The Univariate Polynomial Closure Conditions of All Fully-Parallel Planar Robots Derived From a Single Polynomial

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Abstract—The real roots of the univariate polynomial closure condition of a planar parallel robot determine the solutions of its forward kinematics. This paper shows how the univariate polynomials of all fully-parallel planar robots can be derived directly from that of the widely known 3-RPR robot by simply formulating these polynomials in terms of distances and oriented areas. This is a relevant result because it avoids the case-by-case treatment that requires different sets of variable eliminations to obtain the univariate polynomial of each fully-parallel planar robot.

Index Terms—Planar robots, forward kinematics, position analysis, planar Gough-Stewart platforms.

I. INTRODUCTION

A fully-parallel planar robot consists of a *moving platform* connected to a *fixed base* by three serial kinematic chains, or *legs*. Each chain consists of three independent 1-degree-of-freedom lower pair joints, one of which is driven by an *actuator*. Since the displacement of the moving platform is confined to a plane, only revolute (R) and prismatic (P) pairs are considered. Then, the topology of each serial kinematic chain can be described by three letters. There are seven possible combinations: RRR, RPR, RRP, PRR, RPP, PRP, and PPR. The chain PPP is not considered because three P pairs represent three translations in the plane which cannot be independent. The actuated joint is identified by underlining it. Then, since any of the three joints can be actuated, there are twenty one possible legs which can be grouped in four leg architectures —known as the RR, PR, RP and PP legs (see Table I)— attending to the sequence of passive joints. The number of passive prismatic joints in a loop cannot be more than three, otherwise the robot would gain one degree of freedom. This fact limits the number of possible PP legs to one. Then, since there are $\binom{18}{3} = 1140$ feasible combinations of legs of type RR, PR, and RP, and $3 \cdot \binom{18}{2} = 513$ feasible combinations of one leg of type PP with any two legs of the other three types, the total number of different planar fully-parallel robots is 1653. When the three actuators are locked, the robot becomes a rigid structure provided that it is not in a singularity. This permits to classify these 1653 robots into 10 classes attending to the topology of the resulting structure (see Table II). Up to this point, we have followed the usual approach used in robotics to present all possible fully-parallel planar robots but observe that the resulting structures are nothing more than the 5-link Assur kinematic chains, also known as the planar Assur II groups [1].

The forward kinematics of parallel robots consists in finding the possible poses of the mobile platform, for specified values of the actuated joint coordinates, with respect to the fixed base. Thus, it reduces to the position analysis of one of the ten possible structures in Table II. Numerical solutions to this problem are enough for many applications but yield little insight into the problem. The alternative are the exact methods which rely on the computation of a univariate

TABLE I
PLANAR FULLY-PARALLEL ROBOT LEG TYPES

Type RR	Type RP	Type PR	Type PP
R̲RR	RP̲R	R̲PR	
RR̲R	RP̲P	P̲RR	
RR̲R	R̲PP	P̲RR	RPP̲
RP̲R	P̲RP	PRP̲	PR̲P
P̲RR	R̲RP	P̲PR	PPR̲
RR̲P	RR̲P	PP̲R	
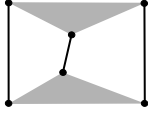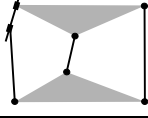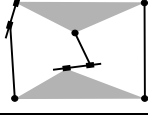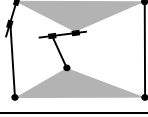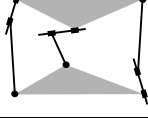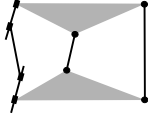
polynomial thus providing what it is usually called a *closed-form solution* to the problem.

In 1987, Li and Matthew solved the position analysis problem of the ten 5-link Assur kinematic chains in closed form for the first time [2]. Their approach consisted in realizing that every Assur II group consisted of two kinematically independent loops which can be classified into only six types: RRRR, RPRR, RPPR, RPRP, RRPP, and RPPP. Then, they reduced the problem to obtain the loop equations for these six loops in general form and compute the resultant in a single variable for the ten feasible combinations. Although outstanding in many ways for its time (the authors even envisaged the possibility of applying their results to planar and spherical robots), this work has been overlooked by the robotics community. In 1996, Merlet tackled the same problem from a different point of view which resulted in a case-by-case analysis [3]. Other solutions for some 5-link Assur kinematic chains have been presented, at least, in [4], [5], [6], [7]. The development of a remarkable unified formulation for the forward kinematics of all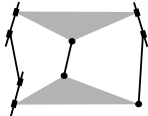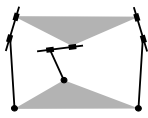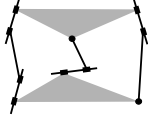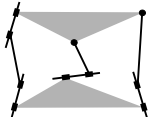 fully-parallel planar robots started in 1995 with Husty's first use of the Grünwald-Blaschke kinematic mapping to solve the forward kinematics of the 3-RPR robot [8]. This formulation was thereafter extended by Hayes, Chen, Zsombor-Murray, and Husty himself who presented their results in a series of publications that culminated with a recent monograph [9], [10], [11], [12], [13], [14], [15], [16], [17]. The approach followed by these authors is based on examining the motion of each leg separately which can be represented by only three types of surfaces: an hyperbolic paraboloid (for legs of type PR and RP), a hyperboloid of one sheet (for legs of type RR), or a plane (for legs of type PP). Then, the forward kinematics problem boils down to find the points of intersection of these three such surfaces. The result is indeed a uniform procedure for solving the forward kinematics of all fully-parallel planar robots but an elimination process is still required to obtain a univariate polynomial for 11 different cases.

A different unifying approach stems from regarding a translational motion as an infinitely small rotation about a point at infinity. It is well-known that a translation in the direction (u_x, u_y) may be represented as a rotation about the ideal point given in homogeneous coordinates by $(-u_y, u_x, 0)$. This is probably the most simple unifying approach to deal with revolute and prismatic joints simultaneously but, using the standard formulations such as those based on independent loop equations, it is difficult to be accommodated. This paper is essentially devoted to show how a coordinate-free formulation based on distances and oriented areas provides a framework within which this idea can be easily applied. This will allow us to conclude that the univariate polynomial of the 3-RPR robot contains all the necessary and sufficient information for solving the forward kinematics of all fully-parallel planar robots.

This paper is organized as follows. Section II briefly reviews the basics of deriving the univariate polynomial closure condition of the 3-RPR robot in terms of distances and oriented areas. Section III shows how to transform any other fully-parallel robot with prismatic joints into a 3-RPR robot with some revolute joint centers located

TABLE II
THE 10 FULLY-PARALLEL PLANAR ROBOT FAMILIES

Robot family (# of different robot topologies)	Associated structure	Leg types
I (56)		RR-RR-RR
II (252)		RP-RR-RR PR-RR-RR
III (216)		RP-PR-RR
IV (252)		RP-RP-RR PR-PR-RR
V (252)		RP-RP-PR PR-PR-RP
VI (63)		PP-RR-RR
VII (216)		PP-RR-RP PP-RR-PR
VIII (112)		RP-RP-RP PR-PR-PR
IX (108)		PP-PR-RP
X (126)		PP-RP-RP PP-RP-PR

at infinity. Section IV shows through examples how to solve — using the univariate polynomial of the 3-RPR robot— the position analysis of different fully-parallel planar robots. Finally, we conclude in Section V with a summary of the main contributions and prospects for future research.

II. THE UNIVARIATE POLYNOMIAL CLOSURE CONDITION OF THE 3-RPR ROBOT IN TERMS OF DISTANCES AND ORIENTED AREAS

In what follows, P_i will denote a point, $\overline{P_i P_j}$ the segment defined by P_i and P_j, and $\triangle P_i P_j P_k$ the triangle defined by P_i, P_j, and P_k.

Moreover, $\mathbf{p}_{ij} = \overrightarrow{P_i P_j}$ and $s_{i,j} = d_{i,j}^2 = \|\mathbf{p}_{i,j}\|^2$.
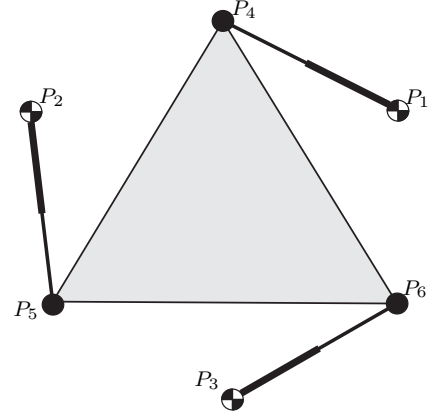

Fig. 1. The general 3-RPR planar robot platform.

Figure 1 shows a general 3-RPR planar robot platform. The center of the three grounded passive revolute joints, ◓, define the base oriented triangle $\triangle P_1 P_3 P_2$ and the three moving passive revolute joints centers, ●, the moving oriented triangle $\triangle P_6 P_4 P_5$. The squared lengths of the active prismatic joints are $s_{1,4}$, $s_{2,5}$, and $s_{3,6}$. Much has been written about this parallel robot because of its practical interest, mechanical simplicity, and rich mathematical properties. A review of the different techniques for solving the forward kinematics of this robot can be found in [18] where it is shown that this problem can be solved by finding the roots of:

$$s_{2,5} = \det\left(\mathbf{I} - \mathbf{Z}_{1,3,2}\mathbf{Z}_{1,6,3} - \mathbf{Z}_{6,4,5}\,\mathbf{Z}_{6,1,4}\right) s_{1,6}, \quad (1)$$

where

$$\mathbf{Z}_{i,j,k} = \frac{1}{2\,s_{i,j}} \left[\begin{array}{cc} s_{i,j} + s_{i,k} - s_{j,k} & -4\,A_{i,j,k} \\ 4\,A_{i,j,k} & s_{i,j} + s_{i,k} - s_{j,k} \end{array} \right]$$

with

$$A_{i,j,k} = \pm\frac{1}{4}\sqrt{(s_{i,j}+s_{i,k}+s_{j,k})^2 - 2\,(s_{i,j}{}^2 + s_{i,k}{}^2 + s_{j,k}{}^2)}$$

being the oriented area of $\triangle P_i P_j P_k$ (defined as positive if P_i, P_j, and P_k are ordered counter-clockwise, and negative otherwise).

Equation (1) expresses the set of values of $s_{1,6}$ compatible with the fixed lengths of the active joints, the base and moving platform side lengths, and the signs of the oriented triangles $\triangle P_1 P_3 P_2$ and $\triangle P_6 P_4 P_5$. The expansion of the determinant in this equation yields a scalar radical equation as a function of the unknown squared distance $s_{1,6}$ which, after clearing radicals, finally yields the univariate polynomial closure condition of the robot, a 6th-degree polynomial equation in $s_{1,6}$ which will be denoted by $\Gamma(s_{1,6}) = 0$. The real roots of this polynomial determine the solutions of the forward kinematics. The interested reader is referred to [18] for details on the above derivation.

III. REPLACING REVOLUTE BY PRISMATIC JOINTS

In this section, we will consider the case in which the three revolute joints connected to the moving platform are replaced with prismatic joints. We will proceed progressively by replacing first one, then two, and finally the three revolute joints. At the end, it will become clear that all other cases can be easily deduced from this analysis.
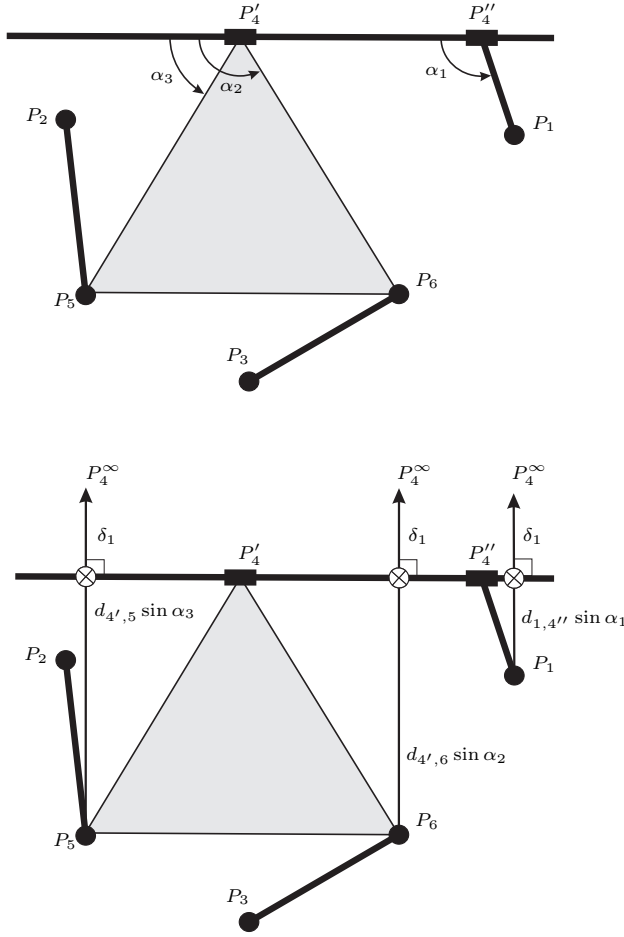
Fig. 2. One revolute joint in the moving platform is substituted by a prismatic joint.

A. Replacing one revolute joint

Let us suppose that the revolute joint centered at P_4 in Fig. 1 is replaced with a prismatic joint, as shown in Fig. 2(top), such that P_4 is split into P_4' and P_4''. This new joint is placed at fixed orientations with respect to the links connected to them. Once an orientation is assigned to the prismatic joint axis with respect to its adjacent links, a set of orientation angles can be defined (in this case α_1, α_2 and α_3) and, as a consequence, an oriented distance can be defined between P_1, P_5 and P_6 and the prismatic joint axis. This defines a set of new points on this axis: those that realize the minimum distance to P_1, P_5 and P_6 which are denoted by $\otimes$ in Fig. 2(bottom). Note that the prismatic joint imposes the alignment of these points but, for the moment, let us suppose that they all are located at the same distance, say δ_i, from P_4^∞. This would imply that they would lie on a circle but, if $\delta_i \to \infty$, they would again lie on a line as imposed by the prismatic join. Actually, the presented geometric transformation simply consists in replacing P_4 by P_4^∞ and substituting the distances between P_4 and P_1, P_5, and P_6, by

$$d_{4,1} = \delta_1 + d_{1,4''} \sin \alpha_1, \quad (2)$$
$$d_{4,5} = \delta_1 + d_{4',5} \sin \alpha_3, \quad (3)$$
$$d_{4,6} = \delta_1 + d_{4',6} \sin \alpha_2, \quad (4)$$

respectively.

It is worth noting that, after the described geometric transformation, it might happen that the orientation of $\triangle P_4^\infty P_5 P_6$ have changed with

respect to that of $\triangle P_4 P_5 P_6$. This has to be taken into account in the univariate polynomial by changing the sign of $A_{6,4,5}$ if needed.

After the introduced changes, the univariate polynomial closure condition can be rewritten as a polynomial in δ_1, that is,

$$\Gamma(s_{1,6}) \Big|_{\substack{s_{1,4} = (\delta_1 + d_{1,4''} \sin \alpha_1)^2 \\ s_{4,5} = (\delta_1 + d_{4',5} \sin \alpha_3)^2 \\ s_{4,6} = (\delta_1 + d_{4',6} \sin \alpha_2)^2}} = \sum_{i=0}^{n} \gamma_i(s_{1,6}) \delta_1^i = 0. \quad (5)$$

Now, factoring out δ_1^n in the above closure condition, we get

$$\delta_1^n \left(\gamma_n(s_{1,6}) + \frac{\gamma_{n-1}(s_{1,6})}{\delta_1} + \frac{\gamma_{n-2}(s_{1,6})}{\delta_1^2} + \dots \right) = 0. \quad (6)$$

Then, the closure condition can be expressed as:

$$\gamma_n(s_{1,6}) + \frac{\gamma_{n-1}(s_{1,6})}{\delta_1} + \frac{\gamma_{n-2}(s_{1,6})}{\delta_1^2} + \dots = 0. \quad (7)$$

Since δ_1 is finally made to tend to infinity, we conclude that the univariate polynomial closure condition for the resulting robot, after the introduction of a prismatic joint, is $\gamma_n(s_{1,6}) = 0$.

B. Replacing two revolute joints

Fig. 3. Two revolute joints connected to the moving platform are substituted by prismatic joints.

Now, let us suppose that the revolute joint centered at P_5 is also replaced with a prismatic joint as indicated in Fig. 3. Following the same reasoning used above, this is equivalent to replacing P_5 with a point at infinity, P_5^∞, and substituting the distances between P_5 and P_2, and P_6, by

$$d_{5,2} = \delta_2 + d_{5'',2} \sin \alpha_6, \quad (8)$$
$$d_{5,6} = \delta_2 + d_{5',6} \sin \alpha_4, \quad (9)$$

with $\delta_2 \to \infty$, respectively. To obtain the substitution for $d_{4,5}$, observe that the angle formed by the two prismatic joints is $\alpha_5 - \alpha_3$. Then,

$$\angle P_5^\infty P_6 P_4^\infty = \pi - \alpha_5 + \alpha_3.$$

Therefore, using the law of cosines, we conclude that:

$$d_{4,5}^2 = d_{5,6}^2 + d_{4,6}^2 - 2 d_{5,6} d_{4,6} \cos(\pi - \alpha_5 + \alpha_3)$$
$$= (\delta_2 + d_{5',6} \sin \alpha_4)^2 + (\delta_1 + d_{4',6} \sin \alpha_2)^2$$
$$+ 2(\delta_2 + d_{5',6} \sin \alpha_4)(\delta_1 + d_{4',6} \sin \alpha_2) \cos(\alpha_5 - \alpha_3).$$

C. Replacing three revolute joints

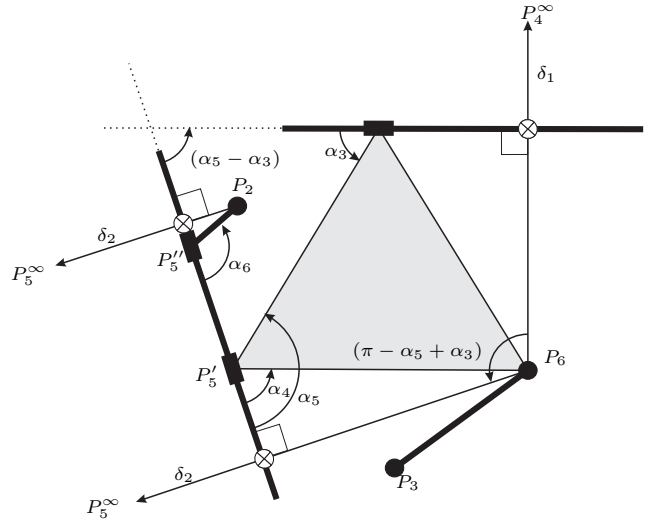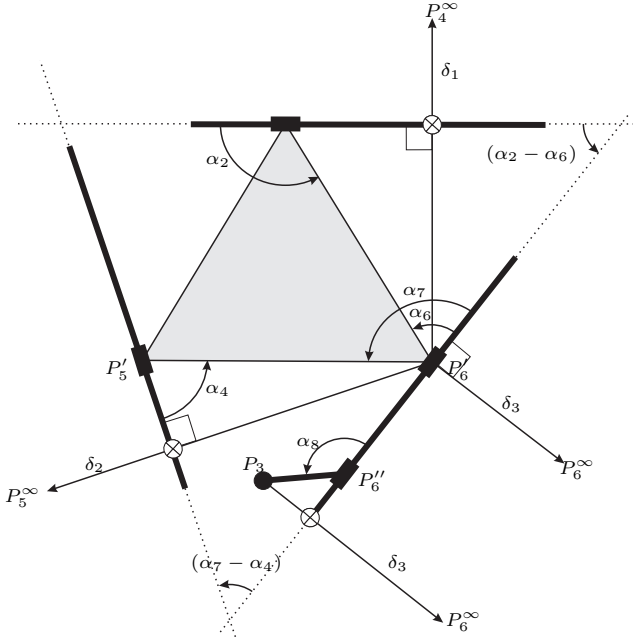with $\delta_1 \to \infty$ and $\delta_2 \to \infty$.

Fig. 4. The three revolute joints connected to the moving platform are substituted by prismatic joints.

Finally, let us assume that the revolute joint centered at P_6 is also replaced with a prismatic joint with the orientation angles indicated in Fig. 4. Using the same reasoning as above, this is equivalent to take P_6 to infinity and substitute the distance between P_6 and P_3 by

$$d_{3,6} = \delta_3 + d_{3,6''}\sin\alpha_8. \tag{10}$$

In this case, to obtain the substitutions for the distances between P_6 and P_5, and P_4, observe that the angles formed by the new prismatic joint axis with those defined by the other two are $\alpha_7 - \alpha_4$ and $\alpha_2 - \alpha_6$ (see Fig. 4). Then,

$$\angle P_5^\infty P_6' P_6^\infty = \pi - \alpha_7 + \alpha_4,$$
$$\angle P_6^\infty P_6' P_4^\infty = \pi - \alpha_2 + \alpha_6.$$

Consequently, using the law of cosines,

$$
\begin{aligned}
d_{5,6}^2 =& (\delta_2 + d_{5',6'}\sin\alpha_4)^2 + \delta_3^2 \\
& + 2(\delta_2 + d_{5',6'}\sin\alpha_4)\delta_3\cos(\alpha_7 - \alpha_4),
\end{aligned} \tag{11}
$$
$$
\begin{aligned}
d_{6,4}^2 =& (\delta_1 + d_{4',6'}\sin\alpha_2)^2 + \delta_3^2 \\
& + 2(\delta_1 + d_{4',6'}\sin\alpha_2)\delta_3\cos(\alpha_2 - \alpha_6),
\end{aligned} \tag{12}
$$

and

$$d_{4,5}^2 = d_{5,6}^2 + d_{4,6}^2 + 2\,d_{5,6}d_{4,6}\cos(\alpha_5 + \alpha_3). \tag{13}$$

The required substitutions for all other cases can be derived following the same procedure. Table III compiles them all thus concluding that the univariate polynomials of all fully-parallel planar robots can be deduced from the univariate polynomial of the 3-RPR robot when expressed in terms of distances and oriented areas. The degree of the resulting univariate polynomial for each family, that is, the maximum number of solutions of the forward kinematics, is also included in Table III and denoted by FK. The practical consequences of the presented formulation are better understood through an example.

IV. EXAMPLE

Let us consider the planar 3-RPR parallel robot in Fig. 1. As an example, let us set $s_{1,2} = 16$, $s_{1,3} = 65$, $s_{2,3} = 73$, $s_{5,6} = 25$, $s_{4,6} = 25$, $s_{4,5} = 36$, and squared input joints $s_{1,4} = 1$, $s_{2,5} = 121$, and $s_{3,6} = 169$. Let us also assume that the orientation of $\triangle P_1 P_2 P_3$ and $\triangle P_6 P_4 P_5$ are positive. Substituting these values in $\Gamma(s_{1,6})$, the following univariate polynomial is obtained

$$
\begin{aligned}
& s_{1,6}^6 - 293.1486\, s_{1,6}^5 + 54084.9111\, s_{1,6}^4 - 3.5587\, 10^6\, s_{1,6}^3 \\
& + 1.0004\, 10^8\, s_{1,6}^2 - 1.2240\, 10^9\, s_{1,6} + 5.3843\, 10^9.
\end{aligned}
$$

This polynomial has a double real root at 32.0000. The corresponding solution of the forward kinematics, for the case in which $P_1 = (0,0)^T$, $P_2 = (4,0)^T$, and $P_3 = (1,8)^T$, appears in the first row of Fig. 5. This example was studied in [18] to compare the different formulations for solving the forward kinematic of the 3-RPR parallel robot presented in [19], [20], and [8], with that based on distances and oriented areas.

When the revolute joint centered at P_4 is replaced with a prismatic joint, a RPP-RPR-RPR planar robot is obtained. This kind of robot belongs to the type II robot family in Table III. If the orientation of the passive prismatic joint with respect to its adjacent links, according to the notation used in Table III, is given by

$$\alpha_1 = \frac{3}{4}\pi,\ \alpha_2 = \frac{5}{4}\pi + \arctan\frac{4}{3},\ \text{and}\ \alpha_3 = \frac{5}{4}\pi,$$

then

$$d_{1,4} = \delta_1 + \frac{1}{\sqrt{2}},\ d_{4,5} = \delta_1 - 3\sqrt{2},\ \text{and}\ d_{4,6} = \delta_1 - \frac{7}{\sqrt{2}}.$$

Substituting these distances in $\Gamma(s_{1,6})$, while keeping all other distances unaltered, and computing the leading coefficient of the resulting polynomial in δ_1, we get the univariate polynomial

$$
\begin{aligned}
& s_{1,6}^6 - 672.0638\, s_{1,6}^5 + 1.4983\, 10^5\, s_{1,6}^4 - 1.4372\, 10^7\, s_{1,6}^3 \\
& + 6.1729\, 10^8\, s_{1,6}^2 - 1.2023\, 10^{10}\, s_{1,6} + 8.7934\, 10^{10}.
\end{aligned}
$$

The real roots of this polynomial are 32.0000, 112.2332, 141.1726, and 342.8691. The corresponding solutions of the forward kinematics appear in the second row of Fig. 5.

Now, if the revolute joint centered at P_6 is also replaced with a prismatic joint, a RPP-RPP-RPR robot is obtained. This kind of robot belongs to the type IV robot family in Table III. If this new prismatic joint orientation with respect to its adjacent links, according to the notation of Table III, is given by

$$\alpha_4 = 2\pi - \arctan\frac{4}{3},\ \alpha_5 = \pi + \arctan\frac{4}{3},$$

and

$$\alpha_6 = \pi + \arctan\frac{12}{5},$$

then

$$
\begin{aligned}
d_{1,4} &= \delta_1 + \frac{1}{\sqrt{2}}, \\
d_{4,5} &= \delta_1 - 3\sqrt{2}, \\
d_{5,6} &= \delta_2 - 4, \\
d_{4,6}^2 &= \delta_1^2 + \delta_2^2 + \sqrt{2}\,\delta_1\delta_2 - 10\sqrt{2}\,\delta_1 - 14\,\delta_2 + 58, \\
d_{3,6} &= \delta_2 - 12, \\
d_{1,6} &= \delta_2 - t.
\end{aligned}
$$

Substituting these values in $\Gamma(s_{1,6})$, while keeping all other distances unchanged, and computing the leading coefficient of the resulting polynomial in δ_1, and then the leading coefficient of the re-

TABLE III
DISTANCE SUBSTITUTIONS FOR EACH ROBOT FAMILY

Family (FK)	Associated structure	Substitutions
I (6)		None
II (6)		$d_{1,4} = \delta_1 + d_{1,4'} \sin\alpha_1$ $d_{4,6} = \delta_1 + d_{4'',6} \sin\alpha_2$ $d_{4,5} = \delta_1 + d_{4'',5} \sin\alpha_3$
III (6)		$d_{1,4} = \delta_1 + d_{1,4'} \sin\alpha_1$ $d_{4,6} = \delta_1 + d_{4'',6} \sin\alpha_2$ $d_{4,5} = \delta_1 + d_{4'',5} \sin\alpha_3$ $d_{1,3} = \delta_2 + d_{1,3'} \sin\alpha_4$ $d_{2,3} = \delta_2 + d_{2,3'} \sin\alpha_5$ $d_{3,6} = \delta_2 + d_{3'',6} \sin\alpha_6$
IV (4)		$d_{1,4} = \delta_1 + d_{1,4'} \sin\alpha_1$ $d_{4,5} = \delta_1 + d_{4'',5} \sin\alpha_3$ $d_{5,6} = \delta_2 + d_{5,6''} \sin\alpha_4$ $d_{4,6}^2 = d_{4,5}^2 + d_{5,6}^2 + 2\,d_{4,5}\,d_{5,6}\,\cos(\alpha_2 - \alpha_5)$ $d_{3,6} = \delta_2 + d_{3,6'} \sin\alpha_6$ $d_{1,6} = \delta_2 + t$
V (4)		$d_{1,4} = \delta_1 + d_{1,4'} \sin\alpha_1$ $d_{4,5} = \delta_1 + d_{4'',5} \sin\alpha_3$ $d_{5,6} = \delta_2 + d_{5,6''} \sin\alpha_4$ $d_{4,6}^2 = d_{4,5}^2 + d_{5,6}^2 + 2\,d_{4,5}\,d_{5,6}\,\cos(\alpha_2 - \alpha_5)$ $d_{3,6} = \delta_2 + d_{3,6'} \sin\alpha_6$ $d_{1,6} = \delta_2 + t$ $d_{1,2} = \delta_3 + d_{1,2''} \sin\alpha_7$ $d_{2,3} = \delta_3 + d_{2'',3} \sin\alpha_8$ $d_{2,5} = \delta_3 + d_{2',5} \sin\alpha_9$

TABLE III (CONT'D)
DISTANCE SUBSTITUTIONS FOR EACH ROBOT FAMILY

Family (FK)	Associated structure	Substitutions
VI (2)		$d_{1,4}^2 = \delta_1^2 + \eta^2 - 2\,\delta_1\,\eta\,\cos(\alpha_6 - \alpha_1)$, with $\eta = \delta_2 + d_{1'',4'}\,\sin\alpha_6$ $d_{4,6} = \delta_1 + d_{4'',6}\,\sin\alpha_2$ $d_{4,5} = \delta_1 + d_{4'',5}\,\sin\alpha_3$ $d_{1,2} = \delta_2 + d_{1',2}\,\sin\alpha_4$ $d_{1,3} = \delta_2 + d_{1',3}\,\sin\alpha_5$ $d_{1,6} = \delta_2 + t$
VII (2)		$d_{1,4}^2 = \delta_1^2 + \eta^2 - 2\,\delta_1\,\eta\,\cos(\alpha_6 - \alpha_1)$, with $\eta = \delta_2 + d_{1'',4'}\,\sin\alpha_6$ $d_{4,6} = \delta_1 + d_{4'',6}\,\sin\alpha_2$ $d_{1,2} = \delta_2 + d_{1',2}\,\sin\alpha_4$ $d_{1,3} = \delta_2 + d_{1',3}\,\sin\alpha_5$ $d_{1,6} = \delta_2 + t$ $d_{5,6} = \delta_3 + d_{5',6}\,\sin\alpha_8$ $d_{4,5}^2 = d_{4,6}^2 + d_{5,6}^2 - 2\,d_{4,6}\,d_{5,6}\,\cos(\alpha_7 - \alpha_3)$ $d_{2,5} = \delta_3 + d_{2,5''}\,\sin\alpha_9$
VIII (2)		$d_{1,4} = \delta_1 + d_{1,4'}\,\sin\alpha_1$ $d_{3,6} = \delta_2 + d_{3,6'}\,\sin\alpha_6$ $d_{1,6} = \delta_2 + t$ $d_{4,5}^2 = \delta_3^2 + \eta_1{}^2 - 2\,\delta_3\,\eta_1\,\cos(\alpha_7 - \alpha_3)$, with $\eta_1 = \delta_1 + d_{4'',5'}\,\sin\alpha_3$ $d_{5,6}^2 = \delta_3^2 + \eta_2{}^2 - 2\,\delta_3\,\eta_2\,\cos(\alpha_4 - \alpha_8)$, with $\eta_2 = \delta_2 + d_{5',6''}\,\sin\alpha_4$ $d_{4,6}^2 = d_{4,5}^2 + d_{5,6}^2 + 2\,d_{4,5}\,d_{5,6}\,\cos(\alpha_2 - \alpha_5)$ $d_{2,5} = \delta_3 + d_{2,5''}\,\sin\alpha_9$
IX (1)		$d_{1,4}^2 = \delta_1^2 + \eta^2 - 2\,\delta_1\,\eta\,\cos(\alpha_6 - \alpha_1)$, with $\eta = \delta_2 + d_{1'',4'}\,\sin\alpha_6$ $d_{4,6} = \delta_1 + d_{4'',6}\,\sin\alpha_2$ $d_{1,2} = \delta_2 + d_{1',2}\,\sin\alpha_4$ $d_{1,6} = \delta_2 + t$ $d_{5,6} = \delta_3 + d_{5',6}\,\sin\alpha_8$ $d_{4,5}^2 = d_{4,6}^2 + d_{5,6}^2 - 2\,d_{4,6}\,d_{5,6}\,\cos(\alpha_7 - \alpha_3)$ $d_{2,5} = \delta_3 + d_{2,5''}\,\sin\alpha_9$ $d_{2,3} = \delta_4 + d_{2,3'}\,\sin\alpha_{11}$ $d_{1,3}^2 = d_{1,2}^2 + d_{2,3}^2 - 2\,d_{1,2}\,d_{2,3}\,\cos(\alpha_{10} - \alpha_5)$ $d_{3,6} = \delta_4 + d_{3'',6}\,\sin\alpha_{12}$
X (1)		$d_{1,4}^2 = \delta_1^2 + \eta^2 - 2\,\delta_1\,\eta\,\cos(\alpha_6 - \alpha_1)$, with $\eta = \delta_2 + d_{1'',4'}\,\sin\alpha_6$ $d_{4,6} = \delta_1 + d_{4'',6}\,\sin\alpha_2$ $d_{4,5} = \delta_1 + d_{4'',5}\,\sin\alpha_3$ $d_{1,6} = \delta_2 + t$ $d_{2,3} = \delta_3 + d_{2,3'}\,\sin\alpha_8$ $d_{3,6} = \delta_3 + d_{3'',6}\,\sin\alpha_9$ $d_{1,2}^2 = \delta_4^2 + \eta_1{}^2 - 2\,\delta_4\,\eta_1\,\cos(\alpha_{10} - \alpha_4)$, with $\eta_1 = \delta_2 + d_{1',2''}\,\sin\alpha_4$ $d_{2,3}^2 = \delta_4^2 + \eta_2{}^2 + 2\,\delta_4\,\eta_2\,\cos(\alpha_{11} - \alpha_8)$, with $\eta_2 = \delta_3 + d_{2'',3'}\,\sin\alpha_8$ $d_{1,3}^2 = d_{1,2}^2 + d_{2,3}^2 - 2\,d_{1,2}\,d_{2,3}\,\cos(\alpha_7 - \alpha_5)$ $d_{2,5} = \delta_4 + d_{2',5}\,\sin\alpha_{12}$

$s_{1,6} = 32.0000$

$s_{1,6} = 32.0000$

$s_{1,6} = 112.2332$

$s_{1,6} = 141.1726$

$s_{1,6} = 342.8691$

$t = 3.9529$

$t = 4.0000$

$t = 11.0000$

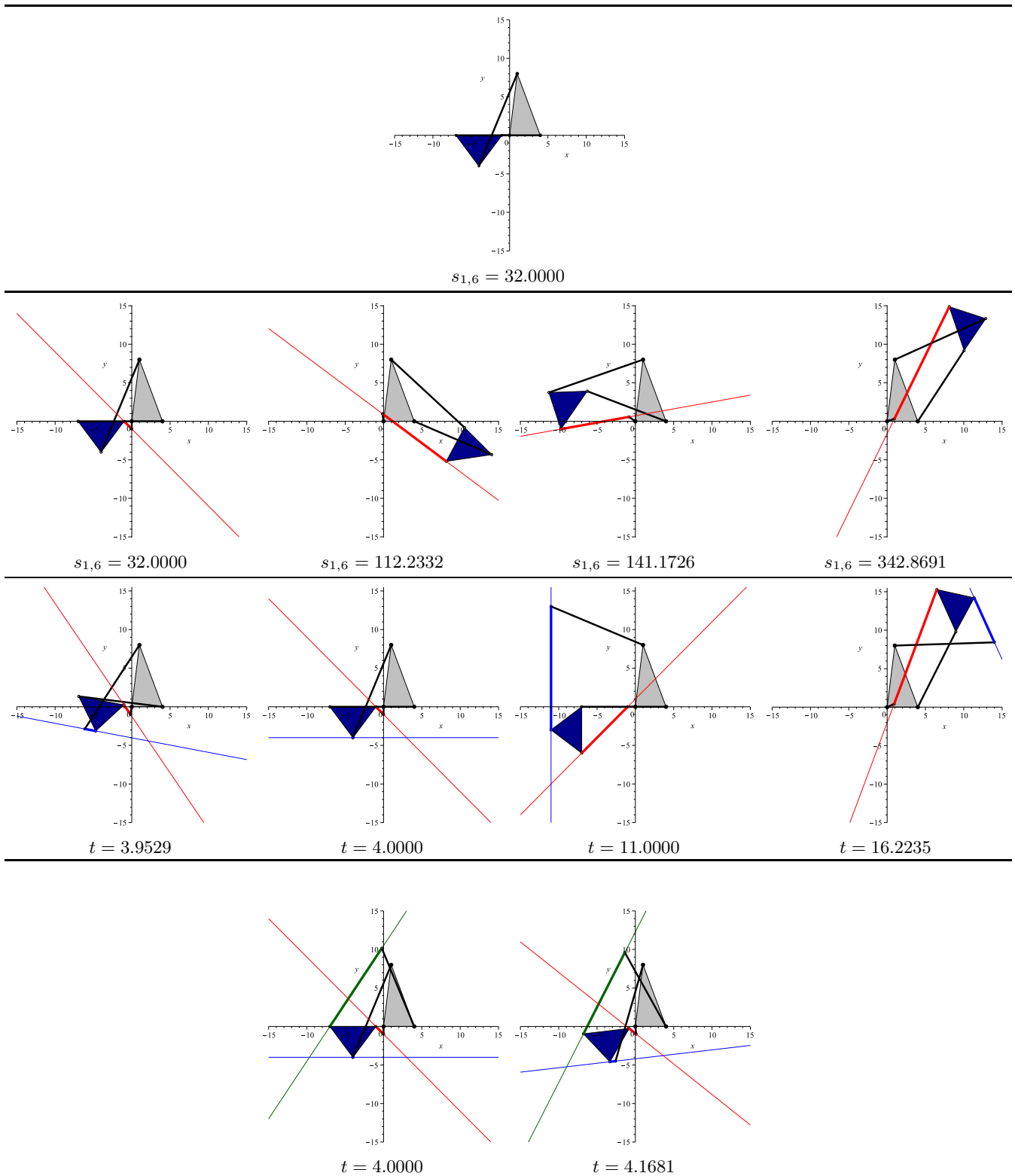$t = 16.2235$

$t = 4.0000$

$t = 4.1681$

Fig. 5. The solutions of the forward kinematics of the analyzed fully-parallel planar robots (see the Maple Worksheet in the supplementary downloadable multimedia material for the numerical computations).

sulting polynomial in δ_2, we get the univariate polynomial

$$t^4 - 35.1765\,t^3 + 410.7778\,t^2 - 1849.7255\,t + 2821.7516,$$

where t is the oriented distance between P_1 and the axis of the prismatic joint that substitutes the revolute joint centered at P_6.

The real roots of this polynomial are 3.9529, 4.0000, 11.0000, and 16.2235. The corresponding solutions of the forward kinematics appear in the third row of Fig. 5.

Finally, if the revolute joint centered at P_5 is also replaced with a prismatic joint, a 3-RPP planar robot platform is obtained. This kind of robot platform belongs to the type VIII robot family in Table III. If the orientation of this prismatic joint with respect to its adjacent links, according to the notation of Table III, is given by

$$\alpha_7 = 2\pi - \arctan\frac{3}{2},$$
$$\alpha_8 = 2\pi - \arctan\frac{3}{2} - \arctan\frac{4}{3},$$
$$\alpha_9 = \pi + \arcsin\frac{3}{\sqrt{13}},$$

then

$$d_{1,4} = \delta_1 + \frac{1}{\sqrt{2}},$$
$$d_{3,6} = \delta_2 - 12,$$
$$d_{1,6} = \delta_2 - t,$$
$$d_{4,5}^2 = \delta_1^2 + \delta_3^2 + \sqrt{\frac{2}{13}}\,\delta_1\delta_3 - 6\sqrt{2}\,\delta_1 - \frac{6}{\sqrt{13}}\,\delta_3 + 18,$$
$$d_{5,6}^2 = \delta_2^2 + \delta_3^2 + \frac{4}{\sqrt{13}}\,\delta_2\delta_3 - 8\,\delta_2 - \frac{16}{\sqrt{13}}\,\delta_3 + 16,$$
$$d_{4,6}^2 = \delta_1^2 + \delta_2^2 + 2\delta_3^2 + \sqrt{\frac{2}{13}}\,\delta_1\delta_3 + \frac{4}{\sqrt{13}}\,\delta_2\delta_3$$
$$- 6\sqrt{2}\,\delta_1 - 8\,\delta_2 - \frac{22}{\sqrt{13}}\,\delta_3 + 34 + \sqrt{2}\,d_{4,5}\,d_{5,6},$$
$$d_{2,5} = \delta_3 - \frac{33}{\sqrt{13}}.$$

Substituting these values in $\Gamma(s_{1,6})$, while keeping all other distances unaltered, and iteratively computing the leading coefficients of the resulting polynomial in δ_1, and then in δ_2, and finally in δ_3, we get the univariate polynomial

$$t^2 - 8.1681\,t + 16.6723.$$

The real roots of this polynomial are 4.000 and 4.1681. The corresponding solutions of the forward kinematics appear in the last row of Fig. 5.

V. CONCLUSION

Regarding a translational motion as an infinitely small rotation about a point at infinity has been a common device to analyze some simple kinematics problems. Applying it to the position analysis of multiloop linkages did not seem to provide any advantage with respect to existing approaches. Nevertheless, it has been shown that, when this idea is combined with a formulation based on distances and oriented areas, the result is a powerful tool that allowed us to conclude that the univariate polynomial closure condition of the 3-RPR robot contains the necessary and sufficient information to solve the position analysis of all fully-parallel robots. No new sets of variable eliminations are required.

Moreover, it is worth reminding that the singularities of a fully-parallel planar robot can be obtained by computing the discriminant of its univariate polynomial. Then, according to the presented results,

these singularities could be obtained from those of the 3-RPR robot through a limit process. In this sense, it can be said that the presented ideas have far-ranging implications as they can be applied to solve other problems than those tackled in this article.

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