Modelling the Dynamic Relationship between Systematic Default and Recovery Risk

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Learning without thought is labour lost; thought without learning is perilous.

— Confucius
To my family
Abstract

Default correlation modelling is becoming the most popular problem in the field of credit derivatives pricing. An increase in default risk would cause the recovery rate to change correspondingly. Correlation between default and recovery rates has a noticeable effect on risk measures and credit derivatives pricing.

After an introduction, we review the most recent literature covering default correlation and the relationship between default and recovery rates. We adopt the copula methodology to focus on estimating the default correlations rather than focus on modelling probabilities of default, we then use stress testing to compare the distributions of the probability of default under different copula functions. We develop a Gamma-Beta model to link the recovery rate directly with the individual probability of default, this is instead of an extended one factor model to relate them by a systematic common factor. One factor models are re-examined to explore correlated recovery rates under three distributions: the Logit-normal, the Normal and the Log-normal. By analyzing the results respectively obtained from these two classes of modelling scheme, we argue that the direct dependence (Gamma-Beta) model behaves better, in estimating the recovery rate given individual probability of default and in suggesting a better indication of their relationship. Finally, we apply default correlation and the correlated recovery rate to portfolio risk modelling. We conclude that if the recovery rates are independent stochastic variables, the expected losses in a large portfolio might be underestimated because the uncorrelated recovery risks can be diversified, so the correlation between default rate and recovery risk can not be neglected in the applications.

Here, we believe the first time, the recovery rate depends on individual default probability by means of a closed formula.
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Chapter 1

Introduction

The credit derivatives market was established in the early 1990s. An increasing variety of research topics is created as the credit market grows rapidly. The pricing of credit derivatives and the management of credit risk attract much interest from practitioners. In particular, managing credit portfolio risk becomes a research focus in this area.

Credit risk is a risk of loss resulting from the occurrence of a default when an obligor fails to make a promised payment. The credit risk of an asset comes mainly from three components:

(1) Probability of default

The probability of default refers to the probability that an obligor falls into default. There are several ways to define defaults. The classical definition relates a firm’s liabilities with its assets. When the asset value can not fully pay off the debt at maturity, a default happens. Another definition describes a default event that is determined by a certain threshold at any time. How to treat the occurrence of defaults is a vital element affecting the credit derivatives modelling and the risk management.

(2) Recovery risk

If a default event happens, the losses of an investor become a key
question of concern. Recovery risk arises from the uncertain quantity of the payoff that an investor actually receives given a default event. The positive recovery of a defaultable asset is viewed as the expected value of the payoff at the occurrence of a default. Usually, this recovery risk is determined by recovery rate. The proper framework to model recovery rate becomes an important research topic.

(3) Exposure at default

Exposure at default (EAD) is usually treated as an estimated amount to which a counterparty is exposed at the time of a default. All possible losses at default are estimated upon the EAD. Calculation of EAD is often guided by the relevant regulators.

The defaults tend to cluster in both credit derivatives and corporate bond markets. This clustering phenomenon is a significance in the risk management and pricing of credit derivatives. There has been much literature concentrating on modelling the first component of credit risk, probability of default. Yet, there is a little work focusing directly on default correlations, which plays a prominent role in default risk, so estimating the correlation parameter is a question of major concern. In this research, we use copula methodology to model the probability of default, we then use explicit estimation functions to obtain the estimates of default correlation. At the same time, stress testing and Bayes approach are applied to compare the advantages and disadvantages of different copula functions.

Many traditional credit risk models focus mainly on default risk and usually treat recovery rate as a constant or as a stochastic random variable independent of default probability, however, empirical evidence indicates that recovery rates tend to decline during the high default periods in an economic recession. Failure to consider this correlation in a portfolio risk model would lead to the risk being underestimated. Therefore, the relationship between probability of default and recovery rate can not be neglected. Recently, the analysis of this association has attracted much research attention.
An increase in default risk would cause the recovery rate to change correspondingly. Such correlation between default and recovery rates would have a noticeable effect on risk measures and credit derivatives pricing.

One factor models are used to capture systematic risk in default rates and simultaneously in recovery rates. In a general sense, the macroeconomic factor not only drives the defaults, but also determines the recoveries. We explore the extended one factor models based on different assumptions for the recovery rate distribution proposed by Frye (2000c) and Düllmann and Trapp (2004). By extending the direct dependence ideas addressed in Altman, Resti and Sironi (2001), we construct a Gamma-Beta model to state that the recovery rate directly depends on individual probability of default.

In this thesis, we contribute the following:

(1) We provide the estimates of default correlations implemented on the latest data base from Moody's by relaxing the assumption of fixed degrees of freedom in the T copula. The Bayes results show us that for very low rating classes, the T copula model behaves better than the Gaussian copula, however, for other higher ratings, there is little difference between these two copulas. For the investment grade group, the Gaussian copula plays a much better role. This contribution allows us to avoid the model risk inherent in the choice of different copulas and provides appropriate modelling guidelines in the fields of credit derivatives pricing and portfolio risk management.

(2) One factor models are re-examined to explore correlated recovery rates under Logit-normal, Normal and Log-normal distributions based on the more recent data from Moody's. The closed formulae for the expected recovery rates conditional on defaults are derived. This proves that the expected recoveries are overestimated if we consider the recovery rates irrespective of whether the defaults occur.

(3) As a key contribution in this research, we construct a Gamma-Beta model to state that the recovery rate directly depends on the individual
probability of default. The recovery rate is firstly introduced to directly depend on individual default probability in a closed formula.

(4) The two-sample Kolmogorov-Smirnov test confirms that the direct dependence model explains the relationship between recovery and default rates more adequately than the one factor model.

(5) If the recovery rate is taken as an independent stochastic variable, the uncorrelated recovery risks can be diversified in a large portfolio. The correlations between default and recovery rates can not be neglected in applications. By testing the tail distributions under two different frameworks, we argue that a fatter tail of loss distribution can be captured by the direct dependence model.

The thesis consists of six chapters. Chapter 2 reviews the most recent literature on credit modelling including two classic frameworks, intensity models and structure models. Most of the work about default correlations and correlated recovery rates is developed from these two frameworks. We also describe the work involving default correlation and the relationship between default and recovery rates. In Chapter 3, we explain the use of copula methodology to model default correlation in detail, and then give the explicit closed formula of estimation procedure. The models are implemented empirically. Finally, we compare correlation parameters inferred from different copulas by using stress testing. In Chapter 4, we re-examine the existing models to estimate the correlation parameter relating default and recovery variables based on more recent data. Next, a new model to describe such a link is set up theoretically. Then, we implement the direct dependence model and obtain optimal parameter estimates. Detailed estimation methods are introduced as well. We use the Jarque-Bera and two-sample Kolmogorov-Smirnov tests to compare different distributions and examine how the simulated results are close to the empirical observations under different models. Chapter 5 applies the previous results given in Chapter 3 and Chapter 4. We analyze the portfolio loss distributions under different frameworks. Simula-
tion results lead us to draw the conclusions in Chapter 6. Finally we conclude and provide some suggestions for possible future research in this field.
Chapter 2

Literature Review

Approaches to describe the default process mainly use the structural model or the intensity based model. The structural model considers economic arguments and a default event occurs when the debt is not able to be fully paid off by the asset value at maturity, that is to say, the borrower’s assets fall below its liabilities. The Merton model (1974) is thought of as the first structural model to determine the time of default using a firm’s structural variables. The intensity based approach models a firm’s default by introducing some unpredictable Poisson-like events directly, rather than by providing economic arguments. In this approach, defaults have been usually treated as a jump diffusion process with given intensity. Black and Cox (1976) advanced a certain threshold to determine a default happening. Default can occur at any time in intensity-based models, while in structural models default is usually assumed to only happen at maturity, such that, for Merton model the default time $\tau$ can be written as

$$\tau = T \mathbf{1}_{\{V_T < F\}} + \infty \mathbf{1}_{\{V_T \geq F\}},$$

where $F$ refers to the bond face value at maturity $T$, and $V_T$ asset value. The indicator function $\mathbf{1}_{\{V_T \geq F\}}$ represents

$$\mathbf{1}_{\{V_T \geq F\}} = \begin{cases} 1 & \text{if } V_T \geq F \\ 0 & \text{if } V_T < F \end{cases}.$$
For the Black and Cox model, the default time \( \tau \) is defined as
\[
\tau = \inf \{ t \geq 0 : V_t \leq \bar{v}(t) \},
\]
where \( \bar{v}(t) = K e^{-\gamma(T-t)} \) with \( K \) and \( \gamma \) constants and \( K e^{-\gamma T} < V_0 \).

In the following sections, we recall these two classes of framework and review how they can be applied to default correlation modelling.

### 2.1 Credit Derivatives modelling

#### 2.1.1 Intensity Models

In intensity-based models, the default arrival time \( \tau \) is assumed to be the first jump time of the Poisson process \( N_t \). The exponential distribution with a random variable \( T \) and a parameter \( \lambda \) has the following formula:
\[
P(T \leq t) = 1 - e^{-\lambda t}, \quad t \leq 0.
\]
A random variable \( T \) satisfies:
\[
P(T \leq s + t | T > s) = P(T \leq t), \quad \forall s, t \leq 0.
\]
So that the default probability \( P(s, T) \) can be given as
\[
P(s, T) = P(\tau \leq T | \tau > s) = 1 - e^{-\lambda(T-s)}.
\] (2.1)

If the intensity \( \lambda \) is not a constant, it is a function depending on the parameter \( t \), and then (2.1) can be extended by
\[
P(s, T) = P(\tau \leq T | \tau > s) = 1 - \exp \left( - \int_{s}^{T} \lambda(u) du \right).
\] (2.2)

Suppose a random variable \( X \) follows a Poisson process with intensity parameter \( \lambda \), and then \( X = K \) has the following probability
\[
P(X = K) = e^{-\lambda} \frac{\lambda^K}{K!}, \quad K = 0, 1, 2, \ldots.
\]

The Poisson process \( N_t \) has these properties:
• The probability of a jump occurring in the interval \([t, t + \Delta t]\) is independent of a jump on or before time \(t\), which implies that the Poisson process \(N_t\) has independent increments.

• Given exactly one jump happening in the interval \([t, t + \Delta t]\), the jump time has the uniform distribution on \([t, t + \Delta t]\).

• The expectation of \(\Delta N\) is equal to the intensity rate \(\lambda\) timing \(\Delta t\).

In the case that the density parameter in a Poisson process is a function of time \(\lambda(t)\) instead of a constant \(\lambda\), we have the definition 2.1.1 to specify the Poisson increment property under the assumption of a stochastic intensity process (see Zheng, 2005).

**Definition 2.1.1** If the intensity \(\lambda_t\) in Poisson process \(N_t\) satisfies \(\mathcal{F}_t\) predictable, then \(N_s - N_t\) has a Poisson distribution with parameter \(\int_t^s \lambda(u) du\), so that

\[
P(N_s - N_t = n|\mathcal{F}_t) = \mathbb{E}\left( \frac{1}{n!} \left( \int_t^s \lambda(u) du \right)^n \exp\left( - \int_t^s \lambda(u) du \right) \biggm| \mathcal{F}_t \right).
\]

Given the default time \(\tau > s\) and all known information at time \(s\), the survival probability \(\bar{P}(t, s)\) from \(t\) to \(s\) is obtained by

\[
\bar{P}(t, s) = P(\tau > s|\mathcal{F}_t) = P(N_s - N_t = 0|\mathcal{F}_t) = \mathbb{E}\left( \exp\left( - \int_t^s \lambda(u) du | \mathcal{F}_t \right)\right).
\]

In Zheng (2005), the default probability in a very small interval time \(\Delta t\) is given by

\[
P(\tau < t + \Delta t|\mathcal{F}_t) = \mathbb{E}\left( 1 - \exp\left( - \int_t^{t+\Delta t} \lambda(u) du \right) \biggm| \mathcal{F}_t \right) \approx \lambda(t)\Delta t.
\]

In the Cox, Ingersoll and Ross (1985) (CIR) models, the intensity process \(\lambda_t\) follows a mean reverting process:

\[
d\lambda_t = k(\eta - \lambda_t)dt + \sigma \sqrt{\lambda_t}dW_t, \quad (2.3)
\]
where a constant $\eta$ is a long-run rate of time change, $\lambda$ reverts to $\eta$ at a rate $k$, $W_t$ is a standard Brownian motion process, and $k, \eta, \sigma$ are positive constants. Then the survival probability conditional on $\tau > t$ has the following closed formula:

$$\bar{P}(t, s) = \exp\left(\alpha(s - t) + \beta(s - t)\lambda(t)\right).$$

The details of the CIR model are discussed in Cox, Ingersoll and Ross (1985).

### 2.1.2 Structural Models

We concentrate on structural models to capture the default correlation by considering the firm’s asset value.

The structural approach is to model default events in terms of fundamental firm variables based on economic arguments. It assumes that complete knowledge of all the firm’s assets and liabilities is available so that a default is predictable.

The model proposed by Merton (1974) is typical. It assumes the firm is financed by equity and a zero coupon bond, and also assumes that the default can only occur at the maturity of the zero coupon bond, no matter what level the firm’s asset value reaches before the default time. The total asset value of the firm can be expressed as follows:

$$dV_t = rV_t dt + \sigma V_t dW_t,$$

(2.4)

where $\sigma$ is the constant volatility of the asset price, $r$ a fixed interest rate, and $W_t$ the standard Brownian motion. Thus

$$V_t = V_0 e^{(r - \frac{1}{2} \sigma^2) t + \sigma W_t},$$

(2.5)

where $V_0$ is the positive value of the firm at time zero. The value of the asset is the sum of the values of the debt and the equity. Default occurs when the firm can not pay off its debt fully at maturity time.

The Merton model assumes that the firm pays the bond investors before it pays the equity investors, so the value of zero coupon bond $\bar{P}$ at maturity
time $T$ can be denoted by

$$
\bar{P}(T, T) = \min(F, V_T) = F - \max(F - V_T, 0), \quad (2.6)
$$

where $F$ is the face value of the zero coupon bond, $V_T$ the asset value at time $T$ stated in (2.5). Therefore, the equity value $S$ at time $T$ is equal to

$$
S_T = V_T - \bar{P}(T, T) = \max(V_T - F, 0), \quad (2.7)
$$

which is similar to the expression for a European call option. Following the Black-Scholes European call option formula, we have the value of the equity $S_t$ at time $t \leq T$,

$$
S_t = V_t \Phi(d_1) - Fe^{-r(T-t)}\Phi(d_2),
$$

where

$$
d_1 = \ln(V_t/F) + \frac{(r + \frac{1}{2}\sigma^2)(T-t)}{\sigma \sqrt{T-t}} \quad \text{and} \quad d_2 = d_1 - \sigma \sqrt{T-t}.
$$

Thus, the zero coupon bond value $\bar{P}(t, T)$ at time $t \leq T$ can be given by

$$
\bar{P}(t, T) = V_t - S_t = V_t \Phi(-d_1) + Fe^{-r(T-t)}\Phi(d_2).
$$

In the Merton model, the default only could happen at maturity $T$ of the zero coupon bond, so the default time $\tau$ is given by

$$
\tau = T 1_{\{V_T < F\}} + \infty 1_{\{V_T \geq F\}}.
$$

The bond value $\bar{P}(T, T)$ in (2.6) can be given as

$$
\bar{P}(T, T) = V_T 1_{\{\tau = T\}} + F 1_{\{\tau = \infty\}}.
$$

The risk-neutral probability of default at time $t \leq T$ is denoted by

$$
P(\tau = T \mid \mathcal{F}_t) = P(V_T < F \mid \mathcal{F}_t) = P(V_t e^{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)} < F \mid \mathcal{F}_t) = P(\sigma(W_T - W_t) < -\ln(V_t/F) - (r - \frac{1}{2}\sigma^2)(T-t)) = \Phi(-d_2).
$$
2.1 Credit Derivatives modelling

Since the Merton model assumes the firm only defaults at the maturity of the zero coupon bond, which is in general not true, other more complicated structural models are proposed based on different assumptions about default time. In the First Passage Time model, the asset value and the default boundary are assumed to follow some stochastic processes, and the default is assumed to occur at the time when the two processes first cross each other. The default time \( \tau \) is given by

\[
\tau = \inf\{t \geq 0 : X_1^t \leq X_2^t\},
\]

where \( X_1^t \) and \( X_2^t \) are two stochastic processes respectively referring to asset value and default boundary. In the Leland and Toft (1996) model, the default time \( \tau \) is defined as

\[
\tau = \inf\{t \geq 0 : V_t \leq \bar{v}\},
\]

where \( \bar{v} \) is assumed to be a constant with \( \bar{v} < V_0 \). However, in the Black and Cox (1976) model, the boundary level is assumed to be \( \bar{v}(t) \), which is a variable changing over time \( t \). The default time \( \tau \) is given by

\[
\tau = \inf\{t \geq 0 : V_t \leq \bar{v}(t)\},
\]

where \( \bar{v}(t) = Ke^{-\gamma(T-t)} \) with constants \( K \) and \( \gamma \) and \( Ke^{-\gamma T} < V_0 \).

The structural approach usually models the market value of a firm’s assets, and also assumes that the firm defaults if the reference firm’s outstanding debts hit a default boundary. The relationship between the value of a firm’s assets and the value of a firm’s equity leads the possibility of simultaneously modelling the equity value and the firm’s credit quality using structural models. Therefore, default correlation can be inferred from a correlation between returns on different assets using a structural model with one factor being the asset value of the reference firm.
2.2 Theories and Models for Default Correlation

Although these two classes of models were initially proposed to capture the individual firm’s default phenomena, the basic frameworks to model the dependency of the defaults are generally developed from them. The modelling of default correlations between different firms is extended from single firm cases based on these two frameworks.

There are several different ways to produce default correlations among firms based on intensity models. The first and very basic one is the conditional independent defaults (CID) approach, which is also linked to factor models. Duffie and Singleton (1999) introduced the joint jumps in the default process of a firm to extend CID models, but there are some difficulties in estimation and calibration.

An extension of this approach is the infectious defaults model developed by Davis and Lo (1999, 2000) and Jarrow and Yu (2001). The default intensities are treated as the joint jumps in a discrete amount. The calibration of this model is very complicated, and sometimes impossible. This model has not been fitted to market observations over a fixed time horizon.

Another modelling framework evolved from intensity-based models was addressed by Duffie and Singleton (1999) and Kijima (2000). The different processes separately representing the joint default events are introduced to allow the stronger dependency of the defaults. An intensity must be specified to denote each possible joint event of defaults. This approach overcomes the drawback of lower default correlations, but causes the number of these events to increase dramatically with the number of obligors. This approach assumes that the default events occur to the several obligors at same time, which is unrealistic in the real market.

The other popular approach is the use of copula functions. A copula is a function that formulates a multivariate joint distribution to link two or more univariate marginal distributions. Given the marginal functions and
a correlation structure, we can specify a joint distribution function with a copula, so that the copula is a very convenient way to transform this dependence frame.

The static copula model is based on the original time horizon. In a static model, the default structure is determined by a single variable value for the whole life of model, thus the static copula can not capture default contagion. Gaussian copula models were proposed by Vasicek (1987), Li (2000) and Gregory and Laurent (2005). They still focus on static modelling framework. For the default contagion problem, it is necessary to extend this methodology to a dynamic copula function. Dynamic models are very important for the valuation of some structures, for example, options on CDOs.

The development of a specific dynamic model was considered by Hull and White (2007). A jump process is assumed to represent the default probability for an individual obligor. In a general framework, the modeled distributions of default probability remain the same for all obligors. This development is based on structural models.

The dynamic intensity-based models describe the hazard rates by correlated diffusion processes, but it is difficult to calibrate the model to market data. Schönbucher and Schubert (2001) provided a continuous time dynamic model. The default probability of all obligors is continuously consistent with the observations of the default occurred in the other obligors. Combined with the intensity-based models for individual obligors, copula functions are used to structure the dependency of the times of default.

A one-period model, Credit Metrics by JP Morgan (1997) is designed to manage market risk via rating changes. The approach of Credit Metrics has been refined by Belkin et al (1998), Finger (1999) and Lucas et al (1999). Later, Moody’s (2004) argues that these approaches have not used the transition data optimally. They proposed a directional rating transition matrix (DRTM) model, which sets up the joint rating transition matrix only taking the directions of rating changes into account. The direction includes three states: no movement, upgrade and downgrade. Since rating transitions hap-
pen far more frequently than actual defaults, a much richer set of data can be provided in this way. However, it does not take default contagion into account.

DRTM is a methodology to derive asset correlations by using the rating transitions. There exists a relationship between asset correlation and default correlation, so in this way, the default correlation can be inferred when the asset correlation is given.

For the modelling framework, let us also consider the possible data sources on which the approaches are mainly based.

There are three data sources usually used in the recently developed models (Schönbucher, 2006):

- Direct historical observations of default

They are based on real default data, rather than inferred from other sources, so the advantage is that they directly describe the modelling problem and can be easily interpreted. However, the disadvantage is that defaults are fairly rare events, so that it is difficult to draw inferences across a wide range of industries, of default probabilities, or regions. In many cases direct data are not available.

- Credit spreads

Credit spreads contain much more information about the default risk, and changes in credit spreads infer changes in the riskiness of these investments. Thus if there exists strong correlation between the credit spreads of two obligors, it is reasonable to believe that the defaults of these obligors are also correlated strongly.

However, it is difficult to decompose credit spreads into a default probability component and other components based on liquidity and other aspects.

- Equity price
Equity price data is quite easy to obtain and usually of better quality than credit spreads. However, the connection between equity prices and credit risk is not very obvious. It is not easy to explain the market credit spreads through a theoretical model constructed to capture the link between them.

We can see that none of these sources is perfect, so the choice of the data sources depends on where we focus our modelling.

2.3 Relationship between Default and Recovery Rates

During the past few years, some approaches to modelling the correlation of recovery rates with the probability of default are appearing, and also associated with empirical investigations. These models cover Frye (2000a and 2000b), Jarrow (2001), Altman, Resti and Sironi (2001), Altman et al (2005).

Frye (2000a and 2000b) applied a conditional approach and proposed that a single macroeconomic factor links the probability of default and the recovery rate. These models assume that a business cycle causing more default events might also lead the recovery rate to decline. The distribution of recovery rate differs over distinct default clustering periods. In this framework, the systematic factor representing the economic state determines both default and recovery rates, therefore, the correlation between them is inferred from their dependence structure on the same common factor.

This class of models is extended from one factor conditional models. It implies that when a default occurs on a loan, the recovery rate is based on the value of its collateral, meanwhile, the state of economy affects the value of assets as well as the value of collateral. If the economy is in a boom cycle, recovery rates tend to increase due to the decreasing of default rates. Theoretically, there exists a negative correlation between default and
recovery rates.

Frye (2000b) analyzed this extended one factor model empirically by testing historical data from the U.S. corporate bond market. A strong negatively correlated relationship between default and recovery rates has been observed from the results. He drew a conclusion that recoveries of bonds would decline during a normal business cycle by 20-25 percent when a recession occurs.

Jarrow (2001) proposed a new approach to estimate default probability and recovery rate, in this model, these two variables are correlated by the macroeconomic factor as discussed in Frye (2000a and 2000b). However, this methodology separately identifies the recovery rate and probability of default by explicitly importing equity prices into the estimation process. In addition, Jarrow (2001) introduced a liquidity premium in the estimation, which is thought of as an essential due to the highly volatile yield spreads between risky and government securities.

Altman et al (2001) addressed another interpretation for the relationship between default and recovery rates. They found consistent results with Frye’s that a strong negative correlation does exist, but a single systematic factor is not enough to predict the changes of recovery rate. They advanced a supply and demand framework that drives the trends of recovery rates rather than a common economy factor to determine the dependence of the two variables.

The supply of defaultable securities seems to exceed their demand in high default periods, which affects the performance of a secondary market, therefore, the recovery rates tend to decline. Altman et al (2005) suggested that recovery rates can be expressed as a function depending on supply and demand of the securities, in which the probability of default plays a significant role.

According to the literature so far, most studies concentrate on linking the default and recovery rates via an indirect factor, e.g. economic performance. However, credit ratings also influence the changes of recovery rates, in this case, a single systematic factor is less predictive in the dependent structure between these two variables. Instead, we would consider a more direct way to
model the recoveries depending on the probabilities of default in this paper. Altman et al (2001) addressed this possible idea, but have not explored it further. We are going to present an explicit expression for recovery rates directly conditional on the individual probability of default, which is our key contribution in this paper.

The distribution that a stochastic recovery rate follows is another aspect affecting portfolio loss. Frye (2000c) assumed recovery rates follow a normal and a log-normal distributions, while Düllmann and Trapp (2004) extended the assumption to the logit-normal distribution, and then made a comparison between them. We see that there are not any significant differences of recovery estimators under these different assumptions, but they do have different influences on the tails of the loss distributions.

2.4 Summary

In this chapter, we review the literature involving credit derivatives modelling, and look at the more recent work on default correlations and the relationship between default and recovery rates.

The structural model is based on economic arguments. A default is treated as the event when the asset value is not able to fully pay off the debt at the maturity. Merton (1974) firstly addressed this framework to determine the default time from a firm’s structural variables. In intensity models, the unexpected events are directly introduced to represent a firm’s default. Black and Cox (1976) advanced that a certain default threshold is able to determine the default occurring at any time. Cox, Ingersoll and Ross (1985) proposed a more complicated model by introducing a mean reverting process describing a default.

Most of the work to analyze correlated default probabilities and correlated recovery rates is developed from the intensity models and the structural models.

There are different ways to capture correlations of defaults based on in-
tensity models. The conditional independent defaults (CID) approach is firstly to model the correlated defaults. This approach is also linked to the structural models. Davis and Lo (1999, 2000) and Jarrow and Yu (2001) extended the CID model to deal with the default intensity with joint jumps at the default occurring time. Duffie and Singleton (1999) and Kijima (2000) used the separate process to represent the joint default events. Another popular and important approach is the use of copula functions. The univariate marginal distributions can be linked to the joint multivariate distributions by a copula function. Vasicek (1987), Li (2000) and Gregory and Laurent (2005) proposed the Gaussian copula models to focus on the static modelling framework. Hull and White (2007) extended the static copulas to a specific dynamic model. Schönbucher and Schubert (2001) provided a continuous time dynamic model to connect the default dependencies with the joint dynamic default intensities.

With the development of correlated defaults modelling, the correlation of recovery rates with the default probability was attracted much concerns during the past few years. The models focus on the extended one factor conditional framework. A business cycle causing more defaults is assumed to lead the recovery rate to decline. Jarrow (2001) proposed an approach to import the equity prices into the estimation and introduced a liquidity premium in addition. Altman et al (2001) advanced a supply and demand framework that drives the trends of recovery rates rather than a common economy factor to determine their dependency.

According to the literature on correlated recovery rates so far, most studies concentrate on connecting the default and the recovery via an indirect factor. Altman et al (2001) proposed the idea to link the recovery rates directly with default probability, but have not explored the model further. In our work, we are going to extend this framework and set up a direct model to capture the dynamic relationship between the default and the recovery rates.
Chapter 3

Default Correlation Modelling

Default correlation describes the dependency of defaults for different firms over the same period. The firms in the same region or in the same industry have a tendency to experience economic problems at the same time since similar external macroeconomic conditions tend to have an impact. In the pricing of credit derivatives, modelling default dependency becomes an important and demanding problem.

Default correlation modelling is based on the way the probability of default is treated. In intensity models, default probability is assumed to be a stochastic process depending on macroeconomic variables, so that the correlation of defaults can be generated from the random variables denoting the business cycle. Structural models assume that the value of a firm’s asset follows a stochastic process. If an asset value falls below a given default level, a default event occurs. The correlation of defaults under the structural framework is produced by modelling the correlated stochastic processes representing the assets in different companies.

This chapter begins with a preliminary review on the probability of default. After this, the default correlation modelling is discussed under the different frameworks theoretically and empirically.
3.1 Preliminaries

3.1.1 Probability of Default

Let us define the prices of non-defaultable zero coupon bonds as $B(t, T)$, and the prices of defaultable zero coupon bonds as $\overline{B}(t, T)$. To ensure no arbitrage probability exists, the value of the defaultable bonds must be required to be less than non-defaultable bonds with the same maturity time, that can be expressed as:

$$0 \leq \overline{B}(t, T) < B(t, T) \quad \forall \quad t < T.$$

If we assume the recovery rate is zero, the price of a defaultable bond $\overline{B}$ at time $t$ is written:

$$I(t)\overline{B}(t, T) = \begin{cases} \overline{B}(t, T) & \text{if } \tau > t, \\ 0 & \text{if } \tau \leq t, \end{cases}$$

where $\tau$ is the time of default. The payoff of the zero coupon bonds at the maturity time $T$ is as follows:

$$\text{Payoff} = 1_{\{\tau > T\}} = \begin{cases} 1 & \tau > T \quad \text{if default after } T, \\ 0 & \tau \leq T \quad \text{if default before } T. \end{cases}$$

We can get the following fundamental relationship between defaultable bonds and non-defaultable bonds:

$$\overline{B}(t, T) = \mathbb{E}\left[e^{-\int_t^T r(s)ds} \cdot I(T)\right] = \mathbb{E}\left[e^{-\int_t^T r(s)ds} \mathbb{E}[I(T)]\right] = B(t, T)\mathbb{E}[I(T)] = B(t, T)P_s(t, T), \quad (3.1)$$

where $P_s(t, T)$ is the survival probability in the interval $[t, T]$. So the survival probability from $t$ to $T$ can be given as:

$$P_s(t, T) = \frac{\overline{B}(t, T)}{B(t, T)},$$
and the probability of default (PD) over \([t, T]\) is

\[
PD(t, T) = 1 - P_s(t, T).
\]

The survival probability \(P_s(t, T)\) is a positive decreasing function of \(T\).

### 3.1.2 Motivation

Modelling default correlation is linked to a general demand for developing market-based credit risk management. Basically, the individual probability of default does not significantly affect the whole risk of a portfolio since the risks can be diversified if there are not exceptionally large exposures. However, if there exists a strongly systematic dependency among individual defaults, a portfolio will be exposed to a very high risk. Consistently modelling the default correlation is an essential part of credit derivatives pricing and portfolio risk management.

Let us look at the evidence of observed default rates during the period of 1920-2006. Figure 3.1 shows us that there exist some clusters of high numbers of defaults observed within the different credit rating classes (B & Ba). If defaults have no correlation, the historical data should behave like the simulated data obtained by simulating independent defaults of 2000 obligors with individual default probability 4.09% (B) and 1.31% (Ba). We can see the majority of historical default rates are far from the average default rate of the companies. We also observe that the default rates between different credit rating classes are correlated strongly. For example, during the periods 1930-1935 and 1980-1995, the defaults occurred more frequently across both credit ratings B and Ba (Figure 3.1). Therefore, the correlation between individual default rates is an important factor in explaining historical default events.

With reference to portfolio credit risk modelling, the basic properties of a good model are stated in Schönbucher (2006).

- Default dependence: The model must be able to develop dependent defaults for a portfolio.
Figure 3.1: Average historical default rates 1920-2006 compared with simulated paths of the independent defaults

- **Dynamic property:** It should be capable of modelling the number of defaults as well as the timing of defaults, the model should not just fix a time horizon with only one step.

- **Clustering:** The model should reproduce some periods in which defaults tend to cluster. That is, several defaults occur close to each other, but not at the same time.

- **Estimation of parameters:** The number of parameters to describe the default dependence structure has to be limited and should not grow dramatically with the number of obligors for the purpose of easy calibration.
The feasibility of implementation is a very basic requirement for the model.

So our major task is to model default correlations while capturing the above properties.

3.2 Factor Models

3.2.1 Description

A one factor model is very basic and easily understood. The common factor $Y$ and specific factor $\varepsilon_n$ are correlated by the coefficient $\rho$. Defaults are driven when the firm’s asset value drops below a threshold at the time horizon $T$.

The value of asset return can be written as:

$$V_n(T) = \sqrt{\rho} \cdot Y + \sqrt{1 - \rho} \cdot \varepsilon_n,$$

where $Y$ is the systematic factor, and $\varepsilon_n$ is an idiosyncratic factor with a standard normal distribution. A linear correlation coefficient $\rho$ relates the asset values of two obligors. An indicator of the business cycle can be represented by the systematic risk factor $Y$, and the idiosyncratic factor $\varepsilon_n$ is viewed as a firm-specific factor such as the quality of the individual firm.

The individual conditional default probability $p(y)$ is the probability that the firm’s value $V_n(T)$ falls below the barrier $K$, given that the systematic factor $Y$ takes the value $y$:

$$p(y) = P(V_n(T) < K | Y = y)$$

$$= P(\sqrt{\rho} \cdot Y + \sqrt{1 - \rho} \cdot \varepsilon_n < K | Y = y)$$

$$= P\left(\varepsilon_n < \frac{K - \sqrt{\rho}Y}{\sqrt{1 - \rho}} \bigg| Y = y\right)$$

$$= \Phi\left(\frac{K - \sqrt{\rho}y}{\sqrt{1 - \rho}}\right).$$

Conditional on the systematic factor $Y$, the firm’s values and defaults are independent. Therefore, we also call this model the Conditionally Independent
Default (CID) model. Here, the barrier level \( K \) is defined by \( K = \Phi^{-1}(p_n) \) with individual default probability \( p_n \).

For having exactly \( n \) defaults, the probability based on the conditional probabilities is:

\[
P(M = n) = \int_{-\infty}^{+\infty} P(M = n | Y = y) \phi(y) dy
\]

where \( M \) denotes the number of defaults. Then, we substitute (3.3) for \( p(y) \) in (3.4). For the distribution of \( M \), we can have

\[
P(M \leq m) = \sum_{n=0}^{m} P(M = n).
\]

We invoke the theorem 3.2.1 to calculate the probability of default.

**Theorem 3.2.1 (Strong Law of Large Numbers)** If there is an infinite sequence of random variables \( X_1, X_2, \ldots, X_n, \ldots \), which are identically and independently distributed with \( \mathbb{E}(|X_i|) < +\infty \), then

\[
P\left( \lim_{n \to +\infty} X_n = \mu \right) = P\left( \lim_{n \to +\infty} \frac{1}{n} \sum_{i=1}^{n} X_i = \mu \right) = 1,
\]

where \( \mu = \mathbb{E}(X_i) \).

If the number of obligors \( N \) tends to infinity, we denote the fraction of the defaulted obligors in the portfolio as \( L, L = \frac{M}{N} \), and have

\[
P(L = p(y)|Y = y) = 1
\]

by applying Strong Law of Large Numbers. So, formula (3.5) can be extended as

\[
P(L \leq l) = \int_{-\infty}^{+\infty} P(L \leq l | Y = y) \phi(y) dy
\]

\[
= \int_{-\infty}^{+\infty} P(L = p(y) \leq l | Y = y) \phi(y) dy
\]

\[
= \int_{\Phi^{-1}(l)}^{+\infty} \phi(y) dy = \Phi(y^*),
\]

where \( \Phi \) is the cumulative distribution function.
where \( y^\ast \) is derived from \( p(y) \leq l \), such that

\[
\Phi\left( \frac{\Phi^{-1}(p_n) - \sqrt{\rho}y}{\sqrt{1 - \rho}} \right) \leq l \Leftrightarrow y \geq \frac{1}{\sqrt{\rho}} \left( \Phi^{-1}(p_n) - \sqrt{1 - \rho} \Phi^{-1}(l) \right) = -y^\ast.
\]

So, \( y^\ast \) is set to

\[
y^\ast = \frac{\sqrt{1 - \rho} \Phi^{-1}(l) - \Phi^{-1}(p_n)}{\sqrt{\rho}},
\]

and then (3.6) can be written as

\[
P(L \leq l) = \Phi\left( \frac{\sqrt{1 - \rho} \Phi^{-1}(l) - \Phi^{-1}(p_n)}{\sqrt{\rho}} \right).
\]

### 3.2.2 Application

Using the formula (3.7), we can test the effects of different correlations on default frequency (see Schönbucher, 2006). To simplify, we assume the individual default probability \( p_n = 5\% \), which means \( K = \Phi^{-1}(5\%) \) for all obligors \( N = 100 \), and then consider the default loss distribution respectively under different correlations \( \rho = 0, 0.01, 0.1, 0.3, 0.5 \). Figure 3.2 shows the probability of default changing with different correlations based on one-factor model.

From Figure 3.2, we can easily observe that increasing default correlation leads to the density becoming more positively skewed and fatter right hand tails. Looking at the logarithmic figure on the bottom, high correlation has a pronounced effect on the default distribution in its right hand tail. In contrast, the probabilities decrease very quickly for independence and very low correlations.

So, if \( p_n \) and \( K \) are given, the correlation coefficient can be estimated by Maximum Likelihood or Least Squares methods.

Intuitively, the factor model captures default dependency, has a closed formula for the default distribution allowing an easily built and implementable model.
Chapter 3. Default Correlation Modelling

Figure 3.2: Probability of default based on one-factor model under the different correlations

3.3 Copula Models

3.3.1 Definitions and Properties

Copula functions are ideal tools to describe how a dependent default framework can be characterized by the given individual term structure of default risk.
When we want to capture the dependent default times of some obligors, we have to solve two problems. On the one hand a single default term structure is demanded and on the other hand the default dependency among different obligors needs to be modeled as well. A copula function combines these two tasks perfectly. We may construct a multivariate distribution with different margins and the dependence structure obtained from a given copula function (Schönbucher, 2006).

**Definition 3.3.1 (Copula)** A function $C : [0, 1]^n \rightarrow [0, 1]$ is defined a $n$-dimensional copula if the following conditions are satisfied:

- Random variables $U_1, \ldots, U_n$ take values in $[0, 1]$ with their distribution function $C$.
- $C$ has uniform marginal distributions, for all $i \leq n$, $u_i \in [0, 1]$ 
  $$C(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i$$
- For all $(v_1, \ldots, v_n) \in [0, 1]^n$, $C(v_1, \ldots, v_n) = 0$ if any $v_i = 0$.

The essence of the copula functions is Sklar’s theorem which gives their properties.

**Theorem 3.3.1 (Sklar)** If random variables $X_1, X_2, \ldots, X_n$ have marginal distribution functions $F_1, F_2, \ldots, F_n$ and joint distribution function $F$. Then $F$ has the unique $n$ dimensional copula representation:

$$F(x_1, x_2, \ldots, x_n) = C(F_1(x_1), F_2(x_2), \ldots, F_n(x_n))$$

and the copula $C$ can be expressed as:

$$C(u_1, u_2, \ldots, u_n) = F(F_1^{-1}(u_1), F_2^{-1}(u_2), \ldots, F_n^{-1}(u_n))$$

If $F_1, F_2, \ldots, F_n$ are continuous, then $C$ is unique. The copula $C(\cdot)$ is the distribution function of random variables $U_1 = F_1(X_1), U_2 = F_2(X_2), \ldots, U_n = F_n(X_n)$. 
This theorem tells us that there exists a corresponding copula function for any multivariate distribution function on $\mathbb{R}^n$ and also suggests that the univariate margins and dependence scheme can be separately described by a copula.

**Theorem 3.3.2 (Fréchet-Hoeffding Bounds)** If $C$ is a $n$-dimensional copula, then for all $(v_1, \ldots, v_n) \in [0, 1]^n$, $C(v_1, \ldots, v_n)$ has the following boundaries:

$$W(v_1, \ldots, v_n) \leq C(v_1, \ldots, v_n) \leq M(v_1, \ldots, v_n),$$

where

$$W(v_1, \ldots, v_n) = \max(v_1 + v_2 + \ldots + v_n - n + 1, 0),$$
$$M(v_1, \ldots, v_n) = \min(v_1, v_2, \ldots, v_n).$$

Fréchet-Hoeffding bounds provide the largest possible positive and negative dependent structures.

Copula functions can capture the dependency of random variables, which can not be implied by linear correlation. In a real market, the copula framework is easy to build by mathematical analysis, however, the practical implementation of the model with appropriate parameters is a tough problem. The parameters needing to be estimated depend on the choice of different families of copulas. We have to consider some convenient copula functions with restricted parameters.

Popular copula functions include the Gaussian, Student-T and Archimedean copulas.

### 3.3.2 Copula Functions

Let us consider the static copula model, which mainly depends on the original inputs. (See Schönbucher, 2006)

**Definition 3.3.2 (Static Copula Model)**

- $T$: a fixed time horizon
3.3 Copula Models

- \( \{1, \ldots, N\} \): a set of obligors
- \( p_i, i \leq N \): the individual default probability
- \( \{U_1, \ldots, U_n\} \): random variables taking values in \([0, 1]\) such that \( C(u_1, \ldots, u_n) \) is their distribution function

We consider some copula models to derive the density functions with respect to the default rates. The density functions are used to estimate the default correlations inferred from the different copula functions.

**Gaussian Copula**

The Gaussian copula represents the multivariate normal distribution with mean zero and covariance matrix \( \Sigma \) for random variable \( U_1, U_2, \ldots, U_n \), such that

\[
C(u_1, \ldots, u_n) = P(U_1 < u_1, \ldots, U_n < u_n) = \Phi_n(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n), \Sigma),
\]

where \( \Sigma \) is defined as

\[
\Sigma = \begin{pmatrix}
1 & \rho_{12} & \ldots & \rho_{1n} \\
\rho_{21} & 1 & \ldots & \rho_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\rho_{n1} & \rho_{n2} & \ldots & 1
\end{pmatrix}
\]

**Assumption 3.3.1 (Gaussian Copula)**

- The asset values of \( n \) obligors are driven by one systematic factor \( Y \) and an idiosyncratic factor \( \varepsilon_i \)
  
  \[
  V_i = \sqrt{\rho_i}Y + \sqrt{1 - \rho_i}\varepsilon_i \quad \forall i \leq N,
  \]

  where \( Y \) and \( \varepsilon_i \) are i.i.d. standard normal distributed.

- Default of obligor \( i \) is triggered by \( V_i \leq K_i \). The barrier \( K_i \) is chosen as \( K_i = \Phi^{-1}(p_i) \).
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- The covariance matrix of $V_i$ and $V_j$ is given by

\[
\Sigma = \begin{pmatrix}
1 & \sqrt{\rho_1 \rho_2} & \cdots & \sqrt{\rho_1 \rho_n} \\
\sqrt{\rho_1 \rho_2} & 1 & \cdots & \sqrt{\rho_2 \rho_n} \\
\cdots & \cdots & \cdots & \cdots \\
\sqrt{\rho_1 \rho_n} & \sqrt{\rho_2 \rho_n} & \cdots & 1
\end{pmatrix}.
\]

So, the Gaussian copula $C(\cdot)$ is extended from a one-factor model by substituting covariance matrix $\Sigma$ for the linear correlation $\rho$. If we set $U_i = \Phi(V_i)$, we have

\[
P(U_i \leq p_i) = P(\Phi(V_i) \leq p_i) = P(V_i \leq \Phi^{-1}(p_i))
\]

and

\[
C(u_1, \ldots, u_n) = C(p_1, \ldots, p_n) = P(U_1 \leq p_1, \ldots, U_n \leq p_n) = \Phi_n(\Phi^{-1}(p_1), \ldots, \Phi^{-1}(p_n), \Sigma).
\]

Let us return to the loss distribution (3.7) discussed in the context of a one-factor model. If we set $\rho_1 = \rho_2 = \ldots = \rho_n = \rho$, the multivariate Gaussian copula is consistent with the one-factor model, and the density function with respect to the default rate $l$ can be derived:

\[
g_{\text{Gaussian}}(l) = \frac{\sqrt{1 - \rho}}{\sqrt{\rho}} \exp \left\{-\frac{1}{2} \left( \frac{\sqrt{1 - \rho} \Phi^{-1}(l) - \Phi^{-1}(p_n))^2}{\rho} + \frac{1}{2}(\Phi^{-1}(l))^2 \right) \right\},
\]

where $p_n$ and $l$ respectively denote individual default probability and the fraction of defaulted obligors in total ones.

**Student-T Copula**

The multivariate Student-T Copula is defined as

\[
C(u_1, u_2, \ldots, u_n) = t_v(t_v^{-1}(u_1), \ldots, t_v^{-1}(u_n), \Sigma).
\]

Similar to Gaussian copulas, the Student-T copula can be obtained from a multivariate Student-T distribution with mean zero and covariance matrix $\Sigma$. 

Assumption 3.3.2 (Student-T Copula)

- The asset values of $n$ obligors are driven by two systematic factors $Y, Z$ and an idiosyncratic factor $\varepsilon_i$ 

$$X_i = \sqrt{\rho_i}Y + \sqrt{1-\rho_i}\varepsilon_i \quad \forall i \leq N$$

$$V_i = \frac{\sqrt{\nu}X_i}{\sqrt{Z}}, \quad \forall i \leq N$$

where $Y$ and $\varepsilon_i$ are i.i.d. standard normal distributed, $Z$ follows $\chi^2$ distribution with freedom degree $\nu$.

- Systematic factors $Y, Z$ and idiosyncratic factor $\varepsilon_i$ are independent.

- Default of obligor $i$ is triggered by $V_i \leq K_i$. The barrier $K_i$ is chosen as $K_i = t_{\nu}^{-1}(p_i)$.

- The covariance matrix of $V_i$ and $V_j$ is given by 

$$\Sigma = \begin{pmatrix}
1 & \sqrt{\rho_1\rho_2} & \ldots & \sqrt{\rho_1\rho_n} \\
\sqrt{\rho_1\rho_2} & 1 & \ldots & \sqrt{\rho_2\rho_n} \\
\vdots & \vdots & \ddots & \vdots \\
\sqrt{\rho_1\rho_n} & \sqrt{\rho_2\rho_n} & \ldots & 1
\end{pmatrix}$$

Since the asset value depends on two common factors, Student-T copula model is actually a two-factor model. Similarly, we get 

$$C(u_1, \ldots, u_n) = C(p_1, \ldots, p_n) = t_{\nu}(t_{\nu}^{-1}(p_1), \ldots, t_{\nu}^{-1}(p_n), \Sigma)$$

by assuming $U_i = t_{\nu}(V_i)$.

For the T-copula function, there are two independent systematic factors, $Y \sim N(0, 1)$ and $Z \sim \chi^2(\nu)$, so the conditional default probability is subject
to two factors,

\[ p(y, z) = P \left( \frac{\sqrt{\nu}y}{\sqrt{z}} + \frac{(1 - \rho)\nu \varepsilon_i}{\sqrt{z}} \leq t_\nu^{-1}(p_i) \right) \]

\[ = P \left( \varepsilon_i \leq \frac{t_\nu^{-1}(p_i)}{\sqrt{1 - \rho}} \frac{\sqrt{z}}{\sqrt{\nu}} - \sqrt{\rho y} \right) \]

\[ = \Phi \left( \frac{t_\nu^{-1}(p_i)\sqrt{z}}{\sqrt{\nu}} - \sqrt{\rho y} \right). \quad (3.11) \]

We can get the loss distribution function under the T-copula,

\[ P(L \leq l) = \int_0^{+\infty} \int_{-\infty}^{+\infty} P(p(Y, Z) \leq l | Y = y, Z = z) \phi(y) dF(z) \]

\[ = \int_0^{+\infty} \Phi \left( \frac{1}{\sqrt{1 - \rho}} \left( \sqrt{1 - \rho \Phi^{-1}(l)} - t_\nu^{-1}(p_i) \frac{\sqrt{z}}{\sqrt{\nu}} \right) \right) dF(z) \]

and the density function with respect to \( l \)

\[ g_T(l) = \frac{1}{\sqrt{1 - \rho}} \int_0^{+\infty} \phi \left( \frac{1}{\sqrt{1 - \rho}} \left( \sqrt{1 - \rho \Phi^{-1}(l)} - t_\nu^{-1}(p_i) \frac{\sqrt{z}}{\sqrt{\nu}} \right) \right) \frac{1}{\phi(\Phi^{-1}(l))} dF(z), \quad (3.12) \]

where \( F \) is \( \chi^2 \) distribution function with degree of freedom \( \nu \).

**Archimedean Copula**

The class of multivariate Archimedean copulas is defined as

\[ C(u_1, u_2, \ldots, u_n) = \phi^{-1} \left( \sum_{i=1}^{n} \phi(u_i) \right), \]

where the generator function \( \phi \) is continuous, strictly decreasing and convex with \( \phi(0) = \infty \) and \( \phi(1) = 0 \). There are three widely used Archimedean copulas.

- **Clayton Copula**: \( \phi(t) = t^{-\theta} - 1, \theta \geq 0 \)

\[ C(u_1, u_2, \ldots, u_n) = \left( \sum_{i=1}^{n} u_i^{-\theta} - n + 1 \right)^{-\frac{1}{\theta}} \]
3.3 Copula Models

- Gumbel Copula: \( \phi(t) = (-\ln t)^\theta, \theta \geq 1 \)

\[
C(u_1, u_2, \ldots, u_n) = \exp \left\{ - \left( \sum_{i=1}^{n} (-\ln u_i)^\theta \right)^{\frac{1}{\theta}} \right\}
\]

- Frank Copula: \( \phi(t) = -\ln \frac{e^{-\theta t} - 1}{e^{-\theta} - 1}, \theta \in \mathbb{R} \setminus \{0\} \)

\[
C(u_1, u_2, \ldots, u_n) = -\frac{1}{\theta} \ln \left\{ 1 + \frac{\prod_{i=1}^{n} (e^{-\theta u_i} - 1)}{(e^{-\theta} - 1)^{n-1}} \right\}
\]

With respect to the Archimedean copula, we assume obligor \( i \) defaults when \( V_i \leq p_i \), so that

\[
C(u_1, u_2, \ldots, u_n) = C(p_1, \ldots, p_n) = \phi^{-1} \left( \sum_{i=1}^{n} \phi(p_i) \right).
\]

Marshall and Olkin (1988) proposed that the Archimedean copula random variables \( X_i \) can be driven by a uniform distributed \( U_i \) in \([0, 1]\) and a mixture variable \( Y \),

\[
X_i = \phi^{-1} \left( -\frac{1}{Y} \ln U_i \right). \quad (3.13)
\]

The mixture variable \( Y \) satisfies

\[
E(e^{-sY}) = \int_{0}^{+\infty} e^{-sy}dG(y) = \phi^{-1}(s),
\]

where \( G \) is the distribution function of \( Y \), which is independent of \( U_i \). This means that the Archimedean copula function also can be viewed as a one-factor model with one systematic factor \( Y \) and idiosyncratic factor \( U_i \).

Let us re-consider the conditional default probability \( p(y) \) stated in (3.3) by applying the Archimedean variable \( X_i \) in (3.13),

\[
p(y) = P(V_i(T) \leq p_i | Y = y)
= P(\phi^{-1} \left( -\frac{1}{y} \ln U_i \right) \leq p_i)
= P \left( -\frac{1}{y} \ln U_i \geq \phi(p_i) \right)
= P(U_i \leq \exp \{-y\phi(p_i)\})
= \exp \{-y\phi(p_i)\}. \quad (3.14)
\]
The loss distribution for a large group under Archimedean copula approach is given by

\[
P(L \leq l) = \int_{-\infty}^{+\infty} P(p(y) \leq l | Y = y) dG(y) = \int_{-\infty}^{+\infty} P(\exp(-y\phi(p_i)) \leq l) dG(y) = \int_{-\ln l \phi(p_i)}^{\infty} dG(y) = 1 - G(-\frac{\ln l}{\phi(p_i)}).
\]

The density function is

\[
g_{\text{Arch}}(l) = \frac{1}{l \phi(p_i)} g\left(-\frac{\ln l}{\phi(p_i)}\right),
\]

where \(g\) is the density function of mixture variable \(Y\).

In particular, for Clayton copula, the density function is specified as

\[
g_{\text{Clayton}}(l) = \frac{1}{l(p_i^{-\theta} - 1)} f\left(-\frac{\ln l}{p_i^{-\theta} - 1}\right),
\]

where \(f\) is Gamma density function.

### 3.4 Implementation

#### 3.4.1 Empirical Analysis of Default Rates

We analyze historical default rates data from 1920 to 2006. The original source is Moody’s 20th annual survey of corporate defaults and recovery rates\(^1\). In total, Moody’s data refers to the credit events occurring over 18,000 corporates where long term public debts were sold between 1920 and 2006. There are 3,600 long term bond defaults covered by the data. Moody’s considers both rated and non-rated issuers.

Moody’s defines default as one of three types of credit events:

• An interest or principal experiences a missed or delayed disbursement;

• The timely payment of interest or principal is blocked by bankruptcy, administration or other legal receivership;

• A distressed exchange occurs where a new security or package of securities amounting to a diminished financial obligation is offered to the debt holders by the issuer, or the exchange is intended to help the borrower avoid default.

All of the above credit events are supposed to capture the change in relationship between the debt issuer and debt holders from the originally contracted one.

The historical average default and rating migration rates are calculated by taking the weighted average of groups of issuers, which are called cohorts categorized by equally spaced time intervals. The weights correspond to the number of issuers in each cohort and rating category.

Table 3.1: Descriptive corporate default rates in percentage (1920-2006)

<table>
<thead>
<tr>
<th></th>
<th>Aaa</th>
<th>Aa</th>
<th>A</th>
<th>Baa</th>
<th>Ba</th>
<th>B</th>
<th>Caa-C</th>
<th>All Rated</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.000</td>
<td>0.057</td>
<td>0.090</td>
<td>0.272</td>
<td>1.071</td>
<td>3.587</td>
<td>13.60</td>
<td>1.087</td>
</tr>
<tr>
<td>Median</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.649</td>
<td>2.133</td>
<td>7.917</td>
<td>0.654</td>
</tr>
<tr>
<td>St.Dev.</td>
<td>N/A</td>
<td>0.177</td>
<td>0.264</td>
<td>0.477</td>
<td>1.649</td>
<td>4.246</td>
<td>16.89</td>
<td>1.365</td>
</tr>
<tr>
<td>Min</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>Max</td>
<td>0.000</td>
<td>0.833</td>
<td>1.700</td>
<td>1.972</td>
<td>11.11</td>
<td>19.72</td>
<td>100.0</td>
<td>8.403</td>
</tr>
</tbody>
</table>

Annual Issuer Weighted

Table 3.2 and Table 3.3 show us average one-year rating migration rates with monthly cohort spacing during two different sample periods, 1920-2006 and 1970-2006. Each row denotes the start rating in each group at the beginning of one year. Each column indicates the corresponding rating group or default at the end of one year. Each cell entry, excluding the last Default column, is the average rate of issuers holding the row rating at the beginning
Table 3.2: Average one-year corporate rating migration rates in percentage (1920-2006)

<table>
<thead>
<tr>
<th></th>
<th>Aaa</th>
<th>Aa</th>
<th>A</th>
<th>Baa</th>
<th>Ba</th>
<th>B</th>
<th>Caa</th>
<th>Ca-C</th>
<th>Default</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aaa</td>
<td>87.68</td>
<td>7.377</td>
<td>0.852</td>
<td>0.160</td>
<td>0.022</td>
<td>0.001</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>Aa</td>
<td>1.083</td>
<td>85.96</td>
<td>6.455</td>
<td>0.678</td>
<td>0.173</td>
<td>0.037</td>
<td>0.001</td>
<td>0.004</td>
<td>0.060</td>
</tr>
<tr>
<td>A</td>
<td>0.077</td>
<td>2.763</td>
<td>85.78</td>
<td>5.233</td>
<td>0.685</td>
<td>0.110</td>
<td>0.019</td>
<td>0.008</td>
<td>0.072</td>
</tr>
<tr>
<td>Baa</td>
<td>0.042</td>
<td>0.292</td>
<td>4.572</td>
<td>81.77</td>
<td>5.059</td>
<td>0.778</td>
<td>0.152</td>
<td>0.017</td>
<td>0.286</td>
</tr>
<tr>
<td>Ba</td>
<td>0.007</td>
<td>0.085</td>
<td>0.506</td>
<td>5.750</td>
<td>74.71</td>
<td>6.745</td>
<td>0.551</td>
<td>0.050</td>
<td>1.307</td>
</tr>
<tr>
<td>B</td>
<td>0.004</td>
<td>0.056</td>
<td>0.176</td>
<td>0.635</td>
<td>6.242</td>
<td>72.45</td>
<td>4.782</td>
<td>0.513</td>
<td>4.085</td>
</tr>
<tr>
<td>Caa</td>
<td>0.000</td>
<td>0.030</td>
<td>0.037</td>
<td>0.230</td>
<td>0.914</td>
<td>7.904</td>
<td>64.36</td>
<td>3.619</td>
<td>12.47</td>
</tr>
<tr>
<td>Ca-C</td>
<td>0.000</td>
<td>0.000</td>
<td>0.109</td>
<td>0.000</td>
<td>0.445</td>
<td>3.013</td>
<td>7.251</td>
<td>56.74</td>
<td>19.68</td>
</tr>
</tbody>
</table>

and the column rating at the end. We can observe the probability of default as well, which are displayed in the last column of tables. Some issuers in each group are withdrawn ratings (WR) at the end of one year. WR columns are not included in our tables, so that the sum of each row is less than 100%.

From empirical migration rates respectively calculated during whole period 1920-2006 and during modern times between 1970 and 2006, we observed
that the probabilities of default based on credit ratings differ noticeably be-
tween the two samples. Especially for investment grade (such as Baa and
above) issuers, the average default rates in modern times are much lower than
in whole period, but they are not lower for speculative grade ones. This sug-
gests that there has been a structural change in the behaviour of corporate
defaults over the last 86 years.

Given the Figure 3.3, we observe that both in investment grade classes
and in speculative grade classes, the defaults occurred much more frequently
during the periods 1920-1940 and 1970-2006 than during the period 1940-
1970. The apparent clusters of defaults exist. Let us consider the average
corporate default rates across credit rating classes for different time horizons
in Table 3.4.

Table 3.4: Average corporate default rates in percentage

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Aaa</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>Aa</td>
<td>0.209</td>
<td>0.000</td>
<td>0.008</td>
</tr>
<tr>
<td>A</td>
<td>0.344</td>
<td>0.000</td>
<td>0.020</td>
</tr>
<tr>
<td>Baa</td>
<td>0.821</td>
<td>0.018</td>
<td>0.177</td>
</tr>
<tr>
<td>Ba</td>
<td>2.106</td>
<td>0.373</td>
<td>1.156</td>
</tr>
<tr>
<td>B</td>
<td>3.919</td>
<td>1.075</td>
<td>4.998</td>
</tr>
<tr>
<td>Caa-C</td>
<td>12.28</td>
<td>3.396</td>
<td>23.34</td>
</tr>
</tbody>
</table>

According to the noticeable differences of average default rates between
the different sample periods, there exist three apparent regimes 1920-1940,
1940-1970 and 1970-2006. We are able to observe the following facts from
Table 3.4 and Figure 3.3:

- 1920-1940, relative to modern time 1970-2006, these decades have a
  very high probability of default (PD) for investment grade issuers (such
  as Baa and above), but low PDs for speculative grade ones. Figure 3.3
  shows us this fact. Default rates cluster during different sample periods;
Chapter 3. Default Correlation Modelling

Figure 3.3: Empirical annual default rates 1920-2006
• 1940-1970, these decades have quite low PDs for all issuers relative to both its later period 1970-2006 and its previous period 1920-1940. For speculative grade issuers, the PDs in 1940-1970 are much lower than in any other regimes;

• 1970-2006, these decades exhibit a very large gap for PDs between investment and speculative grade issuers relative to any other regimes.

The clustering of default rates depends on different credit ratings, investment grade and speculative grade, during distinct periods. These historical observations tell us default rates are strongly dependent not only among the same credit rating issuers, but also between the different rating groups.

Default correlation is a vital factor in explaining historical default events. If there is the strongly systematic default dependency among individual defaults, it is likely that a very high risk exists in a portfolio.

3.4.2 Estimation Methodologies

We intend to estimate implied default correlations by matching modeled default rates with the empirical observations. We are going to use the most conventional estimation methods namely Maximum Likelihood and Least Squares.

Maximum Likelihood Estimation

Denote \( p(D, \theta) \) as the probability density function of the data set \( D = (y_1, \Lambda, y_n) \), and \( p(D, \theta) \) is known as the likelihood function if it is regarded as a function of \( \theta \), if \( y_i \)s are i.i.d., then

\[
p(D, \theta) = \prod_{i=1}^{n} p(y_i; \theta).
\]

Let \( l_n(\theta) = \log p(D; \theta) \) denote the log-likelihood function. By maximizing \( p(D, \theta) \) or equivalently \( l_n(\theta) \) over \( \theta \), the maximum likelihood estimator \( \hat{\theta} \) for \( \theta \) can be obtained. Naturally, \( \hat{\theta} \) is regarded as the most likely value of \( \theta \).
Chapter 3. Default Correlation Modelling

Least Squares Estimation

Suppose that the data \( D = (y_1, \Lambda, y_n) \) have observations \((y_1, \Lambda, y_n)\) with means \((\mu_1(\theta), \Lambda, \mu_n(\theta))\) with an unknown parameter \(\theta\). The least squares estimator \(\hat{\theta}\) of \(\theta\) is obtained by minimizing the function \(Q\) with respect to \(\theta\):

\[
Q(\theta) = \sum_{i=1}^{n} (y_i - \mu_i(\theta))^2. \tag{3.18}
\]

In the Gaussian Copula, given the density function (3.10) with parameter \(\rho\), we can maximize the likelihood function (3.17) to reach the maximum likelihood estimator \(\hat{\rho}\), which is regarded as the estimated default correlation.

In the Student-T copula, there is an extra parameter \(\nu\) in addition to the correlation \(\rho\) as in the Gaussian copula, so it is much more complicated to estimate parameters in this model. In order to understand the effects of the degree of freedom \(\nu\) on Student-T copula model, let us recall the properties of the T distribution. As we know, the shape of the probability density function is determined by the freedom parameter \(\nu\), and also resembles the bell shape of a standard normally distributed variable. As the degree of freedom increases, the T distribution becomes closer to the standard normal distribution. When \(\nu\) reaches 10, the T distribution is very similar to the normal distribution.

In Figure 3.4, we compare the probability density functions between the standard normal distribution and T distribution with different degrees of freedom 4, 10 and 30. The tail distributions are exhibited on the bottom. We can see that the T distribution assigns more probability to the tails compared with the normal distribution, especially with lower degrees of freedom.

The copula function is used to model the dependence between individual default events. Tail dependence displays the phenomenon of the co-occurrence of extreme events. Corresponding to the heavy tail property of the T distribution, Joe (1997) proved that the T copula is a tail dependent copula, but the Gaussian copula function does not have tail dependence.
3.5 Calibration Results

3.5.1 Calibration by Each Rating Class

In this section, we use the Gaussian and Student-T copulas to calculate joint default correlations.

For the Gaussian copula, random samples are easily generated from the Gaussian distribution. There is only one parameter to be estimated, which makes the calculation process straightforward. Although the estimation procedure for the Student-T copula is much more complicated relative to the Gaussian copula due to the additional parameter \( \nu \), the heavy tail property...
Chapter 3. Default Correlation Modelling

of the T distribution is attractive for modelling the dependency of extreme default events. We are going to choose these two types of copulas to test the effects of default correlation on the distribution of default probability. Furthermore, we will compare the different influences of these two copulas on the probability of default, especially on the extreme cases.

Maximum Likelihood and Least Squares estimation methods are respectively used to determine the parameters within each copula function depending on historical annual default rates.

Assumption 3.5.1 (Calibration)

- The obligors are divided into seven groups based on their credit rating classes from Aaa to Caa-C

- The unconditional default probability \( p_i \) and the corresponding correlation parameters in different copulas are supposed to depend on credit rating classes, that is to say, they are the same value for all obligors within each class \( i, \ i = 1, 2, \ldots, 7 \). For the Gaussian copula, the parameter \( \rho_i \) needs to be calculated; for T copula, another parameter \( \nu_i \) is required to be known in addition to \( \rho_i, \ i = 1, 2, \ldots, 7 \). All parameters are estimated respectively under Maximum Likelihood and Least Squares methods.

- The number of obligors in each rating class is assumed to be infinite \( (N_i \to \infty) \), so that Strong Law of Large Numbers (Theorem 3.2.1) can be applied to calculate the approximate default distributions.

Table 3.5: Empirical default probability for each rating class (1970-2006)

<table>
<thead>
<tr>
<th></th>
<th>Aaa</th>
<th>Aa</th>
<th>A</th>
<th>Baa</th>
<th>Ba</th>
<th>B</th>
<th>Caa-C</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>0.008%</td>
<td>0.02%</td>
<td>0.18%</td>
<td>1.16%</td>
<td>5.00%</td>
<td>23.34%</td>
<td></td>
</tr>
</tbody>
</table>

Due to the apparent regimes existing during the whole historical period 1920-2006 as identified in section 3.4.1, we concentrate on the results for
modern times 1970-2006 instead of analyzing the whole period. This allows us to avoid possible inaccuracy in results due to the long-term business circles.

For each group, we apply corresponding default loss density functions (3.10) and (3.12) to estimate the default correlation $\rho_1$, $\rho_2$ and the degrees of freedom $\nu_1$ for the Gaussian copula and the Student T copula. The log-likelihood function $LL_G(\rho_1)$ for the Gaussian copula is given by

$$LL_G(\rho_1) = \sum_{i=1}^{n} \log (g_{\text{Gaussian}}(l_i|\rho_1))$$

$$= \sum_{i=1}^{n} \left( \log \left( \frac{\sqrt{1 - \rho_1}}{\sqrt{\rho_1}} \right) - \frac{1}{2} \frac{(\sqrt{1 - \rho_1} \Phi^{-1}(l_i) - \Phi^{-1}(p_m))^2}{\rho_1} + \frac{1}{2} (\Phi^{-1}(l_i))^2 \right).$$

The likelihood function $L_T(\rho_2, \nu_1)$ for the Student-T copula is given by

$$L_T(\rho_2, \nu_1) = \prod_{i=1}^{n} g_T(l_i|\rho_2, \nu_1)$$

$$= \prod_{i=1}^{n} \frac{\sqrt{1 - \rho_2}}{\sqrt{\rho_2}} \int_{0}^{+\infty} \phi \left( \frac{1}{\sqrt{\rho_2}} \left( \frac{a - \rho_2 \Phi^{-1}(l_i) - t_{\nu_1}^{-1}(p_m) \sqrt{z}}{\sqrt{\nu_1}} \right) \right) \frac{1}{\phi(\Phi^{-1}(l_i))} dF(z).$$

In formulae (3.19) and (3.20), $l_i$ denotes the observed default rate, $p_m$ is the unconditional default probability in rating class $m$, $F(z)$ is $\chi^2$ distribution function with the degrees of freedom $\nu_1$. We maximize the functions (3.19) and (3.20) numerically respectively in terms of $\rho_1$, $\rho_2$ and $\nu_1$.

The least squares estimation function $ML_G(\rho_3)$ for the Gaussian copula can be written as

$$ML_G(\rho_3) = \sum_{i=1}^{n} (F_e(l_i) - F_G(l_i))^2$$

$$= \sum_{i=1}^{n} \left( F_e(l_i) - \Phi \left( \frac{\sqrt{1 - \rho_3} \Phi^{-1}(l_i) - \Phi^{-1}(p_m)}{\sqrt{\rho_3}} \right) \right)^2,$$

where $F_e(l_i)$ is the empirical cumulative probability of observed default rates $l_i$, $F_G(l_i)$ is the theoretical cumulative probability function under the Gaus-
sian copula. The least squares estimation function $ML_T(\rho_4, \nu_2)$ for the Student T copula is written as

\[
ML_T(\rho_4, \nu_2) = \sum_{i=1}^n (F_e(l_i) - F_T(l_i))^2 \tag{3.22}
\]

where $F(z)$ is $\chi^2$ distribution function with the degrees of freedom $\nu_2$. We minimize the functions (3.21) and (3.22) to obtain the optimal values for $\rho_3$, $\rho_4$ and $\nu_2$. The unconditional default probabilities $p_m$ are given in Table 3.5. The estimates of the default correlation and the degrees of freedom are exhibited in Table 3.6.

From the results (Table 3.6), we can see that the correlation parameters in Caa-C rating class play a significant role in both copulas. For Gaussian copula, the correlations $\rho_i$ decrease first and then increase with respect to credit rating downgrading, while in T copula, parameters in rating Baa, Ba and B are quite lower than in Gaussian copula. The values of estimated degrees of freedom are as high as 30 for most rating classes by Maximum Likelihood method, but they decrease to 10 for lower rating groups B and Caa-C by Least Squares method. In most rating classes, the standard errors by Maximum Likelihood are smaller than by Least squares. We see that the Maximum Likelihood approach brings about more precise results for most rating groups.

Figure 3.5 explains how the degrees of freedom in T Copula affect the likelihood of default rates distribution in Maximum Likelihood methodology. To give intuition, we plot the ratio of the likelihood with each freedom level to the total value with the whole levels. In the speculative grade classes, Ba, B and Caa-C, the freedom degree 30 maximizes the likelihood value significantly, but in the higher ratings Baa and A, when the degree of freedom is greater than 30, the likelihood curve stays very flat with the increased degrees of freedom. In credit rating Baa, the likelihood reaches the maximum
Table 3.6: Estimates of default correlation and degrees of freedom for each rating class

<table>
<thead>
<tr>
<th></th>
<th>Gaussian Copula ($\rho$)</th>
<th>T Copula ($\nu$, $\rho$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ML</td>
<td>LS</td>
</tr>
<tr>
<td>Aa</td>
<td>0.440</td>
<td>0.555</td>
</tr>
<tr>
<td></td>
<td>(0.434)</td>
<td>(0.507)</td>
</tr>
<tr>
<td>A</td>
<td>0.210</td>
<td>0.310</td>
</tr>
<tr>
<td></td>
<td>(0.193)</td>
<td>(0.334)</td>
</tr>
<tr>
<td>Baa</td>
<td>0.090</td>
<td>0.165</td>
</tr>
<tr>
<td></td>
<td>(0.055)</td>
<td>(0.181)</td>
</tr>
<tr>
<td>Ba</td>
<td>0.090</td>
<td>0.145</td>
</tr>
<tr>
<td></td>
<td>(0.038)</td>
<td>(0.098)</td>
</tr>
<tr>
<td>B</td>
<td>0.110</td>
<td>0.095</td>
</tr>
<tr>
<td></td>
<td>(0.045)</td>
<td>(0.034)</td>
</tr>
<tr>
<td>Caa-C</td>
<td>0.170</td>
<td>0.305</td>
</tr>
<tr>
<td></td>
<td>(0.074)</td>
<td>(0.212)</td>
</tr>
</tbody>
</table>

*ML: Maximum Likelihood Estimation*

*LS: Least Squares Estimation*

*Std.Err.: Standard Errors*

value at the degree of freedom 50, which implies that the T copula model gets very close to the Gaussian copula model for higher rating groups, but there is still a difference in estimates of the correlation $\rho$.

Considering the tail dependence property of the T copula, we expect more default probabilities assigned to extreme events by T copula modelling. We will discuss the tail behaviour with stress testing in the later section.

We plot the cumulative frequency of empirical observations and compare with default rates from the copula models using estimated parameters in Table 3.6.

Figure 3.6 and Figure 3.7 show us how the copula with estimated cor-
relation parameters $\rho$ and degrees of freedom $\nu$ can replicate the empirical default rates. The implementation results show that the modelling default rates match historical patterns very well and the two figures from the Gaussian and T copulas look very similar. However, which one of them gives a better indication for our case is hard to say only from the comparison of results. We consider a Bayesian approach to test these two models.

Bayes factor is used to select a model between $M_1$ and $M_2$. It provides a scale of evidence in support of one model versus another. Jeffreys (1961) interpreted the scale of the Bayes factor in Table 3.7.

Let us refer to the Bayes factor $K$ to test Gaussian and T copula models for the underlying distribution of the default rates:

$$
\begin{align*}
M_1 : & \quad \text{T copula model} \\
M_2 : & \quad \text{Gaussian copula model}
\end{align*}
$$

The parameters $\rho$ and $\nu$ in $M_1$ and $M_2$ are supposed to be the estimated
3.5 Calibration Results

One Year Default Probability
Cumulative Frequency
Gaussian Copula
Empirical
ML
LS
Baa
Ba
B
Caa−C

ML: Maximum Likelihood Estimation; LS: Least Squares Estimation

Figure 3.6: Modelling default rates under Gaussian copula compared with empirical observations (1970-2006) for Baa, Ba, B and Caa-C rating classes

Table 3.7: Interpretation of the Bayes factor

<table>
<thead>
<tr>
<th>$K{M_1/M_2}$</th>
<th>Strength of Evidence</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&lt; 1:10$</td>
<td>Strong for $M_2$</td>
</tr>
<tr>
<td>$1:10$ to $1:3$</td>
<td>Moderate for $M_2$</td>
</tr>
<tr>
<td>$1:3$ to $1:1$</td>
<td>Weak for $M_2$</td>
</tr>
<tr>
<td>$1:1$ to $3:1$</td>
<td>Weak for $M_1$</td>
</tr>
<tr>
<td>$3:1$ to $10:1$</td>
<td>Moderate for $M_1$</td>
</tr>
<tr>
<td>$&gt; 10:1$</td>
<td>Strong for $M_1$</td>
</tr>
</tbody>
</table>
values by Maximum Likelihood and Least Squares methods, so the Bayes factor $K$ is given by

$$K = \frac{\mathbb{P}(x/M_1)}{\mathbb{P}(x/M_2)} = \frac{\int \int p(\rho/M_1)p(\nu/M_1)p(x/\rho, \nu, M_1)d\rho d\nu}{\int \int p(\rho/M_2)p(x/\rho, M_2)d\rho}.$$  

(3.23)

Table 3.8 displays the results of the Bayes factor $K$ conditional on the given parameters in Table 3.6.

We found that the factors in the rating classes with Ba and above are below than or very close to 3:1. This indicates little difference between

---

ML: Maximum Likelihood Estimation; LS: Least Squares Estimation

Figure 3.7: Modelling default rates under T copula compared with empirical observations (1970-2006) for Baa, Ba, B and Caa-C rating classes
two copula models. They give very similar distribution probabilities for the given empirical data. For the very low ratings B and Caa-C, the factors are significant greater than 10, which means the T copula model gives the better indications. Especially in the lowest credit rating group Caa-C, the Bayes factor is strongly in support of the T copula model. This result is due to the tail dependence property of T copula, since it is able to capture more extreme defaults in very low rating classes.

### 3.5.2 Calibration by Investment Grade and Speculative Grade Classes

Since the default events in the high rating classes (Aaa or Aa) occurred very rarely, it is not easy to capture the correlated defaults only among these groups. So that, let us consider grouping the obligors into two classes, investment grade class (Baa and above) and speculative grade class (Ba, B and Caa-C). We also regard all of the obligors as one group to estimate the default correlation.

Table 3.9: Empirical default probability for three groups (1970-2006)

<table>
<thead>
<tr>
<th>Investment Grade</th>
<th>Speculative Grade</th>
<th>All Corporate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.07%</td>
<td>3.78%</td>
<td>1.23%</td>
</tr>
</tbody>
</table>

The unconditional default probabilities $p_i$ in the recent period 1970-2006 are given in Table 3.9. With respect to the estimates (Table 3.10), the default correlations $\rho$ modeled by the Gaussian copula are greater than those of the T copula, and the degrees of freedom parameters $\nu$ are lower in the
speculative grade class according to both estimation methods compared with the investment grade class. The smaller values of $\nu$ in speculative grade class represent the fatter tail property relative to investment grade.

Table 3.10: Estimates of default correlation and degrees of freedom for the three groups

<table>
<thead>
<tr>
<th></th>
<th>Gaussian Copula ($\rho$)</th>
<th>T Copula ($\nu$, $\rho$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ML</td>
<td>LS</td>
</tr>
<tr>
<td>Investment Grade</td>
<td>0.080</td>
<td>0.165</td>
</tr>
<tr>
<td></td>
<td>(0.050)</td>
<td>(0.220)</td>
</tr>
<tr>
<td>Speculative Grade</td>
<td>0.105</td>
<td>0.120</td>
</tr>
<tr>
<td></td>
<td>(0.040)</td>
<td>(0.052)</td>
</tr>
<tr>
<td>All Corporates</td>
<td>0.105</td>
<td>0.115</td>
</tr>
<tr>
<td></td>
<td>(0.039)</td>
<td>(0.046)</td>
</tr>
</tbody>
</table>

*ML: Maximum Likelihood Estimation*

*LS: Least Squares Estimation*

*Std.Err.: Standard Errors*

Figure 3.8 tells us that the likelihood value reaches a maximum when the degree of freedom $\nu$ equals 30 for speculative grade obligors. This result is consistent with the discussion in section 3.5.1.

We can see similar implementation curves for Gaussian and T copulas in Figure 3.9 and Figure 3.10. Intuitively, in this case both models replicate historical behaviour better than in the individual rating class case (Figure 3.6 and Figure 3.7). In order to make a comparison between two copula models, we test the Bayes factor $K$ as before.

The Bayes factor $K$ strongly supports the Gaussian model in the investment grade class and is relatively supportive to the T model in the speculative grade, but for all corporate, there is little difference between two models.

The copula approach allows the individual term structure of default risk
3.5 Calibration Results

![Graph showing the relationship between likelihood value and degrees of freedom for three groups.](image)

Figure 3.8: The relationship between likelihood value and degrees of freedom for three groups

<table>
<thead>
<tr>
<th></th>
<th>Investment Grade</th>
<th>Speculative Grade</th>
<th>All Corporate</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K$</td>
<td>$&lt;0.0001$</td>
<td>6.25</td>
<td>0.98</td>
</tr>
</tbody>
</table>

Table 3.11: Bayes factor $K$ for three groups

separated from the default dependency model, and it is possible to extend the result to continuous time to display default contagion. However, we have to face model risk and parameter risk in choosing a specific copula function from many existing models.

3.5.3 Stress Testing

We apply stress testing to the probability of default, PD, to see how the default correlations perform on PDs under some of the most extreme events. We compared PDs at the 99.9th percentile based on copula models with the worst historical observations for each rating class and for grouped investment
Figure 3.9: Modelling default rates under Gaussian copula compared with empirical observations (1970-2006) for three groups

For each rating class, the worst event occurred in the 1980’s for most rating groups during the most recent period 1970-2006, while during the whole sample period 1920-2006, the worst event occurred in the 1930’s. We can see that the worst events clustered across different credit rating classes (Table 3.12).

Comparing the stress testing results displayed in Figure 3.11 and Figure 3.12, we conclude that both copulas, Gaussian and T, are able to capture extreme events for most credit ratings during the recent period 1970-2006, except for the lowest rating class Caa-C. In 1938, the probability of default in rating A reached to 1.7%, which is much higher than the average PD 0.02% (Table 3.5), so that both copulas are unable to capture this most extreme event. It is obvious that the 99.9th percentile default rates with
3.5 Calibration Results

Figure 3.10: Modelling default rates under T copula compared with empirical observations (1970-2006) for three groups

Table 3.12: Worst historical default rates for each rating class

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Aaa</td>
<td>Aa</td>
<td>A</td>
<td>Baa</td>
</tr>
<tr>
<td>Worst PD</td>
<td>0.00%</td>
<td>0.60%</td>
<td>0.25%</td>
<td>1.36%</td>
</tr>
<tr>
<td>Worst PD</td>
<td>0.00%</td>
<td>0.83%</td>
<td>1.70%</td>
<td>1.97%</td>
</tr>
<tr>
<td>Year</td>
<td>N/A</td>
<td>1938</td>
<td>1938</td>
<td>1938</td>
</tr>
</tbody>
</table>

the default correlation estimated by Least Squares method are much higher than with the Maximum Likelihood estimation. The gaps of the correlation between two estimation methodologies are very apparent in Table 3.6. The
Table 3.13: Modelling default rates at the 99.9th percentile for each rating class

<table>
<thead>
<tr>
<th>Class</th>
<th>Aa</th>
<th>A</th>
<th>Baa</th>
<th>Ba</th>
<th>B</th>
<th>Caa-C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian (LS)</td>
<td>1.36%</td>
<td>1.42%</td>
<td>3.45%</td>
<td>11.82%</td>
<td>23.33%</td>
<td>88%</td>
</tr>
<tr>
<td>Gaussian (ML)</td>
<td>1.06%</td>
<td>0.84%</td>
<td>1.85%</td>
<td>7.94%</td>
<td>25.55%</td>
<td>73%</td>
</tr>
<tr>
<td>T(LS)</td>
<td>1.50%</td>
<td>1.50%</td>
<td>3.40%</td>
<td>11.70%</td>
<td>24.80%</td>
<td>88%</td>
</tr>
<tr>
<td>T(ML)</td>
<td>1.20%</td>
<td>0.80%</td>
<td>2.60%</td>
<td>7.30%</td>
<td>27.30%</td>
<td>59%</td>
</tr>
</tbody>
</table>

LS: Least Squares Estimation
ML: Maximum Likelihood Estimation

higher default correlations lead the higher default rates. But the two different copula models have no noticeable effects on the 99.9th percentile results. Two copulas have the very similar extreme values in our case. So, we would like to further think about the tail distributions for even more extreme events.
3.5 Calibration Results

We use the Gaussian and T copula models to calculate the tail distributions, \( P(PD > x) \), where \( x \) is the extreme value of probability of default. In Caa-C group, we consider the cases with \( x \geq 0.7 \). In order to compare the influences of the two copulas on tail distribution, we looked at the ratio of their probabilities. In Figure 3.13, \( x \) is the extreme value of the default rates, and the y axis denotes the ratio of \( P_T(PD > x) \) to \( P_{\text{Gaussian}}(PD > x) \). Here, the degree of freedom \( \nu \) in T copula is equal to 10, and the other correlation parameters are taken from Least Squares results.

The ratios are greater than 1 when the default rates are above 0.7, and then increase significantly if \( x > 0.9 \). So the T copula assigns much more probability to the occurrence of the most extreme cases than Gaussian copula, which is consistent with our expectations corresponding to the properties of T copula function. It also explains the very strong Bayes factor in Caa-C group discussed in section 3.5.1.

By exploring the role of copulas, we find that the default dependency captured by copula functions performs very well in evaluating probability of...
default. It is very likely to capture most dependent default events by both Gaussian and T copulas, but for some extreme cases especially in lower rating classes, the T copula behaves better than the Gaussian copula.

Table 3.14: Worst historical default rates for three groups

<table>
<thead>
<tr>
<th></th>
<th>Investment Grade</th>
<th>Speculative Grade</th>
<th>All Corporate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Worst PD (1970-2006)</td>
<td>0.51%</td>
<td>10.59%</td>
<td>3.91%</td>
</tr>
<tr>
<td>Year</td>
<td>2002</td>
<td>2001</td>
<td>2001</td>
</tr>
<tr>
<td>Worst PD (1920-2006)</td>
<td>1.55%</td>
<td>15.39%</td>
<td>8.40%</td>
</tr>
<tr>
<td>Year</td>
<td>1938</td>
<td>1933</td>
<td>1933</td>
</tr>
</tbody>
</table>

The above stress testing for three groups shows that the two models are able to capture the extreme events not only during the sample period 1970-2006, but also during the whole period 1920-2006. So the Gaussian and the T copula methodologies can model default dependency and evaluate the extreme probability of default.

We are going to apply the methodologies of modelling default correlations discussed above to the application of value-at-risk models in later sections of the thesis.
Table 3.15: Modelling default rates at the 99.9\textsuperscript{th} percentile for three groups

<table>
<thead>
<tr>
<th></th>
<th>Investment Grade</th>
<th>Speculative Grade</th>
<th>All Corporate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian (LS)</td>
<td>1.64 %</td>
<td>22.58 %</td>
<td>10.09 %</td>
</tr>
<tr>
<td>Gaussian (ML)</td>
<td>0.75 %</td>
<td>20.63 %</td>
<td>9.37 %</td>
</tr>
<tr>
<td>T(LS)</td>
<td>1.60 %</td>
<td>21.00 %</td>
<td>9.30 %</td>
</tr>
<tr>
<td>T(ML)</td>
<td>0.80 %</td>
<td>21.10 %</td>
<td>8.50 %</td>
</tr>
</tbody>
</table>

*LS: Least Squares Estimation
ML: Maximum Likelihood Estimation

Figure 3.14: Comparison of the 99.9\textsuperscript{th} PD of copula models with the worst historical observations for three groups

3.6 Summary

In this chapter, we describe the empirical default clustering phenomenon sampled during 1920 and 2006. We find that there exists a strong correlated relationship among defaults at different horizons and across rating classes.
defaults are correlated as well. This is our motivation to analyse by how much these defaults are correlated within each rating group. Thereafter, we discuss the procedure for estimating the default correlation parameters using two different models.

Next, we reviewed the one factor model in a general way, thus model plays a significant role in modelling default probabilities. Under the one factor framework, the copula is the most popular methodology to separate the marginal individual default probabilities from the joint default distribution. In order to make good use of this approach, it is necessary for us to invoke some well-known copula functions and understand their corresponding properties.

After the analysis on different copula functions, we focus on two of them, the Gaussian copula and the Student T copula, to estimate the parameters based on the most recent market data under the different schemes, respectively for each rating class (from Aaa to Caa-C) and for investment grade, speculative grade and all rated classes. Given the calibration results, we conclude that the default dependency structured by copula functions performs very well in evaluating the default probabilities. The modeled distributions with estimates of correlation replicate the empirical observations well. The defaults in the rating class Caa-C are correlated most strongly under both copula models.

Our outcome is partly consistent with the results stated in Xu (2006) who used a different data source, moreover, we extend the estimation approach for the T copula by releasing the assumption of the fixed degree of freedom used in Xu (2006). Furthermore, we adopt the Bayes approach to compare two copula models, and obtain the results that for very low rating classes, for example, B and Caa-C, the T copula model behaves better than Gaussian copula, however, for other higher ratings, there is little difference between these two copulas. This is due to the tail dependence property of T copula. For the investment grade group, the Gaussian copula plays a more convincing role.
3.6 Summary

We use stress testing to examine the effects of different copulas on the extreme cases. As expected, the results show that greater default probabilities inferred by the T copula are assigned to extreme events.

The conclusions obtained in this chapter show us how the Gaussian and T copulas influence the distributions of default probability depending on credit ratings. This allows us to avoid model risk inherent in the choice of different copulas. Certainly, our results provide appropriate modelling guidelines in the fields of the credit derivatives pricing and portfolio risk management.
Chapter 4

Correlations between Default and Recovery Rates

4.1 Motivation

Most models of default risk describe the expected recovery rate as adapted processes under the risk neutral measure. Given a default event, the recovery rate is usually known as a random variable having an expected level for valuation purposes. In a recession, obligors tend to default more frequently than in more prosperous times, meanwhile, recovery rates are likely to go down as the economy worsens. A strong economy and low default rates result in high recovery rates of defaulted debt.

Empirically, we observe that there is a negative correlation between the annual default rate and the annual average recovery rate during 1982 and 2006 as Figure 4.1 and Figure 4.2. Such a link would possibly induce a significant increase in losses implied by credit risk models.

In estimating recovery rates and pricing credit derivatives, most models assume loss distributions based on stochastic or static recovery rates, but uncorrelated with default. During the last two years, there is some research referring to the association between recovery rate and probability of default, but the analysis is very limited. The models mostly focus on the same com-
4.1 Motivation

Figure 4.1: Annual average recovery rates and annual default rates

Figure 4.2: Scatter of annual average recovery rates and annual default rates

mon factor to work on the recovery rate and default probability. However, sometimes the systematic factor is not the only variable to link the recovery
Chapter 4. Correlations between Default and Recovery Rates

with default. We will extend this indirect relationship to a direct one, which describes how the recovery rate depends directly on the default probability.

4.2 Introduction of Recovery Rate Models

We usually treat the payoff of an asset at default with an additional positive recovery rate instead of a zero recovery payoff. Thus, modelling the recovery of a defaultable asset is necessary in the framework of credit derivatives pricing and credit sensitive risks.

Assumption 4.2.1 (General Recovery Framework) The non-default of a defaultable asset is set to be $\bar{p}(t)$, such that, no default happens before the time $t$. If $\tau$ is the default time and $\tau \geq t$, the asset has a payoff of $(1 - \phi(\tau))$ units of account at default time $\tau$, where $\phi(\tau)$ is the stochastic recovery rate with $\mathcal{F}_\tau$ measurable.

The recovery rate $\phi(\tau)$ must be modeled using different families of defaultable assets. Schönbucher (2006) summarized the developed framework of recovery models based on the specification of recovery rates across different pricing problems.

The very basic and particular model is the recovery of par, first advanced by Duffie (1998). Under this framework (RP), we assume that the recovery payoff of a defaultable security can be equal to the multiplication of the claim amount and the recovery rate. Schönbucher (2006) gave the details about the specifications of recovery of par. The defaultable claim amount is defined as $V(t)$, $0 \leq t \leq T$, where $T$ is the maturity time and $t$ is the default time. The recovery rate is set to be $\pi(t)$. We have the following expressions:

- The default payoff at time $t = 0$ is

$$\int_0^T \pi(t)V(t)h(0, t)\overline{B}(0, t)dt,$$

where $h(0, t)$ is the default hazard rate and $\overline{B}(0, t)$ is the price of a zero recovery defaultable bond.
• Given the zero recovery claim value $\bar{p}$, the value of the claim is equal to,

$$p_{RP} = \bar{p} + \int_0^T \pi(t)V(t)h(0,t)B(0,t)dt.$$ 

Another framework is related to recovery of treasury (RT). In the recovery of treasury, the market prices of default free assets are usually used to express the recovery values. The price of a defaultable asset $\bar{p}_{RT}$ under the assumption of recovery of treasury can be given by

$$\bar{p}_{RT} = (1 - \phi(\tau))\bar{p} + \phi(\tau)p,$$

where $\bar{p}$ is the price of the defaultable asset with zero recovery rate, $p$ the price of default free asset and $\tau$ is the default time.

The recovery of par and the recovery of treasury both assume the recovery parameter is time dependent or constant. Under the non-stochastic framework, the analysis can be simplified, but it is not realistic because a significant recovery risk has been ignored. Recovery risk is a very important factor in the work on credit derivatives pricing. Considering the stochastic features of recovery rate, we must modify the general recovery framework stated in assumption 4.2.1 (see Schönbucher, 2006).

**Assumption 4.2.2 (Stochastic Recovery Framework)** The recovery rate $\psi(t)$ is a random variable and forms a point process combined with the default time $\tau$, such that the measure of the process is

$$C(d\varphi, dt) = K(d\varphi)\lambda(t)dt,$$

where $K(d\varphi)$ is the conditional distribution of recovery rate $\psi(t)$ and $\lambda(t)$ is the density of time $t$.

How to model the recovery rate as a stochastic process is the focus that we will now concentrate on. In the following sections, we will discuss the recovery distributions in terms of different stochastic processes. Since, we know the recovery rate is not an independent stochastic process, but correlated with default probabilities, modelling the correlation between recovery rate and default probability is the key problem.
Chapter 4. Correlations between Default and Recovery Rates

4.3 Correlated Recovery Rate Modelling

4.3.1 Extended Factor Model

Based on the one factor model, the default rate is linked to the systematic risk, which represents the business cycle. The potential relationship between the recovery rate and the business cycle is our concern here. If the recovery rate is determined by the systematic risk factor as well, the correlations between the default and recovery rates can be captured by the business cycle indicator.

Let us recall the value of asset return (3.2)

\[ V_n(T) = \sqrt{\rho} \cdot Y + \sqrt{1 - \rho} \cdot \varepsilon_n, \]

where \( Y \) is the systematic factor as an indicator of the business cycle, and \( \varepsilon_n \) is an idiosyncratic factor independent of \( Y \), representing the quality of the individual firm. The random variables \( Y \) and \( \varepsilon_n \) are distributed normally with zero mean and unit standard deviation. The conditional default probability \( p(y) \) given in (3.3) is rewritten as

\[ p(Y = y) = \Phi \left( \frac{\Phi^{-1}(\bar{p}) - \sqrt{\rho}y}{\sqrt{1 - \rho}} \right), \quad (4.1) \]

where \( \Phi^{-1}(\bar{p}) \) is the default threshold.

The recovery rate \( R(X_n) \) conditional on the default of obligor \( n \) is modeled as a function of the random variable \( X_n \), which follows a normal distribution \( N(\mu, \sigma) \) as follows:

\[ X_n(Y) = \mu + \sigma \sqrt{\omega} \cdot Y + \sigma \sqrt{1 - \omega} \cdot z_n, \quad (4.2) \]

where \( Y \) and \( z_n \) are independent random variables with standard normal distribution \( N(0, 1) \). We have the following relationships:

\[
\begin{align*}
\text{correlation}(V_n, Y) &= \sqrt{\rho} \\
\text{correlation}(X_n, Y) &= \sqrt{\omega}.
\end{align*}
\]
We need a distribution for recovery rate on the interval $[0, 1]$ that does not involve many parameters. The class of beta distributions is the conventional choice for uncorrelated recovery rates. As an alternative use of the beta distribution, a transformed normally distributed random variable can be used. We usually call this logit transformation the logit-normal distribution. Considering the property of the thicker tails, we choose the log-normal distribution to model the correlated recovery rate. Another benefit of the use of the log-normal is that the recovery rate can be guaranteed to be positive. For the purpose of comparison, we use the normal distribution as well.

We consider three different distributions for recovery rate $R(X_n)$ depending on $X_n$:

- **Logit-normal distribution**
  \[
  R(X_n) = \frac{\exp(X_n)}{1 + \exp(X_n)} \tag{4.3}
  \]

- **Normal distribution**
  \[
  R(X_n) = X_n \tag{4.4}
  \]

- **Log-normal distribution**
  \[
  R(X_n) = \exp(X_n) \tag{4.5}
  \]

Comparing the three distributions, we strictly have the fact that there are non-negative recovery rates with fat tails under the assumption of log-normal distribution, but it does not meet the requirement to guarantee the recovery rate below the upper bound 1. Under the logit-normal distribution, the recovery rate can be bounded by the unit interval $[0, 1]$. The recovery rate is strictly non-negative, so the log-normal and the logit-normal distributions seem more realistic than the normal distribution. The parameters $\mu$, $\sigma$ and $\omega$ in (4.2) vary with the different distributions of recovery rate.

In this methodology, we suppose that systematic risk plays a major role in determining the recovery rates as well as in affecting the default probability.
The idiosyncratic factors $\varepsilon_n$ and $z_n$ respectively indicate the specific essentials in individual firm $n$ to impact the default probability and recovery rate during the same business cycle.

### 4.3.2 Expected Recovery under a One Factor Model

Let us consider the expected recovery rate under a one factor model. We know the expected recovery can be given by calculating the expectation as:

$$
\mathbb{E}[R] = \int_a^b R(X_n) \phi(X_n) dX_n
$$

$$
= \frac{1}{\sigma \sqrt{2\pi}} \int_a^b R(X_n) \cdot \exp \left( -\frac{(X_n - \mu)^2}{2\sigma^2} \right) dX_n, \quad (4.6)
$$

where the lower and upper bounds $a$ and $b$ are determined by the distributions of recovery rate $R$.

In the logit-normal case, there are no restriction on the random variable $X_n$, but for the normal distribution, the equation (4.6) can be written as

$$
\mathbb{E}[R] = \frac{1}{\sigma \sqrt{2\pi}} \int_0^1 X_n \cdot \exp \left( -\frac{(X_n - \mu)^2}{2\sigma^2} \right) dX_n
$$

$$
= \mu \cdot \left[ \Phi \left( \frac{1 - \mu}{\sigma} \right) - \Phi \left( -\frac{\mu}{\sigma} \right) \right] + \frac{\sigma}{\sqrt{2\pi}} \left[ \exp \left( -\frac{\mu^2}{2\sigma^2} \right) - \exp \left( -\frac{(1 - \mu)^2}{2\sigma^2} \right) \right], \quad (4.7)
$$

for the log-normal distribution, we have

$$
\mathbb{E}[R] = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^0 \exp (X_n) \cdot \exp \left( -\frac{(X_n - \mu)^2}{2\sigma^2} \right) dX_n
$$

$$
= \exp \left( \mu + \frac{\sigma^2}{2} \right) \cdot \Phi \left( -\frac{\mu}{\sigma} - \sigma \right), \quad (4.8)
$$

where $\Phi(\cdot)$ is the cumulative standard normal distribution function. In the logit-normal case, we can calculate the following integration:

$$
\mathbb{E}[R] = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{e^{X_n}}{1 + e^{X_n}} \cdot \exp \left( -\frac{(X_n - \mu)^2}{2\sigma^2} \right) dX_n. \quad (4.9)
$$
However, if we consider the recovery rate for defaults, the formula (4.6) does not provide the correct expected values because it is defined irrespective of whether the default occurs. Therefore, the expected recovery is supposed to be calculated conditional on the default event as $\mathbb{E}[R | V_n < \Phi^{-1}(p)]$ rather than as equation (4.6).

We note that the asset return $V_n$ is correlated with the random variable $X_n$ by $\rho_{vx}$, which can be denoted as:

$$\rho_{vx} = \sqrt{\rho} \cdot \sqrt{\omega}. \quad (4.10)$$

Therefore, $V_n$ can be represented as a linear function of $X_n$ and the standard normal random variable $\eta_n$ independent of $X_n$:

$$V_n = \rho_{vx} \cdot \left( \frac{X_n - \mu}{\sigma} \right) + \sqrt{1 - \rho_{vx}^2} \cdot \eta_n. \quad (4.11)$$

So that, upon the equation (4.11), the conditional default probability can be written as $p(X_n)$ depending on the random variable $X_n$:

$$p(X_n) = \mathbb{P}[V_n < \Phi^{-1}(p) | X_n] = \Phi \left( \frac{\Phi^{-1}(p) - \rho_{vx} \cdot \left( \frac{X_n - \mu}{\sigma} \right)}{\sqrt{1 - \rho_{vx}^2}} \right). \quad (4.12)$$

The expected recovery rate conditional on defaults can be derived as:

$$\mathbb{E}[R | V_n < \Phi^{-1}(p)] = \mathbb{E}[R \cdot \mathbf{1}_{\{V_n < \Phi^{-1}(p)\}}] / \mathbb{P}[V_n < \Phi^{-1}(p)] = \frac{1}{\bar{p} \cdot \sigma \sqrt{2\pi}} \int_a^b R(x) \cdot \exp \left( -\frac{(x - \mu)^2}{2\sigma^2} \right) \mathbb{P}(V_n < \Phi^{-1}(p) | X_n = x) dx. \quad (4.13)$$

Substituting (4.12) into equation (4.13), for the normal distribution, we can
obtain

\[
\mathbb{E}[R|V_n < \Phi^{-1}(\bar{p})] = \frac{1}{\bar{p} \cdot \sigma \sqrt{2\pi}} \int_{0}^{1} x \cdot \exp \left( -\frac{(x - \mu)^2}{2\sigma^2} \right) \cdot \mathbb{P}(V_n < \Phi^{-1}(\bar{p})|X_n = x) \, dx
\]

\[
= \frac{\mu}{\bar{p}} \left[ \Phi_2(\Phi^{-1}(\bar{p}), (1 - \mu)/\sigma, \rho_{vx}) - \Phi_2(\Phi^{-1}(\bar{p}), -\mu/\sigma, \rho_{vx}) \right]
\]

\[
+ \frac{\sigma}{\bar{p} \sqrt{2\pi}} \cdot \Phi \left( \frac{\Phi^{-1}(\bar{p}) + \rho_{vx} \cdot \mu/\sigma}{\sqrt{1 - \rho_{vx}^2}} \right) \cdot \exp \left( -\frac{\mu^2}{2\sigma^2} \right)
\]

\[
- \frac{\sigma}{\bar{p} \sqrt{2\pi}} \cdot \Phi \left( \frac{\Phi^{-1}(\bar{p}) - \rho_{vx} \cdot \frac{1-\mu}{\sigma}}{\sqrt{1 - \rho_{vx}^2}} \right) \cdot \exp \left( -\frac{(1 - \mu)^2}{2\sigma^2} \right)
\]

\[
- \frac{\sigma}{\bar{p} \sqrt{2\pi}} \cdot \rho_{vx} \cdot \exp \left( -\frac{\left(\Phi^{-1}(\bar{p})\right)^2}{2} \right) \cdot \left( \Phi(U) - \Phi(L) \right),
\] (4.14)

where \( \Phi_2(\cdot, \cdot, \cdot) \) denotes the bivariate normal cumulative distribution function with \( \rho_{vx} \) stated in (4.10) and

\[
\left\{ \begin{array}{l}
U = [(1 - \mu)/\sigma - \Phi^{-1}(\bar{p}) \cdot \rho_{vx}] / \sqrt{1 - \rho_{vx}^2} \\
L = [-\mu/\sigma - \Phi^{-1}(\bar{p}) \cdot \rho_{vx}] / \sqrt{1 - \rho_{vx}^2} ;
\end{array} \right.
\]

for the log-normal distribution, we have

\[
\mathbb{E}[R|V_n < \Phi^{-1}(\bar{p})] = \frac{1}{\bar{p} \cdot \sigma \sqrt{2\pi}} \int_{-\infty}^{0} \exp(x) \cdot \exp \left( -\frac{(x - \mu)^2}{2\sigma^2} \right) \cdot \mathbb{P}(V_n < \Phi^{-1}(\bar{p})|X_n = x) \, dx
\]

\[
= \frac{1}{\bar{p}} \exp \left( \mu + \frac{\sigma^2}{2} \right) \Phi_2 \left[ \Phi^{-1}(\bar{p}) - \sigma \cdot \rho_{vx}, -\frac{\mu}{\sigma} - \rho_{vx} \right];
\] (4.15)

for the logit-normal distribution, we have to calculate the integration of

\[
\mathbb{E}[R|V_n < \Phi^{-1}(\bar{p})] = \frac{1}{\bar{p} \cdot \sigma \sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\exp(x)}{1 + \exp(x)} \cdot \exp \left( -\frac{(x - \mu)^2}{2\sigma^2} \right) \cdot \mathbb{P}(V_n < \Phi^{-1}(\bar{p})|X_n = x) \, dx.
\] (4.16)

The deriving process according to the closed formulae (4.14) and (4.15) can be found in detail in Appendix A.
Comparing the expected recovery rate stated in (4.7), (4.8) and (4.9) with (4.14), (4.15) and (4.16) based on the different distributions, we find that the equations (4.14), (4.15) and (4.16) are equivalent to (4.7), (4.8) and (4.9) if the recovery rate is assumed to be uncorrelated with the default probability with $\rho_{tr} = 0$. The calculated results of expected recovery rate conditional on defaults are listed in Table 4.4.

### 4.3.3 Direct Dependence Model

The probability of default for individual obligor can come from two aspects, a long term default probability of the borrower representing the macroeconomic factor and a short term impact indicating the individual factor of each obligor.

It is known that firms with different credit ratings have different default rates on average. From the empirical discussion in section 3.4.1, we see that default rates vary greatly across different rating classes. So we can say the long term default probability may be determined by credit ratings, and the actual default probability of each distinct obligor might change over time with respect to the state of the economy and the firm’s business cycles and individual cash flows. Based on this idea, we can think of the probability of default as the sum of two independent components, each of which is a weighted random variable.

Altman (2001) used this idea to model default probability and then link it to the recovery rate distribution. In this thesis, we extend this framework to a closed formula, which is used to model recovery rate dependent on the default probability directly rather than impose a rank correlation between the recovery rate and the systematic factor as done by Altman.

In addition to the macroeconomic factor, credit ratings might also be an important determinant of the changes in recovery rates. It is impossible to link the relationship between the credit quality and the recovery rates only by the systematic risk factor in the extended one factor model. We try to consider recovery changes depending directly on the default probability for
an individual obligor instead of depending on business cycle variables.

Default probabilities tend to have very low values in most cases, but in some extreme scenarios they go up dramatically. The Gamma distribution with a highly skewed tail on the right can account for this phenomenon in default probabilities.

Based on this property of Gamma distribution, the probability of default can be regarded as a combined Gamma process:

\[ PD_n = \overline{p}_j(w_1 Z + w_2 Z_n), \]

(4.17)

where \( \overline{p}_j \) is the average probability of default in the credit rating class \( j \), which also means the long term default rate for each rating \( j \). \( Z \) is a common factor indicating the macroeconomic state, and \( Z_n \) plays a distinct role for different obligor \( n \). The parameters \( w_1 \) and \( w_2 \) are weighted ratios respectively standing for the common factor \( Z \) and the idiosyncratic factor \( Z_n \) satisfying \( w_1 + w_2 = 1 \) and \( 0 \leq w_1, w_2 \leq 1 \). Both \( Z \) and \( Z_n \) are drawn from Gamma distributions independently.

According to this weighted combined Gamma method, if the state of economy stays in recession, the random variable \( Z \) tends to have a high value and significantly impacts on the default probabilities of most obligors, then would cause PDs increased above their long term values; if the economy prospers, the PDs are lower than their average long term values because of low random values of \( Z \).

The probability density function of the Gamma distributed random variable \( x \) is expressed as

\[ \psi(x; \alpha, \beta) = x^{\alpha - 1} \cdot \frac{e^{-x/\beta}}{\beta^\alpha \cdot \Gamma(\alpha)}, \]

for \( x > 0 \) and \( \alpha, \beta > 0 \) and \( \Gamma(\alpha) \) is the Gamma function.

Let us recall some properties of Gamma distribution:

- Moments
The expected value and variance of a Gamma random variable $X$ with parameters $\alpha$ and $\beta$ are given by

$$E(X) = \alpha \cdot \beta \quad \text{and} \quad Var(X) = \alpha \cdot \beta^2.$$ 

- **Summation**

If $X_i$ is independently distributed with $\Gamma(\alpha_i, \beta)$ for $i = 1, 2, \ldots, N$, then

$$\sum_{i=1}^{N} X_i \sim \Gamma \left( \sum_{i=1}^{N} \alpha_i, \beta \right).$$

- **Scaling**

For any $t > 0$, $tX_i$ has a $\Gamma(\alpha, t\beta)$ distribution.

- **Beta Distribution**

A Beta distributed random variable $z$ has the probability density function of

$$f(z; \alpha, \beta) = \frac{z^{\alpha-1}(1-z)^{\beta-1}}{\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}du}.$$  

If $X$ and $Y$ independently follow $\Gamma(\alpha, \theta)$ and $\Gamma(\beta, \theta)$, then $\frac{X}{X+Y}$ has a Beta distribution with parameters $\alpha$ and $\beta$. The Beta variable has a mean of $\frac{\alpha}{\alpha+\beta}$.

In order to ensure the expected individual default probability $PD_n$ is consistent with the long term probability $\overline{p}_j$, the weighted sum of two Gamma variables should equal one.

For the purpose of simplifying calculation process without loss of generality, we assume that the two random variables in (4.17) are independently Gamma distributed with:

$$Z \sim \Gamma(a, 1/a),$$

$$Z_n \sim \Gamma(a_1, 1/a_1).$$ (4.18)
From the properties of Gamma distribution, the means of $Z$ and $Z_n$ are equal to one. Thus, the expectation of $PD_n$ associated with rating class $j$ is

$$E(PD_n) = \overline{p}_j(w_1 \cdot 1 + w_2 \cdot 1) = \overline{p}_j.$$  

We add a constraint on the relationship between the weights and the Gamma parameters in (4.18):

$$w_1/a = w_2/a_1,$$  

so that $PD_n$ follows a combined Gamma distribution with the shape $a + a_1$ and the scale $\overline{p}_j \cdot w_1/a$ based on their characters of summation and scaling. The variance of $PD_n$ is equal to

$$Var(PD_n) = \overline{p}_n^2(w_1^2/a + w_2^2/a_1) = \overline{p}_n^2 \cdot w_1/a.$$  

Given the assumption of the negative relationship between default and recovery rates, the recovery rate can be thought of as:

$$R_n = \frac{v_n}{v_n + PD_n},$$  

where $v_n \sim \Gamma(b, c)$, independent of $Z, Z_n$, changes with the different individual obligor. The recovery rate directly depends on the default probability, which results in the intuitively reasonable property that different levels of the default rates for different credit ratings influence the recovery rate.

The beta distribution has an appealing property that the random variable falls in an unit interval with a flexible shape characterized by two parameters. For this reason, the Beta distribution is a good way to describe the changes of the recovery rate.

In model (4.20), if we set $c = \overline{p}_j \cdot w_1/a$, then the recovery rate $R_n$ follows the Beta distribution with parameters $b$ and $a + a_1$ provided by the property of Gamma distribution. Thus, the expected value of recovery rate irrespective of whether the default occurs equals to

$$\frac{b}{b + (a + a_1)}.$$
With regard to the expected recovery conditional on the default events, we will discuss it in the section 4.4.2.

It has been noted that some of existing models draw recovery rate from a Beta distribution in the default, but independent of probability of default. These models are usually used to manage portfolio risk, such as computing Value at Risk, including J.P. Morgan’s CreditMetrics (Gupton, Finger and Bhatia, 1997) or McKinsey’s CreditPortfolioView (Wilson, 1997a, 1997b and 1998). In our discussion, the recovery rate is developed by a Beta distribution, moreover, it is negatively correlated with the default probability.

4.4 Implementation

4.4.1 Description of Historical Data

We examine recovery rates for corporate bonds categorized by seniority as well as for bank loans over 1982-2006. The original data source is also based on Moody’s 20th annual survey of corporate defaults and recovery rates.

Table 4.1: Average Corporate Debt Recovery Rates (1982-2006)

<table>
<thead>
<tr>
<th></th>
<th>Issuer Weighted</th>
<th>Value Weighted</th>
</tr>
</thead>
<tbody>
<tr>
<td>Secured Bank Loans</td>
<td>70.41 %</td>
<td>64.68 %</td>
</tr>
<tr>
<td>Senior Secured Bonds</td>
<td>54.44 %</td>
<td>58.70 %</td>
</tr>
<tr>
<td>Senior Unsecured Bonds</td>
<td>38.39 %</td>
<td>37.04 %</td>
</tr>
<tr>
<td>Senior Subordinated Bonds</td>
<td>32.85 %</td>
<td>29.25 %</td>
</tr>
<tr>
<td>Subordinated Bonds</td>
<td>31.61 %</td>
<td>29.54 %</td>
</tr>
<tr>
<td>Junior Subordinated Bonds</td>
<td>24.47 %</td>
<td>17.38 %</td>
</tr>
</tbody>
</table>

*Based on 30-day post-default market prices*

Table 4.1 by Moody’s shows us that recovery rates vary across different seniority and instrument types. Higher seniority implies higher recovery rate.

---

1Corporate Default and Recovery Rates 1920-2006, Global Credit Research, Moody’s Investors Service, February 2007
Table 4.2: Descriptive annual average recovery rates (1982-2006)

<table>
<thead>
<tr>
<th></th>
<th>Median</th>
<th>St.Dev.</th>
<th>99.9th Percentile</th>
</tr>
</thead>
<tbody>
<tr>
<td>All Bonds</td>
<td>44.13 %</td>
<td>9.74 %</td>
<td>59.80 %</td>
</tr>
<tr>
<td>Secured Bank Loans</td>
<td>71.57 %</td>
<td>12.65 %</td>
<td>89.08 %</td>
</tr>
<tr>
<td>Senior Secured Bonds</td>
<td>59.22 %</td>
<td>14.19 %</td>
<td>83.52 %</td>
</tr>
<tr>
<td>Senior Unsecured Bonds</td>
<td>46.15 %</td>
<td>11.30 %</td>
<td>62.75 %</td>
</tr>
<tr>
<td>Senior Subordinated Bonds</td>
<td>42.74 %</td>
<td>11.18 %</td>
<td>67.50 %</td>
</tr>
<tr>
<td>Subordinated Bonds</td>
<td>35.54 %</td>
<td>15.10 %</td>
<td>82.28 %</td>
</tr>
<tr>
<td>Junior Subordinated Bonds</td>
<td>23.72 %</td>
<td>18.88 %</td>
<td>61.88 %</td>
</tr>
</tbody>
</table>

Issuer-weighted based on 30-day post-default market prices

in average, and credit rating is also a determinant of recovery rates at default. The issuer weighted average recovery rate is derived across issuers by averaging mean recovery rates of each issuer. This type of recovery is mainly used for an issue involving well diversified portfolios across issuers. The value weighted average recovery rate is calculated by considering average recovery rates weighted by the face value of all defaulted issuers. These recovery rates usually can be applied to market portfolios. In our models, we focus on analyzing the issuer weighted average recovery rates.

We observe that in Table 4.2, the annual average recovery rates for subordinated bonds reached to 82.28% at 99.9th percentile, which is much higher than annual recovery rates for higher seniority bonds, for example, senior subordinated and senior unsecured bonds. During the sample period 1982-2006, the annual recovery rates for subordinated bonds are very volatile.

4.4.2 Estimation Process

Extended Factor Model

In the extended one-factor models involving systematic risk, there are two sets of parameters. The asset correlation $\rho$ and the threshold parameter $\overline{p}$ are needed firstly. On the side of recovery rate in the form (4.2), $\mu$, $\sigma$ and the
correlation parameter $\omega$ need to be estimated. We carry out the estimation process based on the systematic factor $Y$ inferred from the conditional default probability given in (4.1).

The empirical threshold parameter $\bar{p}$ for each rating class during the sample period 1982-2006 is displayed in Table 4.3. The estimation process of the default correlation $\rho$ is carried out under the Gaussian copula model with the Maximum Likelihood methodology as discussed in Chapter 3.

### Table 4.3: Empirical default probability and estimated default correlation (1982-2006)

<table>
<thead>
<tr>
<th></th>
<th>Aaa</th>
<th>Aa</th>
<th>A</th>
<th>Baa</th>
<th>Ba</th>
<th>B</th>
<th>Caa-C</th>
<th>All Rated</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{p}$</td>
<td>0.00%</td>
<td>0.02%</td>
<td>0.02%</td>
<td>0.21%</td>
<td>1.31%</td>
<td>6.05%</td>
<td>25.13%</td>
<td>1.58%</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.005</td>
<td>0.350</td>
<td>0.195</td>
<td>0.085</td>
<td>0.100</td>
<td>0.125</td>
<td>0.155</td>
<td>0.056</td>
</tr>
</tbody>
</table>

Comparing the default correlations in Table 4.3 with the results in Table 3.6 and Table 3.10, we observe that the estimates are relatively stable during two sample periods, except that for rating class Aa and all rated group the correlations during 1982-2006 are relatively smaller than during 1970-2006. This result is due to the big gap between empirical default probabilities in two sample periods for rating class Aa and all rated group (Table 4.3, Table 3.5 and Table 3.9).

According to the parameters in (4.2), the estimation method proposed by Düllmann and Trapp (2004) is applied. We consider the recovery rates in the defaults, if given the number of defaults $D_t$ in the time $t$, the average recovery rate can be regarded as $\sum_{i=1}^{D_t} R_{ni}/D_t$. Under the logit-normal and log-normal distributions, the recovery rate $R_{ni}$ is taken as $\ln\left(\frac{R_{ni}}{1+R_{ni}}\right)$ and $\ln(R_{ni})$. Conditional on the systematic factor $Y_t$, the random variable $X(Y_t)$ in formula (4.2) during period $t$ has the distribution of

$$X(Y_t) \sim N\left(\mu + \sigma \sqrt{\omega} \cdot Y_t, \sigma^2(1 - \omega)/D_t\right),$$

so that we have the log likelihood function $LL(\mu, \sigma, \omega; r_t, D_t)$ depending on
the observations $r_t$ and $D_t$ with respect to the relevant parameters.

For the Logit-normal distribution, the log likelihood function is given by

$$LL(\mu_1, \sigma_1, \omega_1; r_t, D_t) = \sum_{t=1}^{T} \ln \left( \frac{D_t}{2\pi \sigma_1^2 (1-\omega_1)} \right)^{1/2} r_t^2 (1-r_t)^2 \exp \left( -D_t \left( \ln \left( \frac{r_t}{1-r_t} \right) - \mu_1 - \sqrt{\omega_1} \sigma_1 Y_t \right)^2 \right),$$

(4.21)

where $D_t$ denotes the observed default count in period $t$ and $r_t$ is the historical annual average recovery rate.

In the Normal distribution (4.4), the conditional recovery rate $R(X(Y_t))$ is equal to $X(Y_t)$, and the log likelihood function is written as

$$LL(\mu_2, \sigma_2, \omega_2; r_t, D_t) = \sum_{t=1}^{T} \ln \left( \frac{D_t}{2\pi \sigma_2^2 (1-\omega_2)} \right)^{1/2} \cdot \exp \left( -D_t \left( r_t - \mu_2 - \sqrt{\omega_2} \sigma_2 Y_t \right)^2 \right).$$

(4.22)

In the third model, the recovery rate is assumed to follow log-normal distribution, which results in the following log likelihood function

$$LL(\mu_3, \sigma_3, \omega_3; r_t, D_t) = \sum_{t=1}^{T} \ln \left( \frac{D_t}{2\pi \sigma_3^2 (1-\omega_3)} \right)^{1/2} \cdot \exp \left( -D_t \left( \ln r_t - \mu_3 - \sqrt{\omega_3} \sigma_3 Y_t \right)^2 \right).$$

(4.23)

The parameters $\mu$, $\sigma$ and $\omega$ under these different assumptions can be obtained by maximizing the relevant log likelihood functions (4.21), (4.22) and (4.23).

The estimates in Table 4.4 are determined from observed annual average recovery rates $r_t$ and the historical systematic risk factor $Y_1, \ldots, Y_T$ inferred from default rates for all rated issuers. Here, in the Normal distribution, $\sigma$ is estimated from the historical volatility of annual recovery rates, while in the
Table 4.4: Parameter estimates for recovery rates under extended one-factor model

<table>
<thead>
<tr>
<th></th>
<th>Logit</th>
<th>Normal</th>
<th>Log-normal</th>
<th>Logit</th>
<th>Normal</th>
<th>Log-normal</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>All Bonds</td>
<td></td>
<td>Sr. Secured Bonds</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \mu )</td>
<td>-0.392</td>
<td>0.408</td>
<td>-0.937</td>
<td>0.299</td>
<td>0.567</td>
<td>-0.596</td>
</tr>
<tr>
<td>(Std.Err.)</td>
<td>(1.30%)</td>
<td>(0.30%)</td>
<td>(0.78%)</td>
<td>(1.86%)</td>
<td>(0.42%)</td>
<td>(0.76%)</td>
</tr>
<tr>
<td>( \omega )</td>
<td>0.342</td>
<td>0.339</td>
<td>0.344</td>
<td>0.217</td>
<td>0.225</td>
<td>0.231</td>
</tr>
<tr>
<td>(Std.Err.)</td>
<td>(2.23%)</td>
<td>(2.22%)</td>
<td>(2.23%)</td>
<td>(1.92%)</td>
<td>(1.94%)</td>
<td>(1.98%)</td>
</tr>
<tr>
<td>( \sigma_{hist} )</td>
<td>0.513</td>
<td>0.119</td>
<td>0.306</td>
<td>0.686</td>
<td>0.157</td>
<td>0.276</td>
</tr>
<tr>
<td>Exp.R</td>
<td>0.369</td>
<td>0.367</td>
<td>0.369</td>
<td>0.525</td>
<td>0.522</td>
<td>0.522</td>
</tr>
<tr>
<td>Hist. Ave.</td>
<td>-</td>
<td></td>
<td>0.544</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sr. Unsecured Bonds</td>
<td></td>
<td></td>
<td></td>
<td>Sr. Subordinated Bonds</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \mu )</td>
<td>-0.321</td>
<td>0.428</td>
<td>-0.909</td>
<td>-0.615</td>
<td>0.359</td>
<td>-1.079</td>
</tr>
<tr>
<td>(Std.Err.)</td>
<td>(1.60%)</td>
<td>(0.36%)</td>
<td>(0.98%)</td>
<td>(1.68%)</td>
<td>(0.37%)</td>
<td>(1.06%)</td>
</tr>
<tr>
<td>( \omega )</td>
<td>0.258</td>
<td>0.266</td>
<td>0.239</td>
<td>0.144</td>
<td>0.138</td>
<td>0.151</td>
</tr>
<tr>
<td>(Std.Err.)</td>
<td>(2.08%)</td>
<td>(2.09%)</td>
<td>(2.06%)</td>
<td>(1.68%)</td>
<td>(1.64%)</td>
<td>(1.73%)</td>
</tr>
<tr>
<td>( \sigma_{hist} )</td>
<td>0.561</td>
<td>0.132</td>
<td>0.321</td>
<td>0.531</td>
<td>0.120</td>
<td>0.324</td>
</tr>
<tr>
<td>Exp.R</td>
<td>0.388</td>
<td>0.388</td>
<td>0.386</td>
<td>0.334</td>
<td>0.333</td>
<td>0.332</td>
</tr>
<tr>
<td>Hist. Ave.</td>
<td>0.384</td>
<td></td>
<td></td>
<td>0.329</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\( Exp.R: \) Expected recovery rate  
\( Hist.Ave.: \) Historical average recovery rate  
\( Std.Err.: \) Standard errors

In the case of Logit-normal, \( r_t \) is substituted by \( \ln \left( \frac{r_t}{1-r_t} \right) \), and under the Log-normal assumption \( \ln(r_t) \) is used.

The parameter \( \mu \) can not be compared directly in these three cases. Expected recovery rates are calculated from the equations (4.14), (4.15) and (4.16) based on the estimated parameters. Let us look at the expected re-
covery rate in Table 4.4, there are no significant differences with the different assumptions. Comparing the estimated results with the historical average recovery rate, we observe that the modelling values are very close to the observed ones. Thus, the recovery rate in defaults can be modeled as correlated with default probability under a one factor framework.

**Direct Dependence Model**

From (4.17), there are three required parameters $w_1$, $w_2$ and $p_j$. Here, $p_j$ is regarded as the long term default probability for each rating class $j$.

In the model (4.17), conditional on the systematic factor $Z$, the expected default probability during the period $t$ is given as

$$
\mathbb{E}[PD(Z)] = p_j(w_1 \cdot Z + w_2).
$$

(4.24)

Suppose that the number of individual obligors is large enough in each period $t$, the expected default probability in (4.24) is approximately equal to expected default rate during the time $t$. For annual default rate, we have the variance of $p_j^2 \cdot w_1^2/a$.

In our implementation, we match the variance of modeled default rates to the observed value, which implies that

$$
p_j^2 \cdot w_1^2/a = \text{Var}_{hist},
$$

where $\text{Var}_{hist}$ is the historical variance of annual default rates. Taking the variance constraint together with (4.19), we have the following conditions:

$$
a = \frac{p_j^2 \cdot w_1^2}{\text{Var}_{hist}} \quad \text{and} \quad a_1 = \frac{1 - w_1}{w_1} \cdot a.
$$

(4.25)

So the problem has been simplified to estimate one parameter $w_1$ conditional on the observed default rates. Thus, the distribution of the expected conditional default probability $PD_t(Z)$ is

$$
P[PD_t(Z) \leq l|Z] = \int_0^{+\infty} \mathbb{P}[PD_t(Z) \leq l|Z = z]dF(z)
$$

(4.26)

$$
= \int_0^{+\infty} \mathbb{P}[p_j(w_1z + w_2) \leq l]\varphi(z)dz.
$$
4.4 Implementation

Since
\[ \bar{p}_j(w_1z + w_2) \leq l \iff z \leq \frac{l - \bar{p}_j \cdot w_2}{\bar{p}_j \cdot w_1}, \]
so, if we set
\[ z^* = \frac{l}{\bar{p}_j \cdot w_1} - \frac{w_2}{w_1} = \frac{l}{\bar{p}_j \cdot w_1} - \frac{1}{w_1} + 1, \]
then
\[ \mathbb{P}[PD_t(Z) \leq l | Z] = \int_0^{z^*} \varphi(z; a, 1/a) dz = \Gamma(z^*; a, 1/a) = G(w_1), \]
where \( \Gamma(z^*; a, 1/a) \) is the Gamma distribution function with parameters \( a \) and \( 1/a \). The maximum likelihood function can be written as
\[ \text{ML}(\text{DR}_t; w_1) = \prod \frac{\partial G(w_1)}{\partial w_1} = \prod \left[ \frac{1}{w_1^{a_1}} \left( 1 - \frac{l}{\bar{p}_j} \right) \cdot \varphi(z^*, a, 1/a) \right], \quad (4.27) \]
where \( \varphi(z^*; a, 1/a) \) is the Gamma density function. We maximize the function \( (4.27) \) to gain the optimal estimates \( \hat{w}_1 \) of \( w_1 \). From \( (4.25) \), the estimated values \( \hat{a} \) and \( \hat{a}_1 \) corresponding to \( a \) and \( a_1 \) can be calculated based on \( \hat{w}_1 \).

In the recovery model \( (4.20) \), the distribution parameter \( c \) of \( v_n \) is set to be \( \bar{p}_j \cdot w_1/a \) in order to ensure that the recovery rate follows the Beta distribution with \( b \) and \( a + a_1 \). The expected value of annual recovery rates in defaults conditional on the systematic factor \( Z \) is derived as
\[ \overline{R(Z)} = \mathbb{E}[R(Z)] = \int \int v_n \frac{v_n}{v_n + \bar{p}_j(w_1Z + w_2Z_n)} \varphi(v_n; b, c) \varphi(Z_n; a_1, 1/a_1) dv_n dZ_n, \]
where \( c = \bar{p}_j \cdot w_1/a \). Given the estimates \( \hat{w}_1 \), \( \hat{a} \) and \( \hat{a}_1 \), we only need to determine the shape parameter \( b \) of variable \( v_n \). Thus, the distribution of annual average recovery rate \( \overline{R(Z)} \) can be given as
\[ \mathbb{P} \left( \overline{R(Z)} \leq r | Z \right) = \int_0^{r^*} \varphi(z; \hat{a}, 1/\hat{a}) dz = \Gamma(r^*; \hat{a}, 1/\hat{a}) = H(b), \]
where \( r^* \) is obtained by solving \( (4.28) \) numerically in terms of \( b \). As a result, the likelihood function can be given as \( \prod \frac{\partial H(b)}{\partial b} \). We use numerical methods to get the optimal value of \( b \), shown in Table 4.5.
The weights \( w_1 \) and \( w_2 \) indicate the influences respectively of the business cycle and the individual obligors. If more weight is assigned to the systematic factor \( Z \) than to the idiosyncratic risk \( Z_n \), the probability of default responds more to the business cycle, which causes the strong default correlation and a thick cluster; otherwise, if a significant weight is given to the individual risk variable, the defaults among obligors behave in a relatively uncorrelated manner. Altman and et al (2001) assumed a simple 50 : 50 weighting scheme. We would rather estimate these weights than make a simple assumption. Table 4.5 exhibits the estimates of all parameters for probability of default in all rated class and recovery rate for all bonds. From the relationship of \( a_1 = (1 - w_1) \cdot a/w_1 \), we obtain \( a_1 \) equals to 0.541 based on the estimates in Table 4.5.

It is obvious that if we do not consider whether the defaults have happened, the expected recovery rate is equal to

\[
\frac{b}{b + (a + a_1)} = 42.2\%.
\]

However, the actual recovery rate is supposed to occur conditional on the default events. From the simulated scenarios, the expected value is given as 35.1\%, which is smaller than the unconditional value 42.2\%. This fact is consistent with the results that the one factor model provided. We see that the expected recovery rate under the correlated beta distribution is close to the results implied by extended one factor models (Table 4.4).

Table 4.5: Estimates of parameters under direct dependence model (all rated)

<table>
<thead>
<tr>
<th></th>
<th>( w_1 )</th>
<th>( a )</th>
<th>( b )</th>
<th>Exp.R</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimates</td>
<td>0.678</td>
<td>1.14</td>
<td>1.230</td>
<td>0.351</td>
</tr>
<tr>
<td>(Std. Err.)</td>
<td>(0.008)</td>
<td>(0.009)</td>
<td>(0.097)</td>
<td>-</td>
</tr>
</tbody>
</table>
4.4 Implementation

4.4.3 Simulation Results

Extended Factor Model

We simulate the recovery rate and the default probability simultaneously depending on the same systematic factor $Y_t$ at each time $t$ by using the recovery estimates in Table 4.4 and the default parameters for all ratings.

We compare the modeled average recovery rates at the 99.9th percentile to see how the three different distributions work on the correlated recovery rates in extreme cases. The 99.9th percentile annual average recovery rates for all bonds and senior secured bonds are very similar to the historical observations (Table 4.2), but for lower secured groups the simulated results can not arrive at the extreme values in the sampling period 1982-2006. We see that in log-normal case the values at 99.9th percentile are greatest, so as expected, the log-normal distributed recovery rates have the fatter tail compared with the other two distributed recovery rates.

The part $\sigma \sqrt{\omega}$ reflects the sensitivity of recovery rate to the systematic factor $Y$. For the normal distribution, the sensitivity $\partial R / \partial Y$ is exactly equal to $\sigma \sqrt{\omega}$; for the logit-normal, it is $\sigma \sqrt{\omega} \cdot \exp \left[ X(Y) \right] \left[ 1 + \exp \left( X(Y) \right) \right]^{-2}$; for the log-normal, it is $\sigma \sqrt{\omega} \cdot \exp \left[ X(Y) \right]$. We find that except for the normal distribution, the sensitivity is not only determined by the multiple of the two parameters, but also determined by the systematic variables.

In Table 4.6, we compare the values of $\sigma \sqrt{\omega}$ directly under the normal distribution, and then discover that the recovery rates for senior secured bonds are more sensitive to the changes of macroeconomic factors, while the business cycle has less influence on lower secured groups.

We carry out a Jarque-Bera test in order to look at which assumption of distributions can explain the recovery rates better. The Jarque-Bera test is a goodness of fit measure to test the null hypothesis that a given sample comes from a set of random variables with normal distribution. The alternative
Table 4.6: Simulation results for annual average recovery rates in percentage

<table>
<thead>
<tr>
<th></th>
<th>99.9th Percentile</th>
<th>( \sigma \cdot \sqrt{\omega} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Logit</td>
<td>Normal</td>
</tr>
<tr>
<td>All Bonds</td>
<td>57.30</td>
<td>57.68</td>
</tr>
<tr>
<td>Sr. Secured Bonds</td>
<td>77.54</td>
<td>78.83</td>
</tr>
<tr>
<td>Sr. Unsecured Bonds</td>
<td>58.51</td>
<td>59.09</td>
</tr>
<tr>
<td>Sr. Subordinated Bonds</td>
<td>48.86</td>
<td>48.60</td>
</tr>
<tr>
<td>Subordinated Bonds</td>
<td>52.60</td>
<td>52.84</td>
</tr>
</tbody>
</table>

The hypothesis is that a given sample is not drawn from a normal distribution. The Jarque-Bera test normally is performed for small samples with unknown parameters, which is the key reason why we choose this type of goodness of fit measures. Cromwell et al. (1994) gave a detailed discussion on it. This test is carried out on annual average recovery rates. Under the logit-normal distribution, recovery rates \( R \) are considered to be the transformation \( \ln\left(\frac{R}{1-R}\right) \); under the log-normal distribution, recovery rates are taken as \( \ln(R) \). The test results are exhibited in Table 4.7.

Table 4.7: Jarque-Bera test for recovery rates

<table>
<thead>
<tr>
<th></th>
<th>All Bonds</th>
<th>Sr. Secured Bonds</th>
<th>Subordinated Bonds</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Logit</td>
<td>Normal</td>
<td>Log</td>
</tr>
<tr>
<td>( h )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( p)-value</td>
<td>0.75</td>
<td>0.79</td>
<td>0.27</td>
</tr>
<tr>
<td>Jbstat</td>
<td>0.47</td>
<td>0.38</td>
<td>1.45</td>
</tr>
</tbody>
</table>

Significance Level: 5%

\( Jbstat \): the value of the Jarque-Bera test statistic

\( p\)-value: the probability of obtaining a test statistic

\( h \): logical value

\( h = 0 \) can not reject the null hypothesis at 5% significance

\( h = 1 \) reject the null hypothesis at 5% significance
In the groups of all bonds and senior secured bonds, the logical values of $h$ are zero for the three distributions. Thus, the null hypothesis that the given sample of recovery rates follows the three distributions can not be rejected with the higher $p$-values. However, for the subordinated bonds, the assumptions of logit-normal and normal distributed recovery rates are rejected. The $p$-value of 35% indicates that the assumption of log-normal distribution in this group is more adequate than the other two assumptions. We observe the empirical recovery rates for the subordinated bonds in Figure 4.5 have some extreme values relative to for the other groups, therefore, the log-normal distribution with a fatter tail may explain the recoveries better in the subordinated group. But, due to the advantage of logit-normal with the values being bounded by 0 and 1, the logit-normal distribution has been used widely in preference. Such results for the all bonds are consistent with the results obtained in Düllmann and Trapp (2004). The test results for the senior secured bonds show us that the three distributions are barely determined which one is better.

Figure 4.3, 4.4 and 4.5 present the simulated pairs for average recovery rate and default probability. The simulated average recovery rates are weighted upon default counts over each period $t$. We see that the points in subordinated figure are more scattered than in other figures since the recovery rates behave in higher volatility for subordinated bonds (Table 4.4).

Intuitively, the modeled curves replicate the trends of the historical relationship. However, it is impossible to capture extreme points especially in the group of subordinated bonds (Figure 4.5). The Figures also show us if the default rate stays at a very low level, the simulated recovery rates are below the observed values.

We estimate the recovery rates implied by models based on the systematic factors calculated from the observed default rates. The curves replicate the historical pattern very well (Figure 4.6).

The cumulative distributions of recovery rate under distinct levels of probability of default are exhibited in Figure 4.7. With the lower PDs, log-normal
distribution has the stronger effects on tails; while with the higher default probability it has no significant effects on that.

The modeled annual recovery rates are compared with the observed values in Figure 4.8. The models replicate the patterns of distribution of empirical average rates in each business cycle. The logit-normal, normal and log-normal distributions indicate very similar influences on annual recovery rates.

Figure 4.3: Comparison between simulated and empirical default rates and recovery rates (all bonds)
4.4 Implementation

We use a numerical measure to test how close the two distributions in Figure 4.3, 4.4 and 4.5 are. The two-sample Kolmogorov-Smirnov test\(^2\) is based on the null hypothesis that two sample data are drawn from the same continuous distribution. The alternative hypothesis is that the distributions of two samples are different. We apply the two-sample Kolmogorov-Smirnov test.

Figure 4.5: Comparison between simulated and empirical default rates and recovery rates (Subordinated)
Figure 4.6: Modelling annual average recovery rates conditional on systematic risk $Y$

Figure 4.7: Recovery rate cumulative distribution conditional on probability of default
test to compare the distributions of the empirical and the simulated recovery rates in order to determine if the empirical recoveries can be modeled under our discussed framework.

The results of the two-sample Kolmogorov-Smirnov test are displayed in Table 4.8. The higher p-values indicate the closer relationship between the empirical and the simulated curves in Figure 4.3, 4.4 and 4.5. For the group of all bonds, the simulated recovery rates are close to the empirical observations with the high p-values of 21%, 21% and 20%. We can not reject the null hypothesis that these two sample data come from the same distribution. There are no significant differences under the three distributions. The logit-normal and normal distributions drive a slightly closer match than the log-normal distribution. For the other two groups in Figure 4.4 and 4.5, the simulated distributions seem to be different from the empirical one. For the
4.4 Implementation

Table 4.8: Two-sample Kolmogorov-Smirnov test for one factor model

<table>
<thead>
<tr>
<th></th>
<th>All Bonds</th>
<th>Sr. Secured Bonds</th>
<th>Subordinated Bonds</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Logit</td>
<td>Normal</td>
<td>Log</td>
</tr>
<tr>
<td>h</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>p-value</td>
<td>0.21</td>
<td>0.21</td>
<td>0.20</td>
</tr>
<tr>
<td>KSstat</td>
<td>0.21</td>
<td>0.21</td>
<td>0.21</td>
</tr>
</tbody>
</table>

Significance Level: 5%

KSstat: the value of the two-sample Kolmogorov-Smirnov test statistic

p-value: the probability of obtaining a test statistic

h: logical value

h = 0 can not reject the null hypothesis at 5% significance

h = 1 reject the null hypothesis at 5% significance

subordinated bonds, the modeled curves barely replicate the observations. One of the reasons why the relationship in the groups of senior secured bonds and subordinated bonds can not be captured comes from the inconsistent data between the recovery rates and the probabilities of default. Due to the lack of historical data, we applied the default rates within the class of all rating bonds in the modelling for the different groups of recovery rates.

The Jarque-Bera test is performed based on the normal distribution with unknown parameters, while the two-sample Kolmogorov-Smirnov measure in our discussion can be treated as comparing the empirical sample with the simulated sample distributed under the estimated parameters. This explains that the p-values in all cases under the Jarque-Bera test (Table 4.7) are higher than under the two-sample Kolmogorov-Smirnov test (Table 4.8). Overall, the numerical tests confirm that the correlated recovery rates modeled under the extended one factor framework have a close match to the empirical observations within the proper groups.
Direct Dependence Model

We simulate the probability of default and the correlated recovery rate simultaneously under the direct dependence model as well as under the extended one factor model. The results exhibit that the modelling curves match the empirical observations very well.

The simulated scatter in Figure 4.9 representing the relationship between two variables is flatter than the curve generated by extended one factor models discussed in Figure 4.3. This fact indicates that at the same default levels the recovery rates modeled under the correlated Beta distribution have lower values than under the factor models. We respectively compare the cumulative probability distributions of simulated annual recovery rate and default probability with their relevant empirical observations in Figure 4.10 and Figure 4.11.

![Figure 4.9: Comparison between simulated and empirical default rates and recovery rates under Gamma-Beta distribution](image)

We use the two-sample Kolmogorov-Smirnov measure as before to test
Figure 4.10: Cumulative probability distribution of average recovery rates

Table 4.9: Two-sample Kolmogorov-Smirnov test for direct dependence model

<table>
<thead>
<tr>
<th>h</th>
<th>Recovery Rate</th>
<th>Probability of Default</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>p-value</td>
<td>KSstat</td>
</tr>
<tr>
<td>0</td>
<td>0.534</td>
<td>0.158</td>
</tr>
<tr>
<td>0</td>
<td>0.915</td>
<td>0.109</td>
</tr>
</tbody>
</table>

how the modeled recovery rates and default probabilities are close to the empirical values under the direct dependence model. The test results are shown in Table 4.9. The logical values of $h$ are zero for the recovery rates and the probabilities of default. The null hypothesis that the simulated and the empirical samples are drawn from the same distribution can not be rejected in both cases.
Comparing the $p$-value of 53.4% in Table 4.9 with the $p$-values for the group of all bonds in Table 4.8, we find that the $p$-value testing the recovery rates under the direct dependence model is more than 50% higher than the $p$-values under the one factor framework. We conclude that the new developed model gives much closer matching with the observed data and suggests a better indication for the relationship between the recoveries and the default probabilities (Figure 4.9 and Figure 4.10).

The prominently high $p$-value of 91.5% indicates that the distribution of default probability implied by Gamma model closely matches to the empirical distribution. The two-sample Kolmogorov-Smirnov testing confirms that the direct dependence model explains the relationship between recovery and default rates more adequately than the one factor model.
4.5 Summary

In this chapter, we review the development of recovery modelling, and then recall the extended one factor model to link the probability of default and the recovery rate. A Gamma-Beta function is set up to reflect their direct relationship such that the changes of recovery depend on the individual default probability rather than on the business cycle variables.

We assume that the recovery rate follows stochastic processes dependent on default probability through the macroeconomic factor inferred from the one factor model. The Logit-normal, Normal and Lognormal distributions are considered as well as described in Düllmann and Trapp (2004). We examine the more recent data from Moody’s, which is different from the original data source used in Düllmann and Trapp (2004). The closed formulae for the expected recovery rates conditional on defaults are derived. We conclude that the expected recovery rates in defaults can be estimated by the one factor model. If we do not consider whether the defaults occur, the expected recoveries are overestimated.

We use a Jarque-Bera test to compare the three distributions. In most cases, the null hypothesis that a given sample of recovery rates follows an assumed distribution can not be rejected. The log-normal distribution is the best fitting in the subordinated group. The logit-normal distribution is applied widely in reality due to its advantage of values being bounded by 0 and 1.

The two-sample Kolmogorov-Smirnov test is carried out to examine how the simulated results are close to the empirical observations under different models. The results of testing indicate that the simulated curves have a close match to the empirical ones for the group of all bonds under two different frameworks. The logit-normal and normal distributions achieve a slightly closer match than the log-normal distribution.

Comparing the test results between two frameworks, the $p$-value under the direct dependence model is more than 50% higher than the $p$-values under the one factor framework. We argue that the direct dependence model gives a
much closer match between the simulated values and the observed data. The distribution of default probability implied by Gamma model closely matches to the empirical distribution. The direct dependence model explains the relationship between default and recovery rates more adequately than the one factor model.

The direct dependence model is applied to calculate the value at risk to be compared with the existing one factor models in the next chapter.
Chapter 5

Applications

In portfolio credit risk management, some existing models treat the recovery rate in defaults as a constant. In some models, recovery rate is modeled as a stochastic variable, but independent of the probability of default. For large portfolios, the profit and loss (P&L) has the almost identical distribution under both assumptions, constants and random variables, irrespective of the assumed distributions of recovery rate. Properly incorporating correlated recovery rates into existing models is our task in this chapter. We apply the default correlation and the correlated recovery models developed in the previous chapters to test how the existing models of portfolio risk are improved upon our developed approaches in this thesis.

5.1 Theoretical VaR Models

Value at Risk (VaR) is a statistical method to measure risk based on current positions. Given a level of confidence, the worst loss that would not be exceeded over a horizon is summarized by VaR. It measures risk in a single number, which is the greatest advantage of this essential tool.

**Definition 5.1.1 (Value at Risk)** At the confidence level \( \alpha \in (0, 1) \) the Value at Risk of the portfolio is given as the number \( l \) such that the loss \( L \)
exceeds \( l \) with the probability no larger than \( 1 - \alpha \),

\[
VaR_\alpha = \inf\{l \in \mathbb{R}, \mathbb{P}(L > l) \leq 1 - \alpha\}. \tag{5.1}
\]

In terms of the distribution function \( F_L \), (5.1) can be rewritten as

\[
VaR_\alpha = \inf\{l \in \mathbb{R}, 1 - F_L(l) \leq 1 - \alpha\} = \inf\{l \in \mathbb{R}, F_L(l) \geq \alpha\}.
\]

The tail of the overall loss distribution is determined by the dependence among obligors in large credit portfolios. It is necessary that a risk measure must have proper theoretical properties to express the skewness of the credit loss distribution for a particular dependent portfolio.

Let us consider a portfolio consisting of \( n \) defaultable firms and fix a time horizon \([t, t + \Delta t]\) with corresponding indicator vector \((D^t_1, \ldots, D^t_n)\), where \( D^t_i \) is a Bernoulli random variable for obligor \( i \) at time \( t \) with values in \([0, 1]\). \( D^t_i \) indicates whether the default has occurred to obligor \( i \) before time \( t \), such that

\[
D^t_i = \begin{cases} 
1 & \text{default} \\
0 & \text{non-default} 
\end{cases}
\]

In the default case, the amount of loss induced by obligor \( i \) is set to be \( M_i \).

The aggregate loss of this credit portfolio is given as:

\[
L_t = \sum_{i=1}^{n} M_i D^t_i.
\]

Frey and McNeil (2002) summarized some standard industry models, such as CreditMetrics and CreditPortfolioView model, as Bernoulli Mixture Models.

**Definition 5.1.2 (Bernoulli Mixture Model)** Given a \( p \)-dimensional factor vector \( \Psi = (\Psi_1, \ldots, \Psi_p) \), the random vector \( D = (D_1, \ldots, D_n) \), \( p < n \), follows a Bernoulli mixture model, if there exist functions \( Q_i : \mathbb{R}^p \to [0, 1], 1 \leq i \leq n \), satisfying

\[
\mathbb{P}(D_i = 1|\Psi) = Q_i(\Psi),
\]

where \( \{D_i\} \) are independent Bernoulli random variables.
In CreditMetrics models, default is treated as occurring if the firm’s asset value drops below its liabilities. Let us consider a random vector $X = (X_1, \ldots, X_m)$ with a multivariate normal distribution. $X_i$ denotes a latent variable of the $i^{th}$ firm and is determined by macroeconomic factors. $X$ is modeled as a linear factor function with $p$-dimensions, such that

$$X_i = a_i' \Theta + \sigma_i \varepsilon_i,$$

where $\Theta$ follows a $p$-dimensional Gaussian vector $\Theta \sim N_p(0, \Omega)$, $\varepsilon_i$ are independent random variables with standard normal distribution and $\sigma_i$ and $a_i = (a_{i1}, \ldots, a_{ip})'$ are given constants.

The threshold level is defined as a vector $(T_1, \ldots, T_n)$. The obligor $i$ defaults as $X_i$ falls below the threshold $T_i$,

$$D_i = 1 \iff X_i \leq T_i.$$  

The CreditMetrics model is based on this same structural framework, but only differs in the calibration and interpretation.

In CreditMetrics model, a single deterministic threshold is substituted by a multi-state framework. The partitioned latent variables $X_i$ respectively express downgrading of credit ratings and default clustering situations. The threshold levels are defined by matching probabilities of default and credit rating transitions to the historical market data.

The heterogeneous portfolio under different default probabilities with different correlations between obligors can be modeled by the general form of Bernoulli Mixture models. To separate a large portfolio into a number of homogeneous portfolios is straightforward. Mathematically, the default indicators $\{D_i\}$ is supposed to be exchangeable, such that

$$(D_1, \ldots, D_n) = (D_{\Pi(1)}, \ldots, D_{\Pi(n)}),$$

where $(\Pi(1), \ldots, \Pi(n))$ is any permutation of $(1, \ldots, n)$. The functions $Q_i$ in definition 5.1.2 should be identical to ensure the Bernoulli Mixture model is exchangeable. In the identical case, the random variable $Q$ is set to be

$$Q := Q_i(\Psi)$$
and for $\mathbf{d} = (d_1, \ldots, d_n)$ in $\{0, 1\}^n$, the probability of default indicator conditional on the random variable $Q$, $\mathbb{P}(\mathbf{D}|Q)$, is deduced as

$$ \mathbb{P}(\mathbf{D} = \mathbf{d}|Q) = Q^{\sum_{i=1}^n d_i} (1 - Q)^{n - \sum_{i=1}^n d_i}. $$

In the particular case $D_i = 1$, the probability conditional on $Q$ is equal to $Q$. Frey and McNeil (2002) summarized the distributions of $Q$ in the homogeneous portfolios for some common industry models.

- **CreditRisk⁺**: the random variable $Q$ is assumed to be

  $$ Q = 1 - \exp(-Y), $$

  where the factor variable $Y$ is Gamma distributed with two parameters $a$ and $b$, for example, $Y \sim \Gamma(a, b)$.

- **CreditMetrics**: $Q$ is set to be equal to $\Phi(Z)$ with

  $$ Z \sim N(\mu, \sigma^2). $$

  In the model (5.2), the correlation between any asset values $X_i$ and $X_j$ is specified as

  $$ \rho = \frac{\sigma^2}{1 + \sigma^2}. $$

- **CreditPortfolioView**: $Q$ is assumed to have a logit-normal distribution,

  $$ Q = \frac{\exp(Z)}{1 + \exp(Z)}, $$

  where $Z \sim N(\mu, \sigma^2)$.

For large homogeneous portfolios, the mixture distribution of $Q$ is used to drive the distribution of the portfolio loss. Suppose a series of exposures $\{M_i\}_{i \in \mathbb{N}}$ with mean $\mu_M$, the loss of $n$ obligors in the default over the period $[t, t + \Delta t]$ is calculated by

$$ L^{(n)} = \sum_{i=1}^n M_i D_i. $$
5.2 Implementation Results

Proposition 5.1.3 The Value at Risk of a portfolio consisting of \( n \) obligors is denoted as \( \text{VaR}_\alpha(L^{(n)}) \) at \( \alpha \) confidence level. Let continuous function \( \alpha \to q_\alpha(Q) \) be the quantile function satisfying

\[
F(q_\alpha(Q) + \delta) > \alpha, \quad \delta > 0,
\]

where \( F \) is the distribution function of the variable \( Q \). Then we have the following relationship:

\[
\lim_{n \to \infty} \frac{1}{n} \text{VaR}_\alpha(L^{(n)}) = \mu_M q_\alpha(Q). \tag{5.3}
\]

Formula (5.3) has been proved in Frey and McNeil (2001). In large portfolios, the Value at Risk can be calculated approximately as

\[
\text{VaR}_\alpha(L^{(n)}) \approx n \cdot \mu_M q_\alpha(Q).
\]

5.2 Implementation Results

We run Monte Carlo simulations on a specified portfolio and compare the expected losses obtained under the following different assumptions (a)-(d). The relationship between default probability and recovery rate is treated in four ways:

(a) Recovery rate is fixed as a constant, for example, 40%, uncorrelated with default probability;

(b) Recovery rate is assumed to follow stochastic processes, uncorrelated with the default process, respectively under Logit-normal, Normal, Lognormal and Beta distributions;

(c) Stochastic recovery rate under Logit-normal, Normal and Lognormal distributions is assumed to be correlated with the probability of default;

(d) Recovery rate is assumed to directly depend on the probability of default under the Gamma-Beta assumption rather than be assumed to be linked with PD by the macroeconomic factor.
We aim to compare the expected loss among different approaches and expect to find the negative correlation between recovery rate and default probability which will drive the fat tail of risk distribution, while under assumptions (a) and (b), the risk would be underestimated relative to under (c) and (d). As discussed in section 4.3, we examine if only the systematic factor is able to capture this dependence. Value at Risk is expected to exhibit a fatter tail under the approach (d) than under the approach (c). We are going to show by how much the independent framework underestimates the risk relative to the dependent methodology.

We consider a portfolio consisting of 1000 obligors divided into different credit rating classes. For the four approaches, the same portfolio is carried out to the implementation. We use the same average default rates during 1982 and 2006 to calculate the thresholds with respect to different rating classes. The simulated outcomes are shown in Table 5.2.

In the extended one factor model, we use the parameters for default probability in Table 4.3. For recovery rate models, the estimates for the group of all bonds are applied in simulation process.

Table 5.1: Parameter estimates for each rating class under direct dependence model

<table>
<thead>
<tr>
<th></th>
<th>Aa</th>
<th>A</th>
<th>Baa</th>
<th>Ba</th>
<th>B</th>
<th>Caa-C</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1$</td>
<td>0.986</td>
<td>0.986</td>
<td>0.986</td>
<td>0.993</td>
<td>0.886</td>
<td>0.971</td>
</tr>
<tr>
<td>(Std.Err.)</td>
<td>(0.001)</td>
<td>(0.283)</td>
<td>(0.609)</td>
<td>(0.517)</td>
<td>(0.104)</td>
<td>(0.323)</td>
</tr>
<tr>
<td>$a$</td>
<td>1.14</td>
<td>1.14</td>
<td>1.14</td>
<td>1.14</td>
<td>1.14</td>
<td>1.14</td>
</tr>
<tr>
<td>(Std.Err.)</td>
<td>(0.611)</td>
<td>(0.542)</td>
<td>(0.486)</td>
<td>(0.393)</td>
<td>(0.475)</td>
<td>(0.424)</td>
</tr>
<tr>
<td>$b$</td>
<td>0.770</td>
<td>0.770</td>
<td>0.770</td>
<td>0.760</td>
<td>0.850</td>
<td>0.780</td>
</tr>
<tr>
<td>(Std.Err.)</td>
<td>(2.836)</td>
<td>(0.689)</td>
<td>(0.501)</td>
<td>(0.361)</td>
<td>(0.139)</td>
<td>(0.176)</td>
</tr>
</tbody>
</table>

In the direct dependence model, the distribution parameter $a$ of systematic factor $Z$ is selected by matching the variance of historical default rates in the class of all rated bonds. All other parameters vary within each rating
We use the same parameters for the recovery distributions under the uncorrelated and correlated assumptions, so that the expected loss can be compared directly. Table 5.2 exhibits the profit and loss of a specified portfolio at the different percentiles. Let us look at the third row in the table, it means that the expected loss of the portfolio will not exceed 145.02 with the probability of 99.9% if the recovery rate is assumed to be fixed at 40%. If we assume the recovery rate follows a log-normal distribution dependent on the default probability, the expected loss increases to 181.48, which indicates that we have the confidence of 99.9% to expect the portfolio loss to be less than 181.48.

Table 5.2: P & L comparison conditional on recovery rate distribution

<table>
<thead>
<tr>
<th>Recovery Distribution</th>
<th>Constant Fixed 40%</th>
<th>Uncorrelated Stochastic Recovery</th>
<th>Correlated Stochastic Recovery</th>
</tr>
</thead>
<tbody>
<tr>
<td>MAX</td>
<td>196.86</td>
<td>186.77</td>
<td>182.95</td>
</tr>
<tr>
<td>99.97%</td>
<td>163.98</td>
<td>163.23</td>
<td>163.26</td>
</tr>
<tr>
<td><strong>99.9%</strong></td>
<td><strong>145.02</strong></td>
<td><strong>147.68</strong></td>
<td><strong>148.12</strong></td>
</tr>
<tr>
<td>99.5%</td>
<td>115.50</td>
<td>113.87</td>
<td>114.47</td>
</tr>
<tr>
<td>99%</td>
<td>101.76</td>
<td>99.42</td>
<td>99.07</td>
</tr>
<tr>
<td>95%</td>
<td>66.60</td>
<td>66.37</td>
<td>66.52</td>
</tr>
<tr>
<td>90%</td>
<td>53.11</td>
<td>52.78</td>
<td>52.82</td>
</tr>
<tr>
<td>84%</td>
<td>43.02</td>
<td>42.59</td>
<td>42.67</td>
</tr>
<tr>
<td>50%</td>
<td>17.46</td>
<td>17.15</td>
<td>17.05</td>
</tr>
<tr>
<td>16%</td>
<td>2.04</td>
<td>1.83</td>
<td>1.73</td>
</tr>
<tr>
<td>10%</td>
<td>0.00</td>
<td>0.00</td>
<td>-0.06</td>
</tr>
<tr>
<td>5%</td>
<td>-3.78</td>
<td>-4.02</td>
<td>-4.01</td>
</tr>
<tr>
<td>1%</td>
<td>-11.28</td>
<td>-11.79</td>
<td>-11.82</td>
</tr>
<tr>
<td>0.5%</td>
<td>-14.52</td>
<td>-15.51</td>
<td>-15.52</td>
</tr>
<tr>
<td>0.1%</td>
<td>-22.26</td>
<td>-22.51</td>
<td>-22.72</td>
</tr>
<tr>
<td>0.03%</td>
<td>-27.21</td>
<td>-28.08</td>
<td>-28.19</td>
</tr>
<tr>
<td>MIN</td>
<td>-35.34</td>
<td>-37.82</td>
<td>-37.90</td>
</tr>
<tr>
<td>MEAN</td>
<td>22.54</td>
<td>22.23</td>
<td>22.25</td>
</tr>
</tbody>
</table>

It is obvious that the values increase from left to right in Table 5.2. There are no dramatic increases from (a) to (b), while the values with the correlated
phenomenon in (c) are noticeably different from other two cases (a) and (b) especially above 99.5% percentile. We can see if the recovery rate is taken as an independent stochastic variable, the uncorrelated recovery risks can be diversified in a large portfolio, so the correlation between default rate and recovery risk becomes an important factor affecting the whole portfolio risk, and can not be neglected in the computation of the expected losses.

Let us return to the last column in Table 5.2, we observe that the expected losses under the direct dependence model are increased compared to the extended one-factor models at very high percentiles. That the recovery rate directly depends on the changes of probability of default leads the changes of recovery rates not only driven by business cycles, but also driven by other factors, like credit ratings.

Figure 5.1 exhibits the cumulative distributions of profit and loss (P&L) for a specified portfolio conditional on different dependence frameworks. We find that Gamma-Beta function is able to imply higher expected losses at the high percentile than the extended one-factor models.

![Figure 5.1: Comparison of cumulative probability distribution of P&L](image-url)
5.2 Implementation Results

We compare the tail distributions of the $P\&L$ under the direct dependence and the one factor models. In Figure 5.2, $x$ is the extreme value of the expected losses, and the y axis denotes the ratio of $P(P\&L > x)$ by the Gamma-Beta distribution to $P(P\&L > x)$ by the normal distribution. We consider the extreme values $x$ that are higher than the simulated losses at the 99.9th percentile. We look at the ratio of their probabilities to compare the tail distributions driven by two different models.

The ratios increase as the expected losses increase. We see that if the extreme losses are greater than 209, the ratios are strictly greater than 1. If the greatest loss of 241.8 happens, the ratio of their probabilities arrives at 4. So the direct dependence model assigns much more probability to the occurrence of the most extreme losses than the one factor model, which is consistent with our previous results about the comparison of the correlated recovery modelling in section 4.4.3. We conclude that the fatter tail of the loss distribution can be captured by the direct dependence model.
5.3 Summary

In this chapter, we apply the models developed in Chapter 3 and Chapter 4 to analyze the portfolio loss distributions under different frameworks. We use the Monte Carlo simulations on a specified portfolio under different assumptions of the relationship between default probability and recovery rate.

The simulation outcomes show that there is little difference in $P&L$ between the case of constant recoveries and the case of independent random variables, while the values under the assumption of correlated recoveries are noticeably different from the other two cases. We draw a conclusion that if the recovery rate is taken as an independent stochastic variables, the uncorrelated recovery risks can be diversified in a large portfolio, so the correlation between default rate and recovery risk can not be neglected in the applications.

The expected losses at high percentile under the Gamma-Beta function are higher than under the extended one-factor models. By testing the tail distributions under two different frameworks, we argue that a fatter tail of loss distribution can be captured by the direct dependence model.
Chapter 6

Conclusion

In this thesis, we consider two problems of correlated defaults and correlated recoveries arising from credit derivatives and corporate bond markets, we then apply the approaches developed to portfolio risk modelling.

Firstly, we use the Gaussian and the T copulas to obtain the estimates of default correlation. The defaults in the rating class Caa-C are correlated most strongly under both copula models. Furthermore, we obtain that for very low rating classes, the T copula model behaves better than Gaussian copula, however, for other higher ratings, there is little difference between these two copulas. The stress testing results show that more default probabilities inferred by the T copula are assigned to extreme events.

The conclusions show us how the Gaussian and T copulas influence the distributions of default probability depending on credit ratings. This contribution allows us to avoid model risk inherent in the choice of different copulas and provides appropriate modelling guidelines in the fields of the credit derivatives pricing and portfolio risk management.

Secondly, we explore the extended one factor models based on different assumptions for the recovery rate distribution. As a key contribution in this research, we construct a Gamma-Beta model to demonstrate that the recovery rate directly depends on individual probability of default.

One factor models are re-examined to explore correlated recovery rates
under Logit-normal, Normal and Log-normal distributions based on the more recent data from Moody’s. Closed formulae for the expected recovery rates conditional on defaults are derived. This proves that if we do not consider whether the defaults occur, the expected recoveries are overestimated.

The Jarque-Bera test shows that, in most cases, the null hypothesis that a given sample of recovery rates follows an assumed distribution can not be rejected. The log-normal distribution is the best fitting in the subordinated group.

The two-sample Kolmogorov-Smirnov test indicates that the simulated curves have a close match to the empirical ones for the group of all bonds under two different frameworks. The logit-normal and normal distributions drive a slightly closer match than the log-normal distribution. The direct dependence model explains the relationship between recovery and default rates more adequately than the one factor model.

Lastly, we apply the developed models to portfolio loss distributions under different frameworks. The simulation outcomes show that the correlation between the default and recovery rates can not be neglected because if the recovery rate is taken as independent stochastic variables, the uncorrelated recovery risks can be diversified in a large portfolio. By testing the tail distributions under two different frameworks, we argue that a fatter tail of loss distribution can be captured by the direct dependence model.

In this thesis, for the first time the recovery rate is formulated to directly depend on individual default probability in a closed formula.

6.1 Summary

We analyze historical annual default rates within each rating class from 1920 to 2006 and average annual recovery rates across seniorities and instrument types from 1982 to 2006. The original source comes from Moody’s 20th annual survey of corporate defaults and recovery rates.

Firstly, we examine historical data statistically and find that defaults clus-
ter in three distinct regimes 1920-1940, 1940-1970 and 1970-2006. In these three regimes, the default rates vary across different rating classes. The empirical evidence further proves that default correlation plays a vital role in determining default probabilities. From the statistical descriptions on the default rates and annual average recovery rates during relevant sample period, it is easy to observe that there does exist a strong negative relationship between them. So, the descriptive analysis provides us the motivations for our modelling.

Secondly, we use the Gaussian copula and Student T copula to model the default clusters. Both copula functions are derived from one factor models and then give us the conditional probability of default depending on the systematic factor driven by different business cycles. We estimate correlation parameters for different schemes, respectively for each rating class (from Aaa to Caa-C) and for investment grade, speculative grade and all rated classes. Given calibration results, the default correlation within the rating class Caa-C is most significant under both copula models. The estimated degrees of freedom in Student T copula can reach 30 for most higher rating classes, which implies that two copulas are very similar for groups with better credit quality, while, within lower rating classes, the degree of freedom decreases to 10. Stress testing tells us that more default probabilities inferred by Student T copula are assigned to the extreme cases. The Bayes factor is used to testify two copula models. The results show that within Baa and above classes two copulas give us the very similar replications for empirical data. But for very low rating grade classes, for example, B and Caa-C, the Bayes factor strongly supports the T copula model. This outcome is due to the tail dependence property of T copula. For investment grade group, we see that the Gaussian copula models default probability well.

By exploring the role of copulas, we find that the default dependency structured by copula functions performs very well in evaluating the probability of default. Most dependent default events are captured by both Gaussian and T copulas, but for some extreme cases especially in lower rating classes,
the T copula behaves better than Gaussian copula.

Thirdly, in Chapter 4, we review the development of recovery modelling, and then recall the extended one factor model to link the probability of default and the recovery rate. A Gamma-Beta function is set up to reflect their direct relationship such that the changes of recovery depend on the individual default probability rather than on the business cycle variables.

We assume recovery rate follows a stochastic process dependent on default probability through a macroeconomic factor inferred from one factor models. Recovery rates are considered to be distributed with Logit-normal, Normal and Lognormal distributions. We apply maximum likelihood estimation to calculate the relevant parameters of recovery rates conditional on average default rates and default correlations obtained in the copula sections. Closed formulae for the expected recovery rates conditional on defaults are derived. We conclude that if we do not consider whether the defaults occur under the one factor model, the expected recoveries are overestimated.

A Jarque-Bera test is used to compare three distributions. In most cases, the null hypothesis that a given sample of recovery rates follows an assumed distribution can not be rejected. The log-normal distribution is the best fitting in the subordinated group. The logit-normal distribution is applied widely in reality due to its advantage of values being bounded by 0 and 1.

The two-sample Kolmogorov-Smirnov test is carried out to examine how the simulated results are close to the empirical observations under different models. The testing results indicate that the logit-normal and normal distributions drive a slightly closer match than the log-normal distribution. The numerical results confirm that the direct dependence model gives a much closer match between the simulated values and the observed data and suggests that the direct dependence model explains the relationship between recovery and default rates more adequately than the one factor model.

Lastly, in Chapter 5, Monte Carlo simulations on a specified portfolio have been carried out under different frameworks to detect the expected losses. We treat the relationship between default probability and recovery
rate in four ways: (a) uncorrelated with constant recovery; (b) uncorrelated with stochastic recovery; (c) correlated under an extended one factor model and (d) correlated under a direct dependence model. From the simulation outcomes, we can conclude that if the recovery rate is taken as stochastic variable, but independent of each other, the uncorrelated recovery risks can be diversified in a large portfolio, so the correlation between default rate and recovery risk can not be neglected in the applications. The expected losses at high percentile under the Gamma-Beta function are higher than under the extended one-factor models. The test of tail distribution tells us that a fatter tail of loss distribution can be captured by the direct dependence model. Thus, we contribute a new approach to associate the recovery rate with the individual probability of default.

The dynamic relationship between default risk and recovery risk is a key determinant in portfolio risk modelling and credit derivatives. The approaches discussed in the thesis will be useful in the relevant fields theoretically and practically.

6.2 Future Work

Using the copula approach, we have to face model risk and parameter risk in choosing copula functions. In many cases, the estimation of too many parameters makes application infeasible. Although a dynamic intensity-based model is able to capture the impact of the timing of defaults, the implementation for dynamic models is a challenge left to us. More work can be done by extending the methods discussed upon the availability of market data.

The direct dependence model involves a very complicated estimation procedure with respect to Gamma and Beta distributions. It is not obvious how to observe the distributions of annual average recovery rate under this framework. According to the expected recovery rate in defaults, we only give the simulation results, but do not provide the exact formula. This model needs to be developed in further research.
Appendix A

Expected Recovery Rate

In the one factor framework, the expected conditional recovery rate can be written as

$$E[R(X_n)|V_n < \Phi^{-1}(\bar{p})] = E[R(X_n) \cdot 1_{\{V_n < \Phi^{-1}(\bar{p})\}}]/\bar{p},$$  \hspace{1cm} (A.1)

where the function $1_{\{V_n < \Phi^{-1}(\bar{p})\}}$ indicates whether the default occurs.

A.1 Normal Distribution

$$E[R(X_n) \cdot 1_{\{V_n < \Phi^{-1}(\bar{p})\}}] = \frac{1}{\sigma\sqrt{2\pi}} \int_0^1 x \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} \mathbb{P}(V_n < \Phi^{-1}(\bar{p})|X_n = x) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\mu/\sigma}^{1-\mu/\sigma} (\sigma x_1 + \mu) \cdot \exp\left(-\frac{x_1^2}{2}\right) \cdot \mathbb{P}(\rho_{vx} \cdot x_1 + \sqrt{1-\rho_{vx}^2} \cdot \eta < \Phi^{-1}(\bar{p})) dx_1$$

$$= \mu \cdot \frac{1}{2\pi} \int_{-\mu/\sigma}^{1-\mu/\sigma} \exp\left(-\frac{x_1^2}{2}\right) \int_{-\infty}^{\Phi^{-1}(\bar{p})} \frac{1}{\sqrt{1-\rho_{vx}^2}} \cdot \exp\left(-\frac{(\eta - \rho_{vx} \cdot x_1)^2}{2(1-\rho_{vx}^2)}\right) d\eta dx_1$$

$$+ \frac{\sigma}{\sqrt{2\pi}} \int_{-\mu/\sigma}^{1-\mu/\sigma} x_1 \cdot \exp\left(-\frac{x_1^2}{2}\right) \cdot \Phi\left(\frac{\Phi^{-1}(\bar{p}) - \rho_{vx} \cdot x_1}{\sqrt{1-\rho_{vx}^2}}\right) dx_1$$

$$= \mu \cdot \frac{1}{2\pi \sqrt{1-\rho_{vx}^2}} \int_{-\mu/\sigma}^{1-\mu/\sigma} \int_{-\infty}^{\Phi^{-1}(\bar{p})} \exp\left(-\frac{\eta^2 - 2\rho_{vx}\eta x_1 + x_1^2}{2(1-\rho_{vx}^2)}\right) d\eta dx_1 + A$$

$$= \mu \left[\Phi_2(\Phi^{-1}(\bar{p}), (1-\mu)/\sigma, \rho_{vx}) - \Phi_2(\Phi^{-1}(\bar{p}), -\mu/\sigma, \rho_{vx})\right] + A. \hspace{1cm} (A.2)"
The notation $\Phi_2(\cdot, \cdot, \cdot)$ in (A.2) denotes the bivariate normal cumulative distribution function with $\rho_{vx}$. Then, let us consider the part $A$ in (A.2).

$$A = \frac{\sigma}{\sqrt{2\pi}} \int_{-\mu/\sigma}^{1-\mu/\sigma} \left[ -\Phi\left( \frac{\Phi^{-1}(\bar{p}) - \rho_{vx} x_1}{\sqrt{1 - \rho_{vx}^2}} \right) \right] d\left( e^{-\frac{x_1^2}{2}} \right)$$

$$= \frac{\sigma}{\sqrt{2\pi}} \left[ \frac{\rho_{vx}^2}{2\pi \sigma^2} \Phi\left( \frac{\Phi^{-1}(\bar{p}) + \rho_{vx} \mu/\sigma}{\sqrt{1 - \rho_{vx}^2}} \right) - e^{-\frac{(1-\mu)^2}{2\sigma^2}} \Phi\left( \frac{\Phi^{-1}(\bar{p}) - \rho_{vx}(1 - \mu)/\sigma}{\sqrt{1 - \rho_{vx}^2}} \right) \right]$$

$$+ \frac{\sigma}{2\pi} \left( -\frac{\rho_{vx}}{\sqrt{1 - \rho_{vx}^2}} \right) \int_{-\mu/\sigma}^{1-\mu/\sigma} \exp\left( -\frac{x_1^2}{2} \right) \exp\left( -\frac{(\Phi^{-1}(\bar{p}) - \rho_{vx} x_1)^2}{2(1 - \rho_{vx}^2)} \right) dx_1. \quad (A.3)$$

$$B = \frac{\sigma}{2\pi} \left( -\frac{\rho_{vx}}{\sqrt{1 - \rho_{vx}^2}} \right) \int_{-\mu/\sigma}^{1-\mu/\sigma} \exp\left( -\frac{\Phi^{-1}(\bar{p})^2}{2} \right) \exp\left( -\frac{(x_1 - \Phi^{-1}(\bar{p}) \cdot \rho_{vx})^2}{2(1 - \rho_{vx}^2)} \right) dx_1$$

$$= -\frac{\sigma}{\sqrt{2\pi}} \rho_{vx} \cdot \exp\left( -\frac{(\Phi^{-1}(\bar{p})^2}{2} \right) \int_{L}^{U} \frac{1}{\sqrt{2\pi}} e^{-\frac{x_2^2}{2}} dx_2$$

$$= -\frac{\sigma}{\sqrt{2\pi}} \rho_{vx} \cdot \exp\left( -\frac{(\Phi^{-1}(\bar{p})^2}{2} \right) \cdot [\Phi(U) - \Phi(L)], \quad (A.4)$$

where

$$\begin{align*}
U &= \left[ (1 - \mu)/\sigma - \Phi^{-1}(\bar{p}) \cdot \rho_{vx} \right] / \sqrt{1 - \rho_{vx}^2} \\
L &= -\frac{\mu}{\sigma} - \Phi^{-1}(\bar{p}) \cdot \rho_{vx} \right] / \sqrt{1 - \rho_{vx}^2}.
\end{align*}$$

Finally, substituting (A.4), (A.3) and (A.2) into (A.1), we can obtain the expected recovery rate conditional on the defaults under normal distribution.
A.2 Lognormal Distribution

\[ E[R(X_n) \cdot 1_{(V_n < \Phi^{-1}(\bar{p})]}] = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{0} e^x \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} \cdot \mathbb{P}(V_n < \Phi^{-1}(\bar{p})|X_n = x) dx \]

\[ = \frac{e^{\mu + \frac{\sigma^2}{2}}}{\sigma \sqrt{2\pi}} \int_{-\infty}^{0} \exp\left(\frac{x^2}{2} - \frac{(x - \mu - \sigma^2)^2}{2\sigma^2}\right) \mathbb{P}\left(\frac{\rho_{vx}(x - \mu)}{\sigma} + \sqrt{1 - \rho_{vx}^2} \cdot \eta < \Phi^{-1}(\bar{p})\right) dx \]

\[ = \exp(\mu + \frac{\sigma^2}{2}) \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\mu - \sigma}{\rho_{vx}}} \exp(\frac{-x^2}{2}) \mathbb{P}(\rho_{vx} \cdot x + \sqrt{1 - \rho_{vx}^2} \cdot \eta < \Phi^{-1}(\bar{p}) - \rho_{vx}\sigma) dx_3 \]

\[ = \exp(\mu + \frac{\sigma^2}{2}) \cdot \frac{1}{2\pi \sqrt{1 - \rho_{vx}^2}} \int_{-\infty}^{\Phi^{-1}(\bar{p}) - \rho_{vx}\sigma} \int_{-\infty}^{\Phi^{-1}(\bar{p}) - \rho_{vx}\sigma} \exp\left(-\frac{\eta^2 - 2\rho_{vx}\eta x_3 + x_3^2}{2(1 - \rho_{vx}^2)}\right) d\eta dx_3 \]

\[ = \exp(\mu + \frac{\sigma^2}{2}) \Phi_2(\Phi^{-1}(\bar{p}) - \rho_{vx} \cdot \sigma, -\frac{\mu}{\sigma} - \sigma, \rho_{vx}) \quad (A.5) \]

We substitute (A.5) into (A.1) to obtain the formula to expect the average recovery rate conditional on the defaults under log-normal distribution.
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