



Partial differential equations

Almost global well-posedness of Kirchhoff equation with Gevrey data

L'équation de Kirchhoff avec données de Gevrey est presque globalement bien posée

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ABSTRACT

The aim of this note is to present the almost global well-posedness result for the Cauchy problem for the Kirchhoff equation with large data in Gevrey spaces. We also briefly discuss the corresponding results in bounded and in exterior domains.

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R É S U M É

Le propos de cette Note est d'énoncer que l'équation de Kirchhoff avec des données grandes dans les espaces de Gevrey est presque globalement bien posée. Nous discutons aussi brièvement les résultats correspondants dans les domaines bornés et les domaines extérieurs.

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Version française abrégée

Dans cette note, nous considérons le problème de Cauchy pour l'équation de Kirchhoff

$$\begin{cases} \partial_t^2 u - (1 + \int_{\mathbb{R}^n} |\nabla u(t, y)|^2 dy) \Delta u = 0, & t > 0, \quad x \in \mathbb{R}^n, \\ u(0, x) = u_0(x), \quad \partial_t u(0, x) = u_1(x), & x \in \mathbb{R}^n. \end{cases} \quad (1)$$

Cette équation a une longue histoire. Elle est apparue comme un modèle pour le mouvement d'une chaîne dans le livre [10] de Kirchhoff. Bernstein a prouvé dans [2] l'existence d'une solution analytique globale en temps sur un intervalle de la ligne réelle. Arosio et Spagnolo ont discuté dans [1] des solutions analytiques en dimensions spatiales plus élevées. D'Ancona et Spagnolo prouvent dans [4] que l'équation de Kirchhoff dégénérée est analytiquement bien posée, en liaison avec un travail

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de Kajitani et Yamaguti [9]. Dans la classe quasi-analytique, il est connu que des solutions globales existent – voir [7,15,16]. De plus, Manfrin [11] a montré l’existence de solutions globales en temps dans des espaces de Sobolev correspondant à des données non analytiques ayant un trou spectral – voir aussi [8] et Ghisi et Gobino [6,5].

Pour les petites données, des résultats beaucoup plus nombreux sont disponibles. Ici, nous nous référons à [12], qui donne une revue détaillée de la littérature sur les solutions de (1) ainsi que les résultats sur les petites données les plus récents. Dans cette note, nous sommes intéressés à l’existence de solutions pour les grandes données. Aussi nous référons-nous simplement à [13] pour un survol des grandes et surtout des petites données. Ces résultats seront publiés dans [14].

1. Introduction

In this paper, we consider the Cauchy problem for the Kirchhoff equation:

$$\begin{cases} \partial_t^2 u - (1 + \int_{\mathbb{R}^n} |\nabla u(t, y)|^2 dy) \Delta u = 0, & t > 0, \quad x \in \mathbb{R}^n, \\ u(0, x) = u_0(x), \quad \partial_t u(0, x) = u_1(x), & x \in \mathbb{R}^n. \end{cases} \tag{2}$$

This equation has a long history, having appeared as a model for the string motion in the book [10] of Kirchhoff. Bernstein [2] proved for it the existence of global in time analytic solutions on an interval of the real line. Arosio and Spagnolo [1] discussed analytic solutions in higher spatial dimensions, and in [4] D’Ancona and Spagnolo proved analytic well-posedness for the degenerate Kirchhoff equation, with a related work by Kajitani and Yamaguti [9]. In the quasi-analytic class, it is known that global solutions exist, see [7,15,16]. Moreover, Manfrin [11] showed the existence of time global solutions in Sobolev spaces corresponding to non-analytic data having a spectral gap, see also [8]. For the existence of local solutions in Gevrey spaces, we refer the reader to the results of Ghisi and Gobino (see [6,5]).

For small data, many more results are available. Here we refer the reader to [12], where a detailed literature review for solutions to (2) is given, as well as the most recent results on small data. In this note, we are interested in the existence of solutions for large data, so we only refer to [13] for a detailed survey dealing with large, but especially with small data. The results announced in this note will appear in [14].

2. Statement of results

In this section, we discuss the existence of solutions to (2) for Gevrey data. Let us recall the definition of the Gevrey class of L^2 type. For $s \geq 1$, we denote by $\gamma_{L^2}^s = \gamma_{L^2}^s(\mathbb{R}^n)$ the Roumieu–Gevrey class defined by

$$\gamma_{L^2}^s = \bigcup_{\eta > 0} \gamma_{\eta, L^2}^s,$$

where $f \in \gamma_{\eta, L^2}^s$ if

$$\int_{\mathbb{R}^n} e^{\eta|\xi|^{1/s}} |\widehat{f}(\xi)|^2 d\xi < \infty.$$

Here $\widehat{f}(\xi)$ is the Fourier transform of $f(x)$. The class $\gamma_{L^2}^s$ is endowed with the inductive limit topology. In particular, if $s = 1$, then $\gamma_{L^2}^1(\mathbb{R}^n)$ is the class \mathcal{A}_{L^2} of the analytic functions on \mathbb{R}^n . The spaces γ_{η, L^2}^s are normed spaces with norms

$$\|f\|_{\gamma_{\eta, L^2}^s} = \left[\int_{\mathbb{R}^n} e^{\eta|\xi|^{1/s}} |\widehat{f}(\xi)|^2 d\xi \right]^{1/2}.$$

We also denote

$$\|(f, g)\|_{\gamma_{\eta, L^2}^s \times \gamma_{\eta, L^2}^s} = \left[\int_{\mathbb{R}^n} e^{\eta|\xi|^{1/s}} \{ |\widehat{f}(\xi)|^2 + |\widehat{g}(\xi)|^2 \} d\xi \right]^{1/2}$$

for $\eta > 0$.

A special feature of the equation (2) is its Hamiltonian structure: for the energy

$$\mathcal{H}(u; t) := \frac{1}{2} \left\{ \|\nabla u(t)\|_{L^2}^2 + \|\partial_t u(t)\|_{L^2}^2 \right\} + \frac{1}{4} \|\nabla u(t)\|_{L^2}^4,$$

we have

$$\mathcal{H}(u; t) = \mathcal{H}(u; 0)$$

on the interval of the existence of solutions.

The following result gives the almost global existence for Gevrey solutions for (2) with large Gevrey data.

Theorem 1. Let $T > 0$ and $s > 1$. Let $M > 2$, $R > 0$ and denote $\eta_0(M, R, T) = 2sM^2e^{4M^2}RT^{1+\frac{1}{s}} + 4M^2$. Let $u_0, u_1 \in \gamma_{L^2}^s$. Assume that for some $\eta > \eta_0(M, R, T)$, we have

$$2\mathcal{H}(u; 0) < \frac{M^2}{4} - 1,$$

$$\left\| ((-\Delta)^{3/4}u_0, (-\Delta)^{1/4}u_1) \right\|_{\gamma_{\eta, L^2}^s \times \gamma_{\eta, L^2}^s}^2 \leq R.$$

Then the Cauchy problem (2) admits a unique solution $u \in C^1([0, T]; \gamma_{L^2}^s)$.

We note that the smallness of data is not required in **Theorem 1**: the size of the data is measured by constants M and R that are allowed to be large. However, an interesting feature is that the regularity of the data is related to the size, although this regularity is measured within the same Gevrey class $\gamma_{L^2}^s$. So, we can informally describe conditions of **Theorem 1** by saying that ‘the larger the data is the more regular it has to be’.

The statement concerning the regularity in the class $\gamma_{L^2}^s$ can be refined. Namely, we have the following remark.

Remark 2. Assume the conditions of **Theorem 1**. Then the solution u satisfies

$$u \in \bigcap_{j=0}^1 C^j \left([0, T]; (-\Delta)^{-(3/4)+(j/2)} \gamma_{\eta', L^2}^s \cap (-\Delta)^{-(1/2)+(j/2)} \gamma_{\eta', L^2}^s \right),$$

with

$$\eta' = \eta - \eta_0(M, R, T) > 0.$$

We conclude this note by giving an outline of the proof of **Theorem 1**. To begin with, we consider the linear Cauchy problem

$$\partial_t^2 v - c(t)\Delta v = 0, \quad t \in (0, T), \quad x \in \mathbb{R}^n, \quad (3)$$

with the same data (u_0, u_1) as in the nonlinear Cauchy problem (2). The coefficient $c(t)$ is assumed to be of Lipschitz regularity on $[0, T]$, and satisfy the conditions

$$1 \leq c(t) \leq M \quad \text{on } [0, T],$$

$$|c'(t)| \leq \frac{K}{(T-t)^{1+\frac{1}{s}}} \quad \text{on } [0, T],$$

for some $M > 2$ and $K > 0$. Then, after using the energy estimates for solutions to (3) (in fact, a refined version of energy estimates from Colombini, Del Santo and Kinoshita [3]), and choosing the constant K such that

$$K = M^2 e^{4M^2} RT^{1+\frac{1}{s}},$$

we get

$$|\tilde{c}'(t)| \leq \frac{K}{(T-t)^{1+\frac{1}{s}}} \quad \text{on } [0, T],$$

where we put

$$\tilde{c}(t) = \sqrt{1 + \int_{\mathbb{R}^n} |\nabla v(t, x)|^2 dx}.$$

Thanks to the continuity argument, we can also prove that

$$2\mathcal{H}(v; t) \leq \frac{M^2}{4} - 1 \quad \text{on } [0, T].$$

Thus, the Schauder–Tychonoff fixed point theorem allows us to conclude that the solution v of the linear Cauchy problem (3) with data (u_0, u_1) is also the solution to the nonlinear Cauchy problem (2).

The argument in the proof of **Theorem 1** is also applicable to the initial-boundary value problems in an open set Ω of \mathbb{R}^n : bounded domains and exterior domains with analytic boundary. The results can be proved by the Fourier series expansion method in bounded domains, and by the generalised Fourier transform method in exterior domains, respectively. It is known from the spectral theorem that a self-adjoint operator on a separable Hilbert space is unitarily equivalent to a multiplication operator on some $L^2(\mathcal{M}, \mu)$, where (\mathcal{M}, μ) is a measure space. Then $L^2(\Omega)$ is unitarily equivalent to $L^2(\mathbb{R}^n)$. This means that the Fourier transform method in \mathbb{R}^n is available for the L^2 space on an open set Ω in \mathbb{R}^n ; any multiplier acting on $L^2(\mathbb{R}^n)$ is unitarily transformed into a multiplier acting on $L^2(\Omega)$. For more details, we refer the reader to [14].

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