Interactive buckling in long thin-walled rectangular hollow section struts

Jiajia Shen, M. Ahmer Wadee*, Adam J. Sadowski

Department of Civil and Environmental Engineering, Imperial College London,
South Kensington Campus, London SW7 2AZ, UK

Abstract

An analytical model describing the nonlinear interaction between global and local buckling modes in long thin-walled rectangular hollow section struts under pure compression founded on variational principles is presented. A system of nonlinear differential and integral equations subject to boundary conditions is formulated and solved using numerical continuation techniques. For the first time, the equilibrium behaviour of such struts with different cross-section joint rigidities is highlighted with characteristically unstable interactive buckling paths and a progressive change in the local buckling wavelength. With increasing joint rigidity within the cross-section, the severity of the unstable post-buckling behaviour is shown to be mollified. The results from the analytical model are validated using a nonlinear finite element model developed within the commercial package ABAQUS and show excellent comparisons. A simplified method to calculate the local buckling load of the more compressed web undergoing global buckling and the corresponding global mode amplitude at the secondary bifurcation is also developed. Parametric studies on the effect of varying the length and cross-section aspect ratio are also presented that demonstrate the effectiveness of the currently developed models.

Keywords: Mode interaction; Global and local buckling; Variational principles; Thin-walled structures;

1. Introduction

Thin-walled plated structures are widely used in current structural engineering practice owing to their mass efficiency and relative ease of manufacture. Buckling instabilities are practically always the governing failure mode of such structures [1, 2, 3]; moreover, compression members made from slender plate elements are vulnerable to a variety of different buckling phenomena [4, 5, 6, 7, 8]. The interaction between individual modes can lead to a profound change in the post-buckling behaviour, even though these modes may

*Corresponding author

Email addresses: j.shen14@imperial.ac.uk (Jiajia Shen), a.wadee@imperial.ac.uk (M. Ahmer Wadee), a.sadowski@imperial.ac.uk (Adam J. Sadowski)
be stable when triggered in isolation. In particular, such systems can exhibit a violent destabilization after the peak load is reached \[9, 10, 11\] and, in certain parametric ranges, have been demonstrated to be highly sensitive to initial geometric imperfections \[12, 13, 14, 15, 16, 17, 18, 19\].

Early work on the interactive buckling of columns was conducted by van der Neut \[20\]. Using Koiter’s theory \[21\], van der Neut developed a relatively simplified model that comprised two load-carrying flanges and a pair of rigid webs with no longitudinal stiffness. Moreover, the webs provided simply-supported boundary conditions to the flanges. Using this model, the initial post-buckling behaviour and imperfection sensitivity were investigated. The study identified the potentially dangerous consequences of columns exhibiting interactive buckling. However, owing to increased technical complexity, the presented model assumed no interaction between the individual plate elements that make up the cross-section.

Recently, Wadee and his collaborators have developed a series of analytical models using variational principles to investigate the interactive buckling of open-section compression members, i.e. I-section struts \[10, 22\] and stiffened plates \[11\]. These analytical models show very good quantitative comparisons with FE results and existing experimental results \[9\]. Moreover, snap-backs in the response, showing sequential destabilization and re-stabilization and a progressive spreading of the local buckling mode from mid-span, known as cellular buckling \[23\], are captured well by these models, which have also been observed in physical experiments on thin-walled struts \[4\] and beams \[24\].

A number of experimental studies \[7, 25\] also exist on rectangular hollow section columns, in which an interaction between the local and the global buckling modes was observed. However, these studies primarily focused upon the ultimate capacity and the behaviour in the inelastic range; only very limited details on the actual mechanism of the mode interaction were presented.

In order to investigate how the interaction between individual plate elements affects the nonlinear interactive buckling behaviour, the aforementioned analytical models for stiffened plates and I-section struts were extended to include rotational springs at the junctions within the cross-sections \[26, 27\]. In both cases, a rapid erosion of the snap-backs in the equilibrium path was observed with the increase of the rotational rigidity at the junction and there was generally an increase in the residual post-buckling load carrying capacity.

The present work aims to investigate the global–local mode interaction in rectangular hollow section struts. In comparison with open-section members, the interaction between individual plates in a closed-section strut tends to be more significant. The principal focus currently is on how this affects the nonlinear behaviour, thus leading to a better understanding of the underlying mechanics. An analytical model describing the behaviour of a thin-walled rectangular hollow section strut with semi-rigid cross-section joints under axial compression is developed founded on variational principles. The primary aim is to analyse the interaction of global and local buckling modes for the case where global buckling is critical. A relationship describing how the cross-section joint rigidity affects the properties of the system is obtained explicitly from the developed equilibrium equations. These
equations are solved using numerical continuation techniques through the well-known software AUTO-07P [28]. The resulting equilibrium paths are presented for various different cases and potentially dangerous unstable interactive buckling is found. The analytical results show excellent comparisons throughout the post-buckling range with numerical results obtained using a nonlinear finite element (FE) model developed within the commercial package ABAQUS [29]. A simplified method to predict the local buckling load of the more compressed web undergoing global buckling and the corresponding amplitude at the secondary bifurcation is developed based on the validated analytical model. A couple of parametric studies concerning the geometric properties are also presented that also successfully validate the simplified methodology.

The current work facilitates a better understanding of how the interaction between the individual plates affects the post-buckling response of rectangular hollow section struts and in future would allow for imperfection sensitivity studies to be conducted such that more robust design guidance can be established.

2. Development of the analytical model

A thin-walled simply supported rectangular hollow section strut of length $L$, loaded by an axial force $P$ at the centroid of the cross-section is considered, as shown in Figure 1. The web depth and thickness are $d$ and $t_w$ respectively; the flange width and thickness are $b$ and $t_f$ respectively. The joints between the webs and the flanges are assumed to be semi-rigid and connected by a rotational spring with stiffness $c$. It should be stressed that when $c \to \infty$, it converges to a rigid joint; when $c \to 0$, it converges to a pinned joint. The strut material is assumed to be linearly elastic, homogeneous and isotropic with Young’s modulus $E$, Poisson’s ratio $\nu$ and shear modulus $G = E / [2(1 + \nu)]$. It is assumed that global buckling occurs about the weak axis of bending.

Figure 1: (a) Plan view of the rectangular hollow section strut of length $L$ under the concentric axial load $P$. The lateral and longitudinal coordinates are $x$ and $z$ respectively. (b) Cross-section geometry of the strut with semi-rigid joints including definitions of the rotational stiffness at junctions; the vertical axis coordinate is $y$.

are $b$ and $t_f$ respectively. The joints between the webs and the flanges are assumed to be semi-rigid and connected by a rotational spring with stiffness $c$. It should be stressed that when $c \to \infty$, it converges to a rigid joint; when $c \to 0$, it converges to a pinned joint. The strut material is assumed to be linearly elastic, homogeneous and isotropic with Young’s modulus $E$, Poisson’s ratio $\nu$ and shear modulus $G = E / [2(1 + \nu)]$. It is assumed that global buckling occurs about the weak axis of bending.
2.1. Modal descriptions

The formulation begins with the definition of both the global and the local modal displacements based on a recent purely numerical study [30]. Since previous studies [31, 32, 10, 11, 26, 27] have clearly demonstrated that it is essential to include the shear strain contributions into the total potential energy formulation to model the interactive buckling behaviour, Timoshenko beam theory is assumed currently. The global mode is decomposed into two components: a pure lateral displacement $W$ and a pure rotation of the plane sections $\theta$, see Figure 2, known as the ‘sway’ and ‘tilt’ modes [33, 31] respectively.

Figure 2: (a) Sway and tilt components of the global buckling mode bending about the weak axis $y$. (b) Out-of-plane local mode in the flanges $w_f(x, z)$ and in the more compressed web $w_{wc}(y, z)$. Also shown are the in-plane local mode in the flanges $u_f(x, z)$ and in the more compressed web $u_{wc}(y, z)$.

The global buckling lateral displacement $W$ and the corresponding rotation $\theta$ are defined by the following expressions:

$$W(z) = -q_s L \sin \left( \frac{\pi z}{L} \right), \quad \theta(z) = -q_t \pi \cos \left( \frac{\pi z}{L} \right), \quad (1)$$
where $q_s$ and $q_t$ are the generalized coordinates for the sway and tilt modes respectively. The shear strain in the flanges from global buckling is given by the following expression:

$$\gamma_{xz} = \frac{dW}{dz} - \theta = - (q_s - q_t) \pi \cos \frac{\pi z}{L}.$$  \hspace{1cm} (2)

In the current study, the focus is on the cases where global buckling occurs first and so the local displacement in the less compressed web is assumed to be zero. The local buckling mode, including out-of-plane and in-plane displacement components, shown in Figure 2(b), is defined with the following variables:

$$w_f(x, z) = f_f(x) w(z), \quad w_{wc}(y, z) = f_{wc}(y) w(z),$$  \hspace{1cm} (3)

$$u_f(x, z) = g_f(x) u(z), \quad u_{wc}(y, z) = g_{wc}(y) u(z),$$  \hspace{1cm} (4)

where $f$ and $g$ are the cross-section components for the out-of-plane and in-plane displacement components respectively; $w(z)$ and $u(z)$ are the longitudinal out-of-plane and in-plane displacement components respectively.

From an earlier numerical study [30], it was determined that the cross-section shape functions for the out-of-plane and the in-plane components, $f$ and $g$, are approximately the same. Therefore, currently, these components are in fact assumed to be the same, i.e. $g_f(x) = f_f(x)$ and $g_{wc}(y) = f_{wc}(y)$. The cross-section components, $f_f(x)$ and $f_{wc}(y)$, as shown in Figure 4(a), are estimated by applying appropriate kinematic and static boundary conditions for each plate in conjunction with the Rayleigh–Ritz method. It is assumed that $f_{wc}$ has the functional form that is derived from the conditions of a simply-supported strut (a cosine wave) and a beam under pure bending (a parabola) such that the cases for a fully pinned joint or a joint that rotates as a rigid body can be modelled, thus:

$$f_{wc} = B_0 \cos \left( \frac{\pi y}{d} \right) + (1 - B_0) \left( 1 - \frac{4y^2}{d^2} \right).$$  \hspace{1cm} (5)

For $f_f$, the functional form is derived from a beam with one end clamped and the other end simply-supported with an end moment arising from the transfer of moment at a non-pinned joint. This naturally leads to a cubic polynomial form:

$$f_f = A_0 \left( x + \frac{b}{2} \right) + A_1 \left( x + \frac{b}{2} \right)^2 + A_2 \left( x + \frac{b}{2} \right)^3.$$  \hspace{1cm} (6)

The coefficients $B_0$ in $f_{wc}$ and $A_0$, $A_1$ and $A_2$ in $f_f$ are determined by applying appropriate boundary conditions at the junctions. The form of $f_{wc}$ automatically satisfies the natural boundary conditions for the web displacement function, i.e. $f_{wc}(\pm d/2) = 0$. Since global buckling occurs first and the resulting less compressed web is assumed to have zero out-of-plane displacement, the flanges near the less compressed side also have zero out-of-plane displacement. Therefore, the junction between the less compressed web and the flanges is assumed to be rigid, as shown in Figure 3(a). At the junction between the
Figure 3: Semi-rigid joints with corresponding kinematic and static boundary conditions at the web–flange junctions. (a) Cross-section component of the local mode in the flanges $f_f(x)$ and the more compressed web $f_{wc}(y)$; the stiffness of the rotational spring at the joints is $c_\theta$. (b) Kinematic boundary condition at the junction; $\theta_f$ and $\theta_w$ are the rotations of the flange and the more compressed web at the junction respectively. (c) Equilibrium condition at the junction; $M_f$ and $M_w$ are the bending moments in the flange and the more compressed web at the junction respectively. (d) Equivalent rotational springs with stiffness $c_\theta$ attached to the more compressed web.

less compressed web and flanges, $x = -b/2$ and $y = \pm d/2$, the boundary condition for the flanges are:

$$f_f(-b/2) = f'_f(-b/2) = 0,$$

where the prime denotes differentiation with respect to $x$.

Another boundary condition can be obtained by considering moment continuity at the junction between the flanges and the more compressed web given that there is a rotational spring of stiffness $c_\theta$ present, as shown in Figure 3(b–c). Hence, the following boundary conditions need to be satisfied:

$$M_f(x = b/2) + M_{wc}(y = -d/2) = c_\theta(\theta_w - \theta_f),$$

where:

$$M_f(x = b/2) = \left[ D_f \left( \frac{\partial^2 w_f}{\partial x^2} + \nu \frac{\partial^2 w_f}{\partial z^2} \right) \right]_{x=b/2},$$

$$M_{wc}(y = -d/2) = \left[ D_w \left( \frac{\partial^2 w_{wc}}{\partial y^2} + \nu \frac{\partial^2 w_{wc}}{\partial z^2} \right) \right]_{y=-d/2},$$

$$\theta_w = \frac{df_{wc}}{dy} \bigg|_{y=-d/2}, \quad \theta_f = \frac{df_f}{dx} \bigg|_{x=b/2},$$

with $D_f = Et_f^3/[12(1 - \nu^2)]$ and $D_w = Et_w^3/[12(1 - \nu^2)]$ being the flexural rigidities of the individual flanges and webs respectively.
As for the purely pinned or rigid joint, one more boundary condition at the more compressed web and flange junction can be obtained. For the purely pinned joint case, the flanges do not buckle, hence:

$$\theta_f = f'_f(x = b/2) = 0.$$ (12)

For the case where the joint rotates as a rigid body, the rotation of the more compressed web and the less compressed web are the same, hence:

$$\theta_f = f'_f(x = b/2) = \theta_w = f'_{wc}(y = -d/2).$$ (13)

These four equations above, i.e. Eqs. (7–8), (12) and (13), can resolve the four undetermined coefficients in $f_{wc}$ and $f_f$ for the pinned and rigid joint cases respectively.

However, for the semi-rigid joint case, the fourth boundary condition cannot be obtained directly as for the pinned and rigid joint cases above. When the more compressed web buckles, both the flanges and the joint rotational springs provide the web with restraints. Therefore, by isolating the more compressed web plate, the total rotational stiffness provided by the flanges together with the rotational spring can be replaced by an equivalent rotational spring $c_{gt}$, as shown in Figure 3(d). Moreover, since the flanges and the rotational springs are effectively in series, the following standard relationship can be used:

$$\frac{1}{c_{gt}} = \frac{1}{c_f} + \frac{1}{c_l},$$ (14)

where $c_l$ is the equivalent rotational stiffness accounting for the rotational restraint provided by the flanges, as shown in Figure 4(a). In the rigid joint case, where $c_0 \to \infty$ and $c_{gt} = c_l$,

Figure 4: (a) Equivalent rotational springs with stiffness $c_f$ on the more compressed web provided by the connecting flange. (b) Cross-section component of the local mode in the flanges $f_f(x)$ and the more compressed web $f_{wc}(y)$ for the rigid joint case due to the rotation of the flange and the more compressed web at the junction, $\theta_w$. (c) Free-bodies of the more compressed web–flange junctions; $M_{fw}$ is the bending moment within the flange and the more compressed web at the junction.

the rotational stiffness $c_l$ can be determined by considering the continuity of moment and
rotation at the junctions, as shown in Figure 1(b–c), hence:

\[
\bar{M}_{fw} = c_f \bar{\theta}_w = \left[ D_w \left( \frac{\partial^2 w_{wc}}{\partial y^2} + \nu \frac{\partial^2 w_{wc}}{\partial z^2} \right) \right]_{y = -d/2},
\]

where:

\[
\bar{\theta}_w = \left. \frac{\partial w_{wc}}{\partial y} \right|_{y = -d/2} = w \left[ f'_{wc} (y = -d/2) \right],
\]

\[
\left. \frac{\partial^2 w_{wc}}{\partial y^2} \right|_{y = -d/2} = w \left[ f''_{wc} (y = -d/2) \right],
\]

\[
\left. \frac{\partial^2 w_{wc}}{\partial z^2} \right|_{y = -d/2} = \dot{\omega} \left[ f_{wc} (y = -d/2) \right] = 0.
\]

Substituting the displacement function for the more compressed web in the rigid joint case, the rotational stiffness \( c_f \) can be obtained:

\[
c_f = \frac{4D_f}{b}
\]

and an expression for the equivalent rotational spring stiffness \( c_{\theta_f} \) can be expressed thus:

\[
c_{\theta_f} = \frac{c_{\theta}}{c_{\theta}/c_f + 1}.
\]

Therefore, the final boundary condition to determine the undetermined parameters is given by the relationship:

\[
c_{\theta_f} \theta_w = \left[ D_w \left( \frac{\partial^2 w_{wc}}{\partial y^2} + \nu \frac{\partial^2 w_{wc}}{\partial z^2} \right) \right]_{y = -d/2}.
\]

Based on these conditions, the coefficients \( A_i \), where \( i = \{0, 1, 2\} \) and \( B_0 \) can be determined:

\[
A_0 = 0,
\]

\[
A_1 = -\frac{2\pi \bar{c}_\theta (\bar{c}_\theta + 2)}{b^2 \phi_c (\bar{c}_\theta + 1) \left( \pi \phi_c \phi^3_c \bar{c}_\theta - 4\phi_c \phi^3_c \bar{c}_\theta - 2\bar{c}_\theta - 2 \right)},
\]

\[
A_2 = \frac{2\pi \bar{c}_\theta (\bar{c}_\theta + 2)}{b^3 \phi_c (\bar{c}_\theta + 1) \left( \pi \phi_c \phi^3_c \bar{c}_\theta - 4\phi_c \phi^3_c \bar{c}_\theta - 2\bar{c}_\theta - 2 \right)},
\]

\[
B_0 = -\frac{2}{\pi \phi_c \phi^3_c \bar{c}_\theta - 4\phi_c \phi^3_c \bar{c}_\theta - 2\bar{c}_\theta - 2}.
\]

where \( \bar{c}_\theta = c_\theta/c_f, \phi_t = t_t/t_w \) and \( \phi_c = d/b \). It should be stressed that when \( c_\theta \rightarrow \infty \), \( f_t \) and \( f_{wc} \) both converge to the rigid joint case, i.e. \( \theta_t = f'_t (x = b/2) = \theta_w = f'_w (y = -d/2) \); when \( c_\theta \rightarrow 0 \), \( f_t \) and \( f_{wc} \) both converge to the pinned joint case, i.e. \( f_t = 0 \) and \( f_{wc} = \cos (\pi y/d) \).
2.2. Total potential energy

The total potential energy $V$ comprises the contributions from the strain energy $U$ stored from the global bending of the strut, axial and shear stresses in the whole cross-section, the local bending of the flanges and the more compressed web, the rotational springs and the work done by the external load $PE$, where $E$ is the total end-shortening.

The only contribution to the global bending strain energy $U_{b,o}$ is from the webs through the sway mode, since the membrane strain energy contributions from the flanges and webs account for the effect of global bending through the tilt mode, as shown in Figure (a). Therefore, the global bending strain energy, $U_{b,o}$, can be expressed thus:

$$U_{b,o} = 2 \int_0^L \frac{EI_w}{2} \ddot{W}^2 \, dz = EI_w \int_0^L q_z^2 \frac{\pi^4}{L^2} \sin^2 \frac{\pi z}{L} \, dz,$$

where $EI_w = Edt^3/12$ is the flexural rigidity about the local weak neutral axis of the web and dots represent differentiation with respect to $z$. The factor of 2 is included to account for both webs.

The local bending strain energy stored in both flanges and the more compressed web can be determined with the following standard expressions:

$$U_{b,fl} = D_f \int_0^L \int_{-b/2}^{b/2} \left\{ \left( \frac{\partial^2 w_f}{\partial z^2} + \frac{\partial^2 w_f}{\partial x^2} \right)^2 - 2(1 - \nu) \left[ \frac{\partial^2 w_f}{\partial z^2} \frac{\partial^2 w_f}{\partial x^2} - \left( \frac{\partial^2 w_f}{\partial z \partial x} \right)^2 \right] \right\} \, dx \, dz,$$

$$U_{b,wcl} = \frac{D_w}{2} \int_0^L \int_{-d/2}^{d/2} \left\{ \left( \frac{\partial^2 w_{wc}}{\partial z^2} + \frac{\partial^2 w_{wc}}{\partial y^2} \right)^2 \right\} \, dy \, dz$$

Since it is assumed that there is no buckling displacement in the less compressed web, it naturally follows that there is zero local bending strain energy in that element.

The membrane strain energy in the flanges $U_{m,fl}$ is derived from considering the direct strains ($\varepsilon$) and the shear strains ($\gamma$). The complete direct strain expression for the flanges can be written as:

$$\varepsilon_{z,f} = \frac{\partial u_t}{\partial z} + \frac{\partial u_t}{\partial z} + \frac{1}{2} \left( \frac{\partial w_f}{\partial z} \right)^2 - \Delta,$$

where the first term is from the global mode and $u_t = x\theta(z)$, being the ‘tilt’ in-plane displacement; the second and third terms are the local components obtained based on von Kármán plate theory; the final term is the purely in-plane compressive strain. The corresponding shear strain component can be written thus:

$$\gamma_{xx,f} = \frac{\partial u_t}{\partial x} + \frac{\partial W}{\partial z} - \theta + \frac{\partial w_f}{\partial x} \frac{\partial w_t}{\partial z}$$
From the previous numerical study [30], the transverse strain component was shown to be very small when compared with the other two components, a finding that also coincides with earlier analytical work [34], hence it is not included presently. The current nonlinear formulation is deemed sufficient since it is well-known that it accounts for local deflections that are of the same order of magnitude as the plate thickness. The complete expression for the membrane strain energy stored in both flanges can be written thus:

\[
U_{m,f} = 2 \int_0^L \int_{-t_f/2}^{t_f/2} \int_{-b/2}^{b/2} \frac{1}{2} \left( E \varepsilon_{z,f}^2 + G \gamma_{xy,f}^2 \right) \, dx \, dy \, dz. \tag{31}
\]

The membrane strain energy stored in the more compressed web also comprises direct and shear strain energy contributions. As for the less compressed web, the expression is more straightforward since it is assumed that there are no local buckling related terms. The complete expressions for the direct strain in the more and less compressed webs are:

\[
\varepsilon_{z,wc} = \varepsilon_{z,wco} - \Delta + \frac{\partial u_{wc}}{\partial z} + \frac{1}{2} \left( \frac{\partial w_{wc}}{\partial z} \right)^2, \tag{32}
\]

\[
\varepsilon_{z,wt} = \varepsilon_{z,wto} - \Delta, \tag{33}
\]

where the direct strains from the global mode, i.e. \( \varepsilon_{z,wco} \) and \( \varepsilon_{z,wto} \), can be written thus:

\[
\varepsilon_{z,wco} = -\frac{b}{2} \theta = -q_t b \pi^2 \frac{\pi}{2L} \sin \frac{\pi z}{L}, \tag{34}
\]

\[
\varepsilon_{z,wto} = +\frac{b}{2} \theta = q_t b \pi^2 \frac{\pi}{2L} \sin \frac{\pi z}{L}. \tag{35}
\]

Unlike the flanges, the shear strain within the webs only contain terms from the local mode. The shear strain in the less compressed web is zero and the shear strain in the more compressed web can be written thus:

\[
\gamma_{yz,wc} = \frac{\partial w_{wc}}{\partial y} + \frac{\partial w_{wc}}{\partial y} \frac{\partial w_{wc}}{\partial z}. \tag{36}
\]

The transverse strain is once again neglected in the current formulation for the same reasons as outlined above, thus the membrane strain energy contributions from the webs \( U_{m,wc} \) and \( U_{m,wt} \) can be given respectively thus:

\[
U_{m,wc} = \int_0^L \int_{-t_w/2}^{t_w/2} \int_{-d/2}^{d/2} \frac{1}{2} \left( E \varepsilon_{z,wc}^2 + G \gamma_{yz,wc}^2 \right) \, dy \, dx \, dz, \tag{37}
\]

\[
U_{m,wt} = \int_0^L \int_{-t_w/2}^{t_w/2} \int_{-d/2}^{d/2} \frac{1}{2} E \varepsilon_{z,wt}^2 \, dy \, dx \, dz. \tag{38}
\]

The strain energy stored in the rotational springs, \( U_{sp} \), accounting for the web–flange joints in the side of the more compressed web, is given by the following expression:

\[
U_{sp} = 2 \int_0^L \frac{1}{2} \frac{c_2}{2} \left( \left. \frac{\partial w_f}{\partial x} \right|_{x=b/2} - \left. \frac{\partial w_{wc}}{\partial y} \right|_{y=d/2} \right)^2 \, dz. \tag{39}
\]
The factor of 2 is included to account for the rotation of both corners, as shown in Figure (a).

The total end-shortening $E$ comprises components from pure squash, the global sway mode and the local in-plane displacement. Hence, the work done by the external load $P$ is given by the expression:

$$P \mathcal{E} = P \int_0^L \left( \frac{g_s \pi^2}{2} \cos^2 \frac{\pi z}{L} + \Delta - \Delta_m \right) \, dz, \quad (40)$$

with:

$$\Delta_m = \frac{\left[ 2 \{ g_t \}_x + \{ g_{wc} \}_y \right] \dot{u}}{2 (b + d)}, \quad (41)$$

where $\{ g_t \}_x$ and $\{ g_{wc} \}_y$ are definite integrals with respect to their corresponding subscript $x$ or $y$, thus:

$$\{ g_t \}_x = \int_{-b/2}^{b/2} g_t \, dx, \quad \{ g_{wc} \}_y = \int_{-d/2}^{d/2} g_{wc} \, dy. \quad (42)$$

In summary, the total potential energy $V$ can be expressed by the summation of all the strain energy terms minus the work done by the external load:

$$V = U_{b,o} + U_{m,f} + U_{m,wc} + U_{m,wt} + U_{b,fl} + U_{b,wcl} + U_{sp} - P \mathcal{E}. \quad (43)$$

2.3. Governing equations

By performing the calculus of variations on the total potential energy $V$, the governing equations of equilibrium can be obtained. The integrand of the total potential energy $V$ can be written as a Lagrangian ($\mathcal{L}$) of the form thus:

$$V = \int_0^L \mathcal{L} (\ddot{w}, \dot{w}, w, \dot{u}, u, z) \, dz. \quad (44)$$

The equilibrium states of the system can be obtained by invoking the condition that $V$ is stationary by setting the first variation of $V$, i.e. $\delta V$, to zero, where:

$$\delta V = \int_0^L \left( \frac{\partial \mathcal{L}}{\partial \ddot{w}} \delta \ddot{w} + \frac{\partial \mathcal{L}}{\partial \dot{w}} \delta \dot{w} + \frac{\partial \mathcal{L}}{\partial w} \delta w + \frac{\partial \mathcal{L}}{\partial \dot{u}} \delta \dot{u} + \frac{\partial \mathcal{L}}{\partial u} \delta u \right) \, dz. \quad (45)$$

Since $\delta \ddot{w} = d(\delta \dot{w})/dz$, $\delta \dot{w} = d(\delta w)/dz$ and $\delta \dot{u} = d(\delta u)/dz$, integration by parts allows the development of the Euler–Lagrange equations for $w$ and $u$, resulting in a fourth order nonlinear ordinary differential equation (ODE) in $w$ and a second order nonlinear ODE in
Moreover, equilibrium also requires the minimization of $V$ with respect to the generalized coordinates $q_s$, $q_t$, and $\Delta$, leading to three integral equations:

$$\frac{\partial V}{\partial q_s} = \pi^2 Gt L (q_s - q_t) + \frac{\pi^4 EI_w q_s}{L} - P \frac{\pi^2 L q_s}{2}$$

$$- 2\pi Gt \int_0^L \left[ \{ g_t \}_x u + \{ f_t f_t \}_x \tilde{w} \right] \cos \frac{\pi z}{L} \, dz = 0,$n

$$\frac{\partial V}{\partial q_t} = \frac{\pi^4 ET_w b^2 d_t}{4L} \left[ 1 + \frac{\phi_k}{3\phi_c} \right] - \pi^2 Gt L (q_s - q_t)$$

$$+ 2\pi Gt \int_0^L \left[ \{ g_t \}_x u + \{ f_t f_t \}_x \tilde{w} \right] \cos \frac{\pi z}{L} \, dz$$

$$- \frac{\pi^2 E b t w}{2L} \int_0^L \left\{ \{ g_{wc} \}_y + \frac{4\phi_k}{b} \{ x g_t \}_x \right\} \tilde{u}$$

where $K$ is the coefficient of the linear term $w$, sometimes also referred to as the foundation term, which is well known to affect the local buckling load [31]. Currently, it comprises a plate-related term $K_p$ and a spring-related term $K_s$, i.e. $K = K_p + K_s$.
\[
\begin{align*}
\partial V \over \partial \Delta &= -Et_\text{w} \int_0^L \left\{ \left( g_{w\text{c}} y + 2\phi_t \left\{ g \right\}_x \right) \dot{u} + \left[ \frac{1}{2} \left\{ f_{w\text{c}} \right\}_y + \phi_t \left\{ f \right\}_x \right] \ddot{w}^2 \right\} \sin \frac{\pi z}{L} \, dz, \\
+ 2Et_\text{w}dL\Delta \left( 1 + \frac{\phi_t}{\phi_c} \right) - PL &= 0.
\end{align*}
\]

Since the strut is an integral member, Eq. (50) provides a relationship between \( q_s \) and \( q_t \) before the local mode is triggered, i.e. when \( u = w = \dot{w} = 0 \):

\[ q_s = (1 + s) q_t, \tag{52} \]

where:

\[ s = \frac{\pi^2 E b^2}{4GL^2} \left( \frac{1}{3} + \frac{\phi_c}{\phi_t} \right). \tag{53} \]

The boundary conditions for \( w \) and \( u \) and their derivatives are for a simple-support at \( z = 0 \) and for symmetry at \( z = L/2 \):

\[ w(0) = \ddot{w}(0) = \ddot{w}(L/2) = \dot{w}(L/2) = u(L/2) = 0. \tag{54} \]

A further boundary condition can be obtained from minimizing \( V \) and it is a condition that relates to matching the in-plane strain at the ends:

\[
\begin{align*}
\dot{u}(0) \left\{ g_{w\text{c}} y + 2\phi_t \left\{ g \right\}_x \right\} + \frac{\ddot{w}^2(0)}{2} \left\{ g_{w\text{c}} f_{w\text{c}} y + 2\phi_t \left\{ g f \right\} \right\}_x \\
- \Delta \left\{ g_{w\text{c}} y + 2\phi_t \left\{ g \right\}_x \right\} + \frac{P \left\{ g_{w\text{c}} y + 2\phi_t \left\{ g \right\}_x \right\}}{2Et_\text{w} d (1 + \phi_t/\phi_c)} &= 0.
\end{align*}
\]

Linear eigenvalue analysis for the perfect column is conducted to determine the critical load for global buckling \( P^\text{C}_o \). This is achieved by considering the condition where the Hessian matrix \( V_{ij} \) is singular when \( q_s = q_t = w = u = 0 \), where:

\[
V_{ij} = \begin{bmatrix}
\frac{\partial^2 V}{\partial q_s^2} & \frac{\partial^2 V}{\partial q_s \partial q_t} \\
\frac{\partial^2 V}{\partial q_t \partial q_s} & \frac{\partial^2 V}{\partial q_t^2}
\end{bmatrix},
\tag{56}
\]

which produces the following expression:

\[
P^\text{C}_o = \frac{2\pi^2 E I_\text{w}}{L^2} + \frac{\pi^2 E t_b^3}{2 (1 + s) L^2} \left( \frac{1}{3} + \frac{\phi_c}{\phi_t} \right). \tag{57}
\]

Note that if Euler–Bernoulli bending theory had been assumed, the shear modulus \( G \rightarrow \infty \), which implies that \( s \rightarrow 0 \), and \( P^\text{C}_o \) would reduce to the classical Euler load, as would be expected.
Table 1: Geometric properties of the rectangular hollow section strut in the numerical example, selected to ensure global buckling is critical.

<table>
<thead>
<tr>
<th>Length</th>
<th>Flange width</th>
<th>Web depth</th>
<th>Flange thickness</th>
<th>Web thickness</th>
</tr>
</thead>
<tbody>
<tr>
<td>L</td>
<td>b</td>
<td>d</td>
<td>t_f</td>
<td>t_w</td>
</tr>
<tr>
<td>5250 mm</td>
<td>60 mm</td>
<td>120 mm</td>
<td>1 mm</td>
<td>1 mm</td>
</tr>
</tbody>
</table>

3. Numerical results

In this section, representative numerical examples from the analytical model with a varying rotational stiffness $c_{\theta}$ are presented. The geometric properties of the example strut are presented in Table 1. The Young’s Modulus $E$ and Poisson’s ratio $\nu$ of the material are chosen to be 210 kN/mm$^2$ and 0.3 respectively. For the case where $c_{\theta} = 0$, the theoretical buckling stresses and critical mode are presented in Table 2. The global buckling stress is calculated using Eq. (57) where $\sigma_{C} = P_{C}^o / A$ and $A$ is the total area of the cross-section. The local buckling stress is estimated by using the classical plate buckling stress $\sigma_{C, w}^l = k_p D_w \pi^2 / (d^2 t_w)$ for the webs and $\sigma_{C, f}^l = k_d D_f \pi^2 / (b^2 t_f)$ for the flanges. Since it is initially assumed that $c_{\theta} = 0$, all the plates have effectively pinned edges and the length of the strut is much larger than the width of the plate, $k_p = 4$ is adopted for the plate buckling coefficient to obtain a lower bound. With the increase of the rotational stiffness $c_{\theta}$, the local buckling load $P_{C}^l$ would also increase whereas the global buckling load $P_{C}^o$ would remain the same. Therefore, the selected geometric dimensions and material properties ensure that global buckling is always critical for any positive value of $c_{\theta}$ in the examples presented.

The system of nonlinear ordinary differential equations is solved numerically using the continuation and bifurcation software AUTO [28]. The software is not only capable of solving the nonlinear ordinary differential equations numerically, but it also maintains the intrinsic bifurcational structure of the solutions. Moreover, importantly, the software can switch between, as well as trace, different equilibrium paths, allowing the behaviour evolution of the geometrically perfect cases to be studied.

The solution strategy for using AUTO is shown diagrammatically in Figure 5. The critical buckling load $P_{C}^o$ is obtained explicitly from Eq. (57). Using the continuation method, the generalized global sway mode amplitude $q_s$ is first varied, while $P = P_{C}^o$, to obtain the secondary bifurcation points $S_i$, where the first one ($S_1 \equiv S$) pinpoints the location where interactive buckling is practically triggered. Subsequently, the second run is started at the secondary bifurcation point $S$ using the branch switching facility within the software and $P$ is varied to compute the interactive buckling path. With the increase
Figure 5: Numerical continuation procedure for determining the interactive buckling equilibrium path for perfect struts where global buckling is critical. The thicker solid line shows the actual solution path. Circles marked C and S\(i\) represent the critical and secondary bifurcation points respectively. The generalized coordinate of the sway mode at the secondary bifurcation point is defined as \(q^S\).

of \(c_\theta\), the value of \(q_s\) at the secondary bifurcation point \(q^S_s\) increases due to the higher local buckling stress for the more compressed web relative to the constant global buckling load.

However, before conducting numerical continuation in AUTO, the effects of the rotational springs on the nonlinear ODEs are investigated. The explicit spring related term in the ODEs is the coefficient of linear term \(w\), defined as \(K\) in Eq. (46). Figure 6(a) shows the relationship between the individual components of \(K\), i.e. \(K_p\) and \(K_s\) defined in Eq. (48), while varying the normalized joint rigidity \(\bar{c}_\theta\). It can be seen that the plate-related term \(K_p\) rises with the increase of the rotational spring stiffness. Moreover, it reaches a plateau with the value being the same as the rigid joint case when the normalized joint stiffness \(\bar{c}_\theta\) is close to 7. As for the spring-related term \(K_s\), it reaches a peak and then decays to zero as \(\bar{c}_\theta\) is increased further. This can be attributed to the fact that the rotation of the flange and the more compressed web at the joint are approximately the same and thus \(f'_{wc}|_{y=d/2} = f'_{wt}|_{x=b/2}\). Based on this analysis, struts with the cross-section joint stiffness \(c_\theta\) listed in Table 3 are used in the subsequent numerical study.

Table 3: Rotational stiffness \(c_\theta\) and the corresponding normalized stiffness \(\bar{c}_\theta\) values used in the numerical studies.

<table>
<thead>
<tr>
<th>(c_\theta) (Nm/m)</th>
<th>0</th>
<th>160.26</th>
<th>641.03</th>
<th>2564.10</th>
<th>∞</th>
<th>(\bar{c}_\theta)</th>
<th>0</th>
<th>0.125</th>
<th>0.5</th>
<th>2</th>
<th>∞</th>
</tr>
</thead>
</table>

Unstable post-buckling behaviours arising from the triggering of interactive buckling are observed in all example struts presented in Figure 7. Figure 7(a) shows that the severe snap-back phenomenon in the normalized load \(p = P/P^C_{o}\) versus the normalized end-shortening \(E/L\) relationship is mollified with the increasing joint rigidity \(\bar{c}_\theta\). Moreover, stiffer joints within the cross-section also lead to a higher residual post-buckling capacity. A gradual transition from highly unstable behaviour to less unstable behaviour can be
Figure 6: (a) The influence of the rotational spring stiffness on the coefficient of the linear term $w$ in Eq. 46 for the example strut with the properties listed in Table 1. The quantities $K_p$ and $K_s$ are the plate and spring related terms respectively, given in Eq. (48). (b) The normalized values of the coefficient of the linear term $\bar{K}$ and the generalized coordinate of global sway mode $\bar{q}_s^S$ versus the normalized joint rigidity $\bar{c}_\theta$.

observed. At the same load level in the post-buckling range, a higher joint rigidity case corresponds to larger global and local mode amplitudes, as shown in Figure 7(b, c). The generalized global mode amplitude to trigger the mode interaction $q_s^S$ increases significantly especially between $\bar{c}_\theta = 0$ and $\bar{c}_\theta = 0.5$. The rate of increase in $q_s^S$, however, begins to reduce significantly as $\bar{c}_\theta$ is increased further; for example, the equilibrium path of the case with $\bar{c}_\theta = 2$ is very similar to the rigid case. By normalizing $K$ and $q_s^S$ between the pinned and rigid cases, thus:

$$
\bar{K} = \frac{K - K_0}{K_R - K_0}, \quad \bar{q}_s^S = \frac{q_s^S - q_{s0}^S}{q_{sR}^S - q_{s0}^S},
$$

where $K_0$ is the coefficient of the linear term for the pinned case ($\bar{c}_\theta = 0$); $K_R$ is the coefficient of the linear term for the rigid case ($c_\theta \to \infty$); $q_{s0}^S$ is the value of $q_s^S$ for the pinned case ($\bar{c}_\theta = 0$); and $q_{sR}^S$ is the value of $q_s^S$ for the rigid case ($c_\theta \to \infty$). It can be observed in Figure 7(b) that $\bar{K}$ and $\bar{q}_s^S$ are practically identically distributed with $\bar{c}_\theta$, which is entirely logical since $K$ is known to control the local buckling load [31] and that in turn controls when the secondary bifurcation occurs.

From the solutions of the out-of-plane components of the local mode $w$, a wavelength variation is observed with the progress of interactive buckling, as shown in Figure 8. The initially localized buckling mode spreads outwards from the mid-span of the column, developing with more peaks and troughs alongside a clear reduction in wavelength as the modal amplitude becomes larger and the load drops in the post-buckling range. Since the
global buckling mode amplitude at the secondary bifurcation is relatively larger for struts with a higher joint rigidity, the local buckling profile is initially more localized at mid-span at $p = 0.995$. Moreover, the higher joint rigidity also leads to a smaller wavelength at the same load level. This is in accord with results in previous work on I-section struts [27] and on struts on elastic softening–hardening foundations [36].

Three dimensional representations of the numerical solutions with $\bar{c}_\theta = 0$, $\bar{c}_\theta = 0.5$ and the rigid case ($\bar{c}_\theta \to \infty$) at load levels $p = 0.995$ and $p = 0.790$ in the post-buckling range are shown in Figure 9. It should be emphasized that there is no buckle in the flanges for the $\bar{c}_\theta = 0$ case, as shown in Figure 9(a), owing to the lack of interaction between adjacent plates within the cross-section.
Figure 8: Numerical solutions of the normalized local out-of-plane displacement in the more compressed web $w_{wc}/t_w$ for the cases where: (a) $\bar{c}_\theta = 0$ (pinned), (b) $\bar{c}_\theta = 0.5$ and (c) $\bar{c}_\theta \to \infty$ (rigid). The left and right columns correspond to the normalized load in the post-buckling range $p = 0.995$ and $p = 0.790$ respectively. Note that the longitudinal coordinates are normalized with respect to half the strut length $z = 2z/L$ and that the buckling wavelengths are reduced at the lower load.

4. Finite element model

The commercial finite element (FE) package ABAQUS [29] has been demonstrated to be an excellent tool to conduct nonlinear post-buckling analysis, especially with its implementation of the Riks arc-length method [38]. In previous works, it has been used successfully for mode interaction problems on stainless steel I-section columns [39], stainless steel box section columns [25], linear elastic stiffened plates [26] and linear elastic I-section columns [27]. Therefore, it is implemented currently to establish a purely numerical model for the strut with semi-rigid joints with the same structural properties as the analytical model shown in Figure 1(a) together with the properties presented in Table 1. The purpose of the FE model is that it is used as a benchmark to validate the analytical model.

4.1. Element type and meshing scheme

The struts are modelled using the four-node, reduced-integration S4R shell elements [29]. Previous research studies [40, 10, 26, 25] have demonstrated that this element can model plate buckling problems with very good accuracy. Since the wavelength of the local buckling mode is much smaller than that of the global one, a meshing scheme suitable for capturing the local buckling mode naturally would be sufficiently good to capture global buckling. Moreover, to increase computational accuracy, the shape of the element should be made to be as square as possible [41]. A sensitivity analysis was conducted to find an acceptable meshing scheme that not only yields accurate results but is also computationally efficient. As demonstrated in [30], the meshing scheme of 20 elements per wavelength yields a more than satisfactory result.
Figure 9: 3D visualization of the numerical solutions from the analytical model, plotted using MATLAB. The results are shown for the post-buckling equilibrium states for the cases where: (a) $\bar{c}_\theta = 0$ (pinned), (b) $\bar{c}_\theta = 0.5$ and (c) $\bar{c}_\theta \to \infty$ (rigid) from the top to the bottom row respectively. The left and right columns correspond to the normalized load in the post-buckling range $p = 0.995$ and $p = 0.790$ respectively. Note that the deformations shown have been amplified by a factor of 5 and the longitudinal coordinate ($z$) has been scaled by a factor of 0.25, both to aid visualization. All dimensions are in millimetres.
4.2. Strut modelling

To capture the local deformation of each individual plate and the effects of the rotational stiffness at the junction, each plate is modelled separately. The nodes at the junctions of the webs and the flanges are defined and labelled separately but share the same coordinates. The translational degrees of freedom (DOFs) of these nodes are then tied together but the rotational DOFs are not, thus approximating a pinned joint. A rotational spring element, ‘SPRING2’ in the ABAQUS element library, is then introduced to connect the end nodes of each flange and web, as shown in Figure 10(a). As in the analytical model, with the varying stiffness of the spring element, cross-section joint properties ranging from pinned to rigid can be modelled.

Unlike the rotational springs in the analytical model, however, the rotational springs are discretely distributed in the FE model, as shown in Figure 10(b). To make the springs equivalent in these two cases, the following relationship is applied:

\[ k_\theta = \frac{c_\theta L}{m_l + 1}, \]

where \( c_\theta \) and \( L \) were defined earlier in the analytical formulation; \( k_\theta \) is the rigidity of an individual rotational spring in the FE model with its units being Nmm and \( m_l \) is the number of elements along the length of the strut. As for the two limiting joint cases, i.e. the pinned and rigid cases, special treatments are adopted. In the FE model, if \( k_\theta \) is set to be zero, the strut would in fact be a mechanism and hence would not satisfy static equilibrium. Therefore, \( k_\theta \) is set to be a very small nominal value of \( 10^{-6} \) Nmm such that potential kinematic mechanisms can be avoided and the pinned case is essentially satisfied numerically. As for the rigid joint case, the rotational DOFs at the junctions of the webs and flanges are tied together in the same way as the translational DOFs.

In terms of modelling the simply-supported boundary condition, a reference point is introduced at the cross-section centroid of the loaded section and linked to the loaded edges via rigid body kinematic coupling. This ensures that all the boundary and loading
conditions, defined at the reference point, are uniformly transmitted to the whole section and the pinned–roller support assumption is therefore satisfied. The translational DOFs in the x and y directions at the reference point are restrained. Moreover, symmetry boundary conditions are adopted at mid-span to reduce computational effort. The compressive axial load is applied to the column by specifying a concentrated force to the reference point in the z-direction.

4.3. Analysis procedure

It is not possible to analyse perfect post-buckling behaviour in ABAQUS directly owing to the discontinuous pitchfork bifurcation response at the initial instability. Therefore, it is necessary to transform the bifurcation problem into a continuous one by introducing an initial perturbation in the geometry [42], the analysis procedure comprising two stages. In the first stage, a linear eigenvalue analysis is performed to obtain the buckling loads and modes, some of which are introduced as a perturbation in the second stage of analysis. In the second stage, the Riks arc-length method [38] is used to conduct the post-buckling analysis with initial geometric perturbations. The scale factors for the global and local buckling modes are set to $10^{-6}L$ and $10^{-3}t_w$ respectively. These values are sufficiently small to ensure that the response essentially mimics the perfect case as far as possible, without encountering the pitchfork bifurcations that would lead to convergence problems. A study of the sensitivity to more realistic imperfection sizes is left for future work.

5. Validation and discussion

A linear eigenvalue analysis is first conducted in ABAQUS to obtain the global buckling load of the FE model with the same geometric properties as the analytical model. The global critical load is found to be $18.90 \text{kN}$, approximately $0.21\%$ smaller than the analytical solution for $P_c$ using Eq. (57). The insignificant error is postulated to be derived from the global mode displacement field assumption in Eq. (1).

In terms of the nonlinear behaviour, equilibrium paths obtained from both the analytical and FE models show excellent agreement for all rotational stiffness cases listed in Table 3 with the analytical model exhibiting a very slightly stiffer response in the advanced post-buckling range. Three typical cases, i.e. $\bar{c}_\theta = 0$ (pinned), $\bar{c}_\theta = 0.5$ and $\bar{c}_\theta \to \infty$ (rigid), are presented and discussed currently, as shown in Figure 11. From both the $p-q_s$ and $w_{wc,max}-q_s$ equilibrium diagrams, it can be observed that the values of $q_s^2$ for all cases are extremely close between the two models. As for the local–global mode relationship, the present model matches better with FE results, when compared to previous studies on I-section struts [27, 22] using the same methodology. Apart from the fact that the cross-section functions $f_t, f_{wc}, g_t$ and $g_{wc}$ have assumed forms, an additional source for the stiffer response of the analytical model is derived from the underlying assumption that the neutral axis location remains unchanged. In fact, the neutral axis would move to the less compressed web side when local buckling occurs in the more compressed web and flanges; however, it is worth emphasising that presently the errors are fundamentally small.
Figure 11: Comparison of the post-buckling equilibrium paths for the cases where: (a) \( \overline{c}_\theta = 0 \) (pinned), (b) \( \overline{c}_\theta = 0.5 \), and (c) \( \overline{c}_\theta \to \infty \) (rigid) from the analytical (solid line) and FE (dashed line) models. Graphs of the normalized load ratio \( p = P / P_C \) versus the normalized end-shortening \( \varepsilon / L \) in the first column, the generalized coordinate of the sway mode \( q_s \) in the second column, and the normalized maximum amplitude of local deflection in the more compressed web \( w_{wc,\text{max}} / t_w \) in the third column; the fourth column shows \( w_{wc,\text{max}} / t_w \) versus \( q_s \).

Figure 12 shows the evolution of the cross-section deformation at mid-span for the cases where \( \overline{c}_\theta = 0 \) (pinned), \( \overline{c}_\theta = 0.5 \), and \( \overline{c}_\theta \to \infty \) (rigid). The excellent comparisons throughout validate the effectiveness of the assumed cross-sectional shape functions. At \( \overline{c}_\theta = 0 \), there is no out-of-plane displacement in the flanges. With the increase of the joint rigidity \( \overline{c}_\theta \), the flanges bulge increasingly and finish with the rotation at the joint being equal to the more compressed web, i.e. \( \theta_f(x = b/2) = \theta_{wc}(y = d/2) \), as shown in Figure 12(c). A small difference in the more compressed web deflection can be observed between the FE and the analytical results in the advanced post-buckling range; the difference increases as \( \overline{c}_\theta \) is increased, as shown in the fourth column of Figure 12(b, c). Moreover, the discrepancy in the less compressed web is caused by the large amount of bending in that web \( \theta_e \), which is currently not included as an extra local displacement function in the analytical model.
Figure 12: Local deformation of the cross-section for the example struts at mid-span at different load levels for the cases where: (a) \( \bar{c}_\theta = 0 \) (pinned), (b) \( \bar{c}_\theta = 0.5 \), and (c) \( \bar{c}_\theta \to \infty \) (rigid) from the analytical (solid line) and FE (dashed line) models. Note that the displacements shown have been amplified by a factor of 20 to aid visualization.
Figure 13 shows the comparison for the normalized solutions of the out-of-plane displacemnt in the more compressed web $w_{wc}/t_w$ for the cases where $\bar{c}_\theta = 0$ (pinned), $\bar{c}_\theta = 0.5$, and $\bar{c}_\theta \to \infty$ (rigid), at $p = 0.950$ and $p = 0.790$ respectively. An excellent comparison is observed in all the cases, especially for the case where $\bar{c}_\theta = 0$. However, the slight error increases with the commensurate increase of the joint rigidity and the progression of interactive buckling, as described for Figure 12. In the analytical model, the profile of the local mode is assumed to be the same in the whole strut and the amplitude is allowed to vary, see Eqs. (3) and (4). This assumption holds true for the $\bar{c}_\theta = 0$ case, where the profile is always the sine function, since the flanges provide no rotational restraints to the buckled web. For the cases where $\bar{c}_\theta > 0$, the cross-section profile depends on the bending moment at the web–flange junctions, which varies along the length. Figure 12 demonstrates that the currently assumed shape function matches well with the cross-section deformation at mid-span. Therefore, the errors, although very small, become increasingly larger towards the ends, where the profile is slightly different from that at mid-span.

Since the FE package ABAQUS can output the strain energy in individual plates and the work done by load as standard, it provides an additional perspective for validating the analytical model. Figure 14 presents the comparisons between the components of the potential energy during the loading for the cases where $\bar{c}_\theta = 0$ (pinned), $\bar{c}_\theta = 0.5$ and $\bar{c}_\theta \to \infty$ (rigid). As for the energy in the analytical model, the strain energy in the flanges $U_f$ comprises the local bending energy $U_{b,f}$, given in Eq. (21), and the membrane strain energy $U_{m,f}$, given in Eq. (31). The strain energy in the more compressed web $U_{wc}$ comprises half of the total global bending energy $U_{b,g}/2$, given in Eq. (26), the local bending energy $U_{b,wcl}$, given in Eq. (28), and the membrane strain energy $U_{m,wc}$, given in Eq. (37). The strain energy in the less compressed web $U_{wt}$ comprises half of the total global bending
energy $U_{b,o}/2$, given in Eq. (26), and the membrane strain energy $U_{m,wt}$, given in Eq. (38). The work done by load term $PE$ is given in Eq. (40).

There are three individual stages that may be observed in the energy relationships versus the generalized coordinate of the sway mode $q_s$. The first stage corresponds to the purely axial deformation of the struts under compression before the buckling load is reached. The second stage is where pure global buckling is triggered and the third stage is where interactive buckling progresses with the simultaneous increase of the global and local modes. Except for the strain energy in the more compressed web, an energy reduction can be observed at the initial stage of interactive buckling. For the strain energy in the flanges and the work done by the load, the reduction corresponds to the ‘snap-back’ that features in the load–end-shortening relationship shown in Figure 7(a) for small values of $\bar{c}_\theta$. With the increase of the joint rigidity, the reduction diminishes; for the rigid joint case,
the energy reduction is essentially negligible, which perhaps explains why no snap-back is observed for that case.

The comparisons of the strain energy terms and the work done by the load between the FE and the analytical models are generally excellent for all values of \( \bar{c}_\theta \). The main source of discrepancy resides in the strain energy stored in the less compressed web \( U_{wt} \), with the relative errors being 40%, 33% and 28% at \( q_s = 0.01 \) for \( \bar{c}_\theta = 0 \) (pinned), \( \bar{c}_\theta = 0.5 \) and \( \bar{c}_\theta \to \infty \) (rigid) cases respectively. However, the proportion of the strain energy stored in the less compressed web compared to the total strain energy stored in the rectangular hollow section strut, \( U_{wt}/U \), is relatively small; for the rigid joint case, \( U_{wt}/U \approx 0.07 \) when \( q_s = 0.01 \). Therefore, the errors for the entire system due to errors from the less compressed web are in fact below 3%. Referring to Figure 7(d), at the same value of \( q_s \), the local mode amplitude is higher for smaller values of \( \bar{c}_\theta \). Therefore, any neutral axis movement would be larger for smaller values of \( \bar{c}_\theta \). This contributes to the reason why the error in the strain energy stored in the less compressed web is the largest in the effective pinned joint case at the same value of \( q_s \). The neutral axis movement due to the plate buckling and the assumed cross-section shape functions (see Figure 12) are the two principal factors that are postulated to be responsible for the small overall discrepancy in the strain energy of the more compressed and less compressed webs. All of these factors taken together lead to a very marginally stiffer response in the analytical model, but it is not particularly large and is only really significant in the far-field post-buckling range. Hence, it may be concluded that the developed analytical model has been validated and may now be exploited further.

6. Simplified approach to predicting the location of secondary bifurcation

From the numerical results, as presented in Figure 7, unstable post-buckling equilibrium paths were observed after the secondary bifurcation point and the severely unstable behaviour is somewhat mollified with the increase of \( \bar{c}_\theta \), which in turn shows an increase in the generalized coordinate of the sway mode at the secondary bifurcation point \( q_s^5 \). Earlier studies [20, 13, 4, 6, 17] on thin-walled structures susceptible to mode interaction also show that the imperfection sensitivity decreases with the relative increase of \( q_s^5 \) since the proximity of the critical and secondary bifurcation points is reduced. A simplified approach to predicting \( q_s^5 \) based on the validated analytical model is presented currently since this quantity provides a valuable indication of the potential sensitivity to imperfections.

When the strut buckles in a purely global mode, the direct strain \( \varepsilon_{z,wc} \) can be written as Eq. (32) by assuming the local buckling components are zero:

\[
\varepsilon_{z,wc} = \varepsilon_{z,wco} - \Delta,
\]

where \( \varepsilon_{z,wco} \) is obtained from Eq. (34); \( \Delta \) is also obtained by assuming the local buckling components in Eq. (51) are zero. Since the transverse strain is neglected, the compressive stress in the more compressed web \( \sigma_{z,wc} \) can be written thus:

\[
\sigma_{z,wc} = E\varepsilon_{z,wc} = -\frac{\pi^2 Eb_l}{2L} \sin \frac{\pi z}{L} - \frac{P_o^c}{2t_wd(1 + \phi_l/\phi_c)}.
\]
From the numerical results in Figure 8, the local mode is initially localized. Instead of analysing the whole web with the entire strut length, a plate element at mid-span with length \( l_e \) is isolated to compute the approximate local buckling coefficient \( k_p \), as shown in Figure 15. It is assumed that within this plate element, the axial stress is constant along the length with the value of the direct stress at mid-span. Therefore, when the direct stress in the more compressed web \( \sigma_{z,wc} \) reaches the local buckling stress, \( \sigma_{wcl}^c = k_p \pi^2 E/[12(1 - \nu^2)(d/t_w)^2] \), it may be assumed that interactive buckling will also be triggered.

Since the cross-section shape function for the more compressed web has already been obtained, with reference to Eq. 5 and Figure 3(d), the local buckling coefficient \( k_p \) may be calculated by applying minimum potential energy principles on the isolated plate element of the more compressed web. The buckled displacement field is thus assumed to be:

\[
 w_{wc}(y, z) = Q f_{wc}(y) \sin \frac{\pi z}{l_e},
\]

where \( Q \) is a new generalized coordinate representing the amplitude of the local buckling mode within the plate element shown in Figure 15.

The strain energy \( U \) in the plate element comprises two components: the strain energy stored from local buckling \( U_{b,wcl} \) and the strain energy stored in the equivalent rotational springs \( U_{sp,\theta f} \):

\[
 U_{b,wcl} = \frac{D_w}{2} \int_0^{l_e} \int_{-d/2}^{d/2} \left\{ \left( \frac{\partial^2 w_{wc}}{\partial z^2} + \frac{\partial^2 w_{wc}}{\partial y^2} \right)^2 \right. \\
 - 2(1 - \nu) \left. \left[ \frac{\partial^2 w_{wc}}{\partial z^2} \frac{\partial^2 w_{wc}}{\partial y^2} - \left( \frac{\partial^2 w_{wc}}{\partial z \partial y} \right)^2 \right] \right\} \text{d}y \text{d}z,
\]
\[
U_{sp,\theta f} = 2 \int_0^{l_e} \frac{1}{2} c_{\theta f} \left( \frac{\partial w_{wc}}{\partial y} \right)_{y=-d/2}^2 \, dz,
\]
(64)

where the expression for \(c_{\theta f}\) was presented in Eq. (20). The work done by load term is given by the following standard expression:

\[
P\Delta = \frac{\sigma_{wc}^C t_w}{2} \int_0^{l_e} \int_{-d'/2}^{d'/2} \left( \frac{\partial w_{wc}}{\partial z} \right)^2 \, dy \, dz.
\]
(65)

The total potential energy can thus be written as:

\[
V = U_{h,wc1} + U_{sp,\theta f} - P\Delta,
\]
(66)

and by setting \(\partial V/\partial Q = 0\) for equilibrium, the following expression for \(k_p\) is obtained:

\[
k_p = a_0 + a_1 \phi_t^2 + \frac{1}{\phi_t^2},
\]
(67)

where \(\phi_t = l_e/d\) with \(a_0\) and \(a_1\) being constants that are functions of \(\phi_c, \phi_t\) and \(\bar{c}_\theta\), thus:

\[
a_0 = \frac{10 \left\{ 4 \phi_c \phi_t^3 \bar{c}_\theta \left[ \phi_c \phi_t^3 \bar{c}_\theta \left( 5 \pi^2 - 48 \right) + 3 (\pi^2 - 8) (\bar{c}_\theta + 1) \right] + 3 \pi^2 (\bar{c}_\theta + 1)^2 \right\}}{4 \phi_c \phi_t^3 \bar{c}_\theta \left[ \phi_c \phi_t^3 \bar{c}_\theta \left( \pi^4 + 15 \pi^2 - 240 \right) + 15 (\pi^2 - 8) (\bar{c}_\theta + 1) \right] + 15 \pi^2 (\bar{c}_\theta + 1)^2},
\]
(68)

\[
a_1 = \frac{15 \left\{ 4 \phi_c \phi_t^3 \bar{c}_\theta \left[ \phi_c \phi_t^3 \bar{c}_\theta \left( \pi^2 - 8 \right) + (\pi^2 - 4) (\bar{c}_\theta + 1) \right] + \pi^2 (\bar{c}_\theta + 1)^2 \right\}}{4 \phi_c \phi_t^3 \bar{c}_\theta \left[ \phi_c \phi_t^3 \bar{c}_\theta \left( \pi^4 + 15 \pi^2 - 240 \right) + 15 (\pi^2 - 8) (\bar{c}_\theta + 1) \right] + 15 \pi^2 (\bar{c}_\theta + 1)^2}.
\]
(69)

Defining \(\phi_t = (a_1)^{-1/4}\), an expression for the minimum value of \(k_p\) is found:

\[
k_p = a_0 + 2 \sqrt{a_1}.
\]
(70)

By referring to the relationship between \(q_s\) and \(q_t\) given in Eq. (52), an explicit expression for the secondary bifurcation point \(q_s^S\) is duly obtained:

\[
q_s^S = \frac{2 \left( \sigma_{wc1}^C - \sigma_o^C \right) (1 + s) L}{\pi^2 E b}.
\]
(71)

7. Parametric studies

To validate the simplified approach, a couple of parametric studies are presented where the length and the cross-section aspect ratios are varied. The results from the simplified approach are compared to the full analytical model solved using numerical continuation in AUTO.
7.1. Length variation

The strut geometries have the same cross-section properties as shown in Table 1 and the joint rigidity values are $\bar{c}_\theta = \{0, 0.5, 1, \infty\}$. The length of the struts is varied from the case where global buckling is marginally critical to $L = 7200$ mm. The comparison of the generalized coordinate of the sway mode at the secondary bifurcation point $q_s^S$ between the full analytical model using numerical continuation and the simplified approach using Eq. (71) is shown in Figure 16. With the increase of the length $L$, $q_s^S$ increases; the simplified approach predicts $q_s^S$ with good accuracy and is always on the safe side. The source of difference in the prediction is derived from the fact that the simplified model assumes that the stress is constant along the length of the plate element. However, the stress distribution is effectively a combination of the uniform stress from the axial load and the superposition of the sine function from global buckling, as given in Eq. (61).
7.2. Cross-section aspect ratio variation

For the cross-section aspect ratio parametric study, the geometric properties of the struts are shown in Table 4. The cross-section aspect ratio $\phi_c$ ranges from 1 to 2.5; the

Table 4: Geometric properties of the rectangular hollow section struts in the parametric study, selected to ensure global buckling is critical. The flange width $b = 60$ mm and the wall thickness $t_t = t_w = 1$ mm throughout.

<table>
<thead>
<tr>
<th>Cross-section aspect ratio $\phi_c$</th>
<th>Web depth $d$ (mm)</th>
<th>Length $L$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>60</td>
<td>2430</td>
</tr>
<tr>
<td>1.25</td>
<td>75</td>
<td>3120</td>
</tr>
<tr>
<td>1.5</td>
<td>90</td>
<td>3830</td>
</tr>
<tr>
<td>1.75</td>
<td>105</td>
<td>4540</td>
</tr>
<tr>
<td>2</td>
<td>120</td>
<td>5250</td>
</tr>
<tr>
<td>2.5 tall</td>
<td>150</td>
<td>6700</td>
</tr>
</tbody>
</table>

width of the flange $b$ is fixed and the wall thickness is fixed and uniform throughout the cross-section. The length of each strut is selected to ensure that global buckling is marginally critical ($P_{Co}/P_{Cl} \approx 0.995$) for the pinned joint case. Since $q_s$ is related to the strut length, the current focus is on the local buckling coefficient $k_p$ at the secondary bifurcation point.

Table 5: Comparison of the local buckling coefficient $k_p$ for the more compressed web at the secondary bifurcation point from the full analytical model, $k_{p,AUTO}$, solved using numerical continuation and the approximation presented in Eq. (70) from the pinned case ($\dot{\theta} = 0$) to rigid case ($\dot{\theta} \rightarrow \infty$) for different cross-section aspect ratios.

<table>
<thead>
<tr>
<th>$\phi_c$</th>
<th>Ranges</th>
<th>$k_{p,AUTO}$</th>
<th>$k_{p,Eq}/k_{p,AUTO}$</th>
<th>Mean</th>
<th>COV</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.01 → 5.03</td>
<td>0.968 → 0.998</td>
<td>0.980</td>
<td>1.16%</td>
<td></td>
</tr>
<tr>
<td>1.25</td>
<td>4.01 → 5.32</td>
<td>0.941 → 0.998</td>
<td>0.959</td>
<td>2.22%</td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>4.01 → 5.47</td>
<td>0.936 → 0.997</td>
<td>0.956</td>
<td>2.38%</td>
<td></td>
</tr>
<tr>
<td>1.75</td>
<td>4.01 → 5.56</td>
<td>0.938 → 0.998</td>
<td>0.957</td>
<td>2.31%</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>4.01 → 5.63</td>
<td>0.941 → 1.000</td>
<td>0.960</td>
<td>2.28%</td>
<td></td>
</tr>
<tr>
<td>2.5</td>
<td>4.01 → 5.76</td>
<td>0.947 → 0.998</td>
<td>0.964</td>
<td>1.93%</td>
<td></td>
</tr>
</tbody>
</table>

$k_p$ at the secondary bifurcation point from the full analytical model solved by numerical continuation and the approximation presented in Eq. (70). In the same way, as shown in the length parameter study results, the simplified method is demonstrated to predict $k_p$ with very good accuracy yet being always on the safe side for the cases studied. Defining $k_{p,Eq}$ as the prediction of $k_p$ from the simplified method using Eq. (70) and $k_{p,AUTO}$ as the value of $k_p$ from the full analytical model, for each cross-section case, the mean value
of $k_{p, EQ}/k_{p, AUTO}$ ranges between 0.956 and 0.980 and the maximum COV (coefficient of variation) is 2.38%. With the increase of the aspect ratio $\phi_c$, an increase in $k_p$ is observed due to the rotational restraint provided by the relatively narrower flanges. Therefore, a larger cross-section aspect ratio would lead to a relatively higher post-buckling strength.

Since cases with rigid joints ($\bar{c}_g \to \infty$) and uniform thickness ($\phi_t = 1$) are most common in practice, a power series approximation for Eq. (70) can be derived to order $\phi_c^2$ for such cases:

$$k_p = 4.33 + 0.76\phi_c - 0.10\phi_c^2.$$  \hspace{1cm} (72)

The comparison between this function and Eq. (70) is shown in Figure 17 and can be seen to be practically perfect for the range shown.

![Figure 17: The relationship between the local buckling coefficient $k_p$ and the cross-section aspect ratio $\phi_c$ for the rigid joint case from the simplified method using Eq. (70) and the curve fit function given in Eq. (72).](image)

8. Concluding remarks

A nonlinear analytical model describing the interactive buckling of a thin-walled rectangular hollow section strut with varying rigidities of the web–flange joints under pure compression has been developed using variational principles. Numerical examples, focusing on cases where global buckling is critical, have been presented and validated using the FE package ABAQUS. Unstable post-buckling behaviour due to mode interaction was observed. A progressive change in the local buckling mode is identified in terms of both the wavelength and the amplitude. As far as the authors are aware, it is the first time that this has been demonstrated in rectangular hollow section struts. With the increase of the cross-section joint rigidity, a transition from highly unstable to more mildly unstable
post-buckling behaviour is observed. The excellent comparisons between the analytical and FE results validate the effectiveness of the presented methodology.

A simplified method to predict the local buckling coefficient in the more compressed web and the global buckling amplitude at the secondary bifurcation point is proposed based on the validated analytical model; it is demonstrated to be simple, yet safe and accurate for the cases studied. Work is currently being conducted to extend the model to describe the scenarios where local buckling is critical and for struts with initial geometric imperfections. The ultimate aim of this work is to provide guidance to engineers for designing thin-walled rectangular hollow section struts with geometries that are susceptible to modal interactions.

Acknowledgement

Financial support for Jiajia Shen was provided by the Imperial PhD scholarship scheme.

References


