\alpha(t,\vec{x}),\Psi^+_\beta(t,\vec{y})\} = 0 \qquad (A.3)$$

$$\text{(b)}\ \{\Psi_\alpha(t,\vec{x}),\Psi^+_\beta(t,\vec{y})\} = \delta_{\alpha\beta}\delta^3(\vec{x}-\vec{y})$$

gives the C.C.R between any two fermion bilinears:

$$[\bar{\Psi}(0,\vec{x})\Gamma_1\Psi(0,\vec{x}),\bar{\Psi}(0,\vec{y})\Gamma_2\Psi(0,\vec{y})] = \qquad (A.4)$$

$$=\bar{\Psi}(0,\vec{x})(\Gamma_1\gamma_0\Gamma_2-\Gamma_2\gamma_0\Gamma_1)\Psi(0,\vec{x})\delta^3(\vec{x}-\vec{y})$$

from which, by taking $\Gamma_1=\gamma_i$, $\Gamma_2=\gamma_j$, and using the identity $\varepsilon_{ilm}\gamma_5\gamma_0\gamma^i=-\sigma_{lm}$ ($\sigma_{lm}\equiv \frac{1}{2}i[\gamma_l,\gamma_m]$) we get:

$$[V_i(0,\vec{x}),V_j(0,\vec{y})]_{canonical}= 2i\varepsilon_{ijk}A^k(0,\vec{x})\delta^3(\vec{x}-\vec{y}) \qquad (A.5)$$

Appendix B

As was mentioned on several occasions in the text, quantum field theory singularities force us to look for a careful definition of a commutator. Such a definition is provided by the BJL prescription, according to which a matrix element of the commutator of two local operators A and B, between two arbitrary physical states is given by (ref. 3-10, 3-16):

$$\lim_{q_0\to i\infty} \langle\alpha| \int d^4x\; e^{iq\cdot x} T(A(x)B(0))\,|\beta\rangle = \tag{B.1}$$

$$= \left(\frac{1}{-iq_0}\right) \langle\alpha| \int d^3x\; e^{-i\vec{q}\cdot\vec{x}} [A(0,\vec{x}),B(0)]\,|\beta\rangle + O\left(\frac{1}{q_0^2}\right)$$

When Feynman rules are used to evaluate the T-product on the l.h.s. of (B.1), pure q_0 polynomials are to be dropped, since they arise from the difference between the Lorentz non covariant T-product which is used to define the BJL limit and the covariant object one gets from Feynman rules and uses in the actual calculation. On the other hand, terms which contain $\ln(q_0)$ or $q_0\ln(q_0) = \dfrac{q_0^2\ln(q_0)}{q_0}$ are interpreted respectively as logarithmic and quadratic divergences in the commutator.

There exist several formal proofs of the definition (B.1) (see for instance ref. 2-13). They serve to show that the BJL definition agrees with more naive ways of calculating commutators when no singularities are encountered.

Finally, by an obvious generalization of (B.1), a double commutator of three local operators can be defined as the coefficient of the $\frac{1}{q_0 p_0}$ term in an expansion of their three-point function, reached through a successive limit:

$$\lim_{q_0 \to i\infty} \lim_{p_0 \to i\infty} q_0 p_0 \, T(p,q) = - \int d^3x \, d^3y \, e^{-i\vec{p}\cdot\vec{x}} \, e^{-i\vec{q}\cdot\vec{y}} \langle\alpha| [B(0,\vec{y}),[A(0,\vec{x}),C(0,\vec{0})]] |\beta\rangle \qquad (B.2)$$

where:

$$T(p,q) \equiv \int d^4x \, d^4y \, e^{ipx} \, e^{iqy} \langle\alpha| TA(x)B(y)C(0) |\beta\rangle \qquad (B.3)$$

Appendix C

In this appendix we show in some detail, that no local modifications of H(x) and G(x) can remove the "G-independent" anomaly in Poincaré algebra, discussed in section 3.2 . By a local counter term (c.t.) we mean a term which is a finite polynomial in the basic variables, which we take to be:

$$\left(\Pi^{\substack{L\\R}}(x)\pm\partial_x\Phi^{\substack{L\\R}}(x)\right),\ A(x),\ E(x) \tag{C.1}$$

The restrictions we impose on these c.t.'s are the following:

1. G(x) is to be modified only by an A(x)-dependent term.
2. No change is to be made in the fermionic content of the theory.
3. The vector Schwinger model (normal 2-dim QED) expressions, should be recovered smoothly in the limit $(e_L^2-e_R^2)\to 0_+$.
4. $[H_M,G_M(x)]=0$
5. No new constraints should be generated.

We now construct the general form of H_M and $G_M(x)$, compatible with the above requirements, when e_L^2 is close enough to e_R^2. It is useful to recall the mass dimensions of various quantities:

$$[H(x)]=[G(x)]=2; \tag{C.2}$$

$$\left[\Pi^{\substack{L\\R}}(x)\pm\partial_x\Phi^{\substack{L\\R}}(x)\right]=[E(x)]=[\partial_x]=[e_L]=[e_R]=1;$$

$$[A(x)]=0;$$

Using the above set of assumptions and dimensional considerations we get:

$$G_M(x) = G(x)+e^2 G_1(A(x))+e\partial_x G_2(A(x)) \equiv G(x)+M_G(x) \qquad (C.3)$$

$$H_M(x)\equiv H(x)+e^2 H_1(A(x))+\partial_x H_2\left(A(x),E(x),\Pi^{L}_{R}(x)\pm\partial_x \Phi^{L}_{R}(x)\right)+ \qquad (C.4)$$
$$+eE(x)H_3(A(x))+U_H(A(x))G(x) \equiv H(x)+M_H(x)$$

Where $e\equiv\sqrt{(e_L^2- e_R^2)}$. Note that we allow a term proportional to the constraint to appear in $H_M(x)$. We have used G(x) in this term, because $U_H(G_M-G)$ can be absorbed in the other terms in H_M. We now proceed to evaluate the relevant commutators from the canonical commutation relations given in (3.8-9). We introduce the following notation:

$$\frac{\delta}{\delta A(y)} F(A(y)) \equiv F'(A(y)) \qquad (C.5)$$

Then the basic commutators give:

$$\text{(a)} \quad \left[E(x),F(A(y))\right]= iF'(A(y))\delta(x-y) \qquad (C.6)$$

$$\text{(b)} \quad \left[F(A(x)),\partial_y E(y)\right]= -i\partial_y\left(F'(A(x))\delta(x-y)\right) =$$

$$= iF'(A(x))\delta'(x-y)$$

From our fourth demand we get the following set of equations:

$$\text{(a)}\ G_1'=0\ ;\ \text{(b)}\ G_2'=H_3'\ ; \qquad \text{(C.7)}$$
$$\text{(c)}\ -H_1'' + \frac{1}{2}(H_3'^{2})' - U_H' + U_H''G_1=0\ ;$$

Next we want to calculate P_M from:

$$-iP_M = -\int dxdy\ x[H_M(x),H_M(y)]-i\int dx\ U_P(x)G_M(x) \equiv P+M_P \qquad \text{(C.8)}$$

where P is given by eq's (3.20) and (3.22). P has the following commutation relations with the basic variables:

$$\text{(a)}\ [P,\Pi^{\stackrel{L}{R}}(x)\pm\partial_x\Phi^{\stackrel{L}{R}}(x)] = i\partial_x(\Pi^{\stackrel{L}{R}}(x)\pm\partial_x\Phi^{\stackrel{L}{R}}(x)) \qquad \text{(C.9)}$$

$$\text{(b)}\ [P,A(x)] = i\partial_x A(x)$$

$$\text{(c)}\ [P,E(x)] = i\partial_x E(x) + ie^2 A(x)$$

where (C.9c) is anomalous. We therefore want M_P to commute with the scalar variables and with A(x), and to give $[M_P,E(x)] = -ie^2 A(x)$. Note that (C.9) has already fixed for us $U_P(x) = -A(x)$. Before we proceed we note that H_2 decomposes as follows:

$$H_2(x) = H_2^{\stackrel{L}{R}}(A(x))(\Pi^{\stackrel{L}{R}}(x)\pm\partial_x\Phi^{\stackrel{L}{R}}(x))+H_2^E(A(x))E(x)+ \qquad \text{(C.10)}$$
$$+\ eH_2^E(A(x))$$

From the demand that M_P should commute with the scalar variables we get:

$$\text{(a)}\ H_2^{\overset{L}{R}}{}'=0\ ;\ \text{(b)}\ \left[H_2^E(U_H'-A)+\frac{1}{2}(U_H^2)'\right]'=0\ ; \tag{C.11}$$

$$\text{(c)}\ \left[H_2^E(U_H'+A)+\frac{1}{2}(U_H^2)'\right]'=0\ ;$$

Substracting (C.11b) from (C.11c) we get $(H_2^E A)'=0$, from which by locality of H_2^E we get:

$$H_2^E=0 \tag{C.12}$$

Substituting back into (C.11) we get:

$$\frac{1}{2}(U_H^2)''=0\ ; \tag{C.13}$$

From $[M_P, A(x)]=0$ we get:

$$H_2^A{}'= U_H H_3' \tag{C.14}$$

Finally from the demand on the commutator of M_P with E(x) we get:

$$(U_H H_1'-H_2^A{}' H_3)'+ \frac{1}{2}(U_H^2)'-(AG_1)'= -A \tag{C.15}$$

Using (C.7a) and (C.14), (C.15) becomes:

$$U_H'\left(H_1' - \frac{1}{2}(H_3^2)'\right) + U_H\left(H_1'' - \frac{1}{2}(H_3^2)''\right) + \frac{1}{2}(U_H^2)' - G_1 = -A \qquad \text{(C.16)}$$

The solution of (C.13) is:

$$U_H^2(A) = c_0 + c_1 A \qquad \text{(C.17)}$$

But from locality of U_H, $c_1=0$. Substituting $U_H'=0$, (C.7c) becomes:

$$-H_1'' + \frac{1}{2}(H_3^2)'' = 0 \qquad \text{(C.18)}$$

(C.16) now gives:

$$G_1 = A \qquad \text{(C.19)}$$

which cannot be satisfied because of (C.7a).

Appendix D

In this appendix we give some details of the d=4 BJL calculations involving the square diagram.

In a calculation of the l.h.s. of (B.1), some preliminary steps can be taken before actually doing any Feynman integral. In this we follow the method employed by Jo (ref. 3-6), and we give here a short summary of the main idea.

Our commutators are expected to separate into a sum of a canonical term plus anomalous contributions. The canonical terms are defined to be those which we get by naive application of the canonical commutation relations that the basic fields satisfy, ignoring divergences of composite operators. The canonical results relevant to our calculation are:

$$\text{(a)}\quad [H_F, J_0^5(\vec{x})] = [\int d^3y(-i\bar{\Psi}(\vec{y})\gamma_i\partial^i\Psi(\vec{y})), J_0^5(\vec{x})] = \qquad \text{(D.1)}$$

$$= i\partial^i J_i^5(\vec{x})$$

$$\text{(b)}\quad [P_F^i, J_i^5(\vec{x})] = [\int d^3y(i\Psi^+(\vec{y})\partial^i\Psi(\vec{y})), J_i^5(\vec{x})] = -i\partial^i J_i^5(\vec{x})$$

$$\text{(c)}\quad [H_F, \vec{\nabla}\cdot\vec{E}(\vec{x})] = [\int d^3y(-i\bar{\Psi}(\vec{y})\gamma_i\partial^i\Psi(\vec{y})), \vec{\nabla}\cdot\vec{E}(\vec{x})] = 0$$

$$\text{(d)}\quad [\int d^3y\; J_i^5(\vec{y})A^i(\vec{y}), \vec{\nabla}\cdot\vec{E}(\vec{x})] = i\partial^i J_i^5(\vec{x})$$

As a consequence we expect:

$$\lim_{k_{10}\to i\infty}(-ik_{10})\langle 0|\int d^4x\, e^{-ik_1\cdot x}\, T\left(H_F J_0^5(x)\right)|\gamma(k_2,\varepsilon_2)\gamma(k_3,\varepsilon_3)\rangle = \tag{D.2}$$

$$=\int d^3x\, e^{i\vec{k}_1\cdot\vec{x}}\left\{-k_1^j\langle 0|J_j^5(\vec{x})|\gamma\gamma\rangle + \text{anomalous terms}\right\}$$

One of the aims of the preliminary steps mentioned is to separate the canonical contribution (the first term on the r.h.s. of (D.2)) from the rest, before explicitly evaluating the Feynman diagram. This means changing the order by taking the BJL limit before doing the loop integral. Taking the limit first is a safe operation only when the resulting expression is not superficially divergent (both in UV and IR regions). In this spirit the following operation and obvious generalizations of it are also safe:

$$\lim_{k_{10}\to i\infty}\frac{1}{k_{10}}\text{(a superficially log-div integral)} = 0 \tag{D.3}$$

$$\text{If } \lim_{k_{10}\to i\infty}\frac{1}{k_{10}}\text{(integrand)} = 0$$

Our starting Feynman expression for the square is superficially linearly divergent. Taking the BJL limit before doing the integral is therefore not safe. If one still insists on doing so, this procedure will pick up only the canonical terms in the commutator, and miss the anomalous ones. The way to proceed is to make use of algebraic identities which separate out convergent bits of the starting expression. Two useful identities are:

$$\text{(a)}\quad \frac{1}{\not r+\not p} = \frac{1}{\not r} - \frac{1}{\not r}\not p\frac{1}{\not r} + \frac{1}{\not r+\not p}\not p\frac{1}{\not r}\not p\frac{1}{\not r} \qquad \text{(D.4)}$$

$$\text{(b)}\quad \frac{1}{\not r+\not p} = \frac{1}{\not r} + \left(\frac{1}{\not r+\not p} - \frac{1}{\not r}\right)$$

Thinking of r as a loop momentum and of p as some linear combination of external momenta, the l.h.s. of (D.4) is a typical fermion propagator appearing in the expression for the loop. Then, the second and the third terms on the r.h.s. of (D.4a) and similarly the second term on the r.h.s. of (D.4b). have an improved UV behaviour. However, one should be careful not to use repeatedly the identities in (D.4) too many times or the resulting single terms may become superficially IR divergent.

A typical square diagram, relevant for evaluating the T-product on the l.h.s. of (D.2) is shown in Fig.3b . From Feynman rules we get:

$$\text{(a)}\quad D_1 = \varepsilon^i(k_2)\varepsilon^j(k_3)\int\frac{d^4r}{(2\pi)^4}(-1)\,\mathrm{tr}\Big[\frac{i}{\not r+\not k_2}(-ie\gamma_i\gamma_5)\frac{i}{\not r}\gamma_0\gamma_5 \qquad \text{(D.5)}$$

$$\frac{i}{\not r-\not k_1}(-ie\gamma_j\gamma_5)\frac{i}{\not r-(\not k_1+\not k_3)}\Big(-\frac{1}{2}\gamma_\ell(2r+k_2-k_3-k_1)^\ell\Big)\Big]$$

We have to add to it:

(b) $D_2 = D_1(j\to 0,\ k_1\to k_3$, in the trace)

(c) $D_3 = D_1(0\to i,\ k_1\to k_2$, in the trace) and three more diagrams with the fermionic arrows reversed in each of the D_i's. These diagrams are equal to those given above due to charge conjugation. Therefore the final expression for our T-product is $2(D_1+D_2+D_3)$.

D_2 and D_3 contain the canonical contributions. After making use of the above mentioned algebraic identities we get:

$$\lim_{k_{10}\to i\infty}(-ik_{10})2(D_2+D_3)= ie^2\varepsilon^i(k_2)\varepsilon^j(k_3)\lim_{k_{10}\to i\infty} \tag{D.6}$$

$$\{2[S+\sum_{m=1}^{2}\Lambda_m^a+\sum_{m=1}^{5}\Lambda_m^b+\sum_{m=1}^{8}\Lambda_m^c+\sum_{m=1}^{3}\Lambda_m^d+\sum_{m=1}^{3}\Lambda_m^e]$$

$$+\int\frac{d^4r}{(2\pi)^4}\,tr[\frac{1}{\not r+\not k_2}\gamma_i\frac{1}{\not r}\gamma_j\frac{1}{\not r-\not k_3}\gamma_\ell\gamma_5](k_1+k_2+k_3)^\ell$$

$$+\int\frac{d^4r}{(2\pi)^4}\,tr[\frac{1}{\not r}\gamma_i\frac{1}{\not r-\not k_2}\gamma_j\frac{1}{\not r-\not k_2-\not k_3}\gamma_\ell\gamma_5](k_1-k_2-k_3)^\ell\}$$

where the Λ_m's and S will be defined below. In the last two lines of (D.6), we have (almost) separated the canonical contribution. The canonical term is basically the AAA triangle. Demanding full Bose-symmetry (i.e. symmetry under the exchange of any two of the three photon legs) completely fixes the ambiguity in the definition of AAA. In bringing the last two terms in (D.6) to this symmetric form, we pick up surface terms from a shift in the loop momentum. The last two terms in (D.6) therefore give us:

$$2ie^2 \varepsilon^i(k_2)\varepsilon^j(k_3)\{\int \frac{d^4 r}{(2\pi)^4} \mathrm{tr} \tag{D.7}$$

$$[\frac{1}{\not r+\not p} \gamma_i \frac{1}{\not r+\not p-\not k_2} \gamma_j \frac{1}{\not r+\not p-\not k_2-\not k_3} \gamma_\ell \gamma_5] k_1^\ell +$$

$$+ \frac{-1}{2\pi^2} \varepsilon_{oij\ell} [\frac{1}{8} k_{20}(k_1-k_2-k_3)^\ell + \frac{1}{12}(k_{20}-k_{30})k_1^\ell]\}$$

where $p \equiv \frac{2}{3}k_2 + \frac{1}{3}k_3$.

An important technical comment is in place here. The last term on the r.h.s. of (D.7):

$$- \frac{i}{12\pi^2} \varepsilon^i(k_2)\varepsilon^j(k_3)\varepsilon_{oij\ell} k_1^\ell (k_{20}-k_{30})$$

comes from shifting the "partially Bose symmetric" form of the triangle (i.e. symmetry in only one pair of photonic legs) to the completely Bose symmetric form. It reflects an ambiguity in the definition of the canonical part of the commutator. However, the two commutators we are interested in, namely, (3.35) and (3.48), are insensitive to the way we choose to define the triangle, as long as we use the same definition, since their canonical parts vanish. The last term in (D.7) was therefore systematically omitted from (D.1a,b,d).

In general, the regularization in our calculations is effected through keeping track of shifts in loop momentum, and through symmetric integrations (see appendix E). In this way, linear divergences disappear, leaving finite surface terms. The way to define and evaluate surface terms is briefly reviewed in appendix E. In the course of calculation one also encounters integrals which are superficially log-divergent. To evaluate these we need to introduce a cutoff. After doing so one finds that in fact the divergences cancel, leaving well-defined and cutoff-independent results, as should be expected for an anomaly.

We now give the definition of the various terms in (D.6). S is a surface term given by:

$$S=\int\frac{d^4r}{(2\pi)^4}\,tr[\frac{1}{\not r}\,\gamma_i\,\frac{1}{\not r-\not k_2}\,\gamma_j\,\frac{1}{\not r-\not k_2-\not k_3}\,\gamma_\ell\gamma_5]r^\ell-(r\rightarrow r+k_2)= \qquad (D.8)$$

$$=\frac{1}{12\pi^2}\varepsilon_{oij\ell}[-k_{20}(k_2+k_3)^\ell+k_{30}k_2^\ell]$$

The various Λ_m's are defined as follows:

$$(a)\ \Lambda_1^a\equiv\int\frac{d^4r}{(2\pi)^4}\,tr[\frac{1}{\not r}\,\gamma_i\,\frac{1}{\not r}\,\gamma_j\,\frac{1}{\not r-\not k_1-\not k_3}\,\gamma_\ell\gamma_5]r^\ell \qquad (D.9)$$

$$(b)\ \Lambda_2^a\equiv-\int\frac{d^4r}{(2\pi)^4}\,tr[\frac{1}{\not r}\,\gamma_j\,\frac{1}{\not r}\,\gamma_\ell\,\frac{1}{\not r+\not k_1}\,\gamma_i\gamma_5]r^\ell$$

$$\text{(a)}\quad \Lambda_1^b \equiv -\int\frac{d^4r}{(2\pi)^4}\,\mathrm{tr}\Big[\frac{1}{\not{r}}\,\not{k}_2\,\frac{1}{\not{r}}\,\gamma_i\,\frac{1}{\not{r}}\,\gamma_j\,\frac{1}{\not{r}-\not{k}_1-\not{k}_3}\,\gamma_\ell\gamma_5\Big]r^\ell \qquad \text{(D.10)}$$

$$\text{(b)}\quad \Lambda_2^b \equiv -\int\frac{d^4r}{(2\pi)^4}\,\mathrm{tr}\Big[\frac{1}{\not{r}}\,\gamma_i\,\frac{1}{\not{r}}\,\gamma_j\,\frac{1}{\not{r}}\,\tilde{\not{k}}_1\,\frac{1}{\not{r}-\not{k}_1-\not{k}_3}\,\gamma_\ell\gamma_5\Big]r^\ell$$

$$\text{(c)}\quad \Lambda_3^b \equiv -\int\frac{d^4r}{(2\pi)^4}\,\mathrm{tr}\Big[\frac{1}{\not{r}}\,\not{k}_2\,\frac{1}{\not{r}}\,\gamma_j\,\frac{1}{\not{r}}\,\gamma_\ell\,\frac{1}{\not{r}+\not{k}_1}\,\gamma_i\gamma_5\Big]r^\ell$$

$$\text{(d)}\quad \Lambda_4^b \equiv -\int\frac{d^4r}{(2\pi)^4}\,\mathrm{tr}\Big[\frac{1}{\not{r}}\,\gamma_i\,\frac{1}{\not{r}}\,\gamma_j\,\frac{1}{\not{r}}\,\gamma_\ell\,\frac{1}{\not{r}+\not{k}_1}\,\tilde{\not{k}}_1\gamma_5\Big]r^\ell$$

$$\text{(e)}\quad \Lambda_5^b \equiv -\int\frac{d^4r}{(2\pi)^4}\,\mathrm{tr}\Big[\frac{1}{\not{r}}\,\gamma_j\,\frac{1}{\not{r}}\,(\not{k}_2+\not{k}_3)\,\frac{1}{\not{r}}\,\gamma_\ell\,\frac{1}{\not{r}+\not{k}_1}\,\gamma_i\gamma_5\Big]r^\ell$$

$$\text{(a)}\quad \Lambda_1^c \equiv -\int\frac{d^4r}{(2\pi)^4}\,\mathrm{tr}\Big[\frac{1}{\not{r}}\,\gamma_i\,\frac{1}{\not{r}}\,\gamma_j\,\frac{1}{\not{r}}\,\not{k}_3\,\frac{1}{\not{r}}\,\tilde{\not{k}}_1\,\frac{1}{\not{r}-\not{k}_1-\not{k}_3}\,\gamma_\ell\gamma_5\Big]r^\ell \qquad \text{(D.11)}$$

$$\text{(b)}\quad \Lambda_2^c \equiv -\int\frac{d^4r}{(2\pi)^4}\,\mathrm{tr}\Big[\frac{1}{\not{r}}\,\not{k}_2\,\frac{1}{\not{r}}\,\gamma_j\,\frac{1}{\not{r}}(\not{k}_2+\not{k}_3)\frac{1}{\not{r}}\,\gamma_\ell\,\frac{1}{\not{r}+\not{k}_1}\,\gamma_i\gamma_5\Big]r^\ell$$

$$\text{(c)}\quad \Lambda_3^c \equiv -\int\frac{d^4r}{(2\pi)^4}\,\mathrm{tr}\Big[\frac{1}{\not{r}}\,\gamma_i\,\frac{1}{\not{r}}\,\gamma_j\,\frac{1}{\not{r}}(\not{k}_2+\not{k}_3)\frac{1}{\not{r}}\,\gamma_\ell\,\frac{1}{\not{r}+\not{k}_1}\,\tilde{\not{k}}_1\gamma_5\Big]r^\ell$$

$$\text{(d)}\quad \Lambda_4^c \equiv \int\frac{d^4r}{(2\pi)^4}\,\mathrm{tr}\Big[\frac{1}{\not{r}}\,\not{k}_2\,\frac{1}{\not{r}}\,\gamma_i\,\frac{1}{\not{r}}\,\gamma_j\,\frac{1}{\not{r}}\,\tilde{\not{k}}_1\,\frac{1}{\not{r}-\not{k}_1-\not{k}_3}\,\gamma_\ell\gamma_5\Big]r^\ell$$

(e) $$\Lambda_5^c \equiv -\int\frac{d^4r}{(2\pi)^4}\,\mathrm{tr}\Big[\frac{1}{\not{r}}\,\gamma_i\,\frac{1}{\not{r}}\,\not{k}_2\,\frac{1}{\not{r}}\,\gamma_j\,\frac{1}{\not{r}}\,\gamma_\ell\,\frac{1}{\not{r}+\not{k}_1}\,\tilde{\not{k}}_1\gamma_5\Big]r^\ell$$

(f) $$\Lambda_6^c \equiv -\int\frac{d^4r}{(2\pi)^4}\,\mathrm{tr}\Big[\frac{1}{\not{r}}\,\not{k}_2\,\frac{1}{\not{r}}\,\not{k}_2\,\frac{1}{\not{r}}\,\gamma_j\,\frac{1}{\not{r}}\,\gamma_\ell\,\frac{1}{\not{r}+\not{k}_1}\,\gamma_i\gamma_5\Big]r^\ell$$

(g) $$\Lambda_7^c \equiv \int\frac{d^4r}{(2\pi)^4}\,\mathrm{tr}\Big[\frac{1}{\not{r}}\,\not{k}_2\,\frac{1}{\not{r}}\,\not{k}_2\,\frac{1}{\not{r}}\,\gamma_i\,\frac{1}{\not{r}}\,\gamma_j\,\frac{1}{\not{r}-\not{k}_1-\not{k}_3}\,\gamma_\ell\gamma_5\Big]r^\ell$$

(h) $$\Lambda_8^c \equiv -\int\frac{d^4r}{(2\pi)^4}\,\mathrm{tr}\Big[\frac{1}{\not{r}}\,\gamma_j\,\frac{1}{\not{r}}(\not{k}_2+\not{k}_3)\frac{1}{\not{r}}(\not{k}_2+\not{k}_3)\frac{1}{\not{r}}\,\gamma_\ell\,\frac{1}{\not{r}+\not{k}_1}\,\gamma_i\gamma_5\Big]r^\ell$$

Where $\tilde{\not{k}}_1 \equiv \not{k}_1 - k_{10}\gamma_0$.

(a) $$\Lambda_1^d \equiv \int\frac{d^4r}{(2\pi)^4}\,\mathrm{tr}\Big[\frac{1}{\not{r}}\,\gamma_i\,\frac{1}{\not{r}}\,\gamma_j\,\frac{1}{\not{r}-\not{k}_1-\not{k}_3}\,\gamma_\ell\gamma_5\Big]\frac{1}{2}(k_2-k_1-k_3)^\ell \qquad (D.12)$$

(b) $$\Lambda_2^d \equiv -\int\frac{d^4r}{(2\pi)^4}\,\mathrm{tr}\Big[\frac{1}{\not{r}}\,\not{k}_2\,\frac{1}{\not{r}}\,\gamma_i\,\frac{1}{\not{r}}\,\gamma_j\,\frac{1}{\not{r}-\not{k}_1-\not{k}_3}\,\gamma_\ell\gamma_5\Big]\frac{1}{2}(k_2-k_1-k_3)^\ell$$

(c) $$\Lambda_3^d \equiv -\int\frac{d^4r}{(2\pi)^4}\,\mathrm{tr}\Big[\frac{1}{\not{r}}\,\gamma_i\,\frac{1}{\not{r}}\,\gamma_j\,\frac{1}{\not{r}-\not{k}_3}\,\tilde{\not{k}}_1\,\frac{1}{\not{r}-\not{k}_1-\not{k}_3}\,\gamma_\ell\gamma_5\Big]\frac{1}{2}(k_2-k_1-k_3)^\ell$$

(a) $$\Lambda_1^e \equiv -\int\frac{d^4r}{(2\pi)^4}\,\mathrm{tr}\Big[\frac{1}{\not{r}+\not{k}_1}\,\gamma_i\,\frac{1}{\not{r}}\,\gamma_j\,\frac{1}{\not{r}-\not{k}_2-\not{k}_3}\,\gamma_\ell\gamma_5\Big]\frac{1}{2}(k_1-k_2-k_3)^\ell \qquad (D.13)$$

(b) $$\Lambda_2^e \equiv -\int\frac{d^4r}{(2\pi)^4}\,\mathrm{tr}\Big[\frac{1}{\not r+\not k_1}\,\not\varepsilon_1\,\frac{1}{\not r}\,\gamma_i\,\frac{1}{\not r}\,\gamma_j\,\frac{1}{\not r}\,\gamma_\ell\gamma_5\Big]\frac{1}{2}(k_1-k_2-k_3)^\ell$$

(c) $$\Lambda_3^e \equiv -\int\frac{d^4r}{(2\pi)^4}\,\mathrm{tr}\Big[\frac{1}{\not r+\not k_1}\,\gamma_i\,\frac{1}{\not r}\,\not\varepsilon_2\,\frac{1}{\not r}\,\gamma_j\,\frac{1}{\not r}\,\gamma_\ell\gamma_5\Big]\frac{1}{2}(k_1-k_2-k_3)^\ell$$

D_1 will contribute only to the anomalous terms. After some algebraic manipulations we get:

$$\lim_{k_{10}\to i\infty}(-ik_{10})2D_1 = 2ie^2\varepsilon^i(k_2)\varepsilon^j(k_3)\lim_{k_{10}\to i\infty}k_{10}\cdot\Big[\sum_{m=1}^{6}\Lambda_m^f + \sum_{m=1}^{3}\Lambda_m^g\Big] \qquad \text{(D.14)}$$

(a) $$\Lambda_1^f \equiv \int\frac{d^4r}{(2\pi)^4}\,\mathrm{tr}\Big[\frac{1}{\not r}\,\gamma_i\,\frac{1}{\not r}\,\gamma_0\,\frac{1}{\not r-\not k_1}\,\gamma_j\,\frac{1}{\not r-\not k_1}\,\gamma_\ell\gamma_5\Big]r^\ell \qquad \text{(D.15)}$$

(b) $$\Lambda_2^f \equiv \int\frac{d^4r}{(2\pi)^4}\,\mathrm{tr}\Big[\frac{1}{\not r}\,\gamma_i\,\frac{1}{\not r}\,\gamma_0\,\frac{1}{\not r-\not k_1}\,\gamma_j\,\frac{1}{\not r-\not k_1}\,\not k_3\,\frac{1}{\not r-\not k_1}\,\gamma_\ell\gamma_5\Big]r^\ell$$

(c) $$\Lambda_3^f \equiv -\int\frac{d^4r}{(2\pi)^4}\,\mathrm{tr}\Big[\frac{1}{\not r}\,\not k_2\,\frac{1}{\not r}\,\gamma_i\,\frac{1}{\not r}\,\gamma_0\,\frac{1}{\not r-\not k_1}\,\gamma_j\,\frac{1}{\not r-\not k_1}\,\gamma_\ell\gamma_5\Big]r^\ell$$

(d) $$\Lambda_4^f \equiv -\int\frac{d^4r}{(2\pi)^4}\,\mathrm{tr}\Big[\frac{1}{\not r}\,\not k_2\,\frac{1}{\not r}\,\gamma_i\,\frac{1}{\not r}\,\gamma_0\,\frac{1}{\not r-\not k_1}\,\gamma_j\,\frac{1}{\not r-\not k_1}\,\not k_3\,\frac{1}{\not r-\not k_1}\,\gamma_\ell\gamma_5\Big]r^\ell$$

(e) $$\Lambda_5^f \equiv \int\frac{d^4r}{(2\pi)^4}\,\mathrm{tr}\Big[\frac{1}{\not r+\not k_2}\,\not k_2\,\frac{1}{\not r}\,\not k_2\,\frac{1}{\not r}\,\gamma_i\,\frac{1}{\not r}\,\gamma_0\,\frac{1}{\not r-\not k_1}\,\gamma_j\,\frac{1}{\not r-\not k_1}\,\gamma_\ell\gamma_5\Big]r^\ell$$

(d) $\Lambda_6^f \equiv \int \frac{d^4r}{(2\pi)^4} \mathrm{tr}[\frac{1}{\not{r}+\not{k}_1}\gamma_i \frac{1}{\not{r}+\not{k}_1}\gamma_0 \frac{1}{\not{r}}\gamma_j \frac{1}{\not{r}-\not{k}_3}\not{k}_3 \frac{1}{\not{r}}\not{k}_3 \frac{1}{\not{r}}\gamma_\ell\gamma_5]r^\ell$

(a) $\Lambda_1^g \equiv \int \frac{d^4r}{(2\pi)^4} \mathrm{tr}[\frac{1}{\not{r}+\not{k}_1}\gamma_i \frac{1}{\not{r}+\not{k}_1}\gamma_0 \frac{1}{\not{r}}\gamma_j \frac{1}{\not{r}}\gamma_\ell\gamma_5]\frac{1}{2}(k_2-k_1-k_3)^\ell$ (D.16)

(b) $\Lambda_2^g \equiv \int \frac{d^4r}{(2\pi)^4} \mathrm{tr}[\frac{1}{\not{r}+\not{k}_1}\gamma_i \frac{1}{\not{r}+\not{k}_1}\gamma_0 \frac{1}{\not{r}}\gamma_j \frac{1}{\not{r}}\not{k}_3 \frac{1}{\not{r}}\gamma_\ell\gamma_5]\frac{1}{2}(k_2-k_1-k_3)^\ell$

(c) $\Lambda_3^g \equiv -\int \frac{d^4r}{(2\pi)^4} \mathrm{tr}[\frac{1}{\not{r}}\not{k}_2 \frac{1}{\not{r}}\gamma_i \frac{1}{\not{r}}\gamma_0 \frac{1}{\not{r}-\not{k}_1}\gamma_j \frac{1}{\not{r}-\not{k}_1}\gamma_\ell\gamma_5]\frac{1}{2}(k_2-k_1-k_3)^\ell$

Most of the Λ_m's are superficially linearly or logarithmically divergent, and Λ_m^a are superficially quadratically divergent. As explained, the linear divergences disappear after a shift to a symmetric origin. The quadratic and log divergent terms in Λ_m^a, Λ_m^d, Λ_m^e, Λ_m^g cancel for each Λ_m separately, when the integrals are defined according to the symmetric integration formulas (E.1), and the traces over the Dirac γ-matrices are taken. For Λ_m^b, Λ_m^c and $k_{10}\Lambda_m^f$, the cancellation is less trivial, in the sense that it occurs only when all the different contributions from these terms are summed together. To treat the superficially log-div parts, a cutoff is introduced and extra care is needed in order not to miss various finite terms which are left at the end.

Let us illustrate the last points in a particular example. In order to calculate the Λ^b_m's we need to calculate:

$$I \equiv \int \frac{d^4r}{(2\pi)^4} tr[\frac{1}{\not r} \gamma_\nu \frac{1}{\not r} \gamma_\eta \frac{1}{\not r} \gamma_j \frac{1}{\not r - \not p} \gamma_\ell \gamma_5] r^\ell \qquad (D.17)$$

where $p \equiv k_1 + k$, and k has no dependence on r or k_1. After introducing a Feynman parameter and shifting the r-integral, we get:

$$I = I_1 + \text{a surface term} \qquad (D.18)$$

$$I_1 \equiv 3\int_0^1 dx(1-x)^2 \int \frac{d^4r}{(2\pi)^4} \left\{ \frac{tr[(\not r + x\not p)\gamma_\nu(\not r + x\not p)\gamma_\eta(\not r + x\not p)\gamma_j(\not r - (1-x)\not p)\gamma_\ell\gamma_5](r+xp)^\ell}{[r^2 + x(1-x)p^2]^4} \right\}$$

The linearly divergent term in I_1 ($\not r \not r \not r \not r r^\ell$) drops out because it's odd under $r \to -r$.

There are four superficially log-div terms. The $\not r \not r \not r \not r x p^\ell$ term is zero due to the Dirac trace and the symmetric integration. The other three terms, involving r^ℓ, do not vanish. After using (E.1), we are left with an r-integral of the form:

$$I_{4r4} \equiv \int \frac{d^4r}{(2\pi)^4} \frac{r^2 r^2}{[r^2 + x(1-x)p^2 + i\varepsilon]^4} \qquad (D.19)$$

where we have put back the $i\varepsilon$, so far omitted from the fermion propagators. Since the integrand is a scalar we can replace $\int d^4 r \to 4\pi\int_0^\infty r^2 dr \int_{-\infty}^\infty dr_0$ (here $r^2 \equiv \vec{r}^{\,2}$). We then perform the r_0 integration along the contour dictated by the $i\varepsilon$. The result of this integral is of course finite, but the remaining integral is log divergent. After introducing a cutoff Λ, we finally get for (D.19):

$$I_{4r4} = \frac{i}{16\pi^2} \ln\left(\frac{\Lambda^2}{-x(1-x)p^2}\right) - \frac{34i}{192\pi^2} \qquad \text{(D.20)}$$

We now separate all finite contributions from (D.20) which may be important in the limit $k_{10} \to i\infty$:

$$I_{4r4} = \frac{i}{16\pi^2} \ln\left(\frac{\Lambda^2}{-k_{10}^2}\right) - \frac{34i}{192\pi^2} - \frac{i}{16\pi^2} \ln(x(1-x)) - \frac{i}{8\pi^2} \frac{k_0}{k_{10}} + \qquad \text{(D.21)}$$
$$+ O\left(\frac{1}{k_{10}^2}\right)$$

The first term on the l.h.s. of (D.21) is the "true" divergent part, and it cancels among the different Λ_m's as mentioned above. For future reference, we denote this term by D.

The finite r-integrals and the various finite contributions from the log-div parts, stay finite in the BJL limit and after integrating over the Feynman parameters. In what follows, we give a list of key intermediate results. They were derived both by a "hand calculation" and by using the algebraic manipulation program "REDUCE" (ref. 3-14).

(a) $\int \frac{d^4r}{(2\pi)^4} \mathrm{tr}[\frac{1}{\not r} \gamma_\rho \frac{1}{\not r} \gamma_\mu \frac{1}{\not r-\not p} \gamma_\ell \gamma_5] r^\ell =$ (D.22)

$$= -\frac{1}{24\pi^2} \varepsilon_{o\rho\mu\ell} p^o p^\ell$$

(b) $\lim_{k_{10}\to i\infty} \int \frac{d^4r}{(2\pi)^4} \mathrm{tr}[\frac{1}{\not r} \gamma_\nu \frac{1}{\not r} \gamma_\eta \frac{1}{\not r} \gamma_\kappa \frac{1}{\not r-\not p} \gamma_\ell \gamma_5] r^\ell =$

$$= \frac{1}{6} iD\{p^\alpha[3g^\ell_\kappa \varepsilon_{\nu\eta\alpha\ell} + 3\varepsilon_{\nu\eta\kappa\alpha} - g^\ell_\eta \varepsilon_{\nu\alpha\kappa\ell} + g^\ell_\nu \varepsilon_{\eta\alpha\kappa\ell}] - 3p^\ell \varepsilon_{\nu\eta\kappa\ell}\} +$$

$$+ \frac{1}{288\pi^2}[19p_o \varepsilon_{\nu\eta\kappa o} + p^\ell(7g_{\kappa o}\varepsilon_{\nu\eta o\ell} + 23g_{\eta o}\varepsilon_{\nu o\kappa\ell} + 11g_{\nu o}\varepsilon_{o\eta\kappa\ell})] +$$

$$+ \frac{1}{24\pi^2} k_o \varepsilon_{\eta o\kappa\nu}$$

(c) $\lim_{k_{10}\to i\infty} \int \frac{d^4r}{(2\pi)^4} \mathrm{tr}[\frac{1}{\not r} \gamma_\nu \frac{1}{\not r} \gamma_\eta \frac{1}{\not r} \gamma_\kappa \frac{1}{\not r} \gamma_\ell \frac{1}{\not r-\not p} \gamma_\lambda \gamma_5] r^\ell =$

$$= \frac{1}{3} iD[2g^\ell_\kappa \varepsilon_{\lambda\ell\eta\nu} + 2g^\ell_\eta \varepsilon_{\lambda\kappa\ell\nu} - 3\varepsilon_{\lambda\kappa\eta\nu}] +$$

$$+ \frac{1}{288\pi^2}[19(g_{\kappa o}\varepsilon_{o\lambda\eta\nu} + g_{\lambda o}\varepsilon_{o\kappa\eta\nu}) - 13(g_{\eta o}\varepsilon_{o\lambda\kappa\nu} + g_{\nu o}\varepsilon_{o\lambda\kappa\eta})]$$

(d) $\lim_{k_{10}\to i\infty} \int \frac{d^4r}{(2\pi)^4} \mathrm{tr}[\frac{1}{\not r-\not k_1} \gamma_\sigma \frac{1}{\not r+\not q} \gamma_\rho \frac{1}{\not r} \gamma_\mu \gamma_5] =$

$$= \frac{1}{16\pi^2}[-2k_1^\ell \varepsilon_{\ell\sigma\rho\mu} + q^\alpha(-\varepsilon_{\sigma\alpha\rho\mu} - g_{\mu o}\varepsilon_{\sigma\alpha\rho o} + g_{\rho o}\varepsilon_{\sigma\alpha o\mu} + 3g_{\sigma o}\varepsilon_{o\alpha\rho\mu}) +$$

$$+ q_o \varepsilon_{\sigma o\rho\mu}]$$

(e) $$\lim_{k_{10}\to i\infty} \int \frac{d^4r}{(2\pi)^4} \mathrm{tr}\left[\frac{1}{\not r-\not k_1}\gamma_\sigma \frac{1}{\not r}\gamma_\rho \frac{1}{\not r}\gamma_\mu \frac{1}{\not r}\gamma_\nu\gamma_5\right] =$$

$$= -\frac{1}{16\pi^2}\left[-\varepsilon_{\sigma\rho\mu\nu} - g_{\nu o}\varepsilon_{\sigma\rho\mu o} + g_{\mu o}\varepsilon_{\sigma\rho o\nu} + g_{\rho o}\varepsilon_{\sigma o\mu\nu} + 3g_{\sigma o}\varepsilon_{o\rho\mu\nu}\right]$$

(f) $$\lim_{k_{10}\to i\infty} k_{10}\int \frac{d^4r}{(2\pi)^4} \mathrm{tr}\left[\frac{1}{\not r}\not k_2 \frac{1}{\not r}\gamma_i \frac{1}{\not r}\gamma_0 \frac{1}{\not r-\not k_1}\gamma_j \frac{1}{\not r-\not k_1}\gamma_\ell\gamma_5\right] r^\ell =$$

$$= \frac{1}{3} iD\varepsilon_{oij\ell} k_{10} k_2^\ell$$

(g) $$\lim_{k_{10}\to i\infty} k_{10}\int \frac{d^4r}{(2\pi)^4} \mathrm{tr}\left[\frac{1}{\not r}\gamma_i \frac{1}{\not r}\not k_2 \frac{1}{\not r}\gamma_0 \frac{1}{\not r-\not k_1}\gamma_j \frac{1}{\not r-\not k_1}\gamma_\ell\gamma_5\right] r^\ell =$$

$$= -\frac{1}{3} iD\varepsilon_{oij\ell} k_{10} k_2^\ell$$

(h) $$\lim_{k_{10}\to i\infty} k_{10}\int \frac{d^4r}{(2\pi)^4} \mathrm{tr}\left[\frac{1}{\not r}\not k_2 \frac{1}{\not r}\not k_2 \frac{1}{\not r}\gamma_i \frac{1}{\not r}\gamma_0 \frac{1}{\not r-\not k_1}\gamma_j \frac{1}{\not r-\not k_1}\gamma_\ell\gamma_5\right] r^\ell =$$

$$= -\frac{1}{48\pi^2}\varepsilon_{oij\ell} k_{20} k_2^\ell$$

(i) $$\lim_{k_{10}\to i\infty} k_{10}\int \frac{d^4r}{(2\pi)^4} \mathrm{tr}\left[\frac{1}{\not r}\gamma_i \frac{1}{\not r}\not k_2 \frac{1}{\not r}\not k_2 \frac{1}{\not r}\gamma_0 \frac{1}{\not r-\not k_1}\gamma_j \frac{1}{\not r-\not k_1}\gamma_\ell\gamma_5\right] r^\ell =$$

$$= \frac{7}{48\pi^2}\varepsilon_{oij\ell} k_{20} k_2^\ell$$

(j) $\lim_{k_{10}\to i\infty} k_{10}\int\frac{d^4r}{(2\pi)^4}\,tr[\frac{1}{\not r}\,\not k_2\,\frac{1}{\not r}\,\gamma_i\,\frac{1}{\not r}\,\gamma_0\,\frac{1}{\not r-\not k_1}\,\gamma_j\,\frac{1}{\not r-\not k_1}\,\not k_3\,\frac{1}{\not r-\not k_1}\,\gamma_\ell\gamma_5]r^\ell =$

$= \frac{7}{48\pi^2}\varepsilon_{oij\ell}(k_{30}k_2^\ell - k_{20}k_3^\ell)$

(k) $\lim_{k_{10}\to i\infty} k_{10}\int\frac{d^4r}{(2\pi)^4}\,tr[\frac{1}{\not r}\,\gamma_i\,\frac{1}{\not r}\,\not k_2\,\frac{1}{\not r}\,\gamma_0\,\frac{1}{\not r-\not k_1}\,\not k_3\,\frac{1}{\not r-\not k_1}\,\gamma_j\,\frac{1}{\not r-\not k_1}\,\gamma_\ell\gamma_5]r^\ell =$

$= \frac{5}{48\pi^2}\varepsilon_{oij\ell}(k_{30}k_2^\ell - k_{20}k_3^\ell)$

(l) $\int\frac{d^4r}{(2\pi)^4}\,tr[\frac{1}{\not r+\not k_1}\,\gamma_\sigma\,\frac{1}{\not r+\not k_1}\,\gamma_\rho\,\frac{1}{\not r}\,\gamma_\mu\,\frac{1}{\not r}\,\gamma_\nu\gamma_5] =$

$= \frac{1}{24\pi^2}\,\frac{1}{k_1^2}\,[-5k_1^2\varepsilon_{\nu\mu\rho\sigma}+k_1^\alpha(-2k_{1\mu}\varepsilon_{\alpha\nu\rho\sigma}-2k_{1\sigma}\varepsilon_{\alpha\rho\nu\mu}+$

$+\ 2k_{1\rho}\varepsilon_{\alpha\mu\sigma\nu}+2k_{1\nu}\varepsilon_{\alpha\sigma\mu\rho})]$

(m) $\lim_{k_{10}\to i\infty} k_{10}\int\frac{d^4r}{(2\pi)^4}\,tr[\frac{1}{\not r+\not k_1}\,\gamma_i\,\frac{1}{\not r+\not k_1}\,\gamma_0\,\frac{1}{\not r}\,\gamma_j\,\frac{1}{\not r}\,\not k_3\,\frac{1}{\not r}\,\gamma_\ell\gamma_5] =$

$= \frac{1}{8\pi^2}\,k_{30}\varepsilon_{oij\ell}$

(n) $\lim_{k_{10}\to i\infty} k_{10}\int\frac{d^4r}{(2\pi)^4}\,tr[\frac{1}{\not r+\not k_1}\,\gamma_j\,\frac{1}{\not r+\not k_1}\,\gamma_0\,\frac{1}{\not r}\,\not k_2\,\frac{1}{\not r}\,\gamma_i\,\frac{1}{\not r}\,\gamma_\ell\gamma_5] =$

$= \frac{1}{8\pi^2}\,k_{20}\varepsilon_{oij\ell}$

(o) $$\lim_{k_{10}\to i\infty}\int\frac{d^4r}{(2\pi)^4}\,\mathrm{tr}\Big[\frac{1}{\not r}\,\not k_2\,\frac{1}{\not r}\,\gamma_i\,\frac{1}{\not r}\,\gamma_m\,\frac{1}{\not r-\not k_1}\,\gamma_j\,\frac{1}{\not r-\not k_1}\,\gamma_\ell\gamma_5\Big]r^\ell=$$

$$= -\Big(-\frac{1}{3}iD + \frac{11}{288\pi^2}\Big)k_{20}\,\varepsilon_{oijm}$$

In the above formulas, $p = (k_1+k)$, and k,q, do not depend on either r or k_1. D denotes the "true" log-divergent part (the first term on the r.h.s. of (D.21)). With these formulas one should be able to reproduce (3.51). It should be remembered that in some cases a rearrangement of the starting expression may be needed in order to match it with the form given in (D.22). Useful properties in this context are the charge conjugation identity (ref. 3-15 ch.7), the cyclic property of the trace, etc. Note also, that we have dropped non-divergent polynomials in k_{10}, as prescribed by the BJL definition.

We have shown in some detail how to calculate (3.51a). The calculation of (3.51c) follows the same path. The expression analogous to D_1 will be:

$$D_1^P = \varepsilon^i(k_2)\varepsilon^j(k_3)\int\frac{d^4r}{(2\pi)^4}(-1)\,\mathrm{tr}\Big[\frac{i}{\not r+\not k_2}(-ie\gamma_i\gamma_5)\frac{i}{\not r}\gamma_\ell\gamma_5 \quad \text{(D.23)}$$
$$\frac{i}{\not r-\not k_1}(-ie\gamma_j\gamma_5)\frac{i}{\not r-(\not k_1+\not k_3)}\Big(\frac{1}{2}\,\gamma_0(2r+k_2-k_3-k_1)^\ell\Big)\Big]$$

The diagram associated with (3.51b) contains a photon propagator. In the Weyl gauge the photon propagator is given by:

$$D_{ij}(p) = \frac{i\delta_{ij}}{p^2} + O\left(\frac{1}{p_0^4}\right) \qquad (D.24)$$

Since the basic square is at most linearly divergent, one can see, by using a relation similar to (D.3), that the higher order terms in (D.24) cannot contribute to the BJL limit. In order to calculate (3.51b) we therefore start from the expression:

$$D_1^E = ik_{10}(-ik_1^m)\,\frac{i}{k_1^2}\,\varepsilon^i(k_2)\varepsilon^j(k_3) \qquad (D.25)$$

$$\int\frac{d^4r}{(2\pi)^4}(-1)\mathrm{tr}\left[\frac{i}{\not r+\not k_2}(-ie\gamma_i\gamma_5)\frac{i}{\not r}(ie\gamma_m\gamma_5)\right.$$

$$\left.\frac{i}{\not r-\not k_1}(-ie\gamma_j\gamma_5)\frac{i}{\not r-(\not k_1+\not k_3)}\left(-\frac{1}{2}\gamma_\ell(2r+k_2-k_3-k_1)^\ell\right)\right]$$

Again, most of the intermediate results needed for the calculation are given in (D.22). However, there is an important difference between the calculation of (3.51a,c) and that of (3.51b). In the previous cases, we were looking for a term in the basic square which behaves like $\frac{1}{k_{10}}$, while here, because of the photon propagator, we are looking for the term which behaves like 1. Therefore, unlike the previous cases, the present result is sensitive to the ambiguities of the square, which are reflected in the possibility of adding or substracting polynomials in the external momenta. Again, we impose Bose symmetry. The expression for the T-product based on D_1^E is not symmetric under $k_2 \leftrightarrow k_3$, $i \leftrightarrow j$, due to surface terms. These terms can be easily calculated and then one can read off the polynomial that should be added in order to symmetrize.

Appendix E

1. Symmetric integration formulas.

$$\text{(a)}\quad \int d^4r\, r_\alpha r_\beta f(r^2) = \frac{1}{4}\, g_{\alpha\beta}\int d^4r\, r^2 f(r^2) \qquad \text{(E.1)}$$

$$\text{(b)}\quad \int d^4r\, r_\alpha r_\beta r_\gamma r_\delta f(r^2) = \frac{1}{24}(g_{\alpha\beta}g_{\gamma\delta}+ g_{\alpha\gamma}g_{\beta\delta}+ g_{\alpha\delta}g_{\beta\gamma}) \int d^4r\, r^2 r^2 f(r^2)$$

$$\text{(c)}\quad \int d^4r\, r_\alpha r_\beta r_\gamma r_\delta r_\varepsilon r_\eta f(r^2) = \frac{1}{192}\{g_{\alpha\beta}(g_{\gamma\delta}g_{\varepsilon\eta}+ g_{\gamma\varepsilon}g_{\delta\eta}+ g_{\gamma\eta}g_{\delta\varepsilon})$$
$$+ \text{4 similar terms with } \beta\leftrightarrow\gamma,\beta\leftrightarrow\delta,\beta\leftrightarrow\varepsilon,\beta\leftrightarrow\eta)\}\int d^4r\, r^2 r^2 r^2 f(r^2)$$

2. Evaluation of surface integrals.

Surface terms arising in our calculations are of the form:

$$\int\frac{d^4r}{(2\pi)^4}[f(x,p,r+a) - f(x,p,r)] = \qquad \text{(E.2)}$$

$$=\int\frac{d^4r}{(2\pi)^4}(\exp(a\cdot\partial)-1)f(x,p,r) = a_\kappa\int\frac{d^4r}{(2\pi)^4}\frac{\partial}{\partial r_\kappa} f(x,p,r)$$

$$+ \frac{1}{2}\, a_\kappa a_\eta \int\frac{d^4r}{(2\pi)^4}\frac{\partial}{\partial r_\kappa}\frac{\partial}{\partial r_\eta} f(x,p,r) + O\left(\frac{\partial}{\partial r}\right)^3$$

where x is a possible Feynman parameter, p an external momentum, and a_k depends on some external momenta and on x. The function f will in general be a ratio of two polynomials in r. It is useful to work with an f chosen in such a way that its denominator depends on r only through r^2 (see for example (D.18)). We then perform the integral in a finite spherical 4-volume defined by $r^2=R^2$ in Euclidean space. After that, the limit $R \to \infty$ is taken. A useful basic result is:

$$\int_D d^4r\, \partial_\nu (r_\mu g(r^2)) = i2\pi^2 (\frac{1}{4} g_{\mu\nu}) R^4 g(R^2) \qquad (E.3)$$

where the domain of integration D is the 4-sphere r=R. This result can be generalized to $r_\mu r_\alpha r_\beta \ldots . f(r^2)$, in a straightforward way (for the next non-zero case, $r_\mu r_\alpha r_\beta g(r^2)$, one should replace $\frac{1}{4} g_{\mu\nu}$ on the r.h.s. of (E.3), by the tensor structure on the r.h.s. of (E.1b) etc.). For evaluation of the the second order (in derivatives) term in (E.2), the following result is useful:

$$\int_D d^4r\, \partial^\kappa \partial^\nu (r^\gamma r^\lambda g(r^2)) = i2\pi^2 \{ (g^{\kappa\gamma} g^{\nu\lambda} + g^{\kappa\lambda} g^{\nu\gamma}) \qquad (E.4)$$

$$[\frac{1}{4} R^4 g(R^2) + \frac{1}{24} R^5 \frac{\partial}{\partial R} g(R^2)] + g^{\nu\kappa} g^{\gamma\lambda} \frac{1}{24} R^5 \frac{\partial}{\partial R} g(R^2) \}$$

F I G U R E S

Figure Captions

Fig 1. : a. A bubble diagram describing a 2-current Green's function in a free fermionic theory.

b. The bubble diagram in a d=2 theory with a good anomaly.

c. The bubble diagram in a d=2 theory with a bad anomaly.

Fig 2. : The AVV triangle diagram

Fig 3. : a. The AAA triangle diagram in axial massless QED.

b. A square diagram contributing to $[H_F, J_0^5(\vec{x})]$.

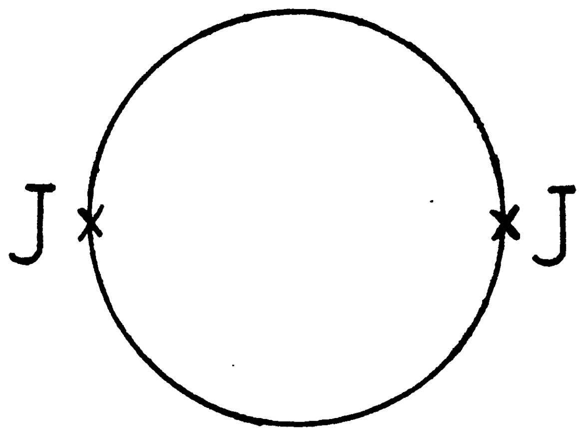

Figure 1a

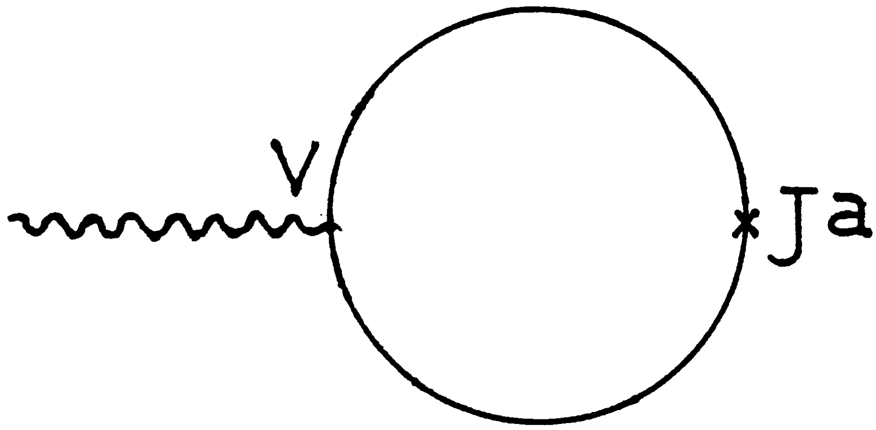

Figure 1b

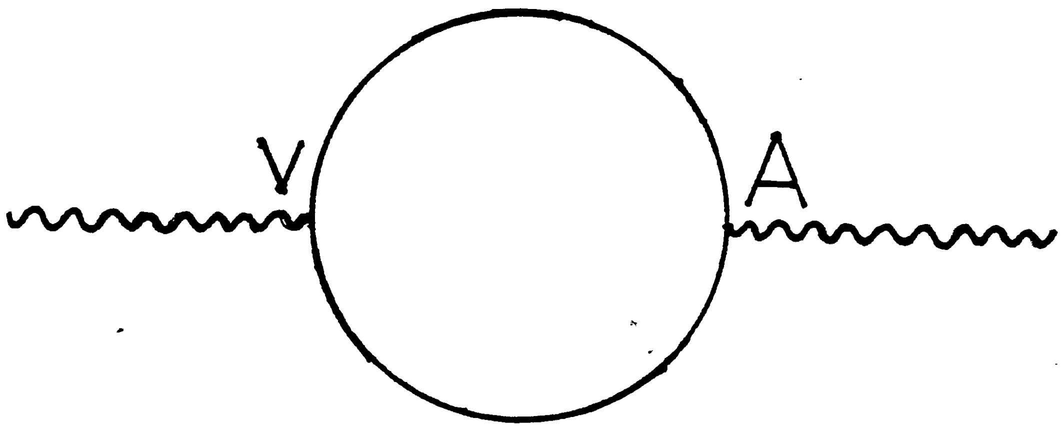

Figure 1c

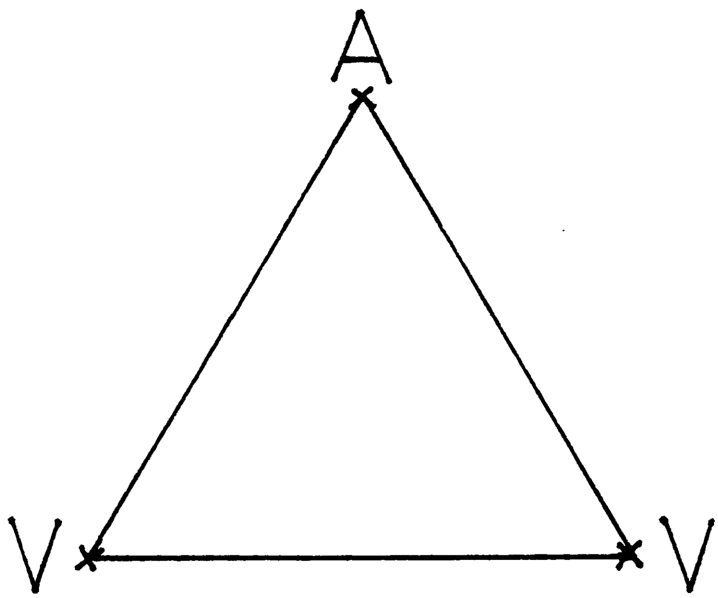

Figure 2

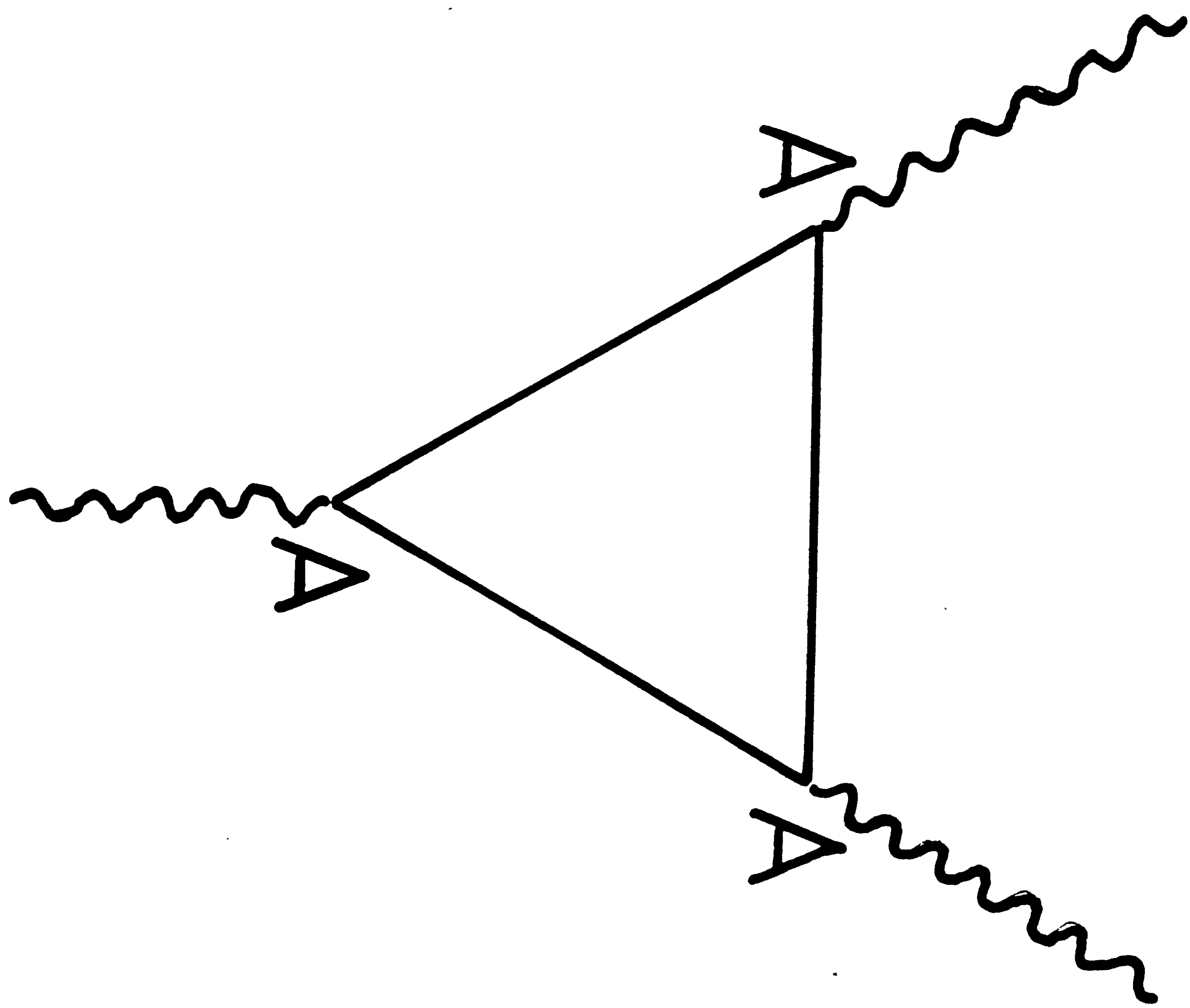

Figure 3a

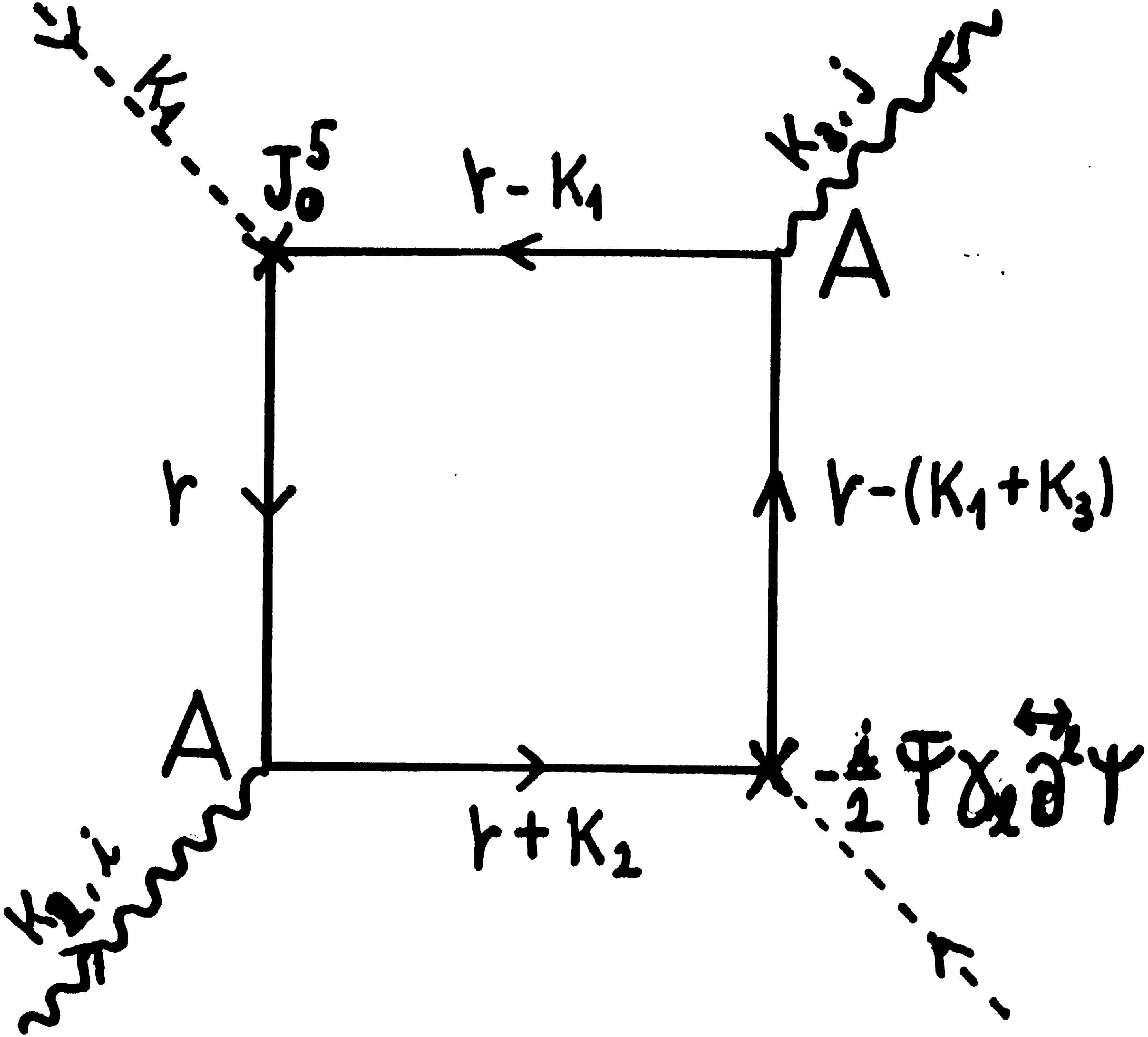
π
K_1
J_0^5
$r - K_1$
K_3, j
A
r
$r-(K_1+K_3)$
A
$-\frac{i}{2}\bar{\Psi}\gamma_2\overleftrightarrow{\partial}^2\Psi$
$r+K_2$
K_2, i
π

Figure 3b