A laboratory investigation concerning the superharmonic free wave suppression in shallow and intermediate water conditions

David Aknin, Johannes Spinneken

Imperial College London, Department of Civil and Environmental Engineering, London SW7 2AZ, UK

ARTICLE INFO

Keywords:
- Piston-type wavemaker
- Position control
- Force control
- Free wave suppression

ABSTRACT

This paper concerns laboratory wavemaking in shallow and intermediate water conditions. A comparison is made between two wave generation techniques, a first based on controlling the wavemaker displacement, and a second based on controlling the wavemaker force. Nonlinear wave generation in position control is well understood, and many laboratories rely on established second-order or Stream-function inputs. In deep water, using flap-type wavemakers, a force-control approach based on a linear demand signal was recently shown to offer benefits in terms of wave quality. The shallow water operation of such force-control strategies is less certain, which motivates the present study.

To investigate the influence of the water depth on this type of control, a range of generation scenarios is considered, including small amplitude and large amplitude regular waves. Adopting both supporting calculations and experimental evidence, the work demonstrates that first-order force-based wave generation in shallow water suffers from similar limitations as first-order position control. This principally concerns the contamination of the testing area due to unwanted free waves, where the present focus is placed on the superharmonic range.

The main advance of the work lies in the solutions it offers to overcome this free wave contamination. A number of nonlinear wave solutions upon which force-based generation can be based are discussed, and a suitable methodology is proposed and validated for each case. The developed methodology allows for high quality wave generation, whilst maintaining the benefit of active wave absorption. The work is timely in the sense that it responds to two recent developments. First, the majority of wavemaking facilities commissioned over the past two decades are computer controlled, and active absorption has become commonplace. The work presented offers solutions highly relevant to such installations. Second, developments particularly in offshore wind, have seen many new structures placed in relatively shallow-water depth. It is essential that the model testing of such structures adequately accounts for the issues and solutions presented herein.

1. Introduction

The derivation of any wavemaker theory involves considering the motion of the wavemaker and its influence on the surrounding fluid. An ideal approach would be to design a wavemaker geometry and motion that match the desired fluid kinematics exactly. However, for practical purposes, wavemaker geometries generally consist of flat wavemakers, having a single translational or rotational degree of freedom. The mismatch between the ideal and the practical design leads to two principal issues:

(i) The mismatch between the wavemaker velocity and the fluid (wave) kinematics throughout the water column leads to the existence of evanescent wave modes. Fortunately, in the limit of linear waves, these modes are only significant in the vicinity of the wavemaker and do not propagate into the testing area. However, they must be understood if considering the applied fluid load on the wavemaker.

(ii) More importantly to nonlinear wavemaking research, a combination of the geometric mismatch and a mismatch in terms of the description of the harmonic content leads to the generation of spurious or unwanted free wave components. These components may arise both at low frequencies (subharmonics) and high frequencies (superharmonics). Given that these spurious components propagate freely, they inevitably contaminate the testing area.

To overcome (ii) above, a number of nonlinear wavemaking
approaches have been proposed in the past. Traditional Stokes-type wavemaker theories rely on a perturbation expansion of the relevant boundary conditions. In addition to expanding the free-surface boundary conditions about the mean water level, the wavemaker displacement must be expanded about its mean position. A first-order Stokes-type solution to the wavemaking problem was presented by Havelock [1], and later validated experimentally by Ursell et al. [2]. At second order, wave-wave interactions give rise to both self-interactions and cross-interactions. These second-order interaction terms can occur as superharmonics and subharmonics. Ottesen-Hansen et al. [3] and Sand and Donslund [4] established that a second-order correction is required for the successful generation of long waves, particularly due to the importance of subharmonics. Sulisz and Hudspeth [5] developed a complete second-order solution for regular waves based on a matched eigenfunction methodology. The second-order formulation used within the present paper originates from the framework established by Schäffer [6], who combined much of the earlier work in a single consistent analytical formulation.

Goring and Raichlen [7] suggested that effective generation of long waves requires a different approach. Unlike Stokes-type expansions of the wavemaker displacement about its mean position, Goring and Raichlen [7] established a methodology where the wavemaker motion is considered relative to the generated wave kinematics. This approach was also demonstrated in the related development of several nonlinear wavemaker theories, including the Cnoidal approach by Goring [8].

In addition to accurate wave generation, active absorption is essential to any modern wavemaking apparatus. Wave energy is generally reflected from downstream beaches, and any test model placed within a laboratory wave flume or wave basin also reflects some wave energy back towards the wavemaker. The purpose of an absorbing wavemaker is to remove this unwanted reflected wave energy, hence maintaining an incident wave field of consistent quality. Research on active laboratory wave absorption led to a number of strategies, including: (i) position-controlled absorption techniques and (ii) force-controlled absorption techniques.

Position-controlled absorption was the first strategy to be developed, and is most commonly based on the water surface elevation recorded on the front face of the wavemaker feeding back into the system [9]. The body of work by Schäffer and Jakobsen [10,11] and Zhang and Schäffer [12] considers simultaneous nonlinear wave generation and active wave absorption. In doing so, Zhang and Schäffer [12] also demonstrated the operation of a Stream-function wavemaker theory is appropriate to both shallow and intermediate water conditions. This approach inherits from Stream-function wave theory its high performance for a wide range of conditions [13]. In particular, Stream-function wavemaker theory was shown to offer benefits over Cnoidal wavemaker theory in intermediate water, and to be more accurate than Stokes second-order wavemaker theory in deep water [12]. Most recently, Lykke Andersen et al. [14] considered the steps necessary to extend simultaneous generation and absorption to second order, and demonstrated the active absorption of highly nonlinear regular waves.

Salter [15] developed a force-controlled absorption technique, based upon the applied hydrodynamic force feeding back into the system. Salter [15] argued that a methodology based on force feedback allows for the determination of the average water conditions across the wavemaker front. In this context, it is also of critical importance that force sensors are not affected by the chemical conditions of the fluid and can be considered calibration free. Designing force-controlled laboratory wavemakers is not without challenges. To ensure that the fluid force is only recorded at the wavemaker front face, dry-backed or displacement-type wavemakers are required. A dry-backed operation is easily achieved for flap-type wavemakers, where a gusset is inserted between adjacent wavemakers or walls. For piston-type wavemakers, so called displacement-type constructions require a more elaborate design to ensure that no waves are generated at the wavemaker rear face. This type of wavemaker design leads to a slightly larger fluid gap under the piston, which may affect fluid leakage. For the purpose of the present work, an additional board was fitted to the front of the wavemaker, ensuring that this gap is minimised. Further detail concerning this gap is given in the discussion in Section 4.3.1.

Spinneken and Swan [16] derived an analytical second-order theory for actively absorbing force-controlled wavemakers; an accompanying experimental verification being presented in Spinneken and Swan [17]. Spinneken and Swan [16,17] also provided a correction force to cancel the spurious wave content at second order. However, they demonstrated that in many practical wave conditions a first-order force demand signal suffices, eliminating the need for a second-order demand. Spinneken and Swan [16,17] focused on flap-type wavemakers in deep-to-intermediate water conditions. In contrast, the present investigation concerns the shallow-to-intermediate water range, adopting piston-type wavemakers, and focusing on a regular wave analysis. The paper continues as follows. Section 2 provides the theoretical background and extends the mathematical formulation as required. Section 3 subsequently presents a brief theoretical comparison between position control and force control. A laboratory investigation in Section 4 demonstrates the problem of spurious wave contamination in wave flumes, and evaluates the practical success of a number of nonlinear wave generation approaches. Conclusions are finally drawn in Section 5.

2. Background

2.1. The wavemaking boundary value problem

Fig. 1 illustrates a piston-type wavemaker located in a two-dimensional wave flume. A Cartesian coordinate system $(x, z)$ is aligned such that $x = 0$ defines the mean position of the wavemaker, and $z = 0$ defines the still water level in water of depth $h$. Waves generated by the wavemaker propagate into the positive $x$-direction and the instantaneous surface elevation is defined by $\eta(x, t)$, where $t$ denotes time. The wavemaker is defined by the distance $d$ between the bed and the bottom of the wavemaker, with the horizontal time-varying position of the wavemaker being denoted as $X(t)$.

Assuming the flow to be inviscid and irrotational, a velocity potential $\phi$ may be introduced. This velocity potential must satisfy mass continuity throughout the fluid domain, which is commonly expressed through Laplace's equation as

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0.$$  \hspace{1cm} (1)

Furthermore, the wavemaking problem defined in Fig. 1 can be fully specified by the following boundary conditions:

![Fig. 1. Definition of a piston-type wavemaker.](image-url)
(a) A bed boundary condition, whereby the vertical fluid velocity on the horizontal and impermeable bed must be equal to zero.

(b) A lateral boundary condition, whereby the horizontal fluid velocity on the vertical and impermeable wavemaker must be equal to the wavemaker velocity.

(c) The Kinematic Free-Surface Boundary Condition (KFSBC), whereby the water surface profile is assumed to be a streamline.

(d) The Dynamic Free-Surface Boundary Condition (DFSBC), whereby the pressure on the instantaneous water surface is constant and equal to the atmospheric pressure.

Expressing these boundary conditions as a set of nonlinear equations yields

\[
\frac{\partial \phi}{\partial t} = 0 \quad \text{at} \quad z = -h, \tag{2a}
\]

\[
\frac{\partial \phi}{\partial t} = \begin{cases} \frac{dX}{dt} & \text{at} \quad X = X(t) \quad \text{for} \quad -(h - d) \leq z \leq \eta(X(t), t) \\ 0 & \text{at} \quad x = 0 \quad \text{for} \quad -h \leq z < -(h - d), \end{cases} \tag{2b}
\]

\[
\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial x} - \frac{\partial \eta}{\partial x} \frac{\partial \phi}{\partial x} = \eta(x, t) \text{and} \tag{2c}
\]

\[
\frac{\partial \phi}{\partial t} = -i g \frac{\partial \eta}{\partial z} - \frac{1}{2} \left( \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial y} \right)^2 \right) \frac{\partial \phi}{\partial x} = \eta(x, t), \tag{2d}
\]

where Eqs. (2a)–(2d) correspond to conditions (a)–(d) respectively.

2.2. Linear wavemaker theory

Linear wavemaker theory relies on a Taylor series expansion of Eqs. (2a) to (2d), retaining only expansion terms up to the first order of wave steepness. For brevity, the details of this expansion are omitted here, and the reader is referred to Schäffer [6]. The first-order wavemaker motion is defined as

\[
X^{(1)}(t) = R[-i X_c e^{i \omega t}], \tag{3}
\]

where \(X_c\) represents the complex amplitude of the wavemaker displacement, \(R\) denotes real part, \(i = \sqrt{-1}\), and the superscript (1) denotes first order. Subject to this wavemaker displacement, the first-order boundary value problem can be solved to obtain the first-order velocity potential as

\[
\psi^{(1)}(x, z, t) = R \left\{ \sum_{j=0}^{\infty} c_j \omega \frac{c_j \cosh(k_j z + h)}{k_j \cosh(k_j h)} e^{i(\omega t - k_j x)} \right\}, \tag{4}
\]

and the first-order water surface elevation as

\[
\eta^{(1)}(x, t) = \sum_{j=0}^{\infty} c_j e^{i(\omega t - k_j x)}. \tag{5}
\]

The wavenumbers \(k_j\) are the solutions to the dispersion equation

\[
\omega^2 = g k_j \tanh(k_j h), \tag{6}
\]

where \(g\) denotes acceleration due to gravity. This overall solution is composed of a single progressive wave mode for \(j = 0\) and an infinite series of evanescent components for \(j \geq 1\). For the progressive mode, the wavenumber \(k_0\) is real, whereas \(k_j\) are imaginary with \(i k_j > 0\).

In defining the velocity potential (Eq. (4)), a so-called wave field coefficient \(c_j\) is introduced. Adopting a series of orthogonal functions \((\sin, \cosh)\), the wave field coefficient for piston-type wavelmakers may be obtained analytically, and is given by

\[
c_j = \frac{2 \sinh(k_j h)(\sinh(k_j h) - \sinh(k_j d))}{k_j h + \sinh(k_j h) \cosh(k_j h)}, \tag{7}
\]

where \(j = 0\) again concerns the progressive wave mode and \(c_0\) is real. In contrast, \(c_j\) for \(j \geq 1\) is imaginary and describes the evanescent modes with \(i k_j > 0\).

Comparison of Eqs. (3) and (5) relates the magnitude of the first-order wavemaker displacement \(X_c\) to first-order progressive wave amplitude \(A\) as

\[
|A| = c_0 |X_c|. \tag{8}
\]

Furthermore, it is important to note that the phase shift between the wavemaker displacement and the progressive wave on the wavemaker front face is 90°. This ensures that the first-order progressive wave elevation is in phase with the first-order wavemaker velocity.

2.3. Second-order position-control wavemaker theory

Introducing the second-order potential as \(\psi^{(2)}\), the boundary conditions (2a) to (2d) may be expanded to second order, and a full solution may be derived analytically. The present solution relies on the work by Schäffer [6] for continuous wavemaker geometries. This is justified given that the board fitted to the front of the wavemaker extends almost entirely to the bed. If the wavemaker is non-continuous (large \(d\) in Fig. 1) then the recent contribution by Pezzutto [18] provides an improved formulation.

In the interest of brevity, the second-order development is omitted here, and reference is made to Schäffer [6]. At this stage, a brief discussion concerning the key solution properties is, however, instructive. For ease of analytical treatment, Schäffer [6] separated the full second-order potential \(\psi^{(2)}\) into three distinct contributions:

(i) A potential \(\psi^{(2)}_I\), defining the second-order wave-wave interaction arising in the absence of the wavemaker; this part being equivalent to the potential derived in Sharma and Dean [19].

(ii) A potential \(\psi^{(2)}_S\), defining the spurious or unwanted free wave arising due to the presence of the wavemaker; the purpose of any nonlinear wavemaker theory being to suppress this unwanted potential.

(iii) A potential \(\psi^{(2)}_F\), defining the additional free-wave correction due to a prescribed second-order motion of the wavemaker.

Throughout the solution presented by Schäffer [6], the superscript \(\pm\) indicates that interactions between the wave components arise both as superharmonics (+ or sum terms) and subharmonics (− ordifference terms). With the present work concerning regular waves, only the superharmonic or sum terms are of direct interest. For the purpose of eliminating the spurious or unwanted term due to \(\psi^{(2)}_S\), Schäffer [6] derived a wavemaker displacement correction signal given by

\[
X^{(2)}_s = \mathcal{F}^S \mathcal{A}^2, \tag{9}
\]

where a full expression for \(\mathcal{F}^S\) is provided in Schäffer [6]. The second-order wavemaker displacement due to \(X^{(2)}_s\) produces an additional free wave \(\eta^{(2)}_F\), which is equal in magnitude but of opposite sign to \(\eta^{(2)}_S\), hence eliminating the unwanted component \(\eta^{(2)}_S\).

2.4. Stream-function position-control wavemaker theory

Zhang and Schäffer [12] first derived a Stream-function wavemaker theory, relying on an ad-hoc combination of linear fully dispersive wavemaker theory and nonlinear wave generation. Mathematically, this theory is developed in two steps as follows:

(i) In a first step, the wavemaker displacement required for this shallow water theory, \(X^{(s)}(t)\), is obtained excluding the effect of wave dispersion. The displacement \(X^{(s)}(t)\) is obtained from the numerical solution of
\[
\frac{dX(t)}{dt} + \omega_0 X(t) = U(X^o(t), t),
\]

which describes the required velocity boundary condition at the wavemaker. Indeed, Eq. (10) ensures that the wavemaker velocity, \(dX^o(t)/dt\), and the nonlinear depth-averaged horizontal fluid velocity, \(U(X^o(t), t)\), match. The additional proportional term, \(\omega_0 X(t)\), describes a first-order high-pass filter of characteristic angular frequency \(\omega_0\). The sole purpose of this filter is to avoid any small low-frequency drift of the wavemaker. The depth-averaged horizontal fluid velocity required in Eq. (10) may be obtained by considering continuity expressed as

\[
U(x, t) = \frac{cn(x, t)}{h + \eta(x, t)},
\]

from which it is evident that the solution relies on an a priori knowledge of the water surface elevation \(\eta(x, t)\) and the wave phase velocity \(c = \omega / k\). In the context of Stream-function wavemaker theory, both the phase velocity and the surface elevation are provided by Stream-function wave theory. However, this methodology is universal in the sense that it can be based on a water surface elevation predicted by any regular wave theory.

(ii) In a second step, a correction term is applied to account for wave dispersion, providing the final wavemaker displacement as the convolution product

\[
X_0(t) = \int_{-\infty}^{t} X^o(t - \tau) \lambda(\tau) d\tau,
\]

where \(t_0\) represents the width of the convolution window and \(\lambda(\tau)\) is the inverse Fourier transform of the dispersion correction term in the frequency domain. For further details concerning this convolution integral the reader is referred to Zhang and Schäffer [12]. The signal \(X_0(t)\) provides the desired time-domain demand applied to the wavemaker.

2.5. Force-controlled wave generation and absorption

A force-control strategy fundamentally relies on the measurement of both the wavemaker velocity and the wavemaker force; the product of these two quantities being directly related to the absorbed power. In the context of wave energy conversion, Falnes [20] showed that two conditions must be met to maximise the absorbed power: a phase condition and a magnitude condition. To achieve these two conditions in practice, a real-time absorption controller is commonly adopted, continuously monitoring both the force and the velocity, and acting upon changes in the incident wave field to maintain optimum absorption at all times. A number of alternative methodologies exist for the implementation of force-based absorption controllers:

(a) a discrete frequency implementation, utilising a frequency estimator [21].
(b) a multiplication factor applied to the position and velocity feedback signals [22].
(c) an Infinite Impulse Response filter (IIR), approximating the ideal transfer function [15]. Spinneken and Swan [24] provide a direct comparison between the methodologies outlined above, and conclude that (c) provides the most effective and robust control strategy. Additional considerations for the implementation of active absorption in force-control are provided in Salter [15,23] and Maisondieu and Clement [25]. These highlight that different control approaches require a different level of implementation accuracy and effort, including the requirement for very high numerical precision (to avoid rounding errors) in the IIR case. The required precision is readily achieved with modern computing systems, and the IIR methodology is applied throughout the present study. Maisondieu and Clement [23] and Spinneken and Swan [25] also note that the transfer function in force control is purely anti-causal. This leads to challenges in formulating an optimum absorption filter, which must necessarily be causal. A comparison between a number of approaches that can yield such a causal filter is given in Spinneken and Swan [25].

Generally speaking, the IIR filter methodologies are based upon an impedance matching procedure, ensuring that the two aforementioned conditions of optimum absorption are met. The advantage of using an IIR filter approach lies in the inherent causality of the resulting digital filter [25]. The overall control strategy is outlined in Fig. 2, where the absorption controller (or filter) \(Z_f(\omega)\) is applied to the wavemaker’s velocity signal. If \(Z_f(\omega)\) is the impedance characterising the dynamics of the wavemaker, Falnes [20] demonstrated that optimum absorption occurs if

\[
Z_f = Z_d^*.
\]

where * denotes the complex conjugate. Following Spinneken and Swan [16], the dynamic wavemaker impedance \(Z_d\) is readily shown to be

\[
Z_d(\omega) = i\omega[M + m(\omega)] + d(\omega) + \frac{K}{i\omega},
\]

where \(M\) is the mass of the wavemaker and \(K\) is the wavemaker buoyancy stiffness due the immersed part of the sector-carrier wavemaker [15]. The added mass \(m(\omega)\) and the radiation damping \(d(\omega)\) are directly related to the radiation force arising as a consequence of the generated wave; their analytic expressions for a piston-type wavemaker being provided in Appendix A.

If \(C(\omega)\) expresses the wavemaker motion controller, and \(G(\omega)\) relates the drive motor torque to the wavemaker velocity, the block diagram shown in Fig. 2 can be expressed as the following transfer function

\[
B(\omega) = \frac{X(\omega)}{F(\omega)} = \frac{1}{i\omega} \frac{C(\omega)G(\omega)}{1 + C(\omega)G(\omega)(Z_f(\omega) + Z_d(\omega))},
\]

where \(B(\omega)\) relates the force demand signal \(F(\omega)\) to the wavemaker position \(X(\omega)\). Assuming a sufficiently high feedback correction.
through the controller \( C(\omega) \), the transfer function \( B(\omega) \) may be approximated by

\[
B(\omega) \approx \frac{1}{i \omega Z_f(\omega) + Z_g(\omega)},
\]

(16)

which avoids the need for an explicit formulation of \( C(\omega) \) and \( G(\omega) \). To avoid wavemaker drift, an additional high pass filter (not shown in Fig. 2) is required, as already discussed by Milgram [9]. This high pass filter has a very low cut-off frequency, and does not affect the wavemaker operation in the wave generation frequency band.

2.6. First-order force-control wavemaker theory

In the context of position control, Section 2.2 introduced the progressive wave field coefficient \( c_0 \), relating the magnitude of the first-order wavemaker displacement, \( x_0 \), to the first-order progressive wave amplitude, \( A \). In contrast, force control concerns the demand force \( F \), and the relation to the first-order progressive wave amplitude \( A \) may be introduced as

\[
A = c_F F,
\]

(17)

where the coefficient \( c_F \) is defined as

\[
c_F = B(\omega)c_0.
\]

(18)

It is important to note that the force coefficient or transfer function \( c_F \) contains both real and imaginary components. As a result, the phase relation between the wavemaker demand (now the force \( F \)) and the progressive wave amplitude is not as straightforward as in position control. Furthermore, the transfer function \( c_F \) does not only depend on the wavemaker geometry, but also on the parameters of the absorption filter \( Z_f(\omega) \) and the dynamic transfer function \( Z_g(\omega) \).

2.7. Second-order force-control wavemaker theory

In the context of force-controlled wave generation, an effective correction of the spurious free wave requires accounting for the force feedback at second order. Indeed, even a linear or sinusoidal wavemaker displacement induces a second-order hydrodynamic force. This force, in turn, feeds back into the system and leads to the generation of an additional spurious wave. In some cases, particularly for flap-type wavemakers, this additional spurious free wave has been shown to (partially) cancel the unwanted free wave due to \( \phi^{(22)} \) [17]. However, this may not always be the case, so that an expression for the additional correction term in force control must be sought.

First, the force required to achieve a spurious wave compensation in the absence of the nonlinear force-feedback component is

\[
F^{(2)}_{x_0} = \mathcal{F} \frac{A^2}{hB(2\omega)}.
\]

(19)

This equation directly arises from Eq. (9), and would lead to the desired second-order wavemaker displacement \( x^{(2)} \) if the nonlinear force-feedback was neglected. However, the nonlinear force-feedback, introduced as \( F^{(2)}_f \), must also be taken into consideration as noted above; a full expression for \( F^{(2)}_f \) being provided in Appendix B. A correct cancellation of the spurious free wave in force control can now be performed by imposing a combined correction of both the spurious free wave due to the velocity potential \( \phi^{(22)} \), and the spurious free wave caused by the second-order force feedback \( F^{(2)}_f \). The combined force correction is given by

\[
F^{(2)}_x = \mathcal{F} \frac{A^2}{hB(2\omega)} + F_f^{(2)} = F^{(2)}_{x_0} + F_f^{(2)}
\]

(20)

which is consistent with the formulation provided by Spinneken and Swan [16]. Spinneken and Swan [16] showed that the force correction \( F^{(2)}_f \) is relatively small, as the two terms shown in Eq. (20) approximately cancel over a wide range of practical wavemaking frequencies.

Unfortunately, cancellation of these two terms is not observed for piston-type wavemakers; a detailed discussion of the behaviour of \( F^{(2)}_f \) being provided as part of both the theoretical (Section 3) and the laboratory (Section 4) investigation.

2.8. Stream-function force-control wavemaker theory

A force-control Stream-function wavemaking theory does not presently exist; the purpose of the present section being to derive a suitable mathematical expression for this type of control. Building upon the Stream-function based wavemaker displacement \( x_0(\tau) \) introduced in Section 2.4, the implementation of Stream-function wavemaker theory in force control concerns the force demand required to achieve \( x_0(\tau) \). As explained in Section 2.7, any nonlinear force demand must take into account the nonlinear forces introduced through the force feedback path. In contrast to the second-order discussion in Section 2.7, the Stream-function force correction is not limited to harmonics arising at the second order. Indeed, the time-domain force demand, \( f(t) \), is defined as

\[
f(t) = \int_{-\infty}^{t} p(X_0(s), z) ds,
\]

(21)

where the pressure \( p(x, z) \) is given by the unsteady form of Bernoulli’s equation as

\[
p(x, z) = -\rho gh - \rho \frac{\partial \phi}{\partial t} - \frac{1}{2} \rho \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial z} \right)^2,
\]

(22)

with \( \rho \) denoting the water density. Within this latter expression for the pressure, the velocity potential \( \phi \) must be derived directly from Stream-function wave theory; a detailed account of the practical application of Eq. (21) and (22) being given in Section 3. In considering the force in Eq. (21), it is important to note that \( f(t) \) only concerns the hydrodynamic force. To obtain the overall demand force, the absorption feedback must also be taken into consideration (Section 2.7).

3. Theoretical comparison of nonlinear wavemaking theories

3.1. Introduction

This section concerns a theoretical comparison of second-order and Stream-function wavemaker theory. The second-order wavemaker theory by Schäffer [6] directly provides the first-order and second-order wavemaker displacements. In contrast, Stream-function wavemaker theory provides the time-history of the wavemaker position, where all frequency components (first harmonic, second harmonic and higher harmonics) are represented in a single signal. For a nonlinear theory of this type, it is important to distinguish between orders and harmonics, as a particular harmonic may arise at several orders. For example, the second harmonic may arise due to interactions at the second and the fourth order. However, if it can be assumed that fourth-order interactions are small, the influence of the fourth order on the second harmonic may also be assumed to be small. Nevertheless, to be entirely clear, all comparisons to follow are based upon the harmonic rather than the order.

A direct comparison of the first- and second-harmonic quantities predicted by Stream-function and second-order wavemaker theories is instructive, particularly for shallow water waves where Stream-function wavemaker theory is expected to provide more accurate predictions. The first-harmonic wavemaker displacement is addressed in Section 3.2. The second-harmonic displacement is first considered in position control (Section 3.3), with an extended discussion relating to force control presented in Section 3.4.
3.2. First-harmonic wavemaker displacement

Fig. 3 shows the normalised first-harmonic wavemaker displacement, $X^{(1)}/X_0$, predicted by both second-order and Stream-function wavemaker theories, where $X_0$ defines the linear wavemaker demand. In the context of the second-order theory, the first-harmonic wavemaker motion is not affected; this case being represented by the horizontal line at unity in Fig. 3. In contrast, Stream-function wavemaker theory contains terms beyond second order, which may give rise to changes at the first harmonic. This is demonstrated by the four steepness cases considered with (i) $Ak = 0.05$ (solid black line), (ii) $Ak = 0.15$ (dark grey line), (iii) $Ak = 0.25$ (light grey line) and (iv) $S = 1$ (dashed line). The parameter $S$ relevant to case (iv) is introduced as

$$S = 4A_{G_{	ext{slow}}}$$

where the full expression for $G_{	ext{slow}}$ is given in Schäffer [6]. The purpose of the parameter $S$ is to define the range of applicability of second-order wavemaker theory, where $S = 1$ defines the extreme upper limit.

In considering Fig. 3, it is apparent that $S=1$ corresponds to a very small wave steepness $Ak$ in shallow water conditions. To be entirely clear, the points at which the solid lines intersect with the dashed line defines the smallest $kh$ at which second-order theory may be applied for a given wave steepness. For example, the black line ($Ak = 0.05$) and the dashed line cross at $kh \approx 0.58$, so that second-order theory may not be applied for $kh < 0.58$ if $Ak = 0.05$. For $Ak = 0.15$ (dark grey line) the relative depth is limited to $kh > 0.9$, which increases further to $kh = 1.18$ for $Ak = 0.25$.

Fig. 3 also indicates that the first-harmonic motion is significantly affected when adopting a Stream-function based theory. For a steepness of $Ak = 0.15$ (dark grey line), the first-harmonic wavemaker motion is reduced by in excess of 15% for $kh \leq 0.6$. This clearly highlights the importance of higher-order terms in shallow water, which will be assessed further as part of the experimental investigation.

3.3. Second-harmonic wavemaker displacement in position control

Fig. 4 concerns both the magnitude (part (a)) and the phase (part (b)) of the normalised second-harmonic wavemaker displacement, $X^{(2)}/X_0$, required to eliminate the spurious free wave content (Section 2.3). Within Fig. 4, all solid lines refer to second-order calculations, while the dashed lines represent Stream-function theory. A number of distinct wave steepness cases are considered, including $Ak = 0.05$, $Ak = 0.15$, $Ak = 0.25$ and $S = 1$. The similarities and differences in the magnitude of $X^{(2)}/X_0$, Fig. 4(a), based upon the two theories are as follows:

(i) For the range $0.8 \leq kh \leq 1.4$, second-order theory and Stream-function theory lead to very similar results.

(ii) As $kh$ increases beyond 1.4, some discrepancies arise between the two theories. In this context, it is important to note that Stream-function wavemaker theory also includes terms beyond second order. For example, at fourth order, additional second-harmonic wave components arise. In deep water and with $S=1$, the steepness of the wave cases is such that second-harmonic, fourth-order terms become important, which explains the difference between the two theories for $kh > 1.4$ and $S=1$.

(iii) For $kh < 0.8$, the second-harmonic wavemaker displacement predicted by second-order theory increases rapidly. It is well known that second-order theory leads to over predictions for combinations of small $kh$ and large $Ak$. This over prediction is particularly evident for $Ak = 0.15$ and $Ak = 0.25$. As a result, Stream-function theory is likely to be more reliable for $kh < 0.8$, particularly as $S$
(iv) For $S = 1$ (light grey line), the calculations by the two theories are very similar for $kh < 1.4$. This is as expected, given that $S = 1$ defines the limit of validity for second-order theory. Indeed, the close match between the two light grey lines provides an independent measure of the range of applicability of second-order wavemaker theory in shallow water conditions.

The phase of $X^{(2)}$, Fig. 4(b), is readily explained. First, the phase information does not depend upon the wave steepness $Ak$, so that only two lines are shown. As before, the solid line corresponds to second-order theory, and the dashed line corresponds to Stream-function theory. In the case of Stream-function wavemaker theory, evanescent modes are not taken into consideration, so that the phase of $X^{(2)}$ is constant at $\pi/2$. In contrast, the increasing significance of the evanescent wave field with $kh$ leads to a phase shift in $X^{(2)}$ if based upon second-order theory. The phase shift between the two theories becomes apparent from $kh \approx 0.8$, and increases to as much as 0.6 rad for $kh = 1.8$. This indicates that Stream-function wavemaker theory may not be effective in deeper water, where the evanescent wave field must be taken into consideration. Nevertheless, Zhang and Schäffer [12] observed that Stream function wavemaker theory is also applicable in deep water conditions (up to $kh \approx 4.5$), but deteriorates for highly-nonlinear waves. This is primarily due to the assumption of linear wave theory within the dispersion correction.

### 3.4. Second-harmonic force in force control

Fig. 5 concerns the comparison of the two wavemaking theories under force control. The second-harmonic wavemaker displacement required to compensate for the spurious wave field is identical to that presented in Fig. 4. However, in the force-control case, the demand quantity is a second-harmonic force. The normalised magnitude of this force, $F^{(2)}_{c}/F^{(1)}$, is shown in Fig. 5(a), with the phase information provided in Fig. 5(b). The overall comparison between second-order theory and Stream-function theory is very similar to that observed in the context of the wavemaker displacement (Fig. 4). The key difference between position control and force control is that the latter must take into account the nonlinear hydrodynamic feedback force. This is perhaps best observed in the phase of $F^{(2)+}_{c}$, Fig. 5(b), as follows:

(i) The phase in Fig. 5(b) is quite different from the phase observed on Fig. 4(b). The phase difference between position control and force control is approximately $\pi/2$ in shallow water ($kh < 0.6$). The wave field in shallow water is dominated by the progressive wave field, and the evanescent modes are relatively small. As a result, the wavemaker force in shallow water is dominated by the associated radiation damping, which is in phase with the wavemaker velocity. The phase difference between the wavemaker displacement and the wavemaker velocity explains the phase shift of approximately $\pi/2$ for $kh < 0.6$.

(ii) The phase for both theories increases with $kh$. Given that Stream-function wavemaker theory does not account for the evanescent wave field, this cannot be due to the evanescent modes. In fact, the observed phase shift in part relates to the wavemaker inertia (an additional force) and in part to the nonlinear hydrodynamic feedback.

(iii) The increasing phase difference between the two theories (solid line and dashed lines in Fig. 5(b)) is once again attributed to the increasing importance of the evanescent wave field with increasing $kh$.

In summary, the above discussion highlighted that both second-order and Stream-function wavemaker theories have common ranges of applicability, but also exhibit distinct limitations. Stream-function wavemaker theory may offer some advances in shallow water; this being demonstrated experimentally as follows.

### 4. Experimental investigation

#### 4.1. Introduction

Building upon the theoretical comparison presented in Section 3, the experimental investigation is then structured as follows:

(i) Position-controlled wave generation is considered in Section 4.3, addressing the following key quantities:

(a) An experimental assessment of the first-order wave field coefficient $c_0$ is presented in Section 4.3.1. All subsequent hydrodynamic quantities (surface elevation, velocity potential, forces) rely on this fundamental coefficient. Crucially, at second order, the coefficient $c_0$ is being squared, and any first-order inaccuracies are further amplified.

(b) An assessment of the spurious free wave content $X^{(2)+}/X^{(1)+}$ and the second-order correction term $X^{(2)+}$ is presented in...
for $3.5 \text{h}$, taken into consideration the initial for $[\text{ ]} 1$ and $\text{ } [\text{ ]}$. A minimum distance of $3.5 \text{h}$ was considered su-

$\text{kh}^2 \text{h} \text{A}^3 \text{h}_k$.

The small-amplitude set is included for comparison to linear wavemaker theory, particularly the wave

Section 4.3.2.

(c) The spurious wave content established under (b) is compared to the Stream-function approach in Section 4.3.3.

(ii) An equivalent series of steps (i) (a)–(c), but incorporating force-feedback control, is presented in Section 4.4 and 4.5.

All experiments shown herein were conducted in the Coastal Wave Flume located in the Hydrodynamics Laboratory of the Department of Civil and Environmental Engineering at Imperial College London. This wave flume is equipped with a $0.6 \text{ m}$ wide displacement-type piston wavemaker, as well as a highly optimised parabolic beach located $23 \text{ m}$ downstream of the wavemaker. For the purpose of the present work, the water depth was held constant at $h = 0.6 \text{ m}$ throughout the flume. The experimental investigation concerns the wave cases noted in Table 1, which includes three sets: (i) a small-amplitude set, (ii) a base nonlinear set and (iii) an extended nonlinear set. For each case considered, Table 1 provides the non-dimensional water depth $\text{S}$, the frequency $f = \omega / (2\pi)$, the nonlinearity parameter $\text{Ak}$, the steepness $A_k$ and the actual wave amplitude $A$.

<table>
<thead>
<tr>
<th>$\text{kh}$</th>
<th>$f$</th>
<th>$\text{S}$</th>
<th>$\text{Ak}$</th>
<th>$A$</th>
<th>$\text{S}$</th>
<th>$\text{Ak}$</th>
<th>$A$</th>
<th>$\text{S}$</th>
<th>$\text{Ak}$</th>
<th>$A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.50</td>
<td>20/64</td>
<td>0.33</td>
<td>0.012</td>
<td>14</td>
<td>1</td>
<td>0.036</td>
<td>43</td>
<td>1.5</td>
<td>0.092</td>
<td>90</td>
</tr>
<tr>
<td>0.56</td>
<td>22/64</td>
<td>0.33</td>
<td>0.016</td>
<td>17</td>
<td>1</td>
<td>0.048</td>
<td>51</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.62</td>
<td>24/64</td>
<td>0.33</td>
<td>0.021</td>
<td>20</td>
<td>1</td>
<td>0.061</td>
<td>60</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.68</td>
<td>26/64</td>
<td>0.33</td>
<td>0.026</td>
<td>23</td>
<td>1</td>
<td>0.077</td>
<td>68</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.74</td>
<td>28/64</td>
<td>0.33</td>
<td>0.032</td>
<td>26</td>
<td>1</td>
<td>0.095</td>
<td>77</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.80</td>
<td>30/64</td>
<td>0.33</td>
<td>0.038</td>
<td>29</td>
<td>1</td>
<td>0.114</td>
<td>86</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.86</td>
<td>32/64</td>
<td>0.33</td>
<td>0.045</td>
<td>31</td>
<td>1</td>
<td>0.136</td>
<td>94</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.93</td>
<td>34/64</td>
<td>0.32</td>
<td>0.050</td>
<td>32</td>
<td>0.945</td>
<td>0.150</td>
<td>97</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.00</td>
<td>36/64</td>
<td>0.27</td>
<td>0.050</td>
<td>30</td>
<td>0.819</td>
<td>0.150</td>
<td>90</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.07</td>
<td>38/64</td>
<td>0.24</td>
<td>0.050</td>
<td>28</td>
<td>0.718</td>
<td>0.150</td>
<td>84</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.15</td>
<td>40/64</td>
<td>0.21</td>
<td>0.050</td>
<td>26</td>
<td>0.637</td>
<td>0.150</td>
<td>78</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.23</td>
<td>42/64</td>
<td>0.19</td>
<td>0.050</td>
<td>24</td>
<td>0.572</td>
<td>0.150</td>
<td>73</td>
<td>0.858</td>
<td>0.225</td>
<td>109</td>
</tr>
<tr>
<td>1.32</td>
<td>44/64</td>
<td>0.17</td>
<td>0.050</td>
<td>23</td>
<td>0.519</td>
<td>0.150</td>
<td>68</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.40</td>
<td>46/64</td>
<td>0.16</td>
<td>0.050</td>
<td>21</td>
<td>0.476</td>
<td>0.150</td>
<td>64</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.50</td>
<td>48/64</td>
<td>0.15</td>
<td>0.050</td>
<td>20</td>
<td>0.441</td>
<td>0.150</td>
<td>60</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.60</td>
<td>50/64</td>
<td>0.14</td>
<td>0.050</td>
<td>19</td>
<td>0.412</td>
<td>0.150</td>
<td>56</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.70</td>
<td>52/64</td>
<td>0.13</td>
<td>0.050</td>
<td>18</td>
<td>0.389</td>
<td>0.150</td>
<td>53</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.81</td>
<td>54/64</td>
<td>0.12</td>
<td>0.050</td>
<td>17</td>
<td>0.370</td>
<td>0.150</td>
<td>50</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.93</td>
<td>56/64</td>
<td>0.12</td>
<td>0.050</td>
<td>16</td>
<td>0.354</td>
<td>0.150</td>
<td>47</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.18</td>
<td>60/64</td>
<td>0.11</td>
<td>0.050</td>
<td>14</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.45</td>
<td>64/64</td>
<td>0.11</td>
<td>0.050</td>
<td>12</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.75</td>
<td>68/64</td>
<td>0.10</td>
<td>0.050</td>
<td>11</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.07</td>
<td>72/64</td>
<td>0.10</td>
<td>0.050</td>
<td>10</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.41</td>
<td>76/64</td>
<td>0.10</td>
<td>0.050</td>
<td>9</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.78</td>
<td>80/64</td>
<td>0.10</td>
<td>0.050</td>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.16</td>
<td>84/64</td>
<td>0.10</td>
<td>0.050</td>
<td>7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.57</td>
<td>88/64</td>
<td>0.10</td>
<td>0.050</td>
<td>7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.99</td>
<td>92/64</td>
<td>0.10</td>
<td>0.050</td>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.43</td>
<td>96/64</td>
<td>0.10</td>
<td>0.050</td>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.89</td>
<td>100/64</td>
<td>0.10</td>
<td>0.050</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6.38</td>
<td>104/64</td>
<td>0.10</td>
<td>0.050</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6.88</td>
<td>108/64</td>
<td>0.10</td>
<td>0.050</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For $kh > 0.9$ and $S = 1$ for $kh < 0.9$. As a result, these cases lie within the range of validity of second-order theory. In contrast, $S$ is close to or in excess of unity for the extended nonlinear set; these cases being included to highlight the limitations of second-order theory (Section 4.5).

4.2. Data recording and analysis

Given the nature of the relatively small quantities involved at higher orders, it is important that all instrumentation is of the utmost precision. The water surface elevation was recorded by seven tensioned wire gauges of wire diameter 0.5 mm, with the first wave gauge located at $x_p = 3.5 \text{ h}$. A minimum distance of $3.5 \text{ h}$ was considered sufficiently far downstream of the wavemaker to neglect the effect of any evanescent modes. Subsequent wave gauges were located with a spacing of $\Delta x = h/3$. As a result, the final wave gauge is located at $x = 5.3 \text{ h}$. With the beach located at $x = 35 \text{ h}$, a time window was calculated for each of the cases in Table 1 for which the wave record is entirely free of reflections. The start time of this window was calculated such that all wave components had sufficient time to propagate to a measurement location of $x = 5.3 \text{ h}$. The end time of this window was determined by the fastest wave component (the fundamental wave) reaching the laboratory beach with its re-

It is also important to clarify the choice of quantities that feed into
the data analysis procedures. The base frequency \( f \) (Table 1) was assumed as highly accurate, which is underpinned by many of our previous observations, including Spinneken and Swan [17]. At second order, the wave dispersion equation is not affected by the wave steepness, such that the fundamental wavenumber \( k \) is constant for each fundamental frequency \( f \). The bound wave in the analysis procedure was then assumed to occur at \( 2 \omega \) and \( 2k \). In contrast, the second-order free wavenumber, \( k_x \), was calculated as a solution to \((2\omega)^2 = g \cdot \tan(k h)\). To verify that amplitude dispersion does not affect the results presented, the wavenumber for each case was also calculated using Stream function theory. In the base nonlinear set (Table 1), the largest departure between the linear and the nonlinear wavenumber was found to be \( 1.8\% \), which occurs for the case with \( kh=0.93 \) and \( Ak=0.15 \). This departure leads to a small phase shift for each wavelength, which is important for the results discussed in Section 4.3.2.

The wavemaker position was recorded using a high-precision laser displacement sensor directed towards the wavemaker front face. To ensure that small inaccuracies in the feedback control do not affect the data and conclusions presented here, this actual measured wavemaker displacement was used for the purpose of all data analyses. This externally measured wavemaker displacement was also used to examine the quality of the servo drive system. For example, it was verified that a sinusoidal demand leads to a displacement free of second-harmonic content, which is crucial for the investigation of nonlinear wavemaker theories.

4.3. Experimental validation of position control

4.3.1. Progressive wave field coefficient

The progressive wave field coefficient \( c_0 \) relates, at first order, the wavemaker displacement to the generated progressive wave amplitude. The accuracy of the coefficient \( c_0 \) is crucial in operating a wavemaker as it provides the wavemaker displacement required to generate a wave at a specific amplitude. Within the second-order potentials (\( \phi^{(1)} \) and \( \phi^{(2)} \)), Schäffer [6] the coefficient \( c_0 \) is squared, and the success of any second-order correction relies on an accurate knowledge of \( c_0 \).

Therefore, an experimental assessment of the magnitude and phase of \( c_0 \) was conducted first.

Fig. 6 shows four sample surface elevations, \( p(t)/h_{1/2} \), corresponding to a subset of the small-amplitude cases noted in Table 1. For the avoidance of any doubt, the normalisation is denoted as \( A_{1/2} \), corresponding to the linearly calculated wave amplitude. The figure is presented as four parts, with (a) \( kh=0.56 \), (b) \( kh=1 \), (c) \( kh=1.4 \) and (d) \( kh=1.8 \). In each case, the experimental data due to a wavemaking demand of \( X_u = A/c_0 \) is shown by the discrete symbols \( \bigcirc \). The black solid line provides a theoretical comparison evaluated using Stream-function theory, where the amplitude of this wave was computed from first-order wavemaker theory \( (A = c_0 X_u) \). In considering the data in Fig. 6, it is clear that the experimentally achieved surface profiles are consistently smaller than those predicted by first-order wavemaker theory.

To investigate this further, Fig. 7 shows the magnitude (part (a)) and the phase (part (b)) of the experimentally observed coefficient \( c_0 \), where the data corresponds to the full set of the small-amplitude cases (Table 1). The first-harmonic wavemaker displacement \( X_u \) and the wave amplitude \( A \) were obtained by applying a Discrete Fourier Transform (DFT) to the respective experimental recordings, retaining the first-harmonic information only. The magnitude of the experimentally observed coefficient \( c_0 \) was then calculated as the magnitude ratio of \( |A|/|X_u| \); the phase being obtained by subtracting the respective phases \( \arg(A) \) and \( \arg(X_u) \).

In considering the phase of \( c_0 \) in Fig. 7(b), a good agreement between the theoretical prediction and the experimentally observed data is found throughout. The theoretical phase of the first-order wavemaker displacement is \( \pi/2 \) rad. The constant phase is a direct consequence of the progressive wave being in phase with the wavemaker’s velocity, which is matched well by the experimental data. The maximum departure between the theoretical prediction and the experimental observation is 0.22 rad (Fig. 7(b)).

In contrast to the phase information, significant departures arise in the magnitude of \( c_0 \) as shown in Fig. 7(a). Within Fig. 7(a), the solid line represents the theoretical magnitude of \( c_0 \) based on Eq. (7). The discrete points (symbol \( \bigcirc \)) represent the experimental data, which are consistently lower than the predicted values. Indeed, the maximum departure is 16%, the average departure is 15%, and the theoretical coefficient consistently overpredicts the amplitude of the generated wave field. A similar trend was reported by Ursell et al. [2] who obtained an estimation of \( c_0 \) that was 10% lower than the theoretical prediction. According to Ursell et al. [2] such a discrepancy may be due to:

(i) Fluid leakage past the wavemaker due to the gaps between the flume and the wavemaker edges. In the context of the present experimental investigation, an optimisation of the wavemaker geometry was carried out by reducing all side gaps from approximately 20 mm to less than 5 mm. As a result, fluid leakage is expected to only have a very minor role in explaining the observed departures. The reduction in the wavemaker gap led to a small improvement in \( c_0 \) from an average departure of 17% to the average departure of 15% observed in Fig. 7.

(ii) Finite-amplitude effects. To address this, the experimental comparison of the wave field coefficient \( c_0 \) was also undertaken for the base nonlinear set noted in Table 1. No marked difference between the small-amplitude set and the base nonlinear set could be observed; the departure of \( c_0 \) being 15% in average in both cases.

The general trend of achieving wave amplitudes lower than predicted theoretically was also noted by Schäffer [6], although they did not establish a fixed percentage. Furthermore, some cases reported by Schäffer [6] also match the theoretical prediction well, see Fig. 10 in Schäffer [6]. It is suspected that the departures observed here are associated with viscous effects. In part, this may be due to skin-friction as the fluid moves vertically along the front of the wavemaker. A second contribution is likely to be due to small vortices being shed at the base of the wavemaker. The wavemaker side gaps were reduced to 5 mm, and the gap underneath the wavemaker is also approximately 5 mm when the wavemaker is in its mean position. However, over the full wavemaker displacement for the largest wave case, this bottom gap increases up to 15 mm, as the sector-carrier motion describes an arc of finite radius. Both effects (skin friction and vortex generation) lead to a dissipation of a part of the wavemaker-induced energy.

A detailed investigation of non-potential wavemaker effects lies outside the scope of the present paper. Nevertheless, for the purpose of the experimental investigations to follow, a corrected progressive wave field coefficient \( c_0' \) was introduced as the reference for the application of linear wavemaking theory. This coefficient \( c_0' \) is represented by the dashed line in Fig. 7(a) and was designed to be both proportional to the theoretical magnitude of \( c_0 \) and to match the experimental magnitude of \( c_0 \). These two conditions provide a coefficient \( c_0' \) which is given by \( c_0' = 0.85c_0 \). Fig. 6 demonstrates the applicability of the modified wave field coefficient \( c_0' \), where the experimental data with symbol * is due to a wavemaker demand of \( X_u = A/c_0' \).

The coefficient \( c_0 \) was also investigated through an independent set of experiments undertaken in the Sediment Transport Flume at Plymouth University. The additional experiments (not shown herein) confirmed a departure of 15% from the theoretically predicted value of \( c_0 \). It should be noted that this departure is larger than reported by, for example, Ursell et al. [2]. The authors would like to encourage other laboratories to report their practical values of \( c_0 \) in future publications.
4.3.2. First-order and second-order control

Adopting the modified coefficient \( c_{0}' \), the base nonlinear test cases described in Table 1 were generated. The test cases were generated twice, first based upon a first-order control signal and subsequently based upon a second-order control signal. Fig. 8 concerns a comparison of the generated surface elevations, recorded at \( x = 3.5 h \), with (a) \( k h = 0.56 \), (b) \( k h = 1 \), (c) \( k h = 1.4 \) and (d) \( k h = 1.8 \). Within Fig. 8, the data with symbol \( \circ \) correspond to first-order control, and the data with symbol * relate to second-order control. In addition, the black solid line indicates the theoretical Stream-function solution for comparison.

Considering the case with \( k h = 0.56 \) first (Fig. 8(a)), it is clear that first-order control leads to a significant departure from the target regular wave elevation; this being due to the presence of a pronounced spurious free wave. In contrast, the second-order control case is in close agreement with the theoretical prediction, providing evidence for the successful suppression of the spurious free wave content. Similar arguments apply to the case with \( k h = 1 \) in Fig. 8(b). For \( k h = 1.4 \), Fig. 8(c), the second-order control case is markedly different from first-order control, but it is difficult to argue which of these two represents a better match to the theoretical prediction. In the final case, Fig. 8(d) with \( k h = 1.8 \), it is difficult to distinguish between the two cases based upon a visual comparison alone.

To provide further evidence of the success of second-order control, an extended analysis was undertaken on the recordings at all seven wave gauges. The second harmonic of the surface elevation recorded at each of the seven wave gauges contains information about both the bound and the free wave content. To separate these two contributions, the methodology introduced by Lin and Huang [26] was applied to each of the wave cases. Within this analysis, a DFT is applied to the recorded water surface elevation, and a least-square method leads to the separation of the free and bound components. The analysis by Lin and Huang [26] may also be used to separate the incident and reflected wave components at each harmonic. However, this was not required in the present context as our choice of time window (Section 4.2) ensured that the recording was free of wave reflections. As a result, only the incident free and bound harmonic components are non-zero here. In the analysis to follow, the phase reference for all wave components was taken as the wavemaker rest position, \( x = 0 \).

To ensure that this separation analysis does not suffer from any singularities for a fixed wave gauge spacing of \( \Delta x = h/3 \), a sensitivity analysis similar to that proposed in Isaacson [27] was undertaken. Assuming a maximum error of 5 mm in the spacing between individual wave probes, and 0.5 mm in the surface elevation recording, yields a maximum relative error of under 10% for the worst possible error.
Due to limitations in the control. In this context, \( c_0 \) corresponds to the absence of any spurious wave content. Moreover, \( c_0 \) is a small quantity at \( k = 2 \), which is the desired condition. For \( k = 2 \), position-control demand (symbol \( \bigcirc \), based upon \( X^{(1)} \)) and a second-order position-control demand (symbol \( \bigstar \), based upon \( X^{(1)} + X^{(2)} \)). The first-order case, based on \( X^{(1)} \) alone, directly corresponds to the ratio of the magnitude data shown in Fig. 9(a) and Fig. 10(a), and any departures observed in Fig. 11 are directly attributed to the departures observed in Fig. 9(a) and Fig. 10(a).

The experimentally observed phase shift has important implications on the spurious wave free correction. Fig. 11 shows the non-dimensional spurious wave content \( A^{(2)/k} / A^{(1)/k} \) for both a linear position-control demand (symbol \( \bigcirc \), based upon \( X^{(1)} \)) and a second-order position-control demand (symbol \( \bigstar \)). The first-order case, based on \( X^{(1)} \) alone, directly corresponds to the ratio of the magnitude data shown in Fig. 9(a) and Fig. 10(a), and any departures observed in Fig. 11 are directly attributed to the departures observed in Fig. 9(a) and Fig. 10(a).

In considering the second-order control case (symbol \( \bigstar \)), this approach is successful in cancelling the spurious free wave content for most of the cases with \( k h < 1.2 \). To be entirely clear, a spurious wave content of \( A^{(2)/k}/A^{(1)/k} = 0 \) corresponds to the absence of any spurious free wave \( \eta^{(2)} = 0 \), which is the desired condition. For \( 0.5 < k h < 1.2 \), the average remaining spurious content is approximately 10%, so that the spurious free wave only corresponds to one tenth of the bound content.

In contrast, the cancellation of the spurious free wave is not entirely successful beyond \( k h = 1.2 \), which may be due to two different effects:

(i) The phase of the correction term \( \eta^{(2)} \) is derived from the phase of the theoretical spurious free wave \( \eta^{(2)} \). Given the large discrepancies in the phase of the spurious free wave observed in Fig. 10(b), it is likely that the theory is unable to provide for a complete spurious wave suppression in intermediate water depth conditions due to a phase mismatch.

(ii) The wavemaker may be unable to correctly reproduce the desired second-order correction term \( \eta^{(2)} \) due to limitations in the control. In this context, it is important to highlight that the required second-harmonic wavemaker displacement \( X^{(2)} \) is a small quantity at relatively high frequency.

To identify the primary cause of the departures in intermediate water, an additional set of test cases was considered. Within this additional test series, the desired correction term was produced in the absence of the first-harmonic wave; the wavemaker displacement being equal to \( X^{(2)} \) only. The results of this verification (not shown herein) provided an excellent agreement between the predicted values and the experimental data, confirming that the wavemaker is indeed capable of producing the desired correction term. The quality of the servo system was also verified by confirming that the second-harmonic motion is correctly represented when the first-harmonic motion is present, which was indeed the case. This confirms that (ii) above is not responsible for the unsuccessful cancellation of the spurious free wave in intermediate water. As a consequence, the inability to correctly predict the phase of the second-order free wave, (i) above, is the most probable reason for the failure of the second-order wavemaker theory in intermediate water conditions. An experimentally derived solution to this issue will be presented in Section 4.3.4.

At this point, it is also important to re-emphasize that the success of the second-order correction in Fig. 11 partially hinges on the empirical correction factor \( c_0 = 0.85 c_1 \). Additional experiments adopting \( c_0 \) combination and wave case. The values used in this error analysis are believed to be conservative, which renders 10% as the upper limit for the relative error.

Fig. 9 shows the amplitude and the phase of the bound wave component, \( A^{(2)} \), under first-order position control. In considering this data, it should first be noted that the discontinuity at \( k h \approx 0.9 \) relates to the change from \( \Delta \kappa = 0.15 \) to \( S = 1 \) (Table 1). Taken as a whole, the bound wave content illustrated in Fig. 9 matches the theoretical prediction well. The bound wave amplitude is in average 13% higher than the prediction by Schäffer [6], with a largest departure of 48%. However, it should be noted that this largest departure arises for the shortest wave case (\( k h \approx 2 \)), where the absolute magnitude of the bound wave content is relatively small. A good match can also be observed between the measured bound wave phase and the prediction by Schäffer [6], Fig. 9(b). The predicted wave phase is \( \pi \) rad, and the experimentally observed phase closely matches this prediction, with a maximum departure of 0.27 rad.

The free wave amplitude, \( A^{(2)} \), under first-order position control is shown in Fig. 10(a), and is seen to be consistently lower than the theoretical prediction by Schäffer [6]. In average, the spurious free wave content is 24% lower than expected. Perhaps more importantly, some very large departures can be observed in the spurious free wave phasing, Fig. 10(b). Indeed, the measured spurious free wave phase only matches the prediction in shallow water, where its value is approximately zero. In deeper water conditions (larger \( k h \)), the experimentally observed phase increases significantly more rapidly than expected from the theoretical prediction, reaching a phase in excess of 4 rad for \( k h = 2 \). It should be noted that this phase shift is far in excess of what could be explained by nonlinear amplitude dispersion. Indeed, the phase error introduced through amplitude dispersion should not exceed 0.1 rad per wavelength for cases with \( k h > 1 \). Given that the wave probes are located within three wavelengths of even the shortest wave case, any errors introduced via amplitude dispersion should not exceed 0.3 rad, which does not explain the phase departures observed in Fig. 10(b).

However, it should be noted that this largest departure arises for the

**Fig. 7.** Progressive wave field coefficient with (a) magnitude and (b) phase showing \( \bigcirc \) experimental data — theoretical prediction of \( c_0 \) and \( \bigstar \) adjusted coefficient \( c_0' = 0.85 c_0 \). (a) Magnitude of \( c_0 \) (b) Phase of \( c_0 \).
directly were also undertaken, and the resulting spurious free wave content under second-order control was found to be larger than that shown in Fig. 11 for the vast majority of wave cases. In considering Fig. 11, a comparison to Schäffer [6] is also instructive. Schäffer [6] showed significant improvements by the use of second-order wave generation up to $kh = 4.4$. It is, however, difficult to make direct comparisons, as Schäffer [6] did not explicitly separate the free and bound wave harmonics, but instead relied on a comparison of the total second-harmonic content at four wave probes.

### 4.3.3. Stream-function based correction

Following the experimental assessment of the second-order theory by Schäffer [6], the present section highlights the differences that arise when the nonlinear position-control demand is based upon the Stream-function wavemaker theory by Zhang and Schäffer [12]. Fig. 12 shows the non-dimensional superharmonic spurious free wave content $\eta_{n,2}/A_{Hn}$, making a direct comparison between (i) the theoretically expected content under first-order control (black line), (ii) the experimental data relating to a first-order demand signal (symbol ○) and (iii) the experimental data relating to the Stream-function demand (symbol *). Observing the data in Fig. 12, it is evident that the spurious free wave content based upon the Stream-function demand is similar to the spurious free wave content based on the second-order theory (Fig. 11). In close similarity to the second-order demand, the Stream-function wavemaker theory succeeds in cancelling the spurious free wave content for $kh < 1.2$, but is unable to provide a full compensation beyond $kh = 1.2$. The causes for this discrepancy may be two-fold:

(i) The data provides further evidence for the effects discussed in Section 4.3.2. Indeed, the experimental data confirms that the unsuccessful cancellation of the spurious free wave for $kh > 1.2$ is not specific to second-order wavemaker theory. As a result, this phenomenon is not directly related to the type of theoretical expansion, but is likely to be caused by the underlying assumption of potential flow theory.

(ii) In considering the development of Stream-function wavemaker theory (Section 2.4), it is important to note that all evanescent mode interactions are neglected. With the evanescent modes becoming more relevant for increasing $kh$, the departures observed in Fig. 12 may hence partially be attributed to this latter cause, but the effects noted in the context of second-order theory remain relevant.
Sections 4.3.2 and 4.3.3 demonstrated that both second-order control and Stream-function control are unable to fully eliminate the spurious free wave for $k h > 1.2$; this being due a phase shift in the underlying free wave $\eta^{(22)}$ (Fig. 10(b)). As a result, an adequate phase correction in the second-order wavemaker displacement must be introduced. Two methodologies were considered:

(I) The corrected phase of the transfer function $\mathcal{F}^+$ is based upon the experimentally obtained phase of the spurious free wave $\eta^{(22)}$ as shown in Fig. 10(b).

(II) The corrected phase of the transfer function $\mathcal{F}^+$ is obtained through an approach that seeks to minimise the spurious free wave content at each frequency. In practice, the spurious wave content was minimised experimentally for the three select cases of frequency $k h = 0.5$, $k h = 1$ and $k h = 1.6$. Based on these three cases, a polynomial regression provided the corrected phase for the remaining frequencies of interest.

In both Methods I and II the magnitude of $\mathcal{F}^+$ remains unchanged. A polynomial regression provided a sixth- and a fourth-order polynomial approximation for Method I and II respectively (Table 2). These
Polynomials were subsequently used to determine the phase for any case in the range $0.504 \leq kh \leq 1.926$. Fig. 13 shows the phase of the transfer function $\mathcal{F}^+$ as provided by Schäffer [6] (solid line), the phase according to Method I (line $\cdot\cdot$), and the phase according to Method II (line $\cdot$). Method I effectively describes the phase shift observed previously in Fig. 10(b). In contrast, Method II leads to a substantially reduced phase shift, which lies much closer to the analytical prediction by Schäffer [6].

Table 2

<table>
<thead>
<tr>
<th>Order</th>
<th>Method I (rad)</th>
<th>Method II (rad)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>58.882</td>
<td>0.924</td>
</tr>
<tr>
<td>1</td>
<td>$-315.047$</td>
<td>4.943</td>
</tr>
<tr>
<td>2</td>
<td>666.139</td>
<td>$-9.283$</td>
</tr>
<tr>
<td>3</td>
<td>$-719.058$</td>
<td>7.062</td>
</tr>
<tr>
<td>4</td>
<td>424.855</td>
<td>$-1.658$</td>
</tr>
<tr>
<td>5</td>
<td>$-130.559$</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>16.312</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 14 shows the spurious free wave content for the corrected phase Methods I (symbol $+$) and II (symbol $\ast$). For $kh \approx 0.6$, the two methods yield very similar results, which is as expected, given the phase of approximately zero in Fig. 13. For $kh \geq 0.8$, Method I fails to successfully correct for the spurious wave content. Indeed, this method leads to a significantly larger spurious wave than the uncorrected second-order compensation. Method II is also successful for $kh$ beyond 0.8, and delivers the most consistent results of all methods considered thus far.

In considering the above evidence, it is surprising that the optimal phase of the transfer function $\mathcal{F}^+$ does not correspond to the phase of the spurious free wave observed in Fig. 10(b). The term $\mathcal{F}^+$ is derived such that the correction wave $\eta^{(23)+}$ is exactly out of phase with the spurious free wave $\eta^{(22)+}$. As a result, any phase shift in $\eta^{(22)+}$ should also apply to $\eta^{(23)+}$ and therefore $\mathcal{F}^+$. Having carefully examined all potential causes for this effect, small reflections from the laboratory beach can be ruled out due to the choice of analysis window (Section 4.2), and phase accumulations due to amplitude dispersion are very small given the measurement locations ($3.5 \leq x \leq 5.3$ h). The analysis here is also very similar to that adopted in Spinneken and Swan [17], where very good agreement with the underlying theory was established. As a result, the phase departures must be due to nonlinearities that are currently not included in the modelling, which will be examined as part of our future modelling efforts. Nevertheless, Method II provides an alternative means of obtaining the optimum phase of $\mathcal{F}^+$.

4.4. Experimental validation of force control

4.4.1. First-order control

Under force control, the demand to the wavemaker is represented by a force rather than a position. To achieve a particular (desired) wave amplitude in the wave flume, the force transfer function must first be computed. This transfer function incorporates the following components: (i) the wavemaker mass or inertia, (ii) the wavemaker spring stiffness, (iii) the wavemaker radiation damping and (iv) the wavemaker added mass. Further details of this transfer function were previously discussed in Spinneken and Swan [28], and the analytic expressions for the wavemaker hydrodynamic coefficients are provided in Appendix A.

From the expressions in Appendix A (Eqs. (A.2) and (A.3)) it is clear that the progressive wave field coefficient $c_0$ directly affects the wavemaker’s hydrodynamic coefficients. As a result, any inaccuracy in
$c_0$ is also reflected in an inaccurate prediction of the wavemaker force. To overcome this, the modified coefficient $c_0'$ was once again adopted. Based upon $c_0'$, the force demand was calculated entirely analytically, without any further modifications.

Fig. 15 shows the surface elevations generated from a first-order force demand, concerning a subset of the small-amplitude wave cases (Table 1) with (a) $kh = 0.56$ and $S = 0.33$, (b) $kh = 1$ and $Ak = 0.05$, (c) $kh = 1.4$ and $Ak = 0.05$, (d) $kh = 1.8$ and $Ak = 0.05$. As before, the theoretical Stream-function solution is provided for reference, and the agreement between the experimental data and the analytical expression is evident. As a result, Fig. 15 confirms that the analytical expressions provided for the hydrodynamic coefficients are indeed correct. However, these expressions must take into consideration the reduction in $c_0'$; the modified coefficient $c_0'$ being adopted for all force-control experiments shown herein. Additional experiments for the small-amplitude cases noted in Table 1 confirmed that the generated wave amplitudes in first-order force-control were consistently within 5% of the theoretical prediction.

### 4.4.2. Second-order forces

If the wavemaker operates under force control, the additional nonlinear force feedback must be taken into consideration (Section 2.7 and 3.4). This additional force feedback arises due to second-harmonic forces acting on the wavemaker, even when the wavemaker motion is entirely sinusoidal. To provide a full compensation of the spurious free wave content, both the phasing and the magnitude of this additional nonlinear force feedback must be predicted accurately; the purpose of the present section being to validate the theoretical expressions derived in Section 2.7.

The second-harmonic force feedback can be determined by (i) measuring the force applied to the wavemaker load cell or by (ii) evaluating the (nonlinear) wavemaker displacement. For the purpose of the present investigation, the latter approach was chosen due to the high accuracy of the laser displacement sensor. Fig. 16 concerns the second-harmonic of the wavemaker displacement, where the wavemaker is operated using a first-order or sinusoidal force demand. As outlined above, this second-harmonic displacement arises due to the second-harmonic forcing on the wavemaker. Both the experimentally obtained magnitude (Fig. 16(a)) and the phase (Fig. 16(b)) are shown (symbol ○), and a direct comparison is made to the theoretical expressions derived in Section 2.7 (solid lines). The discontinuity at $kh = 0.9$ is once again due to the change from $Ak = 0.15$ to $S = 1$.

Taken as a whole, the trends predicted analytically match the data relatively well. However, the magnitude is over-predicted for $kh < 1.1$ and under-predicted for $kh > 1.1$. The observed discrepancies are relatively significant in terms of their percentage departure, but actually corresponds to a maximum absolute difference of only 1.1 mm; this maximum departure being observed for $kh = 0.8$. A difference of order 1 mm is believed to have an insignificant impact...
on the wave quality as it is small when compared to the wavemaker stroke (108 mm for $kh = 0.8$) and also small when compared to the second-order wavemaker correction term (7.5 mm for $kh = 0.8$). The phasing of the additional second-harmonic displacement is predicted relatively well, with a maximum difference of 0.5 rad.

At this stage it should also be noted that the data in Fig. 16 provide the first direct evidence of the occurrence of second-harmonic wavemaker displacements under first-order force control. Spinneken and Swan [16] derived the theoretical model to explain the additional second-harmonic displacement. However, the experimental data by Spinneken and Swan [17] did not provide an explicit confirmation of the nonlinear displacement. Indeed, the discussion in Spinneken and Swan [17] focused on the observed surface elevation, and did not address the wavemaker displacement. As a result, the data in Fig. 16 provide additional evidence of the general force-feedback model developed by Spinneken and Swan [16].

**4.4.3. Second-order control**

To achieve effective second-order force control, the effects noted in the context of second-order position control must be taken into consideration. Specifically, Section 4.3.4 demonstrated that a phase correction must be applied to the transfer function $F^{*}$. This phase correction, based on the experimental optimisation approach (Method II in Section 4.3.4), was once again adopted for the purpose of the present experiments. To be entirely clear, no further modifications to the phase correction were made, and $F^{*}$ was identical to that adopted in the context of position control. The function $F^{*}$ was then used in the expression for the second-order correction term, Eq. (20). Method I was also tested but did not lead to satisfactory free wave suppression.

Adopting Method II, Fig. 17 illustrates the water surface elevations for the four select cases as before. Within Fig. 17, the first-order force control case is included as the data with symbol C, and second-order force control is represented by the data points *. For all four cases shown, the experimentally observed surface elevations under second-order control closely match the theoretical Stream-function prediction (solid line).

Fig. 18 shows the spurious free wave content for the full set of nonlinear base cases, and includes both first-order and second-order force control. In close similarity to position control, the theoretical prediction (solid line) over-estimates the spurious free wave content. Nevertheless, the second-order correction is successful in eliminating a significant part of the spurious free wave content for most cases. Second-order control is particularly effective in the range $0.8 < kh < 1.4$, where the remaining free wave content may be as low as 5%. The free wave content of up to 30% for $kh < 0.8$ is most probably caused by an over-prediction of the second-order wavemaker displacement due to the nonlinear force feedback (Fig. 16). Indeed, this implies that the second-order force feedback is slightly over-corrected for $kh < 0.8$. For $kh > 1.5$, the small remaining spurious content is likely to be caused by an under-prediction of the second-order wavemaker displacement (Fig. 16). Setting aside these two small issues, the wave quality achieved under second-order force control is considered very good.

**4.4.4. Stream-function control**

Fig. 19 provides the spurious free wave content relating to force-controlled Stream-function wavemaking (symbol *). As before, the first-order force-control data (symbol C) are also shown. The experimental data relating to the Stream-function demand lead to very little spurious wave content throughout. In considering this data, it should be stressed that the only empirical correction relates to the wave field coefficient $C_{0}^{*}$, with no further phase modification at second harmonic.

Indeed, the excellent match of the data in Fig. 19 is somewhat surprising, given that position-control Stream-function wavemaking exhibited some small remaining departures (Fig. 12). While the comparison between these two cases (Figs. 19 and 12) highlights some variability in the application of nonlinear wavemaking theories, the data on Fig. 19 nonetheless provides evidence of the success of nonlinear force-feedback control. The remaining free wave content is <30% for all cases considered, with an average content of approximately 15%.

**4.5. Extended nonlinear set**

The above discussion highlighted that both second-order control and Stream-function control are suitable for the generation of mildly-nonlinear waves in relatively shallow water. To ensure that the range of validity of second-order control was maintained, all of the previous wave cases remained within the bounds of $S \leq 1$ (Table 1). A nonlinearity parameter of $S > 1$ indicates that the underlying second-order theory no longer applies.

To examine the influence of both large $S$ and $Ah$, an extended set of wave cases was considered. The extended set concerns two cases, as noted in Table 1, characterised by (i) $kh = 0.61$ and $S = 1.5$ (or $Ah = 0.092$) and (ii) $kh = 1.23$ and $Ah = 0.225$ (or $S = 0.858$). The reasons for choosing these two cases are as follows. Case (i) presents a case with $S > 1$, where the maximum wave steepness was limited by the available Stroke of the wavemaker. In effect, this case is the steepest...
Fig. 17. Water surface elevation $\eta(t)/\eta_{\text{in}}$ with Theoretical Stream-function solution and experimental data based upon □ first-order force control and * second-order force control. (a) $kh = 0.56$ and $S = 1$ (b) $kh = 1$ and $Ak = 0.15$ (c) $kh = 1.4$ and $Ak = 0.15$ (d) $kh = 1.8$ and $Ak = 0.15$.

Fig. 18. Spurious free wave content in force control for — theoretical content under first-order control and experimental data based on □ linear and * second-order wavemaker theory.

Fig. 19. Spurious free wave content in force control with — theoretical content under first-order control and experimental data based on □ first-order control and * Stream-function control.
shallow-water case than can be produced in the Coastal Wave Flume. Case (ii), with $k h = 1.23$, lies in the intermediate water depth regime. For $A k = 0.225$, the onset of wave breaking was observed, which limits the steepness for this particular case.

Both cases were produced in position control and force control. With the findings being very similar, only the force control data are reported here. Fig. 20 concerns the water surface elevation for both cases (i) and (ii), and relates to (a, c) second-order force control and (b, d) Stream-function force control. Considering case (i) in Figs. 20 (a) and (b), it is clear that first-order control (symbol ○) leads to a very pronounced free wave content. Expressed in terms of the ratio of free and bound second-harmonic wave, first-order control leads to a spurious wave content of $A^{2^{\text{nd}}} / A^{2^{\text{nd}}} = 0.61$.

This spurious free wave content reduces to $A^{2^{\text{nd}}} / A^{2^{\text{nd}}} = 0.24$ under second-order control (Fig. 20(a)) and $A^{2^{\text{nd}}} / A^{2^{\text{nd}}} = 0.11$ under Stream-function control (Fig. 20(b)). Given that $S = 1.5$, the difference between the two control modes is surprisingly small, indicating that second-order control may perhaps be stretched beyond $S = 1$. It is also important to re-emphasise that this wave case relates to the largest wave possible within the stroke limitation of the wavemaker. As a result, it may be argued that second-order control is adequate for the large majority of the regular wave generation range in a medium-sized coastal wave facility. However, for facilities with increased stroke capabilities, this may not apply.

Case (ii) with $A k = 0.225$ (but $S < 1$) in Fig. 20 (c) and (d) relates to a steep near breaking case. A first-order control signal leads to $A^{2^{\text{nd}}} / A^{2^{\text{nd}}} = 0.31$, which is approximately 15% less than theoretically predicted (Fig. 19). Both second-order control and Stream-function control lead to a small yet significant improvement, with $A^{2^{\text{nd}}} / A^{2^{\text{nd}}} = 0.24$ and $A^{2^{\text{nd}}} / A^{2^{\text{nd}}} = 0.17$ respectively. The difference between the two control methods is again relatively small, and it may be argued that second-order control is sufficiently accurate (Fig. 20(c)).

5. Conclusions

The present study investigated the generation of nonlinear regular waves in shallow and intermediate water conditions. Ultimately, the main advance lies in the development and validation of nonlinear generation techniques under force control. To ensure a direct comparison to the more established position-control operation, this case was examined first. In considering nonlinear position control, two empirical correction factors were found to be necessary. First, a magnitude correction of 15% was required in the first-order progressive wave field coefficient; this correction factor being constant over a wide range of water depths and wave steepnesses. Second, a phase correction proved crucial in the formulation of the second-order control signal. This

---

**Fig. 20.** Water surface elevation $\eta / A$ based on (a, c) second-order force control and (b, d) Stream-function force control showing theoretical Stream-function solution ○ experimental first-order control and * experimental nonlinear control. (a) $k h = 0.61$ and $S = 1.5$, second order (b) $k h = 0.61$ and $S = 1.5$, Stream function (c) $k h = 1.23$ and $A k = 0.225$, second order (d) $k h = 1.23$ and $A k = 0.225$, Stream function.
phase shift was found to be negligible in shallow water conditions, but plays an important role in the generation of high-quality waves in intermediate water depths. The methodology developed herein, combining theoretical predictions with empirical corrections, enables the successful compensation of the spurious free wave content in position control.

Based upon the position-control findings, high-quality force-controlled wave generation can also be achieved in both shallow and intermediate water. In this context, an important difference between flap-type and piston-type wavemakers was established. Previous work demonstrated that flap-type wavemakers may often rely on a linear force-demand signal. This is not the case for piston-type wavemakers. In the flap-type case, a nonlinear force-feedback term was previously shown to lead to an inherent self-correction of the spurious free wave content. In contrast, the nonlinear force-feedback arising in the piston wavemaker case is relatively small, and self-correction was not observed. The reason for this partially lies in the fact that piston wavemakers are associated with significant inertia, such that the nonlinear hydrodynamic forcing at the second-harmonic leads to little wavemaker motion. As a result, piston-type force-controlled wave-makers exhibit a spurious free wave content similar to that observed under position control. Given this relatively large free wave content, a nonlinear correction force is required.

To address this, both second-order and Stream-function correction terms were derived and experimentally verified. These nonlinear approaches enable force-controlled generation of comparable wave quality to position control. Furthermore, the work also demonstrated that second-order control may indeed be adequate for the large majority of the practical wave generation range in a typical coastal wave facility. Nevertheless, Stream-function control remains important for waves of very high steepness or located in shallow water conditions ([12]).

Acknowledgement

The authors would like to acknowledge the support provided by Edinburgh Designs Ltd, particularly the access to their control system and many fruitful technical discussions with Douglas Rogers. The work presented herein has also received financial support through UK EPSRC Grant EP/J010197/1.

Appendix A. Added mass and radiation damping for a piston-type wavemaker

The first-order hydrodynamic force $F_{\phi 1}$ acting on a piston-type wavemaker is expressed as the sum of a radiation damping term and an added mass term. While the radiation damping term is induced by the progressive wave field, the added mass term represents the forcing due to the evanescent wave field. If $X^{(1)}$ is the first-order wavemaker displacement, $F_{\phi 1}$ is given by

$$F_{\phi 1}^{(1)} = d_i \alpha X^{(1)} - m_i \alpha^2 X^{(1)}, \quad (A.1)$$

where the radiation damping $d_i$ is defined by

$$d_i = \frac{\partial^2 \rho \omega}{\partial \omega} \tanh k_0 h$$

and the added mass $m_i$ is equal to

$$m_i = -i \frac{\partial \rho}{\partial \omega} \sum_{j=1}^{\infty} \frac{c_j}{k_j} \tanh k_j h. \quad (A.2)$$

A full derivation of these expressions can be found in Spinneken [29].

Appendix B. Second-order wavemaker forces for a piston-type wavemaker

For ease of notation, and to provide a solution consistent with that in Schäffer [6], the second-order force expressions given herein are provided for the general case of an irregular wave solution. Within the irregular wave notation, wave components $n$ and $m$ form interaction pairs, and summations take place over $1 \leq n \leq N$ and $1 \leq m \leq M$, where $N$ and $M$ are positive integers. To apply the irregular wave solution to a regular wave case, only one single component $n = m$ is included, such that $N = M = 1$.

Following Spinneken [29], the second-order hydrodynamic force $F_{\phi 2}$ acting upon a piston-type wavemaker may be expressed as

$$F_{\phi 2}^{(2)} = F_{\phi 1}^{(2)} + F_{\phi 2}^{(2) +} + F_{\phi 2}^{(2) -} + F_{\phi 2}^{(2) \pm} + F_{\phi 2}^{(2) -}$$

where the individual force components are noted below. The full mathematical expressions required to obtain these forcing terms can be found in Spinneken [29], and, for brevity, only the final results are included here. For the regular wave cases discussed in the present study, the plus or sum terms will give rise to an interaction term at the second-harmonic of the fundamental wave component, while the difference terms will give rise to a mean forcing term.

The first-order unsteady wave acceleration integrated to the first-order free surface leads to a forcing component

$$F_{\phi 1}^{(2)} (t) = i R \left\{ \sum_{i=1}^{N} \sum_{m=1}^{\infty} \sum_{k_{nm}} \delta_m \rho \phi X^{(1)} \sum_{j=1}^{\infty} \sum_{c_{j}} \sum_{n} \sum_{m} \sum_{c_{j}} \sum_{n} \sum_{j} c_{m} k_{nm}^{*} e^{i(n \alpha_{m} - m \alpha_{n})} \right\} \quad (B.2)$$

where $\delta_m$ is 0.5 for $n = m$ and 1 for $n \neq m$. The changing effective wetted surface of the wavemaker introduces the second-order force component

$$F_{\phi 2}^{(2) +} (t) = i R \left\{ \sum_{i=1}^{N} \sum_{m=1}^{\infty} \sum_{k_{nm}} \delta_m \rho \phi X^{(1)} \sum_{j=1}^{\infty} \sum_{c_{j}} \sum_{n} \sum_{m} \sum_{c_{j}} \sum_{n} \sum_{j} c_{m} k_{nm}^{*} e^{i(n \alpha_{m} - m \alpha_{n})} \right\}$$

which is equal to $-F_{\phi 2}^{(1) \pm} / 2$. The force associated with the horizontal velocity term is

$$F_{\phi 2}^{(2) -} (t) = i R \left\{ \sum_{i=1}^{N} \sum_{m=1}^{\infty} \sum_{k_{nm}} \delta_m \rho \phi X^{(1)} \sum_{j=1}^{\infty} \sum_{c_{j}} \sum_{n} \sum_{m} \sum_{c_{j}} \sum_{n} \sum_{j} c_{m} k_{nm}^{*} k_{nm} e^{i(n \alpha_{m} - m \alpha_{n})} \right\} \quad (B.3)$$

(130)
where
\[
\Gamma_{j}^{c} = \begin{cases} 
-\frac{1}{2k_{p}} \{ \sinh k_{p}h \cosh k_{p}d \sinh k_{p}d \cosh k_{p}d \} \\
+ k_{p}(h-d) \text{ for } m\text{ and } j = l \\
-\frac{1}{k_{p} - k_{m}^{\pm}} \{ k_{p} \sinh k_{p}h \cosh k_{m}^{\pm}h - k_{m}^{\pm} \cosh k_{p}h \sinh k_{m}^{\pm}h \} \\
- k_{p} \sinh k_{p}d \cosh k_{m}^{\pm}d + k_{m}^{\pm} \cosh k_{p}d \sinh k_{m}^{\pm}d \text{ otherwise.} 
\end{cases}
\] (B.5)

Similarly, the force associated with the vertical velocity term is
\[
F_{i}^{(2)}(\omega^{2}) = R\left\{ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \delta_{mp} \rho g X_{n}X_{m}^{*} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \frac{c_{jn}k_{jn}^{m} - k_{jn}^{m} - k_{jn}^{m}}{\cosh k_{p}h \sinh k_{j}^{m}h} \phi_{j}^{m} e^{i(\omega_{j} \pm \omega_{m})t} \right\} 
\] (B.6)

where
\[
\Gamma_{j}^{v} = \begin{cases} 
-\frac{1}{2k_{p}} \{ \sinh k_{p}h \cosh k_{p}d \sinh k_{p}d \cosh k_{p}d \} \\
+ k_{p}(h-d) \text{ for } m\text{ and } j = l \\
-\frac{1}{k_{p} - k_{m}^{\pm}} \{ k_{p} \cosh k_{p}h \sinh k_{m}^{\pm}h - k_{m}^{\pm} \cosh k_{p}h \sinh k_{m}^{\pm}h \} \\
- k_{p} \cosh k_{p}d \sinh k_{m}^{\pm}d + k_{m}^{\pm} \cosh k_{p}d \sinh k_{m}^{\pm}d \text{ otherwise.} 
\end{cases}
\] (B.7)

and
\[
\sigma = \begin{cases} 
-1 \text{ if } \omega > 0 \text{ for subharmonics} \\
1 \text{ otherwise.} 
\end{cases}
\] (B.8)

The force induced by the second-order bound wave potential is given by
\[
F_{i}^{(2)}(\omega^{2}) = R\left\{ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \delta_{mp} \rho g X_{n}X_{m}^{*} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \frac{H_{jn}^{\pm} - c_{jn}k_{jn}^{m} - k_{jn}^{m}}{D_{jn}^{\pm} \sinh k_{j}^{m}h} \phi_{j}^{m} e^{i(\omega_{j} \pm \omega_{m})t} \right\} 
\] (B.9)

where
\[
H_{jn}^{\pm} = (\omega_{j} \pm \omega_{m}) \left( \pm \frac{k_{p} \sinh k_{p}h \cosh k_{m}^{\pm}h - k_{m}^{\pm} \cosh k_{p}h \sinh k_{m}^{\pm}h}{\cosh k_{p}h \sinh k_{m}^{\pm}h} \right) 
\] (B.10)

and \(D_{jn}^{\pm} = g(k_{p} \pm k_{m}^{\pm}) \text{ tanh}(k_{p} \pm k_{m}^{\pm})h - (\omega_{p} \pm \omega_{m})^{2}.\) (B.11)

Finally, the force induced by the spurious free wave potential is
\[
F_{i}^{(2)}(\omega^{2}) = R\left\{ 2 \sum_{p=0}^{\infty} \frac{c_{p}^{(22)} \pm}{(K_{p}^{\pm})^{2}} \phi_{p}^{m} e^{i(\omega_{p} \pm \omega_{m})t} \right\} 
\] (B.13)

where
\[
\Gamma_{p}^{c} = \frac{\sinh(K_{p}^{\pm}h) - \sinh(K_{p}^{\pm}d)}{\cosh(K_{p}^{\pm}h)} 
\] (B.14)

and \(K_{p}^{\pm}\) is the solution of
\[
(\omega_{p} \pm \omega_{m})^{2} = K_{p}^{\pm} g \tanh K_{p}^{\pm} h. 
\] (B.15)

The wave field coefficient \(c_{p}^{(22)} \) is again taken from Schäffer [6] as
\[
c_{p}^{(22)} = \frac{\delta_{mp}(\omega_{p} \pm \omega_{m}) \cosh^{2}(K_{p}^{\pm}h)}{g^{2} \delta_{mp}^{2} \Lambda(K_{p}^{\pm})^{2}} \left\{ \sum_{j=1}^{m} \sum_{l=1}^{m} \frac{c_{jn}k_{jn}^{m} \sinh k_{j}^{m}h}{(k_{p} \pm k_{m}^{\pm})^{2} - (K_{p}^{\pm})^{2}} \right\} 
\] (B.16)

where
\[
\Lambda(K_{p}^{\pm}) = \frac{1}{2} \{ K_{p}^{\pm} h + \sinh K_{p}^{\pm} h \cosh K_{p}^{\pm} h \}. 
\] (B.17)
References