Approximate Optimal Control via Measurement Feedback for a Class of Nonlinear Systems

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Abstract: The approximate optimal control problem via measurement feedback for input-affine nonlinear systems is considered in this paper. In particular, a systematic method is provided for constructing stabilising output feedbacks that approximate - with the optimality loss explicitly quantifiable - the solution of the optimal control problem by requiring only the solution of algebraic equations. In fact, the combination of a classical state estimate with an additional dynamic extension permits the construction of a dynamic control law, without involving the solution of any partial differential equation or inequality. Moreover, provided a given sufficient condition is satisfied, the dynamic control law is guaranteed to be (locally) stabilising. A numerical example illustrating the method is provided.

Keywords: Dynamic output feedback, nonlinear control systems, control system design, feedback stabilization, feedback control methods

1. INTRODUCTION

One of the fundamental problems in control theory is that of designing controllers to stabilise an equilibrium of a dynamical system. Other common tasks include output regulation, robust control and optimal control. In many situations the control design must be done subject to partial knowledge of the state of the system, i.e. by using available measurements. Given a dynamical system and access to its output only it is, in the general nonlinear setting, a challenging task to design stabilising output feedback, let alone solve more complex problems, such as designing an optimal control or disturbance attenuating control actions. In fact, designing an optimal control is not a trivial task, even when access to the entire state of the system is available. Herein, we consider a general, input-affine, nonlinear system and we consider the problem of designing stabilising and optimal measurement feedback for the system.

For linear systems, subject to observability and controllability requirements being satisfied, stabilising output feedbacks can be designed by means of a linear observer and a linear state feedback (see, for example, Chapter 15 of Rugh [1996]). For nonlinear systems, however, the problem is somewhat more complicated. Differently from the case with linear systems, the observer design and state feedback design cannot, in general, be done in two independent steps for nonlinear systems (see, for example, Khalil [1996]). This makes the problem of designing stabilising output feedback a challenging one, and as a consequence it has received much attention in the literature. Semiglobal stabilisation using output feedback is considered in Teel and Pray [1994, 1995]. In Pray and Jiang [1993] conditions for stabilization via output feedback are provided for a class of nonlinear systems. Many approaches to the design of controllers using output feedback for nonlinear systems rely on the use of high-gain observers as in Lee and Khalil [1997] or high-gain feedback as in Pray and Jiang [2004]. In Esfandiari and Khalil [1992] stabilising output feedback for fully linearisable systems are constructed using singular perturbation theory and high-gain feedback. Approximate optimal control via measurement feedback for nonlinear systems is still an open problem.

In many practical applications full access to the state describing the system is not available. The design of feedback controllers must then be based on output measurements. A few examples of applications include control of underwater vehicle propellers (see Fossen and Blanke [2000]), control of certain mechanical systems such as the so-called rotational-translational actuator (see Jiang and Kanellopoulos [2000]) and control of robotic manipulators (see Lee and Khalil [1997]).

A method for designing robust, stabilising controllers using measurement feedback for nonlinear systems is presented in Isidori and Astolfi [1992]. The approach relies on solving a system of two partial differential equations (PDEs), the solutions of which are used to construct a Lyapunov function employed then as a certificate of asymptotic stability. However, obtaining a solution of the system of PDEs is a nontrivial task.

Systems of $N > 0$ coupled PDEs characterise the solution of $N$-player differential games (see, for instance Basar and Olsder [1999], Starr and Ho [1969]). In Mylvaganam et al.
systematic methods of constructing approximate solutions for systems of PDEs are provided. The methods rely on solving a system of algebraic equations and, using dynamic state feedback, control strategies which satisfy partial differential inequalities (PDIs) in place of the PDEs are constructed. Moreover, it is shown that the control strategies satisfying the PDIs constitute an approximate solution for the underlying differential game and the system of PDEs characterising its solution.

This paper takes inspiration from Isidori and Astolfi [1992] and Mylvaganam et al. [2015] to provide a novel, systematic method of constructing stabilising feedback based only on available output measurements. In addition to stabilising an equilibrium of the nonlinear system, the proposed output feedback constitutes an approximate solution of an optimal control problem via measurement feedback. The method relies on the solution of algebraic equations and the introduction of a “two-step” augmentation of the state. A dynamic control law is constructed which ensures, subject to a sufficient conditions which are identified herein, (local) asymptotic convergence of the origin of the extended state under the dynamic control law.

The remainder of the paper is organised as follows. The problem of designing a stabilising output feedback controller and the optimal control problem via measurement feedback are defined in Section 2. The algebraic $P$ matrix solution, which is a mathematical tool central to the proposed method, is then introduced in Section 3. The main results of the work, namely a constructive method for designing stabilising output feedback for nonlinear systems which, additionally, approximates the solution of the optimal control problem, are presented in Section 4 before an illustrative numerical example is given in Section 5. Finally, concluding remarks and directions for future work are provided in Section 6.

2. PROBLEM FORMULATION

Consider a general, input-affine dynamical system described by equations of the form

$$
\dot{x} = f(x) + g(x)u, \\
y = h(x),
$$

(1)

where $x(t) \in \mathbb{R}^n$ denotes the state, $u(t) \in \mathbb{R}^m$ denotes the control input and $y(t) \in \mathbb{R}^p$ denotes the output of the system, and $f : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ are smooth mappings and $g : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ is a smooth matrix-valued function. The following standing assumption is considered throughout the remainder of the paper.

Assumption 1. The origin of $\mathbb{R}^n$ is an equilibrium of (1) with $u(t) \equiv 0$, i.e. $f(0) = 0$.

As a consequence of Assumption 1, there exists a (continuous) matrix-valued mapping $F : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ such that $f(x) = F(x)x$ for all $x \in \mathbb{R}^n$. Note that while the mapping $F(x)$ is not uniquely defined, $F(0)$ is unique.

The problem of designing a stabilising output feedback and the optimal control problem via measurement feedback are defined in the following two statements.

Problem 1. Determine an output feedback control law described by the equations

$$
u \geq 0, \quad \nu(t) \in \mathbb{R}^n, \quad \gamma : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^m \quad \text{and} \quad \delta : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^p \quad \text{smooth mappings satisfying} \quad \gamma(0,0) = 0 \quad \text{and} \quad \delta(0,0) = 0, \quad \text{such that the zero equilibrium of the closed-loop system (1)-(2), namely}

$$
\dot{x} = f(x) + g(x)\gamma(\eta, h(\eta)), \\
\dot{\eta} = \delta(\eta, h(\eta)),
$$

(3)

is (locally) asymptotically stable.

Problem 2. Let $q : \mathbb{R}^n \to \mathbb{R}_+, \quad q(0) = 0$, be a positive definite function. Determine an output feedback control law described by (2) such that the cost functional

$$
J(x(0), u) = \frac{1}{2} \int_0^\infty \left( q(x(t)) + \| u(t) \|^2 \right) dt,
$$

(4)

is minimised along the trajectories of the closed-loop system (3).

The function $q(x)$ represents a running cost and in the remainder of the paper the following standing assumption is made.

Assumption 2. The scalar function $q(x)$ is at least locally quadratic, namely there exists a matrix-valued function $Q : \mathbb{R}^n \to \mathbb{R}^{n \times n}$, such that $q(x) = x^\top Q(x)x$ and $Q(x) > 0$, for all $x \in \mathbb{R}^n$.

The objective of the remainder of this paper is to provide a constructive method of designing output feedback of the form (2) which, under suitable conditions (essentially related to asymptotic stability properties of (2) in the absence of external signals), yields a solution to Problem 1. Moreover, it is demonstrated that the same feedback provides an approximate solution for Problem 2. Obtaining a solution of Problem 2 is in fact, in the nonlinear setting, a daunting task even letting $h(x) \equiv x$. Herein, we provide an approximate solution for the optimal control problem with measurement feedback and we provide explicit values of the performance loss, measured in terms of the cost functional (4). In the performance loss, we distinguish and identify the marginal contributions due to an approximate scheme for state-feedback optimal control, on the one hand, and due to the presence of an output function $y = h(x)$, on the other hand.

3. ALGEBRAIC $P$ SOLUTION

Achieving disturbance attenuation and internal stability of nonlinear, input-affine systems using measurement feedback is considered in Isidori and Astolfi [1992]. The authors demonstrate that through solving a system of two PDEs it is possible to design a dynamic output feedback which stabilises an equilibrium of the system while guaranteeing a certain disturbance attenuation level. Interestingly, whereas the solution of one of the PDEs is used to design the output feedback, the second PDE is an analysis tool used to guarantee (local) asymptotic stability. Since closed-form solutions to PDEs are often not readily available, we propose an alternative method for designing stabilising output feedback, which builds on the core ideas from Isidori and Astolfi [1992], Sassano and Astolfi [2012], Mylvaganam et al. [2015].
A systematic method for constructing stabilising output feedback for the class of nonlinear systems described by (1) is provided in Section 4. The method relies on the introduction of an augmented dynamical system and the notion of an algebraic $\bar{P}$ matrix solution, which has been introduced in Sassano and Astolfi [2012] for solving optimal control problems and in Mylvagamnam et al. [2015] for N-player differential games. The augmented dynamical system and the properties of the algebraic $\bar{P}$ matrix solution are exploited to design extended value functions which, by construction, satisfy two PDIs from which we infer (local) asymptotic stability. The augmented dynamical system consists of a dynamic extension variable and a state estimate, and the method proposed can be analysed in terms of traverse stability (see Andrieu et al. [2016]).

Consider the augmented state $x_e = (x, \hat{x}, \xi)$, where $\hat{x}(t) \in \mathbb{R}^n$ is an estimate of the state $x$, similar to what is used in Isidori and Astolfi [1992], and $\xi(t) \in \mathbb{R}^n$ is a dynamic extension similar to what is used in Sassano and Astolfi [2012], Mylvagamnam et al. [2015]. The dynamics of the augmented state is given by

$$\dot{x}_e = f_e(x_e)$$

(5) with

$$f_e(x_e) = \begin{bmatrix} f(x) \\ f(\hat{x}) - G(\hat{x})(h(\hat{x}) - h(x)) \\ \beta(\hat{x}, \xi) \\ \xi \end{bmatrix} + \begin{bmatrix} g(x)u \\ g(\hat{x})u \\ 0 \\ 0 \end{bmatrix},$$

(6)

where $G : \mathbb{R}^n \to \mathbb{R}^{n \times p}$ is a smooth matrix-valued function to be determined. Note that, by the properties of the dynamic output feedback discussed in Problem 1, namely that $\gamma(0, 0) = 0$ and $\delta(0, 0) = 0$, the system (5) in closed loop with the control law $u = -g(x)P(x) + \hat{x}(\xi, y)$ possesses an equilibrium point at the origin. As a consequence, similarly to Assumption 1, there exists a matrix-valued function $F_e : \mathbb{R}^{3n} \to \mathbb{R}^{3n \times 3n}$ such that $f_e(x_e) = F_e(x_e)x_e$. Note that the function $F_e$ depends on the control input $u$. Finally, the linear approximations of the systems (1) and (5) about the origin are given by

$$\dot{x} = Ax + Bu,$$

and

$$\dot{x}_e = A_ex_e + B_eu,$$

respectively, where

$$A = \frac{\partial f}{\partial x}_{|x=0} = F(0),$$

$$B = g(0)$$

and

$$A_e = \frac{\partial f_e}{\partial x_e}_{|x_e=0}, \quad B_e = [B^T, B^T, 0]^T.$$

In the following the control input is designed such that it is a function of the estimate of the state $\hat{x}$ and the dynamic extension $\xi$ only, i.e. $u = u(\hat{x}, \xi)$.

Different notions of the algebraic $\bar{P}$ matrix solution are used in Sassano and Astolfi [2012], Mylvagamnam et al. [2015] to systematically design (dynamic) feedback laws. In the following section it is demonstrated that a modified notion of the algebraic $\bar{P}$ matrix solution is instrumental in designing output feedbacks for the class of nonlinear systems described by (1) which solves Problem 1 essentially by approximating the solution of Problem 2. To this end, let $q_e(x_e) = x_e^T Q_e(x_e)x_e$, where $Q_e : \mathbb{R}^{3n} \to \mathbb{R}^{3n \times 3n}$ and $q_e(x_e) \geq 0$. An algebraic $\bar{P}$ matrix solution for Problem 1 and Problem 2 is defined as follows.

**Definition 1.** Consider the system (1). Let $\Sigma : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ be matrix-valued functions such that $\Sigma(x) = \Sigma(x)^T \geq 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$ and $\Sigma(0) \geq 0$. Similarly, let $\Sigma_e : \mathbb{R}^{3n} \to \mathbb{R}^{3n \times 3n}$, be matrix-valued functions such that $\Sigma_e(x) = \Sigma_e(x)^T \geq 0$ for all $x \in \mathbb{R}^{3n} \setminus \{0\}$ and $\Sigma_e(0) \geq 0$. Finally, let $R = R^T > 0$, $Q = Q(0)$ and $Q_e = Q_e(0)$.

The pair of $\Sigma^1$ matrix-valued functions $P : \mathbb{R}^n \to \mathbb{R}^{n \times n}$, $P_e = P(x)^T$ and $P_e : \mathbb{R}^{3n} \to \mathbb{R}^{3n \times 3n}$, $P_e(x) = P_e(x)^T$ is said to be an $\bar{X}$-algebraic $\bar{P}$ matrix solution $1$ of Problem 1 provided the following conditions hold.

(i) For all $x \in X \subseteq \mathbb{R}^n$, $i = 1, 2$, $j \neq i$,

$$P(x)F(x) + F(x)^T P(x) - P(x)g(x)g(x)^T P(x) + Q(x) + \Sigma(x) = 0,$$

(7)

and

$$P_e(x)F_e(x) + F_e(x)^T P_e(x) + Q_e(x) + \Sigma_e(x) = 0,$$

(8)

where $F_e$ is the matrix-valued mapping corresponding to the input $u = -g(x)^T P(x)(\dot{x} + R(\hat{x} - \xi))$.

(ii) $P(\bar{0}) = P$, such that $\bar{P} > 0$, with $\bar{P}$, the symmetric solution of the algebraic Riccati equation

$$\bar{P}A + A^T \bar{P} - PBB^T \bar{P} + \bar{Q} + \bar{\Sigma} = 0,$$

(9)

and $P_e(\bar{0}) = \bar{P}_e$, such that $\bar{P}_e > 0$, with $\bar{P}_e$, the symmetric solution of the Lypanunov equation

$$\bar{P}_e A_e + A_e^T \bar{P}_e + Q_e + \bar{\Sigma}_e = 0,$$

(10)

where $A_e$ is the matrix describing the linear approximation of the system $2$ (6) with the input $u = -g(x)^T P(x)(\dot{x} + R(\hat{x} - \xi))$.

If $x \in \mathbb{R}^n$, i.e. $X = \mathbb{R}^n$, then $P$ is said to be algebraic $\bar{P}$ matrix solutions for Problem 1.

In the remainder of the paper it is assumed that an algebraic solution exists.

**Remark 1.** The condition (10) is a Lypanunov equation and obtaining a solution for this equation requires $3$ $\sigma(A_e) \subseteq \mathbb{C}^-$. This is similar to the second PDE introduced in Isidori and Astolfi [1992] in the context of disturbance attenuation, which in the case of linear systems, reduces to a Lypanunov equation.

**4. STABILISING OPTIMAL CONTROL VIA MEASUREMENT FEEDBACK**

The notion of an algebraic $\bar{P}$ solution can be used to systematically design an output feedback which is such that the zero equilibrium of the closed-loop system (1) is asymptotically stable.

Let $P(x)$ and $P_e(x_e)$ denote an algebraic $\bar{P}$ solution and consider the dynamic variable $\xi_e(t) \in \mathbb{R}^{3n}$ with

$$\dot{\xi}_e = \beta_e(\xi_e, x_e),$$

(11)
where \( \beta_\ell : \mathbb{R}^p \times \mathbb{R}^n \rightarrow \mathbb{R}^{3n} \) is to be defined. Consider the functions
\[
V(x, \xi) = \frac{1}{2} x^T P(\xi) x + \frac{1}{2} ||x - \xi||_R^2,
\]
and
\[
V_e(x, \xi, \xi) = \frac{1}{2} x_e^T P_e(\xi) x_e + \frac{1}{2} ||x_e - \xi_e||_R^2,
\]
with \( R_e = R^+ \) (recall that \( R = R^+ > 0 \) also), defined in the extended state spaces \( (x, \xi) \) and \((x_e, \xi_e)\), respectively. Let \( \Psi(x, \xi) \) denote the Jacobian of \( \frac{1}{2} P(\xi)x \) with respect to \( \xi \) and let \( \Phi(x, \xi) \) denote a matrix-valued function such that \( x^T(P(x) - \Psi(\xi))(x) \). Similarly, let \( \Psi_e(x, \xi, \xi) \) denote the Jacobian of \( \frac{1}{2} P_e(\xi_e)x_e \) with respect to \( \xi_e \) and let \( \Phi_e(x_e, \xi, \xi) \) denote a matrix-valued function such that \( x_e^T(P_e(x_e) - P_e(\xi_e)) = (x - \xi) \). \( \Phi_e(x_e, \xi, \xi) \).

For clarity of presentation the following notation is defined at this stage. Let \( \hat{x} = (\hat{x}, \hat{\xi}, \xi) \),
\[
\begin{align*}
\alpha(\xi, x) &= -g(x)^T (P(\xi)x + R(x - \xi)), \\
\beta(\xi, x) &= -\kappa(\Psi(x, \xi)x - R(x - \xi)), \\
\beta_e(\xi_e, x_e) &= -\kappa_e(\Psi_e(x_e, \xi_e)x_e - R_e(x_e - \xi_e)).
\end{align*}
\]
and
\[
G(\hat{x}) = \tilde{G} \left( \frac{\partial h(\hat{x})}{\partial \hat{x}} \right)^T,
\]
with \( \tilde{G} \in \mathbb{R}_+ \), and let \( N_h \) denote the set
\[
N_h = \{(x, \hat{x}) \in \mathbb{R}^n \times \mathbb{R}^n : (h(\hat{x}) - h(x)) \frac{\partial h(\hat{x})}{\partial \hat{x}} = 0, \hat{x} \neq x\}.
\]

Let \( P \) and \( P_e \) denote an algebraic solution, i.e. matrix-valued functions satisfying (7), (8), (9) and (10).

The following statement outlines a systematic method using the algebraic solution to design a dynamic output feedback. Sufficient conditions for guaranteeing (local) asymptotic stability of the zero equilibrium of the closed-loop system are provided.

**Theorem 1.** Consider the system (1) and let the matrix-valued function \( P \) and \( P_e \) satisfy the algebraic equations (7)-(10). There exists \( \tilde{k} > 0 \) and \( \tilde{k}_e > 0 \) and a non-empty set \( \Omega \in \mathbb{R}^{3n} \) containing the origin such that, for all \( k \geq \tilde{k} \) and \( k_e \geq \tilde{k}_e \), the dynamic output feedback
\[
\begin{align*}
u &= \alpha(\xi, \hat{x}), \\
\dot{\hat{x}} &= f(\hat{x}) + g(\hat{x}) u + G(\hat{x})(y - h(\hat{x})), \\
\dot{\xi} &= \beta(\xi, \hat{x}),
\end{align*}
\]
defined in (14)-(16), is such that the partial differential inequalities
\[
I(x, \xi, \hat{x}, \xi) \leq \frac{1}{2} (g(x) - q_e(x)) + \frac{\partial V}{\partial \xi} f(x) + \frac{1}{2} \beta(\xi, \hat{x}) \leq 0,
\]
and
\[
\frac{\partial V_e}{\partial \xi} f_e(x_e) + \frac{\partial V_e}{\partial \xi} \beta(\hat{x}, \xi_e) + \frac{1}{2} q_e(x_e) - \frac{\partial V}{\partial \xi} (\beta(\xi, \hat{x}) - \beta(\xi, x)) \leq 0,
\]
as required to ensure global asymptotic stability of the zero equilibrium of the closed-loop system (5)-(18) in the neighbourhood \( \Omega \). Moreover, provided
\[
\begin{align*}
(h(\hat{x}) - h(x)) \frac{\partial h(\hat{x})}{\partial \hat{x}} f(\hat{x}) + g(\hat{x}) a(\xi, \xi) \\
+ (x^T \Psi(x, \xi) - (x, \xi)^T R) (\beta(\xi, \hat{x}) - \beta(\xi, x)) \leq 0,
\end{align*}
\]
for all \( (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \), there exists \( G > 0 \) such that the dynamic output feedback (18) renders the zero equilibrium of the extended state \( (x, \hat{x}, \xi, \xi) \) (locally) asymptotically stable.

**Remark 2.** In Isidori and Astolfi [1992] a stabilising output feedback is designed using two partial differential equations. The approach taken in this paper is such that the dynamic control laws (18) satisfy the partial differential inequalities (19) and (20) in place of two partial differential equations. This can be interpreted as constructing approximate solutions for partial differential equations for which the level of approximation can be increased by means of the Lyapunov function \( V_q \).

**Remark 3.** The algebraic P matrix solution consists of two key components, namely \( P(x) \) and \( P_e(x_e) \). The two components have different roles: whereas \( P(x) \) and the dynamic extension \( \xi(t) \) are used directly in the feedback design (18) as well as for analysis purposes, \( P_e(x_e) \) and \( \xi_e \) are used solely for analysis purposes to establish stability properties of the closed-loop system. It is worth emphasising here that only \( P(x) \) and \( \xi(t) \) are utilised in the feedback design.

**Remark 4.** Interestingly, the dynamics of \( \hat{x} \) alone, given by
\[
\dot{\hat{x}} = f(\hat{x}) + g(\hat{x}) u + G(\hat{x})(y - h(\hat{x})),
\]
which can be loosely interpreted as a nonlinear extension of the well-known Luenberg observer, may not constitute an asymptotic observer for \( x \), or at least it may not be possible to guarantee this property by means of the Lyapunov function \( V_q \). However, the “observer-like” properties of \( \hat{x} \) are obtained in combination with the additional contribution of the dynamic extension \( \xi \) that affects the dynamics of \( \hat{x} \) via the control input \( u \).

We can further demonstrate that the dynamic output feedback (18) constitutes an approximate solution to Problem 2. Moreover, it is demonstrated that the level of approximation can be quantified in terms of two sources of error (with respect to the classic state feedback optimal control problem, i.e. when \( h(x) \equiv x \)), namely the negativity gap in the PDI (19) and the difference between the state estimate \( \hat{x} \) and the actual state of the system.

To this end, let \( c : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \) denote a positive definite function. In particular, let \( c(x, \hat{x}, \xi) = c_2(x, \hat{x}, \xi) + c_2(x, \hat{x}, \xi) \), where \( c_2 = -2Z \), with \( Z \) is as defined in (19), and \( c_2(x, \hat{x}, \xi) = (\alpha(\xi, x) - \alpha(\xi, \hat{x}))^T + \alpha(\xi, \hat{x}) - \frac{1}{2} q_e(x_e) \). The dynamic output feedback (18) is such that the following statement is true.

\footnote{The set of positive real numbers is denoted by \( \mathbb{R}_+ \).}
Theorem 2. Considering the system (1) and the cost functional (4), suppose (21) is satisfied for all \( (x, \dot{x}) \in \mathcal{N}_b \) and suppose (18) is such that the PDIs (19) and (20) are satisfied. Then the dynamic measurement feedback (18) is such that the modified cost functional

\[
\tilde{J}(x(0), \dot{x}(0), \xi(0), u, \dot{\xi}) = \frac{1}{2} \int_0^\infty (q(x(t)) + \|u(t)\|^2 + c(x, \dot{x}, \xi)) dt,
\]

is minimised. □

Remark 5. Theorem 2 implies that the dynamic measurement feedback (18) constitutes an approximate solution to the optimal control problem in Problem 2, where the loss with respect to the optimal state feedback is quantified by the additional cost

\[
\frac{1}{2} \int_0^\infty c(x, \dot{x}, \xi) dt.
\]

The additional cost can be interpreted as the performance loss associated with the approximation, where \( c_2 \) is due to the inequality (19) and \( c_2 \) is due to the error in the state observer, i.e. it is due to the use of output feedback in place of state feedback). ▲

Remark 6. The approximate solution to Problem 2 provided in Theorem 2 is similar in spirit to the approximate solutions provided for optimal control and differential games (with state feedback) in Sassano and Astolfi [2012] and Mylvaganam et al. [2015], respectively. As remarked therein, in the case of state feedback the initial condition of the dynamic extension \( \xi(0) \) can in principle be selected to minimise (23) for a given \( x(0) \). However, differently from Sassano and Astolfi [2012], Mylvaganam et al. [2015], since full knowledge of the state \( x(0) \) is not available in the setting considered in this paper, the initial conditions of \( \dot{x}(0) \) and \( \xi(0) \) cannot be “optimally” selected. ▲

5. SIMULATIONS

Consider a planar system, described by the state \( x = [x_1, x_2]^\top \in \mathbb{R}^2 \), with the dynamics

\[
\dot{x} = \left[ a_1 x_1^2 + a_2 x_2 \right] x_1 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u,
\]

where \( a_1 > 0 \) and \( a_2 < 0 \) are scalar parameters and \( u \in \mathbb{R} \) is the control input. The output of the system is \( y = x_1 \) and it is of interest to design an output feedback to minimise a cost function of the form (4) with \( q(x) = c_1 x_1^2 + c_2 x_2^2 \), where \( c_1 > 0 \) and \( c_2 > 0 \), are scalar quantities.

Let \( a_1 = 1 \), \( a_2 = -1 \) and \( c_1 = c_2 = 1 \). The matrix-valued function

\[
P(x) = \text{diag} \left\{ p_1 (1 + x_1^2), p_2 \right\},
\]

with \( p_1 = 8 \) and \( p_2 = 2 \) is such that the conditions (7) and (9) in the definition of the algebraic \( P \) matrix solution are satisfied with

\[
\Sigma(x) = \begin{bmatrix} 47 + 96 x_2^2 + 48 x_1^4 + 16 x_1^2 + 16 x_2^2 \\ 16 + 16 x_1^2 \\ 7 \end{bmatrix},
\]

which is positive definite for all \( x \in \mathbb{R}^2 \). Thus, following Remark 3, \( P(x) \) given in (25) corresponds to the “design component” of the algebraic \( P \) matrix solution.

Let \( \dot{x} = [\dot{x}_1(t), \dot{x}_2(t)]^\top \) and \( \dot{\xi} = [\dot{\xi}_1(t), \dot{\xi}_2(t)]^\top \). Consider \( P(x) \) in (25), let \( R = I \) and consider the corresponding

\[\footnotesize
\begin{align*}
\text{Fig. 1. The trajectory of the state } x. \text{ The circular and square markers denote the initial and final states.}
\end{align*}
\]

\[\footnotesize
\begin{align*}
\text{Fig. 2. The time histories } e_1(t) \text{ (top) and } e_2(t) \text{ (bottom).}
\end{align*}
\]

closed-loop (extended) system (6)-(18). In this case, the feedback (18) is given by

\[
\dot{x} = \left[ \begin{array}{c} (a_1 + \dot{x}_1^2) \dot{x}_1 + G(x_1 - \dot{x}_1) + u \\ a_2 \dot{x}_2 + u \end{array} \right],
\]

\[
\dot{\xi} = -k \left[ \begin{array}{c} \xi_1 \dot{x}_1 + \dot{x}_1 - \xi_1 \\ \dot{x}_2 - \xi_2 \end{array} \right].
\]

Consider the case in which \( x(0) = [-10, 20]^\top \) and the initial conditions of the observer and the dynamic extension are \( \dot{x}(0) = [0, 0] \) and \( \xi(0) = [20, 0] \), respectively. The remaining control parameters are selected as \( k = 1 \) and \( G = 200 \).

The trajectory of the state \( x \) is shown in Figure 1 where the initial and final states are denoted by the circular and square solid markers, respectively. The time histories of the first and second components of the error between the observer and the state, i.e. \( e(t) = \dot{x} - x = [e_1(t), e_2(t)]^\top \), are shown in the top and bottom plots of Figure 2, respectively. Finally, the time histories of the components of the dynamic extension are shown in Figure 3, where the solid and dashed lines represent \( \xi_1 \) and \( \xi_2 \), respectively. Clearly, the extended state converges to zero and it follows from Theorem 2 that the dynamic feedback law constitutes an approximate solution for the optimal control problem Problem 2.

6. CONCLUSION

Optimal control via measurement feedback is considered for general, nonlinear systems in this paper. Drawing inspi-
ration from Isidori and Astolfi [1992], Mylvaganam et al. [2015], a novel method for designing stabilising output feedback which approximates the solution of the optimal control problem is provided. The proposed method relies on a so-called algebraic matrix solution which is used to design the output feedback and to systematically construct a Lyapunov function which allows to establish asymptotic stability of the origin of the closed-loop system. The resulting control laws constitute an approximate solution of the optimal control problem and, moreover, the level of approximation is quantified. Finally, the method is illustrated on a simple numerical example.

Directions for future research include taking into consideration possible disturbances influencing the system (1) and establishing some level of robustness to external disturbances, similarly to what has been done in Isidori and Astolfi [1992] and Isidori [1994]. It is also of interest to consider a variety of numerical examples and explore different areas of applications for the developed theory. An area of particular interest is multi-agent systems: output feedback for problems such as those considered in Mylvaganam et al. [2014], Mylvaganam and Astolfi [2016], Mylvaganam and Astolfi [2015] will be studied.

REFERENCES


