Character ratios for finite groups of Lie type, and applications

Martin W. Liebeck

1. Introduction

For a finite group G, a *character ratio* is a complex number of the form $\frac{\chi(x)}{\chi(1)}$, where $x \in G$ and χ is an irreducible character of G. Upper bounds for absolute values of character ratios, particularly for simple groups, have long been of interest, for various reasons; these include applications to covering numbers, mixing times of random walks, and the study of word maps. In this article we shall survey some results on character ratios for finite groups of Lie type, and their applications. Character ratios for alternating and symmetric groups have been studied in great depth also – see for example [**32, 33**] – culminating in the definitive results and applications to be found in [**20**]; but we shall not discuss these here.

It is not hard to see the connections between character ratios and group structure. Here are three well known, elementary results illustrating these connections. The first two go back to Frobenius. Denote by $\mathrm{Irr}(G)$ the set of irreducible characters of G.

Lemma 1.1. *Let G be a finite group, and $x \in G$. The number of pairs $(g,h) \in G \times G$ such that $[g,h] = x$ is equal to*

$$|G| \sum_{\chi \in Irr(G)} \frac{\chi(x)}{\chi(1)}.$$

In particular, x is a commutator if and only if the above sum is nonzero.

This can be found in [**11**, p.13]. A proof of the next result can be found in [**1**, p.43].

Lemma 1.2. *Let G be a finite group, and let C be a conjugacy class in G with representative g. For a positive integer k, and an element $x \in G$, the number of solutions to the equation $g_1 \cdots g_k = x$ with $g_i \in C$ for all i is equal to*

$$\frac{|C|^k}{|G|} \sum_{\chi \in Irr(G)} \frac{\chi(g)^k \chi(x^{-1})}{\chi(1)^{k-1}}.$$

Another way of expressing this result is to say that if $P_k(x)$ is the probability that $g_1 \cdots g_k = x$, where each g_i is chosen uniformly at random in C, then

$$P_k(x) = \frac{1}{|G|}\left(1 + \sum_{\chi\neq 1} \frac{\chi(g)^k \chi(x^{-1})}{\chi(1)^{k-1}}\right). \tag{1}$$

So roughly speaking, if one can show that the character ratio $\frac{\chi(g)}{\chi(1)}$ is suitably "small", then P_k will be close to the uniform distribution U on G.

Exploiting (1), one can establish the following upper bound lemma for the random walk on G based on the conjugacy class C: this random walk starts at the identity, and at each step walks from an element g to gc, where c is chosen uniformly at random from C. This lemma was first proved by Diaconis and Shashahani in [**5**]. In the statement, $||\ \ ||$ denotes the l_1 norm, so that

$$||P_k - U|| = \sum_{x\in G} |P_k(x) - U(x)|.$$

LEMMA 1.3. *Let $C = g^G$ be a conjugacy class of a finite group G, and let $P_k(x)$ be the probability of arriving at x after k steps of the random walk on G based on C. Then*

$$||P_k - U||^2 \leq \sum_{1\neq\chi\in Irr(G)} \left|\frac{\chi(g)}{\chi(1)}\right|^{2k} \chi(1)^2.$$

In the rest of the article we shall present some results on character ratios of finite groups of Lie type, and show how the above three lemmas, together with a host of other methods, can be used to give applications in a wide variety of contexts.

In Section 2 we present some older results, and in Sections 3 and 4 we describe a recent contribution and its applications. Section 5 contains a discussion of some of the ideas in the proof of the new result.

2. Previous results

We shall use the following notation for groups of Lie type in this section. Let K be an algebraically closed field of characteristic $p > 0$, and let $\bar{G}$ be a simple algebraic group over K, of simply connected type. Let F be a Frobenius endomorphism of $\bar{G}$ such that the fixed point group $\bar{G}^F = G(q)$ is a quasisimple group of Lie type over $\mathbb{F}_q$, where $q = p^a$. We define the rank of $G(q)$ to be the rank of the algebraic group $\bar{G}$. For example, we could have $\bar{G} = SL_n(K)$ and $G(q) = SL_n(q)$ or $SU_n(q)$, both of rank $n-1$.

2.1. Character degrees. We begin with some results about the nontrivial irreducible character degrees of $G(q)$. A great deal is known about these. They are polynomials in q (in $\sqrt{q}$ for Suzuki and Ree groups), and the degrees of these polynomials are at least the rank of $\bar{G}$. A classic paper with explicit lower bounds for character degrees is [**18**]. More recently, *gap* results for degrees have appeared: in such results, a polynomial $f(q)$ is specified, usually of much larger degree than that of the smallest nontrivial character, and the irreducible characters of degree less than $f(q)$ are classified explicitly. See [**36**] for a survey of such results. Here is an example, taken from [**16**, 6.2].

THEOREM 2.1. ([**16**]) *Suppose $G = Sp_{2n}(q)$ with q even and $n \geq 4$. There is a collection $\mathcal{W}$ of $q+3$ irreducible characters of G such that if $1 \neq \chi \in \mathrm{Irr}(G)\backslash\mathcal{W}$, then*

$$\chi(1) \geq \frac{(q^{2n}-1)(q^{n-1}-1)(q^{n-1}-q^2)}{2(q^4-1)}.$$

The characters in $\mathcal{W}$ are well understood: their degrees are all of the order of q^{2n-1}, and information about their values is given in [**16**].

We shall also refer later to the following asymptotic result concerning character degrees. For a finite group G and a real number s, define the following "zeta function":

$$\zeta_G(s) = \sum_{\chi \in Irr(G)} \chi(1)^{-s}. \tag{2}$$

THEOREM 2.2. ([**29**])

(i) *Let $G(q)$ be as above, and let h be the Coxeter number of $\bar{G}$. If $s > \frac{2}{h}$, then*
$$\zeta_{G(q)}(s) \to 1 \quad as \quad q \to \infty.$$
(ii) *Fix $t > 0$. Then there is an integer $r(t)$ such that for groups $G = G(q)$ of rank $r \geq r(t)$,*
$$\zeta_{G(q)}(t) \to 1 \quad as \quad |G| \to \infty.$$

The proof of this result uses the theory of Lusztig series of characters (see [**6**] for an exposition); in (i), the constant $\frac{2}{h}$ is sharp, since $G(q)$ has $\sim q^r$ irreducible characters of degree $\sim q^N$, where r is the rank and N the number of positive roots of $\bar{G}$, and $\frac{2}{h}$ is equal to $\frac{r}{N}$.

2.2. Character ratios. We begin with a trivial result on character ratios, which nevertheless is sometimes quite useful.

LEMMA 2.3. *If G is a finite group, $x \in G$, and $\chi \in \mathrm{Irr}(G)$, then $\left|\frac{\chi(x)}{\chi(1)}\right| \leq \frac{|C_G(x)|^{1/2}}{\chi(1)}$.*

This is clear, since $|\chi(x)| \leq |C_G(x)|^{1/2}$ by the orthogonality relations.

The theory of character values for groups of Lie type is much more complicated than that of degrees, and there are still unsolved problems about these. Moreover, even in cases where values are known in principle, they are not given explicitly in a form that can be used to estimate or bound character ratios. One case where this has been done is the paper of Hildebrand [**17**], where character ratios for transvections in $SL_n(q)$ are considered, but for general elements, such explicit calculations are neither appetising not feasible.

The first important general results on character ratios were proved by Gluck. Here is a summary of his results from [**13**].

THEOREM 2.4. ([**13**]) *Let $G = G(q)$ as above, let $x \in G\backslash Z(G)$ and $1 \neq \chi \in \mathrm{Irr}(G)$.*

(i) *If x is a unipotent element, then*

$$\left|\frac{\chi(x)}{\chi(1)}\right| \leq \begin{cases} \frac{1}{\sqrt{q}-1}, & \text{if } q > 4 \\ \frac{3}{4}, & \text{if } q \leq 4. \end{cases}$$

(ii) *If x is a non-unipotent element, then*

$$\left|\frac{\chi(x)}{\chi(1)}\right| \leq \begin{cases} \frac{9}{q}, & \text{if } q > 9 \\ \frac{19}{20}, & \text{if } q \leq 9. \end{cases}$$

Notice that the bound $\frac{1}{\sqrt{q}-1}$ is sharp, as can be seen from the character table of $SL_2(q)$ (given, for example, in [**8**]). The proof in [**13**] is inductive, based on restricting characters to an appropriate parabolic subgroup; although it takes a lot of effort, it is "elementary", in the sense that it does not use Delgne-Lusztig theory.

While Gluck's result leads to some nice consequences when combined with the lemmas in Section 1 (see Subsection 2.3 below), it does not lead to optimal results on random walks. For these, Gluck [**14**] proved the following result, which is asymptotically stronger than Theorem 2.4. In the statement, for an element $g \in GL(V)$, we write $[V,g]$ for the commutator space of g on V.

THEOREM 2.5. ([**14**]) *Suppose $G(q)$ is a quasisimple classical group, with natural module V of dimension n, and let d be a positive integer. There is a positive number $\gamma = \gamma(d,q)$ such that for any $g \in G(q)$ with $\dim[V,g] \leq d$, and any $1 \neq \chi \in \mathrm{Irr}(G(q))$,*

$$\left|\frac{\chi(g)}{\chi(1)}\right| < \chi(1)^{-\gamma/n}.$$

Another result of this flavour was proved in [**21**, 4.3.6]: namely,

$$\frac{|\chi(g)|}{\chi(1)} < q^{-\sqrt{\mathrm{supp}(g)}/481}, \tag{3}$$

where $\mathrm{supp}(g)$ is the codimension of the largest eigenspace of g on V.

The above were the main results in the literature on character ratios of which I am aware, until the new result which we shall discuss in Section 3.

2.3. Applications. Here we discuss some applications of the above results on character ratios. Some further applications will be given in Section 4.

2.3.1. *Commutators in simple groups.* Lemma 1.1 was one of the main tools in the proof of the following result.

THEOREM 2.6. **(The Ore Conjecture)** *Every element of every non-abelian finite simple group is a commutator.*

This conjecture emerged from a 1951 paper of Ore [**31**], after which many partial results were obtained, notably those of Thompson [**35**] for special linear groups, and of Ellers and Gordeev [**9**] proving the result for groups of Lie type over sufficiently large fields $\mathbb{F}_q$ ($q \geq 8$ suffices). The proof was finally completed in [**26**]. One of the main strategies was to show that for an element g of a finite simple group G,

$$\sum_{1 \neq \chi \in Irr(G)} \frac{|\chi(g)|}{\chi(1)} < 1. \tag{4}$$

It then follows from Lemma 1.1 that g is a commutator in G.

Here is a sketch of the proof from [**26**] of Theorem 2.6 for the family of symplectic groups $G = Sp_{2n}(2)$. The argument proceeds by induction. The base cases for the induction are $Sp_{2n}(2)$ with $n \le 6$, and these were handled computationally; of course $Sp_2(2)$ and $Sp_4(2)$ are non-perfect, so Theorem 2.6 does not apply to them.

Let $g \in G$, and write g in block-diagonal form

$$g = \begin{pmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & X_k \end{pmatrix} \in Sp_{2n_1}(2) \times \cdots \times Sp_{2n_k}(2) < G,$$

where $\sum n_i = n$, this decomposition being as refined as possible. If each X_i is a commutator in $Sp_{2n_i}(2)$ then g is a commutator in G. Hence induction gives the conclusion except when either

(1) $k = 1$, or

(2) one of the factors $Sp_{2n_i}(2)$ is $Sp_2(2)$ or $Sp_4(2)$.

We call g *unbreakable* if (1) or (2) holds for every such block-diagonal decomposition of g. Thus to prove the theorem for this case it suffices to show that every unbreakable element g of $G = Sp_{2n}(2)$ with $n \ge 7$ is a commutator.

The first step is to prove that the unbreakable element g has small centralizer, namely

$$|C_G(g)| < 2^{2n+15}.$$

For example, if g is unipotent its unbreakability means that it can have few Jordan blocks, and the possiblities for the centralizers of such elements are given by [**27**, Chapter 7].

Next, Theorem 2.1 shows that there is a collection $\mathcal{W}$ of 5 irreducible characters of G such that

$$\chi(1) \ge \frac{1}{30}(2^{2n} - 1)(2^{n-1} - 1)(2^{n-1} - 4) \text{ for } 1 \ne \chi \in \mathrm{Irr}(G) \setminus \mathcal{W}.$$

Set

$$\Sigma_1(g) = \sum_{\chi \in \mathcal{W}} \frac{|\chi(g)|}{\chi(1)}, \;\; \Sigma_2(g) = \sum_{1 \ne \chi \in \mathrm{Irr}(G) \setminus \mathcal{W}} \frac{|\chi(g)|}{\chi(1)}.$$

Letting $k(G)$ denote the number of conjugacy classes of G, it follows from [**12**, 3.13] that $k(G) \le (15.2) \cdot 2^n$. Also $\sum_{\chi \in Irr(G)} |\chi(g)|^2 = |C_G(g)|$ by the orthogonality relations, from which the Cauchy-Schwartz inequality implies that

$$\sum_{\chi \in Irr(G)} |\chi(g)| \le k(G)^{1/2} |C_G(g)|^{1/2}.$$

Plugging all this into the expression defining $\Sigma_2(g)$, we obtain

$$\Sigma_2(g) < \frac{30\sqrt{15.2} \cdot 2^{n/2} \cdot |C_G(g)|^{1/2}}{(2^{2n} - 1)(2^{n-1} - 1)(2^{n-1} - 4)} < \frac{30\sqrt{15.2} \cdot 2^{n/2} \cdot 2^{n+7.5}}{(2^{2n} - 1)(2^{n-1} - 1)(2^{n-1} - 4)} < 0.6.$$

Bounding $\Sigma_1(g)$ depends on some detailed analysis of the values $\chi(g)$ for the characters $\chi \in \mathcal{W}$, from which one shows that $\Sigma_1(g) < 0.2$.

Hence $\Sigma_1(g) + \Sigma_2(g) < 0.8$, which implies that (4) holds, and hence g is a commutator, as required.

This example gives the flavour of the proof of Theorem 2.6, but it must be said that other families of classical groups over small fields do not yield as easily as this. Indeed the unitary groups presented too many technical obstacles to be handled in this fashion, and a completely different method was used for these in [**26**].

2.3.2. *Width and covering numbers.* For a subset C of a finite group G, and a positive integer k, define $C^k = \{c_1c_2\cdots c_k : c_i \in C\}$. If G is non-abelian simple (or quasisimple) and $C = g^G$ is a non-central conjugacy class, then $C^k = G$ for some k (see [**1**]), and we call the minimal such k the *width* of G with respect to C, written $\mathrm{width}(G, g)$. The *covering number* $\mathrm{cn}(G)$ is defined by

$$\mathrm{cn}(G) = \max\left(\mathrm{width}(G, g) : g \in G \setminus Z(G)\right).$$

Thus $\mathrm{cn}(G)$ is the minimal positive integer m such that $C^m = G$ for all non-central conjugacy classes C of G.

Example Let $G = PSL_n(q)$. The following assertions are proved in [**23, 24**].

1. If $g_1 \in G$ is a transvection, then any product of k conjugates of g_1 fixes an $(n-k)$-space, and so $\mathrm{width}(G, g_1) \geq n$. In fact equality holds, provided $n \geq 3$ and $q \geq 4$.
2. If $g_2 \in G$ is a single unipotent Jordan block, then $\mathrm{width}(G, g_2) = 2$ or 3, provided $q \geq 4$.
3. The covering number $\mathrm{cn}(G) = n$, provided $n \geq 3, q \geq 4$.

The next result uses Gluck's character ratio bound in Theorem 2.4 to prove bounds for covering numbers of all groups of Lie type.

Proposition 2.7. *There is an absolute constant K such that if $G = G(q)$ is a quasisimple group of Lie type of rank r, then $\mathrm{cn}(G) \leq Kr^2$.*

Proof. Let $C = g^G$ be a non-central conjugacy class in G, and let $x \in G$. By Lemma 1.2, if for some positive integer k we show that

$$\Sigma := \left| \sum_{1 \neq \chi \in Irr(G)} \frac{\chi(g)^k \chi(x^{-1})}{\chi(1)^{k-1}} \right| < 1,$$

then $x \in C^k$. By Theorem 2.4, there is a constant $c < \min(3, q^{1/2})$ such that $\left|\frac{\chi(g)}{\chi(1)}\right| \leq \frac{c}{q^{1/2}}$ for all nontrivial irreducible characters χ of G, and hence

$$\Sigma \leq \sum_{\chi \neq 1} \left|\frac{\chi(g)}{\chi(1)}\right|^k \chi(1)^2 \leq \sum_{\chi \neq 1} \left(\frac{c}{q^{1/2}}\right)^k \chi(1)^2.$$

Since $\sum \chi(1)^2 = |G| < q^{4r^2}$, it follows that $\Sigma < 1$ provided $k \geq Kr^2$ for a suitable constant K. The conclusion follows. ∎

The bound in the proposition is quadratic in the rank; this is not the correct order of magnitude – linear bounds can be found in [**10, 22**]. A general upper bound

$$\mathrm{width}(G, g) < C\frac{\log|G|}{\log|g^G|}$$

is proved in [**28**], where C is an absolute constant. None of the above results have sharp, or close to sharp, constants in the bounds; we shall give some sharp constants in Section 4 below.

Notice that the above proof in fact shows that if $P_k(x)$ is the probability that a product of k random conjugates of g is equal to x, and we define

$$||P_k - U||_\infty = |G| \max_{x\in G} |P_k(x) - U(x)|, \tag{5}$$

then $||P_k - U||_\infty \to 0$ as $q \to \infty$, provided $k \geq Kr^2$ (see (1)). We express this by saying that $C^k = G$ *almost uniformly pointwise* as $q \to \infty$.

2.3.3. *Random walks.* Let $G = G(q)$ be a quasisimple group of Lie type, let $C = g^G$ be a non-central conjugacy class, and let P_k be the probability distribution on G after k steps of the random walk on G based on C (as defined in Section 1). Define the *mixing time* to be the smallest integer $t = T(G, g)$ such that $||P_t - U|| < \frac{1}{e}$. (Then for $k \geq t$ we have $||P_k - U|| < e^{-k/t}$.)

The proof of Proposition 2.7, together with Lemma 1.3, shows that the mixing time of the random walk on $G(q)$ based on any conjugacy class is bounded by a quadratic function of the rank. In [**14**], Gluck does better than this for certain classes in classical groups:

PROPOSITION 2.8. *Let $G = G(q)$ be a classical group with natural module V of dimension n, let d be a positive integer, and let $g \in G(q)$ with $\dim[V, g] \leq d$. Define $\gamma = \gamma(d, q)$ as in Theorem 2.5. Then for $|G|$ sufficiently large, the mixing time $T(G, g) \leq 2\gamma^{-1}n$.*

Proof. By Lemma 1.3 and Theorem 2.5,

$$\begin{aligned} ||P_k - U||^2 &\leq \sum_{1\neq\chi\in Irr(G)} \left|\frac{\chi(g)}{\chi(1)}\right|^{2k} \chi(1)^2 \\ &\leq \sum_{1\neq\chi\in Irr(G)} \chi(1)^{-2k\gamma/n}\chi(1)^2 \\ &= \zeta_G(\tfrac{2k\gamma}{n} - 2) - 1, \end{aligned}$$

where ζ_G is as in (2). Now the conclusion follows from Theorem 2.2. ∎

We remark that this result does not give a true linear bound for the mixing time in all cases, since γ depends on q; but when q is fixed and $n \to \infty$, for example, it does give a linear bound. If instead of Theorem 2.5 we use the bound (3) in the above proof, writing $s = \text{supp}(g)$ we obtain

$$||P_k - U||^2 \leq q^{-2k\sqrt{s}/481} \sum \chi(1)^2 \leq q^{-2k\sqrt{s}/481}|G| \leq q^{-2k\sqrt{s}/481} q^{n^2},$$

which gives a bound for the mixing time

$$T(G, g) \leq \frac{250n^2}{\sqrt{s}}.$$

3. A new result

We now present a recent result on character ratios, proved in [**3**]. It applies to a slightly broader class of groups than the quasisimple groups $G(q)$ considered in Section 2 – for example, it applies to $GL_n(q)$ as well as $SL_n(q)$. (Note that Gluck's results [**13, 14**] also apply to the broader class.) Let $\bar{G}$ be a connected reductive algebraic group over an algebraically closed field of characteristic $p > 0$, such that the commutator subgroup $\bar{G}'$ is simple, and let $G(q) = \bar{G}^F$ where F is a Frobenius endomorphism of $\bar{G}$. We assume that the characteristic p is *good* for $\bar{G}$ (meaning that $p \neq 2$ for types B_n, C_n, D_n; $p \neq 2,3$ for exceptional types, and also $p \neq 5$ for type E_8). For a Levi subgroup $\bar{L}$ of $\bar{G}$ that is not a maximal torus, and an element $u \in \bar{L}$, write $\dim u^{\bar{L}}$ for the dimension, as an algebraic variety, of the $\bar{L}$-conjugacy class of u. Define

$$\alpha(\bar{L}) = \max \left(\frac{\dim u^{\bar{L}}}{\dim u^{\bar{G}}} : u \text{ unipotent}, 1 \neq u \in \bar{L} \right),$$

and if L is a maximal torus, set $\alpha(\bar{L}) = 0$.

Example Let $\bar{G} = SL_3(K)$ and let $\bar{L}$ be the Levi subgroup consisting of block diagonal matrices (A, λ) for $A \in GL_2(K)$, $\lambda \det(A) = 1$. There is one class of non-identity unipotent elements in $\bar{L}$, represented by $u = (J_2, 1)$, where J_2 denotes a 2×2 Jordan block, and so

$$\alpha(\bar{L}) = \frac{\dim u^{\bar{L}}}{\dim u^{\bar{G}}} = \frac{2}{4}.$$

Here is the new result.

Theorem 3.1. ([**3**]) *Let $G = G(q)$ as above, and suppose $g \in G$ is an element such that $C_G(g) \leq \bar{L}^F$, where $\bar{L}$ is an F-stable Levi subgroup of an F-stable parabolic subgroup of $\bar{G}$. Then for any non-linear irreducible character $\chi \in \mathrm{Irr}(G)$,*

$$|\chi(g)| \leq f(r) \cdot \chi(1)^{\alpha(\bar{L})},$$

where $f(r)$ depends only on the rank r of $\bar{G}$.

Remarks

(1) As an example, for $G = SL_3(q)$ the theorem applies to all classes of G except
  (a) unipotent elements, and
  (a) regular semisimple elements with centralizer of order $q^2 + q + 1$.
  For instance, for $g = \mathrm{diag}(\lambda, \lambda, \lambda^{-2}) \in G$, the centralizer $C_{\bar{G}}(g) = \bar{L} \cong GL_2(K)$ is an F-stable Levi subgroup of an F-stable parabolic, and $\alpha(\bar{L}) = \frac{1}{2}$ by the previous example, so the theorem says that there is an absolute constant c such that

(6) $$|\chi(g)| \leq c\chi(1)^{\frac{1}{2}}$$

  for all $1 \neq \chi \in \mathrm{Irr}(G)$. Below we give the values of some of the irreducible characters on this class in the case where $q \not\equiv 1 \bmod 3$, using the character table of G in [**34**]:

$\chi(1)$	$q(q+1)$	q^2+q+1	q^3	q^3-1	$q(q^2+q+1)$	$\cdots$
$\chi(g)$	$q+1$	$(q+1)\omega+\omega'$	q	$(q-1)\omega$	$(q+1)\omega+q\omega'$	$\cdots$

  In the table, ω and ω' denote certain roots of unity. From the table we see that the exponent $\frac{1}{2}$ in (6) is sharp.

(2) There are many other examples where the bound $\alpha(\bar{L})$ in the theorem is sharp, or almost sharp. The easiest character to use to see this is the *Steinberg character St*, which on semisimple elements g takes values

$$St(g) = \pm|C_G(g)|_p.$$

For example, if $G = SL_n(q)$ and $g = (\lambda I_{n-1}, \mu) \in G \setminus Z(G)$, then $C_{\bar{G}}(g) = \bar{L} \cong GL_{n-1}$, and

$$St(1) = q^{n(n-1)/2},\ |St(g)| = |GL_{n-1}(q)|_p = q^{(n-1)(n-2)/2} = St(1)^{\frac{n-2}{n}},$$

while $\alpha(\bar{L}) = \frac{n-1}{n-2}$.

As another example, suppose $n = mk$ with $m, k \geq 2$, and that $\lambda_1, \dots, \lambda_m$ are distinct nonzero elements of $\mathbb{F}_q$ and let g be the element $\mathrm{diag}(\lambda_1 I_k, \cdots, \lambda_m I_k) \in G = GL_n(q)$. Then $C_{GL_n}(g) = \bar{L} = (GL_k)^m$ and

$$|St(g)| = |GL_k(q)^m|_p = q^{mk(k-1)/2} = St(1)^{\frac{k-1}{mk-1}},$$

while $\alpha(\bar{L}) = \frac{1}{m}$ for this Levi subgroup. This is close to the exponent $\frac{k-1}{mk-1}$ for k large and m fixed.

As a final example, let G be the exceptional group $E_8(q)$, and suppose $g \in G$ is a semisimple element with centralizer a Levi subgroup $\bar{L}$ of type E_7. Then

$$|St(g)| = |E_7(q)|_p = q^{63} = St(1)^{\beta},$$

where $\beta = \frac{63}{120}$, while $\alpha(\bar{L}) = \frac{17}{29}$.

(3) How restrictive is the condition on $C_G(g)$ in the hypothesis of the theorem? Well, for example if $G = GL_n(q)$ then all elements g satisfy the hypothesis except for those having semisimple part s such that $C_G(s) \cong GL_a(q^b)$ for some a, b with $ab = n$; this includes unipotent elements (for which $a = n, b = 1$). One can similarly enumerate the exceptions for other types.

(4) For exceptional groups, the values $\alpha(\bar{L})$ are computed explicitly in [**3**]. For example, for $G = E_8(q)$, they are as follows:

$\bar{L}'$	E_7	D_7	$\triangleright E_6$	D_6	A_7	rest
$\alpha(\bar{L})$	$\frac{17}{29}$	$\frac{9}{23}$	$\frac{11}{29}$	$\frac{9}{29}$	$\frac{15}{56}$	$\leq \frac{1}{4}$

(5) For G of classical type, explicit values of $\alpha(\bar{L})$ are only obtained in [**3**] for certain specific Levi subgroups $\bar{L}$ (such as GL_{n-1} and GL_m^k as mentioned above); but the following general bound is proved for all Levi subgroups $\bar{L}$:

$$\alpha(\bar{L}) \leq \frac{1}{2}\left(1 + \frac{\dim \bar{L}}{\dim \bar{G}}\right).$$

(6) Finally, we comment on the function $f(r)$ in the conclusion of Theorem 3.1. An explicit $f(r)$ is obtained in [**3**]; in particular, if the rank $r \geq 9$ and $q \geq r^2 + 1$, one can take

$$f(r) = 2^{2r+\sqrt{2r}+3}(r!)^2.$$

We shall offer some comments on the proof of Theorem 3.1 in Section 5.

For $G = SL_n(q)$ or $GL_n(q)$, several further results are proved in [**3**]. One is the direct analogue of Theorem 3.1 for *Brauer* characters χ of G. The other is the following, which applies to *all* elements of G, not just those satisfying the centralizer hypothesis of the theorem.

THEOREM 3.2. ([**3**]) *There is a function $h : \mathbb{N} \to \mathbb{N}$ such that the following statement holds. For any $n \geq 5$, any prime power q, any irreducible complex* *character φ of $G := GL_n(q)$ or $SL_n(q)$, and any non-central element $g \in G$,*

$$|\varphi(g)| \leq h(n) \cdot \varphi(1)^{1-\frac{1}{2n}}.$$

4. Applications of Theorem 3.1

We shall describe some applications of Theorem 3.1 – some with the flavour of the applications in Section 2.3, and some with different flavours. Details can be found in [**3**].

4.1. Width and mixing time. Theorem 3.1 has some quite striking applications in this direction. Here is one such.

PROPOSITION 4.1. *Let $G = G(q)$ be an exceptional group of Lie type, and suppose $g \in G$ is such that $C_G(g) \leq \bar{L}^F$, where $\bar{L}$ is an F-stable Levi subgroup of an F-stable parabolic of $\bar{G}$. Write $C = g^G$.*

(i) *For sufficiently large q, the mixing time $T(G, g) \leq 3$.*
(ii) *$C^6 = G$ almost uniformly pointwise as $q \to \infty$; in particular,* $\text{width}(G, g) \leq 6$ *for large q.*

Proof. For (i), Lemma 1.3 together with Theorem 3.1 gives

$$\begin{aligned} ||P_k - U||^2 &\leq \textstyle\sum_{\chi \neq 1} \left| \frac{\chi(g)}{\chi(1)} \right|^{2k} \chi(1)^2 \\ &\leq f(r)^{2k} \textstyle\sum_{\chi \neq 1} \chi(1)^{2k(\alpha-1)+2} \\ &= f(r)^{2k} \left(-1 + \zeta_G(2k(1-\alpha) - 2)\right), \end{aligned}$$

where $\alpha = \alpha(\bar{L})$. Consider for example $G = E_8(q)$. By Remark (4) after Theorem 3.1, we have $\alpha \leq \frac{17}{29}$. Taking $k = 3$, check that $2k(1-\alpha) - 2 \geq 6.\frac{12}{29} - 2 > \frac{2}{h}$ holds, where $h = 30$ is the Coxeter number of G. Hence the conclusion of (i) holds for type E_8 by Theorem 2.2. Other exceptional types are handled in the same way.

Part (ii) is proved in similar fashion to (i), using (instead of Lemma 1.3) the bound

$$||P_k - U||_\infty \leq \sum_{\chi \neq 1} \left| \frac{\chi(g)}{\chi(1)} \right|^k \chi(1)^2$$

which follows from (1), where the ∞-norm is as defined in (5). ∎

Note that there are elements g for which the mixing time $T(G, g)$ is greater than 2 – for example, elements for which $\dim g^{\bar{G}} < \frac{1}{2} \dim \bar{G}$ – so the bound in part (i) of the proposition is sharp.

Theorem 3.1 leads to similar bounds for mixing times and width for classical groups: if $G = G(q)$ is classical of rank r, and $g \in G$ is as in Proposition 4.1, then for large q,

$$T(G, g) \leq r + 2, \quad \text{and also } T(G, g) \leq \left\lceil \left(2 + \frac{2}{h}\right) \frac{\dim \bar{G}}{\dim \bar{G} - \dim \bar{L}} \right\rceil.$$

Also Theorem 3.2 leads to a bound of $T(G, g) \leq 2n + 3$ for *all* non-central elements g in $G = SL_n(q)$, for large q.

4.2. Representation varieties. Let Γ be a finitely presented group, and K an algebraically closed field of characteristic $p > 0$. The *representation variety* of Γ in dimension n over K is defined to be

$$R_{n,K}(\Gamma) := \mathrm{Hom}(\Gamma, GL_n(K)).$$

For $q = p^a$, the $\mathbb{F}_q$-points of this variety are $\mathrm{Hom}(\Gamma, GL_n(q))$. For certain finitely presented groups Γ, this finite space can be studied using character-theoretic methods, and in particular Theorem 3.1 can be applied to estimate its size. The groups Γ in question are the *Fuchsian groups*. Recall that a co-compact Fuchsian group Γ of genus g, having d elliptic generators of orders $m_1, \ldots, m_d$, has a presentation of the form

$$\langle a_1, b_1, \ldots, a_g, b_g,\ x_1, \ldots, x_d \mid x_1^{m_1} = \cdots = x_d^{m_d} = 1,\ x_1 \cdots x_d \prod_1^g [a_i, b_i] = 1 \rangle,$$

where the *measure* of Γ is

$$\mu(\Gamma) = 2g - 2 + \sum_{i=1}^{d} \left(1 - \frac{1}{m_i}\right) > 0$$

(see [**2**]). Here for notational convenience we are assuming that Γ is orientation-preserving. We also assume that Γ is not virtually abelian, which means that $2g + d \geq 3$.

Examples of such Fuchsian groups include surface groups (where $d = 0$) and triangle groups (where $g = 0, d = 3$).

In order to illustrate how Theorem 3.1 can be applied in this area, we sketch a proof of the following result.

Proposition 4.2. *There are constants $N(\Gamma), M(\Gamma)$ depending only on Γ, and a function $k : \mathbb{N} \to \mathbb{N}$ such that the following holds. For any $n \geq N(\Gamma)$, and any prime power $q > k(n)$ such that $q \equiv 1 \bmod m_i$ for all i,*

$$|\mathrm{Hom}(\Gamma, GL_n(q))| > q^{-M(\Gamma)} |GL_n(q)|^{\mu(\Gamma)+1}.$$

Sketch Proof. Take n large, and let $G = SL_n(q)$. Fix i and write $n = km_i + s$, where $0 \leq s < m_i$. Let $\lambda_1, \ldots, \lambda_{m_i}$ be the m_i^{th} roots of 1 in $\mathbb{F}_q$, and define

$$g_i = \mathrm{diag}\,(\lambda_1 I_{k+1}, \ldots, \lambda_s I_{k+1},\ \lambda_{s+1} I_k, \ldots, \lambda_{m_i} I_k) \in GL_n(q).$$

We can choose g_i to lie in G except in one particular case (when $s = 0$, m_i is even and k is odd) which is dealt with using a slight variant of the method to follow. So assume $g_i \in G$. We have $C_{GL_n(q)}(g_i) = \bar{L}^F$, where $\bar{L}$ is the Levi subgroup

$$\bar{L} = GL_{k+1}^s \times GL_k^{m_i - s}$$

of $GL_n(K)$.

The first step of the proof is to establish that $\alpha(\bar{L}) \leq \frac{1}{m_i}$ for this Levi subgroup. Given this, Theorem 3.1 implies that

$$|\chi(g_i)| < f(n) \chi(1)^{\frac{1}{m_i}} \tag{7}$$

for all $\chi \in \mathrm{Irr}(G)$.

Let $C_i = g_i^G$. Calculation gives

$$|C_i| = |G : C_G(g_i)| > q^{-m_i} |G|^{1 - \frac{1}{m_i}}. \tag{8}$$

Let $\mathbf{C} = (C_1, \ldots, C_d)$ and define

$$\mathrm{Hom}_{\mathbf{C}}(\Gamma, G) = \{\phi \in \mathrm{Hom}(\Gamma, G) : \phi(x_i) \in C_i \text{ for } i = 1, \ldots, d\}.$$

An extension of Lemma 1.2 gives the formula

$$|\mathrm{Hom}_{\mathbf{C}}(\Gamma, G)| = |G|^{2g-1}|C_1| \cdots |C_d| \sum_{\chi \in Irr(G)} \frac{\chi(g_1) \cdots \chi(g_d)}{\chi(1)^{d-2+2g}}. \tag{9}$$

Now (7) gives

$$\sum_{\chi \neq 1} \frac{|\chi(g_1) \cdots \chi(g_d)|}{\chi(1)^{d-2+2g}} < f(n)^d \sum_{\chi \neq 1} \chi(1)^{-\mu(\Gamma)},$$

so provided $\mu(\Gamma) > \frac{2}{h} = \frac{2}{n}$, Theorem 2.2 implies that the above sum tends to 0 as $q \to \infty$. Hence for large q, (8) and (9) give

$$|\mathrm{Hom}_{\mathbf{C}}(\Gamma, G)| > \frac{1}{2}|G|^{2g-1}|C_1| \cdots |C_d| > |G|^{\mu(\Gamma)+1} q^{-M(\Gamma)}$$

for a suitable constant $M(\Gamma)$, as required. ■

Now a simple application of the Lang-Weil estimate [**19**] for the number of $\mathbb{F}_q$ points in a K-variety implies that the dimension of the representation variety $R_{n,K}(\Gamma)$ is at least the degree of the leading term in the right hand side of the inequality in Proposition 4.2, so that

$$\dim R_{n,K}(\Gamma) \geq n^2 \left(\mu(\Gamma) + 1\right) - M(\Gamma).$$

Further argument gives the following quite precise estimate in [**3**] for the dimension.

THEOREM 4.3. *If $\mu(\Gamma) > 2$ and $n \geq N(\Gamma)$, then*

$$\dim R_{n,K}(\Gamma) = n^2 \left(\mu(\Gamma) + 1\right) - c,$$

where $-1 \leq c \leq \mu(\Gamma) + 1 + \sum m_i$.

Estimates are also proved in [**3**] for the dimensions of the varieties $\mathrm{Hom}(\Gamma, G(K))$ for all the other types of simple algebraic groups $G(K)$.

4.3. Random generation. For a finite group G, and a finitely generated group Γ, define $P_\Gamma(G)$ to be the probability that a randomly chosen homomorphism in $\mathrm{Hom}(\Gamma, G)$ is surjective. For example, if Γ is the free group F_2 of rank 2,

$$P_{F_2}(G) = \mathrm{Prob}\left(G = \langle x, y \rangle \text{ for random } x, y \in G\right),$$

while if Γ is a triangle group $\langle x_1, x_2, x_3 : x_i^{m_i} = 1,\ x_1x_2x_3 = 1\rangle$, then

$$P_\Gamma(G) = \mathrm{Prob}\left(G = \langle g_1, g_2, g_3 \rangle\right)$$

for random $g_1, g_2, g_3 \in G$ satisfying the triangle relations.

There is a large literature concerning the behaviour of these random generation probabilities $P_\Gamma(G)$ for various Γ when G is a finite simple group – see [**25**] for a survey. When $G = G(q)$ is of Lie type and Γ is a Fuchsian group, many new results of this type are obtained in [**3**]. Here is one such result.

THEOREM 4.4. ([**3**]) *Let* Γ *be a Fuchsian group, and assume that* $\mu(\Gamma) > \max\left(2,\ 1 + \sum \frac{1}{m_i}\right)$. *Then for* $n > N(\Gamma)$,

$$P_\Gamma(SL_n(q)) \to 1$$

where the limit is taken as $q \to \infty$ *through prime powers that are congruent to* 1 *modulo* m_i *for all* i. *In particular, for sufficiently large such* q, $SL_n(q)$ *is generated by a tuple of elements satisfying the defining relations of* Γ.

An obvious observation starts off the proof of this: if a homomorphism in $\mathrm{Hom}(\Gamma, G)$ is not surjective, then it maps Γ into some maximal subgroup of G, and hence

$$1 - P_\Gamma(G) \le \frac{\sum_{M \ max\ G} |\mathrm{Hom}(\Gamma, M)|}{|\mathrm{Hom}(\Gamma, G)|}.$$

We showed in Proposition 4.2 how character theory can be applied to obtain bounds for $|\mathrm{Hom}(\Gamma, G)|$, and this is one ingredient of the proof of the theorem.

5. Remarks on the proof of Theorem 3.1

In this final section we sketch some of the ideas involved in the proof of Theorem 3.1. These are easiest to describe for the case where $G = GL_n(q)$, so we focus most of the discussion on this case.

Let $G = G(q) = \bar{G}^F$ be as in Theorem 3.1, and let $g \in G$ be such that $C_G(g) \le \bar{L}^F$, where $\bar{L}$ is an F-stable Levi subgroup of an F-stable parabolic $\bar{P}$. Write $L = \bar{L}^F, P = \bar{P}^F$, and let $P = QL$ where Q is the unipotent radical of P.

Now let χ be an irreducible character of G. As in [**6**, p.49], denote the *Harish Chandra restriction* of χ to L by $\psi = {}^*R_L^G(\chi)$, defined as follows, for $l \in L$:

$$\psi(l) = \frac{1}{|Q|} \sum_{u \in Q} \chi(ul).$$

The condition $C_G(g) \le L$ implies that $\psi(g) = \chi(g)$, hence in particular

(A) $|\chi(g)| \le \psi(1) = {}^*R_L^G(\chi)(1)$.

Next, standard results on Deligne-Lusztig characters $R_L^G(\chi)$ in [**4**, Chapter 9] can be used to prove

(B) The number of irreducible constituents of the character $\psi = {}^*R_L^G(\chi)$ of L is at most $A(r)$, a function depending only on the rank r of $\bar{G}$.

Indeed, one can take $A(r)$ to be $|W(\bar{G})|^2$, where $W(\bar{G})$ is the Weyl group of $\bar{G}$.

The next steps of the proof in [**3**] for general $G = G(q)$ are rather technical and complicated, so at this point we focus the discussion on the case where $G = GL_n(q)$. For this group the irreducible characters were found by Green [**15**]; to describe them we shall adopt the notation of Dipper and James [**7**, 4.7].

For elements s_1, s_2 lying in a finite extension of $\mathbb{F}_q$, write $s_1 \sim s_2$ if they are roots of the same irreducible polynomial over $\mathbb{F}_q$. Let $\mathcal{S}$ be a set of $\sim$ class representatives and select a total order $\le$ on $\mathcal{S}$. Define an *index* to be a symbol

$$\iota = \left(\begin{array}{ccc|ccc} d_1 & \cdots & d_N & k_1 & \cdots & k_N \\ s_1 & \cdots & s_N & \lambda^{(1)} & \cdots & \lambda^{(N)} \end{array} \right) \tag{10}$$

such that for all i,

(i) $s_i \in \mathcal{S}$, s_i has degree d_i over $\mathbb{F}_q$, and $s_1 < s_2 < \cdots < s_N$,
(ii) $k_i > 0$ and $\lambda^{(i)}$ is a partition of k_i,
(iii) $\sum_1^N d_i k_i = n$.

The indices correspond bijectively with the conjugacy classes of $G = GL_n(q)$. Indeed, the index (10) corresponds to a class with representative

$$v(\iota) = su \tag{11}$$

where the semisimple part s has k_i diagonal $d_i \times d_i$ blocks corresponding to s_i for each i, and the unipotent part u has Jordan decomposition determined by the partitions $\lambda^{(1)}, \ldots \lambda^{(N)}$.

For each index ι as in (10) there is a corresponding irreducible character of $G = GL_n(q)$, defined as follows. The basic case is that in which $dk = n$ and the index is

$$\left(\begin{array}{c|c} d & k \\ s & \lambda \end{array} \right).$$

In this case there is an irreducible character $S(s, \lambda)$ of $GL_n(q)$ of degree

$$q^{d \sum_k (k-1)\lambda_k} \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q - 1)}{\prod_h (q^{hd} - 1)} \tag{12}$$

where the product in the denominator is over the hook lengths h in the Young tableau corresponding to λ.

For a general index (10), the tensor product $\bigotimes_1^N S(s_i, \lambda_i)$ is a character of the Levi subgroup $\prod GL_{d_i k_i}(q)$ of G; extend this to a character of a parabolic of G with this Levi subgroup, and induce to G. This character

$$\left(\bigotimes_1^N S(s_i, \lambda_i) \right) \uparrow G \tag{13}$$

is the irreducible character of G corresponding to the index ι in (10). Call this character χ_v, where $v = v(\iota)$ as in (11).

It is apparent from the above that $\chi_v(1)$ is a monic polynomial in q. A key fact we need is the following:

(C) The degree of the polynomial $\chi_v(1)$ in q is equal to $\frac{1}{2} \dim v^{\bar{G}}$, where $\bar{G} = GL_n(K)$.

This seemingly miraculous fact follows quickly from the case where $\chi_v = S(s, \lambda)$, in which case it can be verified directly using the formula (12). It is a special case of a much more general observation of Lusztig (see [**30**, (13.4.3)]).

We now return to consideration of the Levi subgroup L of $G = GL_n(q)$ (where $C_G(g) \leq L$). Here L is a direct product $\prod_1^k GL_{n_i}(q)$, and as above its irreducible characters take the form χ_u, where $u = (u_1, \ldots, u_k) \in L$. The last fact we require is

(D) If χ_u is a constituent of $\psi = {}^*R_L^G(\chi_v)$, then $u^{\bar{G}}$ is contained in the closure of $v^{\bar{G}}$.

The proof of this reduces to the case where $L = GL_a(q) \times GL_b(q)\,(a + b = n)$, $\chi_v = S(s, \lambda)$ and $\chi_u = (S(s, \alpha) \otimes S(s, \beta)) \uparrow G$, where α, β are partitions of a, b and

λ is a partition of n; here (D) amounts to showing the $\lambda \trianglerighteq (\alpha, \beta)$ in the dominance order on partitions of n.

Given the facts (A) – (D), we can deduce Theorem 3.1 for $G = GL_n(q)$ as follows. Let $g \in G$ with $C_G(g) \leq L \leq P$ as above, and let $\chi = \chi_v \in \mathrm{Irr}(G)$. By (C),

$$\chi_v(1) \approx q^{\frac{1}{2}\dim v^{\bar{G}}}. \tag{14}$$

Let $\psi = {}^*R_L^G(\chi_v)$. Then by (A),

$$|\chi_v(g)| \leq \psi(1). \tag{15}$$

If χ_u is an irreducible constituent of ψ, then by (D),

$$\dim u^{\bar{G}} \leq \dim v^{\bar{G}}. \tag{16}$$

By (C), $\chi_u(1) \approx q^{\frac{1}{2}\dim u^{\bar{L}}}$. Now recall the definition of $\alpha = \alpha(\bar{L})$ in Section 3. From this it follows that $\dim u^{\bar{L}} \leq \alpha \dim u^{\bar{G}}$. Hence by (14) and (16),

$$\chi_u(1) \lesssim q^{\frac{\alpha}{2}\dim u^{\bar{G}}} \leq q^{\frac{\alpha}{2}\dim v^{\bar{G}}} \approx \chi_v(1)^{\alpha}.$$

Hence by (B),

$$\psi(1) \leq A(r)\chi_v(1)^{\alpha},$$

which implies the conclusion of Theorem 3.1 by (15).

References

[1] Z. Arad and M. Herzog (eds.), *Products of Conjugacy Classes in Groups*, Springer Lecture Notes **1112**, Springer-Verlag, Berlin, 1985.

[2] A.F. Beardon, *The Geometry of Discrete Groups*, Graduate Texts in Mathematics **91**, Springer-Verlag, New York, 1983.

[3] R. Bezrukavnikov, M. W. Liebeck, A. Shalev, and P. H. Tiep, Character ratios, growth and random walks on finite groups of Lie type, preprint, 2016.

[4] R.W. Carter, *Finite Groups of Lie type: Conjugacy Classes and Complex Characters*, Wiley, Chichester, 1985.

[5] P. Diaconis and M. Shahshahani, Generating a random permutation with random transpositions, *Z. Wahrscheinlichkeitstheorie Verw. Gebiete* **57** (1981), 159-179.

[6] F. Digne and J. Michel, *Representations of Finite Groups of Lie Type*, London Math. Soc. Student Texts 21, Cambridge Univ. Press, 1991.

[7] R. Dipper and G.D. James, Identification of the irreducible modular representations of $GL_n(q)$, *J. Algebra* **104** (1986), 266–288.

[8] L. Dornhoff, *Group Representation Theory, Part A*, Marcel Dekker, 1971.

[9] E.W. Ellers and N. Gordeev, On the conjectures of J. Thompson and O. Ore, *Trans. Amer. Math. Soc.* **350** (1998), 3657–3671.

[10] E.E. Ellers, N. Gordeev and M. Herzog, Covering numbers for Chevalley groups, *Israel J. Math.* **111** (1999), 339–372.

[11] F. G. Frobenius, Über Gruppencharaktere, Sitzber. Preuss. Akad. Wiss. (1896) 985–1021; reprinted in Gesammelte Abhandlungen, Vol. 3 (Springer, Heidelberg, 1968) 1–37.

[12] J. Fulman and R. Guralnick, Bounds on the number and sizes of conjugacy classes in finite Chevalley groups with applications to derangements, *Trans. Amer. Math. Soc.* **364** (2012), 3023–3070.

[13] D. Gluck, Sharper character value estimates for groups of Lie type, *J. Algebra* **174** (1995), 229–266.

[14] D. Gluck, Characters and random walks on finite classical groups, *Adv. Math.* **129** (1997), 46-72.

[15] J.A. Green, The characters of the finite general linear groups, *Trans. Amer. Math. Soc.* **80** (1955), 402–447.

[16] R. Guralnick and P.H. Tiep, Cross characteristic representations of even characteristic symplectic groups, *Trans. Amer. Math. Soc.* **356** (2004), 4969–5023.
[17] M. Hildebrand, Generating random elements in $SL_n(\mathbb{F}_q)$ by random transvections, *J. Alg. Comb.* **1** (1992), 133–150.
[18] V. Landazuri and G.M. Seitz, On the minimal degrees of projective representations of the finite Chevalley groups, *J. Algebra* **32** (1974), 418–443.
[19] S. Lang and A. Weil, Number of points of varieties over finite fields, *Amer. J. Math.* **76** (1954), 819-827.
[20] M. Larsen and A. Shalev, Characters of symmetric groups: sharp bounds and applications, *Invent. Math.* **174** (2008), 645–687.
[21] M. Larsen, A. Shalev, and Pham Huu Tiep, The Waring problem for finite simple groups, *Annals of Math.* **174** (2011), 1885–1950.
[22] R. Lawther and M.W. Liebeck, On the diameter of a Cayley graph of a simple group of Lie type based on a conjugacy class, *J. Comb. Theory Ser. A* **83** (1998), 118–137.
[23] A. Lev, Products of cyclic similarity classes in the groups $GL_n(F)$, *Linear Alg. Appl.* **202** (1994), 235–266.
[24] A. Lev, The covering number of the group $PSL_n(F)$, *J. Algebra* **182** (1996), 60–84.
[25] M.W. Liebeck, Probabilistic and asymptotic aspects of finite simple groups, *Probabilistic group theory, combinatorics, and computing*, 1–34, Lecture Notes in Math. **2070**, Springer, London, 2013.
[26] M. W. Liebeck, E. A. O'Brien, A. Shalev and P. H. Tiep, The Ore conjecture, *J. Eur. Math. Soc.* **12** (2010), 939–1008.
[27] M.W. Liebeck and G.M. Seitz, *Unipotent and nilpotent classes in simple algebraic groups and Lie algebras*, Amer. Math. Soc. Surveys and Monographs **180** (2012).
[28] M.W. Liebeck and A. Shalev, Diameters of simple groups: sharp bounds and applications, *Annals of Math.* **154** (2001), 383–406.
[29] M.W. Liebeck and A. Shalev, Character degrees and random walks in finite groups of Lie type, *Proc. London Math. Soc.* **90** (2005), 61–86.
[30] G. Lusztig, *Characters of Reductive Groups over a Finite Field*, Annals of Math. Studies **107**, Princeton Univ. Press, Princeton, 1984.
[31] O. Ore, Some remarks on commutators, *Proc. Amer. Math. Soc.* **2** (1951), 307–314.
[32] A. Rattan and P. Śniady, Upper bound on the characters of the symmetric groups for balanced Young diagrams and a generalized Frobenius formula, *Adv. Math.* **218** (2008), 673–695.
[33] Y. Roichman, Upper bound on the characters of the symmetric groups, *Invent. Math.* **125** (1996), 451–485.
[34] W.A. Simpson and J.S. Frame, The character tables for $SL(3,q)$, $SU(3,q^2)$, $PSL(3,q)$, $PSU(3,q^2)$, *Canad. J. Math.* (1973) **25**, 486–494.
[35] R.C. Thompson, Commutators in the special and general linear groups, *Trans. Amer. Math. Soc.* **101** (1961), 16–33.
[36] P.H. Tiep, Low dimensional representations of finite quasisimple groups, *Groups, combinatorics and geometry* (Durham, 2001), 277–294, World Sci. Publ., River Edge, NJ, 2003.

Department of Mathematics, Imperial College, London SW7 2BZ, UK
m.liebeck@imperial.ac.uk