Cascades of iISS and Strong iISS systems with multiple invariant sets

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***Abstract*— In recent papers [1,2], the notions of Input-to-State Stability (ISS) and Integral ISS (iISS) have been generalized for systems evolving on manifolds and having multiple invariant sets, i.e. multistable systems. The well-known property of conservation of ISS under cascade interconnection has also been proven true for multistable systems in different scenarios [3]. Unfortunately, multistability hampers a straightforward extension of analogous conservation properties for integral ISS systems. By means of counterexamples, this work highlights the necessity of the additional assumptions which yield the conservation of the iISS and Strong iISS properties in cascades of multistable systems. In particular, a characterization of the invariant set of the cascade is provided in terms of its finest possible decomposition.**

I. INTRODUCTION

The study of stability and robustness of cascades of nonlinear systems have long been studied and have led to several constructive design methods such as backstepping and forwarding ([1] and [2]). In the analysis of cascades, the ISS approach has proven to be a well-suitable tool mainly due to its input-to-state formulation based on injection and dissipation gains. It is in fact well known that the ISS property is preserved under cascade interconnection [3].

At the same time, the study of nonlinear systems which exhibit a variety of non-trivial dynamical behaviors - multiple equilibria, periodicity, almost-periodicity - has great importance to several scientific disciplines ranging from mechanics and electronics to biology and neuroscience. In particular, the research in systems biology motivates the analysis of the stability and robustness properties of cascades of multistable systems. Such properties can be characterized in a novel ISS framework [4] where the ISS notion has been generalized for systems evolving on manifolds and having a decomposable invariant set. Interestingly enough, such generalized ISS is still preserved under cascade interconnections and this conservation property also provides a fine structure on the invariant set resulting from the interconnection [5].

Despite its usefulness, the notion of ISS might sometimes be too strong a requirement for nonlinear systems, and the same holds in the multistable context. This fact motivates the consideration of weaker notions of stability, iISS [6] and Strong iISS [7], whose related generalizations in the multistable context have been characterized in [8].

The cascade interconnection preserves the iISS property under suitable conditions on the matching between the dissipation gain of the driving system and the injection gain of the driven system, as in [9] and [10]. Conversely, Strong iISS of cascades can be verified with no additional conditions other than Strong iISS being satisfied by each subsystem. This paper investigates the preservation of the iISS and Strong iISS properties in the context of multistable systems. In particular, we focus on the necessity of the additional requirements that multistability demands in order to infer the preservation of the two properties. Moreover, we provide a fine description of the structure of the invariant set resulting from the interconnection.

Notation: The symbol $\mathfrak{d}[x_1, x_2]$ denotes the Riemannian distance between two points x_1 and x_2 of a Riemannian manifold M. For a set $S \subset M$ define $|\cdot|_S$ as $|x|_S = \inf_{a \in S} \mathfrak{d}[x, a]$. For a compact set $A \subset M$, $\eth(A)$ and $\operatorname{int}\{A\}$ respectively denote the boundary and the interior of A. We denote the differential [11, Definition 4.2.5] of a smooth function $V : M \to \mathbb{R}$ by the covector field $dV : M \to T^\star M$, and then we denote the Lie derivative [11, Definition 4.2.6] of V along a vector field $X : M \to TM$ at a point $x \in M$ by $\mathfrak{L}_X V(x) := dV(X)(s) = dV(x) \cdot X(x)$, namely the action of dV on X at x. Notations $|\cdot|$ and $|\cdot|_{\mathfrak{g}}$ respectively indicate the standard Euclidean norm and the norm induced by the Riemannian metric $\mathfrak{g}$ on $T_x M$, i.e. $|v|_{\mathfrak{g}} := \sqrt{\mathfrak{g}_x(v, v)}$ for any $v \in T_x M$. For a measurable function $d : \mathbb{R}_+ \to \mathbb{R}^m$ define its infinity norm over the time interval $[t_1, t_2]$ as $\left\|d_{[t_1,t_2]}\right\| = \operatorname{ess\,sup}_{t_1 \le t \le t_2} |d(t)|$, and denote $\|d\| := \left\|d_{[0,+\infty)}\right\|$. Define the infinity norm of $d(\cdot)$ with respect to a compact set S as follows: $\left\|d_{[t_1,t_2]}\right\|_S = \operatorname{ess\,sup}_{t_1 \le t \le t_2} |d(t)|_S$, and define $\|d\|_S := \left\|d_{[0,+\infty)}\right\|_S$. We define the saturation function as $\operatorname{sat}(x) := \operatorname{sign}\{x\} \min\{1, |x|\}$. Given two continuous functions $f, g : \mathbb{R} \to \mathbb{R}$, expression $f(s) = \mathcal{O}(g(s))$ as $s \to 0^+$ will indicate that $\limsup_{s \to 0^+} |f(s)/g(s)| < +\infty$.

II. PRELIMINARY DEFINITIONS

The subject of our study is the cascade interconnection of two or more nonlinear systems which exhibit multistable behavior. We are going to illustrate first the class of multistable systems of interest, and then we will address the cascade composition of such systems.

Let M be a geodesically complete, connected, n-dimensional Riemannian manifold without boundary. Let D be a closed subset of $\mathbb{R}^m$ containing the origin. Consider now the general nonlinear autonomous system:

$$\dot{w}(t) = F(w(t), d(t)), \tag{1}$$

and its autonomous counterpart:

$$\dot{w}(t) = F(w(t), 0), \tag{2}$$

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where $F(w, d) : M \times D \to T_w M$ is a locally Lipschitz continuous mapping and $d(\cdot)$ is any locally essentially bounded and measurable input signal taking values in D. We denote with $\mathcal{M}_D$ such class of input signals. We denote with $W(t, w; d)$ the uniquely defined solution of (2) at time t fulfilling $W(0, w; d) = w$, under the input $d(\cdot)$. Typically, all α- and ω-limit sets are assumed to be compact. We then introduce the following notions of $\mathcal{W}$-limit set and decomposition.

Definition 1 ($\mathcal{W}$-limit set): Let $\mathcal{W} \subset M$ be a compact invariant set containing all the α- and ω-limit sets of (2), i.e. $\alpha(w) \cup \omega(w) \subset \mathcal{W} \, \forall w \in M$. The set $\mathcal{W}$ is then called a $\mathcal{W}$-limit set for (2).

Definition 2 (Decomposition): Let $\mathcal{W} \in M$ be a $\mathcal{W}$-limit set for (2). A *decomposition* of $\mathcal{W}$ is a finite, disjoint family of compact invariant sets $\mathcal{W}_1, \ldots, \mathcal{W}_K$ (called *atoms* of the decomposition) such that: $\mathcal{W} = \bigcup_{i=1}^{K} \mathcal{W}_i$.
For an invariant set Λ, its attracting and repulsing subsets are defined as follows:

$$\mathfrak{A}(\Lambda) = \{w \in M : |W(t, w, 0)|_\Lambda \to 0 \text{ as } t \to +\infty\},$$
$$\mathfrak{R}(\Lambda) = \{w \in M : |W(t, w, 0)|_\Lambda \to 0 \text{ as } t \to -\infty\}.$$

Define a relation on two atoms $\mathcal{W}_i$ and $\mathcal{W}_j$ by $\mathcal{W}_i \prec \mathcal{W}_j$ whenever $\mathfrak{A}(\mathcal{W}_i) \cap \mathfrak{R}(\mathcal{W}_j) \neq \emptyset$.

Definition 3 (r-cycle, 1-cycle, filtration): Let $\mathcal{W}_1, \ldots, \mathcal{W}_K$ be a decomposition of $\mathcal{W}$, then:

1) An r-cycle ($r \geq 2$) is an ordered r-tuple of distinct indexes $i_1, \ldots, i_r$ such that $\mathcal{W}_{i_1} \prec \cdots \prec \mathcal{W}_{i_r} \prec \mathcal{W}_{i_1}$.
2) A 1-cycle is an index i such that $[\mathfrak{R}(\mathcal{W}_i) \cap \mathfrak{A}(\mathcal{W}_i)] - \mathcal{W}_i \neq \emptyset$.
3) A filtration ordering is a numbering of the $\mathcal{W}_i$ so that $\mathcal{W}_i \prec \mathcal{W}_j \Rightarrow i \leq j$.

Existence of an r-cycle for (2) with $r \geq 2$ is equivalent to existence of a heteroclinic cycle, and existence of a 1-cycle implies existence of a homoclinic orbit. Typically, system (2) is required to satisfy the following assumption on $\mathcal{W}$.

Assumption 1 (No cycle condition): The $\mathcal{W}$-limit set for (2) admits a finite decomposition, i.e. $\mathcal{W} = \bigcup_{i=1}^{k} \mathcal{W}_i$, for some non-empty disjoint compact sets $\mathcal{W}_i$, which shows no cycle between the $\mathcal{W}_i$s and which satisfies a filtration ordering, as detailed in Definitions 2 and 3. Under the specified assumptions, the set $\mathcal{W}$ is said to *satisfy the no-cycle condition under the flow of (2)*.

We recall here the definitions of iISS and Strong iISS for multistable systems as (1).

Definition 4: System (1) is said to satisfy the *iISS property for multistable systems* if it satisfies:

- the zero-global attractivity (zero-GATT) property, namely $\lim_{t \to +\infty} |W(t, w; 0)|_\mathcal{W} = 0$ for all $x \in M$.
- the uniform bounded-energy bounded-state (UBEBS) property, namely there exist $\mathcal{K}_\infty$ functions α, γ, σ and some positive constant c_u such that $\alpha(|W(t, w; d)|_\mathcal{W}) \leq \gamma(|w|_\mathcal{W}) + \int_0^t \sigma(|d(s)|) ds + c_u$ for all $t \geq 0$, all $w \in M$, and all $d(\cdot) \in \mathcal{M}_D$.

It has been established in [8] that a necessary and sufficient condition for iISS in multistable systems is the existence of a smooth iISS-Lyapunov function, namely the existence of a smooth function $V : M \to \mathbb{R}_{\geq 0}$, a continuous positive-definite function ϖ, and $\mathcal{K}_\infty$ functions $\underline{\alpha}, \gamma$ such that the following inequalities hold for all $w \in M$ and all $d \in D$:

$$\underline{\alpha}(|w|_\mathcal{W}) \leq V(w)$$
$$\mathfrak{L}_{f(w,d)} V(w) \leq -\varpi(|w|_\mathcal{W}) + \gamma(|d|).$$

Note that, by virtue of [11, Proposition 4.2.10], we have

$$\frac{d}{dt} V\left(W(t, w; \tilde{d})\right) = \mathfrak{L}_{f(w, \tilde{d}(t))} V\left(W(t, w; \tilde{d})\right),$$

for all $t \geq 0$, all $w \in M$, and all $\tilde{d} \in \mathcal{M}_D$.

Another necessary and sufficient condition for iISS in multistable systems is given by the Bounded-Energy Strongly-Convergent-State (BESCS) property [8], namely the existence of a $\mathcal{K}_\infty$ function σ for which the following implication holds for any $w \in M$ and any $d(\cdot) \in \mathcal{M}_D$:

$$\int_0^{+\infty} \sigma(|d(s)|) \, ds < +\infty \;\Rightarrow\; \lim_{t \to +\infty} |W(t, w, d)|_\mathcal{W} = 0. \tag{3}$$

Definition 5 (Strong iISS for multistable systems): System (1) is said to be *Strongly iISS for multistable systems* if it satisfies the iISS property for multistable systems and, furthermore, it satisfies the asymptotic gain (AG) property with respect to small inputs, namely there is a function $\eta \in \mathcal{K}$ and a positive constant R such that:

$$\|d\| \leq R \;\Rightarrow\; \limsup_{t \to +\infty} |W(t, w; d)|_\mathcal{W} \leq \kappa(\|d\|), \tag{4}$$

for all $w \in M$ and all $d(\cdot) \in \mathcal{M}_D$.

III. Cascade of integral ISS systems - driving system is multistable

We are now going to address the cascade interconnection of such multistable systems. Let M_x and M_z be two geodesically complete, connected, Riemannian manifolds without boundary which have dimension n_x and n_z respectively. Let D be a closed subset of $\mathbb{R}^m$ containing the origin. Consider the cascade system:

$$\dot{x}(t) = g(x(t), d(t)) \tag{5a}$$
$$\dot{z}(t) = f(z(t), x(t), d(t)), \tag{5b}$$

where $g(x, d) : M_x \times D \to T_x(M_x)$ and $f(z, x, d) : M_z \times M_x \times D \to T_z(M_z)$ are two locally Lipschitz continuous mappings, and $d(\cdot) \in \mathcal{M}_D$. We respectively denote with $X(t, x; d)$ and $Z(t, z; X, d)$ the uniquely defined solutions of (5a) and (5b) at time t fulfilling $X(0, x; d) = x$ and $Z(t, z; X, d)$, under the inputs $X(\cdot)$ and $d(\cdot)$. Finally, we denote by $y = (x, z) \in M_x \times M_z$ the joint state and by $Y(t, y; d)$ the uniquely defined solution of (5) at time t fulfilling $Y(0, y; d) = y$ under the input $d(\cdot)$.

We also consider the unperturbed cascade system:

$$\dot{x}(t) = g(x(t), 0) \tag{6a}$$
$$\dot{z}(t) = f(z(t), x(t), 0). \tag{6b}$$

In this Section, we derive sufficient conditions for the iISS stability of the cascade system (5), when both the driving system (6a) and the driven system (6b) satisfy the iISS property for multistable systems. In particular, we assume that the invariant set of the driving system (6) comprises fixed points only, as follows.

Assumption 2: (*Multistability without cycles of driving system*) Assume that all α- and ω-limit sets of (6a) are compact, then let $\mathcal{W}_x$ denote a $\mathcal{W}$-limit set of the driving system (6a). Assume that $\mathcal{W}_x$ satisfies the no-cycle condition under the flow of the driving system (6a) and that each atom of the decomposition of $\mathcal{W}_x$ is a singleton, namely

$$\mathcal{W}_{x,i} = \{x_i\} \text{ with } x_i \in M_x \text{ for all } i = 1, \ldots, K.$$

Remark 1: Restricts the analysis to the case of driving systems whose invariant set consists of fixed points only yields a natural generalization of the Theorems in [9], [10], [12] to the multistability context. Conversely, as shown in [5], the case of periodic orbits and almost-periodic attractors requires a number of additional assumptions on the stability properties of the driven system (e.g. incremental ISS), and it will be tackled in a subsequent publication.

We define for all $i = 1, \ldots, K$ the limit system:

$$\dot{z}(t) = F_i(z(t)) := f(z(t), x_i, 0) \ , \ \ i = 1, \ldots, K \ , \tag{7}$$

obtained by applying the constant input $x(t) \equiv x_i$ for all $t \geq 0$.

Assumption 3: (*No-cycle condition of each limit system*) For all $i \in \{1, \ldots, K\}$, assume that all α- and ω-limit sets of (7) are compact, then let $\mathcal{W}_z^{(i)}$ denote a $\mathcal{W}$-limit set of (7) for $i = 1, \ldots, K$. Each set $\mathcal{W}_z^{(i)}$ is assumed to satisfy the no-cycle condition under the flow of the limit system (7).

Theorem 1: Let Assumption 2 and 3 hold. If:

- the driving system (6a) is iISS in the multistable sense with respect to the set $\mathcal{W}_x$ and input $d(\cdot)$, namely there exists a constant $c_x \geq 0$, a proper positive-definite function $V_x : M_x \to \mathbb{R}^+$, a positive definite function ϖ_x, and $\mathcal{K}_\infty$ functions $\underline{\alpha}_x, \bar{\alpha}_x, \gamma_x$ which satisfy the following inequalities:
$$\underline{\alpha}_x(|x|_{\mathcal{W}_x}) \leq V_x(x) \leq \bar{\alpha}_x\left(|x|_{\mathcal{W}_x}\right) + c_x \tag{8}$$
$$\mathfrak{L}_{g(x,d)} V_x(x) \leq -\varpi_x(|x|_{\mathcal{W}_x}) + \gamma_x(|d|); \tag{9}$$
- for all $i = 1, \ldots, K$, the driven system (6b) is iISS in the multistable sense with respect to the set $\mathcal{W}_z^{(i)}$ and input $d(\cdot)$, $\mathfrak{d}\left[X(\cdot), x_i\right]$, namely there exist a constant $c_{z,i} \geq 0$ a proper positive-definite function $V_{z,i} : M_z \to \mathbb{R}^{\geq}$, a positive definite function $\varpi_{z,i}$, and $\mathcal{K}_\infty$ functions $\underline{\alpha}_{z,i}, \bar{\alpha}_{z,i}, \gamma_{z,i}, \gamma_{d,i}$ which satisfy the following inequalities:
$$\underline{\alpha}_{z,i}(|z|_{\mathcal{W}_z^{(i)}}) \leq V_{z,i}(z) \leq \bar{\alpha}_{z,i}(|z|_{\mathcal{W}_z^{(i)}}) + c_{z,i} \tag{10}$$
$$\begin{aligned} \mathfrak{L}_{f(z,d)} V_{z,i}(z) \leq &-\varpi_{z,i}(|z|_{\mathcal{W}_z^{(i)}}) + \\ &\gamma_{z,i}\left(\mathfrak{d}[x, x_i]\right) + \gamma_{d,i}(|d|); \end{aligned} \tag{11}$$
- for all $i = 1, \ldots, K$, it holds that $\gamma_{z,i}(s) = \mathcal{O}\left(\varpi_x(s)\right)$ as $s \to 0^+$;

then the set

$$\mathcal{W}_\Theta := \bigcup_{i=1}^{K} \Big(\{x_i\} \times \mathcal{W}_z^{(i)} \Big) \tag{12}$$

qualifies as a $\mathcal{W}$-limit set for the cascade (5) which satisfies the no-cycle condition under the flow of (6). Moreover, the cascade (5) is iISS in the multistable sense with respect to the set $\mathcal{W}_\Theta$ and input $d(\cdot)$.

Proof: The driving system is iISS, and thus satisfies the BESCS property (3) for some $\mathcal{K}_\infty$ function σ_x. Let $\gamma_d(s) := \max_{i=1,\ldots,K}\{\gamma_{d,i}(s)\}$ for all $s \geq 0$. Moreover, by local Lipschitz continuity of f, we may consider the following function:

$$\tilde{\rho}(r, s) := \max_{|z|+|x| \leq r, |d| \leq s} \left|f(z, x, d) - f(z, x, 0)\right|_{\mathfrak{g}} \tag{13}$$

Note that $\tilde{\rho}(r, s)$ is continuous, nondecreasing with respect to each argument and vanishes for $s = 0$. By virtue of [6, Corollary IV.5], there exist two $\mathcal{K}_\infty$ functions ρ_z and ρ_d such that:

$$\begin{aligned} \left|f(x, d)\right|_{\mathfrak{g}} &\leq \left|f(z, x, d) - f(z, x, 0)\right|_{\mathfrak{g}} + \left|f(z, x, 0)\right|_{\mathfrak{g}} \\ &\leq \rho_z(|z| + |x| + 1)\rho_d(|d|) + \left|f(z, x, 0)\right|_{\mathfrak{g}}. \end{aligned} \tag{14}$$

Let σ be the $\mathcal{K}_\infty$ function defined as $\sigma(s) := \max\{\gamma_x(s), \sigma_x(s), \gamma_d(s), \rho_d(s)\}$. We are now going to prove the BESCS property of cascade (5) by using function σ. To this aim, pick any $(x, z) \in M_x \times M_z$ and any $d(\cdot) \in \mathcal{M}_D$ such that:

$$\int_0^{+\infty} \sigma(|d(s)|)\, ds < +\infty. \tag{15}$$

For ease of notation, denote $X(t) := X(t, x; d)$. By virtue of (15), integration of (9) over the time interval $[0, +\infty)$ yields:

$$\int_0^{+\infty} \varpi_x(|X(s)|_{\mathcal{W}_x})\, ds \leq V_x(x) + \int_0^{+\infty} \gamma_x(|d(s)|)\, ds < +\infty. \tag{16}$$

Since $\gamma_{z,i}(s) = \mathcal{O}\left(\varpi_x(s)\right)$ as $s \to 0^+$, there exist two constants $\bar{s}, k > 0$ such that $\gamma_{z,\bar{i}}(s) \leq k\varpi_x(s)$ for all $s \in [0, \bar{s}]$. The BESCS implies $\lim_{t\to+\infty} |X(t)|_{\mathcal{W}_x} = 0$, and thus there exists $\bar{i} \in \{1, \ldots, K\}$ such that $X(t) \to x_{\bar{i}}$ as $t \to +\infty$. Since the equilibria x_is are isolated, there exists a time t_0 such that $\mathfrak{d}\left[X(t), x_{\bar{i}}\right] = |X(t)|_{\mathcal{W}_x}$ for all $t \geq t_0$. Since $X(t) \to x_{\bar{i}}$, there exists $T > t_0$ such that $\mathfrak{d}\left[X(s),\, x_{\bar{i}}\right] \leq \bar{s}$ for all $t \geq T$. It then follows that:

$$\begin{aligned} \int_0^{+\infty} \gamma_{z,\bar{i}}\left(\mathfrak{d}\left[X(s),\, x_{\bar{i}}\right]\right)\, ds &\leq \int_0^{T} \gamma_{z,\bar{i}}\left(\mathfrak{d}\left[X(s),\, x_{\bar{i}}\right]\right)\, ds \\ &+ \int_T^{+\infty} k\, \varpi_x(|X(s)|_{\mathcal{W}_x})\, ds < +\infty, \end{aligned} \tag{17}$$

where last inequality holds true due to continuity of $X(t, x; d)$ and (16). By virtue of (15) and (17), integration of (11) yields the following inequality:

$$\begin{aligned} &\underline{\alpha}_{z,\bar{i}}\left(|Z(t, z; X, d)|_{\mathcal{W}_z^{(\bar{i})}}\right) + \int_0^t \varpi_{z,\bar{i}}(|Z(s, z; X, d)|_{\mathcal{W}_z^{(\bar{i})}})\, ds \\ &\leq V_{z,\bar{i}}(Z(t, z; X, d)) + \int_0^t \varpi_{z,\bar{i}}(|Z(s, z; X, d)|_{\mathcal{W}_z^{(\bar{i})}})\, ds \\ &\leq V_{z,\bar{i}}(z) + \int_0^{+\infty} \gamma_{z,\bar{i}}\left(\mathfrak{d}\left[X(s),\, x_{\bar{i}}\right]\right)\, ds \\ &\quad + \int_0^{+\infty} \gamma_{d,\bar{i}}(|d(s)|)\, ds \leq c_z < +\infty, \end{aligned} \tag{18}$$

for some $c_z > 0$ and for all $t \geq 0$. Since c_z does not depend upon t, inequality (18) implies

$$\int_0^{+\infty} \varpi_{z,\bar{i}}(|Z(s,z;X,d)|_{\mathcal{W}_z^{(\bar{i})}})\, ds < +\infty, \tag{19}$$

and boundedness of trajectories for all times, namely:

$$|Z(t,z;X,d)|_{\mathcal{W}_z^{(\bar{i})}} \leq \underline{\alpha}_{z,\bar{i}}^{-1}(c_z) \text{ for all } t \geq 0. \tag{20}$$

From this point onwards, the proof follows along the lines of [8, Lemma 8].

Claim 1: $\liminf_{t\to+\infty} |Z(t,z;X,d)|_{\mathcal{W}_z^{(\bar{i})}} = 0$.

Proof: Assume by contradiction that $\liminf_{t\to+\infty} |Z(t,z;X,d)|_{\mathcal{W}_z^{(\bar{i})}} = \varepsilon > 0$. Therefore, there exists a time $T_\varepsilon > 0$ such that $|Z(t,z;X,d)|_{\mathcal{W}_z^{(\bar{i})}} \geq \varepsilon/2$ for all $t \geq T_\varepsilon$. Since boundedness of trajectories (20) also holds for all $t \geq 0$, it makes sense to define

$$\Omega := \min\left\{\varpi_{z,\bar{i}}(s) \text{ with } \frac{\varepsilon}{2} \leq s \leq \underline{\alpha}_{z,\bar{i}}^{-1}(c_z)\right\}$$

and then observe that:

$$\int_{T_\varepsilon}^{+\infty} \varpi_{z,\bar{i}}(|Z(s,z;X,d)|_{\mathcal{W}_z^{(\bar{i})}})\, ds \geq \int_{T_\varepsilon}^{+\infty} \Omega = +\infty.$$

The latter inequality contradicts (19). ■

We are now going to prove that $\limsup_{t\to+\infty} |Z(t,z;X,d)|_{\mathcal{W}_z^{(\bar{i})}} = 0$. Assume by contradiction that there exists $\bar{\varepsilon} > 0$ and a diverging sequence of times $\{\bar{t}_{n,\bar{\varepsilon}}\}_{n\in\mathbb{N}}$ such that:

$$|Z(\bar{t}_{n,\bar{\varepsilon}},z;X,d)|_{\mathcal{W}_z^{(\bar{i})}} > \bar{\varepsilon}. \tag{21}$$

By virtue of Claim 1, we can select $\varepsilon := \bar{\varepsilon}/2$ so as to obtain the sequence $\left\{\underline{t}_{n,\bar{\varepsilon}/2}\right\}_{n\in\mathbb{N}}$ such that:

$$|Z(\underline{t}_{n,\bar{\varepsilon}/2},z;X,d)|_{\mathcal{W}_z^{(\bar{i})}} \leq \bar{\varepsilon}/2. \tag{22}$$

We can therefore select a subsequence of $\left\{\underline{t}_{n,\bar{\varepsilon}/2}\right\}_{n\in\mathbb{N}}$, say $\{t_m\}_{m\in\mathbb{N}}$, such that, for all $m \in \mathbb{N}$, at least one element of $\{\bar{t}_{n,\bar{\varepsilon}}\}_{n\in\mathbb{N}}$ belongs to the interval $[t_m, t_{m+1}]$. In other words, we select the t_ms in such a way to obtain at least one "spike" of $|Z(t,z;X,d)|_{\mathcal{W}_z^{(\bar{i})}}$ in the interval $[t_m, t_{m+1}]$. We then make the following definitions for all $m \in \mathbb{N}$. Let $t_{m,B}$ denote the first occurrence of $\bar{t}_{n,\bar{\varepsilon}}$ in the interval $[t_m, t_{m+1}]$, namely the time at which the first "spike" occurs in the interval $[t_m, t_{m+1}]$. Let $t_{m,A} \in [t_m, t_{m,B}]$ denote the last time that the state $Z(t,z;X,d)$ leaves the set $\mathcal{P} := \left\{w \in M_z \mid |w|_{\mathcal{W}_z^{(\bar{i})}} \leq \bar{\varepsilon}/2\right\}$, and thus we have

$$|Z(t,z;X,d)|_{\mathcal{W}_z^{(\bar{i})}} \geq \bar{\varepsilon}/2. \tag{23}$$

for all $t \in [t_{m,A}, t_{m,B}]$.

Claim 2: $\lim_{m\to+\infty} (t_{m,B} - t_{m,A}) = 0$.

Proof: It follows along the lines of [8, Claim 1]. ■

Define $z_{m,A} = Z(t_{m,A},z;X,d)$ and $z_{m,B} = Z(t_{m,B},z;X,d)$. By definition, we have, for all $m \in \mathbb{N}$:

$$|z_{m,A}|_{\mathcal{W}_z^{(\bar{i})}} = \bar{\varepsilon}/2 \text{ and } |z_{m,B}|_{\mathcal{W}_z^{(\bar{i})}} \geq \bar{\varepsilon}. \tag{24}$$

The distance of a solution of a differential equation from an initial condition is bounded from above, and for the couple $(z_{m,A}, z_{m,B})$ it reads as:

$$\mathfrak{d}\,[z_{m,B}, z_{m,A}] \leq \int_{t_{m_A}}^{t_{m_B}} |f\,(Z(s,z;X,d), X(s,x;d), d(s))|_{\mathfrak{g}}\, ds \tag{25}$$

Boundedness of trajectories for all times implies that $Z(\cdot)$ and $X(\cdot)$ belong to a compact set $\mathcal{X} \subset M$ for all $t \geq 0$. Then, by virtue of (14), inequality (25) is rewritten as

$$\mathfrak{d}\,[z_{m,B}, z_{m,A}] \leq \bar{F}_0(t_{m,B} - t_{m,A}) + \bar{\rho}_z \int_{t_{m,A}}^{t_{m,B}} \rho_d(|d(s)|)\, ds. \tag{26}$$

with $F_0 := \max_{x,z\in\mathcal{X}} \{|f(z,x,0)|_{\mathfrak{g}}\}$ and $\bar{\rho}_z := \max_{x,z\in\mathcal{X}} \{\rho_z(|z| + |x| + 1)\}$ By virtue of Claim 2 and finiteness of the integral of ρ_d from (15), it follows that

$$\lim_{m\to+\infty} \mathfrak{d}\,[z_{m,B}, z_{m,A}] = 0, \tag{27}$$

which implies

$$\lim_{m\to+\infty} |z_{m,A}|_{\mathcal{W}_z^{(\bar{i})}} = \lim_{m\to+\infty} |z_{m,B}|_{\mathcal{W}_z^{(\bar{i})}},$$

thus representing a contradiction with (24). A proof of the no-cycle condition for $\mathcal{W}_\Theta$ follows along the same lines as in [5, Theorem 3.1]. ■

Definition 6: Given a $\mathcal{K}$ function α, a $\mathcal{K}$ function $\gamma(\cdot)$ is said to be a *class-$\mathcal{H}_\alpha$* function if it satisfies

$$\int_0^1 \frac{(\gamma \circ \alpha)(s)}{s}\, ds < +\infty.$$

If α is the identity, then $\gamma(\cdot)$ is said to be a *class-$\mathcal{H}_I$* function.

Theorem 2: Let Assumption 2 and 3 hold. If:

- the driving system (6a) satisfies the zero-GATT property with respect to the set $\mathcal{W}_x$ and, moreover, for all $i = \{1, \ldots, K\}$ there exists a closed neighborhood $\mathcal{U}_i$ of $\mathcal{W}_{x,i}$ such that:
$$|X(t,x;0)|_{\{x_i\}} \leq \alpha\left(e^{-kt}\beta(|x|_{\{x_i\}})\right), \tag{28}$$
for all $t \geq 0$ and all $x \in \mathcal{U}_i \cap \mathfrak{A}(\mathcal{W}_i)$, and with functions $\alpha, \beta \in \mathcal{K}_\infty$, and a positive constant k;
- for all $i \in \{1, \ldots, K\}$, the driven system (6b) satisfies the integral ISS property wrt to input $\mathfrak{d}\,[X(\cdot), x_i]$ and the invariant set $\mathcal{W}_z^{(i)}$, namely there exists a proper positive-definite function $V_{z,x_i} : M_z \to \mathbb{R}^+$, a positive definite function ϖ_{z,x_i}, functions $\underline{\alpha}_{z,x_i,1}, \gamma_{z,x_i} \in \mathcal{K}_\infty$ which satisfy the following inequalities:
$$\underline{\alpha}_{z,x_i,1}(|z|_{\mathcal{W}_z^{(i)}}) \leq V_{z,x_i}(z) \tag{29}$$
$$\mathfrak{L}_{f(z,d)} V_{z,x_i}(z) \leq -\varpi_{z,x_i}(|z|_{\mathcal{W}_z^{(i)}}) + \gamma_{z,x_i}\left(\mathfrak{d}[x, x_i]\right); \tag{30}$$
- for all $i = 1, \ldots, K$, it holds that γ_{z,x_i} is a class-$\mathcal{H}_\alpha$ function with α as in (28);

then the set $\mathcal{W}_\Theta$ qualifies as a $\mathcal{W}$-limit set for the cascade (5), and satisfies the zero-GATT property and the no-cycle condition under the flow of (6).

Corollary 1: Let the hypothesis of Theorem 2 hold with α being the identity function and γ_{z,x_i} being a locally

Lipschitz function in a neighborhood of zero. Then the set $\mathcal{W}_\Theta$ qualifies as a $\mathcal{W}$-limit set for the cascade (5), and satisfies the zero-GATT property and the no-cycle condition under the flow of (6).

IV. Cascade of Strong iISS systems - driving system is multistable

In this Section, a sufficient condition for the preservation of the Strong iISS stability property of the cascade system (5) is obtained. For ease of presentation, we let $f(z,x,d) = f(z,x)$. We recall that compactness of $\mathcal{W}_x$ implies the existence of a function $\nu_3 \in \mathcal{K}_\infty$ and of a constant $c_3 > 0$ such that:

$$|x| \leq \nu_3(|x|_{\mathcal{W}_x}) + c_3. \tag{31}$$

We are now ready to state the main result of this Section.

Theorem 3: Let Assumptions 2 and 3 hold. If:

- the driving system (5a) is Strongly iISS wrt input d and with input threshold R_x;
- the driven system (5b) is Strongly iISS wrt input $|X(\cdot)|$ and with input threshold R_z;
- it holds that $R_z = c_3 + \tilde{c}$ for some $\tilde{c} > 0$.

then the set $\mathcal{W}_\Theta$ qualifies as a $\mathcal{W}$-limit set for (6) and satisfies the no-cycle condition under the flow of (6). Moreover, the cascade (5) is Strongly iISS wrt the set $\mathcal{W}_\Theta$ and input d, and with input threshold $R := \min \left\{\kappa_x^{-1}\left(\nu_3^{-1}(\tilde{c})\right), R_x\right\}$, where $\kappa_x \in \mathcal{K}_\infty$ denotes the AG gain of the driving system for small inputs.

Proof: Step 0: zero-GATT of $\mathcal{W}_\Theta$

Let $d(t) \equiv 0$ for all $t \geq 0$. Let y denote the joint initial conditions (x,z). Since the driving system satisfies the zero-GATT property, there exists $T_0(x) > 0$ such that:

$$\left\| X(\cdot, x; 0)_{[T_0(x), +\infty)} \right\|_{\mathcal{W}_x} < \nu_3^{-1}(\tilde{c}). \tag{32}$$

Combining (32) and (31) yields: $|X(t,x;0)| \leq \tilde{c} + c_3 < R_z$ for all $t \geq T_0$ which, together with the AG property for small inputs of the driven system, yields:

$$\limsup_{t\to+\infty} |Z(t,z;X(\cdot))|_{\mathcal{W}_z} \leq \kappa_z(R_z), \tag{33}$$

and, in particular, there exists constants $\varepsilon > 0$ and $T_\varepsilon(x,z) \geq T_0(x)$ such that $Y(t,y;0) \in \mathcal{Y}$ for all $t \geq T_\varepsilon(x,z)$, with $\mathcal{Y} := \left\{(x,z) \in M_x \times M_z \mid |x|_{\mathcal{W}_x} \leq \nu_3^{-1}(\tilde{c}), |z|_{\mathcal{W}_z} \leq \kappa_z(R_z) + \varepsilon\right\}$, which proves ultimate boundedness of trajectories. Due to $|X(t,x;0)|_{\mathcal{W}_x} \to 0$ as $t \to +\infty$, there exists an index $i \in \{1,\ldots,K\}$ such that $\mathfrak{d}\left[X(t,x;0), x_i\right] \to 0$ as $t \to +\infty$. We can then denote with Φ_i the semiflow of $\dot{z}(t) = f\left(z(t), X(t,x;0)\right)$ and Θ_i the semiflow of $\dot{z}(t) = f\left(z(t), x_i\right)$, and then notice that Φ_i is asymptotically autonomous with limit flow Θ_i [13]. Assumption 3 ensures that $\mathcal{W}_z^{(i)}$ is nonempty, compact, and connected, and moreover is invariant and chain recurrent for Θ_i. Ultimate boundedness of trajectories ultimately ensures compact closure of all trajectories in $\mathcal{Y}$. We can then invoke [13, Theorem 1.8] to estabilish global attractiveness of $\mathcal{W}_z^{(i)}$ under the semiflow Φ_i. By iterating the same reasoning over all is, we conclude that the set $\mathcal{W}_\Theta$ as in (12) is globally attractive for the autonomous cascade (6). A proof of the no-cycle condition for $\mathcal{W}_\Theta$ follows along the same lines as in [5, Theorem 3.1].

Step 1: ISS wrt small inputs

Let $d(\cdot)$ be an input signal such that $\|d\| \leq R$. Since the driving system satisfies the AG property wrt small inputs, there exists $T_1(x,d) > 0$ such that

$$\left\| X(\cdot, x; d)_{[T_1(x,d), +\infty)} \right\|_{\mathcal{W}_x} < \nu_3^{-1}(\tilde{c}). \tag{34}$$

By adopting a similar reasoning as in Step 0, it is possible to prove ultimate boundedness of trajectories, namely the existence of constants $\varepsilon > 0$ and $T_\varepsilon(x,z,d) > T_1(x,d) > 0$ such that $Y(t,y;d) \in \mathcal{Y}$ for all $t \geq T_\varepsilon(x,z,d) > 0$. Moreover, as shown in the proof of [4, Claim 4], the zero-GATT property of $\mathcal{W}_\Theta$ ensures the existence of a smooth Lyapunov function $U : M_x \times M_z \to \mathbb{R}_{\geq 0}$ and functions $\alpha, \tilde{\alpha}, \delta \in \mathcal{K}_\infty$ such that along the trajectories of (5) it holds:

$$\begin{aligned} &\alpha(|y|_{\mathcal{W}_\Theta}) \leq U(y) \\ &\dot{U}(y,d) \leq -\tilde{\alpha}\left(|y|_{\mathcal{W}_\Theta}\right) + \delta\left(1 + |y|_{\mathcal{W}_\Theta}\right) \delta\left(|d|\right). \end{aligned} \tag{35}$$

Given ultimate boundedness of trajectories and compactness of $\mathcal{W}_\Theta$, it holds that $\delta\left(1 + |y(t)|_{\mathcal{W}_\Theta}\right) \leq \bar{\delta}$, for some constant $\bar{\delta}$ and for all $t \geq T_\varepsilon(x,z,d)$. Estimate (35) can thus be rewritten as

$$\dot{U}(y,d) \leq -\tilde{\alpha}\left(|y|_{\mathcal{W}_\Theta}\right) + \bar{\delta}\,\delta\left(|d|\right). \tag{36}$$

Estimate (36) shows that U qualifies as an ISS Lyapunov function for system (5), for all $t \geq T_\varepsilon(x,z,d)$ and $\|d\| \leq R$, and thus implies the AG property:

$$\limsup_{t\to+\infty} |Y(t,y;d(\cdot)|_{\mathcal{W}_\Theta} \leq \eta\left(\|d\|\right), \tag{37}$$

with $\eta \in \mathcal{K}_\infty$.

Step 2: iISS

Given iISS of the driving system, BESCS is satisfied for some $\mathcal{K}_\infty$ function $\tilde{\sigma}$. Pick any $x \in M_x$. Pick any input $d(\cdot) \in \mathcal{M}_D$ such that $\int_0^{+\infty} \tilde{\sigma}(|d(s)|) < +\infty$. Then there exists an index $i \in \{1,\ldots,K\}$ such that $\mathfrak{d}\left[X(t,x;0), x_i\right] \to 0$ as $t \to +\infty$. In Step 0 we have proved, under the very same condition, that the set $\mathcal{W}_z^{(i)}$ satisfies the zero-GATT property for the semiflow Φ_i. By iterating the same reasoning over all is, we conclude that the left-hand side of (3) implies convergence of trajectories to $\mathcal{W}_\Theta$ for all initial conditions $y = (x,z) \in M_x \times M_z$, namely that

$$\int_0^{+\infty} \tilde{\sigma}\left(|d(s)|\right) ds < +\infty \;\Rightarrow\; \limsup_{t\to+\infty} |Y(t,y;d)|_{\mathcal{W}_\Theta} = 0. \tag{38}$$

Inequality (38) proves the BESCS property for system (5). By a straightforward application of [8, Theorem 1], we conclude the UBEBS property for system (5), thus estabilishing iISS. ∎

Corollary 2: Let Assumptions 2 and 3 hold. If:

- the driving system (5a) satisfies the zero-GATT property wrt the set $\mathcal{W}_x$;

- the driven system (5b) is Strongly iISS with input threshold R_z and wrt input $|X(\cdot)|$;
- $R_z = c_3 + \tilde{c}$ for some $\tilde{c} > 0$.

then the cascade system (5) satisfies the zero-GATT property wrt to the set $\mathcal{W}_\Theta$, which qualifies as a $\mathcal{W}$-limit set for (6) and satisfies the no-cycle condition under the flow of (6).

Corollary 3: Let Assumptions 2 and 3 hold. If:

- the driving system (5a) is iISS wrt the set $\mathcal{W}_x$ and input d;
- the driven system (5b) is Strongly iISS with input threshold R_z and wrt input $|X(\cdot)|$;
- $R_z = c_3 + \tilde{c}$ for some $\tilde{c} > 0$.

then the cascade system (5) is iISS wrt to input d and the set $\mathcal{W}_\Theta$, which qualifies as a $\mathcal{W}$-limit set for (6) and satisfies the no-cycle condition under the flow of (6).

V. Cascade of integral ISS systems - Examples

A. Example 1

Consider the following set of differential equations:

$$\dot{x} = \frac{-x^3 + x}{1 + (-x^3 + x)^2} + d \tag{39a}$$

$$\dot{z} = \frac{-z + x}{1 + z^2} + d. \tag{39b}$$

Subsystem (39a) satisfies the iISS property with respect to the set $\mathcal{W}_x = \{-1, 0, 1\}$. Indeed, let $V_x(x) := \ln\left(1 + \frac{1}{4}(x-1)^2(x+1)^2\right)$. Observe that $V_x(x) \geq \alpha(|x|_{\mathcal{W}})$ with $\alpha(s) := \ln(1 + s^2/4)$ but there exists no class-$\mathcal{K}_\infty$ function of $|x|_{\mathcal{W}}$ bounding $V_x(x)$ from above due to $V_x(0) > 0$. In fact, Lyapunov functions for multistable systems typically attain local minima at sinks at local maxima at sources [4]. By taking the time derivative of V_x along the trajectories of (39a), we obtain:

$$\begin{aligned} \dot{V}_x &= -\frac{(-x^3+x)^2}{(1 + (-x^3+x)^2)\left(1 + \frac{1}{4}(x-1)^2(x+1)^2\right)} \\ &\quad + \frac{1}{\left(1 + \frac{1}{4}(x-1)^2(x+1)^2\right)}\, d \\ &\leq -\varpi_x(|x|_{\mathcal{W}_x}) + |d|, \end{aligned}$$

with $\varpi_x(r) := r^2 \,/\, (4(1 + r^6))$. Furthermore, subsystem (39b) is iISS. Indeed, let $V_{z,i}(z) = \ln\left(1 + \frac{1}{2}(z - x_i)^2\right)$. By taking the time derivative of $V_{z,i}$ along the trajectories of (39b), we obtain:

$$\begin{aligned} dV_z\left(\frac{-z+x}{1+z^2} + d\right) &= \frac{(z - x_i)(-z + x_i - x_i + x)}{(1 + (z-x_i)^2/2)\,(1+z^2)} \\ &\quad + \frac{z - x_i}{1 + (z-x_i)^2/2}\, d \\ &\leq -\frac{(z-x_i)^2}{(1 + (z-x_i)^2/2)\,(1+z^2)} + \gamma_{z,i}(|x - x_i|) + |d|, \end{aligned}$$

with $\gamma_{z,i}(r) := r^2$. Since $\lim_{r\to 0^+} \gamma_{z,i}(r)/\varpi_x(r) = 1$ for all $i \in \{1,2,3\}$, we conclude by virtue of Theorem 1 that the cascade (39) is iISS wrt to input d and $\mathcal{W}_\Theta = \{(1,1), (0,0), (-1,-1)\}$. Note that both subsystems are iISS but not Strong iISS with respect to d, as it has been shown in [7, Example 1].

B. Example 2

In this example, we show that the driven system being iISS wrt a single input equilibrium is not sufficient for global convergence of the cascade. In fact, convergence of all limit equations is required. Consider the set of differential equations:

$$\dot{x} = -x^3 + x \tag{40a}$$

$$\dot{z} = -\mathrm{sat}(z) + \frac{1}{2}\, z\, x. \tag{40b}$$

It is immediate to check that although subsystem (40b) is integral ISS wrt the equilibrium $z = 0$ and the input x, convergence is not global for the limit systems $\dot{z} = -\mathrm{sat}\{z\} + \frac{1}{2}z$ and $\dot{z} = -\mathrm{sat}\{z\} + \frac{1}{2}z$ respectively obtained by setting $x(t) \equiv 1$ and $x(t) \equiv -1$ for all $t \geq 0$, and thus we cannot conclude iISS of cascade (40).

VI. Conclusion

In this work, we have addressed the preservation of the iISS and Strong iISS properties for multistable systems under cascade interconnection. We have shown that most results available in the literature continue to hold in the context of multistable systems, under suitable additional requirements. Further work could address the feedback interconnection of multistable iISS and Strong iISS systems as well as an extension of Theorem 2 for the case of the driving system admitting inputs.

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