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ABSTRACT

Two degree-of-freedom closed spatial linkages can be useful in the design of robotic devices for spatial rigid-body guidance or manipulation. One of the simplest linkages of this type, without any passive degree of freedom on its links, is the revolute-spherical-revolute-spherical (RSRS) four-bar spatial linkage. Although the RSRS topology has been used in some robotics applications, the kinematics study of this basic linkage has unexpectedly received little attention in the literature over the years. Counteracting this historical tendency, this work presents the derivation of the general implicit equation of the surface generated by a point on the coupler link of the general RSRS spatial mechanism. Since the derived surface equation expresses the Cartesian coordinates of the coupler point as a function only of known geometric parameters of the linkage, the equation can be useful for instance in the process of synthesizing new devices. The steps for generating the coupler surface, which is computed from a distance-based parametrization of the mechanism and is algebraic of order twelve, are detailed and a web link where the interested reader can download the full equation for further study is provided. It is also shown how the celebrated sextic curve of the planar four-bar linkage is obtained from this RSRS dodecic.

I. INTRODUCTION

When the end-links of two revolute-spherical (RS) serial chains are rigidly connected, an RSSR four-bar spatial linkage is obtained. This closed chain mechanism has mobility two and, regardless of the considered kinematic inversion, one of these degrees of freedom (DOF) corresponds to the passive rotation of the link connecting the two spherical pairs about the axis defined by their centers of rotation. The kinematic analysis and synthesis of this spatial linkage has been object of extensive study over the years. The literature on the topic is indeed vast, but a good introduction to relevant results and its applications can be found in [1 - 4].

The linkage resulting from the topological permutation of one of the revolute-spherical couples in the RSSR mechanism corresponds to a four-bar spatial linkage of mobility two without any passive degree of freedom on its links: the spatial RSRS linkage, where all kinematic inversions are equivalent (Fig. 1). Surprisingly, this 2-DOF four-bar spatial linkage has received little attention in the literature except for cursory

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where geometric implicit determines expressions interesting, equations parametrization of [12], relevant points the linkages use environment These are for mechanism presented implicit straightforwardly several spatial can in the assembly. This result can be assembled. Thus, for instance, this surface equation can be useful in the process of synthesizing new devices based on the RSRS topology for any application that implies spatial rigid-body guidance or manipulation.

Parametric equations of coupler curves/surfaces of planar and spatial linkages are in general simpler to derive than implicit equations –think for example in the coupler curve equation of the planar four-bar linkage in parametric form. The deduction of implicit expressions for the motion of coupler links is indeed far more complicated. These kinds of equations are important in theoretical kinematics of mechanisms for several reasons, for instance: implicit equations allow the determination of singular points –e.g., ordinary double points– in the workspace of the mechanism and other algebraic, geometric, and kinematic properties of it –e.g. circularity– [4, 11]. Moreover, in implicit expressions the point membership determination is trivial; then it can be straightforwardly detected if a point belongs to the workspace of the linkage. This is a relevant question for any designer and specialized software can use the property to guide the design not only of the mechanism itself but also of the surrounding environment to avoid potential collisions. Other aspects that show the relevance of implicit coupler equations include the synthesis of mechanisms for known workspaces [12], the development of alternative approaches for kinematic synthesis [13], and their use in the validation of numerical approaches for the computation of the workspace of linkages and robotic systems –e.g., [14, 15].

The RSRS coupler surface, which is computed from a distance-based parametrization of the mechanism, is algebraic of order twelve –i.e. a dodecic surface. The highest-order term of the equation is \( \frac{1}{16} s_{1,2}^4 A_{4,5,6}^4 (x^2 + z^2)^2 (x^2 + y^2 + z^2)^4 \), where \( s_{1,2} \) and \( A_{4,5,6} \) correspond to known squared distance and area values of the spatial linkage. The details for generating such surface are provided, as well as a web link where the interested reader can download the full equation for further study. The presented result is additionally validated by showing how the celebrated sextic curve of the planar four-bar linkage is obtained from the derived RSRS dodecic. Both results as well as the distance-based technique introduced for the computation of coupler equations of spatial linkages constitute original contributions to theoretical kinematics of mechanisms.

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II. THE RSRS COUPLER SURFACE

Using a distance-based parametrization; a link connecting two skew revolute axes can be modelled by taking two points on each of these axes and connecting them all with edges to form a tetrahedron. Similarly, a link connecting a revolute axis and a ball joint can be modelled by taking two points on the axis and the center of rotation of the spherical pair and connecting them all with edges to form a triangle. Then, a general RSRS mechanism can be modelled as the bar-and-joint framework involving 7 vertices and 13 edges shown in Fig. 1, where \( P_7 \) corresponds to a point on the coupler link.

Given the point sequence \( P_i, P_j, P_k, P_l, P_m \) in some Euclidean space, the Cayley-Menger determinant of the involved points is defined as

\[
D(i, j, k, l, m) = -\frac{1}{16} |\Delta_{l,j,k,l,m}| = -\frac{1}{16} \begin{vmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 0 & s_{i,j} & s_{i,k} & s_{i,l} \\
1 & s_{i,j} & 0 & s_{j,k} & s_{j,l} \\
1 & s_{i,k} & s_{j,k} & 0 & s_{k,l} \\
1 & s_{i,l} & s_{j,l} & s_{k,l} & 0 \\
1 & s_{i,m} & s_{j,m} & s_{k,m} & s_{l,m}
\end{vmatrix},
\]

where \( s_{i,j} = d_{i,j}^2 = \|p_j - p_i\|^2 \) is the squared distance between \( p_i \) and \( p_j \), with \( p_i \) being the position vector of point \( p_i \) in the global reference frame. Equation (1) is a quadratic polynomial in \( s_{l,m} \) that via Laplace expansion can be expressed as

\[
D(i, j, k, l, m) = -\frac{1}{16} (a_{i,j,k,l,m} s_{l,m}^2 + b_{i,j,k,l,m} s_{l,m} + c_{i,j,k,l,m}),
\]

with the coefficients \( a_{i,j,k,l,m} \), \( b_{i,j,k,l,m} \), and \( c_{i,j,k,l,m} \) as given in Appendix A. For the general point sequence \( P_1, P_2, ..., P_n \), the Cayley-Menger determinant gives \((n - 1)!^2\) times the squared hypervolume of the simplex spanned by the points in \( \mathbb{E}^{n-1} \) [16]. Hence, \( D(i, j, k, l, m) = 0 \) in \( \mathbb{E}^3 \).

According to the notation of Fig. 1, let \( f_1 = D(1,2,7,3,4) \) and \( f_2 = D(5,6,7,3,4) \). Then,

\[
f_1 = a_{1,2,7,3,4} s_{3,4}^2 + b_{1,2,7,3,4} s_{3,4} + c_{1,2,7,3,4} = 0 \quad \text{and} \quad (3)
\]

\[
f_2 = a_{5,6,7,3,4} s_{3,4}^2 + b_{5,6,7,3,4} s_{3,4} + c_{5,6,7,3,4} = 0, \quad \text{(4)}
\]

since points \( P_1, ..., P_7 \) are all embedded in \( \mathbb{E}^3 \). Now, by eliminating \( s_{3,4} \) in the system formed by equations (3) and (4), we get

\[
a_1^2 c_2^2 - 2 a_1 a_2 c_1 c_2 - a_1 b_1 b_2 c_2 + a_1 b_2^2 c_1 + a_2^2 c_1^2 + a_2 b_1^2 c_2 - a_2 b_1 b_2 c_1 = 0,
\]

where \( a_1 = a_{1,2,7,3,4}, b_1 = b_{1,2,7,3,4}, c_1 = c_{1,2,7,3,4}, a_2 = a_{5,6,7,3,4}, b_2 = b_{5,6,7,3,4}, \) and \( c_2 = c_{5,6,7,3,4} \). Equation (5) is a closure polynomial of the RSRS mechanism in terms of the squared lengths of the edges of its associated framework and the unknown squared distances: \( s_{1,7}, s_{2,7}, \) and \( s_{3,7} \). This equation is solely satisfied in the points in \( \mathbb{E}^3 \) where the spatial four-bar linkage can be assembled with the corresponding bar dimensions and orientation of the revolute pairs.

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Finally, to derive the locus of point \( P_7 \) with coordinates \( \mathbf{p}_7 = (x, y, z) \), the location of the ground link has to be defined. Then, we can assume that \( P_1 \) equals the origin of the global reference frame with \( P_2 \) located in the positive side of the \( y \)-axis and that \( P_3 \) is located in the positive quadrant of the \( xy \)-plane. Thus, we have \( \mathbf{p}_1 = (0,0,0) \), \( \mathbf{p}_2 = (0, d_{1,2}, 0) \), and

\[
\mathbf{p}_3 = \left( \frac{2 A_{1,2,3}}{d_{1,2}}, \frac{s_{1,2} - s_{2,3} + s_{1,3}}{2 d_{1,2}}, 0 \right),
\]

where \( A_{1,2,3} = \frac{1}{4} \sqrt{-s_{2,3}^2 + 2 s_{1,2} s_{2,3} + 2 s_{1,3} s_{2,3} - s_{1,3}^2 + 2 s_{1,3} s_{1,2} - s_{1,2}^2} \) is the area of the triangle defined by points \( P_1, P_2, \) and \( P_3 \). Therefore,

\[
\begin{align*}
  s_{1,2} &= x^2 + y^2 + z^2, \\
  s_{2,7} &= x^2 + y^2 + z^2 - 2 d_{1,2} y + s_{1,2}, \quad \text{and} \\
  s_{3,7} &= x^2 + y^2 + z^2 - \frac{4 A_{1,2,3}}{d_{1,2}} x - \frac{s_{1,2} - s_{2,3} + s_{1,3}}{d_{1,2}} y + s_{1,3}.
\end{align*}
\]

Substituting equation (6) into equation (5), fully expanding the result and rearranging terms, we get

\[
\Gamma(x, y, z) \equiv \frac{1}{16} s_{1,2}^4 A_{4,5,6}^2 (x^2 + z^2)^2 (x^2 + y^2 + z^2)^4 + \frac{1}{32} d_{1,2}^2 A_{4,5,6}^2 p_3 (x^2 + z^2) (x^2 + y^2 + z^2)^3 + \frac{1}{143} s_{1,2}^3 p_6 (x^2 + y^2 + z^2)^2 + \frac{1}{212} d_{1,2}^2 p_7 (x^2 + y^2 + z^2)^2 + \frac{1}{216} s_{1,2}^2 q_8 + \frac{1}{214} d_{1,2}^2 q_7 + \frac{1}{215} s_{1,2} q_6 + \frac{1}{214} d_{1,2} q_5 + q_4 + \frac{1}{214} d_{1,2} (s_{1,2} - s_{4,5}) q_3 + \frac{1}{215} s_{1,2} (s_{1,2} - s_{4,5})^2 q_2 + \frac{1}{216} d_{1,2}^2 A_{2,3} A_{5,6,7} q_1 + \frac{1}{216} s_{1,2}^2 A_{5,6,7} (s_{1,2} - s_{4,5})^4 q_0 = 0,
\]

where \( p_i \) and \( q_i \) are homogeneous polynomials of the form \( \sum_{a+b+c=i} a_{a,b,c} x^a y^b z^c \) with at most \( \binom{i+2}{2} \) monomials and \( a_{a,b,c} \) a polynomial in \( s_{1,2}, s_{1,3}, s_{1,4}, s_{2,3}, s_{2,4}, s_{3,5}, s_{3,6}, s_{4,5}, s_{4,6}, s_{4,7}, s_{5,6}, s_{5,7}, \) and \( s_{6,7} \) that can be zero. Additionally,

\[
A_{4,5,6} = \frac{1}{4} \sqrt{-s_{5,6}^2 + 2 s_{5,6} s_{4,6} - s_{4,6}^2 + 2 s_{5,6} s_{4,6} + 2 s_{4,5} s_{5,6} - s_{4,5}^2} \quad \text{and} \quad A_{5,6,7} = \frac{1}{4} \sqrt{-s_{5,6}^2 + 2 s_{5,7} s_{5,6} - s_{5,7}^2 + 2 s_{5,6} s_{6,7} + 2 s_{6,7} s_{5,7} - s_{6,7}^2}.
\]

\( \Gamma(x, y, z) \) is an algebraic surface of degree 12 (a dodecic) that corresponds to the coupler surface, traced by point \( P_7 \), of the general RSRS spatial mechanism. The expressions of the polynomials \( p_i \) and \( q_i \) cannot be included here due to space limitations. However, they can be easily reproduced using a computer algebra system following the steps given above and the interested reader can download the full expression of \( \Gamma(x, y, z) \) from http://www.robot-mechanics.com/rsrsmechanism.html. Examples of the RSRS dodecic are depicted in Fig. 2.

III. THE PLANAR FOUR-BAR SEXTIC

According to the notation of Fig. 1, if

(i) the axes of the revolute pairs, defined by points \( P_1 \) and \( P_2 \), and \( P_5 \) and \( P_6 \), respectively, are parallel and

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(ii) the points \( P_1, P_3, P_4, P_6, \) and \( P_7 \) lie on the \( xz \)-plane, the RSRS spatial mechanism reduces to a planar four-bar linkage, namely the kinematic chain constituted by the revolute centers \( P_1, P_3, P_4, \) and \( P_6 \) with the triangle formed by \( P_4, P_6, \) and \( P_7 \) defining the coupler link. It is well known that the coupler equation of a planar four-bar linkage – i.e., the curve traced by point \( P_7 \) – is a curve of order six [1]. Next we show how such sextic can be obtained from \( \Gamma(x,y,z) \), the RSRS dodecic presented in equation (7).

Using projective geometry arguments, condition (i) implies that the axes of the revolute pairs meet at a point at infinity, say \( P_6 \). Hence, \( d_2 = d_3 = d_4 = d_5 = d_6 = d_7 = \delta \), with \( \delta > 0, \delta \to \infty \). Moreover, condition (ii) implies that the motion of the mechanism is restricted to the surface \( y = 0 \).

After substituting equations (8) and (9) into equation (7), the RSRS coupler surface reduces to a curve that can be rewritten as a radical equation in \( \delta \). Explicitly,

\[
\Gamma(x,y,z)_{y=0, s_{1.2}=\delta^2} = y_{12}(x,z) \frac{\delta^{12}}{\delta} + \sqrt{\frac{s_{1.3} (4 \delta^2 - s_{1.3})}{\delta}} y_{11}(x,z) \frac{\delta^{11}}{\delta^{10}} + y_{10}(x,z) \frac{\delta^{10}}{\delta} + \cdots + \sqrt{\frac{s_{1.3} (4 \delta^2 - s_{1.3})}{\delta}} y_{0}(x,z) \frac{\delta^{12}}{\delta} = 0.
\]

Then, the curve equation can be expressed as

\[
y_{12}(x,z) + \frac{\sqrt{s_{1.3} (4 \delta^2 - s_{1.3})}}{\delta} y_{11}(x,z) + \frac{1}{\delta^{10}} y_{10}(x,z) + \cdots + \frac{\sqrt{s_{1.3} (4 \delta^2 - s_{1.3})}}{\delta^{11}} y_{1}(x,z) + \frac{1}{\delta^{12}} y_{0}(x,z) = 0.
\]

Since \( \delta \to \infty \) and

\[
\lim_{\delta \to \infty} \frac{\sqrt{s_{1.3} (4 \delta^2 - s_{1.3})}}{\delta^n} = \begin{cases} 2^{1/3} \delta_1, & \text{if } n = 1 \\ 0, & \text{if } n > 1 \end{cases}
\]

it is concluded that the curve equation traced by point \( P_7 \) is
\[ y_{12}(x, z) + 2d_{1,3} y_{11}(x, z) = 0. \]  

This curve is algebraic of order twelve, and factorizes as
\[
\Omega(x, z) \equiv y_{12}(x, z) + 2d_{1,3} y_{11}(x, z) = \frac{1}{256} (\psi(x, z) - \phi(x, z) A_{4,6,7})(\psi(x, z) + \phi(x, z) A_{4,6,7}) = 0, \tag{15}
\]

where
\[ A_{4,6,7} = \frac{1}{4} \sqrt{-s_{6,7}^2 + 2s_{4,6}s_{6,7} + 2s_{6,7}s_{4,7} - s_{4,7}^2 + 2s_{4,7}s_{4,6} - s_{4,6}^2} \]

with \( \psi(x, z) \) and \( \phi(x, z) \) as given in Appendix B.

It can be straightforwardly verified that the expression deduced by S. Robertson for the coupler curve of the planar four-bar linkage [17] (a succinct reproduction of this deduction can be found in [1]) is equivalent to \( \psi(x, z) - \phi(x, z) A_{4,6,7} = 0 \). This equation corresponds to the coupler curve of the linkage in the case that, in the triangle associated to the coupler link, \( P_7 \) is located to the left of the vector defined by points \( P_4 \) and \( P_6 \). The equation \( \psi(x, z) + \phi(x, z) A_{4,6,7} = 0 \) corresponds to the coupler curve in the case \( P_7 \) is located to the right of such vector.

From the topology of the RSRS mechanism, according to the notation of Fig. 1, it can seem that a valid configuration of the linkage depends on the orientation of the tetrahedrons \( P_1P_2P_3P_4 \) and \( P_4P_5P_6P_7 \); thus making the condition of equation (5) necessary but not sufficient. Nevertheless, this is not the case. Note that the tetrahedron \( P_1P_2P_3P_4 \) is not a geometric parameter of the RSRS linkage. Then, an RSRS mechanism may have valid configurations with positive and negative orientations of such tetrahedron. In a planar four-bar linkage, this is equivalent to the case of the triangle formed by the crank and the ground link in a given configuration.

The case of the tetrahedron \( P_4P_5P_6P_7 \) is different because it defines the coupler link of the mechanism, being evidently a geometric parameter of the linkage. In this circumstance, observe that at any given configuration of an RSRS mechanism with, for instance, a positive-oriented coupler link tetrahedron, a second valid configuration with opposite orientation can always be obtained since the points \( P_5 \) and \( P_6 \) can be mirrored respect to the plane defined by the points \( P_3 \), \( P_4 \), and \( P_7 \). In other words, a valid configuration of an RSRS mechanism does not depend on the orientation of the coupler link. However, note that this is not the case in a planar four-bar linkage; since, in general, the coupler link cannot be reflected respect to a fixed reference at a particular configuration. This is indeed the reason why equation (15) factorizes in two factors of degree six.

**IV. CONCLUSION**

The derivation of the general implicit equation of the coupler surface of the RSRS spatial mechanism has been presented and its full expression has been made publicly available for those interested in its further study. Despite that this closed chain linkage belongs to the family of four-bar spatial mechanisms, whose members are the simplest movable closed chains and have been the object of extensive analyses over the years, the kinematics study of the RSRS topology has certainly received little attention. Additionally, it has been also shown how the celebrated sextic curve of the planar four-bar...
bar linkage can be obtained from the derived RSRS coupler expression. To the authors’ knowledge, all these results are original contributions to theoretical kinematics of mechanisms; they are obtained thanks to a novel distance-based technique introduced herein to compute implicit coupler equations of spatial linkages. Future work on the topic could focus on determining algebraic, geometric, and kinematic properties of the RSRS mechanism from its coupler equation as well as on exploring the application of the presented technique to the unification of the computation of coupler surfaces of 2-DOF four-bar spatial mechanisms.

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APPENDIX A
This appendix presents the expressions of the coefficients \( a_{i,j,k,l,m} \), \( b_{i,j,k,l,m} \), and \( c_{i,j,k,l,m} \) of equation (2). They can be straightforwardly obtained from iteratively developing \( D(i, j, k, l, m) = -\frac{1}{16} |\Delta_{i,j,k,l,m}| \) according to Laplace along the last row. Thus,

\[
a_{i,j,k,l,m} = -M_{65,55},
\]

\[
b_{i,j,k,l,m} = -M_{61,55} + s_{l,m} M_{62,55} - s_{j,m} M_{63,55} + s_{k,m} M_{64,55} - M_{65,51} + s_{l,l} M_{65,52} - s_{j,l} M_{65,53} + s_{k,l} M_{65,54},
\]

\[
c_{i,j,k,l,m} = -s_{l,l} M_{61,51} + s_{j,l} M_{61,52} - s_{k,l} M_{61,53} + s_{l,m} \left( M_{62,51} - s_{j,l} M_{62,52} + s_{k,l} M_{62,53} \right) - s_{j,m} \left( M_{63,51} - s_{l,l} M_{63,52} + s_{k,l} M_{63,53} \right) + s_{k,m} \left( M_{64,51} - s_{l,l} M_{64,52} + s_{j,l} M_{64,53} \right),
\]

where \( M_{op,qr} \) is the \((q, r)\) minor of the submatrix constructed by removing row \( o \) and column \( p \) from \( \Delta_{i,j,k,l,m} \). For instance,

\[
M_{65,55} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & s_{i,j} & s_{i,k} \\ 1 & s_{i,j} & 0 & s_{j,k} \\ 1 & s_{i,k} & s_{j,k} & 0 \end{bmatrix}.
\]

APPENDIX B
This appendix presents the expressions of the polynomials \( \psi(x, z) \) and \( \phi(x, z) \) of equation (15):

\[
\psi(x, z) = s_{1,3} (x^2 + z^2)^3 - d_{1,3} (s_{4,7} + 3 s_{4,6} - s_{6,7}) x (x^2 + z^2)^2 + r_2 (x^2 + z^2) \\
- d_{1,2} \left( -3 s_{4,7} s_{3,6} - s_{1,4} s_{6,7} + s_{6,7} s_{3,6} - s_{3,6} s_{4,6} - s_{4,7} s_{6,7} - 3 s_{4,6} s_{1,4} + 3 s_{1,4} s_{4,7} + s_{4,6} s_{1,3} - 3 s_{4,6} s_{6,7} - s_{6,7} s_{1,3} + 2 s_{4,6} + 3 s_{4,7} s_{1,3} + s_{6,7}^2 \\
- s_{4,7}^2 \right) x (x^2 + z^2) + c_1 x^2 + c_2 z^2 + d_{1,3} \left( s_{1,4} - s_{4,7} \right) c_3 x \\
+ s_{1,3} s_{6,7} \left( s_{1,4} - s_{4,7} \right)^2,
\]

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\[ \phi(x, z) = -4 \, d_{1,3} \, z \, (x^2 + z^2)^2 + 8 \, s_{1,3} \, x \, z \, (x^2 + z^2) \\
- 4 \, d_{1,3} \left( 2 \, s_{4,6} - s_{6,7} + s_{1,3} - s_{3,6} - s_{4,7} - s_{1,4} \right) z \, (x^2 + z^2) \\
- 8 \, s_{1,3} \left( s_{6,7} + s_{1,4} - s_{4,6} \right) x \, z \\
+ 4 \, d_{1,3} \left( s_{1,3} + s_{6,7} - s_{3,6} \right) (s_{1,4} - s_{4,7}) z, \]  

with  
\[ \begin{align*}
  r_2 &= (-s_{4,7} \, s_{3,6} - s_{4,7} \, s_{4,6} - s_{3,6} \, s_{4,6} + 3 \, s_{4,6} \, s_{1,3} - 2 \, s_{6,7} \, s_{1,3} - s_{1,4} \, s_{6,7} + s_{1,4} \, s_{4,7} \\
  &\quad - s_{4,6} \, s_{6,7} + s_{4,6}^2 - s_{4,6} \, s_{1,4} + 3 \, s_{4,7} \, s_{1,3} + s_{6,7} \, s_{3,6})x^2 \\
  &\quad + (-s_{4,7} \, s_{3,6} - s_{4,6} \, s_{6,7} - s_{1,4} \, s_{6,7} + s_{4,7} \, s_{1,3} - s_{4,7} \, s_{4,6} - s_{3,6} \, s_{4,6} \\
  &\quad + s_{4,6} \, s_{1,3} + s_{1,4} \, s_{4,7} + s_{4,6}^2 - s_{4,6} \, s_{1,4} + s_{6,7} \, s_{3,6})z^2, \\

c_1 &= s_{1,4}^2 \, s_{6,7} + s_{6,7}^2 \, s_{1,4} + s_{1,3} \, s_{6,7}^2 + s_{4,7} \, s_{4,6} \, s_{1,3} - 2 \, s_{4,7} \, s_{1,3} - 3 \, s_{4,6} \, s_{1,4} \, s_{1,3} \\
  &\quad - s_{1,4} \, s_{4,7} \, s_{6,7} + s_{4,6} \, s_{6,7} \, s_{4,7} + s_{1,4} \, s_{1,3} \, s_{6,7} + s_{4,7} \, s_{1,3}^2 + s_{4,7} \, s_{3,6}^2 \\
  &\quad - 2 \, s_{4,6} \, s_{6,7} \, s_{1,3} - s_{4,7} \, s_{1,3} \, s_{6,7} - s_{4,7} \, s_{3,6} \, s_{4,6} - 2 \, s_{4,7} \, s_{1,3} \, s_{3,6} \\
  &\quad - s_{1,4} \, s_{6,7} \, s_{3,6} - s_{1,4} \, s_{4,7} \, s_{3,6} - s_{6,7} \, s_{4,7} \, s_{3,6} + s_{4,7} \, s_{3,6}^2 + 3 \, s_{1,4} \, s_{1,3} \, s_{4,7} \\
  &\quad - s_{1,4} \, s_{6,7} \, s_{4,6} + s_{4,6}^2 \, s_{1,3} + s_{4,6} \, s_{1,4} \, s_{3,6}, \\

c_2 &= -2 \, s_{4,7} \, s_{1,3} \, s_{3,6} - s_{1,4} \, s_{6,7} \, s_{4,6} + s_{4,7} \, s_{1,3}^2 - s_{1,3} \, s_{6,7}^2 + s_{4,7} \, s_{1,3} \, s_{6,7} - s_{1,4} \, s_{4,7} \, s_{6,7} \\
  &\quad + s_{6,7} \, s_{1,4} + s_{4,7} \, s_{3,6}^2 + s_{4,6} \, s_{6,7} \, s_{4,7} - s_{1,4} \, s_{6,7} \, s_{3,6} + s_{4,7} \, s_{3,6} \\
  &\quad - s_{1,4} \, s_{4,7} \, s_{3,6} - 2 \, s_{4,7}^2 \, s_{1,3} - s_{6,7} \, s_{4,7} \, s_{3,6} - s_{4,7} \, s_{3,6} \, s_{4,6} - s_{1,4} \, s_{1,3} \, s_{6,7} \\
  &\quad + 3 \, s_{4,7} \, s_{4,6} \, s_{1,3} + s_{1,4} \, s_{1,3} \, s_{4,7} + s_{4,6} \, s_{1,4} \, s_{3,6} + 2 \, s_{4,6} \, s_{6,7} \, s_{1,3} \\
  &\quad + s_{1,4} \, s_{6,7} - s_{4,6} \, s_{1,3} - s_{4,6} \, s_{1,4} \, s_{1,3}, \\

c_3 &= -2 \, s_{1,4} \, s_{6,7} + s_{4,6} \, s_{6,7} + s_{6,7} \, s_{3,6} - s_{3,6} \, s_{4,6} - s_{4,7} \, s_{1,3} + s_{6,7} \, s_{4,7} - s_{6,7} \, s_{1,3} \\
  &\quad + s_{4,6} \, s_{1,3} + s_{4,7} \, s_{3,6} - s_{6,7}^2. 
\end{align*} \]
REFERENCES


Figure Captions List

Fig. 1 General RSRS spatial mechanism and its associated notation

Fig. 2 Examples of the RSRS dodecic: (a) $s_{1,2} = 4, s_{1,3} = 65, s_{1,4} = 17, s_{2,3} = 65, s_{2,4} = 21, s_{3,5} = 30, s_{3,6} = 19, s_{4,5} = 66, s_{4,6} = 41, s_{4,7} = 42, s_{5,6} = 9, s_{5,7} = 36, s_{6,7} = 49$, (b) $s_{1,2} = 4, s_{1,3} = 73, s_{1,4} = 21, s_{2,3} = 65, s_{2,4} = 33, s_{3,5} = 53, s_{3,6} = 35, s_{4,5} = 69, s_{4,6} = 37, s_{4,7} = 33, s_{5,6} = 14, s_{5,7} = 62, s_{6,7} = 66$, and (c) $s_{1,2} = 4, s_{1,3} = 50, s_{1,4} = 25, s_{2,3} = 50, s_{2,4} = 29, s_{3,5} = 21, s_{3,6} = 34, s_{4,5} = 36, s_{4,6} = 21, s_{4,7} = 22, s_{5,6} = 9, s_{5,7} = 22, s_{6,7} = 21$
Fig. 1
Fig. 2